ACTL2102 Week 1 - Stochastic Processes, Simulation

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Hello!

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ACTL2102 is especially nice because a lot of the harder content is much easily understood “intuitively” as opposed to understanding it rigorously, and so a lot of the harder questions in this course can be solved using intuition rather than following these methods. I will try to show all of these methods, and explain the content as intuitively as possible :p

Stochastic Processes

Definition (Stochastic Process)

A **stochastic process** is a collection of random variables $X(t)$ for $t \in T$, where T is the *index set.*

$X(t)$ can take any value in the *state space* S. Both T and S can be discrete or continuous.

Examples

- Price of stock at time t.
- Dead or Alive status of policyholder at any time throughout the year.
- Dead or Alive status of policyholder at start of the year.

Feature of a Stochastic Process

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Definition (Markov Property)

A stochastic process has the Markov Property if

$$\mathbb{P}(X(t_n) \mid X(t_{n-1}), X(t_{n-2}), \ldots, X(t_1)) = \mathbb{P}(X(t_n) \mid X(t_{n-1})).$$

Monte Carlo Simulation

Monte Carlo Simulation Method

Suppose we have some random variable $\mathbf{X}$ and we wish to estimate the expected value of some $\mathbf{Y} = g(\mathbf{X})$. Then, a possible estimator can be found by;

1. Simulate $\mathbf{X}$ n times,
2. Using each simulation, set $\mathbf{Y}^{(i)} = g\left(\mathbf{X}^{(i)}\right)$,
3. Taking the average of all simulated $\mathbf{Y}^{(i)}$ yields an estimate for the mean of $\mathbf{Y}$.

This should be intuitive, as this is an application of the the law of large numbers !

Inverse Transform Method

Inverse Transform Method

Suppose X has CDF F_X. Then, if $U \sim \mathrm{Unif}(0,1)$, then $F_X^{-1}(U)$ has the same distribution as X.

So, if we want to simulate some random variable X with CDF F,

1. Find F^{-1},
2. Simulate U from $\mathrm{Unif}(0,1)$,
3. Set $\tilde{X} := F^{-1}(U)$ and this is a simulated value for X.

Acceptance Rejection Method

Suppose X has PDF f_X and Y has PDF g_y. Further, suppose we can simulate Y with other methods, but can't for X. Then, we can simulate X with the following method,

1. Find some C such that $C \geq \frac{f(y)}{g(y)}$ for all y.
2. Simulate U from $\mathrm{Unif}(0, 1)$ and $\tilde{Y}$ using the simulation method,
3. If $U \leq \frac{f(\tilde{Y})}{Cg(\tilde{Y})}$, then set $\tilde{X} := \tilde{Y}$ and this is a simulated value of X. Otherwise, go back to 2.

Simulating Discrete Random Variables

If X is a discrete random variable, then

$$X = \begin{cases} x_1 & \text{wp } p_1, \\ x_2 & \text{wp } p_2, \\ \vdots & \\ x_k & \text{wp } p_k. \end{cases}$$

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So, a method for simulating X after generating U from $\mathrm{Unif}(0, 1)$ is to set

$$X = \begin{cases} x_1 & \text{if } U \leq p_1, \\ x_2 & \text{if } p_1 < U \leq p_1 + p_2, \\ \vdots & \\ x_k & \text{if } \sum_{i=1}^{k-1} p_i < U \leq \sum_{i=1}^{k} p_i. \end{cases}$$

This works as U is equally likely to be any number on $[0, 1]$, so the probability $\tilde{X} = x_k$ is p_k in the above process.