ACTL2131 1.1 - Mathematical Methods

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Welcome to ACTL2131!

My name is Tadhg! (tie - gh) (or just T).

Advice for this course:

- Please please **please** keep up!
- That's it!

Also ask many questions on the forum - we are trying to make this the most active forum ever!

There is tutorial participation (5%) - just contribute in groups to get this (7/9).

Probability Theory and Conditional Probability

The sample space Ω is the set of all outcomes in a random experiment.

A probability measure is a function $\mathbb{P}$ that maps subsets of Ω to $[0,1]$. We call $E \subseteq \Omega$ an *event* and denote $\mathcal{F}$ a collection of all events.

- For any event E_i, $0 \leq \mathbb{P}(E_i) \leq 1$.
- $\mathbb{P}(\Omega) = 1$.
- For any events $E_1, E_2, \ldots$ such that $E_i \cap E_j = \phi$ whenever $i \neq j$, (mutually disjoint) we have

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k).$$

The **conditional probability** of A given B is:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Independence

Independence

We say two events A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

or equivalently,

$$\mathbb{P}(A \mid B) = \mathbb{P}(A).$$

In words, two events are independent if the outcome of one doesn't impact the other!

Law of Total Probability

LOTP

If $E_1, E_2, \ldots$ are *mutually disjoint* events and $A \in \mathcal{F}$, then

$$\mathbb{P}(A) = \sum_{k=1}^{\infty} \mathbb{P}(A \cap E_k) = \sum_{k=1}^{\infty} \mathbb{P}(A \mid E_k) \cdot \mathbb{P}(E_k).$$

The latter expression is almost always the one being used.

We can invoke LOTP if we can draw a tree diagram for the problem we are interested in. In fact, LOTP is just summing the branches of this tree diagram.

Bayes' Theorem

Theorem

Suppose E_1, $E_2, \ldots$ partition Ω and let $A \in \mathcal{F}$, then

$$\mathbb{P}(E_k \mid A) = \frac{\mathbb{P}(A \mid E_k) \cdot \mathbb{P}(E_k)}{\sum_{j=1}^{\infty} \mathbb{P}(A \mid E_j) \cdot \mathbb{P}(E_j)}, \quad \forall k \in \mathbb{Z}^+.$$

We can use this theorem if we can draw a tree diagram.
This theorem is conditional probability and LOTP together!

Random Variables and Distributions

A random variable X is a quantity whose value depends on the outcome of a random experiment. It can be discrete or continuous. We define the Cumulative Distribution Function (CDF) for a R.V. X to be

$$F_X(x) = \mathbb{P}(X \leq x).$$

For continuous X, the Probability Density Function (PDF) is

$$f_X(x) = \frac{\partial}{\partial x} F_X(x),$$

and for discrete X, the Probability Mass Function is

$$p_X(x_k) = \mathbb{P}(X = x_k) = F(x_k) - F(x_{k-1}).$$

There are lots of properties, but the most important one is:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1, \quad \sum_{k=0}^{\infty} p_X(x_k) = 1.$$

Moments

Recall that $\mathbb{E}[X] = \mu_X$ is the **expected value of** X, and in general,

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x)\, f_X(x)\, dx, \quad \mathbb{E}[h(X)] = \sum_k h(x_k)\, p_X(x_k).$$

Further,

- $\mathbb{E}[X^k]$ - kth non-central moment.
- $\mathbb{E}[(X - \mu_X)^k]$ - kth central moment.
- $\mathbb{E}\left[\left(\frac{X-\mu_X}{\sigma}\right)^k\right]$ - kth *standardised* central moment.

Variance (σ^2) is the second central moment, skewness (γ) and kurtosis (κ) are the third and fourth standardised central moments, respectively.

Properties of $\mathbb{E}$ *and* Var

For $\mathbb{E}$,

- $\mathbb{E}[mX + b] = m\,\mathbb{E}[X] + b, \quad a, b \in \mathbb{R}$.
- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
- $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$, given X and Y are independent.

For Var,

- $\text{Var}(aX + b) = a^2\,\text{Var}(X), \quad a, b \in \mathbb{R}$.
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$, given X and Y are independent.

Moment Generating Functions

MGF's

We define the MGF of an RV X to be

$$M_X(t) = \mathbb{E}[e^{Xt}].$$

Taking the kth derivative and setting $t = 0$ gives the kth non-central moment.

There is a one-to-one correspondence between random variables and their MGF's. In other words, if two random variables have the same MGF, there are of the same distribution.

Tutorial Questions

(In this order) 1.1.10, 1.1.12, 1.1.6, 1.1.11