ACTL2102 Week 2 - Discrete Markov Chains II

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Some final definitions

Definition (Recurrent)

A state of a Markov chain is said to be recurrent if we will *eventually* transition back to that state.

Definition (Positive Recurrent)

A state of a Markov chain is said to be positive recurrent if it is recurrent and the expected time to return to the state is finite.

Definition (Transient)

A state of a Markov chain is said to be transient if it is possible we never return back to that state.

Some more final definitions

Definition (Period)

The period of state i, denoted $d(i)$, is the GCD of all $n \geq 1$ for which $P_{ii}^n > 0$.

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Definition (Ergodic)

A chain is said to be ergodic if it is aperiodic and positive recurrent.

Limiting Probabilities

We are now interested in the long-term behaviour of Markov chains.

Definition (Limiting Probability)

The limiting probability of a state j from state i is $\lim_{n\to\infty} P_{ij}^n$.

Interpreting this, it is the long-run proportion of time we will spend in state j given we are currently in state i.

Issues with Limiting Probabilities

For some chains, the limiting probabilities are dependent on the initial state i.

For example, consider

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $\lim_{n\to\infty} P_{ij}^n = 1$ if $i = j$ and 0 if $i \neq j$.

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But, let's say we start in state 1 with probability p and state 2 with probability $1 - p$. Then, what is the long-run proportion of time we will spend in state 1?

Stationary Probabilities

Theorem

For an irreducible (one class; all states communicate), ergodic (aperiodic, positive recurrent) Markov chain, the limiting probabilities $\lim_{n\to\infty} P_{ij}^n$ exist and are independent of i.

So, the process "forgets" where we started in the long run, and the probability we are in a state becomes the same.

Example:

$$\begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0 & 0.1 & 0.3 & 0.6 \\ 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix}^\infty =$$

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Stationary Probabilities (cont.)

Given an irreducible ergodic Markov chain with transition matrix $\mathbf{P}$, let $\pi_j = \lim_{n\to\infty} P_{ij}^n$, and $\boldsymbol{\pi} = \begin{pmatrix}\pi_1 & \pi_2 & \dots & \pi_k\end{pmatrix}$. Then,

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}.$$

So, we can find the limiting probabilities by solving the above under the constraint $\sum_{i=1}^{k} \pi_k = 1$.

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If the initial state is chosen such that $\mathbb{P}(X_0 = i) = \pi_i$, then $\mathbb{P}(X_n = i) = \pi_i \quad \forall n \geq 1$.
$\rightarrow$ This is why we also call these limiting probabilities stationary probabilities.

Mean Transitions to Return

Proposition (Mean time between visits to state j)

Let m_j be the expected number of transitions until a Markov chain returns to state j, given it is currently in state j. Then, given they exist and are unique,

$$m_j = \frac{1}{\pi_j}.$$

This is because (for large n; in the long run), $\mathbb{P}(X_n = i) = \pi_i$, so the distribution of number of transitions to reenter state i is a geometric distribution with parameter π_i.