ACTL2131 1.3 - Bivariate Distributions

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Bivariate Distributions

Joint Functions

To describe *multiple* random variables at one time, we use joint distribution and density functions, which are connected by, for two R.V.'s X, Y,

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \, \partial y} F_{X,Y}(x, y).$$

Note that $F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$.

Bivariate Distributions (cont.)

Recall from Week 1 that for some proper density f_X, the space must integrate to 1. This is still the case! But now instead of worrying about 1 variable, we have 2... that is,

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \to \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$$

and,

$$F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v)\, du\, dv.$$

Marginal Densities

What if we don't care about one of the variables? We can integrate/sum it out to get the **marginal** density, which is the density of X as if Y never existed,

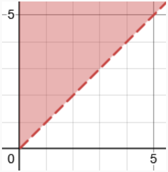
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \quad \text{or} \quad p_X(x) = \sum_{k=0}^{\infty} p_{X,Y}(x, y).$$

This is nothing more than the law of total probability.

Finding Tricky Probabilities

Q: Suppose X, Y has joint density $6e^{-(2x+3y)}, \quad x, y \geq 0$. Find $\mathbb{P}(X < Y)$.

Finding this probability is no different to the univariate case! We just integrate over the area in question. From the diagram, we see $0 \leq x < y$ and $y \geq 0$, giving us the bounds of our double integral,



$$\mathbb{P}(X < Y) = \int_0^{\infty} \int_0^{y} 6e^{-(2x+3y)}\, dx\, dy$$

$$\vdots$$

$$= 0.4.$$

Covariance and Correlation

To quantify the relationship between two variables,

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y],$$

which in turn gives us the actually useful

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

Conditional Distributions

Conditional Density

The conditional density of Y given X is given by

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

Notice the similarity between this and the well-known formula for conditional probability.
Further, this is still a proper density - it must integrate to 1.

Tutorial Questions

1.2.6, 1.2.7, 1.3.2, 1.3.3