Problem 1

a) Find the upper and lower bounds of the symbol error rate

$d_{\text{min}} = 2$ and $M = 8$

Use the following things for upper and lower bound of the error rates:

Upper bound:

$$P_e \leq (M-1)Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right)$$

$$\leq 7 \cdot Q\left(\frac{2}{\sqrt{2N_0}}\right)$$

$$P_e \leq 7 \cdot Q\left(\sqrt{\frac{2}{N_0}}\right)$$

Lower bound:

$$P_e \geq \frac{1}{M}Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right)$$

$$P_e \geq \frac{1}{8}Q\left(\sqrt{\frac{2}{N_0}}\right)$$

b) Find the SNR

First we need to find the average energy of the symbols.

$$\overline{E_s} = \frac{1}{M}\sum_{i=1}^{M} s_i^2$$

For the four outside corners $E_s = (\pm 2)^2 + (\pm\sqrt{3})^2 = 7$ For the two on vertical axis $E_s = (\pm\sqrt{3})^2 = 3$ For the two on horizontal axis $E_s = (\pm\sqrt{1})^2 = 1$

$$\begin{aligned}\overline{E_s} &= \frac{1}{8} \cdot (7+7+7+7+3+3+1+1) \\ &= \frac{36}{8} \\ &= \frac{9}{2}\end{aligned}$$

Signal to noise ratio r_s is

$$r_s = \frac{\overline{E_s}}{N_0}$$

$$\boxed{r_s = \frac{4.5}{N_0}}$$

Problem 2

a) SER of given lines (not optimal)

$$\begin{aligned}P_e &= \frac{1}{M}\sum \mathbb{P}(r \notin D_i | i \text{ is sent}) \\ &= \frac{1}{4}\big(\mathbb{P}(s_1 + n > -2) + \mathbb{P}(s_2 + n < -2) ...\big)\end{aligned}$$

b) Find optimal decision regions

$$\mathbb{P}(s_i)f(r|s_i) \lessgtr \mathbb{P}(s_j)f(r|s_j)$$

This is the PDF

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$
$$f(r|s_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(r-s_i-\mu)^2}{2\sigma^2}\right)$$

Set the neighboring PDFs equal to each other to solve for r. Symbolically solving looks like the following:

$$\mathbb{P}(s_i)f(r|s_i) \gtrless \mathbb{P}(s_j)f(r|s_j)$$

$$\mathbb{P}(s_i)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(r-s_i-\mu)^2}{2\sigma^2}\right) \gtrless \mathbb{P}(s_j)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(r-s_j-\mu)^2}{2\sigma^2}\right)$$
$$\mathbb{P}(s_i)\exp\left(\frac{-(r-s_i-\mu)^2}{2\sigma^2}\right) \gtrless \mathbb{P}(s_j)\exp\left(\frac{-(r-s_j-\mu)^2}{2\sigma^2}\right)$$

$$\frac{\exp\left(\frac{-(r-s_i-\mu)^2}{2\sigma^2}\right)}{\exp\left(\frac{-(r-s_j-\mu)^2}{2\sigma^2}\right)} \gtrless \frac{\mathbb{P}(s_i)}{\mathbb{P}(s_j)}$$

$$\exp\left(\frac{-(r-s_i-\mu)^2+(r-s_j-\mu)^2}{2\sigma^2}\right) \gtrless \frac{\mathbb{P}(s_i)}{\mathbb{P}(s_j)}$$
$$2rs_i - s_i^2 - 2s_i\mu - 2rs_j + s_i^2 + 2s_j\mu \gtrless 2\sigma^2 \cdot \ln\left(\frac{\mathbb{P}(s_i)}{\mathbb{P}(s_j)}\right)$$
$$r(2s_i - 2s_j) \gtrless \left[2\sigma^2 \ln\left(\frac{\mathbb{P}(s_j)}{\mathbb{P}(s_i)}\right) + s_i^2 + 2s_i\mu - s_j^2 - 2s_j\mu\right]$$

$$r \gtrless \frac{\left[2\sigma^2 \ln\left(\frac{\mathbb{P}(s_j)}{\mathbb{P}(s_i)}\right) + s_i^2 + 2s_i\mu - s_j^2 - 2s_j\mu\right]}{2s_i - 2s_j}$$

Plug in the proper values for s_i, s_j, and their probabilities and you'll get the proper decision thresholds.

$$r_{th\_1,2} \gtrless -\frac{1}{4}\sigma^2\ln(3) - 2 + \mu$$
$$r_{th\_2,3} \gtrless 1 + \mu$$
$$r_{th\_3,4} \gtrless -\frac{1}{2}\sigma^2\ln\left(\frac{1}{3}\right) + 3 + \mu$$

c) Solve for SER with the decision thresholds you found in (b)

Use the following:

$$P_e = \frac{1}{4}\Big(\mathbb{P}(s_1 + n > r_{th\_1,2}) + \mathbb{P}(s_2 + n < r_{th\_1,2}) + \mathbb{P}(s_2 + n > r_{th\_2,3}) + \dots$$
$$\mathbb{P}(s_3 + n < r_{th\_2,3}) + \mathbb{P}(s_3 + n > r_{th\_3,4}) + \mathbb{P}(s_4 + n < r_{th\_3,4}) + \Big)$$

$$
\begin{aligned}
P_e = \frac{1}{4}\Bigg(Q\left(\frac{r_{th\_1,2}+4-\mu}{\sigma}\right) + \dots \\
\left(1-Q\left(\frac{r_{th\_1,2}-\mu}{\sigma}\right)\right) + Q\left(\frac{r_{th\_2,3}-\mu}{\sigma}\right) + \dots \\
\left(1-Q\left(\frac{r_{th\_2,3}-2-\mu}{\sigma}\right)\right) + Q\left(\frac{r_{th\_3,4}-2-\mu}{\sigma}\right) + \dots \\
\left(1-Q\left(\frac{r_{th\_3,4}-4-\mu}{\sigma}\right)\right)\Bigg)
\end{aligned}
$$

Problem 3

4-ary constellation is sent with the signal space $s_i(t) = p(t)\cos(2\pi f_c t + \theta_i)$ with θ values $0, \frac{\pi}{4}, \pi, \frac{7\pi}{4}$.

a) Find the representation of the signals in vector form.

b) Find the upper and lower bound of the symbol error rate.

a)

$$S_i = A_i p(t)\cos(2\pi f_c t + \theta_i)$$

Let $A_i = 1$

$$
\begin{aligned}
S_i &= p(t)\cos(2\pi f_c t)\cos(\theta_i) - p(t)\sin(2\pi f_c t)\sin(\theta_i) \\
&= \phi_1 p(t)\cos(\theta_i) - \phi_2 p(t)\sin(\theta_i)
\end{aligned}
$$

Prove the orthogonality if you feel like it

Slide 7-02

$$s_i = \begin{bmatrix} p(t)\cos(\theta_i) \\ -p(t)\sin(\theta_i) \end{bmatrix}$$

So $s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$, $s_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $s_4 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

The s_2 term has the negative instead of s_4 because of the identity used above to split the $cos(2\pi f_c t + \theta_i)$ into two cosine and sine terms for orthogonality.

b)

First find $d_{\min}$ for use in both upper bound and lower bound equations.

$d_{\min}$ is minimum Euclidian distance between any two constellation points.

$$
\begin{aligned}
d_{\min} &= \sqrt{\left(p(t) - p(t)\frac{\sqrt{2}}{2}\right)^2 + \left(p(t)\frac{\sqrt{2}}{2}\right)^2} \\
&= p(t)\sqrt{\left(1 - \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}
\end{aligned}
$$

Using $E_s = p^2(t)$ and therefore $\sqrt{r_s} = \frac{p(t)}{\sqrt{N_0}}$

$$
\begin{aligned}
\frac{d_{\min}}{\sqrt{2N_0}} &= \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}} \cdot \frac{p(t)}{\sqrt{N_0}} \\
&= \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}} \cdot \sqrt{r_s}
\end{aligned}
$$

And that goes inside the Q-function for the bounds:

Upper bound

Use the following for the upper bound:

$$P_e \leq (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Lower bound

Use the following for the lower bound:

$$P_e \geq \frac{1}{M}Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Problem 4

Rectangular 16-QAM setup such that 2A is the min distance between each of the lines.

a) Solve for the Symbol Error Rate

$$P_e = \frac{4(\sqrt{M}-1)}{\sqrt{M}}Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) - \frac{4(\sqrt{M}-1)^2}{M}Q^2\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

$$= 3Q\left(\frac{2A}{\sqrt{2N_0}}\right) - \frac{9}{4}Q^2\left(\frac{2A}{\sqrt{2N_0}}\right)$$

b) Solve for the Bit Error Rate

Bit error rate is approximately calculated below because the 16-QAM constellation is Gray-coded.

$$\begin{aligned}\text{BER} &= \frac{1}{\log_2(M)} \cdot P_e \\ &= \frac{1}{4} \cdot P_e\end{aligned}$$

Problem 5

Two contellations used alternately in the transmission of a signal. Standard QPSK and $\frac{\pi}{4}$-QPSK. Odd index use standard QPSK, and even index use $\frac{\pi}{4}$-QPSK.

Use the low-pass equivalent and show the magnitudes of the real and imaginary parts of $\tilde{\phi}$ line.

Standard QPSK lowpass equivalent:

$\tilde{s_1} = \sqrt{2E_s}$
$\tilde{s_2} = j\sqrt{2E_s}$
$\tilde{s_3} = -\sqrt{2E_s}$
$\tilde{s_4} = -j\sqrt{2E_s}$

$\frac{\pi}{4}$-QPSK lowpass equivalent:

$\tilde{s_1} = \sqrt{E_s} + j\sqrt{E_s}$
$\tilde{s_2} = -\sqrt{E_s} + j\sqrt{E_s}$
$\tilde{s_3} = -\sqrt{E_s} - j\sqrt{E_s}$
$\tilde{s_4} = \sqrt{E_s} - j\sqrt{2_s}$