ACTL2131 1.4 - Functions of Random Variables

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Overview

Suppose we have some random variable X, and we make a new random variable Y that is a function of X, say $Y = g(X)$. We are tasked with the problem of finding the distribution of Y.
This week, we look at ways of achieving this.

CDF Technique

We aim to get $F_Y(y) = \mathbb{P}(Y \leq y)$ as a function we can evaluate, hence,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(g(X) \leq y) \\ &= \mathbb{P}(X \leq g^{-1}(y)) \\ &= F_X(g^{-1}(y)). \end{aligned}$$

Example

Suppose X has CDF F_X and let $Y = 3X$. We seek $F_Y(y) = \mathbb{P}(Y \leq y)$, so

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(3X \leq y) \\ &= \mathbb{P}(X \leq y/3) \\ &= F_X(y/3). \end{aligned}$$

So we can use this to find all probabilities of Y.

Jacobian Transformation

Suppose we are given $U = g_1(X, Y)$, $V = g_2(X, Y)$, and $f_{X,Y}$. We seek $f_{U,V}$.

1. Find functions h_1, h_2 s.t. $h_1(g_1(x, y), g_2(x, y)) = x$ and $h_2(g_1(x, y), g_2(x, y)) = y$.
2. Determine $|J(u, v)|$, i.e. the absolute value of the determinant of the Jacobian.
3. We have $f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v)) \cdot |J(u, v)|$.

It is rote... but there is not much else to this.

If we want to find f_U or f_V, we can integrate out the other variable.

MGF Technique

This is the easiest and most commonly used one!

> There is a one-to-one correspondence between random variables and their MGF's.

In other words, if I can show a random variable has, say, the MGF of a normal distribution with mean μ and variance σ^2, then that R.V. is $\mathcal{N}(\mu, \sigma^2)$!

Example

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and let $Z = (X - \mu)/\sigma$. What is the distribution of Z ?

$$\begin{aligned}
M_Z(t) &= \mathbb{E}\left[e^{Zt}\right] \\
&= \mathbb{E}\left[e^{\frac{X-\mu}{\sigma}t}\right] \\
&= \mathbb{E}\left[e^{\left(\frac{t}{\sigma}\right)X}\right] \cdot \mathbb{E}\left[e^{-\frac{\mu t}{\sigma}}\right] \\
&= M_X\left(\frac{t}{\sigma}\right) \cdot e^{-\frac{\mu t}{\sigma}} \\
&= e^{\mu\frac{t}{\sigma} + \frac{1}{2}\sigma^2\frac{t^2}{\sigma^2}} \cdot e^{-\frac{\mu t}{\sigma}} \\
&= e^{\frac{1}{2}t^2} \\
&\sim \mathcal{N}(0, 1).
\end{aligned}$$

Convolutions

Convolution Formula

Suppose X and Y are **independent** random variables and let $Z = X + Y$. Then,

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx \quad && \text{if continuous,} \\
p_Z(z) &= \sum_{x=0}^{z} p_X(x)\, p_Y(z-x) && \text{if discrete.}
\end{aligned}
$$

Convolutions (cont.)

Note: Recall in the continuous case, the formula is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx.$$

A lot of the time, the range of $z - x$ will **NOT** be perfectly in the domain of f_Y, and blindly plugging it in here will result in an invalid PDF. Here, we must 'split up' the domain of z to consider different bounds of x...

Example

Q: Let $X, Y \sim \mathrm{Unif}(0, 1)$ and suppose X and Y are independent. What is the PDF of $Z = X + Y$?

A: Since X and Y are independent, we use convolutions. Note that if $Z = X + Y = z$, then $z \in (0, 2)$. Further, the domain of f_X, f_Y is $(0, 1)$ so we must place restrictions on x depending on z so that $z - x$ is in this domain.

Example (cont.)

Consider the case when $0 < z < 1$. Then, we require $0 < x < z$ for $z - x$ to be in $(0, 1)$. Otherwise, if $1 < z < 2$, then $z - 1 < x < 1$ for $z - x$ to be in $(0, 1)$. From this,

$$
\begin{aligned}
f_Z(z) &= \begin{cases} \int_0^z 1 \cdot 1 \; dx & \text{if } 0 < z < 1, \\ \int_{z-1}^1 1 \cdot 1 \, dx & \text{if } 1 < z < 2. \end{cases} \\
&= \begin{cases} z & \text{if } 0 < z < 1, \\ 2 - z & \text{if } 1 < z < 2. \end{cases}
\end{aligned}
$$

Order Statistics

Suppose for some R.V. X we have an iid sample of size n, $X_1, X_2, \ldots, X_n$, and we order them from lowest to highest. So,

- $X_{(1)}$ is the smallest,
- $X_{(j)}$ is the jth smallest,
- $X_{(n)}$ is the largest.

Joint Distribution of Order Stats

Intuitively, we can think of $f_{X_{(1)},\dots,X_{(n)}}(y_1, \dots, y_n)$ as

$$\text{“}f_{X_{(1)},\dots,X_{(n)}}(x_1, \dots, x_n) = \mathbb{P}(X_{(1)} = y_1, X_{(2)} = y_2, \dots, X_{(n)} = y_n)\text{.”}$$

That is, the probability that our n sample has the values $y_1, y_2, \dots, y_n$. But there are $n!$ ways of drawing this sequence, and so using the fact that X is iid,

$$f_{X_{(1)},\dots,X_{(n)}}(y_1, \dots, y_n) = n! \cdot f_X(y_1) \cdot f_X(y_2) \cdot f_X(y_n).$$

Distribution of max and min

$F_{X_{(n)}}(x) = \mathbb{P}(X_{(n)} \leq x)$ is the probability the maximum is less than x, so everything is less than x, thus,

$$F_{X_{(n)}}(x) = \mathbb{P}(X_1 \leq x) \dots \mathbb{P}(X_n \leq x) = (F_X(x))^n.$$

$F_{X_{(1)}}(x) = \mathbb{P}(X_{(1)} \leq x)$ is the probability that the minimum value is less than x, so at least one observation is less than x, thus,

$$F_{X_{(1)}}(x) = 1 - \mathbb{P}(X_1 > x) \dots \mathbb{P}(X_n > x) = 1 - (1 - F_X(x))^n.$$

Special Distributions

χ^2 Distribution:

- Sum of n standard normal variables, where n is the 'Degrees of Freedom.' That is, $\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$.

Student's t-Distribution:

- Let $V \sim \chi_n^2$. Then, $T = \frac{Z}{\sqrt{V/n}}$ is said to have a t distribution with n degrees of freedom.
- Generalises the normal distribution to have heavier tails to work with small samples (high uncertainty). When $n > 30$, it is approximately normal.

Snedecor's F Distribution:

- Let $U \sim \chi_{n_1}^2$ and $V \sim \chi_{n_2}^2$. Then, $F = \frac{U/n_1}{V/n_2}$ is said to have an F distribution with n_1 and n_2 degrees of freedom.

All of the above have special use cases that we will get into later...

Tutorial Questions

1.4.2, 1.4.3 (compute 1,2, explain method for 3,4), 1.4.4, 1.4.5