ACTL2102 Week 5 - Poisson Process

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Poisson Process

Definition (Poisson Process)

A counting process $\{N(t) : t \geq 0\}$ is said to be a Poisson process if:

1. $N(0) = 0$,
2. it has independent increments,
3. the number of of events in any inteval of length t has Poisson distribution with mean λt, that is, for all $s, t \geq 0$

$$\mathbb{P}(N(t+s) - N(s) = n) = \frac{(\lambda t)^{-n} e^{-\lambda t}}{n!}, \quad n = 0, 1, \ldots.$$

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Setting the number of events $n := 0$, we recover the exponential distribution with parameter λ.

$\Rightarrow$ The interarrival times (times between consecutive events) are exponentially distributed with rate λ.

Recall the intuition of exponential distribution

Remember, the exponential distribution is the continuous generalisation of the geometric distribution: if in every small interval of time h, there is a λh probability of success, then the time until the first success if exponentially distributed with rate λ.

Alternative Definition

Definition (Poisson Process)

A counting process $\{N(t) : t \geq 0\}$ is said to be a Poisson process if:

1. $N(0) = 0$;
2. it has independent increments;
3. it has stationary increments;
4. for $h \to 0$;

$$\mathbb{P}(N(t+h) - N(t) = 1) = \lambda h + o(h),$$

and

$$\mathbb{P}(N(t+h) - N(t) > 1) = o(h)$$

Interrarrival times

Denote $T_1, T_2, \ldots, T_n$ the interrarival times; T_i is the time between the $i-1$th and ith event for a Poisson process with rate λt. Then, T_i are IID $\mathrm{Exp}(\lambda)$ random variables, and

$$S_n = T_1 + T_2 + \ldots T_n \sim \Gamma(n, \lambda).$$

Sum of Independent Poisson Processes

Theorem

Let $N_1(t)$ and $N_2(t)$ be independent Poisson processes with rate parameters λ_1 and λ_2 respectively. Then,

$$N(t) := N_1(t) + N_2(t)$$

is also a Poisson process, with rate $\lambda_1 + \lambda_2$.

This can be proven via convolutions or MGF, but try to think intuitively why this should be the case.

Thinning of Poisson Processes

Theorem

Let $N(t)$ be a Poisson process with rate λ, and suppose at each event it can be classified as type 1 with probability p, and as type 2 with probability $1 - p$.
Let $N_1(t)$ and $N_2(t)$ be the counting process for Type 1 and 2 events, respectively. Then, $N_1(t)$ and $N_2(t)$ are also Poisson processes, with rates $p\lambda$ and $(1 - p)\lambda$ respectively.

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This, in combination of the previous theorem, we have that $N(t) = N_1(t) + N_2(t)$. Just as we can join Poisson processes together, we can 'thin' them into smaller ones!

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This, in combination of the previous theorem, we have that $N(t) = N_1(t) + N_2(t)$. Just as we can join Poisson processes together, we can 'thin' them into smaller ones!

Again, try think of intuitively why this is the case...

Non-homogenous Poisson Process

Here, we relax the requirement that a Poisson process is homogenous - the rate at which events can occur varies with time.

Definition (Non-homogenous Poisson Process)

A counting process is said to be an non-homogenous Poisson process with intensity function $\lambda(t)$, $t \geq 0$, if

1. $N(0) = 0$;
2. it has independent increments;
3. it has unit jumps, obeying for $h \to 0$

$$\mathbb{P}(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$$

and

$$\mathbb{P}(N(t+h) - N(t) > 1) = o(h).$$

Non-homogenous Poisson process (cont.)

For a non-homogenous Poisson process $N(t)$, we define the mean-value function

$$m(t) = \int_0^t \lambda(s)\, ds.$$

Then,

$$\mathbb{P}(N(t) = n) = \frac{(m(t))^n e^{-m(t)}}{n!},$$

or more generally,

$$\mathbb{P}(N(t) - N(s) = n) = \frac{(m(t) - m(s))^n e^{-(m(t)-m(s))}}{n!}.$$

Compound Poisson Process

Definition (Compound Poisson Process)

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ, and Y_i be IID random variables independent to $N(t)$. Then,

$$X(t) := \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

is said to be a Compound Poisson process.

Heavily used in general insurance $\rightarrow$ number of claims from a portfolio is random, and each claim size is random.

Moments for Compound Poisson Process

Recall from last week's tutorial: if $X(t)$ can be written as

$$X(t) = \sum_{i=1}^{N(t)} Y_i,$$

where $N(t)$ is a Poisson process with rate λ and Y_i are IID and independent to $N(t)$, then,

$$\mathbb{E}[X(t)] = \lambda t \, \mathbb{E}[Y],$$
$$\mathrm{Var}(X(t)] = \lambda t \, \mathbb{E}[Y^2].$$