Project 3 (Group member : Jianqing Li (alone))

Q1) $R_k^{BC} \le I(\underset{\sim}{s}_k : y_k \mid \underset{\sim}{s}_1, \cdots \underset{\sim}{s}_{k-1})$

$= h(y_k \mid \underset{\sim}{s}_1, \cdots, \underset{\sim}{s}_{k-1}) - h(y_k \mid \underset{\sim}{s}_1, \cdots, \underset{\sim}{s}_k)$

$= \log_2 \det(\pi e(\underset{\sim}{I}_M + \underset{\sim}{H}_k(\sum_{i=k}^{K} \underset{\sim}{S}_i)\underset{\sim}{H}_k^H)) -$

$\log_2 \det(\pi e(\underset{\sim}{I}_M + \underset{\sim}{H}_k(\sum_{i=k+1}^{K} \underset{\sim}{S}_i)\underset{\sim}{H}_k^H))$

$= \log_2 \det(\underset{\sim}{I}_M + (\underset{\sim}{I}_M + \underset{\sim}{H}_k(\sum_{i=k+1}^{K} \underset{\sim}{S}_i)\underset{\sim}{H}_k^H)^{-1} \underset{\sim}{H}_k \underset{\sim}{S}_k \underset{\sim}{H}_k^H)$

$= \log_2 \det(\underset{\sim}{I}_N + \underset{\sim}{H}_k^H(\underset{\sim}{I}_M + \underset{\sim}{H}_k(\sum_{i=k+1}^{K} \underset{\sim}{S}_i)\underset{\sim}{H}_k^H)^{-1} \underset{\sim}{H}_k \underset{\sim}{S}_k)$

Q2) $R_k^{MAC} \le I(\underset{\sim}{x}_k, y \mid \underset{\sim}{x}_{k+1}, \cdots, \underset{\sim}{x}_K)$

$= h(y \mid \underset{\sim}{x}_{k+1}, \cdots, \underset{\sim}{x}_K) - h(y \mid \underset{\sim}{x}_k, \underset{\sim}{x}_{k+1}, \cdots, \underset{\sim}{x}_K)$

$= \log_2 \det(\pi e(I_N + \sum_{i=1}^{k} \underset{\sim}{H}_i^H Q_i \underset{\sim}{H}_i)) - \log_2 \det(\pi e(\underset{\sim}{I}_N + \sum_{i=1}^{k-1} \underset{\sim}{H}_i^H Q_i \underset{\sim}{H}_i))$

$= \log_2 \det(\underset{\sim}{I}_N + (\underset{\sim}{I}_N + \sum_{i=1}^{k-1} \underset{\sim}{H}_i^H Q_i \underset{\sim}{H}_i)^{-1} \underset{\sim}{H}_k^H Q_k \underset{\sim}{H}_k)$

$= \log_2 \det(\underset{\sim}{I}_M + \underset{\sim}{H}_k(\underset{\sim}{I}_N + \sum_{i=1}^{k-1} \underset{\sim}{H}_i^H \underset{\sim}{Q}_i \underset{\sim}{H}_i)^{-1} \underset{\sim}{H}_k^H \underset{\sim}{Q}_k)$

Q3) $S = VU^H Q U V^H = VU^H [U \ U'] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^H & Q_{22} \end{bmatrix} \begin{bmatrix} U^H \\ U'^H \end{bmatrix} U V^H$

For $r=N$ since $[U \ U']$ ~~and U are~~ is unitary, U is subunitary

we have $U^H U' = 0 \quad U'^H U = 0 \quad U^H U = \underset{\sim}{I}_r$

Thus, $S = \underset{\sim}{V} [\underset{\sim}{I}_r \ \underset{\sim}{0}] \begin{bmatrix} \underset{\sim}{Q}_{11} & \underset{\sim}{Q}_{12} \\ \underset{\sim}{Q}_{12}^H & \underset{\sim}{Q}_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{I}_r \\ \underset{\sim}{0} \end{bmatrix} \underset{\sim}{V}^H$

$= \underset{\sim}{V} \underset{\sim}{Q}_{11} \underset{\sim}{V}^H$

Finally, we have $tr(S) = tr(\underset{\sim}{Q}_{11} \underset{\sim}{V}^H \underset{\sim}{V}) = tr(\underset{\sim}{Q}_{11})$

$tr(Q) = tr([U \ U'] \begin{bmatrix} \underset{\sim}{Q}_{11} & \underset{\sim}{Q}_{12} \\ \underset{\sim}{Q}_{12}^H & \underset{\sim}{Q}_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}^H \\ \underset{\sim}{U}'^H \end{bmatrix}) = tr(\begin{bmatrix} \underset{\sim}{Q}_{11} & \underset{\sim}{Q}_{12} \\ \underset{\sim}{Q}_{12}^H & \underset{\sim}{Q}_{22} \end{bmatrix})$

$= tr(\underset{\sim}{Q}_{11}) + tr(\underset{\sim}{Q}_{22})$

since $\underset{\sim}{Q}_{11}$, $\underset{\sim}{Q}_{22}$ are positive semi-definite,

We have $tr(\underset{\sim}{S}) = tr(\underset{\sim}{Q}_{11}) \le tr(\underset{\sim}{Q}_{11}) + tr(\underset{\sim}{Q}_{22}) = tr(\underset{\sim}{Q})$

$\Rightarrow tr(\underset{\sim}{S}) \le tr(\underset{\sim}{Q})$

For $r=M$ $\quad S = VU^H Q U V^H \quad tr(S) = tr(VU^H Q U V^H) = tr(U^H Q U V^H V)$

$\hookrightarrow$ U is not only subunitary, $\quad = tr(U^H Q U) = tr(U U^H Q) = tr(\underset{\sim}{Q})$

but also unitary $\quad \Rightarrow tr(\underset{\sim}{S}) = tr(\underset{\sim}{Q})$

Q4)

$$R_k^{MAC} = \log_2 \det\Big(\mathbf{I} + \underbrace{\mathbf{X}_k^{-\frac{1}{2}} \mathbf{H}_k^H \mathbf{E}_k^{-\frac{1}{2}}}_{H_{k,eff}^H} \underbrace{\mathbf{E}_k^{\frac{1}{2}} \mathbf{Q}_k \mathbf{E}_k^{\frac{H}{2}}}_{Q_{k,eff}} \underbrace{\mathbf{E}_k^{-\frac{H}{2}} \mathbf{H}_k \mathbf{X}_k^{-\frac{H}{2}}}_{H_{k,eff}}\Big)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{H}_{k,eff}^H \mathbf{Q}_{k,eff} \mathbf{H}_{k,eff}\right)$$

$$R_k^{BC} = \log_2 \det\left(\mathbf{I} + \mathbf{E}_k^{-\frac{H}{2}} \mathbf{H}_k \mathbf{X}_k^{-\frac{H}{2}} \mathbf{S}_{k,eff} \mathbf{X}_k^{-\frac{1}{2}} \mathbf{H}_k^H \mathbf{E}_k^{-\frac{1}{2}}\right)$$

$$= \log_2 \det\left(\mathbf{I} + H_{k,eff} \mathbf{S}_{k,eff} \mathbf{H}_{k,eff}^H\right) \text{ with } \mathbf{S}_{k,eff} = \mathbf{X}_k^{\frac{H}{2}} \mathbf{S}_k \mathbf{X}_k^{\frac{1}{2}}$$

$$R_k^{MAC} = \log_2 \det\left(\mathbf{I} + \mathbf{V}_k \mathbf{\Sigma}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^H\right)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{\Sigma}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^H \mathbf{V}_k\right)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{\Sigma}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{\Sigma}_k\right)$$

$$R_k^{BC} = \log_2 \det\left(\mathbf{I} + \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{V}_k^H \mathbf{S}_{k,eff} \mathbf{V}_k \mathbf{\Sigma}_k \mathbf{U}_k^H\right)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{\Sigma}_k \mathbf{V}_k^H \mathbf{S}_{k,eff} \mathbf{V}_k \mathbf{\Sigma}_k\right)$$

$$\mathbf{S}_k = \mathbf{X}_k^{-\frac{H}{2}} \mathbf{V}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{V}_k^H \mathbf{X}_k^{-\frac{1}{2}}$$

$$\Downarrow$$

$$\mathbf{S}_{k,eff} = \mathbf{X}_k^{\frac{H}{2}} \mathbf{S}_k \mathbf{X}_k^{\frac{1}{2}} = \mathbf{V}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{V}_k^H$$

$$\Downarrow$$

$$R_k^{BC} = \log_2 \det\Big(\mathbf{I} + \mathbf{\Sigma}_k \overset{\mathbf{I}}{\boxed{\mathbf{V}_k^H \mathbf{V}_k}} \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \overset{\mathbf{I}}{\boxed{\mathbf{V}_k^H \mathbf{V}_k}} \mathbf{\Sigma}_k\Big)$$

$$= \log_2 \det\left(\mathbf{I} + \mathbf{\Sigma}_k \mathbf{U}_k^H \mathbf{Q}_{k,eff} \mathbf{U}_k \mathbf{\Sigma}_k\right)$$

Q5)

order $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$tr(S_1) = 5.3936$

$tr(S_2) = 4.6064$

R_BC = R_MAC =
$\begin{bmatrix} 5.1943 \\ 2.4790 \end{bmatrix}$

order $= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

$tr(S_1) = 3.4113$

$tr(S_2) = 6.5887$

R_BC = R_MAC =
$\begin{bmatrix} 4.1384 \\ 3.5349 \end{bmatrix}$

Q6)

① sum power constraint

$$tr(\underset{\sim}{Q_1}) \leq P_{tx} \qquad tr(\underset{\sim}{Q_2}) \leq P_{tx}$$

$$\alpha\, tr(\underset{\sim}{Q_1}) + (1-\alpha)\, tr(\underset{\sim}{Q_2}) \leq \alpha P_{tx} + (1-\alpha) P_{tx} = P_{tx}$$

$$\alpha \underset{\sim}{Q_1} + (1-\alpha)\underset{\sim}{Q_2} \succeq 0$$

② positive semidefiniteness

$$\underset{\sim}{Q_1} \succeq 0 \iff \underline{x}^H \underset{\sim}{Q_1} \underline{x} \geq 0 \quad \forall x$$

$$\underset{\sim}{Q_2} \succeq 0 \iff \underline{x}^H \underset{\sim}{Q_2} \underline{x} \geq 0 \quad \forall x$$

$$\underline{x}^H(\alpha \underset{\sim}{Q_1} + (1-\alpha)\underset{\sim}{Q_2})\underline{x} = \alpha\, \underline{x}^H \underset{\sim}{Q_1}\underline{x} + (1-\alpha)\underline{x}^H \underset{\sim}{Q_2}\underline{x} \geq 0$$

$$\Updownarrow$$

$$\alpha \underset{\sim}{Q_1} + (1-\alpha)\underset{\sim}{Q_2} \succeq 0$$

Q7)

$$P_{TX} = 10 \quad Q[1] = \begin{bmatrix} 3.0273 & 0.4629 \\ 0.4629 & 3.6667 \end{bmatrix} \quad Q[2] = \begin{bmatrix} 3.2078 & 0.5613 \\ 0.5613 & 0.0982 \end{bmatrix}$$

$$C_{sum} = \cancel{10.2623}\ 10.3623$$

Q8)

$$order = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$S[1] = \begin{bmatrix} 2.4443 & 1.1883 & -1.5921 \\ 1.1883 & 2.8745 & 0.7750 \\ -1.5921 & 0.7750 & 2.0817 \end{bmatrix} \quad S[2] = \begin{bmatrix} 1.3643 & -0.8969 & 0.9385 \\ -0.8969 & 0.5896 & -0.6169 \\ 0.9385 & -0.6169 & 0.6456 \end{bmatrix}$$

$$tr(S[1]) = 7.4005 \qquad tr(S[2]) = 2.5995$$

$$tr(S[1]) + tr(S[2]) = 10$$

$$order = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$S[1] = \begin{bmatrix} 0.7379 & 1.3966 & 0.3470 \\ 1.3966 & 3.1440 & -0.3567 \\ 0.3470 & -0.3567 & 2.2157 \end{bmatrix} \quad S[2] = \begin{bmatrix} 2.8995 & -1.2306 & -1.1805 \\ -1.2306 & 0.5223 & 0.5010 \\ -1.1805 & 0.5010 & 0.4806 \end{bmatrix}$$

$$tr(S[1]) = 6.0976 \qquad tr(S[2]) = 3.9024$$

$$tr(S[1]) + tr(S[2]) = 10$$

Q9)

25dB: ~~24.3939~~ 24.3939 bits

30dB: ~~29.3505~~ 29.3505 bits

slope: $(29.3505 - 24.3939)/(30dB - 25dB) = 0.99132$ bits/dB

Q10)

① User 1 always has larger rate than user 2. ~~but in case~~

② Depending on the encoding order, the relative rates difference between user 1 and user 2 can be changed.

⇒ If user 1 is ~~encoded~~ encoded first, the difference is larger.

⇒ If user 2 is ~~encoded~~ encoded first, the difference is smaller.

③ The two ~~decoding~~ encoded order have larger rate difference at high SNR region.

Q11)

$$\max_{\underset{\sim}{T}_k} \log_2 \det(\underset{\sim}{I}_n + \underbrace{\underset{\sim}{V}_k^H \underset{\sim}{H}_k^H}_{\hat{\underset{\sim}{H}}_k^H} \underset{\sim}{C}_{nk}^{-1} \underbrace{\underset{\sim}{H}_k \underset{\sim}{V}_k}_{\hat{\underset{\sim}{H}}_k} \underset{\sim}{T}_k \underset{\sim}{T}_k^H) \quad \text{s.t. } tr(\underset{\sim}{T}_k \underset{\sim}{T}_k^H) \leq \frac{P_{tx}}{K}$$

Optimal T_k:

$$\underset{\sim}{T}_k \underset{\sim}{T}_k^H = \underset{\sim}{V} \underset{\sim}{\Phi} \underset{\sim}{V}^H \qquad \hat{\underset{\sim}{H}}_k^H \underset{\sim}{C}_{nk}^{-1} \hat{\underset{\sim}{H}}_k = \underset{\sim}{U} \underset{\sim}{\Sigma} \underset{\sim}{U}^H$$

For $\underset{\sim}{T}_k \underset{\sim}{T}_k^H$: $\underset{\sim}{V} = \underset{\sim}{U}$, $\underset{\sim}{\Phi}$ is found by waterfilling

$\underset{\sim}{T}_k$ can be constructed by $\underset{\sim}{T}_k = \underset{\sim}{U} \underset{\sim}{\Phi}^{\frac{1}{2}}$

Q12)

$$\max_{\underset{\sim}{T}} \log_2 \det(\underset{\sim}{I}_{MK} + \underbrace{\underset{\sim}{V}^H \underset{\sim}{H}^H}_{\underset{\sim}{\hat{H}}^H} \underset{\sim}{C}^{-1} \underbrace{\underset{\sim}{H} \underset{\sim}{V}}_{\underset{\sim}{\hat{H}}} \underset{\sim}{T} \underset{\sim}{T}^H) \quad s.t. \ tr(\underset{\sim}{T}\underset{\sim}{T}^H) \leq P_{TX}$$

$\underset{\sim}{T}$ is block diagnal

$\underset{\sim}{\hat{H}}$ is block diagnal, $\underset{\sim}{C}$ is blockdiagnal, $\underset{\sim}{I}_{MK}$ is blockdiagnal

The inverse of a blockdiagnal matrix is block diagnal, since

$$\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B} \end{bmatrix} * \begin{bmatrix} \underset{\sim}{A}^{-1} & 0 \\ \underset{\sim}{0} & \underset{\sim}{B}^{-1} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{I} & 0 \\ 0 & \underset{\sim}{I} \end{bmatrix} = \underset{\sim}{I}$$

The product of blockdiagnal matrices is block diagnal, since

$$\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B} \end{bmatrix} \begin{bmatrix} \underset{\sim}{C} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{D} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}\underset{\sim}{C} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}\underset{\sim}{D} \end{bmatrix}$$

Thus, $\underset{\sim}{\hat{H}}^H \underset{\sim}{C}^{-1} \underset{\sim}{\hat{H}}$ is blockdiagnal, $\underset{\sim}{T}\underset{\sim}{T}^H$ is block diagnol

The EVD of a blockdiagnal matrix is also blockdiagnal, since

$$\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{U}_1 \underset{\sim}{\Lambda}_1 \underset{\sim}{U}_1^H & 0 \\ 0 & \underset{\sim}{U}_2 \underset{\sim}{\Lambda}_2 \underset{\sim}{U}_2^H \end{bmatrix} = \begin{bmatrix} \underset{\sim}{U}_1 & 0 \\ 0 & \underset{\sim}{U}_2 \end{bmatrix} \begin{bmatrix} \underset{\sim}{\Lambda}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}_1^H & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{U}_2^H \end{bmatrix}$$ A, B hermitian.

Thus, the EVD of $\underset{\sim}{\hat{H}}^H \underset{\sim}{C}^{-1} \underset{\sim}{\hat{H}}$ and $\underset{\sim}{T}\underset{\sim}{T}^H$ ~~are~~ are also block diagnal

$\Rightarrow$ apply water filling!

The $\underset{\sim}{T}$ can be found by $\underset{\sim}{T} = \underset{\sim}{U}\underset{\sim}{\Psi}^{\frac{1}{2}}$, where $\underset{\sim}{U}$ is the modal matrix of $\underset{\sim}{\hat{H}}\underset{\sim}{C}^{-1}\underset{\sim}{\hat{H}}$ ($\underset{\sim}{U}$ is also blockdiagnal), $\underset{\sim}{\Psi}$ is the parameter found by ~~waterfing~~ waterfilling. The respective T_k can be found by identifying the coresspording position in the blockdiagnal matrix $\underset{\sim}{T}$.

Q13)

① Optimal power allocation can achieve about twice the code rate of ~~solution~~ ~~2 at low and~~ equal power allocation at low and middle SNR.

② At high SNR, ~~the two schemes tend to give the same performance~~ The difference between optimal power allocation and equal power allocation gets smaller, but optimal power allocation is still better than equal power allocation (about 0.3475 bits better at 40 dB)