Calc2 Notes

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Contents

§1 Calc 1 Review Sheet

This may not have everything, but has a lot of hard to remember stuff (or things I just wanted to try latexing).

§1.1 Limits

Example — Evaluate

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$$

Solution. To solve this, we use the Squeeze Theorem.

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

$$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$$

$$\left\{ \begin{array}{l} \lim_{x\to 0} -x^2 = 0 \\ \lim_{x\to 0} x^2 = 0 \end{array} \right\} \implies \lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$$

□

Remark. Canceling stuff like $(x-a)/(x-a)$ does not matter (for the most part I think) when evaluating limits, even ones that approach a. Most of the time, however, when the top and bottom of the limit are both 0 or ∞, we have to use l'hospitals rule, or sometimes just factoring and canceling suffices.

Special Limits — Special Limits with e:

1. $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$
2. e is the only positive number for where $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$
3. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

this section will not have the stupid epsilon delta definition cuz im too lazy to learn it

§1.2 Derivatives

Definition of the Derivative — The **derivative of $f(x)$ with respect to x** is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Some alternate notations of the derivative of $f(x)$ include

$$f'(x) = y' = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(y\right)$$

And to find the derivative at $x = a$

$$f'(a) = y'\Big|_{x=a} = \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=a} = \frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=a}$$

Remark. A lot of the times, the (x) part can be dropped.

$$f'(x) = f'$$

The Chain Rule — The Chain Rule states that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

The Product Rule —

$$(fg)' = f'g + fg'$$

The Quotient Rule —

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Derivatives of Trig Functions —

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \sec^2(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cot(x)) = -\csc^2(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec(x)) = \sec(x)\tan(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\csc(x)) = -\csc(x)\cot(x)$$

Derivatives of Exponential and Logarithmic functions —

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x\ln(a)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) = \frac{1}{x} \quad x > 0 \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\log_a(x)) = \frac{1}{x\ln(a)}$$

Derivatives of Inverse Trig Functions —

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$

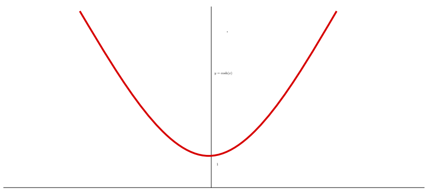
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\csc^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

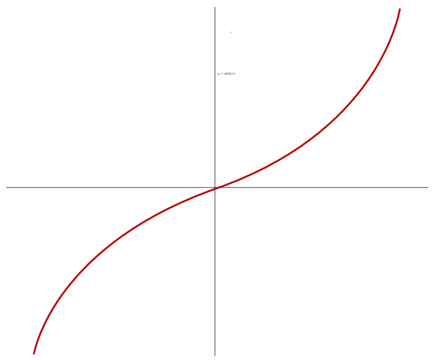
Hyperbolic Functions —

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{1}{\tanh(x)}$$
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

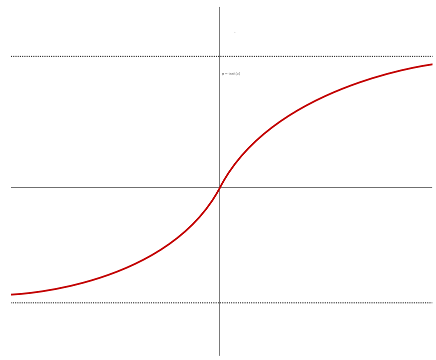
Graphs of Hyperbolic Functions:



y=cosh(x)



y=sinh(x)



y=tanh(x)

Derivatives of Hyperbolic Functions —

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh(x)) = \cosh(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh(x)) = \sinh(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh(x)) = \operatorname{sech}^2(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\coth(x)) = -\operatorname{csch}^2(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech}(x)) = -\operatorname{sech}(x)\tanh(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch}(x)) = -\operatorname{csch}(x)\coth(x)$$

Example — Differentiate
$$y = x^x$$

Solution. How should we do this? We use a technique called **logarithmic differentiation**.

$$y = x^x$$

Notice we can apply the log() operator to get rid of the power.

$$\ln(y) = x\ln(x)$$

Now, differentiating both sides gives

$$\frac{y'}{y} = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1$$

$$\begin{aligned}\implies y' &= y(1+\ln(x)) \\ &= \boxed{x^x(1+\ln(x))}\end{aligned}$$

□

Critical Points — Critical Points are points where

$$f'(c) = 0 \qquad \text{OR} \qquad f'(c) = \text{DNE}$$

Definition —

1. $f(x)$ has an **absolute/global maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in the domain we are working on.
2. $f(x)$ has a **relative/local maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in some open interval around $x = c$.
3. $f(x)$ has an **absolute/global minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in teh domain we are working on.
4. $f(x)$ has a **relative/local minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in some open interval around $x = c$.

Mean Value Theorem — If $f(x)$ is a function that is both

- continuous on the closed interval $[a, b]$.
- differentiable on the open interval (a, b)

Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Newton's Method — If x_n is an approximation of the solution $f(x) = 0$, and $f'(x) \neq 0$, the next approximation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Leibniz Rule — This isn't exactly calc 1, but it is a cool derivative.

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

§1.3 Integrals

Definition — If the **anti-derivative** of $f(x)$ is $F(x)$, then

$$\int f(x)dx = F(x) + c$$

where $\int()$ is the integral symbol, $f(x)$ is the integrand, x is the integration variable, and c is called the constant of integration.

Various Integrals — elow are common integrals:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \qquad n \neq -1$$

Trig and Inverse Trig Functions

$$\int \sin(x)dx = -\cos(x) + c \qquad \int \cos(x)dx = \sin(x) + c$$

$$\int \sec^2(x)dx = \tan(x) + c \qquad \int \sec(x)\tan(x)dx = \sec(x) + c$$

$$\int \csc^2(x)dx = -\cot(x) + c \qquad \int \csc(x)\cot(x)dx = -\csc(x) + c$$

$$\int \frac{1}{x^2+1}dx = \tan^{-1}(x) + c \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c = -\cos^{-1}(x) + c$$

Exponential and Logarithm Functions

$$\int e^x dx = e^x + c \qquad \int a^x dx = \frac{a^x}{\ln(a)} + c \qquad \int \frac{1}{x}dx = \int x^{-1}dx = \ln(|x|) + c$$

Hyperbolic Functions

$$\int \sinh(x)dx = \cosh(x) + c \qquad \int \cosh(x)dx = \sinh(x) + c$$

$$\int \operatorname{sech}^2(x)dx = \tanh(x) + c \qquad \int \operatorname{sech}(x)\tanh(x) = -\operatorname{sech}(x) + c$$

$$\int \operatorname{csch}^2(x)dx = -\coth(x) + c \qquad \int \csc(x)\coth(x)dx = -\operatorname{csch}(x) + c$$

Substitution Rule —

$$\int f\left(g(x)\right)g'(x)dx = \int f(u)du, \qquad \text{where } u = g(x)$$

Notice how this is essentially the ”opposite” of the Chain Rule.

Example — Evaluate

$$\int \frac{1}{\sqrt{1-4x^2}}$$

Solution. Letting

$$u = 2x \qquad \Longrightarrow \qquad dx = \frac{1}{2}du$$

$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du \\ &= \frac{1}{2}\sin^{-1}(u) + c \\ &= \boxed{\frac{1}{2}\sin^{-1}(2x) + c}\end{aligned}$$

□

Example — Evaluate

$$\int \sec(y)dy$$

Solution.

$$\int \sec(y)dy = \int \frac{\sec(y)}{1}\left(\frac{\sec(y)+\tan(y)}{\sec(y)+\tan(y)}\right)dy = \int \frac{\sec^2(y)+\tan(y)\sec(y)}{\sec(y)+\tan(y)}dy$$

Now, using the substitution

$$u = \sec(y) + \tan(y) \qquad \Longrightarrow \qquad du = \left(\sec(y)\tan(y) + \sec^2(y)\right)dy$$

we have

$$\begin{aligned}\int \sec(y)dy &= \int \frac{1}{u}du \\ &= \ln|u| + c \\ &= \boxed{\ln|\sec(y)+\tan(y)| + c}\end{aligned}$$

□

Riemann Sum — For points $x_1^*, x_2^*, \ldots, x_n^*$, which are divided evenly between an interval $[x_0, x_n]$, the area under the equation is approximately

$$A \approx \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

There are three different types of Riemann Sums, including **Left Riemann Sums**, **Right Riemann Sums**, and **Midpoint Riemann Sums** (Names depend on where the f(value) is taken to measure the height of the small rectangles.

The exact area under the curve will be as $n \to \infty$, which leads us to

Definite Integral — Given function $f(x)$ continuous on interval $[a, b]$, then the definite integral of $f(x)$ from a to b (or the area under the curve $f(x)$ from a to b) is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

Fundamental Theorem of Calculus, Pt 1 — If $f(x)$ is continuous on $[a, b]$, then

$$g(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$ and it is differentiable on (a, b) and that

$$g'(x) = f(x)$$

From the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{v(x)}^{u(x)} f(t)dt &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{v(x)}^{a} f(t)dt + \int_a^{u(x)} f(t)dt\right)\\ &= u'(x)f\left(u(x)\right) - v'(x)f\left(v(x)\right)\end{aligned}$$

Example — Differentiate

$$\int_{\sqrt{x}}^{3x} t^2 \sin\big(1+t^2\big)dt$$

Using the formula above,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sqrt{x}}^{3x} t^2 \sin\big(1+t^2\big)dt &= (3)(3x)^2 \sin\left(1+(3x)^2\right)\\ &= \boxed{27x^2 \sin\big(1+9x^2\big) - \frac{1}{2}\sqrt{x}\sin(1+x)}\end{aligned}$$

Fundamental Theorem of Calculus, Pt 2 — If $f(x)$ is a continuous function on $[a, b]$, and $F(x)$ is the anti-derivative for $f(x)$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Notice if $f(x)$ is an even function,

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

and if $f(x)$ is an odd function,

$$\int_{-a}^{a} f(x)dx = 0$$

Note. When using substitutions with definite integrals, make sure to either

1. change the original limits into the substituted variables' limits
2. or take the integral and evaluate the limits with the *original variable*

Example — Evaluate

$$\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx$$

Solution. Using the substitution

$$\begin{aligned} u &= \sin(x) & \implies && du &= \cos(x)\,dx \\ [x_1 &= -\pi, x_2 = \frac{\pi}{2}] & \implies && [u_1 &= \sin(-\pi) = 0, u_2 = \sin\left(\frac{\pi}{2}\right) = 1] \end{aligned}$$

Notice how we shifted the interval to match the substitution we performed. This gives

$$\begin{aligned} \int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx &= \int_0^1 \cos(u)\,du \\ &= \sin(u)\Big|_0^1 \\ &= \sin(1) - \sin(0) \\ &= \boxed{\sin(1)} \end{aligned}$$

□

Note. Pay close attention to integrals, especially ones with x in the denom, such as

$$\int_{-1}^{1} \frac{t}{2-8t^2}dt,$$

and check if they're continuous in the interval of integration (which this one isn't).

Mean Value Theorem — If $f(x)$ is a continuous function on $[a, b]$ then there exists a c in $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Area Between Curves — There are two cases for the area between two functions in the interval $[a, b]$:

$$\begin{aligned} A &= \int_a^b \begin{pmatrix}\text{upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{function}\end{pmatrix} \\ A &= \int_a^b \begin{pmatrix}\text{right}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{left}\\ \text{function}\end{pmatrix} \end{aligned}$$

> **Remark.** A graph is always advised, and be careful of a function being bounded in a curve by itself, and other niche cases like that. Sometimes, working in the form $x = f(y)$ is easier than working in the form $y = f(x)$, especially in the example below.

Method of Disks/Rings — If A is the cross-sectional area of the solid, then

$$A = \pi \left(\begin{pmatrix} \text{outer} \\ \text{radius} \end{pmatrix}^2 - \begin{pmatrix} \text{inner} \\ \text{radius} \end{pmatrix}^2 \right)$$

and

$$V = \int_a^b A(x)dx \quad \text{or} \quad V = \int_a^b A(y)dy$$

Another way to write this is, is the solid of revolution from rotating the continuous curve $f(x)$ around the x-axis from $x = a$ to $x = b$ is

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

Method of Cylinders/Shells —

$$A = 2\pi(\text{radius})(\text{height})$$

§2 Integration

§2.1 Integration by Parts

Integration by Parts is a way to evaluate a lot of integrals that otherwise couldn't be done with a substitution. We all know how to integrate

$$\int xe^{x^2}dx$$

with the simple substitution $u = x^2$, but what about the integral

$$\int xe^{2x}dx?$$

This is an example that can be solved with ibp.
First, lets start off with a proof. We know that

$$(fg)' = f'g + fg' \qquad \text{(the product rule)}$$

Integrating both sides gives

$$\int (fg)'dx = \int f'g + fg'dx = \int f'gdx + \int fg'dx$$

$$\implies \int fg'dx = fg - \int f'gdx$$

And with the substitution $u = f(x)$, $v = g(x)$, we have our formula.

Integration by Parts —

$$\int udv = uv - \int vdu$$

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

This brings us now to our previous problem.

Example — Evaluate

$$\int xe^{2x}dx$$

Solution. We use ibp. Letting

$$u = x \qquad dv = e^{2x}dx$$

$$du = dx \qquad v = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

Now, we have

$$\int xe^{2x}dx = \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}dx$$

$$= \boxed{\left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + c}$$

□

Evaluating the definite integral is the same process;

Example — Evaluate

$$\int_{-2}^{1} xe^{2x}$$

Solution. As we solved before,

$$\int xe^{2x}dx = \left(\frac{x}{2} - \frac{1}{4}\right)e^{2x}$$

From this, we have

$$\begin{aligned}\int_{-2}^{1} xe^{2x}dx &= \left(\left(\frac{x}{2} - \frac{1}{4}\right)e^{2x}\right)\Bigg|_{-2}^{1} \\ &= \frac{1}{4}e^{2} - \left(-\frac{5}{4}e^{-4}\right) \\ &= \boxed{\frac{1}{4}e^{2} + \frac{5}{4}e^{-4}}\end{aligned}$$

□

An important concept is to pick u and dv such that du and v are simple.

Example — Compute

$$\int x\sin(x)dx$$

Solution. In this example, letting $u = \sin(x)$ and $dv = x\,dx$ is the wrong way to approach, as that'll lead to a more complicated integral. We let $u = x$ and $dv = \sin(x)\,dx$.

$$\begin{aligned}\int x\sin(x)\,dx &= -x\cos(x) - \int(-\cos(x))\,dx \\ &= \boxed{-x\cos(x) + \sin(x) + c}\end{aligned}$$

□

§2.2 Trig Functions

Using the trig identities, there are a few neat ways to integrate functions involving trignometry.

Example — Evaluate

$$\int \sin^5(x)dx$$

Solution. Using the identity

$$\sin^2(x) + \cos^2(x) = 1$$

we can rewrite this into

$$\begin{aligned}
\int \sin^5(x)dx = \int \sin^4(x)\sin(x)dx &= \int \left(1-\cos^2(x)\right)^2 \sin(x)dx \\
&= -\int \left(1-u^2\right)^2 du \\
&= -\int u^4 - 2u^2 + 1 du \\
&= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + c \\
&= \boxed{-\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + c}
\end{aligned}$$

□

Using trig identities such as these make integrating trig functions possible.

Useful Trig Identities — Here are a few of the many trig functions that are useful when integrating
Pythagorean

$$\sin^2(x) + \cos^2(x) = 1$$
$$\sec^2(x) - \tan^2(x) = 1$$
$$\csc^2(x) - \cot^2(x) = 1$$

Double Angle

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$$
$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$$
$$\frac{1}{2}\sin(2x) = \sin(x)\cos(x)$$

Product to Sum

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

§2.3 Trig Subtitutions

Not only can those trig identities compute integrals involving trig functions, they can also be used in substitutions to make integrating possible.

Example — Evaluate

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Solution. The denominator, $\sqrt{1-x^2}$, gives us motivation to use the pythagorean identity $\sin^2(x)+\cos^2(x)=1$. Notice that subbing in $x=\sin(u)$ and $x=\cos(u)$ gives the "same" end result.

Letting

$$x=\sin(u) \qquad \Longrightarrow \qquad dx=-\cos(u)\,du$$

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{\cos(u)\,du}{\sqrt{1-\sin^2(u)}}\,du \\ &= \int \frac{\cos(u)}{|\cos(u)|}\,du\end{aligned}$$

How should we get rid of the absolute value? Notice that in the original equation, to stay in the reals, $x\in(-1,1)$, meaning $u\in(-\frac{\pi}{2},\frac{\pi}{2})$, which results in only positive values for $\cos(u)$, making the abs val redundant.

$$\begin{aligned}\int \frac{\cos(u)}{|\cos(u)|}\,du &= \int 1\,du \\ &= u+c \\ &= \sin^{-1}(x)+c\end{aligned}$$

Notice that we just proved an integral that was already stated in Chapter 1. □

The main concept in these substitutions is summed up in the following chart:

Form	Subtitution	Bounds
$\sqrt{x^2-1}$	$x=\sec(u)$	$u\in[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$
$\sqrt{1-x^2}$	$x=\sin(u),\cos(u)$	$u\in[-\frac{\pi}{2},\frac{\pi}{2}],[0,\pi]$
$\sqrt{x^2+1}$	$x=\tan(u)$	$u\in(-\frac{\pi}{2},\frac{\pi}{2})$

The form should also be manipulated in a way such that thee substitution is possible, using stuff like completing the square.

Example — Compute

$$\int \frac{x}{\sqrt{2x^2-4x-7}}\,dx$$

Solution. To be able to subtitute, we must first complete the square.

$$2x^2-4x-7=2(x-1)^2-9$$

From this, we have

$$\int \frac{x}{\sqrt{2x^2-4x-7}}\,dx = \int \frac{x}{\sqrt{2(x-1)^2-9}}\,dx$$

Now, using the substitution $x = 1 + \frac{3}{\sqrt{2}}\sec(u)$,

$$\begin{aligned}
\int \frac{x}{\sqrt{2x^2-4x-7}}\,dx &= \int \frac{1+\frac{3}{\sqrt{2}}\sec(u)}{3\tan(u)}\left(\frac{3}{\sqrt{2}}\sec(u)\tan(u)\right)\,du \\
&= \int \frac{1}{\sqrt{2}}\sec(u) + \frac{3}{2}\sec^2(u)\,du \\
&= \frac{1}{\sqrt{2}}\ln|\sec(u)+\tan(u)| + \frac{3}{2}\tan(u) + c \\
&= \boxed{\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}(x-1)}{3} + \frac{\sqrt{2x^2-4x-7}}{3}\right| + \frac{\sqrt{2x^2-4x-7}}{3} + c}
\end{aligned}$$

□

§2.4 Partial Fractions

Partial Fractions are also a way to reduce, normally factorizable denominators to integrate fractions easier.

Example — Evaluate

$$\int \frac{x^2}{x^2-1}$$

Solution. While ibp is a solution, partial fractions makes it much easier.

$$\begin{aligned}
\int \frac{x^2}{x^2-1}\,dx &= \int 1 + \frac{1}{x^2-1}\,dx \\
&= \int dx + \int \frac{1}{x^2-1}\,dx \\
&= \int dx + \int \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,dx \\
&= \boxed{x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c}
\end{aligned}$$

□

§3 Applications of Integrals