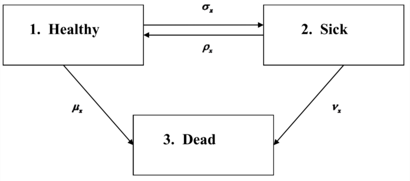
ACTL2102 Week 8 - CTMC: Probability Computations

Tadhg Xu-Glassop

2025T3

CTMC Revision



Remember in the CTMC model,

- q_{ij} represents the instantaneous probability we transition from i to j.
- The time we spend in a state is exponentially distributed with rate $\sum_{k \neq i} q_{ik}$.

Revision: Kolmogorov Equations

Theorem (Kolmogorov Equations)

For a homogenous Markov chain with state space S, for $t \geq 0$ we have

- Kolmogorov's Forward Equation

$$\frac{\partial}{\partial t} P_{ij}(t) = \sum_{k \in S} P_{ik}(t) q_{kj},$$

- Kolmogorov's Backward Equation

$$\frac{\partial}{\partial t} P_{ij(t)} = \sum_{k \in S} q_{ik} P_{kj}(t).$$

Generator Matrix

We have discussed the q_{ij} extensively, and seen how they define the entire process via the KM equations. So, a CTMC can be represented by a matrix of these (similar to $\mathbf{P}$ in DTMC's).

Generator Matrix

We have discussed the q_{ij} extensively, and seen how they define the entire process via the KM equations. So, a CTMC can be represented by a matrix of these (similar to $\mathbf{P}$ in DTMC's).

Definition (Generator Matrix)

For a CTMC with rates q_{ij}, define

$$\mathbf{Q} := \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \ddots & q_{nn} \end{pmatrix}.$$

Note: Since $q_{ii} = -\sum_{k \neq i} q_{ik}$, the rows should sum to 0.

Occupancy Probabilities

Recall we define $P_{ii}(t)$ the probability the process in state i is in state i after t time.

Occupancy Probabilities

Recall we define $P_{ii}(t)$ the probability the process in state i is in state i after t time.

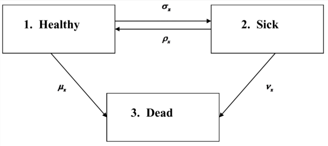
Definition (Occupancy Probabilities)

We define $P_{\overline{ii}}(t)$ the probability the process **remains** in state i over a period of t. This is equivalent to the time spent in the state exceeds t, so

$$P_{\overline{ii}}(t) = \mathbb{P}(T_i > t) = \exp\left\{-t\sum_{k\neq i} q_{ik}\right\}.$$

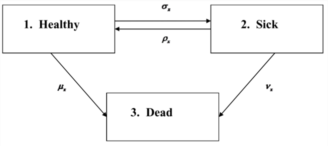
The difference between $P_{ii}(t)$ is that we are allowed to transition out of i and come back, but in $P_{\overline{ii}}(t)$ we must remain there the entire time.

Finding Probabilities



Q: In the above HSD model, what is the probability someone who is initially healthy becomes sick and remains sick until time t?

Finding Probabilities

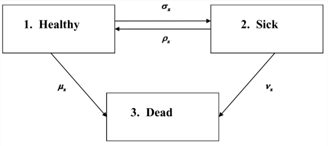


Q: In the above HSD model, what is the probability someone who is initially healthy becomes sick and remains sick until time t?

A: A possible path our process could take is, for some $0 < s < t$,

Stay in H until $s \rightarrow$ Transition to S at $s \rightarrow$ Stay in S from time s until time t.

Finding Probabilities



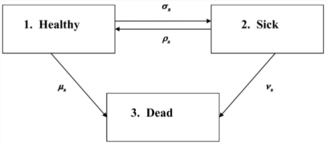
Q: In the above HSD model, what is the probability someone who is initially healthy becomes sick and remains sick until time t?

A: A possible path our process could take is, for some $0 < s < t$,

Stay in H until $s \rightarrow$ Transition to S at $s \rightarrow$ Stay in S from time s until time t.

$$\implies P_{\overline{HH}}(s) \rightarrow q_{HD} \rightarrow P_{\overline{SS}(t-s)}.$$

Finding Probabilities



Q: In the above HSD model, what is the probability someone who is initially healthy becomes sick and remains sick until time t?

A: A possible path our process could take is, for some $0 < s < t$,

Stay in H until $s \rightarrow$ Transition to S at $s \rightarrow$ Stay in S from time s until time t.

$$\Longrightarrow \; P_{\overline{HH}}(s) \rightarrow q_{HD} \rightarrow P_{\overline{SS}(t-s)}.$$

We want to sum over all possible paths, and using the fact that an integral is an infinitesimal sum, this is

$$\int_0^t P_{\overline{HH}}(s)\, q_{HS}\, P_{\overline{SS}}(t-s)\, ds = \int_0^t e^{-s(\mu+\sigma)}\, \sigma\, e^{-(t-s)(\rho+\nu)}\, ds.$$

Occupation Time

Proposition (Occupation Time)

We define $O_i(t)$ the total amount of time spent in state i. We have

$$\mathbb{E}[O_j(t) \mid X(0) = i] = \mathbb{E}\left[\int_0^t 1_{\{ij\}}(s)\, ds\right] = \int_0^t P_{ij}(s)\, ds.$$

Limiting Probabilities

Definition (Limiting Probabilities)

Denote $P_j := \lim_{t\to\infty} P_{ij}(t)$. Then, if the Markov chain with generator matrix $\mathbf{Q}$ is irreducible and positive recurrent, this limit exists, and satisfies

$$\mathbf{Q}\tilde{P} = \mathbf{0},$$

where $\tilde{P} = \begin{pmatrix} P_1 & P_2 & \dots & P_n \end{pmatrix}^T$, subject to $\sum_i P_i = 1$.

Embedded Markov Chain

Consider the CTMC defined by the generator matrix with states $[1, 2, 3]$,

$$Q = \begin{pmatrix} -0.3 & 0.2 & 0.1 \\ 0.5 & -0.8 & 0.3 \\ 0.05 & 0.05 & 1 \end{pmatrix}.$$

Q: Suppose we are in state 1 and we make a transition. What is the probability it is to state 2? Hint: Remember if $T_i \sim \mathrm{Exp}(\lambda_i)$ then $\mathbb{P}(\min_i(X_i) = X_j) = \frac{\lambda_j}{\sum_i \lambda_i}$.

Embedded Markov Chain

Consider the CTMC defined by the generator matrix with states $[1, 2, 3]$,

$$Q = \begin{pmatrix} -0.3 & 0.2 & 0.1 \\ 0.5 & -0.8 & 0.3 \\ 0.05 & 0.05 & 1 \end{pmatrix}.$$

Q: Suppose we are in state 1 and we make a transition. What is the probability it is to state 2? Hint: Remember if $T_i \sim \mathrm{Exp}(\lambda_i)$ then $\mathbb{P}(\min_i(X_i) = X_j) = \frac{\lambda_j}{\sum_i \lambda_i}$.

A: $\frac{0.2}{0.1+0.2} = 2/3$.

Embedded Markov Chain (cont.)

We can define a new process of just the transitions, without considering the time - the transition probabilities will just be these! Specifically,

$$Q = \begin{pmatrix} -0.3 & 0.2 & 0.1 \\ 0.5 & -0.8 & 0.3 \\ 0.05 & 0.05 & 1 \end{pmatrix} \to P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{5}{8} & 0 & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

We call P the **emdedded Markov chain**.

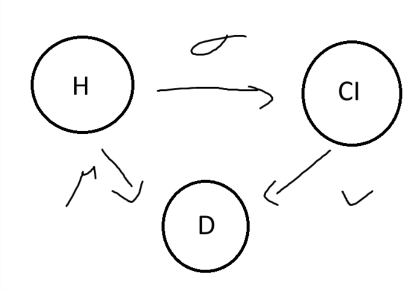
Embedded Markov Chain - long run time spent in a state

From before, suppose we find the embedded markov chain is ergodic and irreducible, and so has stationary probabilities π_i. Then, the proportion of time spent in state i is

$$P_i = \frac{\pi_i/\nu_i}{\sum_j (\pi_j/\nu_j)}.$$

We can view this as multiplying the proportions by the expected time spent in a state, which is $\mathrm{Exp}(\nu_i)$.

Extra Question



Consider the above Healthy-Critical Illness-Dead model. Find $P_{H,CI}(t)$.