ACTL2102 Week 7 - CTMC: Kolmogorov Equations

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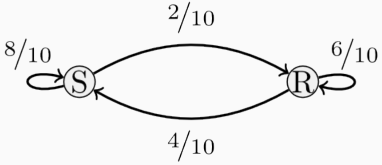
2025T3

Revision of DTMC's



Recall the Discrete Time Markov Chain model - at each step, we have probabilities of transitioning to other states, or remaining in our current state.

Revision of Poisson Process

Further recall our definition of the Poisson process,

Definition (Poisson Process)

A counting process $\{N(t) : t \geq 0\}$ is said to be a Poisson process if:

1. $N(0) = 0$;
2. it has independent increments;
3. it has stationary increments;
4. for $h \to 0$;

$$\mathbb{P}(N(t+h) - N(t) = 1) = \lambda h + o(h),$$

and

$$\mathbb{P}(N(t+h) - N(t) > 1) = o(h)$$

Motivating CTMC's

We want to extend our discrete model where we can define our own state space and transition possibilities to a continuous case; transition can occur at any time, not just periodically.

We can do this by generalising our Poisson process.

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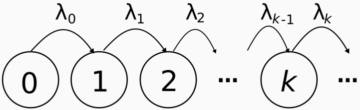
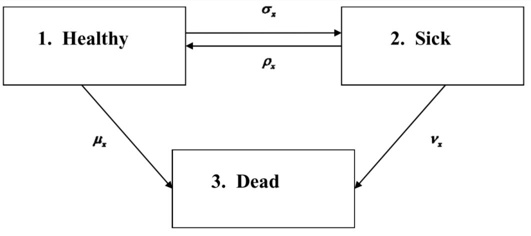
We can visualise it in a very similar way to DTMC's. Recall in a very small amount of time h, the probability we transition to the next state is λ. So, we can represent these infinitesimal transition probabilities with a very similar diagram.



General CTMC's

We can think about CTMC's as Poisson processes, but rather than incrementing by 1 each step, we just have general infinitesimal probabilities between each state.



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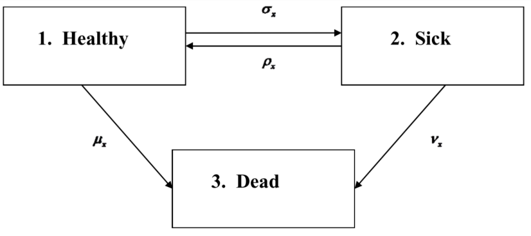

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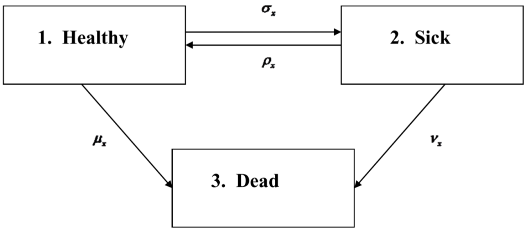

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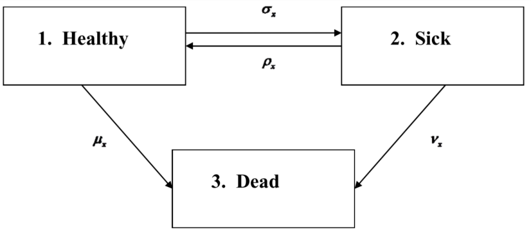

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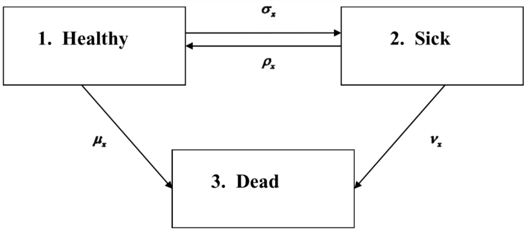

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- What is the total probability of transitioning out of healthy in a small interval *h*? A: $\sigma + \mu$.
- This is the definition of exponential distribution $\implies$ the time we spend in Healthy is $\mathrm{Exp}(\mu + \sigma)$.

Transition Probabilities

Definition

For a continuous time Markov chain X, define

$$P_{ij}(s, s+t) = \mathbb{P}(X(s+t) = j \mid X(s) = i).$$

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Let q_{ij} be the instantaneous probability we transition from state i to j (as before). Then,

$$\frac{\partial}{\partial t} P_{ij}(s, s+t) = \frac{\partial}{\partial t} P_{ij}(t) = q_{ij}.$$

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Intuitively, recall the rate we leave state i is the sum of all rates. So, the rate we stay in the state is the negative of this.

Motivating CK Equations

So its clear that q_{ij} are very important and essentially define a CTMC and by definition, $\frac{\partial}{\partial t} P_{ij}(t) = q_{ij}$ so we just need to compute this!

But how do we find this...

Kolmogorov Equations

Theorem (Kolmogorov Equations)

For a homogenous Markov chain with state space S, for $t \geq 0$ we have

- Kolmogorov's Forward Equation

$$\frac{\partial}{\partial t} P_{ij}(t) = \sum_{k \in S} P_{ik}(t) q_{kj},$$

- Kolmogorov's Backward Equation

$$\frac{\partial}{\partial t} P_{ij(t)} = \sum_{k \in S} q_{ik} P_{kj}(t).$$

So, we can solve for our transition probabilities by solving these differential equations, subject to the intial condition $P_{ij}(0) = 0$ if $i \neq j$ and 1 if $i = j$.