Technische Universität München

MIMO Systems Programming Project 1

Instructions

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Chapter 1

Power Allocation for Point-to-Point MIMO Channels

In the first project, we analyze a point-to-point *multiple-input multiple-output* (MIMO) system as depicted in Fig. 1.1. An N-antenna transmitter sends the signal $\boldsymbol{x} \in \mathbb{C}^N$ via the channel $\boldsymbol{H} \in \mathbb{C}^{M \times N}$ to the M-antenna receiver. The distorted channel output is perturbed by zero-mean circularly symmetric complex additive Gaussian noise $\boldsymbol{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_M, \boldsymbol{C}_{\boldsymbol{n}})$ with covariance $\boldsymbol{C}_{\boldsymbol{n}} \in \mathbb{C}^{M \times M}$. The M-dimensional complex received signal reads as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}. \tag{1.1}$$

For mutual information maximization with limited transmit power, a zero-mean transmit signal $\boldsymbol{x}$ with covariance matrix $\boldsymbol{Q} \in \mathbb{C}^N$ is optimal. The *eigenvalue decomposition* (EVD) of the transmit covariance reads as

$$\boldsymbol{Q} = \boldsymbol{V}\boldsymbol{\Psi}\boldsymbol{V}^{\mathrm{H}} = \boldsymbol{V}\mathrm{diag}(\psi_1, \ldots, \psi_N)\boldsymbol{V}^{\mathrm{H}}, \tag{1.2}$$

where the unitary matrix $\boldsymbol{V} \in \mathbb{C}^{N \times N}$ stems from the EVD

$$\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\boldsymbol{n}}^{-1}\boldsymbol{H} = \boldsymbol{V}\boldsymbol{\Phi}\boldsymbol{V}^{\mathrm{H}} = \boldsymbol{V}\mathrm{diag}(\phi_1, \ldots, \phi_N)\boldsymbol{V}^{\mathrm{H}}. \tag{1.3}$$

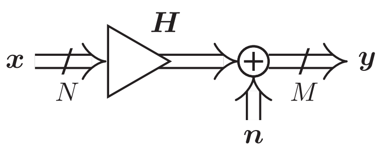

Figure 1.1: Point-to-Point MIMO System

The diagonal elements ψ_i, $i = 1, \ldots, N$, are the power allocations for the individual data streams. Their optimal values depend on the applied performance metric and the imposed transmit power requirements.

The specific aim of this programming project is a numerical analysis and comparison of various power allocation strategies. This comparison is based on the analysis in Problem 2.2 of the MIMO Systems tutorials. Afterwards, different transmission schemes are compared to a SISO system and the diversity gain that can be achieved is analyzed. In the final part, average mutual information maximization is considered.

1.1 Comparison of Power Allocation Schemes

1.1.1 Mutual Information Maximization – Waterfilling Solution

The mutual information maximization for above MIMO point-to-point system is formulated as

$$\max_{\boldsymbol{Q} \succeq \boldsymbol{0}} \log_2 \det(\boldsymbol{I}_N + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{n}}^{-1} \boldsymbol{H} \boldsymbol{Q}) \quad \text{s.t.} \quad \mathrm{tr}(\boldsymbol{Q}) \leq P_{\mathrm{Tx}}, \tag{1.4}$$

where P_{Tx} denotes the maximum average transmit power per channel use. The mutual information maximizing transmit covariance is derived in Section 2.2 of the lecture notes and has the structure as given in (1.2). The solution for ψ_i, $i = 1, \ldots, N$, is given by the waterfilling principle:

$$\begin{aligned} \psi_i &= \max\left(0, \mu' - \frac{1}{\phi_i}\right), \quad i = 1, \ldots, N \\ \sum_{i=1}^{N} \psi_i &= P_{\mathrm{Tx}}. \end{aligned} \tag{1.5}$$

The 'waterlevel' μ' depends on the transmit power, i.e., $\mu' = \frac{1}{K}\left(P_{\mathrm{Tx}} + \sum_{i=1}^{K} \frac{1}{\phi_i}\right)$, where K denotes the number of active streams, that is, $\psi_i > 0$ for $i \in \{1, \ldots, K\}$ and $\psi_i = 0$ for $i > K$, if the eigenvalues ϕ_i, $i = 1, \ldots, N$, are non-increasingly ordered ($\phi_i \geq \phi_j$ for $i \leq j$). The capacity of the MIMO channel can then be found as

$$C = \sum_{i=1}^{N} \log_2(1 + \phi_i \psi_i). \tag{1.6}$$

PROGRAMMING TASK 1

Implement a function that takes the eigenvalues $\phi_1, \ldots, \phi_N$ and the available sum transmit power P_{Tx} as inputs and that computes the optimal waterfilling solution given in (1.5).

1 Deliverables (Matlab code file): `waterfilling.m`
- Function definition:
  `function [psi,mu,K] = waterfilling(phi,Ptx)`

2 Input Specifiations:
- `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$
- `Ptx`: available transmit power

3 Output Specifications:
- `psi`: vector of optimal power allocations $\psi_1^\star, \ldots, \psi_N^\star$
- `mu`: value of the optimal waterlevel $\mu'^\star$
- `K`: number of non-zero data streams K

4 Hint: Make sure that the ordering of the output vector `psi` corresponds to that of the input vector `phi`.

For a comparison with other power allocation strategies the values for P_{Tx} shall be determined where the waterfilling solution switches from K to $K+1$ active streams.

QUESTION 1

Give an expression for P_{Tx} as a function of $\phi_i, i = 1, \ldots, N$, where the waterfilling solution switches from K to $K+1$ active streams. What is the optimal power allocation of the waterfilling solution for $P_{\text{Tx}} \to \infty$?

PROGRAMMING TASK 2

Implement a function that takes the eigenvalues $\phi_1, \ldots, \phi_N$ and calculates the power values P_{Tx} where the waterfilling solution switches from K to $K+1$ active streams for $K = 1, \ldots, N-1$.

1 Deliverables (Matlab code file): `activeStreams_waterfilling.m`
- `function [Ptx] = activeStreams_waterfilling(phi)`

2 Input Specifiations: `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$

3 Output Specifications: `Ptx`: $(N-1) \times 1$ array containing power values where the allocation switches from K to $K+1$ streams

For the next measurement task, consider a system with $N = M = 4$ transmit and receive antennas as given in the file `example_channels.mat` and use the function `activeStreams_waterfilling.m`.

> **QUESTION 2**
>
> Give the transmit powers P_{Tx} when the waterfilling solution switches from K to $K+1$ active streams via calling `[Ptx_K]=activeStreams_waterfilling(phi)` for $K = 1, \ldots, 3$.

1.1.2 MSE Minimizing Power Allocation

In Problem 2.1 of the MIMO Systems tutorial, the solution to the following *minimum mean square error* (MMSE) optimization problem is derived:

$$\min_{\boldsymbol{Q}\succeq\mathbf{0}} \varepsilon(\boldsymbol{Q}), \quad \text{s.t.} \quad \text{tr}(\boldsymbol{Q}) \leq P_{\text{Tx}}, \tag{1.7}$$

where $\varepsilon(\boldsymbol{Q}) = \min(M, N) - M + \text{tr}((\boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^{\text{H}} + \boldsymbol{C}_{\boldsymbol{n}})^{-1}\boldsymbol{C}_{\boldsymbol{n}})$ is the MSE achieved with an MMSE filter at the receiver side and rewritten in terms of the input transmit covariance $\boldsymbol{Q}$ [cf. Problem 2.1 d) of the MIMO Systems tutorials]. The MSE minimizing transmit covariance matrix, i.e., the solution of (1.7), also features the structure of (1.2). The corresponding power allocation is given by

$$\begin{aligned} \psi_i &= \max\left(0, \frac{\bar{\mu}}{\sqrt{\phi_i}} - \frac{1}{\phi_i}\right), \quad i = 1, \ldots, N, \\ \sum_{i=1}^{N} \psi_i &= P_{\text{Tx}}. \end{aligned} \tag{1.8}$$

Here, the common 'level' is $\bar{\mu} = \frac{P_{\text{Tx}} + \sum_{i=1}^{K} \frac{1}{\phi_i}}{\sum_{i=1}^{K} \frac{1}{\sqrt{\phi_i}}}$ for non-increasingly ordered ϕ_i, $i = 1, \ldots, N$, where K is again the number of active streams.

> **PROGRAMMING TASK 3**
>
> Implement a function that takes the values of the eigenvalues $\phi_1, \ldots, \phi_N$ and the available sum transmit power P_{Tx} as inputs and computes the optimal MMSE power allocation given in (1.8).
>
> **1 Deliverables (Matlab code file):** `mmseallocation.m`
>
> - Function definition:
>
> ```
> function [psi,mu,K] = mmseallocation(phi,Ptx)
> ```

2 Input Specifications:
- `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$
- `Ptx`: available transmit power

3 Output Specifications:
- `psi`: vector of optimal power allocations $\psi_1^\star, \ldots, \psi_N^\star$
- `mu`: value of the optimal waterlevel $\bar{\mu}^\star$
- `K`: number of non-zero data streams K

To compare the MMSE power allocation with the waterfilling power allocation, again the range of P_{Tx} shall be determined where K active streams are optimal.

QUESTION 3

Give an expression for P_{Tx} as a function of $\phi_i, i = 1, \ldots, N$, where the MMSE power allocation switches from K to $K+1$ active streams. What is the MMSE optimal power allocation for $P_{\mathrm{Tx}} \to \infty$?

PROGRAMMING TASK 4

Implement a function that takes the eigenvalues $\phi_1, \ldots, \phi_N$ and calculates the powers P_{Tx} where the MMSE allocation switches from K to $K+1$ active streams for $K = 1, \ldots, N-1$.

1 Deliverables (Matlab code file): `activeStreams_mmse.m`
- `function [Ptx] = activeStreams_mmse(phi)`

2 Input Specifiations: `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$

3 Output Specifications: `Ptx`: $(N-1) \times 1$ array containing powers where the allocation switches from K to $K+1$ streams

QUESTION 4

Give the transmit powers P_{Tx} when the MMSE solution switches from K to $K+1$ active streams via calling `[Ptx_K]=activeStreams_mmse(phi)` for $K = 1, \ldots, 3$. Use the file `example_channels.mat` for your measurement.

1.1.3 Optimal Uniform Power Allocation

Besides the waterfilling power allocation, the simple but suboptimal uniform power allocation scheme was considered in Problem 2.2 of the MIMO Systems tutorials. In this scheme, the transmit power is uniformly distributed among the K largest effective channel eigenvalues, i.e.,

$$\psi_i = \begin{cases} P_{\mathrm{Tx}}/K & i \in \{1, \ldots, K\}, \\ 0 & i \in \{K+1, \ldots, N\}. \end{cases} \tag{1.9}$$

The number of active streams K shall be chosen to maximize the achievable rate.

QUESTION 5

Rewrite the achievable rate for K active data streams, i.e.

$$R_K = \log_2 \det(\boldsymbol{I}_N + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C_n}^{-1} \boldsymbol{H} \boldsymbol{Q})$$

in terms of ϕ_i and P_{Tx} when the uniform power allocation from (1.9) is used.

A natural approach to find the rate maximizing K is to test all possibilities in a predefined order. For example, start with $K = 1$, determine whether $R_K > R_{K+1}$ and properly adjust K for the next iteration if the eigenvalues ϕ_i, $i = 1, \ldots, N$ are decreasingly ordered.

PROGRAMMING TASK 5

Implement the uniform power allocation function that takes the values of the eigenvalues $\phi_1, \ldots, \phi_N$ and the available sum transmit power P_{Tx} as inputs and computes the rate maximizing power allocation based on (1.9).

1 Deliverables: `uniform_rate.m`
- Function definitions:
  ```
  function [psi,K] = uniform_rate(phi,Ptx)
  ```

2 Input Specifications:
- `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$
- `Ptx`: available transmit power

3 Output Specifications:
- `psi`: vector of optimal power allocations $\psi_1^\star, \ldots, \psi_N^\star$
- `K`: number of non-zero data streams K

To compare the general behavior of the uniform power allocations with the waterfilling solution, measure the range of P_{Tx} where K active streams are optimal in terms of the mutual information maximization. To this end, consider again the system in `example_channels.mat` with $M = N = 4$ transmit and receive antennas and use the provided function `maxpower_Kstreams.m`[1].

QUESTION 6

Measure the transmit powers when the function `uniform_rate.m`, switches from K to $K+1$ streams by calling `maxpower_Kstreams(phi,K,'uniform_rate')`. Compare the results with those of the waterfilling solution and answer the following three questions.

- Which of the two power allocations switches earlier from K to $K+1$ active streams?
- Is the rate optimal uniform power allocation asymptotically optimal, i.e., does the uniform power allocation achieve the same rate as the waterfilling solution for $P_{\text{Tx}} \to \infty$?

1.1.4 MSE Optimal Power Allocation with Transmit Filter Design

Consider the Point-to-Point MIMO system depicted in Fig. 1.2 with the i.i.d. Gaussian distributed data signal $\boldsymbol{s} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I})$ that is precoded with the precoder $\boldsymbol{T} \in \mathbb{C}^{N \times B}$ and transmitted via the channel $\boldsymbol{H} \in \mathbb{C}^{B \times N}$. The received signal is perturbed by Gaussian noise $\boldsymbol{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{C}_{\boldsymbol{n}})$. The receive filter $\boldsymbol{G} = g\mathbf{I}$ with $g \in \mathbb{C}$ performes a scaling of the received signal. The noise and data signals are uncorrelated, i.e. $\mathrm{E}\left[\boldsymbol{s}\boldsymbol{n}^{\mathrm{H}}\right] = \mathbf{0}$.

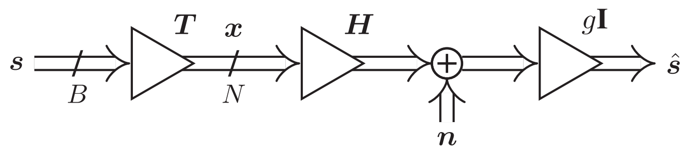

Figure 1.2: Point-to-Point MIMO System

In this problem, the *mean squared error* (MSE) $\varepsilon(\boldsymbol{T}, g) = \mathrm{E}\left[\|\boldsymbol{s} - \hat{\boldsymbol{s}}\|_2^2\right]$ will be

[1]The function `maxpower_Kstreams.m` iteratively performs the power allocation strategy to find the maximum P_{Tx} for the desired K active streams.

minimized with respect to the precoder and equalizer subject to a power constraint

$$\min_{\boldsymbol{T},g} \mathrm{E}\left[\|\boldsymbol{s}-\hat{\boldsymbol{s}}\|_2^2\right] \quad \text{s.t.} \quad \mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right] \le P_{\mathrm{Tx}}. \tag{1.10}$$

QUESTION 7

Give an expression for $\hat{\boldsymbol{s}}$ as a function $\boldsymbol{s}$ and $\boldsymbol{n}$ and calculate $\varepsilon(\boldsymbol{T}, g)$ as a function of $\boldsymbol{T}$, g, and $\boldsymbol{C_n}$.

For a given precoder $\boldsymbol{T}$, the optimal scaling factor g can be found via the unconstrained optimization problem

$$\min_{g} \varepsilon(\boldsymbol{T}, g). \tag{1.11}$$

QUESTION 8

Calculate the optimal choice for g.
Hint: $\varepsilon(\boldsymbol{T}, g)$ is convex in g.

For a given equalizer $g\mathbf{I}$, the optimal precoder $\boldsymbol{T}$ can be found by solving the constrained optimization problem

$$\min_{\boldsymbol{T}} \varepsilon(\boldsymbol{T}, g) \quad \text{s.t.} \quad \mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right] \le P_{\mathrm{Tx}}. \tag{1.12}$$

The optimization problem in (1.12) can be shown to be (hidden) convex. The *Karush-Kuhn-Tucker* (KKT) conditions are therefore necessary and sufficient.

QUESTION 9

Rewrite $\mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right]$ as a function of the precoder $\boldsymbol{T}$ only and give the Lagrangian function $\mathcal{L}(\boldsymbol{T}, \mu)$ for the optimization problem in (1.12), where $\mu \ge 0$ is the Lagrangian multiplier corresponding to the power constraint. State the KKT conditions.

QUESTION 10

Give an expression for $\boldsymbol{T}$ as a function of g and μ.
Hint: Assume that $\mu > 0$.

In order to find μ, the dual feasibility condition can be rewritten to

$$-|g|^2 \operatorname{tr}(\boldsymbol{C_n}) + \mu \operatorname{tr}(\boldsymbol{T}\boldsymbol{T}^{\mathrm{H}}) = 0. \tag{1.13}$$

QUESTION 11

Use (1.13) to find an expression for μ as a function of $\boldsymbol{C_n}$, g, and P_{Tx}, only. Show that $\boldsymbol{T}$ can be written as

$$\boldsymbol{T} = \frac{1}{g}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H} + \frac{\operatorname{tr}(\boldsymbol{C_n})}{P_{\mathrm{Tx}}}\mathbf{I}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}.$$

Use the power constraint to find an expression for g as a function of $\boldsymbol{H}$, $\boldsymbol{C_n}$, and P_{Tx}.
Hint: Assume that the power constraint $\mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right] \leq P_{\mathrm{Tx}}$ is fulfilled with equality.

PROGRAMMING TASK 6

Implement a function that takes a channel $\boldsymbol{H}$, the noise covariance matrix $\boldsymbol{C_n}$, and the available sum transmit power P_{Tx} as inputs and that computes the achievable rate and MSE for the precoder and receive filter determined in Tasks 8 and 11.

1 Deliverables (Matlab code file): `tf_mmseallocation.m`
- Function definition:
  ```
  function [R, MSE] = tf_mmseallocation(H,Cn,Ptx)
  ```

2 Input Specifications:
- `H`: Channel
- `Cn`: Noise covariance matrix
- `Ptx`: available transmit power

3 Output Specifications:
- `R`: Achievable rate

1.1.5 Graphical Comparison of Power Allocation Schemes

To visualize the achievable performance of the discussed power allocation schemes, a script shall be created in the next programming task for plotting the achievable mutual information and MMSE, respectively, versus the average total transmit power P_{Tx} in *decibels* (dB).

PROGRAMMING TASK 7

Modify the two given Matlab scripts `rate_visualization.m` and `mmse_visualization.m` in order to plot the achievable rate versus P_{Tx} in dB for above power allocation schemes. For the waterfilling and MMSE power allocation, mark those values for P_{Tx} where the power allocation switches from K to $K+1$ active streams in the plots.

1 Deliverables:

`rate_visualization.m` and `mmse_visualization.m`

2 Input Specifiations:

- `N`: number of transmit/receive antennas
- `Ptx_dB`: array of available transmit power in dB

3 Internal Specifications:

- `R_waterfilling`: array of achievable rates for the waterfilling solution
- `R_mmse`: array of achievable rates for the MSE minimizing power allocation
- `R_uniform`: array of achievable rates for the uniform power allocation
- `R_tf`: array of achievable rates for the MSE minimizing power allocation with transmit filter design.
- `MMSE_waterfilling`: array of achievable MSE values for the waterfilling solution
- `MMSE_mmse`: array of achievable MSE values for the MSE minimizing power allocation
- `MMSE_uniform`: array of achievable MSE values for the uniform power allocation.
- `MMSE_tf_mmse`: array of achievable MSE values for the MSE minimizing allocation with transmit filter design.

4 Hint(s):

- First, transform the array of dB values for P_{Tx} into the actual transmit powers. Use the channel from `example_channels.mat` and determine $\phi_1, \ldots, \phi_N$ and calculate $\psi_1, \ldots, \psi_N$ for the various schemes. Based on these results, calculate the achievable rates. Finally, calculate the P_{Tx} values (in dB) and the corresponding rates and MMSEs where the waterfilling scheme and the MMSE power allocation increase their number of active streams and mark these points in the plots.

Based on the plots for these scripts, the achieved rate curves of the various power allocation schemes shall be compared for the described system in `example_channels.mat`.

QUESTION 12

Run the Matlab script `rate_visualization.m` to create a plot for the achievable rate with the various power allocation schemes and answer the following questions:

- What are the slopes of the waterfilling and the MMSE power allocation curves for P_{Tx} equal to $30\,\text{dB}$?
- What is the distance in dB between the waterfilling curve and the MMSE curve for a rate of 20 bits per channel use?
- Comment on the high SNR behavior of the uniform power allocation and the MMSE solution with transmit filter design, only.

QUESTION 13

Run the Matlab script `mmse_visualization.m` to create a plot for the achievable MSE with the various power allocation schemes and answer the following questions:

- What are the slopes of the waterfilling and the MMSE power allocation curves for P_{Tx} equal to $30\,\text{dB}$?
- What is the distance in dB between the waterfilling curve and the MMSE power allocation curve for an MSE of 3×10^{-1}?
- Comment on the high SNR behavior of the uniform power allocation and the MMSE solution with transmit filter design, only.