MAE82

Terry Yu

2023

Contents

§1 Ordinary Differential Equations

All ODEs are going to be in the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = F(x)$$

where y is a function of x. Note that all ODEs will have

1. only one independent variable
2. only ordinary derivatives

For a lot of ODEs, the simple solve is by guessing and then plugging in.

First Order ODES

To start, we will only be focusing on first-order ODEs, or differential equations with only $\frac{\mathrm{d}y}{\mathrm{d}x}$. There is a specific type of solvable first order ODE, a seperable ODE by the form of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)q(y)$$

which we can solve by rewriting as

$$\frac{1}{q(y)}\,\mathrm{d}y = p(x)\,\mathrm{d}x$$

and then integrate on both sides.

Example — Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-5}{y^2}$$

Solution. This is equivalent to writing

$$\begin{aligned} y^2\,\mathrm{d}y &= (x-t)\,\mathrm{d}x \\ \int y^2\,\mathrm{d}y &= \int (x-5)\,\mathrm{d}x \\ \frac{y^3}{3} + c_1 &= \frac{x^2}{2} - 5x + c_2 \end{aligned}$$

□

Definition — A first order linear differential equation is one in the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = b(x)$$

Solving these will fall into three categories:
Case 1:

$$a_0(x) = 0$$

This would mean we would get a seperable diffe by the form of

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = b(x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x)}{a_1(x)}$$
$$\boxed{y(x) = \int \frac{b(x)}{a_1(x)}\,\mathrm{d}x + C}$$

<u>Case 2</u>:

$$a_0(x) = {a_1}'(x)$$

Note that this yields, by manipulation

$$\begin{aligned} b(x) &= a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y \\ &= a_1(x)y' + {a_1}'(x)y \\ &= \frac{\mathrm{d}}{\mathrm{d}x}[a_1(x)y] \end{aligned}$$

and integrating both sides gives us

$$\boxed{y = \frac{1}{a_1(x)}\int b(x)\,\mathrm{d}x}$$

Solutions for General First Order Linear DE — Given a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

the solution is

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int \mu(x)Q(x)\,\mathrm{d}x \\ \mu(x) &= e^{\int P(x)\mathrm{d}x} \end{aligned}$$

The $\mu(x)$ is called an *integrating factor*.

Proof. Let both sides of the general form DE be multiplied by a certain $\mu(x)$. We get

$$\mu\frac{\mathrm{d}y}{\mathrm{d}x} + \mu Py = \mu Q$$

Note that if

$$\begin{aligned} \mu P &= \mu' \\ \frac{1}{\mu}\,\mathrm{d}\mu &= P\,\mathrm{d}P \\ \ln(\mu) + C &= \int P\,\mathrm{d}P \\ \mu &= e \end{aligned}$$

□

Definition — The differential form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is exact if there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \qquad \& \qquad \frac{\partial F}{\partial y} = N$$

This implies that

$$F(x,y) = C$$

is the solution for this DE.

Note that with exact equations,

$$\frac{\partial F}{\partial xy} = \frac{\partial F}{\partial yx}$$
$$\frac{\partial}{\partial y}\underbrace{\frac{\partial F}{\partial x}}_{M} = \frac{\partial}{\partial x}\underbrace{\frac{\partial F}{\partial y}}_{N}$$

This means that we can just check if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

to check if its an exact equation, and we can reverse it to solve for $F(x,y)$.

Bernoulli Substitution — Given an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

by letting a certain

$$V = y^{1-n}$$

we can rewrite this equation into the linear DE

$$\frac{1}{1-n}\frac{\mathrm{d}V}{\mathrm{d}x} + P(x)V = Q(x)$$

Note that this is a very specific form; when $n = 0, 1$, it simplifies to a seperable/linear function.
By dividing by y^n on both sides, we get the equation of the form

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x)$$

Seeing that the derivative

$$\frac{\mathrm{d}y^{1-n}}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$$

through chain rule, this gives us motivation for our solution.

Mathematical Models

§2 Linear 2nd Order ODEs

Homogenous 2nd Order ODEs

Homogenous 2nd Order ODE — Homogenous 2nd Order ODEs take the form of

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

Let a certain

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are three cases for solutions (with C_1, C_2 as arbitrary constants):
$b^2 - 4ac > 0$:

$$\boxed{y(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}}$$

$b^2 - 4ac = 0$:

$$\boxed{y(x) = C_1e^{\lambda x} + C_2xe^{\lambda x}}$$

$b^2 - 4ac < 0$:
In this case, let

$$\lambda_1 = \alpha + \beta i$$
$$\lambda_2 = \alpha - \beta i$$

The solution become

$$\begin{aligned} y &= C_1e^{(\alpha+\beta i)x} + C_2e^{(\alpha-\beta i)x} \\ &\boxed{= e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right)} \end{aligned}$$

Proof. Note that the solutions of this equation lie in teh form

$$y = e^{rt}$$

Plugging this into the equation yields

$$\begin{aligned} ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ (ar^2 + br + c)e^{rt} &= 0 \\ \implies e^{rt} = 0 \qquad \text{or} \qquad ar^2 + br + c &= 0 \end{aligned}$$

Other than the redundant solution of $r = -\infty$, the soultions of r can be given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

□

Undertermined Coefficients

Solving the equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(t)$$

is a harder task; it depends on the type of function of $f(t)$. There are three cases:

1. $f(t)$ is Polynomial
2. $f(t)$ is Exponential
3. $f(t)$ is Sinusoidal

Polynomial — Given the equation

$$ay'' + by' + cy = Ct^m$$

We plug in the equation

$$y(t) = A_m t^m + A_{m-1} t^{m-1} + \cdots + A_1 t + A_0$$

and coefficient match to solve for the coefficients of y.

Exponential — Given the equation

$$ay'' + by' + cy = Ce^{mt}$$

Notice how y must be of the form

$$y(t) = Ae^{mt} \qquad (\text{or} \qquad y(t) = Ate^{mt} \qquad y(t) = At^2e^{mt})$$

Plugging in and coefficient matching, we can solve for the coefficients of y.

Note: this doesn't always work – if our guess for the function $y(t)$ is also a solution for the homogenous equation

$$ay'' + by' + cy = 0$$

then our solution isn't valid. To fix this, we can edit our guessed function by multiplying it by t, giving us either

$$y(t) = Ate^{mt} \qquad \text{or} \qquad y(t) = At^2e^{mt}$$

Sinusoidal — Given the equation

$$ay'' + by' + cy = C\sin(mt)$$

Seeing how the derivative of sin is cos, and vice versa, we can plug

$$y(t) = A\sin(mt) + B\cos(mt)$$

into the equation, and coefficient match to get the coefficients of y.

When $f(t)$ is a product of multiple cases, then the solution is a product of the individual roots.

Theorem (Superposition Principle)
If $y_1(t)$ is a solution to

$$ay'' + by' + cy = f(t)$$

and $y_2(t)$ is a solution to

$$ay'' + by' + cy = g(t)$$

then for any constants k_1, k_2, $k_1y_1(t) + k_2y_2(t)$ is a solution to

$$ay'' + by' + cy = f(t) + g(t)$$

Variation of Parameters

Variation of Parameters — To solve a more general differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + q\frac{\mathrm{d}y}{\mathrm{d}x} + ry = g(t)$$

given two complementary solutions $y_1(t)$, $y_2(t)$ for the homogenous counterpart, a particular solution is

$$Y_P(t) = -y_1 \int \frac{y_2 g}{W(y_1, y_2)}\,\mathrm{d}t + y_2 \int \frac{y_1 g}{W(y_1, y_2)}\,\mathrm{d}t$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Proof. □

Mechanical Vibrations

NOTE: all of these methods find the *particular solution* to these non-homogenous equations, not the general solution. The theorem above gives us motivation to show the following:

To find the general solution, we have to $\boxed{\text{find a particular solution to this equation}}$, and <u>combine it</u> with the $\boxed{\text{general solution to its homogenous counterpart}}$.

§3 Chapter 3