Project 1. (Group Member: Jianqing Li (alone))

Q1 For k active stream

$$\gamma_k = \mu' - \frac{1}{\phi_k} > 0 \qquad \gamma_{k+1} = 0 \ \text{since} \ \mu' - \frac{1}{\phi_{k+1}} < 0$$

$$\mu' = \frac{1}{k}\left(P_{tx} + \sum_{i=1}^{k} \frac{1}{\phi_i}\right)$$

$$\frac{1}{\phi_k} < \frac{1}{k}\left(P_{tx} + \sum_{i=1}^{k}\frac{1}{\phi_i}\right) < \frac{1}{\phi_{k+1}}$$

$$\frac{k}{\phi_k} - \sum_{i=1}^{k}\frac{1}{\phi_i} < P_{tx} < \frac{k}{\phi_{k+1}} - \sum_{i=1}^{k}\frac{1}{\phi_i}$$

If $P_{tx} \geq \frac{k}{\phi_{k+1}} - \sum_{i=1}^{k} \frac{1}{\phi_i}$, then there can be k+1 active stream

For $P_{tx} \to \infty$ $\gamma_i = \max\left(0, \frac{1}{k}P_{tx} + \frac{1}{k}\sum_{i=1}^{k}\frac{1}{\phi_i} - \frac{1}{\phi_i}\right)$

$$\lim_{P_{tx}\to\infty} \gamma_i = \frac{P_{tx}}{k}$$

Q2

$$P_{tx} = [0.1504, 2.3402, 26.2019]^T$$

Q3 For k active stream

$$\gamma_k = \frac{\bar{\mu}}{\sqrt{\phi_k}} - \frac{1}{\phi_k} > 0 \qquad \gamma_{k+1} = 0, \ \text{since} \ \frac{\bar{\mu}}{\sqrt{\phi_{k+1}}} - \frac{1}{\phi_{k+1}} < 0$$

$$\bar{\mu} = \frac{P_{tx} + \sum_{i=1}^{k} \frac{1}{\phi_i}}{\sum_{i=1}^{k} \frac{1}{\sqrt{\phi_i}}}$$

$$\frac{1}{\sqrt{\phi_k}} < \frac{P_{tx} + \sum_{i=1}^{k}\frac{1}{\phi_i}}{\sum_{i=1}^{k}\frac{1}{\sqrt{\phi_i}}} < \frac{1}{\sqrt{\phi_{k+1}}}$$

$$\frac{1}{\sqrt{\phi_k}}\sum_{i=1}^{k}\frac{1}{\sqrt{\phi_i}} - \sum_{i=1}^{k}\frac{1}{\phi_i} < P_{tx} < \frac{1}{\sqrt{\phi_{k+1}}}\sum_{i=1}^{k}\frac{1}{\sqrt{\phi_i}} - \sum_{i=1}^{k}\frac{1}{\phi_i}$$

If $P_{tx} \geq \frac{1}{\sqrt{\phi_{k+1}}}\sum_{i=1}^{k}\frac{1}{\sqrt{\phi_i}} - \sum_{i=1}^{k}\frac{1}{\phi_i}$, then there can be k+1 active stream

$P_{tx} \to \infty$ $\lim_{P_{tx}\to\infty} \bar{\mu} = \frac{P_{tx}}{\sum_{i=1}^{k}\frac{1}{\sqrt{\phi_i}}}$ $\lim_{P_{tx}\to\infty} \gamma_i = \frac{P_{tx}}{\sqrt{\phi_i}\sum_{j=1}^{k}\frac{1}{\sqrt{\phi_j}}} = \frac{\bar{\mu}}{\sqrt{\phi_i}}$

Q4

$$P_{tx} = [0.0576, 0.5892, 4.2952]^T$$

Q5:

$R_K = \log_2 \det(I_N + H^H C_n^{-1} H Q)$ $\quad H^H C_n^{-1} H = V \Phi V^H$ $\quad Q = V \Psi V^H$

$= \log_2 \det(I_N + V \Phi V^H V \Psi V^H)$ $\quad \Phi, \Psi$ diagonal

$= \log_2 \det(I_N + \Phi \Psi)$ $\quad$ since $\psi_1 \sim \psi_K = \frac{P_{tx}}{K}$ $\quad \psi_{K+1} \sim \psi_N = 0$

$= \sum_{i=1}^{K} \log_2\left(1 + \phi_i \frac{P_{tx}}{K}\right)$

Q6:

For uniform rate allocation

$K=1$ $\quad P_{tx} = 0.3009$

$K=2$ $\quad P_{tx} = 4.4216$

$K=3$ $\quad P_{tx} = 48.1417$

For waterfilling

$K=1$ $\quad P_{tx} = 0.1504$

$K=2$ $\quad P_{tx} = 2.3402$

$K=3$ $\quad P_{tx} = 26.2019$

- waterfilling solution switches earlier from $K$ to $K+1$

- Yes, since when $P_{tx} \to \infty$, waterfilling solution also allocates the transmission power uniformlly, just like uniform rate allocation

Q7

$\hat{s} = g\,\underset{\sim}{H}\,\underset{\sim}{T}\,\underline{s} + g\,\underline{n}$

$\varepsilon(T,g) = E[\|s-\hat{s}\|_2^2] = E[\|s - g\underset{\sim}{H}\underset{\sim}{T}\underline{s} - g\underline{n}\|_2^2] = E[\|(\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n}\|_2^2]$

$= E[((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n})^H((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n})]$

$= E[tr(((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n})^H((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n}))]$

$= E[tr(((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n})((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n})^H]$

$= tr(E[((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n})((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s} - g\underline{n})^H])$

$= tr(E[(\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})\underline{s}\underline{s}^H(\underset{\sim}{I}_B - g^*\underset{\sim}{T}^H\underset{\sim}{H}^H) + |g|^2\underline{n}\underline{n}^H])$ since $E[sn^H]=0$

$= tr((\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T})(\underset{\sim}{I}_B - g^*\underset{\sim}{T}^H\underset{\sim}{H}^H)) + |g|^2 tr(\underset{\sim}{C}_n)$

$= tr((\underset{\sim}{I}_B - g^*\underset{\sim}{T}^H\underset{\sim}{H}^H)(\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T}) + |g|^2\underset{\sim}{C}_n)$

$= tr(\underset{\sim}{I}_B - g\underset{\sim}{H}\underset{\sim}{T} - g^*\underset{\sim}{T}^H\underset{\sim}{H}^H + |g|^2\underset{\sim}{T}^H\underset{\sim}{H}^H\underset{\sim}{H}\underset{\sim}{T} + |g|^2\underset{\sim}{C}_n)$

$= B - g\,tr(\underset{\sim}{H}\underset{\sim}{T}) - g^*tr(\underset{\sim}{T}^H\underset{\sim}{H}^H) + |g|^2 tr(\underset{\sim}{T}^H\underset{\sim}{H}^H\underset{\sim}{H}\underset{\sim}{T} + \underset{\sim}{C}_n)$

$= B - g\,tr(\underset{\sim}{H}\underset{\sim}{T}) - g^*tr(\underset{\sim}{T}^H\underset{\sim}{H}^H) + gg^*tr(\underset{\sim}{T}^H\underset{\sim}{H}^H\underset{\sim}{H}\underset{\sim}{T} + \underset{\sim}{C}_n)$

Q8:

$\frac{\partial\varepsilon(T,g)}{\partial g} = -tr(\underset{\sim}{H}\underset{\sim}{T}) + g^*tr(\underset{\sim}{T}^H\underset{\sim}{H}^H\underset{\sim}{H}\underset{\sim}{T} + \underset{\sim}{C}_n) \doteq 0$

$g^* = \frac{tr(\underset{\sim}{H}\underset{\sim}{T})}{tr(\underset{\sim}{T}^H\underset{\sim}{H}^H\underset{\sim}{H}\underset{\sim}{T} + \underset{\sim}{C}_n)} \Rightarrow g = \frac{tr(\underset{\sim}{T}^H\underset{\sim}{H}^H)}{tr(\underset{\sim}{T}^H\underset{\sim}{H}^H\underset{\sim}{H}\underset{\sim}{T} + \underset{\sim}{C}_n)}$

since $\varepsilon(T,g)$ is convex in $g$, $g$ is also a global minimum.

$$g = \frac{tr(\underset{\sim}{T}^H\underset{\sim}{H}^H)}{tr(\underset{\sim}{T}^H\underset{\sim}{H}^H\underset{\sim}{H}\underset{\sim}{T} + \underset{\sim}{C}_n)}$$

Q9:

$E[\|x\|^2] = E[tr(\underline{x}^H\underline{x})] = E[tr(\underline{x}\underline{x}^H)] = tr(E[xx^H]) = tr(E[Tss^HT^H]) = tr(TT^H)$

$tr(TT^H) \leq P_{Tx}$

$L(T,\mu) = B - g\,tr(\underset{\sim}{H}\underset{\sim}{T}) - g^*tr(\underset{\sim}{T}^H\underset{\sim}{H}^H) + gg^*tr(\underset{\sim}{T}^H\underset{\sim}{H}^H\underset{\sim}{H}\underset{\sim}{T} + \underset{\sim}{C}_n) + \mu(tr(TT^H) - P_{Tx})$

KKT conditions:

$\frac{\partial L(T,\mu)}{\partial T} = 0 \qquad \mu(tr(TT^H) - P_{Tx}) = 0 \qquad \mu \geq 0$

$tr(TT^H) - P_{Tx} \leq 0$

Q10:

$$\frac{\partial L(T,\mu)}{\partial T} = -g\,\underset{\sim}{H}^T + |g|^2 (\underset{\sim}{T}^H \underset{\sim}{H}^H \underset{\sim}{H})^T + \mu \cdot (T^H)^T \stackrel{!}{=} 0$$

$$T^H(\mu \underset{\sim}{I} + |g|^2 \underset{\sim}{H}^H \underset{\sim}{H}) = g\,\underset{\sim}{H} \qquad T^H = g\,\underset{\sim}{H}(\mu \underset{\sim}{I}_N + |g|^2 \underset{\sim}{H}^H \underset{\sim}{H})^{-1}$$

$$\underset{\sim}{T} = g^*(\mu \underset{\sim}{I}_N + |g|^2 \underset{\sim}{H}^H \underset{\sim}{H})^{-1} \underset{\sim}{H}^H$$

Q11:

$$\mu(tr(TT^H) - P_{TX}) = 0 \qquad \mu > 0 \qquad tr(TT^H) - P_{TX} = 0$$

$$-|g|^2 tr(\underset{\sim}{C}_n) + \mu P_{TX} = 0 \qquad \mu = |g|^2 \frac{tr(\underset{\sim}{C}_n)}{P_{TX}}$$

$$\underset{\sim}{T} = g^*(\mu \underset{\sim}{I}_N + |g|^2 \underset{\sim}{H}^H \underset{\sim}{H})^{-1} \underset{\sim}{H}^H$$

$$= g^*\left(|g|^2 \frac{tr(\underset{\sim}{C}_n)}{P_{TX}} + |g|^2 \underset{\sim}{H}^H \underset{\sim}{H}\right)^{-1} \underset{\sim}{H}^H$$

$$= \frac{g^*}{g g^*}\left(\frac{tr(\underset{\sim}{C}_n)}{P_{TX}} + \underset{\sim}{H}^H \underset{\sim}{H}\right)^{-1} \underset{\sim}{H}^H$$

$$\underset{\sim}{T} = \frac{1}{g}\left(\underset{\sim}{H}^H \underset{\sim}{H} + \frac{tr(\underset{\sim}{C}_n)}{P_{TX}} \underset{\sim}{I}\right)^{-1} \underset{\sim}{H}^H$$

$$tr(\underset{\sim}{T}\underset{\sim}{T}^H) = P_{TX}$$

$$tr\left(\left(\underset{\sim}{H}^H \underset{\sim}{H} + \frac{tr(\underset{\sim}{C}_n)}{P_{TX}} \underset{\sim}{I}\right)^{-1} \underset{\sim}{H}^H \underset{\sim}{H} \left(\underset{\sim}{H}^H \underset{\sim}{H} + \frac{tr(\underset{\sim}{C}_n)}{P_{TX}} \underset{\sim}{I}\right)^{-1}\right) = |g|^2 P_{TX}$$

$$|g|^2 = \frac{1}{P_{TX}} tr\left(\left(\underset{\sim}{H}^H \underset{\sim}{H} + \frac{tr(\underset{\sim}{C}_n)}{P_{TX}} \underset{\sim}{I}_N\right)^{-2} \underset{\sim}{H}^H \underset{\sim}{H}\right)$$

Q12:

- Slopes at 30 dB: waterfilling: 1.3142 bit/channel dB
  MMSE : 1.3142 bit/channel dB
- About 1.8 dB
- For high SNR, uniform power allocation performs as waterfilling. The MMSE solution with transmit filter performs worst. Its rate is smaller than the others

Q13.

- Slopes at 30 dB: waterfilling: −0.0100 /dB
  −0.0057 /dB
- About 2.7 dB
- For high SNR, uniform power allocation performs as good as MMSE with transmit filter.

Although MMSE with transmit filter is MSE larger than MMSE, it performs the second well