ACTL2131 1.5 - Sequences and Convergence of Random Variables

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Chebyshev's Inequality

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For *any* random variable X with mean μ and variance σ^2, we have for any $\epsilon > 0$,

$$\mathbb{P}(|X - \mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$

Setting $\epsilon = k \cdot \sigma$ gives the more common form,

$$\mathbb{P}(|X - \mu| > k \cdot \sigma) \leq \frac{1}{k^2}.$$

Proof (not necessary at all)

Proof.
We prove the latter form given in the previous slide.

$$
\begin{aligned}
\sigma^2 &= \mathbb{E}[|X-\mu|^2] \\
&= \int_{-\infty}^{\infty} |x-\mu|^2 \, f(x)\, dx \\
&= \int_{|x-\mu|<k\sigma} |x-\mu|^2 \, f(x)\, dx + \int_{|x-\mu|\geq k\sigma} |x-\mu|^2 \, f(x)\, dx \\
&\geq \int_{|x-\mu|\geq k\sigma} |x-\mu|^2 \, f(x)\, dx \\
&\geq \int_{|x-\mu|\geq k\sigma} k^2\sigma^2 \, f(x)\, dx \quad \text{using inequality in bound,} \\
&= k^2\sigma^2 \cdot \mathbb{P}(|X-\mu| \geq k\sigma).
\end{aligned}
$$

Rearranging and letting $k = \epsilon/\sigma$ gives the result. $\square$

Convergence

From here, we discuss the behaviour of a sequence of random variables, denoted $X_1, X_2, \ldots, X_n$. We denote the ith element of the sequence X_i.

Convergence in Probability

Convergence in Probability

we say X_n *converges in probability* to the random variable X as $n \to \infty$ iff, for every $\epsilon > 0$,

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0, \qquad \text{as } n \to \infty,$$

and we write $X_n \xrightarrow{p} X$ as $n \to \infty$.

In English, as we observe more and more of our sequence X_n, it will approach a certain value X.

Law of Large Numbers

Let $X_1, X_2, \ldots, X_n$ be independent random variables with common mean and variance, so $\mathbb{E}[X_k] = \mu$ and $\mathrm{Var}(X_k) = \sigma^2$ for $k \in \mathbb{Z}^+$. Then, the sample mean converges in probability to μ, that is,

$$\overline{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k \xrightarrow{p} \mu.$$

This is very intuitive and even more powerful!

Convergence in Distribution

We say X_n converges in distribution to the random variable X as $n \to \infty$ iff, for every x,

$$F_{X_n}(x) \to F_X(x), \quad n \to \infty,$$

and we write $X_n \xrightarrow{d} X$

Again in English, as we observe more and more of our sequence, it will start to follow a particular distribution.

Remark: It doesn't have to be the CDF particularly - we could equivalently show the PDF or the MGF converges to that of a particular distribution.

Motivating the CLT

Let $X \sim \mathrm{Gamma}(1, 1)$, and consider an experiment where we generate an n size sample and compute the sample mean, $\bar{X}$.

If we do this process, say, $1{,}000$ times and plot our results, what will it look like?

Sample Means, $\bar{X}$, $n = 10{,}000$

$X \sim \text{Gamma}(1, 1)$, $n = 10{,}000$

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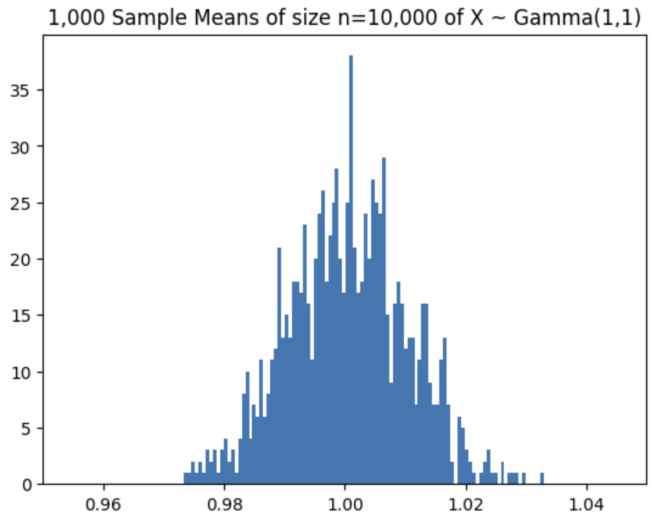

Sample Means, $\bar{X}$, $n = 1{,}000{,}000$

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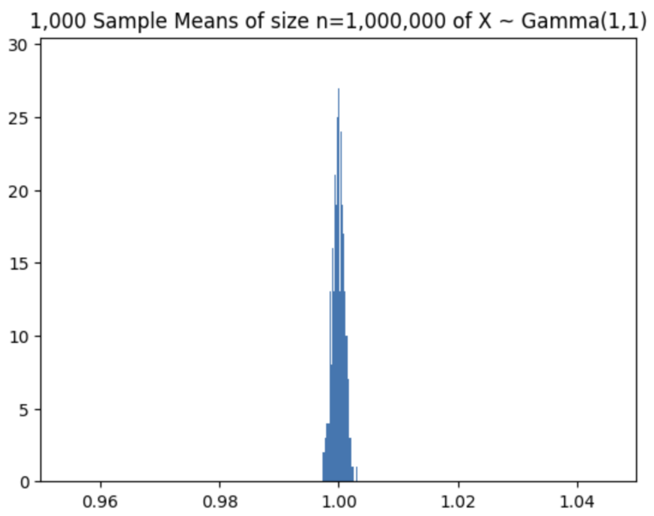


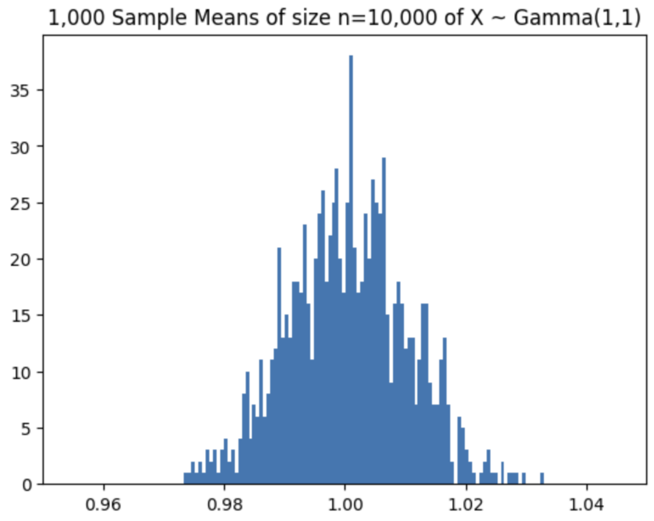
Motivating CLT (cont.)

OK! So increasing the sample size reduces the variance of the sample mean.

But what about changing the variance of the actual random variable itself?

Sample Means, $\bar{X}$, $n = 10{,}000$

$X \sim \mathrm{Gamma}(1,1)$, $n = 10{,}000$ $\qquad \mathbb{E}[X] = 1, \mathsf{Var}(X) = 1.$

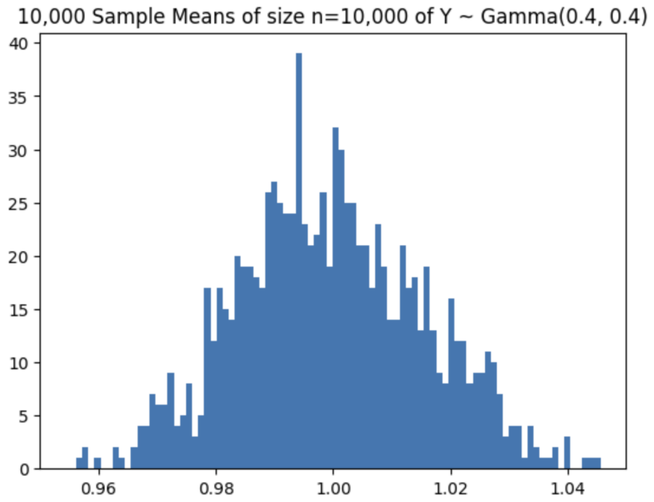


Sample Means, $\bar{Y}$, $n = 10{,}000$

$Y \sim \mathrm{Gamma}(0.4, 0.4), n = 10{,}000 \qquad \mathbb{E}[Y] = 1, \mathsf{Var}(Y) = 2.5$

Sample Means, $\bar{Y}$, $n = 10{,}000$

$Y \sim \mathrm{Gamma}(0.4, 0.4), n = 10{,}000 \qquad \mathbb{E}[Y] = 1, \mathrm{Var}(Y) = 2.5$



Motivating CLT (cont.)

It should be clear that the distribution of $\bar{X}$ must,

- Have mean $\mathbb{E}[X]$,
- Have variance proportional to $\text{Var}(X)$,
- The variance should vanish to 0 as n becomes large,
- The PDF has a familiar bell curve shape...

Central Limit Theorem

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be *independent, identically distributed* (iid) random variables with **finite** mean and variance, μ and σ^2. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then,

$$\bar{X}_n \xrightarrow{d} \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right).$$

Further,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1).$$

Central Limit Theorem (cont.)

It is easy to show (MGF technique), that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $nX \sim \mathcal{N}(n\mu, n^2\sigma^2)$.

Central Limit Theorem (cont.)

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Corollary

Let $X_1, X_2, \ldots, X_n$ be *independent, identically distributed* (iid) random variables with **finite** mean and variance, μ and σ^2. Then,

$$n \cdot \bar{X}_n = \sum_{i=1}^{n} X_i \xrightarrow{d} \mathcal{N}(n\mu, n\sigma^2).$$

Tutorial Questions

1.4.9, 1.5.2, 1.5.7, 1.5.3 (hint: use Chebyshev)