ACTL2102 Week 2 - Exponential Distribution

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Last thing about Markov Chains

Expected Time in Transient States

For a Markov chain with k transient states and, let s_{ij} be the number of expected transitions the chain will spend in state j given it is currently in state i. Then the matrix $S = (s_{ij})_{i,j=1}^{k}$ has

$$S = I + P_T S \quad \Longrightarrow \quad S = (I - P_T)^{-1},$$

where P_T is the probability transition matrix that is just for the transient states.

Exponential Distribution

Definition (Exponential Distribution)

A random variable has exponential distribution with rate parameter $\lambda > 0$ if

$$f_X(x) = \lambda x e^{-x},\ x > 0.$$

Note $\mathbb{E}[X] = \lambda^{-1}$ and $\mathrm{Var}(X) = \lambda^{-2}$.

Exponential Distribution - Memorylessness Property

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The memoryless property makes the most sense when we view $T \sim \mathrm{Exp}(\lambda)$ as a time variable.

For example,

$$\mathbb{P}(T > 3 + 10 \mid T > 10) = \mathbb{P}(T > 100 + 3 \mid T > 100) = \mathbb{P}(T > 3)$$

essentially means that the fact that we've passed 10 or 100 minutes has no impact on the probability T will be another 3 minutes.

Intuition of Exponential Distribution

Recall the Geometric distribution is the distribution of the number of Bernoulli trials required until the first success. We also have that if $X \sim \mathrm{Geo}(p)$, then $\mathbb{E}[X] = p^{-1}$.

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This also explains why the only 2 memoryless distributions are the Geometric and Exponential!

A very very important property of the exp distribution

Proposition

Let $X_1, X_2, \ldots, X_n$ be independent exponential distributions with rate parameter $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Then, $\min(X_1, \ldots X_n) \sim \mathrm{Exp}\left(\frac{1}{\lambda_1+\cdots+\lambda_n}\right)$, and

$$\mathbb{P}(\min(X_1, \ldots X_n) = X_i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}.$$

You can show this by using distribution of min and computing the probability directly, but also try use the intuition from the previous slide to reason why this must also be the case!

Counting Process

Definition (Counting Process)

A stochastic process is said to be a *counting process* $\{N(t) : t \geq 0\}$ if it counts the number of events that occur up to time t.

So, we require it to be non-decreasing wrt t, be integer valued, and have $N(0) = 0$.

Poisson Process

Definition (Poisson Process)

A counting process $\{N(t) : t \geq 0\}$ is said to be a Poisson process if:

1. $N(0) = 0$,
2. it has independent increments,
3. the number of of events in any inteval of length t has Poisson distribution with mean λt, that is, for all $s, t \geq 0$

$$\mathbb{P}(N(t+s) - N(s) = n) = \frac{(\lambda t)^{-n} e^{-\lambda t}}{n!}, \quad n = 0, 1, \ldots .$$

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Setting the number of events $n := 0$, we recover the exponential distribution with parameter λt.
$\Rightarrow$ The interarrival times (times between consecutive events) are exponentially distributed with rate λt.