ISyE 6669 Deterministic Optimization
Lesson 6-2 Modeling with Binary Variables I

Yuang Chen

Georgia Tech Shenzhen Institute

Modeling with Binary Variables

- If A is chosen, then B is chosen.

$$x_A \leq x_B$$

- If A is chosen, then B is not chosen.

$$x_A + x_B \leq 1$$

- If A is chosen, then both B or C are chosen.

$$x_A \leq x_B + x_C$$

- If B or C is chosen, then A is chosen.

$$x_A \geq x_B, \quad x_A \geq x_C$$

Modeling with Binary Variables

- If A is chosen, then both B and C are chosen.

$$x_A \le x_B, \quad x_A \le x_C$$

- If B and C are chosen, then A is chosen.

$$x_A \ge x_B + x_C - 1$$

- A is chosen if and only if B and C are chosen.

$$\begin{aligned} x_A \le x_B, &\quad x_A \le x_C \\ x_A \ge x_B + x_C - 1 & \end{aligned}$$

$$\Downarrow$$

$$x_A = x_B \cdot x_C$$

Modeling with Binary Variables

- If A is chosen, then $x \leq 8$.

$$x \leq 8 + M(1 - y)$$

- If A is chosen, then $x \geq 8$.

$$x \geq 8 - M(1 - y)$$

- If $x < 8$, then A is chosen.

$$x \geq 8 - My$$

- If $x > 8$, then A is chosen.

$$x \leq 8 + My$$

Modeling with Binary Variables

- If $x_1 > 4$, then $x_2 \leq 8$.

$$\begin{cases} x_1 \leq 4 + My \\ x_2 \leq 8 + M(1-y) \\ y \in \{0,1\} \end{cases}$$

- If $x_1 < 4$, then $x_2 \geq 8$.

$$\begin{cases} x_1 \geq 4 - My \\ x_2 \geq 8 - M(1-y) \\ y \in \{0,1\} \end{cases}$$

General "If ... then ... " conditions

If $f(x_1, x_2, ..., x_n) > a$, then $g(x_1, x_2, ..., x_n) \geq b$.

$$\begin{cases} f(x_1, x_2, ..., x_n) \leq a + My \\ g(x_1, x_2, ..., x_n) \geq b - M(1 - y) \\ y \in \{0, 1\} \end{cases}$$

Disjunctive conditions: "Either ... Or ..." conditions

Disjunctive constraints: **at least one** of a set of constraints is satisfied. It's possible the set of constraints are all satisfied.

$$\text{Either A or B} \iff \text{If not A, then B}$$
$$\text{Either A or B} \iff \text{If not B, then A}$$

- Either $x_1 \leq 4$ or $x_2 \leq 8$

$$\text{Either } x_1 \leq 4 \text{ or } x_2 \leq 8$$
$$\Updownarrow$$
$$\text{If } x_1 > 4, \text{ then } x_2 \leq 8$$
$$\Updownarrow$$
$$\begin{cases} x_1 \leq 4 + My \\ x_2 \leq 8 + M(1-y) \\ y \in \{0, 1\} \end{cases}$$

General disjunctive constraints

$$\text{Either } f(x_1, x_2, ..., x_n) \leq a \text{ or } g(x_1, x_2, ..., x_n) \leq b$$

$$\Updownarrow$$

$$\text{If } f(x_1, x_2, ..., x_n) > a, \text{ then } g(x_1, x_2, ..., x_n) \leq b$$

$$\Updownarrow$$

$$\begin{cases} f(x_1, x_2, ..., x_n) \leq a + My \\ g(x_1, x_2, ..., x_n) \leq b + M(1-y) \\ y \in \{0, 1\} \end{cases}$$

Alternative way

Either $f(x_1, x_2, ..., x_n) \leq a$ or $g(x_1, x_2, ..., x_n) \leq b$

$$\begin{cases} \text{If } y = 0, \text{then } f(x_1, x_2, ..., x_n) \leq a \\ \text{If } y = 1, \text{then } g(x_1, x_2, ..., x_n) \leq b \end{cases}$$

Either $f_1(x_1, x_2, ..., x_n) \leq a_1$ or $f_2(x_1, x_2, ..., x_n) \leq a_2$ or $f_3(x_1, x_2, ..., x_n) \leq a_3$

$$\begin{cases} \text{If } y_1 = 1, \text{then } f_1(x_1, x_2, ..., x_n) \leq a_1 \\ \text{If } y_2 = 1, \text{then } f_2(x_1, x_2, ..., x_n) \leq a_2 \\ \text{If } y_3 = 1, \text{then } f_3(x_1, x_2, ..., x_n) \leq a_3 \\ y_1 + y_2 + y_3 \geq 1 \\ y_1, y_2, y_3 \in \{0, 1\} \end{cases}$$

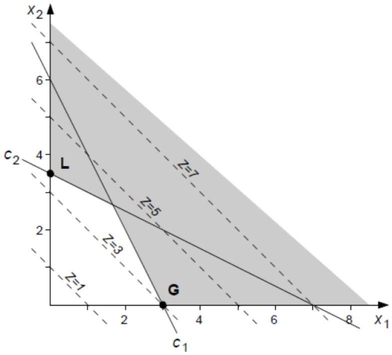
Alternative formulation for disjunctive constraints

Either $f_1(x_1, x_2, ..., x_n) \leq a_1$ or $f_2(x_1, x_2, ..., x_n) \leq a_2$ or $f_3(x_1, x_2, ..., x_n) \leq a_3$

$$\begin{cases} f_1(x_1, x_2, ..., x_n) \leq a_1 + M(1 - y_1) \\ f_2(x_1, x_2, ..., x_n) \leq a_2 + M(1 - y_2) \\ f_3(x_1, x_2, ..., x_n) \leq a_3 + M(1 - y_3) \\ y_1 + y_2 + y_3 \geq 1 \\ y_1, y_2, y_3 \in \{0, 1\} \end{cases}$$

Example

$$\begin{aligned}
\min \quad & x_1 + x_2 \\
\text{s.t.} \quad & \text{either } 2x_1 + x_2 \geq 6 \text{ or } x_1 + 2x_2 \geq 7 \\
& x_1, x_2 \geq 0
\end{aligned}$$



Reformulation to LP

$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \geq 6 - My \\
& x_1 + 2x_2 \geq 7 - M(1 - y) \\
& x_1, x_2 \geq 0 \\
& y \in \{0, 1\}
\end{aligned}
$$

Discrete variables

- A discrete variable has a domain of the form

$$x \in \{a_1, a_2..., a_K\}$$

- If $a_1, a_2..., a_K$ are contiguous integers, we can model as

$$a_1 \le x \le a_k, x \in \mathbb{Z}$$

- Otherwise we can model as

$$x = \sum_{k=1}^{K} a_k y_k, \ \sum_{k=1}^{K} y_k = 1, y_k \in \{0, 1\}, k = 1, .., K$$

Semicontinuous variables

- A semicontinuous variable has a domain of the form

$$x \in [l_1, u_1] \cup [l_2, u_2] \cup \cdots \cup [l_K, u_K]$$

where $l_1 \leq u_1 \leq \cdots \leq l_K \leq u_k$
- One modeling approach for this system is

$$\begin{aligned}
& x = \sum_{k=1}^{K} z_k \\
& l_k y_k \leq z_k \leq u_k y_k \quad k = 1, ..., K \\
& \sum_{k=1}^{K} y_k = 1 \\
& y_k \in \{0, 1\} \quad k = 1, ..., K
\end{aligned}$$

Union of polytopes

- Suppose we require that x satisfies one of the following K systems of inequalities: $A_k x \leq b_k$ for $k = 1, ..., K$
- Suppose that each system defines a polytope, i.e. a bounded polyhedron
- Then a formulation for this system is

$$
\begin{aligned}
& A_k u^k \leq b_k y_k \\
& x = \sum_{k=1}^{K} u^k \\
& \sum_{k=1}^{K} y_k = 1 \\
& y_k \in \{0, 1\}, \forall k = 1, ..., K
\end{aligned}
$$

Set covering problem

There are six cities in Fulton County. The county must determine where to build fire stations. The county wants to build the minimum number of fire stations needed to ensure that at least one fire station is within 15 minutes (driving time) of each city. The times (in minutes) required to drive between the cities are shown in the next table. Decide how many fire stations should be built and where they should be located.

	To					
From	City 1	City 2	City 3	City 4	City 5	City 6
City 1	0	10	20	30	30	20
City 2	10	0	25	35	20	10
City 3	20	25	0	15	30	20
City 4	30	35	15	0	15	25
City 5	30	20	30	15	0	14
City 6	20	10	20	25	14	0

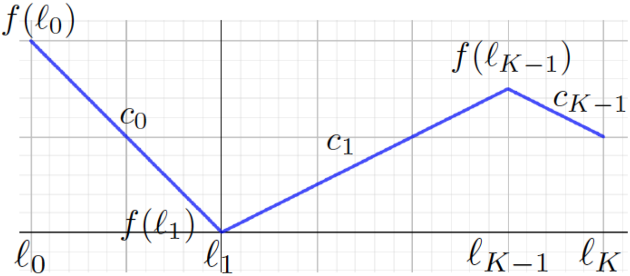
Piecewise linear function

function value at breakpoint

slope

breakpoint

$$f(x) = f(\ell_k) + c_k(x - \ell_k), \ \ell_k \le x \le \ell_{k+1}, \ k = 0, \dots, K-1$$



Wrong modeling approach

Introduce new variables $y_0, \ldots, y_{K-1}$.
Introduce the following constraints:

$$\begin{aligned} &\ell_k y_k \le x \le \ell_{k+1} y_k && k = 0, \ldots, K-1 \\ &\sum_{k=0}^{K-1} y_k = 1 \\ &y_k \in \{0, 1\} && k = 0, \ldots, K-1 \end{aligned}$$

Replace $f(x)$ by $\displaystyle\sum_{k=0}^{K-1} [f(\ell_k) + c_k(x - \ell_k)]\, y_k$

Correct formulation but is **nonlinear** !!!

Piecewise linear function modeling approach

Introduce new variables $u_0, \ldots, u_{K-1}$ and $y_0, \ldots, y_{K-1}$.
Introduce the following constraints:

$$
\begin{aligned}
& x = \sum_{k=0}^{K-1} u_k \\
& \ell_k y_k \le u_k \le \ell_{k+1} y_k \qquad && k = 0, \ldots, K-1 \\
& \sum_{k=0}^{K-1} y_k = 1 \\
& y_k \in \{0, 1\} && k = 0, \ldots, K-1
\end{aligned}
$$

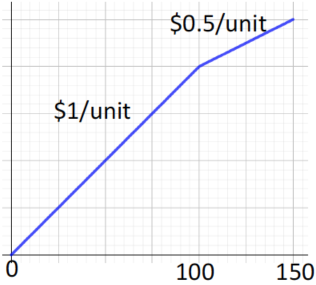
Replace $f(x)$ by

$$\sum_{k=0}^{K-1} [c_k u_k + (f(l_k) - c_k l_k) y_k]$$

Modeling piecewise linear objective

- Decide how much to buy from a set of different vendors (offering quantity discounts) to satisfy a total need

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} f_j(x_j) \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\
& 0 \leq x_j \leq U_j \quad j = 1, \ldots, n
\end{aligned}
$$

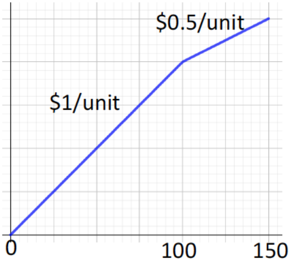


$$
f_j(x_j) = \begin{cases} p_j x_j & 0 \leq x_j \leq L_j \\ p_j L_j + q_j(x_j - L_j) & L_j \leq x_j \leq U_j \end{cases}
$$

Modeling piecewise linear objective

- Decide how much to buy from a set of different vendors (offering quantity discounts) to satisfy a total need

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} [p_j u_j + p_j L_j y_j + q_j (v_j - L_j)] \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\
& 0 \leq x_j \leq U_j \quad j = 1, \ldots, n \\
& x_j = u_j + v_j \quad j = 1, \ldots, n \\
& 0 \leq u_j \leq L_j (1 - y_j) \quad j = 1, \ldots, n \\
& L_j y_j \leq v_j \leq U_j y_j \quad j = 1, \ldots, n \\
& y_j \in \{0, 1\} \quad j = 1, \ldots, n
\end{aligned}
$$



The above objective function is wrong. The right one is

$$
\min \sum_{j=1}^{n} [p_j u_j + q_j v_j + (p_j L_j - q_j L_j) y_j]
$$