ACTL2131 2.3 - Evaluating Point Estimators

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Motivation

So far, we've found ways of estimating unknown parameters θ using estimators, $\hat{\theta}$.

But we can find multiple estimators for the one quantity, (e.g. by finding MME or MLE or guessing your favourite number), so we need a way of assessing how 'good' and estimator is, and comparing different estimators.

Estimators are random - they are based off a sample, which is itself random. So, we can use the same methods of analysing estimators as we would random variables...

Bias, Variance and MSE

> We define the bias of an estimator as
>
> $$\text{Bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta,$$
>
> variance as usual, and Mean Squared Error (MSE) as
>
> $$\text{MSE} = \mathbb{E}[(\theta - \hat{\theta})^2] = \text{Var}(\hat{\theta}) + \text{Bias}^2(\hat{\theta}).$$

We say an estimator is unbiased if $\text{Bias}(\hat{\theta}) = 0$ or $\mathbb{E}[\hat{\theta}] = \theta$.

Intuitively,

- Bias is how wrong we expect our estimator to be;
- Variance is how variable - 'random' our estimator is. The square root gives **standard error**, which is the standard deviatin of an estimator;
- MSE is a measure of the distance between our estimator and the true value.

Bias, Variance and MSE (cont.)

Below is a visualisation of different levels of bias and variance. Try to understand the intuition behind each target.
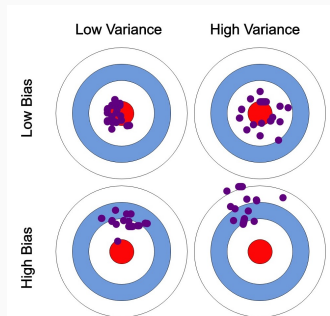
Cramer-Rao Lower Bound

Definition

Define $I_f(\theta)$ the Fisher information of the parameter θ wrt a RV with pdf f, where

$$I_f(\theta) = - \mathbb{E}\left[\frac{\partial^2 \log(f_X(x|\theta))}{\partial \theta^2}\right].$$

Then, the Cramer-Rao Lower Bound for the variance of unbiased estimators for θ is given by

$$\mathrm{CRLB}(\theta) = \frac{1}{n \cdot I_f(\theta)}$$

So, if I have an unbiased estimator $\hat{\theta}$, the smallest possible variance it could have is $\mathrm{CRLB}(\theta)$. If $\mathrm{Var}(\theta) = \mathrm{CRLB}(\theta)$, then this estimator has minimal variance.

Properties of MLE's

> ### Theorem (Asymptotic distribution of MLE's)
>
> Let $\hat{\theta}$ be the MLE for θ. Then
>
> $$\mathbb{E}[\hat{\theta}] \xrightarrow{p} \theta, \qquad \text{and} \qquad \text{Var}(\hat{\theta}) \xrightarrow{p} \text{CRLB}(\theta).$$
>
> Further,
>
> $$\hat{\theta} \xrightarrow{d} \mathcal{N}\left(\theta, \text{CRLB}(\theta)\right).$$

This means we can approximate

$$\frac{\hat{\theta} - \theta}{\sqrt{\text{CRLB}(\theta)}} \sim \mathcal{N}(0, 1),$$

and we can use this to make confidence intervals for θ.

Tutorial Questions

Please do all of the other tutorial questions in the booklet - they are unbelievably important...

2.2.9, 2.2.12, 2.3.3