ACTL2102 Week 2 - Discrete Markov Chains

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Markov Chain

Definition (Markov Property)

A stochastic process has the Markov Property if

$$\mathbb{P}(X(t_n) \mid X(t_{n-1}), X(t_{n-2}), \ldots, X(t_1)) = \mathbb{P}(X(t_n) \mid X(t_{n-1})).$$

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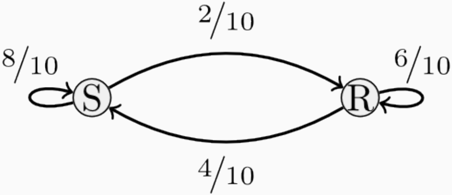
Definition (Markov Chain)

A stochastic process is said to be a **Markov chain** if it has discrete index set AND has the Markov property.

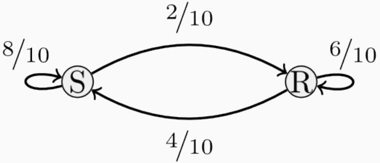
Visualisation



Given you are at some state, the arrows pointing out from the state show the probability of you moving there at the next index; the transition probabilities P_{ij}.

This diagram provides all necessary information - the current state describes the entire distribution and any previous state is irrelevant (Markov property).

Matrix Representation



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The entry in the ith row and jth column is the probability we transition to state j, given we are currently in i. Note that all rows sum to 1.

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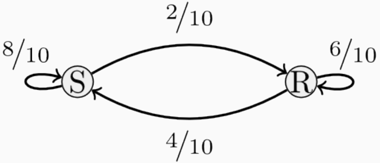

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If P is the transition matrix, then P^2 is the 2-step transition matrix, ..., P^n is the n-step transition matrix.

Classification of States

Definition (Absorbing States)

A state i is said to be an *absorbing state* if $P_{ii} = 1$, so the ith row of the transitino matrix is all 0's except for the ith column, which is 1.

Intuitively, once the process enters this state, it is suck and will forever be in that state.

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Definition (Accessible)

State j is accessible from state i if $P_{ij}^n > 0$ for some $n \geq 0$, and we write $i \to j$. So, we can take a 'path' to get some state i to j.

Classification of States (cont.)

Definition (Communicate)

States i and j communicate if $i \to j$ and $j \to i$, and write $i \leftrightarrow j$.

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Definition (Class)

The *class* of states that communicate with state i is $C(i) = \{j \in S : i \leftrightarrow j\}$.

Note all states in a chain must belong to exactly one class.

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Definition (Irreducible)

A Markov chain is said to be irreducible if all states belong to the same class; i.e. there is only 1 class.

Recurrent v.s. Transient States

Definition (Recurrent)

A state of a Markov chain is said to be recurrent if we will *eventually* transition back to that state.

Definition (Transient)

A state of a Markov chain is said to be transient if it is possible we never return back to that state.