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Q1.

$$R_k^{BC} \le I(s_k ; y_k \mid s_1, \dots, s_{k-1})$$

$$= h(y_k \mid s_1, \dots, s_{k-1}) - h(y_k \mid s_1, \dots s_k)$$

$$= \log_2 \det\left(\pi e\left(\mathbf{I}_m + \mathbf{H}_k \sum_{i=k}^{K} S_i \mathbf{H}_k^H\right)\right) - \log_2 \det\left(\pi e\left(\mathbf{I}_m + \mathbf{H}_k \sum_{i=k+1}^{K} S_i \mathbf{H}_k^H\right)\right)$$

$$= \log_2 \det\left(\mathbf{I}_m + \left(\mathbf{I}_m + \mathbf{H}_k \sum_{i=k+1}^{K} S_i \mathbf{H}_k^H\right)^{-1} \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H\right)$$

Q2.

$$R_k^{MAC} \le I(x_k, y \mid x_{k+1}, \dots, x_K)$$

$$= h(y \mid x_{k+1}, \dots x_K) - h(y \mid x_k, x_{k+1}, \dots x_K)$$

$$= \log_2 \det\left(\pi e\left(\mathbf{I}_N + \sum_{i=1}^{K} \mathbf{H}_i^H \mathbf{Q}_i \mathbf{H}_i\right)\right) - \log_2 \det\left(\pi e\left(\mathbf{I}_N + \sum_{i=1}^{k-1} \mathbf{H}_i^H \mathbf{Q}_i \mathbf{H}_i\right)\right)$$

$$= \log_2 \det\left(\mathbf{I}_N + \left(\mathbf{I}_N + \sum_{i=1}^{k-1} \mathbf{H}_i^H \mathbf{Q}_i \mathbf{H}_i\right)^{-1} \mathbf{H}_k^H \mathbf{Q}_k \mathbf{H}_k\right)$$

Q3. for $r = N$, $U \in \mathbb{C}^{M\times N}$, $U' \in \mathbb{C}^{M\times(M-N)}$

$$S = VU^H Q U V^H$$

$$= VU^H \begin{bmatrix} U & U' \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^H & Q_{22} \end{bmatrix} \begin{bmatrix} U^H \\ U'^H \end{bmatrix} U V^H$$

$$= V \begin{bmatrix} I_N & 0 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^H & Q_{22} \end{bmatrix} \begin{bmatrix} I_N \\ 0 \end{bmatrix} V^H$$

$$= \begin{bmatrix} V I_N Q_{11} & V I_N Q_{12} \end{bmatrix} \begin{bmatrix} I_N \\ 0 \end{bmatrix} V^H$$

$$= V Q_{11} V^H$$

$$tr(S) = tr(VQ_{11}V^H) = tr(Q_{11})$$

Trace of Q reads as:

$$tr(Q) = tr(Q_{11}) + tr(Q_{22})$$

since Q_{11} and Q_{22} are positive semidefinite,

$$tr(S) = tr(Q_{11}) \leqslant tr(Q_{11}) + tr(Q_{22}) = tr(Q)$$

Q4.

$$R_k^{MAC} = \log_2 \det\Big(\mathbf{I} + \underbrace{\mathbf{X}_k^{-\frac{1}{2}} \mathbf{H}_k^H \mathbf{F}_k^{-\frac{1}{2}}}_{\mathbf{H}_{k,eff}^H} \underbrace{\mathbf{F}_k^{\frac{1}{2}} \mathbf{Q}_k \mathbf{F}_k^{\frac{H}{2}}}_{\mathbf{Q}_{k,eff}} \underbrace{\mathbf{F}_k^{-\frac{H}{2}} \mathbf{H}_k \mathbf{X}_k^{-\frac{H}{2}}}_{\mathbf{H}_{k,eff}}\Big)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{V}_k \mathbf{\Sigma}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^H\right)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{\Sigma}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{\Sigma}_k\right) \quad (1)$$

$$R_k^{BC} = \log_2 \det\Big(\mathbf{I} + \underbrace{\mathbf{F}_k^{-\frac{H}{2}} \mathbf{H}_k \mathbf{X}_k^{-\frac{H}{2}}}_{\mathbf{H}_{k,eff}} \mathbf{S}_{k,eff} \underbrace{\mathbf{X}^{-\frac{1}{2}} \mathbf{H}_k^H \mathbf{F}_k^{-\frac{1}{2}}}_{\mathbf{H}_{k,eff}^H}\Big)$$

with $\mathbf{S}_{k,eff} = \mathbf{X}_k^{\frac{H}{2}} \mathbf{S}_k \mathbf{X}_k^{\frac{1}{2}}$

$$\mathbf{S}_k = \mathbf{X}_k^{-\frac{H}{2}} \mathbf{V}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{V}_k^H \mathbf{X}_k^{-\frac{1}{2}}$$

$$\mathbf{S}_{k,eff} = \mathbf{X}^{\frac{H}{2}} \mathbf{X}^{-\frac{H}{2}} \mathbf{V}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{V}_k^H \mathbf{X}_k^{-\frac{1}{2}} \mathbf{X}_k^{\frac{1}{2}}$$

$$= \mathbf{V}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{V}_k^H$$

Bring $\mathbf{S}_{k,eff}$ back to R_k^{BC}:

$$R_k^{BC} = \log_2 \det\left(\mathbf{I} + \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^H \mathbf{V}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{V}_k^H \mathbf{V}_k \mathbf{\Sigma}_k \mathbf{U}_k^H\right)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{U}_k^H\right)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{\Sigma}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{\Sigma}_k\right) \quad (2)$$

compare (1) with (2), $R_k^{BC} = R_k^{MAC}$

Q5.

with order [1,2]

$tr(S_1) = 5,3936$ $\quad$ $tr(S_2) = 4,6064$

$R = [5,1943 \quad 2,4790]$

with order [2,1]

$tr(S_1) = 3,4113$ $\quad$ $tr(S_2) = 6,5887$

$R = [4,1384 \quad 3,5349]$

Q6. $\hat{s}_k = G_k \sum_{i=1}^{K} H_i^H W_i s_i + G_k n$

$$\varepsilon_k = E\left[\|s_k - \hat{s}_k\|_2^2\right]$$

$$= E\left[\left\|s_k - G_k \sum_{i=1}^{K} H_i^H W_i s_i - G_k n\right\|_2^2\right]$$

$$= tr\Big(I - G_k H_k^H W_k - W_k^H H_k G_k^H + G_k \sum_{i=1}^{K} H_i^H W_i W_i^H H_i G_k^H + G_k G_k^H\Big)$$

Q7. Bring $G_{i,MMSE} = W_i^H H_i \left(I + \sum_{l=1}^{K} H_l^H W_l W_l^H H_l\right)^{-1} = W_i^H H_i X^{-1}$

into Q6:

$$\varepsilon_k = tr\Big(I - 2W_k^H H_k X^{-1} H_k^H W_k + W_k^H H_k X^{-1} \sum_{i=1}^{K} H_i^H W_i W_i^H H_i X^{-1} H_k^H W_k + W_k^H H_k X^{-1} X^{-1} H_k^H W_k\Big)$$

$$= tr\Big(I - 2W_k^H H_k X^{-1} H_k^H W_k + W_k^H H_k X^{-1} \underbrace{\Big(\sum_{l=1}^{K} H_l^H W_l W_l^H H_l + I\Big)}_{X} X^{-1} H_k^H W_k\Big)$$

$$= tr\left(I - 2W_k^H H_k X^{-1} H_k^H W_k + W_k^H H_k X^{-1} H_k^H W_k\right)$$

$$= tr\left(I - W_k^H H_k X^{-1} H_k^H W_k\right)$$

Therefore, $\varepsilon = \sum_{k=1}^{K} \varepsilon_k = c + tr(X^{-1})$ with $c \in \mathbb{N}_0$

Q8. $\dfrac{\partial\, tr(\underset{\sim}{X}^{-1})}{\partial \underset{\sim}{Q}_k^T}$

$$= \frac{\partial\, tr\left(\left(\underset{\sim}{I} + \sum_{l=1}^{K} \underset{\sim}{H}_l^H \underset{\sim}{W}_l \underset{\sim}{W}_l^H \underset{\sim}{H}_l\right)^{-1}\right)}{\partial \underset{\sim}{W}_k \underset{\sim}{W}_k^H}$$

$$= -\left(\underset{\sim}{I} + \sum_{l=1}^{K} \underset{\sim}{H}_l^H \underset{\sim}{W}_l \underset{\sim}{W}_l^H \underset{\sim}{H}_l\right)^{-1} \frac{\partial\, tr\left(\underset{\sim}{I} + \sum_{l=1}^{K} H_l^H W_l W_l^H H_l\right)}{\partial W_k W_k^H} \cdot \left(\underset{\sim}{I} + \sum_{l=1}^{K} H_l^H W_l W_l^H H_l\right)^{-1}$$

$$= -\left(\underset{\sim}{I} + \sum_{l=1}^{K} H_l^H W_l W_l^H H_l\right)^{-1} H_k^H H_k \left(\underset{\sim}{I} + \sum_{l=1}^{K} H_l^H W_l W_l^H H_l\right)^{-1}$$

Q9.