Multivariable Calculus

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2021

Contents

my notes are really bad (especially in the first chapter) and not really notes btw i just copied off what the lectures said lmk if there's any typos

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§1 Module 1 (Geometric Preliminaries)

§1.1 Parametrized Curves

If

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

for $\alpha \leq t \leq \beta$, where t is the parameter in time, and every point on this curve is the piont $(x, y) = (f(t), g(t))$

Example —

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$$

$$0 \leq t \leq 2\pi$$

The graph of this is a unit circle, as with $t = 0$, we have the point $(1, 0)$, and so on, until it makes one turn, and it's going counterclockwise at unit speed.

Example —

$$\begin{cases} x = \cos(3t) \\ y = \sin(3t) \end{cases}$$

$$0 \leq t \leq 2\pi$$

This is the exact same thing as before, however it goes 3 times faster and completes a total of 3 spins counterclockwise. Notice how the t is beign multiplied by 3, so its being drawn 3 times faster.

Now how do you sketch a prametrized curve from the equations?

Sketching Parametric Curves

Method 1: Plot points and connect the dots

Method 2:

1. Eliminating the parameter t
2. Using symmetry

Example — Sketch

$$\begin{cases} x = a\cos(t) \\ y = b\sin(t) \end{cases}$$

$$0 \leq t \leq 2\pi$$

By eliminating the paramter t, we have the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Which means this is just an ellipse, making it easier to graph.

Example — Sketch

$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases}$$

$$0 \leq t \leq 2\pi$$

From this, first we eliminate t to have the equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

To sketch this equation out, we look for the range of $0 \leq t \leq \frac{\pi}{2}$, where both $\cos(t)$ and $\sin(t)$ are positive. This gives a function in the first quadrant, and by noticing that the other parts are just that function but reflected across each permutation of the axis, we get what's alled an astroid (graph below).

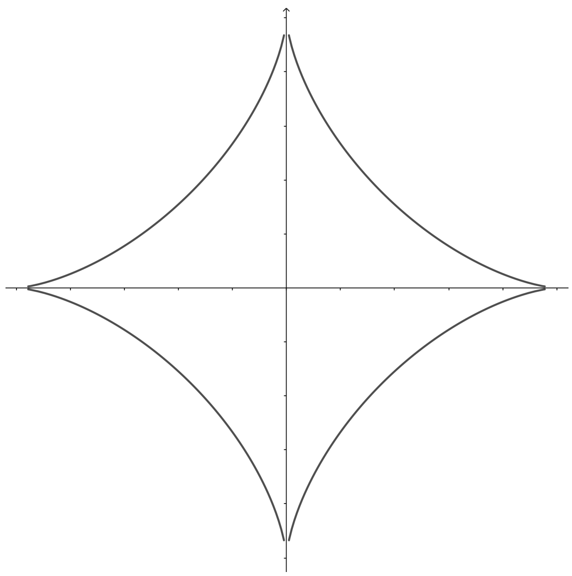


Figure 1: Astroid

Slope of a Parametrized Curve

Slope — Given the curve

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

$$\alpha \leq t \leq \beta$$

$$\text{Slope} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \qquad \text{if } f'(t) \neq 0$$

Proof. If $f'(t) \neq 0$, then locally the curve is a graph $y = h(x)$.

Now we use the Chain Rule.

$$\begin{aligned} & y = h(x) \\ \implies & g(t) = h(f(t)) \\ \implies & g'(t) = \underbrace{\frac{\mathrm{d}h}{\mathrm{d}x}}_{\text{slope}} f'(t) \end{aligned}$$

$$\implies \frac{\mathrm{d}h}{\mathrm{d}x} = \text{slope} = \frac{g'(t)}{f'(t)}$$

□

Remark. If $f'(t) = 0$ and $g'(t) \neq 0$, slop $= \infty$ (tangent line to the curve is vertical)

If $g'(t) = 0$, then you don't know anything, as it just means the point has stopped.

The Cycloid

Definition — A cycloid consists of the positisions of a point on the edge of a wheel of radius R that's rolling, starting at the origin.

Example — Calculate the parametric equation, slope at origin, and concavity of the cycloid.

As the cycloid is a circle rolling, while moving, we should split this up into two motions.

Center of the wheel: say the center of the wheel is moving to the right at R, we can write out

$$\begin{cases} x = Rt \\ y = R \end{cases}$$

$$t = \text{rotation of the wheel in radians}$$

Displacement from center: using basic trig, we can calculate that x has been shifted left $R\sin(t)$ and down $R\cos(t)$. Combining this with the equation frm before, we get our cylcoid eequation as

$$\begin{cases} x = Rt - R\sin(t) \\ y = R - R\cos(t) \end{cases}$$

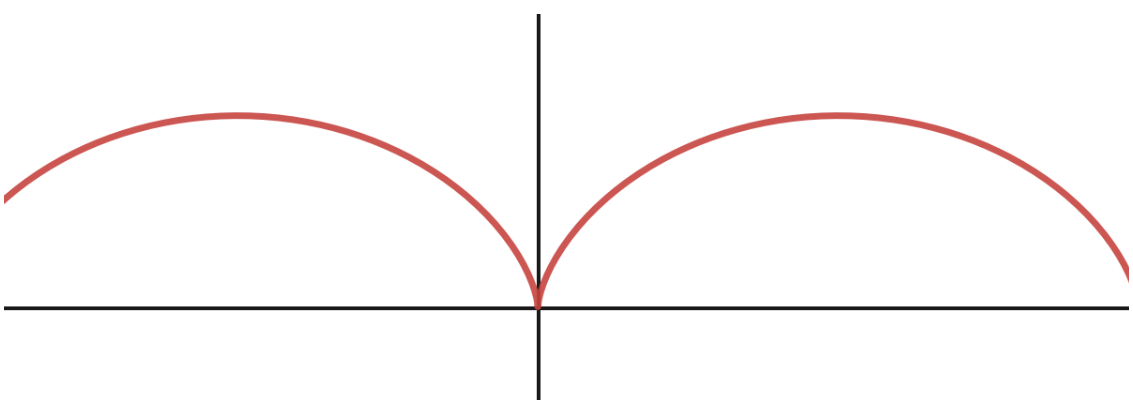


Figure 2: The Cycloid

Now, lets try taking some slopes to understand this curve better.

we have that

$$x' = R - R\cos(t)$$
$$y' = R\sin(t)$$

leading to the slope as

$$\frac{y'}{x'} = \frac{R\sin(t)}{R - R\cos(t)} = \frac{\sin(t)}{1-\cos(t)}$$

Now, what's happening at the origin?

The slope at the origin is

$$\lim_{t\to 0}\frac{\sin(t)}{1-\cos(t)} = \lim_{t\to 0}\frac{\cos(t)}{\sin(t)} = \infty$$

The Cycloid is also concave between the roots of this function. We can check this by finding

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{d(dy/dx)}{dx} \\
&= \frac{d(dy/dx)/dt}{dx/dt} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sin(t)}{1-\cos(t)}\right)}{R(1-\cos(t))} \\
&= \frac{\frac{(1-\cos(t))\cos(t)-\sin(t)(\sin(t))}{(1-\cos(t))^2}}{R(1-\cos(t))} \\
&= \frac{\cos(t)-\cos^2(t)-\sin^2(t)}{R(1-\cos(t))^3} \\
&= \frac{\cos(t)-1}{R(1-\cos(t))^3} \\
&= \frac{-1}{R(1-\cos(t))^2} < 0
\end{aligned}$$

and as the second derivative is negative, it is indeed concave.

Area under Parametrized Curve

The area of a function $h(x)$ from a to b is given by

$$\int_a^b h(x)\,dx$$

Given that $h(x)$ is represented by

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$
$$\alpha \le t \le \beta$$

Then $dx = f'(t)\,dt$ and $h(x) = g(t)$, thus giving us the

Area under parametrized curve —

$$\pm\int_\alpha^\beta yx'\,dt$$

Example — Calculate the area of the astroid defined by

$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases}$$

$$0 \leq t \leq 2\pi$$

Solution. By symmetry, we can just calculate from 0 to π, and multiply that by 2.

$$\begin{aligned}
\text{Area} &= -2\int_0^\pi \sin^3(t)(-3\cos^2(t)\sin(t))\,dt \\
&= 6\int_0^\pi \sin^4(t)\cos^2(t)\,dt \\
&= 6\int_0^\pi \frac{\sin^2(2t)}{4}\left(\frac{1-\cos(2t)}{2}\right)dt \\
&= \boxed{\frac{3\pi}{8}}
\end{aligned}$$

□

Note. Note how we can't take the interval from 0 to 2π, as that'll give us the wrong answer. Sometimes, to find area we have to graph first, and select an appropriate interval.

Length of a parametrized curve

Length —

$$L = \int_\alpha^\beta \underbrace{\sqrt{x'(t)^2 + y'(t)^2}}_{\text{SPEED}}\,dt$$

Proof. Define n parts, where

$$\alpha = t_0 < t_1 < \cdots < t_n = \beta$$

$$\Delta t = t_i - t_{i-1} = \frac{\beta - \alpha}{N}$$

Now we want to take n to infinity, so we have

$$\begin{aligned}
L &= \lim_{n\to\infty}\sum_{i=1}^N \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\
&= \lim_{n\to\infty}\sum_{i=1}^N \Delta t\sqrt{x'(t)^2 + y'(t)^2} \\
&= \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,dt
\end{aligned}$$

□

Fact. Length does not depend on parametrization unless the curve covers itself more than once

Example — Find the lenght of the astroid defined by $x = \cos^3(t)$, $y = \sin^3(t)$, $0 \leq t \leq 2\pi$.

Solution.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-3\cos^2(t)\sin(t))^2 + (3\sin^2(t)\cos(t))^2}\, dt \\ &= \int_0^{2\pi} \sqrt{9\sin^2(t)\cos^2(t)}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} 3\sin(t)\cos(t)\, dt \\ &= 6\sin^2(t)\Big|_0^{\frac{\pi}{2}} \\ &= \boxed{6} \end{aligned}$$

□

Area of surface of revolution

Area —

$$\int_\alpha^\beta 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Proof. As the length is

$$L = \int_\alpha^\beta \underbrace{\sqrt{x'(t)^2 + y'(t)^2}}_{ds=\text{"arc length element"}}\, dt$$

Then the area will be

$$\begin{aligned} A &= \lim_{\Delta S \to 0} \sum \text{Ribbon} \\ &= \int_\alpha^\beta 2\pi y\, ds \\ &= \int_\alpha^\beta 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt \end{aligned}$$

□

Example — Calculate the area of the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution. As we're rotating the circle $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$, we have that

$$\begin{aligned} A &= \int_0^\pi 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &= \int_0^\pi \sin(t)\, dt \\ &= -2\pi\cos(t)\Big|_0^\pi \\ &= \boxed{4\pi} \end{aligned}$$

□

§1.2 Polar Coordinates

Polar Coordinates are denoted with (r, θ) where r is the distance thee point (x, y) is from the origin, and θ is the amount its been rotated. To convert from rectangular to polar you use

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

Notice that θ is only defined modulo 2π, and r can also be negative.

Curves in Polar Coordinates

The curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, has a parametrized equivalent of

$$x = f(\theta)\cos(\theta)$$

$$y = f(\theta)\sin(\theta)$$

$$\alpha \leq \theta \leq \beta$$

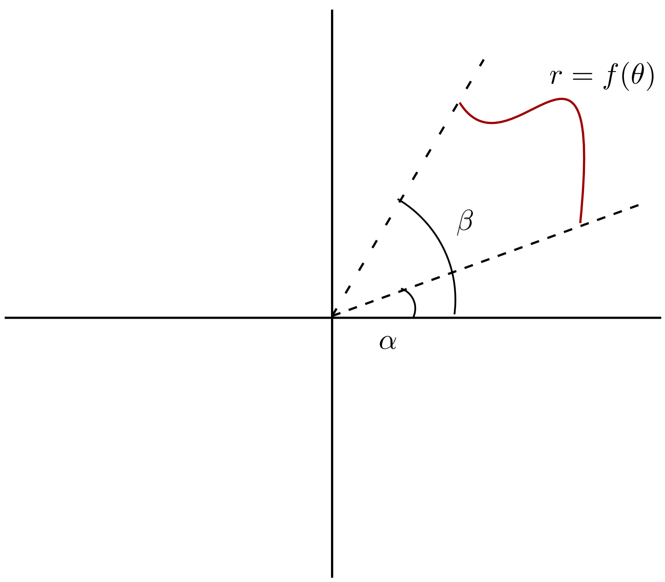


Figure 3: Polar Curve

We can sketch this, once again, by just plotting some points and connecting the dots, or by converting to cartesian.

Note. When not specifying a range for θ, it is assumed that the range is $0 \leq \theta \leq 2\pi$.

Slope of a polar curve

Given the curve $r = f(\theta)$, the parametric form is

$$x = f(\theta)\cos(\theta)$$

$$y = f(\theta)\sin(\theta)$$

This gives the slope as

Slope —

$$\text{Slope} = dy/dx = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}(f(\theta)\sin(\theta))}{\frac{\mathrm{d}}{\mathrm{d}\theta}(f(\theta)\cos(\theta))}$$

Example — Calculate the slope of $r = 1 - 2\cos(\theta)$ at $\theta = \frac{\pi}{2}$

Solution. From this, we can write

$$x = (1 - 2\cos(\theta))\cos(\theta)$$
$$y = (1 - 2\cos(\theta))\sin(\theta)$$

$$\begin{aligned}
\implies \text{Slope} &= \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}\left[(1 - 2\cos(\theta))\sin(\theta)\right]}{\frac{\mathrm{d}}{\mathrm{d}\theta}\left[(1 - 2\cos(\theta))\cos(\theta)\right]} \\
&= \frac{(2\sin(\theta)\sin(\theta) + (1 - 2\cos(\theta))\cos(\theta)}{(2\sin(\theta))\cos(\theta) + (1 - 2\cos(\theta))(-\sin(\theta))} \\
&= \frac{2}{-1} \\
&= \boxed{-2}
\end{aligned}$$

□

Area under the polar curve

Area — The area under the polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, is

$$A = \int_\alpha^\beta \frac{f(\theta)^2}{2}\, d\theta$$

Looking at the polar graph, we can find hte area by dividing it into infinitsimaly small "pie" slices, each with $\Delta\theta$, with a radius of r, so the area of one of those "pie" slices becomes $\frac{r^2\Delta\theta}{2}$. We're taking the limit as this change in θ becomes 0, so we now have

$$\begin{aligned}
\text{Area} &= \lim_{\Delta\theta \to 0} \sum \text{Area (pie slice)} \\
&= \lim_{\Delta\theta \to 0} \sum \frac{r^2\Delta\theta}{2} \\
&= \int_\alpha^\beta \frac{r^2}{2}\, d\theta \\
&= \int_\alpha^\beta \frac{f(\theta)^2}{2}\, d\theta
\end{aligned}$$

Example — Calculate the area enclosed by the inner loop of the curvee $r = 1 - 2\cos(\theta)$.

Solution. Through drawing out the diagram (which can be done yourself), we notice that the area is from $-\frac{\pi}{3} \leq \theta \frac{\pi}{3}$. Notice that we replaced $\frac{5\pi}{3}$ with $-\frac{\pi}{3}$ as taking the integral from $\frac{5\pi}{3}$ will result in the wrong area calculated (the whole thing instead). Now, it's simply taking the integral.

$$\begin{aligned}\text{Area} &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{(1-2\cos(\theta))^2}{2}\,d\theta \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1-4\cos(\theta)+4\cos^2(\theta)}{2}\,d\theta \\ &= \left(\frac{1}{2}\theta - 2\sin(\theta) + \theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= \boxed{\pi - \frac{3\sqrt{3}}{2}}\end{aligned}$$

□

Lenght of Polar Curve

Length —

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,d\theta$$

Solution. There are two "proofs" for this:

Vectors:

As

$$L = \int_{\alpha}^{\beta} \text{Length (Velocity Vector)}\,d\theta$$

we just have to consider the velocity vectors, which is the sum of the two vectors, r, and $\frac{\mathrm{d}r}{\mathrm{d}\theta}$, and using the pythagorean theorem we get our answer.

Cartesian: Using parametric, we have $x = r(\theta)\cos(\theta)$, $y = r(\theta)\sin(\theta)$, where

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos(\theta) - r\sin(\theta) \\ \implies \left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\cos^2(\theta) - 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos(\theta)\sin(\theta) + r^2\sin^2(\theta) \\ \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin(\theta) + r\cos(\theta) \\ \implies \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\sin^2(\theta) + 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin(\theta)\cos(\theta) + r^2\cos^2(\theta)\end{aligned}$$

Using this, we have

$$\begin{aligned}L &= \int_{\alpha}^{\beta} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,d\theta \\ &= \int_{\alpha}^{\beta} \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2}\,d\theta\end{aligned}$$

□

Example — Calculate the length of the curve $r = 2\sin(\theta)$

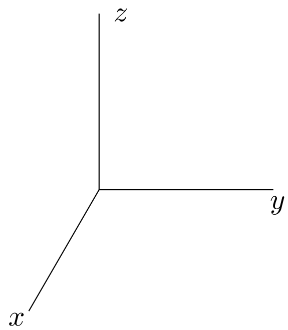
Solution. Notice how our limits of integration are from 0 to π, as that's how much it takes to draw this circle curve exactly once.

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{(2\sin(\theta))^2 + (2\cos(\theta))^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4}\, d\theta \\ &= \boxed{4\pi} \end{aligned}$$

□

§1.3 3-D Space, Vectors, Cross and Dot Product

The 3-D axis are always drawn as shown in the figure below:



Distance

Using the pythagorean theorem, we can see that

Distance — The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Vectors

Definition — In 2d, vectors are denoted by $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$ where $\vec{\mathbf{a}}$ is the vector and $\langle a_1, a_2 \rangle$ is the components which are real numbers.

A vector is thought of as an arrow, where it starts off from a point (tail) with an arrow to another point (head) an amount a_1 right and a_2 up. Vectors have direction and magnitude, and it doesn't matter where it starts, it can be translated around.

Addition — Given vectors $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2 \rangle$, then the sum is

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

The geometric intuition for this is simply appending the vector $\vec{\mathbf{b}}$ after the vector $\vec{\mathbf{a}}$, and the resulting total vector is $\vec{\mathbf{a}} + \vec{\mathbf{b}}$.

Magnitude — The magnitude/length of a vector is

$$|\vec{\mathbf{a}}| = \sqrt{a_1^2 + a_2^2}$$

Multiplication by Scalar — Given a vector $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$ and a scalar c,

$$c\vec{\mathbf{a}} = \langle ca_1, ca_2 \rangle$$

This is like multiplying the length of the vector by c.

Fact. $|c\vec{\mathbf{a}}| = |c||\vec{\mathbf{a}}|$

Triangle Ineq —

$$\left|\vec{\mathbf{a}} + \vec{\mathbf{b}}\right| \leq |\vec{\mathbf{a}}| + |\vec{\mathbf{b}}|$$

Vectors in 3d space is the same thing, with $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$, meaning an arrow in 3d space, and the same rules apply.

Vectors in 3d — Given two vectors

$$\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$$
$$\vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$$

We have

$$\begin{aligned} \vec{\mathbf{a}} + \vec{\mathbf{b}} &= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \\ c\vec{\mathbf{a}} &= \langle ca_1, ca_2, ca_3 \rangle \\ |\vec{\mathbf{a}}| &= \sqrt{a_1^2 + a_2^2 + a_3^2} \\ |c\vec{\mathbf{a}}| &= |c||\vec{\mathbf{a}}| \end{aligned}$$

This is also generalizable to an $n-$dimensional vector.

Dot Product — Given two vectors $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$,

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

(which works up to n dimensions, as long as the two vectors have the same number of components)

Geometric Interpretation: The Dot product represents $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \cos(\Delta\theta)$.

Projects and Orthogonal Components — The projection of a vector $\vec{\mathbf{u}}$ onto a vector $\vec{\mathbf{v}}$ is

$$\text{proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

and the component of vector $\vec{\mathbf{v}}$ that is orthogonal to vector $\vec{\mathbf{u}}$ is given by

$$\text{comp}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \vec{\mathbf{u}} - \text{proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|^2}\right)\vec{\mathbf{v}}$$

The orthagonal projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{a}}$ is $c\vec{\mathbf{a}}$, where $c\vec{\mathbf{a}}$ is a part on $\vec{\mathbf{a}}$ such that $\vec{\mathbf{b}} - c\vec{\mathbf{a}}$ is a vector perpendicular to $\vec{\mathbf{a}}$.

Definition — $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$ means $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.

The dot product is

- Commutative: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$
- Distributive: $\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} + \vec{\mathbf{c}}\right) = \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right) + (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})$

However, the dot product is not associative, so $\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}\right) \neq \vec{\mathbf{c}} \cdot \left(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}\right)$

Example — Prove the Pythagorean Theorem, which states that if $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$ then

$$|\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}} + \vec{\mathbf{b}}|^2$$

Proof.

$$\begin{aligned}
|\vec{\mathbf{a}} + \vec{\mathbf{b}}|^2 &= (\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) \\
&= (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}) + (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) + (\vec{\mathbf{b}} \cdot \vec{\mathbf{a}}) + (\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) \\
&= |\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2
\end{aligned}$$

□

Determinants in 2d and 3d

Determinant of 2d Matrix — Suppose we have a 2d matrix. The determinant is given by

$$\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Geometric Interpretation

With two vectors $\vec{\mathbf{a}} = \langle a_1, a_2\rangle$, $\vec{\mathbf{b}} = \langle b_1, b_2\rangle$, then

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm\text{Area (Paralellogram)}$$

Note. The "+" is used when $\vec{\mathbf{a}}$ points to the right of $\vec{\mathbf{b}}$, and "−" otherwise.

3d Determinants — The detrminant of a 3 by 3 matrix is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a2b_1c_3 - a_3b_2c_1$$

Remark. Remembering this is hard; a much easier way is to remember that the diagonals going topleft to bottom right are positive, other way are negative.

Geometric Interpretation With the 3 vectors $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$, $\vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$, $\vec{\mathbf{c}} = \langle c_1, c_2, c_3 \rangle$, the meaning of the determinant is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \pm\text{Volume (Paralellopiped)}$$

Where the paralellopiped is a paralellogram but in 3d made with the three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$.

Note. Now, the "$+$" is used when $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ satisfy the right hand rule, where using your right hand, curling from $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$, your thumb points toward $\vec{\mathbf{c}}$. The "$-$" is used otherwise.

Note. By drawing the axis as shown in this order, we can see the unit vectors

$$\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle, \qquad \hat{\mathbf{j}} = \langle 0, 1, 0 \rangle, \qquad \hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$$

Satisfy the right hand rule, and that

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Cross Product

1. Inputs two vectors, outputs a vector
2. only works in 3 dimensions

Given two vectors

$$\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle, \qquad \vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$$

and the unit vectors

$$\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle, \qquad \hat{\mathbf{j}} = \langle 0, 1, 0 \rangle, \qquad \hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$$

Cross Product —

$$\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \end{aligned}$$

Example — Compute

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}}$$

and

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}}$$

Solution.

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \langle 1,0,0\rangle \times \langle 0,0,1\rangle = \langle 0,0,1\rangle = \boxed{\hat{\mathbf{k}}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = \langle 0,1,0\rangle \times \langle 1,0,0\rangle = \langle 0,0,-1\rangle = \boxed{-\hat{\mathbf{k}}}$$

□

The Cross product is not commutative, with

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$$

Geometric Interpretation

1. $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is perpendicular to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$
2. $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin(\Delta\theta)$ (The cross product can be thought of as the area of the paralellogram generated by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$)
3. $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ satisfy the right hand rule.

$$\vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \text{Volume of Parellopiped}$$

§1.4 Lines, Planes, and Quadric Surfacese

Lines in 3d Space

Think of these lines as a parameetrized curve in 3d, and each 3d point is an arrow from the origin to that point p.

Parametrization of a line in space

$$\vec{\mathbf{r}} = \vec{\mathbf{r_0}} + t\vec{\mathbf{v}}$$

where

$$\begin{aligned} &\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle \text{ is a point on the line} \\ &t = \text{ time parameter} \\ &\vec{\mathbf{r_0}} = \langle x_0, y_0, z_0\rangle \\ &\vec{\mathbf{v}} = \langle a, b, c\rangle \text{ is a tangent vector to the line} \end{aligned}$$

To describe a line, you need a point, and a tangent vector, and you can write the equation for the line.

Writing this out in coordinates gives us

$$\langle x(t), y(t), z(t)\rangle = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle$$

$$x(t) = x_0 + ta$$

$$y(t) = y_0 + tb$$

$$z(t) = z_0 + tc$$

(These are the same thing, just written in different forms)

Equation of Line through Two Points — The line through two points $\vec{\mathbf{r_0}}$, $\vec{\mathbf{r_1}}$, is

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r_0}} + t(\vec{\mathbf{r_1}} - \vec{\mathbf{r_0}})$$

$$\implies \vec{\mathbf{r}}(t) = (1-t)\vec{\mathbf{r_0}} + t\vec{\mathbf{r_1}}$$

The t in this can be also limited through a certain interval, for example the line between these two points is the same thing as above, but with $0 \leq t \leq 1$, as we chose our vector $\vec{\mathbf{v}}$ to range from $\vec{\mathbf{r_0}}$ to $\vec{\mathbf{r_1}}$.

Planes

We can describe a plane with two things:

1. Point on a plane, $\vec{\mathbf{r_0}}$
2. a vector $\vec{\mathbf{n}}$, a normal (perpendicular) vector to the plane

Any point $\vec{\mathbf{r}}$ is on the point if $(\vec{\mathbf{r}} - \vec{\mathbf{r_0}}) \cdot \vec{\mathbf{n}} = 0$.

If we have $\vec{\mathbf{r_0}} = \langle x_0, y_0, z_0 \rangle$, $\vec{\mathbf{n}} = \langle a, b, c \rangle$, $\vec{\mathbf{r}} = \langle x, y, z \rangle$, we have our equation

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

Evaluating this dot product, we get $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$.

Planes — For $\vec{\mathbf{r_0}} = \langle x_0, y_0, z_0 \rangle$ (a point on the plane), $\vec{\mathbf{n}} = \langle a, b, c \rangle$ (a normal vector to the plane), $\vec{\mathbf{r}} = \langle x, y, z \rangle$ (another point on the plane), a plane can be expressed in two ways, either

$$(\vec{\mathbf{r}} - \vec{\mathbf{r_0}}) \cdot \vec{\mathbf{n}} = 0$$

or

$$ax + by + cz = d$$

where

$$d = ax_0 + by_0 + cz_0$$

Note. Planes through the origin are those with $d = 0$.

Another way to determine a plane is through 3 non-colinear points $\vec{\mathbf{r_0}}, \vec{\mathbf{r_1}}, \vec{\mathbf{r_2}}$.

We need a normal vector, which we find by using the cross product (to find the perpendicular).

$$\vec{\mathbf{n}} = (\vec{\mathbf{r_1}} - \vec{\mathbf{r_0}}) \times (\vec{\mathbf{r_2}} - \vec{\mathbf{r_0}})$$

The equation for the plane is now

Plane with 3 Points — Can be expressed with three point $\vec{\mathbf{r_0}}$, $\vec{\mathbf{r_1}}$, $\vec{\mathbf{r_2}}$ as

$$(\vec{\mathbf{r}} - \vec{\mathbf{r_0}}) \cdot [(\vec{\mathbf{r_1}} - \vec{\mathbf{r_0}}) \times (\vec{\mathbf{r_2}} - \vec{\mathbf{r_0}})] = 0$$

which is equivalent to

$$\det \begin{pmatrix} \vec{\mathbf{r}} - \vec{\mathbf{r_0}} \\ \vec{\mathbf{r_1}} - \vec{\mathbf{r_0}} \\ \vec{\mathbf{r_2}} - \vec{\mathbf{r_0}} \end{pmatrix} = 0$$

> **Remark.** We can also describe this plane as a parrameterized surface, which depends on two parameters. It can be written as
> $$\vec{\mathbf{r}}(t_1, t_2) = \vec{\mathbf{r_0}} + t_1(\vec{\mathbf{r_1}} - \vec{\mathbf{r_0}}) + t_2(\vec{\mathbf{r_1}} - \vec{\mathbf{r_0}})$$
> which gone into more detail later.

Quadric Surface

This is represented by the equation

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0$$

How do we sketch this?

1. Set $z = 0$ to find the intersection with the xy plane
2. Set $y = 0$ to find the intersection with the xz plane
3. Set $x = 0$ to findthe intersection with the yz plane

Examples of quadric surfaces:

Ellipsoid —

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad a, b, c > 0$$

By sketching out the ellipes at $x = 0$, $y = 0$, and $z = 0$, we have the 3d ellispe ovally shape called an ellispoid hitting the x axis at $a, -a$, the y axis at $b, -b$, and z axis at $c, -c$.

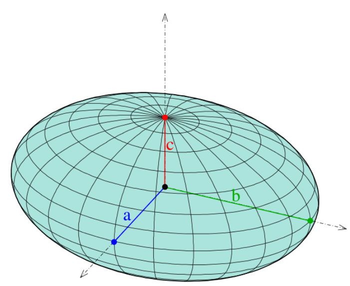


Figure 4: The Ellipsoid

Hyperboloid —

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1 \qquad a, b, c > 0 \quad \text{(not all signs are the same)}$$

Case 1: Two "+" signs

an example of this would be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

By sketching out $x = 0$, $y = 0$, and $z = 0$, we see that there are two hyperbolas and one ellipse, leading to a 3d graph somewhat shown below. This is also called hte hyperboloid of one sheet.

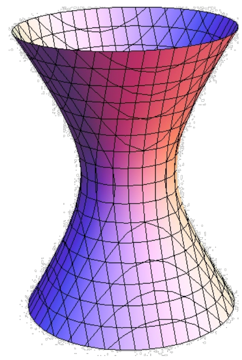


Figure 5: The Hyperboloid

Case 2: Two "−" signs

This would be

$$\frac{-x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

At $z = 0$, we have an empty set, $y = 0$, a hyperbola, and same for $x = 0$. This would produce something looking like the figure below. This is also called the hyperboloid of two sheets.

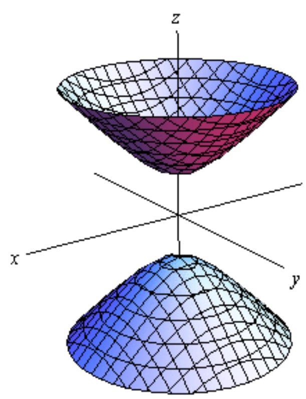


Figure 6: The Hyperboloid

Note. Note that the equation $ax^2 + by^2 + cz^2 + d = 0$, for $a, b, c, d > 0$, even though may look like an ellipsoid or hyperboloid, is actually an empty set, as sum of squares can't be negative.

Cone —

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad \text{(all terms are quadratic)}$$

This is Invariant under Scaling: if (x, y, z) is a solution and λ is a scalar then $(\lambda x, \lambda y, \lambda z)$ is also a solution $\implies$ Surface is a union of lines through the origin.

At $z = 0$, the only solution is $x = 0$ and $y = 0$.

If we set $y = 0$, we get the equation $\frac{z}{c} = \pm\frac{x}{a}$, which is two lines (one with + and one with −).

With $x = 0$, we also get two lines. This gives a full surface like with horizontal slices as ellipses.

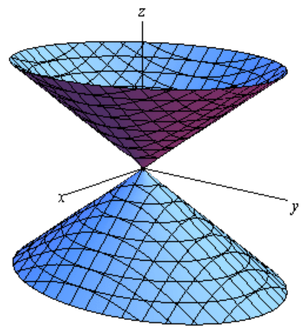


Figure 7: The Cone

Elliptic Paraboloid — This has one linear term, and two positive quadratic terms, for example

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad a, b, c > 0$$

Setting $z = 0$, we only have one point, the origin. Setting z to be a positive constant, we get an ellipse, and we get empty if its a negative constant.

Setting $y = 0$, we get a parabola on the xz plane, and setting $x = 0$ gets the same shape for the yz plane. The whole surface looks something like the thing below.

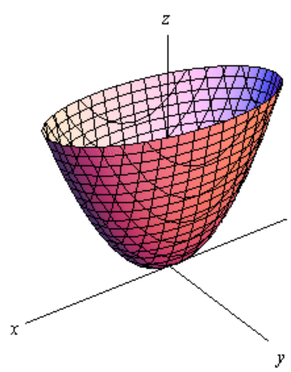


Figure 8: The Elliptic Paraboloid

Hyperbolic Paraboloid — This has one linear term, and two opposite-signed quadratic terms. For example,

$$\frac{z}{c} = -\frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad a, b, c > 0$$

Setting $z = 0$, we get two lines with the equation $\frac{y}{b} = \pm\frac{x}{a}$ Setting z to be a positive constant, we see that we get a hyperbola that intersects y but not x, and vice versa for z as a negative constant.

Setting $y = 0$, we have the equation $\frac{z}{c} = -\frac{x^2}{a^2}$, an upside down parabola.

Setting $x = 0$ gives us $\frac{z}{c} = \frac{y^2}{b^2}$, an upwards parabola.

The image of this is shown below. This shape is sometimes called a saddle.

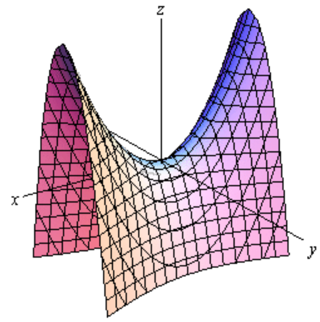


Figure 9: The Hyperbolic Paraboloid

Quadric Cylinder — This has one variable missing, such as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This has the same ellipsoid throughout the whole thing, making a sort of a "cylinder" shape, as shown below.

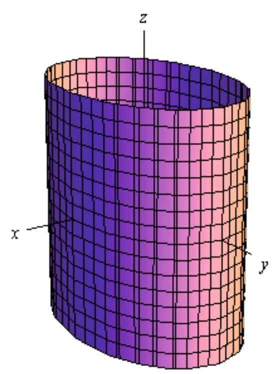


Figure 10: The Quadric Cylinder

General Quadric Surfaces

All those above are "special" quadric shapes. What about a more general one, such as

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + \underbrace{gx + hy + iz}_{\text{get rid of with completing the square}} + j = 0$$

By completing the square to get rid of those, we can simplify the equation.

Example —

$$x^2 + y^2 - 4z^2 + 4x - 6y - 8z = 13$$

We can rewrite this as $(x+2)^2 + (y-3)^2 - 4(z+1)^2 = 22$, by completing the square. By introducing new variables $\overline{x} = x+2$, $\overline{y} = y-3$, $\overline{z} = z+1$, we have

$$\overline{x}^2 + \overline{y}^2 - 4\overline{z}^2 = 22$$

which is a hyperboloid of one sheet, centered in the $\overline{x}$, $\overline{y}$, and $\overline{z}$ axis. Now, we can translate it normally just like in 2d, which is centered at $(-2, 3, -1)$.

This essentially shows us the equation transforms the functions like so:

$$ax^2 + by^2 + cz^2 + \underbrace{dxy + exz + fyz}_{\text{rotation}} + \underbrace{gx + hy + iz}_{\text{translation}} + j = 0$$

§1.5 Vector-valued Functions and Space Curves

Parametrized curves in space and vector-valued functions

Given the parametric form (also known as space curve) of

$$x = f(t), y = g(t), z = h(t), \alpha \leq t \leq \beta$$

We can rewrite this into vector form as

$$\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle = \langle x(t), y(t), z(t)\rangle$$

Example — Sketch $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t\rangle$

As $x^2 + y^2 = 1$, its going to move up in circles while moving upward in the z direction, drawing a helix.

We can think of parameterized vurves as Vector-valued function, $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$.

Limits of Vector-Valued Functions — The limit of a vector function $\vec{\mathbf{r}}(t) = \langle f(t), h(t), g(t)\rangle$ is given as

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

provided the limits of the component functions exist.

Integration of Vector-Valued Functions — Given the vector $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, the integration

$$\int_\alpha^\beta \vec{\mathbf{r}}(t)\, dt = \left\langle \int_\alpha^\beta x(t)\, dt, \int_\alpha^\beta y(t)\, dt, \int_\alpha^\beta z(t)\, dt\right\rangle$$

Note. This is not the "area under the curve", it is a vector.

Definition — Velocity Vector (aka the tangent vector) at time t is

$$\vec{\mathbf{r}}\,'(t) = \lim_{h\to 0} \frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}}{h} = \langle x'(t), y'(t), z'(t)\rangle$$

$\vec{\mathbf{r}}\,'(t)$ points in a direction tangent to the curve, and $|\vec{\mathbf{r}}\,'(t)|$ = speed at time t.

Unit Tangent Vector — The Unit Tangent Vector is just given as the tangent vector scaled to be unit size, given by the formula

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{|\vec{\mathbf{r}}\,'(t)|}$$

Unit Normal Vector — The unit normal vector is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}\,'(t)}{\left|\vec{\mathbf{T}}\,'(t)\right|}$$

Binormal Vector — The binormal vector, the vector orthogonal to both the normal and tangent vector, is defined to be

$$\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)}{|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)|}$$

Tangent Line — This line at time t goes through the point $\vec{\mathbf{r}}(t)$ and has direction $\vec{\mathbf{r}}\,'(t)$, so the equation for this line is

$$\vec{\mathbf{L}}(\vec{\mathbf{s}}) = \vec{\mathbf{r}}(t) + s\vec{\mathbf{r}}\,'(t)$$

Example — Find the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$ at $\langle 1, 0, 0 \rangle$.

Solution. At $t = 0$, $\vec{\mathbf{r}}(t) = \langle 1, 0, 0 \rangle$. This means $\vec{\mathbf{r}}\,'(0) = \langle -\sin(t), \cos(t), 1 \rangle \big|_0 = \langle 0, 1, 1 \rangle$. This means Tangent Line is

$$\boxed{\vec{\mathbf{L}}(s) = \langle 1, 0, 0 \rangle + s \langle 0, 1, 1 \rangle}$$

□

Definition — Acceleration at time is $\vec{\mathbf{r}}\,''(t)$.

An example of this is

$$\begin{aligned} \vec{\mathbf{r}}(t) &= \langle \cos(kt), \sin(kt), 0 \rangle \qquad \text{(a circle)} \\ \vec{\mathbf{r}}\,'(t) &= \langle -k\sin(kt), k\cos(kt), 0 \rangle \\ \vec{\mathbf{r}}\,''(t) &= \left\langle -k^2\cos(kt), -k^2\sin(kt), 0 \right\rangle \end{aligned}$$

Notice that the graph of this curve is a circle, and the velocity vector is pointing tangent to the circle, whereas the acceleration vector is pointing directly inward into the circle, tangent to the tangent line of the circle.

There are three different types of vector multiplication, which leads to three versions of the product rule:

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\vec{\mathbf{r}}(t)\right) = f'(t)\vec{\mathbf{r}}(t) + f(t)\vec{\mathbf{r}}\,'(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{s}}(t)\right) = \vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{s}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{s}}\,'(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathbf{r}}(t)\times\vec{\mathbf{s}}(t)\right) = \vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{s}}(t) + \vec{\mathbf{r}}(t)\times\vec{\mathbf{s}}\,'(t)$$

To prove these, expand in components and use the regular product rule for each component.

Example — Show that if the curve $\vec{\mathbf{r}}(t)$ is on the unit sphere $x^2+y^2+z^2=1$, then $\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t)=0$ for all t.

Solution.

$$\begin{aligned}\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)\right)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\vec{\mathbf{r}}(t)\right|^2\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1)\\ &= 0\end{aligned}$$

□

Definition — The <u>curvature</u> of a curve is

$$\kappa = \left|\frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}s}\right|$$

By the chain rule, we have that

$$\kappa = \left|\frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\vec{\mathbf{T}}/dt}{\mathrm{d}s/dt}\right|$$

As $\frac{\mathrm{d}s}{\mathrm{d}t} = |\vec{\mathbf{r}}\,'(t)|$, we have our final equation

Curvature — The curvature of a curve $\vec{\mathbf{r}}(t)$ given the unit tangent vector $\vec{\mathbf{T}}(t)$ is

$$\kappa(t) = \frac{\left|\vec{\mathbf{T}}\,'(t)\right|}{\left|\vec{\mathbf{r}}\,'(t)\right|^3}$$

Example — Find the curvature of a circle with radius a.

Solution. We have that the circle is defined as $\vec{\mathbf{r}}(t) = \langle a\cos(t), a\sin(t)\rangle$, so $\vec{\mathbf{r}}\,'(t) = \langle -a\sin(t), a\cos(t)\rangle$ and $|\vec{\mathbf{r}}\,'(t)| = a$.

We also have that $\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{|\vec{\mathbf{r}}\,'(t)|} = \langle -\sin(t), \cos(t)\rangle$, so $\vec{\mathbf{T}}\,'(t) = \langle -\cos(t), -\sin(t)\rangle$, and $\left|\vec{\mathbf{T}}\,'(t)\right| = 1$.

Using our curvature formula, this means the curvature is

$$\kappa(t) = \frac{\left|\vec{\mathbf{T}}\,'(t)\right|}{\left|\vec{\mathbf{r}}\,'(t)\right|} = \boxed{\frac{1}{a}}$$

□

Curvature — The Curvature of a curve given by the vector function $\vec{\mathbf{r}}$ is

$$\kappa(t) = \frac{\left|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\right|}{\left|\vec{\mathbf{r}}\,'(t)\right|}$$

Proof. As $\vec{\mathbf{T}} = \frac{\vec{\mathbf{r}}\,'}{|\vec{\mathbf{r}}\,'|}$ and $|\vec{\mathbf{r}}\,'| = \frac{\mathrm{d}s}{\mathrm{d}t}$, we have that

$$\begin{aligned}
\vec{\mathbf{r}}\,' &= \left|\vec{\mathbf{r}}\,'\right|\vec{\mathbf{T}} = \frac{\mathrm{d}s}{\mathrm{d}t}\vec{\mathbf{T}}\,' \\
\vec{\mathbf{r}}\,'' &= \frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbf{r}}\,' = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\vec{\mathbf{T}} + \frac{\mathrm{d}s}{\mathrm{d}t}\vec{\mathbf{T}}\,' \\
\implies \vec{\mathbf{r}}\,' \times \vec{\mathbf{r}}\,'' &= \underbrace{\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\vec{\mathbf{T}} \times \frac{\mathrm{d}s}{\mathrm{d}t}\vec{\mathbf{T}}}_{\vec{\mathbf{T}}\times\vec{\mathbf{T}}=0} + \frac{\mathrm{d}s}{\mathrm{d}t}\vec{\mathbf{T}}\,' \times \frac{\mathrm{d}s}{\mathrm{d}t}\vec{\mathbf{T}} \\
&= \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \left(\vec{\mathbf{T}} \times \vec{\mathbf{T}}\,'\right)
\end{aligned}$$

Now, notice that $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}\,'$ are orthogonal, as by the product rule

$$\vec{\mathbf{T}} \cdot \vec{\mathbf{T}}\,' = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathbf{T}} \cdot \vec{\mathbf{T}}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1) = 0$$

This means that

$$\begin{aligned}
\left|\vec{\mathbf{r}}\,' \times \vec{\mathbf{r}}\,''\right| &= \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \left|\vec{\mathbf{T}} \times \vec{\mathbf{T}}\,'\right| \\
&= \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \left|\vec{\mathbf{T}}\right|\left|\vec{\mathbf{T}}\,'\right| \\
&= \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \left|\vec{\mathbf{T}}\,'\right| \\
\implies \left|\vec{\mathbf{T}}\,'\right| &= \frac{\left|\vec{\mathbf{r}}\,' \times \vec{\mathbf{r}}\,''\right|}{(ds/dt)^2} = \frac{\left|\vec{\mathbf{r}}\,' \times \vec{\mathbf{r}}\,''\right|}{\left|\vec{\mathbf{r}}\,'\right|^2}
\end{aligned}$$

leading to our final equation of

$$\kappa = \frac{\vec{\mathbf{T}}\,'}{\vec{\mathbf{r}}\,'} = \frac{\vec{\mathbf{r}}\,' \times \vec{\mathbf{r}}\,''}{\left|\vec{\mathbf{r}}\,'\right|^3}$$

□

Note that this also has another form, which can be proven easily by substituting $\vec{\mathbf{r}}(x) = x\hat{\mathbf{i}} + f(x)\hat{\mathbf{j}}$ (as we have $\vec{\mathbf{r}}\,'(x) \times \vec{\mathbf{r}}\,''(x) = f''(x)\hat{\mathbf{k}}$ and $|\vec{\mathbf{r}}\,'(x)| = \sqrt{1 + [f'(x)]^2}$).

Curvature of a Function — For the special case of a plane curve with equation $y = f(x)$, the curvature is

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

Curvature is also related to acceleration, with the acceleration of objects commonly split into a tangential component a_T, and a normal component a_N.

Acceleration Pt 2 — With this, acceleration can be written as

$$\vec{\mathbf{a}} = a_T\vec{\mathbf{T}} + a_N\vec{\mathbf{N}}$$

Where $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ are the unit tangential and unit normal vectors to the function.

The acceleration components a_T and a_N are given by

$$a_T = |\vec{\mathbf{v}}(t)|' = \frac{\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}''(t)}{|\vec{\mathbf{r}}'(t)|}$$

$$a_N = \kappa|\vec{\mathbf{v}}(t)|^2 = \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|}$$

Length — The Length of the curve $\vec{\mathbf{r}}(t)$ with $\alpha \leq t \leq \beta$ is

$$\text{Length} = \int_\alpha^\beta \left|\vec{\mathbf{r}}'(t)\right| \, dt$$

Area — The area of the curve $\vec{\mathbf{r}}(t)$ with $\alpha \leq t \leq \beta$ is

$$\text{Area (ribbon)} = \frac{1}{2}\int_\alpha^\beta \left|\vec{\mathbf{r}}(t) \times \vec{\mathbf{r}}'(t)\right| \, dt$$

Fundamental Theorem of Calculus

$$\int_a^b \vec{\mathbf{r}}'(t)\, dt = \vec{\mathbf{r}}(b) - \vec{\mathbf{r}}(a)$$

This means we can integrate velocity to obtain change in position, and acceleration to change in velocity.

§2 Module 2 (Differentiation and Optimization)

§2.1 Functions of Several Variables; Limits and Continuity

Functions with two variables

Let D be a region in $\mathbb{R}^2$.

Definition — A function from D to $\mathbb{R}$, or a "function on D", is a rule f which assigns to each point $(x, y) \in D$ a number $f(x, y) \in \mathbb{R}$.

D is called the domain of f. This is a subset of $\mathbb{R}^2$.

Range $R = \{f(x, y) \mid (x, y) \in D\}$, i.e. the set of all values of f. This is a subset of $\mathbb{R}$.

There are two ways to visualize a function of two variables:

1. Draw the graph.

Graph $= S = \{(x, y, z) \mid (x, y) \in D, z = f(x, y)\}$, which is a surface in $\mathbb{R}^3$.

Example — Graph of $f(x, y) = x - y + 1$ with domain $D = \mathbb{R}^2$

Solution. This graph is the surface of $z = x - y + 1$, which is a plane. We can graph this by testing out stuff, such as $y = x$, where we see $z = 1$, and $y = x + 1$, which we see $z = 0$, which combined make a plane sort of angling downward. □

In general the graph of a linear function $f(x, y) = ax + by + c$ is a plane $z = ax + by + c$.

Example — Graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$ with domain $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$, which is a disc of radius 3 centered at the origin.

Solution. The graph is $z = \sqrt{9 - x^2 - y^2}$. By squaring both sides, we have $x^2 + x^2 + z^2 = 9$, which is a sphere of radius 3 centered at the origin. However, as z is always positive, $z \geq 0$, which is just the upper half of the sphere. □

Note. Vertical line test is the same exact thing for 3d graphs.

Example — Graph of $f(x, y) = \frac{1}{x^2+y^2}$, with domain $D = \mathbb{R}^2 \setminus \{(0, 0)\}$.

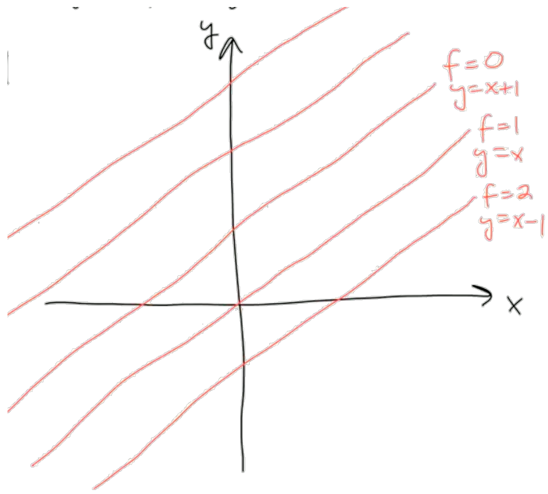
Solution. On the circle $x^2 + y^2 = r^2$, $f = \frac{1}{r^2}$ and is a constant. On large circles, f is small and f is large on small circles. This means the graph is □

2. Level Curves

Level Curve/Set — Let f be a function of two variables. Given k, the set $\{(x, y) \in D \mid f(x, y) = k\}$ is a level curve or level set.

Example — $f(x, y) = x - y + 1$

As $f = 0$ when $y = x + 1$, $f = 1$ when $y = x$, $f = 2$ when $y = x - 1$, then we get the level curve of

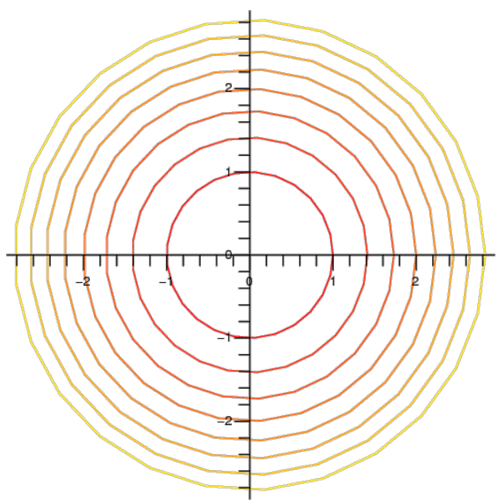


In general, the level set of a plane thats not horizontal or vertical is going to consist of equally spaced parallel lines.

Example — $f(x, y) = \sqrt{9 - x^2 - y^2}$

$f = 0$ when $x^2 + y^2 = 9$, $f = 1$ when $x^2 + y^2 = 8$, $f = 2$ when $x^2 + y^2 = 5$, drawing smaller and smaller circles as f increases, resulting in a sort of shape full of concentric circles.

Near the boundary of the domain, the lines will be closer, and they get further as it gets closer to the origin, describing half of a sphere.



<u>Functions of Three Variables</u>

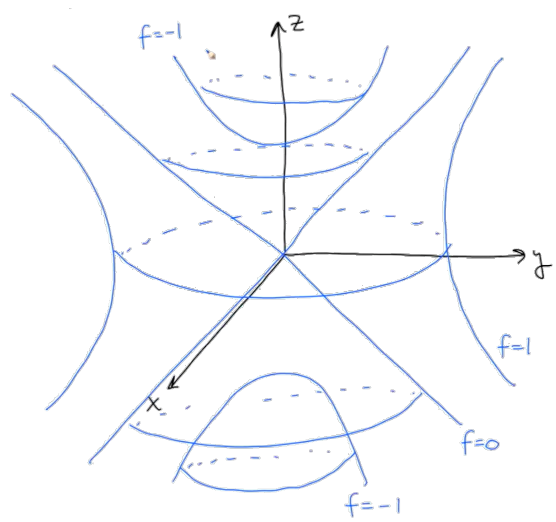
A function f of three variables assigns every number (x, y, z) within the domain D a number $f(x, y, z) \in \mathbb{R}$, meaning D is a subset of $\mathbb{R}^3$.

The graph of f is the set $\{x, y, z, w \in \mathbb{R}^4 \mid w = f(x, y, z)\}$, which is a three dimensional hypersurface in 4d space, which is too hard to draw. We can instead draw <u>level sets/surfaces</u>, with $\{(x, y, z) \in D \mid f(x, y, z) = k\}$.

Example — Sketch $f(x, y, z) = x^2 + y^2 - z^2$.

Setting $f = 0$, we get the equation $x^2 + y^2 = z^2$, which is a cone. Setting $f = 1$, we get $x^2 + y^2 = z^2 + 1$, which is a hyperboloid of one sheet. Setting $f = -1$, we have $x^2 + y^2 = z^2 - 1$, which is a hyperboloid of two sheets.

Looking at the level surface drawn below, we can kind of see how the graph as f is getting larger is moving "outward", from a two sheeted hyperboloid to a cone, then to a one sheeted hyperboloid.



Limits

Let f be a funciton on a domain D in $\mathbb{R}^2$.

Epsilon Delta Definition of Limits — The expression

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

means for every $\epsilon > 0$, there exists $\delta > 0$, such that if $(x, y) \in D$ and $(x, y) \neq (a, b)$ and $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.

Intuitively, you can think of ϵ as a sort of error tolerance, where you want to gurantee that $f(x, y)$ is within ϵ of L.

The definition says that you can guranteee that by having (x, y) to be sufficiently close, or δ away, from (a, b).

This means you can think of δ as how close (x, y) has to be to (a, b) to gurantee that $f(x, y)$ is within the error tolerance of L.

Example — Prove that

$$\lim_{(x,y)\to(0,0)} 2x = 0$$

Solution. Given $\epsilon > 0$, we need to find $\delta > 0$ such that if $(x, y) \neq (0, 0)$ and $\sqrt{x^2 + y^2} < \delta$ then $|2x| < \epsilon$.
Choose $\delta = \frac{\epsilon}{2}$. This works because if $\sqrt{x^2 + y^2} = \frac{\epsilon}{2}$, then $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} < \frac{\epsilon}{2}$ so $|2x| < \epsilon$. □

Properties of Limits — Some properties of limits include

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c$$

Properties of Limits — If both limits on the right are defined for the following, then these properties hold.

$$\lim_{(x,y)\to(a,b)} (f(x,y)+g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$$
$$\lim(fg) = (\lim f)(\lim g)$$
$$\lim(f/g) = \frac{\lim f}{\lim g} \qquad (\text{if } \lim g \neq 0)$$

Properties of Limits — If $h : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then

$$\lim_{(x,y)\to(a,b)} h(f(x,y)) = h(L)$$

Example —

$$\lim_{(x,y)\to(a,b)} e^{x+y} = e^{a+b}$$

as setting $h(t) = e^t$ and $f(x,y) = x+y$ and using the rule above gives us this fact.

Properties of Limits — A function f of two variables is continuous at (a,b) if $f(a,b)$ is defined and

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Things that are continuous include

- Constants
- Coordinate Functions
- Sums/Products/Quotients of continuous functions
- Compositions with continuous functions $h : \mathbb{R} \to \mathbb{R}$

Note. An example of a non-continuous function is

$$f(x,y) = \begin{cases} 1 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

This isn't continuous at $(0,0)$ as $\lim_{(x,y)\to(0,0)} f(x,y) = 1 \neq f(0,0)$ (this is just like in single variable calc).

Proving that Limits do not exist

Properties of Limits — If
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$
and $(x(t), y(t))$ is a parametrized curve with
$$\lim_{t\to t_0} x(t) = a \qquad \lim_{t\to t_0} y(t) = b$$
and $(x(t), y(t)) \neq (a,b)$ for $t \neq t_0$, then
$$\lim_{t\to t_0} f(x(t), y(t)) = L$$

This means that if the limit as (x,y) approach (a,b) of $f(x,y)$ is equal to L, then approaching that point from any other curve should also equal the same L.

This can prove limits doesn't exist as if f has an undefined limit along some curve approaching (a,b), or if f has two different limits along two different curves approaching (a,b), then
$$\lim_{(x,y)\to(a,b)} f(x,y)$$
is not defined.

Example — Prove
$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$
does not exist.

Solution. First, lets try to approach $(0,0)$ from the x-axis. Letting $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$, we have
$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$
Now, lets try approaching $(0,0)$ from the y-axis.
$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1$$
Since $1 \neq -1$,
$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \text{DNE}$$
□

Example — Is
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^4+y^4}$$
defined?

Solution. By approaching on the x-axis, we see that
$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0 = 0$$

By approaching on the y-axis, we see

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} 0 = 0$$

This might make you think the limit may exist. However, lets try approaching this curve along $y = x$, or $x(t) = t$, $y(t) = t$.

$$\lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{t^2}{t^4 + t^4} = \lim_{t\to 0} \frac{1}{2t^2} = \infty$$

This means that this limit does not exist, and shows that one has to approach from all sorts of different curves to check the limit. □

Example — Is

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$$

defined?

Solution. Approaching along the axis yields the limit as 0.

Approachign along the line $y = mx$ where $m \neq 0$, or the parameters $x = t$, $y = mt$.

$$\lim_{t\to 0} f(t, mt) = \lim_{t\to 0} \frac{t(mt)^2}{t^2 + (mt)^4} = \lim_{t\to 0} \frac{m^2t^3}{t^2 + m^4t^4} = \lim_{t\to 0} \frac{m^2 t}{1 + m^4t^2} = 0$$

This means that the limit as it approaches along any line is also 0.

However, approaching on the parabola $x = y^2$ yields a different result.

$$\lim_{t\to 0} f(t^2, t) = \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2}$$

This means that this limit does not in fact exist. □

With these examples, it is seen that limits cannot be concretely proven to do exist with just guessing curves, but it may be a shortcut.

§2.2 Partial Derivatives, Tangent Planes, Linear Approximation

Partial Derivatives

Let f be a function of x and y. With this function, we can find both the change in x, and the change in y, resulting in not one derivative, but two partial derivatives.

Definition — The partial derivative of f with respect to x at (x, y) is

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \text{ if this limit exists}$$

This is the rate at which f changes as we vary x, holding y fixed.

To compute $\frac{\partial f}{\partial x}$, regard y as a constant and differentiate with respect to x.

Definition — The partial derivative of f with respect to y at (x, y) is

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \text{ if this limit exists}$$

This is the rate at which f changes as we vary y, holding x fixed.
Once again, to compute $\frac{\partial f}{\partial y}$, regard x as a constant and differentiate with respect to y.

Example — $f(x, y) = x^3 + y^4 + x^2 y$. Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution. Treating y as a constant, we have

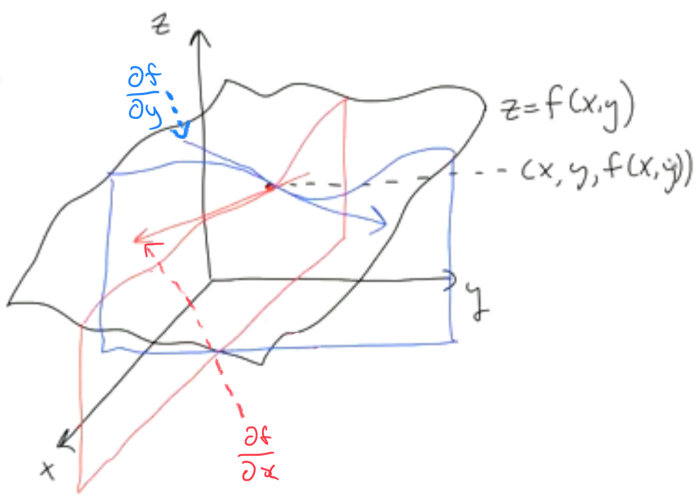
$$\frac{\partial f}{\partial x} = \boxed{3x^2 + 0 + 2xy}$$

Treating x as a constant, we have

$$\frac{\partial f}{\partial y} = \boxed{0 + 3y^3 + 2xy}$$

□

These partial derivatives can be seen geometrically with the figure below.



Suppose we had a function f of x, y, and z. Now we have three partial derivatives.

Partial Derivatives —

$$\frac{\partial f}{\partial x} = f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$
$$\frac{\partial f}{\partial y} = f_y(x, y, z) = \lim_{h \to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$
$$\frac{\partial f}{\partial z} = f_z(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

To compute these, regard the other variables as constants and differentiate with regard to the variable. This is analagous to the partial derivatives from before. Once again, these partials are only defined if the limits exist.

Example — $f(x, y, z) = x^3 + xyz + yz^2$. Calculate the partial derivatives.

Solution.

$$\frac{\partial f}{\partial x} = \boxed{3x^2 + yz + 0}$$
$$\frac{\partial f}{\partial y} = \boxed{0 + xz + z^2}$$
$$\frac{\partial f}{\partial z} = \boxed{0 + xy + 2yz}$$

□

Second Partial Derivatives —

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx} \qquad \frac{\partial^2 f}{\partial x^2} = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx} \qquad \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

This is the same for functions of three variables.

Example — $f(x, y) = x^2 \cos(x + y)$. Calculate the second partial derivatives of f.

Solution. The first partial derivatives are

$$\begin{aligned} f_x &= 2x\cos(x+y) - x^2\sin(x+y) \\ f_y &= -x^2\sin(x+y) \end{aligned}$$

Now, the second partial derivatives are

$$\begin{aligned} f_{xx} &= 2\cos(x+y) - 2x\sin(x+y) - 2x\sin(x+y) - x^2\cos(x+y) \\ f_{xy} &= (f_x)_y = -2x\sin(x+y) - x^2\cos(x+y) \\ f_{yx} &= (f_y)_x = -2x\sin(x+y) - x^2\cos(x+y) \\ f_{yy} &= -x^2\cos(x+y) \end{aligned}$$

□

Notice in this example, $f_{xy} = f_{yx}$. This leads us to the theorem

Theorem 2.1 (Clairaut's Theorem)
If f_{xy} and f_{yx} are defined and continuous in a neighborhood of (x, y), then $f_{xy}(x, y) = f_{yx}(x, y)$.
This holds true for functions of three (or more) variables, with

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad \cdots$$

and higher partial derivatives, such as

$$f_{xyyz} = f_{yxyz} = f_{xzyy} = \cdots$$

Implicit Partial Differentiation

More on this is gone in depth after learning the chain rule. It can be found here.

Instead of the usual explicit formula $z = f(x, y)$, sometimes there are implicit equations in the form

$$F(x, y, z) = 0$$

An example of an implicit equation is $x^2 + y^2 + z^2 = 9$.

Example — Differentiate $x^2 + y^2 + z^2 = 9$.

Solution. There are two ways to solve this: the implicit way, and the way we learned before with explicit functions. We'll show that both of these yield the same answer.

Method 1 (Implicit):

What we want to do to do this implicitly is think of everything in this equation as a function of x and y.

$$\begin{aligned} \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) &= \frac{\partial}{\partial x}(9) \\ 2x + 0 + 2z\frac{\partial z}{\partial x} &= 0 \\ \implies \frac{\partial z}{\partial x} &= \boxed{-\frac{x}{z}} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right) &= \frac{\partial}{\partial y}(9) \\ 0 + 2y + 2z\frac{\partial z}{\partial y} &= 0 \\ \implies \frac{\partial z}{\partial y} &= \boxed{-\frac{y}{z}} \end{aligned}$$

Method 2 (Explicit):

Solving for z, we get $z = \sqrt{9 - x^2 - y^2}$ (for simplicity just doing positive square root).

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{9 - x^2 - y^2}}(-2x) = \frac{-x}{\sqrt{9 - x^2 - y^2}} = \boxed{\frac{-x}{z}}$$

□

Example — Suppose z is defined implicitly by $z^2 + y^2 + xyz^2 = 0$. Calculate $\frac{\partial z}{\partial x}$.

Solution. We solve this by taking the derivative implicity with respect to x.

$$\begin{aligned} \frac{\partial}{\partial x}\left(zx^2 + y^2 + xyz^2\right) &= \frac{\partial}{\partial x}(0) \\ x^2\frac{\partial z}{\partial x} + 2xz + 0 + y\left(2xz\frac{\partial z}{\partial x} + yz^2\right) &= 0 \\ \implies \frac{\partial z}{\partial x} &= \boxed{\frac{-2zx - yz^2}{x^2 + 2xyz}} \end{aligned}$$

□

Tangent Planes

Tangent Line — Given a function $f(x)$, the tangent line at the piont (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0)$$

Tangent Plane — The tangent plane to $z = f(x, y)$ at $(x_0, y_0, z_0 = f(x_0, y_0))$ is

$$z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

We can prove this by writing the slope in the x direction from a certain point (x_0, y_0) is $\frac{\partial f}{\partial x}$ and the slope in the y direction is $\frac{\partial f}{\partial y}$.
From here, we see that the displacement in the z, or $z - z_0$, is equal to the change in the x and the y direction, or $\frac{\partial f}{\partial x}(x - x_0)$ and $\frac{\partial f}{\partial y}(y - y_0)$, and so we get our equation.

Example — Find the tangent plane to $x^2 + y^2 + z^2 = 1$ at (x_0, y_0, z_0).

Solution. With the tangent plane formula, we have

$$\begin{aligned}
&z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0) \\
&z - z_0 = -\frac{x_0}{z_0}(x - x_0) + -\frac{y_0}{z_0}(y - y_0) \\
&z_0(z - z_0) + x_0(x - x_0) + y_0(y - y_0) = 0 \\
&x_0 x + y_0 y + z_0 z = \underbrace{x_0^2 + y_0^2 + z_0^2}_{1}
\end{aligned}$$

$$\implies \boxed{x_0 x + y_0 y + z_0 z = 1}$$

□

Linear Approximation

Linear Approximation — Given a function $f(x)$ and a $x - x_0 = \Delta x$, the linear approximation of $f(x)$ is

$$f(x) = f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Before we talk about linear approximations of functions with 2 variables, we must talk about the differentiability of a function f.

Definition — f is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are defined, and there exists functions ϵ_1, ϵ_2 such that

$$\underbrace{f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}_{\text{equation for tangent plane}} + \epsilon_1(\Delta x, \Delta y)\Delta x + \epsilon_2(\Delta x, \Delta y)\Delta y$$

where

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \epsilon_1(\Delta x,\Delta y) = \lim_{(\Delta x,\Delta y)\to(0,0)} \epsilon_2(\Delta x,\Delta y) = 0$$

Remark. If $\epsilon_1 = \epsilon_2 = 0$ this equation says that the graph $z = f(x,y)$ is the same as the tangent plane to the graph at $(x_0, y_0, z_0 = f(x_0, y_0))$.

This shows that this is deviating from the tangent plane an ϵ_1 and ϵ_2 such that the limit as this approaches 0 when Δx and Δy becomes 0.

The informal definition is just that f is differentiable at (x_0, y_0) means that the graph near that point is well approximated by the tangent plane to the grpah at (x_0, y_0, z_0).

Note. Just noting that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ being defined doesn't necessarily mean that its differentiable. An example of this is shown below.

Example — Is the function $f(x,y) = x^{\frac{1}{3}}y^{\frac{1}{3}}$ differentiable at $(x_0, y_0) = (0,0)$?

Solution. f_x and f_y are both defined, with both equal to 0. However, considering the curve $x = t$, $y = t$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t,t) = \frac{\mathrm{d}}{\mathrm{d}t}t^{\frac{1}{3}}t^{\frac{1}{3}} = \frac{\mathrm{d}}{\mathrm{d}t}t^{\frac{2}{3}} = \frac{2}{3}t^{-\frac{1}{3}}$$

which isn't defined at $t = 0$. Drawing the graph of this also provides a geometric sense of why its not able to be approximated by any plane, so its not differentiable. □

Fact. If f_x and f_y are defined and continuous in a neighborhood of (x_0, y_0), then f is differentiable at the point (x_0, y_0).

Linear Approximation — If f is differentiable at the point (x_0, y_0) then

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

Example — Approximate $\sqrt{(3.012)^2 + (3.997)^2}$.

Solution. We have that

$$f(x,y) = \sqrt{x^2 + y^2} \qquad (x_0, y_0) = (3,4) \qquad \Delta x = 0.12 \quad \Delta y = -0.03$$

and solving for f_x and f_y gives us

$$\begin{aligned} f_x &= \frac{x}{f} \\ f_y &= \frac{y}{f} \end{aligned}$$

Now, plugging into our formula gives

$$\begin{aligned} f(3.012, 3.997) &\approx f(3,4) + f_x(3,4)(0.12) + f_y(3,4)(-0.03) \\ &= 5 + \frac{3}{5}(0.12) + \frac{4}{5}(-0.03) = \boxed{5.0048} \end{aligned}$$

and the "exact" answer by a calculator is 5.004813... □

§2.3 Chain Rule

Multivariable Chain Rule 1 — Suppose f is a differentiable function of x and y, and x and y are differentiable functions of t. Then,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

aka if $z(t) = f(x(t), y(t))$, then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}(x(t), y(t))\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{\mathrm{d}y}{\mathrm{d}t}(t)$$

Intuitively, this makes sense as thinking of f as the height function and thinking of $t \to (x(t), y(t))$ as a parametrized curve, and the chain rule says that the change of the height is from two different places, 1 being from the x contribution, so the $\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$, and one from the y, resulting in the other term.

Proof. If we have $\Delta z = z(t + \Delta t) - z(t)$, or also the "change in z", then

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{f_x \Delta x + f_y \Delta y + \overbrace{\epsilon_1 \Delta x + \epsilon_2 \Delta y}^{\text{limit}=0}}{\Delta t} \\
&= \lim_{\Delta t \to 0} f_x \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} f_y \frac{\Delta y}{\Delta t} \\
&= f_x \frac{\mathrm{d}x}{\mathrm{d}t} + f_y \frac{\mathrm{d}y}{\mathrm{d}t}
\end{aligned}$$

□

Example — Find $\frac{\mathrm{d}z}{\mathrm{d}t}$ for $f(x, y) = y^2 - x^2$ with $x(t) = t$, $y(t) = t$, and $z(t) = f(x(t), y(t))$.

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\
&= (-2x)(1) + (2y)(1)
\end{aligned}$$

Solution. However, as we want this in terms of t, we can substitute the parameterization back in, which results in

$$= -2t + 2t = \boxed{0}$$

□

Example — Find $\frac{\mathrm{d}z}{\mathrm{d}t}$ for $f(x, y) = y^2 - x^2$, $x(t) = t$, $y(t) = t^2$, $z = f(x(t), y(t))$

Solution.

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\
&= (-2x)(1) + (2y)(2t) \\
&= \boxed{(-2t + 4t^3)}
\end{aligned}$$

This is positive if $t > \frac{1}{\sqrt{2}}$, and negative if $0 < t < \frac{1}{\sqrt{2}}$.

Note. It was not necessary to use the chain rule here.

□

Example — Calculate $\frac{\mathrm{d}z}{\mathrm{d}t}$ for $f(x,y) = x^2 + y^2$, $x = e^t$, $y = \ln(t)$, and $z = f(x(t), y(t))$

Solution. Solution 1:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= 2xe^t + 2y(\frac{1}{t}) \\ &= \boxed{2e^{2t} + \frac{2\ln(t)}{t}}\end{aligned}$$

Solution 2: Since $z = f(x(t), y(t))$, we have $z = (e^t)^2 + (\ln(t))^2$, so

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\mathrm{d}z}{\mathrm{d}t}e^{2t} + (\ln(t))^2 \\ &= \boxed{2e^{2t} + \frac{2\ln(t)}{t}}\end{aligned}$$

□

Chain Rule 2 — Suppose

$$\begin{aligned}z &= f(x,y) \\ x &= g(s,t) \\ y &= h(s,t)\end{aligned}$$

Then, the partials of z with respect to s and t are

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example — $z = e^x \sin(y)$, $x = st^2$, $y = s^2t$. Calculate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution.

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= e^x \sin(y)t^2 + e^x \cos(y)2st \\ &= \boxed{e^{st^2}\sin\left(s^2t\right)t^2 + e^{st^2}\cos\left(s^2t\right)2st}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= e^x \sin(y) 2st + e^x \cos(y) s^2 \\ &= \boxed{e^{st^2} \sin\left(s^2 t\right) 2st + e^{st^2} \cos\left(s^2 t\right) s^2}\end{aligned}$$

You can also plug in, just like before, x and y first, and then differentiate. □

General Chain Rule — Suppose that u is a differentiable function of $x_1, \dots, x_n$.

Suppose that each of these variables x_i is a differentiable function of $t_1, \dots, t_m$.

Then for each i from 1 to m,

$$\begin{aligned}\frac{\partial u}{\partial t_i} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x}{\partial t_i} \\ &= \sum_{n=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}\end{aligned}$$

Example — $u = x^3 + xyz + z^3$, $z = s^2 + t^2$, $y = rs$, $z = r^2$. Calculate the partial derivatives.

Solution. In this case, $n = 3$ and $m = 3$. What the chain rule tells us is

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} \\ &= 0 + (xz)(s) + (xy + 3z^2)(2r) \\ &= (s^2 + t^2)(r^2)(s) + ((s^2 + t^2)(rs) + 3(r^2)^2)(2r) \\ &= \boxed{6r^5 + 3r^2 s^3 + 3r^2 st^2}\end{aligned}$$

□

Implicit Partial Differentiation

We can use the chain rule to formally show implicit partial differentiation.

Suppose that z is a function of x, y defined implicitly by $F(x, y, z) = 0$ where F is a differentiable function of 3 variables. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

$$F(x, y, z(x, y)) = 0 \text{ as functions of } x \text{ and } y$$

Now, by the chain rule, we have

$$\begin{aligned}&\frac{\partial}{\partial x} F(x, y, z(x, y)) = \frac{\partial}{\partial x} 0 \\ \implies &\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial y}\frac{\partial z}{\partial x} = 0\end{aligned}$$

We can do the same process for $\frac{\partial z}{\partial y}$.

$$\begin{aligned}&\frac{\partial}{\partial y} F(x, y, z(x, y)) = \frac{\partial}{\partial y} 0 \\ \implies &\frac{\partial F}{\partial y}\underbrace{\frac{\partial x}{\partial y}}_{0} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial y}}_{1} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} = 0\end{aligned}$$

Implicit Partial Differentiation — Given the implicit equation $F(x, y, z) = 0$, $\frac{\partial F}{\partial z} \neq 0$, then the implicit partial derivativees are

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}$$

$$\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}$$

Theorem 2.2 (Implicit Function Theorem)

Suppose F is a differentiable function of x, y, z. Suppose $F(x_0, y_0, z_0) = 0$ and $\frac{\partial F}{\partial z} \neq 0$. Then

- its possible to write z as a function of x, y when (x, y) is in the neighborhood to (x_0, y_0) with z close to z_0 (not rigorous with epsilons)
- z is a differentiable function of x and y.

§2.4 Directional Derivatives and the Gradient Vector

Directional Derivatives

Let f be a function of x, y. Let $\vec{\mathbf{u}} = \langle a, b\rangle$ be a unit vector, or $|\vec{\mathbf{u}}| = 1$.

Definition — The Directional Derivative of f in the direction $\vec{\mathbf{u}}$ at (x, y) is

$$D_{\vec{\mathbf{u}}} f(x, y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x + at, y + bt)$$

This means that geometrically, we're taking the derivative of a line, moving in the direction of $\vec{\mathbf{u}}$, at unit speed, that moves through the point (x, y).

By the Chain Rule, if f is differentiable, we have

$$d_{\vec{\mathbf{u}}} f(x, y) = \frac{\partial f}{\partial x}(x, y) \underbrace{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (x + at)}_{a} + \frac{\partial f}{\partial y}(x, y) \underbrace{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (y + bt)}_{b}$$

leading us to our final equation,

Directional Derivatives in 2D — The derivative from direction of unit vector $\vec{\mathbf{u}} = \langle a, b\rangle$ of function f is

$$D_{\vec{\mathbf{u}}} f = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$$

From this formula, we can see that

$$D_{\langle 1,0\rangle} f = \frac{\partial f}{\partial x}$$

$$D_{\langle 0,1\rangle} f = \frac{\partial f}{\partial y}$$

which makes sense, as that's essentially the definition of the partial derivative.

Similarly, in the 3d plane, if f is a function of x, y, z and $\vec{\mathbf{u}} = \langle a, b, c\rangle$ is a unit vector then

$$D_{\vec{\mathbf{u}}} f(x, y, z) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (f(x + at, y + bt, z + ct))$$

And similarly, by the Chain Rule,

Directional Derivatives in 3D — If f is differentiable and $\vec{\mathbf{u}} = \langle a, b, c\rangle$ is a unit vector in the direction we're taking the derivative of,

$$D_{\vec{\mathbf{u}}} f = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}$$

Gradient

Definition — If f is a function of two variables, define the gradient of f at (x, y) to be

$$\operatorname{grad} f(x, y) = \boldsymbol{\nabla} f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle (x, y)$$

If f is a function of three variables then

$$\boldsymbol{\nabla} f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle (x, y, z)$$

The gradient essentially represents all the partial derivatives as a vector

If f is differentiable and $\vec{\mathbf{u}}$ is a unit vector, then

$$D_{\vec{\mathbf{u}}} f = \vec{\mathbf{u}} \cdot \boldsymbol{\nabla} f$$

This is true for functions of any amount of variables. As we know

$$\vec{\mathbf{u}} = \langle a, b, c\rangle$$
$$\boldsymbol{\nabla} f = \langle f_x, f_y, f_z\rangle$$
$$D_{\vec{\mathbf{u}}} f = af_x + bf_y + cf_z$$

We can see that the equation above is true, with the dot product of $\vec{\mathbf{u}}$ a nd $\boldsymbol{\nabla} f$ equal to $D_{\vec{\mathbf{u}}}$.

The gradient and the directional derivative are functions, and depend on where you are.

Example — Find the gradient of $f(x, y) = xy^3 + \sin(xy)$.

Solution.

$$\boldsymbol{\nabla} f(x, y) = \boxed{\langle y^3 + y\cos(x), 3xy^2 + x\cos(xy)\rangle}$$

Note that to find the directional derivative, you have to actually plug in the value of x and y, then dot product with the direction vector $\vec{\mathbf{u}}$. □

Properties of the Gradient — Let f be a differentiable function of 2 or 3 variables. If $\boldsymbol{\nabla} f \neq 0$ then $\boldsymbol{\nabla} f$ points in the direction in which the direction derivative of f is largest

Proof.

$$\begin{aligned} D_{\vec{\mathbf{u}}} f &= \vec{\mathbf{u}} \cdot \boldsymbol{\nabla} f \\ &= |\vec{\mathbf{u}}||\boldsymbol{\nabla} f| \cos(\theta) \\ &= |\boldsymbol{\nabla} f| \cos(\theta) \\ &\leq |\boldsymbol{\nabla} f| \end{aligned}$$

Equality holds when $\cos(\theta) = 1$, or when $\theta = 0$ when $\vec{\mathbf{u}}$ and $\boldsymbol{\nabla} f$ point in the same direction. □

Properties of the Gradient (Vector Chain Rule) — If $\vec{\mathbf{r}}(t)$ is a vector-valued differentiable function of t, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{\mathbf{r}}(t)) = \boldsymbol{\nabla} f \cdot \vec{\mathbf{r}}\,'(t)$$

Proof. Write $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$. By the Chain Rule,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(\vec{\mathbf{r}}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t), z(t)) \\ &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle \\ &= \boldsymbol{\nabla} f \cdot \vec{\mathbf{r}}\,'(t) \end{aligned}$$

□

Properties of the Gradient — If $\boldsymbol{\nabla} f \neq 0$ then $\boldsymbol{\nabla} f$ is perpendicular to the level sets $\{f = k\}$.

Proof.

$$\boldsymbol{\nabla} f \cdot \vec{\mathbf{r}}\,'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(f(\vec{\mathbf{r}}(t))) = \frac{\mathrm{d}}{\mathrm{d}t}(k) = 0$$

□

Tangent Plane

Given a surface $F(x, y, z) = 0$ and a point (x_0, y_0, z_0), assume that $\boldsymbol{\nabla} F(x_0, y_0, z_0) \neq 0$. Since we know the gradient is the normal vector to the tangent plane, we have

$$\vec{\mathbf{n}} = \boldsymbol{\nabla} F(x_0, y_0, z_0) = \langle F_x, F_y, F_z(x_0, y_0, z_0) \rangle$$

Using the tangent plane formula we learned earlier, we have that

Tangent Plane —

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

This can also be written as if $\nabla F(x_0, y_0, z_0) = \langle a_0, b_0, c_0 \rangle$, then the equation for the tangent plane is

$$a_0 x + b_0 y + c_0 z = d$$

where

$$d = a_0 x_0 + b_0 y_0 + c_0 z_0$$

Example — Find the tangent plane to $x^2 + y^2 + z^2 = 1$ at (x_0, y_0, z_0).

Solution. As we have $F = x^2 + y^2 + z^2 - 1$,

$$F_x = 2x$$
$$F_y = 2y$$
$$F_z = 2z$$

Our equation for the tangent plane is then just

$$\boxed{x_0 x + y_0 y + z_0 z = 1}$$

□

By the Implicit Function Theorem,

If $\nabla F(x_0, y_0, z_0) = k$ where $k \neq 0$, then the level set $\{F = k\}$ is a smooth surface near (x_0, y_0, z_0).

An example of this is

$$F = z^2 - x^2 - y^2$$

The graph of $F = k$ is a cone, with a singularity at $(0, 0, 0)$. The gradient of F is

$$\nabla F = \langle -2x, -2y, 2z \rangle$$

This means that $\nabla F(x_0, y_0, z_0) \neq 0$ unless $(x_0, y_0, z_0) = (0, 0, 0)$. We can see that this is the only place where the cone isn't a smooth surface, showing how that works.

§2.5 Maxima and Minima

Definition — Let f be a function on a domain $D \subset \mathbb{R}^2$, and let $(a, b) \in D$.

- (a, b) is a global maximum (or local extrumum) of f if $f(a, b) \geq f(x, y)$ for all $(x, y) \in D$.
- (a, b) is a local maximum if $f(a, b) \geq f(x, y)$ for all (x, y) in some disk centered at (a, b).
- (a, b) is a global minimum (or local extremum) of f if $f(a, b) \leq f(x, y)$ for all $f(x, y) \in D$.
- (a, b) is a local minimum of f if $f(a, b) \leq f(x, y)$ in some disk centered at $f(a, b)$
- (a, b) is a critical point of f if $f_x(a, b) = f_y(a, b) = 0$

Theorem 2.3

If (a, b) is a local extremum, and (a, b) is not on the boundary of D, and if $f_x(a, b)$ and $f_y(a, b)$ are defined, then $f_x(a, b) = f_y(a, b) = 0$.

Proof. Define $g(x) = f(x, b)$. Then g has a local extremum at a, meaning $g'(a) = f_x(a, b) = 0$, and proving $f_y(a, b) = 0$ is the same process. □

This basically means that if (a, b) is a local extremum, then at least one of the following is true:

1. (a, b) is a critical point of f
2. $f_x(a, b)$ and $f_y(a, b)$ are not both defined
3. (a, b) is on the boundary of D

Example — Find the critical points of $f(x, y) = e^{-x^2-y^2}$.

Solution. A critical point is where both partials are 0.

$$f_x = -2xe^{-x^2-y^2} \qquad f_x = 0 \iff x = 0$$
$$f_y = -2ye^{-x^2-y^2} \qquad f_y = 0 \iff y = 0$$

This means that the only critical point is $(0, 0)$. This is a global maximum, as every other $f(x, y) < f(0, 0)$. □

Example — Find the critical points of $f(x, y) = y^2 - x^2$.

Solution.

$$f_x = -2x \qquad f_x = 0 \iff x = 0$$
$$f_y = -2y \qquad f_y = 0 \iff y = 0$$

This means the only critical point is $(0, 0)$, which is a saddle point as its neither local minimum maximum. □

Example — Find critical points of $f(x, y) = x^2y^2e^{-x^2-y^2}$.

Solution.

$$f_x = 2xy^2(1 - x^2)e^{-x^2-y^2}$$
$$f_y = 2x^2y(1 - y^2)e^{-x^2-y^2}$$

This means the critical points are at

$$\left.\begin{matrix} x = 0, & y = \text{any} \\ y = 0, & x = \text{any} \end{matrix}\right] \text{global minima}$$

$$\underbrace{x = \pm 1, \quad y = \pm 1}_{\text{global maxima}}$$

□

Second Derivative Test — Suppose (a, b) is a critical point of f. Assume f_{xx}, f_{xy}, f_{yy} are defined and continouous in a neighborhood of (a, b). Define

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

If

- $D > 0$, $f_{xx}(a,b)$ and $f_{yy}(a,b) > 0 \implies$ local min
- $D > 0$, $f_{xx}(a,b)$ and $f_{yy}(a,b) < 0 \implies$ local max
- $D < 0 \implies$ saddle point (neither local min or max)
- Test is inconclusive in all other cases

What's happening is since this matrix (also known as the Hessian) is symmetric, the eigenvalues are real. The first case is where both the eigenvalues are positive, second is where both are negative, and third is where one is positive and one is negative. The last case is where there is a zero eigenvalue.

Example — $f(x, y) = x^2 + y^2$ at $(a, b) = (0, 0)$

Solution.

$$\begin{aligned} f_x &= 2x & f_y &= 2y \\ f_{xx} &= 2 > 0 & f_{yy} &= 2 > 0 \\ f_{xy} &= 0 & f_{yx} &= 0 \end{aligned}$$

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

which means that $(0, 0)$ is a local minimum. □

Example — $f(x, y) = -x^2 - y^2$ with $(a, b) = (0, 0)$

Solution.

$$\begin{aligned} f_x &= -2x & f_y &= -2y \\ f_{xx} &= -2 < 0 & f_{yy} &= -2 < 0 \\ f_{xy} &= 0 & f_{yx} &= 0 \end{aligned}$$

$$D = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$$

which means that $(0, 0)$ is a local maximum. □

Example — $f(x, y) = y^2 - x^2$ at $(a, b) = (0, 0)$

Solution.

$$f_x = -2x \qquad f_y = 2y$$
$$f_{xx} = -2 < 0 \qquad f_{yy} = 2 > 0$$
$$f_{xy} = 0 \qquad f_{yx} = 0$$

$$D = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 < 0$$

which means that $(0, 0)$ is a saddle point. □

Example — Find and classify the critical points of

$$f(x, y) = x^5 + y^4 - 5x - 32y - 3$$

Is there a global max? Is there a global min?

Solution. We have $f_x = 5x^4 - 5$, $f_y = 4y^3 - 32$, implying the critical points as $(1, 2)$ and $(-1, 2)$. Writing out the second derivatives gives us $f_{xx} = 20x^3$ and $f_{yy} = 12y^2$ and $f_{xy} = f_{yx} = 0$. Evaluating $(1, 2)$ first gives us

$$D = \begin{vmatrix} 20 & 0 \\ 0 & 48 \end{vmatrix} > 0$$

Which means that $(1, 2)$ is a local minimum. Now, evaluating $(-1, 2)$ gives us

$$D = \begin{vmatrix} -20 & 0 \\ 0 & 48 \end{vmatrix} < 0$$

meaning that $(-1, 2)$ is a saddle. Since theres no local maximum, that means theres no global maximum. There also isn't a global minimum. □

Theorem 2.4 (Extreme Value Theorem)

Let f be a continuous function on $D \subset \mathbb{R}^2$. Assume D is closed and bounded. Then f has a global minimum and a global maximum on D.

What does closed and bounded mean?

Bounded means that there exists a real number $r > 0$ such that D is contained in a disk of radius r centered at the origin (or that D isn't infinitely big).

Closed means (roughly) that D contains all of its bounded points. For examples, if $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$, and this is closed. However, if $D = \{(x, y) \mid x^2 + y^2 < 1\}$, also known as the open unit disk, this is not closed. This means that $f(x, y) = x$ has no global max or global min on the open unit disk.

Example — Find the global maximum of $f = x^2 + y^2 + x^2y + 4$ on the square D with vertices $(\pm 1, \pm 1)$.

Solution. We know that the maximum is either a critical point, or its on the boundary of D.

The partials of this equation are $f_x = 2x(1 + y)$ and $f_y = 2y + x^2$. This gives us the critical points as $(0, 0)$, $(\sqrt{2}, -1)$, and $(-\sqrt{2}, -1)$. However, only $(0, 0)$ is in D, and $f(0, 0) = 4$.

Now lets check the boundaries of D. After checkign $f(x, 1)$, $f(1, x)$, $f(x, -1)$, and $f(-1, x)$, we find that the global maximum is along the upper edge, yielding a value of $f(-1, 1) = f(1, 1) = \boxed{7}$. □

Example — Find the maximum possible volume of a rectangular box with edges parallel to the axes, one corner at $(0,0,0)$, and the opposite corner on the plane $2x+y+z=1$ where $x,y,z \geq 0$.

Solution. We can see that the plane intercepts the axes at $(0,0,1)$, $(\frac{1}{2},0,0)$, and $(0,1,0)$ in a sort of triangle shape. The volume of this box is equal to $xyz = xy(1-2x-y)$, meaning we want to maximize the function

$$f(x,y) = xy(1-2x-y) = xy - 2x^2y - xy^2$$

The partials of this function are $f_x = y - 4xy - y^2 = y(1-4x-y)$ and $f_y = x - 2x^2 - 2xy = x(1-2x-2y)$. As we ignore the values $x=0$ and $y=0$, we have the critical points of $(\frac{1}{6},\frac{1}{3})$. As the boundaries of the domain all result in f being 0, and all partials are defined, this must be the global maximum.

This means that our volume is

$$xyz = \frac{1}{6}\cdot\frac{1}{3}\cdot\frac{1}{3} = \boxed{\frac{1}{54}}$$

□

§2.6 Lagrange Multipliers

Lagrange Multipliers — To minimize or maximize $f(x,y)$ with constraint $g(x,y)=k$, solve the equations

$$\nabla f = \lambda \nabla g$$
$$g = k$$

where λ is a scalar called the lagrange mutiplier, and $\nabla g \neq 0$.

Claim — Suppose (x,y) minimizes or maximizes f subject to the constraint $g(x,y)=k$. Suppose $\nabla g(x,y) \neq 0$ (meaning that its smooth). Then $\nabla f(x,y) = \lambda \nabla g(x,y)$.

Proof. This is true, as we can let $\gamma(t)$ be a parameterized curve on the level set $g=k$ with $\gamma(0)=(x,y)$. Since (x,y) is optimal,

$$\underbrace{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t))}_{\nabla f(x,y)\cdot\gamma'(0)} = 0$$

We also know that $g(\gamma(t)) = k$ for all t, so

$$\underbrace{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g(\gamma(t))}_{\nabla g(x,y)\cdot\gamma'(t)} = 0$$

From this, we know that the gradient of f and the gradient of g are both perpendicular to $\gamma(0)$, so ∇f is just ∇g multiplied by a scalar (as they're pointing in the same direction). This is where the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$ comes from. □

Example — Maximize $f(x,y) = \frac{xy}{2}$ with the constraint $g(x,y) = x^2+y^2 = 100$.

Solution. One solution is solving for x or y with the constraint, but we can also solve with lagrange multipliers.

$$\nabla f = \lambda \nabla g$$

$$\nabla f = \left\langle \frac{y}{2}, \frac{x}{2} \right\rangle \qquad \nabla g = \langle 2x, 2y \rangle$$

$$\implies \left\langle \frac{y}{2}, \frac{x}{2} \right\rangle = \lambda \langle 2x, 2y \rangle$$

Now, we're solving the equations

$$\begin{aligned} \frac{y}{2} &= \lambda 2x \\ \frac{x}{2} &= \lambda 2y \\ x^2 + y^2 &= 100 \end{aligned}$$

Substituting with this system gives us

$$y = 16\lambda^2 y$$

implying (since $y \neq 0$) that $16\lambda^2 = 1$, or $\lambda = \frac{1}{4}$ (lmbda can't be negative, as x and y are both positive). Substituting this back into our first equation, we have $x = y$, and putting this into the constraitn equation gives us

$$\begin{aligned} x^2 + x^2 &= 100 \\ x = y &= \sqrt{50} \end{aligned}$$

giving the total volume as $\boxed{25}$. □

Lagrange Multipliers 3D — To minimize/maximize $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, you have to solve the equations

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g &= k \end{aligned}$$

and just like before, $\nabla g \neq 0$.

Example — Maximize $f = xyz$ with the constraint $g = 2x + y + z = 1$ (with $x, y, z > 0$). (Note: this is the same example as the volume question from here).

Solution. Since we have

$$\nabla f = \langle yz, xz, xy \rangle \qquad \nabla g = \langle 2, 1, 1 \rangle$$

This gives us the system of equations

$$\begin{aligned} yz &= 2\lambda \\ xz &= \lambda \\ xy &= \lambda \\ 2x + y + z &= 1 \end{aligned}$$

We solve this by noticing $z = y = 2x$. Putting this in the constraint equation gives us $x = \frac{1}{6}$, $z = y = \frac{1}{3}$, giving the final volume as

$$\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) = \boxed{\frac{1}{54}}$$

□

Example — Build a topless rectangular box with volume 32000 and the smallest possible area.

Solution. Let the sides of the rectangular box be x, y, and z. The volume of the rectangular box is $g(x, y, z) = xyz = 32000$, and the area of the box is $f(x, y, z) = xy + 2zy + 2yz$ (which we want to minimize).

From this, we have that

$$\nabla f = \langle y + 2z, x + 2z, 2x + 2y\rangle \qquad \nabla g = \langle yz, xz, xy\rangle$$

This gives us the system of equations

$$\lambda yz = y + 2z \tag{1}$$
$$\lambda xz = x + 2z \tag{2}$$
$$\lambda xy = 2x + 2y \tag{3}$$
$$xyz = 32000 \tag{4}$$

Now, we have to solve this system of equations. Subtracting equation (1) from equation (2), we have $(y - x)(1 - \lambda z) = 0$, giving two cases: either $y = x$, or $\lambda z = 1$. However, $\lambda z = 1$ is impossible, as then that would mean $y = y + 2z$ (by equation (1)), which is impossible for positive x, y, z.

Now, as $y = x$, we have, from eq (3), that $4 = \lambda x$. Substituting back into equation (2), we have $x = 2z$. Now, by equation (4), we can substitute in $y = x = 2z$, giving $4z^3 = 32000$, or $(x, y, z) = (40, 40, 20)$. Thus the answer is

$$\text{Area} = (40)(40) + 2(20)(40) + 2(20)(40) = \boxed{3200}$$

Note that this also means $\lambda = \frac{1}{10}$. □

λ sort of describes how the min/max will change as the constraint changes. In the previous problem, if we want the volume of the box to be 32001 instead of 32000, the more area we'll need is approximately $\lambda = \frac{1}{10}$.

Claim — Let $M(k)$ be the minimum value of f subject to the constraint $g = k$. (Suppose that $\nabla g \neq 0$ and the minimum is unique). Then

$$\frac{\mathrm{d}M(k)}{\mathrm{d}k} = \lambda$$

Proof. Let $\vec{\mathbf{r}}(t)$ be a parametrized curve such that $\vec{\mathbf{r}}(k)$ is the point on $g = k$ where f is minimized.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}M(k) = \frac{\mathrm{d}}{\mathrm{d}k}F(\vec{\mathbf{r}}(k)) &= \nabla f \cdot \vec{\mathbf{r}}\,'(k) \\ &= \lambda \nabla g \cdot \vec{\mathbf{r}}\,'(k) \\ &= \lambda \frac{\mathrm{d}}{\mathrm{d}k} g(\vec{\mathbf{r}}(k)) \\ &= \lambda \frac{\mathrm{d}}{\mathrm{d}k} k = \lambda\end{aligned}$$

□

This also makes intuitive sense, as λ describes the scalar from g to f, and so if g changes a certian amt f will change λ times that amt (im not completely sure as i randomly thought of this myself).

Lagrange Multipliers with Two Constraints — Given a function $f(x, y, z)$ with two constraints, $g(x, y, z) = k$ and $h(x, y, z) = c$, then you have to solve the equations

$$\begin{aligned} \nabla f(x_0, y_0, z_0) &= \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0) \\ g(x, y, z) &= k \\ h(x, y, z) &= c \end{aligned}$$

Geometrically, we are looking for the extreme values of f on the intersection of g and h.

Example — Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution. As we have two constraints, we first find the gradients. Let $f = x + 2y + 3z$, $g = x - y + z = 1$, and $h = x^2 + y^2 = 1$.

$$\nabla f = \langle 1, 2, 3 \rangle \qquad \nabla g = \langle 1, -1, 1 \rangle \qquad \nabla h = \langle 2x, 2y, 0 \rangle$$

This gives us the system of equations

$$\begin{aligned} \lambda + \mu 2x &= 1 \\ -\lambda + \mu 2y &= 2 \\ \lambda &= 3 \\ x - y + z &= 1 \\ x^2 + y^2 &= 1 \end{aligned}$$

Solving this system of equations gives us the maximum value of f on the intersection as $\boxed{3 + \sqrt{29}}$. □

§3 Module 3 (Integration)

§3.1 Basics of Double Integrals

The Double Integral — Let $R = [a, b] \times [c, d] = \{(x, y) \mid a \le x \le b, c \le y \le d\}$ denote a rectangle, with the given constraints. (The $\times$ denotes the Cartesian Product).

Suppose we have $f : R \to \mathbb{R}$ (f is defined on this rectangle in the set of real numbers). Then the volume of the region under f is defined as

$$\iint\limits_R f\, dA \qquad \text{(when } f > 0\text{)}$$

To prove this, we are going to divide the rectangle R into n^2 subrectangles, where

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$c = y_0 < y_1 < \cdots < y_n = d$$

Then we have $x_i - x_{i-1} = \frac{b-a}{n} = \Delta x$, and $y_i - y_{i-1} = \frac{d-c}{n} = \Delta y$, and $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ The area of this R_{ij} is equal to $\Delta x \Delta y$. Choose a sample point $(x^*_{ij}, y^*_{ij}) \in R_{ij}$. Now the definition is

Definition —

$$\iint\limits_R f\, dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta x \Delta y$$

This can be thought of as adding the limit of the skyscrapers making up the volume go to infinity, to get the total volume. This leads to the theorem below.

Theorem 3.1

If f is continuous, then

$$\iint\limits_R f\, dA$$

is well defined (exists and doeesn't depend on the sample points you choose).

Basic Properties —

$$\iint\limits_R 1\, dA = \text{Area}(R)$$

$$\iint\limits_R cf\, dA = c \iint\limits_R f\, dA$$

$$\iint\limits_R (f + g)\, dA = \iint\limits_R f\, dA + \iint\limits_R g\, dA$$

If $f \ge g$, for every (x, y) in R then

$$\iint\limits_R f\, dA \ge \iint\limits_R g\, dA$$

Computing Double Integrals

To compute

$$\iint\limits_R f\,dA$$

where $R = [a,b] \times [c,d]$, we slice R into vertical strips, with

$$a = x_0 < x_1 < \cdots < x_n = b \qquad x_i - x_{i-1} = \frac{b-a}{n} = \Delta x$$

The i^{th} vertical strip is $[x_{i-1}, x_i] \times [c,d]$.

This gives us another basic property,

Basic Property —

$$\iint\limits_R f\,dA = \sum_{i=1}^{n} \iint\limits_{[x_{i-1},x_i]\times[c,d]} f\,dA$$

This double integral can then be simplified as

$$\sum_{i=1}^{n} \iint\limits_{[x_{i-1},x_i]\times[c,d]} f\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} \int_c^d f(x_i^*, y)\,dy\Delta x$$

The ending integral in the expression above, shown below,

$$\int_c^d f(x_i^*, y)\,dy$$

is the area under the curve $z = f(x_i^*, y)$ where x_i^* is fixed and Δx is the thickness of a slice.

Now, with the definition of an integral, we have just proven Fubini's Theorem. We can also integrate in the other order, by slicing y into tiny piecese, instead of x.

Theorem 3.2 (Fubini's Theorem)

$$\begin{aligned}\iint\limits_R f\,dA &= \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx \\ &= \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy\end{aligned}$$

This means basically that to calculate double integrals, we have to first calculate the inside with respect to y, assume x is a constant, and then we integrate the outside with respect to x (or the opposite way, whichever way's easier).

Example — Suppose $R = [0,1] \times [0,1]$. Calculate

$$\iint\limits_R xe^{xy}\,dA$$

Solution. By Fubini's Theorem, we have that

$$\iint\limits_R xe^{xy}\,dA = \int_0^1 \left(\int_0^1 xe^{xy}\,dy\right)\,dx$$
$$= \int_0^1 \left(e^{xy}\Big|_{y=0}^{y=1}\right)\,dx$$
$$= \int_0^1 e^x - 1\,dx$$
$$= e^x - x\Big|_{x=0}^{x=1} = \boxed{e-2}$$

or, integrating the other way, that

$$\iint\limits_R xe^{xy}\,dA = \int_0^1 \left(\int_0^1 xe^{xy}\,dx\right)\,dy$$
$$= \int_0^1 \left(\left(\frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}\right)\Big|_{x=0}^{x=1}\right)\,dy$$

which if you do, will yield the same answer, however much more messy. □

Example — Find the volume of the solide region bounded by the elliptic paraboloid $z = 1 + (x-1)^2 + 4y^2$, the planes $x = 3$, $y = 2$, and the coordinate planes.

Solution. As the paraboloid is always positive, it is on the top side of the graph, and bounded by the planes. This means the volume is

$$\iint\limits_R \left(1 + (x-1)^2 + 4y^2\right)\,dA$$

where $R = [0,3] \times [0,2]$. By Fubini's Theorem, we have that this is the same as

$$\int_0^3 \int_0^2 \left(1 + (x-1)^2 + 4y^2\right)\,dy\,dx = \int_0^3 \left(y + (x-1)^2 y + \frac{4}{3}y^3\right)\Big|_{y=0}^{y=2}\,dx$$
$$= \int_0^3 \left(2 + 2(x-1)^2 + \frac{32}{3}\right)\,dx$$
$$= 2x + \frac{2}{3}(x-1)^3 + \frac{32}{3}x\Big|_{x=0}^{x=3}$$
$$= 6 + 6 + 32 = \boxed{44}$$

We can also do it the other way, with it being the same as

$$\int_0^2 \int_0^3 \left(1 + (x-1)^2 + 4y^2\right)\,dx\,dy = \int_0^2 \left(x + \frac{(x-1)^3}{3} + 4y^2 x\right)\Big|_{x=0}^{x=3}\,dy$$
$$= \int_0^2 (3 + 3 + 12y^2)\,dy$$
$$= 6y + 4y^3\Big|_{y=0}^{y=2}$$
$$= 12 + 32 = \boxed{44}$$

□

Double Integrals Over More General Regions

Let D be a closed and bounded region in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$. Then,

$$\iint\limits_D f\,dA = \text{volume under the graph over } D$$

Let R be a rectangle containing D (which works as domain is bounded).

Define a function $F : R \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

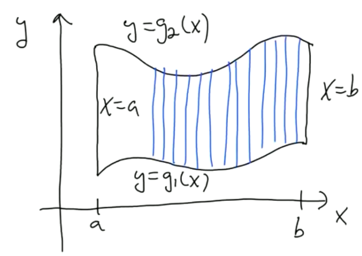
Definition — Then, the definition of the double integral is

$$\iint\limits_D f\,dA = \iint\limits_R F\,dA$$

Now, how do we compute this?

Well, there are two "types" of regions, and two different ways to compute them.

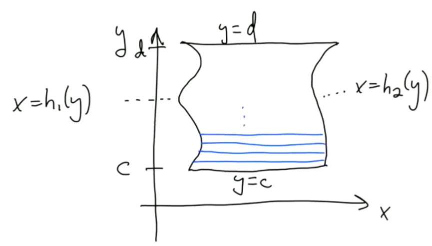
One is where the boundaries for y are defined by two functions, one $g_1(x)$ and another $g_2(x)$, as shown in the picture below. We can get our formula for the double integral by cutting the region into vertical strips.



Double Integrals Pt 1 — With domain $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$ we have

$$\iint\limits_D f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f\,dy\,dx$$

With x being bounded with the functions $h_1(y)$ and $h_2(y)$ (as shown in the image below), we can cut it into horizontal strips and get a similar formula.



This leads us to our second double integral formula,

Double Integrals Pt 2 — With domain $D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$ we have

$$\iint\limits_D f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f\,dx\,dy$$

If a region is both of these types, you can integrate in either order.

Example — Calculate

$$\iint\limits_D x\cos(y)\,dA$$

where D is the region under $y = x^2$ from $x = 0$ to $x = 1$.

Solution. As this D is both types of functions, we can integrate either way.

$$\begin{aligned}\iint\limits_D x\cos(y)\,dA &= \int_0^1 \int_0^{x^2} x\cos(y)\,dy\,dx \\ &= \int_0^1 x\sin(y)\Big|_{y=0}^{y=x^2} dx \\ &= \int_0^1 x\sin\left(x^2\right) dx \\ &= -\frac{1}{2}\cos\left(x^2\right)\Big|_{x=0}^{x=1} \\ &= \boxed{\frac{1}{2} - \frac{1}{2}\cos(1)}\end{aligned}$$

Or, we can do

$$\begin{aligned}\iint\limits_D x\cos(y)\,dA &= \int_0^1 \int_{\sqrt{y}}^1 x\cos(y)\,dy\,dx \\ &= \int_0^1 \frac{1}{2}x^2\cos(y)\Big|_{x=\sqrt{y}}^{x=1} dy \\ &= \int_0^1 \frac{1}{2}\cos(y) - \frac{1}{2}y\cos(y)\,dy\end{aligned}$$

and, evaluating this integral should lead to the same answer (although it is harder). □

Example — Calculate

$$\iint\limits_D xy\,dA$$

where D is the quarter on quadrant 1 of the unit disk.

Solution. This region is both types, once again. This volume can be written as both

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dy\, dx &= \int_0^1 \frac{xy^2}{2}\Big|_{y=0}^{y=\sqrt{1-x^2}} dx \\ &= \int_0^1 \frac{x(1-x^2)}{2}\, dx \\ &= \int_0^1 \frac{x}{2} - \frac{x^3}{2}\, dx \\ &= \frac{x^2}{4} - \frac{x^4}{8}\Big|_{x=0}^{x=1} \\ &= \frac{1}{4} - \frac{1}{8} = \boxed{\frac{1}{8}}\end{aligned}$$

□

Example — Evaluate

$$\int_0^3 \int_{y^2}^9 y\cos\left(x^2\right) dx\, dy$$

Solution. As we can't evaluate integral of $\cos\left(x^2\right)$, we can't use the first type to integrate this (as we did in previous examples). This means we have to change the order of integration, and to do that we have to understand the domain of integration.

This is the region of $y = x^2$ from $y = 0$ to $y = 3$, which is the same as saying the region of $y = \sqrt{x}$ from $x = 0$ to $x = 9$.

Now, we can rewrite and evaluate.

$$\begin{aligned}\int_0^3 \int_{y^2}^9 y\cos\left(x^2\right) dx\, dy &= \int_0^9 \int_0^{\sqrt{x}} y\cos\left(x^2\right) dy\, dx \\ &= \int_0^9 \frac{y^2}{2}\cos\left(x^2\right)\Big|_{y=0}^{y=\sqrt{x}} dx \\ &= \int_0^9 \frac{x}{2}\cos\left(x^2\right) dx \\ &= \frac{1}{4}\sin\left(x^2\right)\Big|_{x=0}^{x=9} \\ &= \boxed{\frac{\sin(81)}{4}}\end{aligned}$$

□

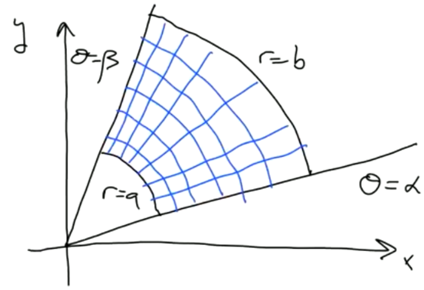
§3.2 Double Integrals in Polar Coordinates, and Surface Area

Double Integrals in Polar Coordinates

We'll start with the "Polar Rectangle", or a region R such that

$$R = \{(r, \theta) \mid \alpha \le \theta \le \beta, a \le r \le b\}$$

as shown in the figure below.



To calculate the double integral $\displaystyle\iint_R f\,dA$, we partition this polar rectangle R into smaller polar rectangles

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta \qquad \Delta\theta = \theta_i - \theta_{i-1} = \frac{\beta - \alpha}{n}$$

$$a = r_0 < r_1 < \cdots < r_n = b \qquad \Delta r = r_i - r_{i-1} = \frac{b-a}{n}$$

Then the tiny polar rectangle region is

$$R_{ij} = \{(r,\theta) \mid \theta_{i-1} \le \theta \le \theta_i, r_{j-1} \le r \le r\}$$

Then, the double integral is

$$\iint_R f\,dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} \text{Area}(R_{ij}) f(r_{ij}^*, \theta_{ij}^*)$$

where $(r_{ij}^*, \theta_{ij}^*) \in R_{ij}$ is a sample point.
Then, we have that the Area of R_{ij} is the difference of two pizza slices, or

$$\begin{aligned}\text{Area}(R_{ij}) &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta \\ &= \frac{1}{2}(r_i + r_{i-1})\Delta r\Delta\theta\end{aligned}$$

Putting this back into the formula for the integral gives us

$$\begin{aligned}\iint_R f\,dA &= \lim_{n\to\infty}\sum_{i,j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)\frac{r_i + r_{i-1}}{2}\Delta r\Delta\theta \\ &= \int_\alpha^\beta \int_a^b f(r,\theta) r\,dr\,d\theta\end{aligned}$$

Double Integral in Polar — Given a region

$$R = \{(r,\theta) \mid \alpha \le \theta \le \beta, a \le r \le b\}$$

the double integral is

$$\iint_R f\,dA = \int_\alpha^\beta \int_a^b f(r,\theta) r\,dr\,d\theta$$

where the r being multiplied can be thought of as a "magnification factor", as the area of a small polar rectangle is approximately $r\Delta r\Delta\theta$.

> **Example —** Find the volume of the region between the surfaces $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

Solution. Visualizing this gives us the area under a sphere, but above a cone. The domain we have to integrate over, or D, is just a circle, where the z values are equal in both equations. Solving for $x^2 + y^2$, we can substitute in r, and get that the radius of this circle domain is $\frac{\sqrt{2}}{2}$.

Now, we just have to subtract the two integrals, where

$$\text{Volume} = \iint_D \underbrace{\sqrt{1 - x^2 - y^2}}_{\text{upper}} - \underbrace{\sqrt{x^2 + y^2}}_{\text{lower}} \, dA$$

Notice, with all the circular stuff, it's easier to evaluate in polar, so lets convert to polar, with $x^2 + y^2 = r^2$.

Now, we have

$$\begin{aligned}
\text{Volume} &= \iint_D \underbrace{\sqrt{1 - x^2 - y^2}}_{\text{upper}} - \underbrace{\sqrt{x^2 + y^2}}_{\text{lower}} \, dA \\
&= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1 - r^2} - r \right) r \, dr \, d\theta \\
&= \int_0^{2\pi} \left(-\frac{1}{3}(1 - r^2)^{\frac{3}{2}} - \frac{r^3}{3} \right) \Bigg|_{r=0}^{r=\frac{1}{\sqrt{2}}} d\theta \\
&= -\frac{1}{3} \int_0^{2\pi} \left((1 - r^2)^{\frac{3}{2}} + r^3 \right) \Bigg|_{r=0}^{r=\frac{1}{\sqrt{2}}} d\theta \\
&= -\frac{1}{3} \int_0^{2\pi} \left(\left(\frac{1}{2}\right)^{\frac{3}{2}} + \left(\frac{1}{\sqrt{2}}\right)^3 - 1 \right) d\theta \\
&= -\frac{1}{3} \int_0^{2\pi} \left(\frac{1}{\sqrt{2}} - 1 \right) d\theta \\
&= \boxed{\frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}} \right)}
\end{aligned}$$

$\square$

> **Example —** Evaluate
> $$A = \int_{-\infty}^{\infty} e^{-x^2} \, dx$$

Solution. This converges, so its integratable. To integrate this, we use a clever trick, by squaring A.

$$\begin{aligned}
A^2 &= \int_{-\infty}^{\infty} e^{-x^2} \, dx \int_{-\infty}^{\infty} e^{-y^2} \, dy \\
&= \iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2} \, dA
\end{aligned}$$

This converges, as $e^{-x^2} e^{-y^2}$ decays rapidly away from the origin (not rigourous).

Now, simplifying and using polar coordinates gives us

$$\begin{aligned}\iint\limits_{\mathbb{R}^2} e^{-x^2}e^{-y^2}\,dA &= \iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\,dA \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta \\ &= \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=\infty}\,d\theta \\ &= \pi\end{aligned}$$

$$\begin{aligned} A^2 &= \pi \\ \implies A &= \boxed{\sqrt{\pi}}\end{aligned}$$

□

IMPORTANT!!!!!! Note that converting $dx\,dy = dA = r\,dr\,d\theta$. DO NOT FORGET THE r BEING MULTIPLIED!!!!! More on magnification factors and their intuition here.

Example — Find the volume of the region obtained from a ball by cutting out a cylinder of height h (and the parts of the ball above and below the cylinder).

Solution. Let r_1 be the radius of the cylinder, and let r_2 be the radius of the ball. The domain is between the circle of r_1 and the circle of r_2. This gives the volume as

$$\begin{aligned}\text{Volume} &= \iint\limits_{r_1\le r\le r_2} \sqrt{r_2^2-x^2-y^2} - \left(-\sqrt{r_2^2-x^2-y^2}\right)\,dA \\ &= \iint\limits_{r_1\le r\le r_2} 2\sqrt{r_2^2-x^2-y^2}\,dA \\ &= 2\int_0^{2\pi}\int_{r_1}^{r_2}\left(\sqrt{r_2^2-r^2}\right) r\,dr\,d\theta \\ &= 2\int_0^{2\pi}\left(-\frac{1}{3}(r_2^2-r^2)^{\frac{3}{2}}\right)\Big|_{r=r_1}^{r=r_2}\,d\theta \\ &= 2\int_0^{2\pi} -\frac{1}{3}\left(-(r_2^2-r_1^2)^{\frac{3}{2}}\right)\,d\theta \\ &= \frac{4\pi}{3}\left(r_2^2-r_1^2\right)^{\frac{3}{2}}\end{aligned}$$

and now, since $r_1^2+\left(\frac{h}{2}\right)^2 = r_2^2$ by the Pythagorean Theorem, we can substitute that in to get

$$\begin{aligned}\frac{4\pi}{3}\left(r_2^2-r_1^2\right)^{\frac{3}{2}} &= \frac{4\pi}{3}\left(\left(\frac{h}{2}\right)^2\right)^{\frac{3}{2}} \\ &= \boxed{\frac{4\pi}{3}\left(\frac{h}{2}\right)^3}\end{aligned}$$

□

Integration Over More General Regions in Polar Coordinates

Double Integral in Polar — With the domain of a polar function such that

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

the double integral is

$$\iint_D f\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r, \theta) r\, dr\, d\theta$$

Example — Calculate

$$\iint_D x^2 + y^2\, dA$$

such that the domain D has $\alpha = 0$, $\beta = 2\pi$, $h_1(\theta) = \theta$, $h_2(\theta) = 2\theta$.

Solution. This is a spiral shape thingy, and evaluating the double integral gives us

$$\begin{aligned}
\iint_D x^2 + y^2\, dA &= \int_0^{2\pi} \int_\theta^{2\theta} r^2 \cdot r\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{r^4}{4}\bigg|_{r=\theta}^{r=2\theta} d\theta \\
&= \int_0^{2\pi} \frac{15}{4}\theta^4\, d\theta \\
&= \frac{3}{4}\theta^5\bigg|_{\theta=0}^{\theta=2\pi} \\
&= \frac{3}{4}(2\pi)^5 = \boxed{24\pi^5}
\end{aligned}$$

□

Surface Area

Let D be a domain in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be differentiable.

To find the area of f over D, we can divide D into small rectangles, with width Δx and height Δy. The part of the graph above this rectangle is approximately a parallelogram.

This parallelogram has vector sides of $\langle \Delta x, 0, f_x \Delta x\rangle$ and $\langle 0, \Delta y, f_y \Delta y\rangle$.

Now, recall that the magnitude of the cross product is the area of the parallelogram, so

$$\begin{aligned}
\text{Area} &= |\langle \Delta x, 0, f_x \Delta x\rangle \times \langle 0, \Delta y, f_y \Delta y\rangle| \\
&= |\langle -f_x \Delta x \Delta y, -\Delta x f_y \Delta y, \Delta x \Delta y\rangle| \\
&= \Delta x \Delta y |\langle -f_x, -f_y, 1\rangle| \\
&= \Delta x \Delta y \sqrt{1 + f_x^2 + f_y^2}
\end{aligned}$$

Now, with this, we have that the area of the graph is

$$\begin{aligned}\text{Area} &= \lim_{\Delta x, \Delta y \to 0} \sum_{\text{rect}} \Delta x \Delta y \sqrt{1 + f_x^2 + f_y^2} \\ &= \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA\end{aligned}$$

Area of 2 Variable Differentiable Function — Given a function $f : D \to \mathbb{R}$ which is differentiable, with D a domain in $\mathbb{R}^2$, the area of the graph of f is given by

$$\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

Example — Find the area of the unit sphere.

Solution. We can split this into two hemispheres, the northern hemisphere and the southern hemisphere. The northern hemisphere is the graph of the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ over $D =$ unit disk.

As we have

$$f_x = \frac{-x}{\sqrt{1 - x^2 - y^2}} \qquad f_y = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

$$\implies 1 + f_x^2 + f_y^2 = \frac{1}{1 - x^2 - y^2}$$

This gives the area of the sphere as

$$\begin{aligned}\text{Area} &= 2 \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA \\ &= 2 \iint_D \sqrt{\frac{1}{1 - x^2 - y^2}}\, dA \\ &= 2 \int_0^{2\pi} \int_0^1 \sqrt{\frac{1}{1 - r^2}}\, r\, dr\, d\theta \\ &= 2 \int_0^{2\pi} -\sqrt{1 - r^2}\Big|_{r=0}^{r=1} d\theta \\ &= 2 \int_0^{2\pi} 1\, d\theta \\ &= \boxed{4\pi}\end{aligned}$$

□

§3.3 Triple Integrals

The Rectangular Box $R = [a, b] \times [c, d] \times [r, s]$, and suppose there's a function $f : R \to \mathbb{R}$. Dividing this box up into

$$a = x_0 < x_1 < \cdots < x_n = b \qquad x_i - x_{i-1} = \Delta x = \frac{b - a}{n}$$

$$c = y_0 < y_1 < \cdots < y_n = d \qquad y_i - y_{i-1} = \Delta y = \frac{d-c}{n}$$

$$r = z_0 < z_1 < \cdots < z_n = s \qquad z_i - z_{i-1} = \Delta z = \frac{s-r}{n}$$

Now, we define

Definition —

$$\iiint\limits_R f\,dV = \lim_{n\to\infty} \sum_{i,j,k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta x \Delta y \Delta z$$

Where $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \in [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{k-1}, z_k]$ (sample point).

Fact. This triple integral is well defined if f is continuous.

Theorem 3.3 (Fubini's Theorem)

$$\begin{aligned}\iiint\limits_R f\,dV &= \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx \\ &= \text{5 other orders}\end{aligned}$$

There are also different types of functions, with the first as the top and bottom bounds being functions $\phi_1(x, y)$ and $\phi_2(x, y)$ in the domain D. or

Triple Integrals — Over the region

$$R = \{(x, y, z) \mid (x, y) \in D, \phi_1(x, y) \le z \le \phi_2(x, y)\}$$

the triple integral is

$$\iiint\limits_R f\,dV = \iint\limits_D \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dA$$

This carries on to the other orders too.

Example — Let E be the solid bounded by the surfaces $x^2 + z^2 = 4$, $y = 0$, $y = 4$.Write $\iiint\limits_E f\,dV$ as an iterated integral.

Solution. This, when drawn, could be seen to be a sideways cylinder bounded from $y = 0$ to $y = 4$. We can write this out iterating over z first, and then x, and then y.

$$\boxed{\int_0^4 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f\,dz\,dx\,dy}$$

We can also write it out as

$$\boxed{\int_{-2}^{2}\int_{0}^{4}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f\,dx\,dy\,dz}$$

□

Example — Compute $\iiint_E x\,dV$ where E is the region bounded by the paraoloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

Solution. This is

$$\int_0^4 \iint_{(y^2+z^2\le\frac{x}{4})} x\,dA\,dx$$

Notice that x is a constant, so this is equal to

$$\begin{aligned}
&= \int_0^4 x \cdot \text{Area}\underbrace{\left(y^2+z^2 \le \frac{x}{4}\right)}_{\text{disk w/ } r=\sqrt{\frac{x}{4}}} dx \\
&= \int_0^4 x \cdot \pi\left(\frac{x}{4}\right) dx \\
&= \frac{\pi x^3}{12}\Bigg|_0^4 = \boxed{\frac{16\pi}{3}}
\end{aligned}$$

We can also evaluate this in different orders, such as

$$\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{4y^2+4z^2}^{4} x\,dx\,dy\,dz$$

This does the exact same thing, although the computation is more tedious. □

Example — Change the order of integration of

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f\,dz\,dy\,dx = \int_?^?\int_?^?\int_?^? f\,dx\,dy\,dz$$

Solution. The Integration Region has that

$$0 \le x \le 1 \qquad \sqrt{x} \le y \le 1 \qquad 0 \le z \le 1-y$$

We can see that $x \le 1$ is redundent, as $\sqrt{x} \le 1$, and that $y \le 1$ is also redundant, as $0 \le 1-y$ is already stated. This gives us

$$0 \le x \qquad \sqrt{x} \le y \qquad 0 \le z \le 1-y$$

Now, as z ranges from 0 to 1, and y ranges from 0 to $1-z$, and x ranges from 0 to y^2, we can rewrite our triple integral for an answer of

$$\int_0^1\int_0^{1-z}\int_0^{y^2} f\,dx\,dy\,dz$$

□

Applications (Mass, Center of Mass, Moment of Inertia) (more here)
Given a solid region E, a function $\rho : E \to \mathbb{R}_{>0}$ which is mass density.

Total Mass —

$$m = \iiint\limits_E \rho \, dV$$

This can kinda be thought of as having a tiny box with volume Δv with mass Δm, and the mass of the box is approximately ρ times ΔV, which you're adding up all of these tiny masses.

Center of Mass — The Center of Mass is $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{1}{m} \iiint\limits_E x\rho \, dV$$

$$\overline{y} = \frac{1}{m} \iiint\limits_E y\rho \, dV$$

$$\overline{z} = \frac{1}{m} \iiint\limits_E z\rho \, dV$$

This triple integral divided by the mass can be thought of as the "weighted average".

Example — Let $E = [0, a] \times [0, a] \times [0, a]$ (cube of side length a), and $\rho = x^2 + y^2 + z^2$. Calculate the center of mass.

Solution. By symmetry, $\overline{x} = \overline{y} = \overline{z} = c$. First, we must find the total mass.

$$\begin{aligned}
m &= \int_0^a \int_0^a \int_0^a x^2 + y^2 + z^2 \, dz \, dy \, dx \\
&= \int_0^a \int_0^a \left(x^2 z + y^2 z + \frac{z^3}{3} \right) \Bigg|_{z=0}^{z=a} dy \, dx \\
&= \int_0^a \int_0^a ax^2 + ay^2 + \frac{a^3}{3} \, dy \, dx \\
&= \int_0^a \left(ax^2 y + \frac{ay^3}{3} + \frac{a^3 y}{3} \right) \Bigg|_{y=0}^{y=a} dx \\
&= \int_0^a a^2 x^2 + \frac{2a^4}{3} \, dx \\
&= \left(\frac{a^2 x^3}{3} + \frac{2a^4 x}{3} \right) \Bigg|_{x=0}^{x=a} \\
&= a^5
\end{aligned}$$

Now, calculating $\overline{x}$ gives us

$$\overline{x} = \frac{1}{m} \int_0^a \int_0^a \int_0^a x \left(x^2 + y^2 + z^2 \right) \, dz \, dy \, dx$$

$$= \frac{1}{a^5} \int_0^a \int_0^a \left(x^3 z + xy^2 z + \frac{xz^3}{3} \right) \bigg|_{z=0}^{z=a} dy\, dx$$
$$= \frac{1}{a^5} \int_-^a \int_0^a ax^3 + axy^2 + \frac{a^3 x}{3}\, dy\, dx$$
$$= \frac{1}{a^5} \int_0^a \left(ax^3 y + \frac{axy^3}{3} + \frac{a^3 xy}{3} \right) \bigg|_{y=0}^{y=a} dx$$
$$= \frac{1}{a^5} \int_{0a}^a a^2 x^3 + \frac{2a^4 x}{3}\, dx$$
$$= \frac{1}{a^5} \left(\frac{a^2 x^4}{4} + \frac{a^4 x^2}{3} \right) \bigg|_{x=0}^{x=a}$$
$$= \frac{1}{a^5} \left(\frac{a^6}{4} + \frac{a^6}{3} \right) = \frac{7}{12} a$$

By symmetry, this gives the center of mass as

$$\boxed{\left(\frac{7}{12} a, \frac{7}{12} a, \frac{7}{12} a \right)}$$

□

This makes sense, as the center of mass is rising the further a point is from the origin, and so the center of mass is above the center of the cube.

§3.4 Triple Integrals in Cylindrical and Spherical Coordinates

Triple Coordinates in Cylindrical Coordinates

Cylindrical Coordinates are like polar coordinates, except with an extra variable z. It is represented by (r, θ, z), where (r, θ) denote a point on the "xy plane" and z being the height of the point.

This gives the the relationships between the two coordinate systems $(x, y, z) \leftrightarrow (r, \theta, z)$ as

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$
$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

Triple Integrals in Cylindrical Coordinates — With a region in 2d as

$$R = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

and a certain 3d region

$$E = \{(r, \theta, z) \mid (r, \theta) \in R, \phi_1(r, \theta) \le z \le \phi_2(r, \theta)\}$$

the triple integral is

$$\iiint_E f\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{\phi_1(r,\theta)}^{\phi_2(r,\theta)} f(r, \theta, z) r\, dz\, dr\, d\theta$$

where the r is the magnification factor.

Example — Let E be the solid region inside the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the surface $z^2 = 4x^2 + 4y^2$. Compute

$$\iiint_E x^2 \, dV$$

Solution. This region is above the disk/cylinder thingy, but below the cone. As there's a lot of circular stuff, it is easiest to convert this to cylindrical coordiates.

By converting this to cylindrical, we have the bounds $r = 1$ from the equation $x^2 + y^2 = 1$, and the bound $z = 2r$ from the equation $z^2 = 4x^2 + 4y^2$, and $z = 0$. This gives the triple integral of

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 \cos^2(\theta) r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^1 2r^4 \cos^2(\theta) \, dr \, d\theta \\ &= \int_0^{2\pi} \frac{2}{5} \cos^2(\theta) \, d\theta \\ &= \frac{2}{5} \left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right) \Bigg|_{\theta=0}^{\theta=2\pi} \\ &= \boxed{\frac{2\pi}{5}} \end{aligned}$$

□

Example — Find the volume of the region which is above the paraboloid $z = x^2 + y^2$ and below the paraboloid $z = 36 - 3x^2 - 3y^2$.

Solution. As, once again, there's a lot of circles, we can convert this to cylindrical coordinates to have a cleaner equation.

The bounds of this equation are, converting to cylindrical, $z = r^2$, $z = 36 - 3r^2$, and $r = 3$. This gives the triple integral as (and triple integrating over 1 will give the volume)

$$\begin{aligned} \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} 1 \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^3 r(36 - 4r^2) \, dr \, d\theta \\ &= \int_0^{2\pi} \left(18r^2 - r^4\right) \Bigg|_{r=0}^{r=3} d\theta \\ &= \int_0^{2\pi} 81 \, d\theta \\ &= \boxed{162\pi} \end{aligned}$$

□

Triple Integrals in Spherical Coordinates

Spherical Coordinates are represented by a point (ρ, θ, ϕ), where θ is the angle from the x axis, and ϕ is the angle from the z axis, and ρ is "r", or the distance from the origin. This makes converting from $(x, y, z) \leftrightarrow (\rho, \theta, \phi)$ with the relationships

$$x = \rho \sin(\phi) \cos(\theta) \qquad y = \rho \sin(\phi) \sin(\theta) \qquad z = \rho \cos(\phi)$$

$$r = \rho \sin(\phi) \qquad \rho^2 = x^2 + y^2 + z^2$$

The ranges of these coordinates are

$$0 \leq \phi \leq \pi \qquad\qquad \rho \geq 0$$

where $\phi = 0$ is the positive z axis, and $\phi = \pi$ is the negative z axis. If $\rho > 0$ is fixed, it gives a sphere, and θ can be thought of as longitude, whereas $\phi \sim$ latitude.

Triple Integrals in Spherical Coordinates — The spherical coordinates "box" is represented by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

Given this, the triple integral over this region is

$$\begin{aligned}\iiint_E f\, dV &= \int_c^d \int_\alpha^\beta \int_a^b f \cdot \left(\rho^2 \sin(\phi)\right)\, d\rho\, d\theta\, d\phi \\ &= \text{ any other 5 orders}\end{aligned}$$

where the $\rho^2 \sin(\phi)$ is the magnification factor.

Example — Compute the total mass of the half ball $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$ with mass density $\mu = \left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}$.

Solution. Recall the formula for mass here. Converting this to spherical coordinates gives us the region $\rho \leq 1$, with $0 \leq \phi \leq \frac{\pi}{2}$, and $0 \leq \theta \leq 2\pi$. This gives us the triple integral

$$\begin{aligned}m &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^3 \cdot \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi \\ &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{1}{6} \sin(\phi)\, d\theta\, d\phi \\ &= \int_0^{\frac{\pi}{2}} \frac{\pi}{3} \sin(\phi)\, d\theta \\ &= -\frac{\pi}{3} \cos(\phi) \Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \\ &= \boxed{\frac{\pi}{3}}\end{aligned}$$

□

Example — Find the volume of the region above the surface $\rho = \frac{\pi}{3}$ and below the surface $\rho = 4\cos(\phi)$.

Solution. By manipulation, we see that this is the area between a cone and a sphere centered at $(0, 0, 2)$ with radius 2. This gives the triple integral as

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_0^{4\cos(\phi)} 1 \cdot \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi &= \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \left(\frac{\rho^3}{3} \sin(\phi)\right) \Big|_{\rho=0}^{\rho=4\cos(\phi)} d\theta\, d\phi \\ &= \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \frac{64}{3} \cos^3(\phi) \sin(\phi)\, d\theta\, d\phi\end{aligned}$$

$$= \int_0^{\frac{\pi}{3}} \frac{128\pi}{3} \cos^3(\phi)\sin(\phi)\,d\phi$$
$$= \left(\frac{-32\pi}{3}\cos^4(\phi)\right)\Bigg|_{\phi=0}^{\phi=\frac{\pi}{3}}$$
$$= -\frac{32\pi}{3}\left(\frac{1}{16}-1\right) = \boxed{10\pi}$$

□

So what is this magnification factor intuitively?
With all these magnification factors when evaluating integrals in polar, cylindrical, and spherical coordinates, why exactly do we need it?

First, imagine this in 2D polar coordinates. A tiny region with a fixed Δr and a $\Delta\theta$ will have more area if it is further from the origin, and less if its closer. To account for this "magnification", we multiply the constants Δr and $\Delta\theta$ by r, to account for this "magnification".

This is the same for all "circular" regions. In the case of Spherical Coordinates, note that as tiny regions of $\Delta\theta$ and $\Delta\phi$ get bigger the further they are from the origin, approximately magnified by $\rho \cdot \rho\sin(\phi)$, so we have to multiply by this.

Another way of thinking of this spherical coordinate example (the thought processes for magnification factors of other coordinates can be thought of the same way) is that the volume of the box (which is length times width times height) is approximately

$$\Delta V \approx (\Delta\rho)\left(\rho\sin(\phi)\Delta\theta\right)(\rho\Delta\phi)$$

and approximating the volume as the change, or Δ, gets small gives us

$$dV = \underbrace{\rho^2\sin(\phi)}_{\text{magnification factor}}\,d\rho\,d\theta\,d\phi$$

§3.5 Change of Variables, Jacobian

Lets say we have a region R in the xy plane, and we want to replace it by another region S in the uv plane. Also suppose theres a transformation T with $x = g(u,v)$ and $y = h(u,v)$ mapping S to R. That is to say, $T : S \to R$.

Functions

1. T is injective if $p \neq q$ are two different points in S, then $T(p) \neq T(q)$
2. T is surjective if every point in R is T of some point in S
3. T is bijective, or a bijection, or a 1-1 correspondence, if T is injective and surjective
4. Composition: If f is a function $f : R \to \mathbb{R}$, $f \circ T : S \to \mathbb{R}$, or

$$(f \circ T)(u,v) = f\left(g(u,v), h(u,v)\right)$$

Proving Change of Variables for Double Integrals
Suppose T is a differentiable bijection transforming a region S in the coordinate plane uv to a region R in the coordinate plane xy with $x = g(u,v)$ and $y = h(u,v)$, with a function $f : R \to \mathbb{R}$. Consider the double integral

$$\iint\limits_R f\,dA$$

which we want to put in terms of the new coordinates u and v. Assume that everything is differentiable.

As we have $f \circ T : S \to R$, we can rewrte this integral into

$$\iint_R f\,dA = \iint_S (f \circ T) \left[\substack{\text{magnification}\\ \text{factor}}\right] dA$$

where the magnification factor is how much a tiny rectangle in S has been magnfied to be transformed into R, or in other words, if A is a rectangle in S with width Δu and length Δv,

$$\text{Magnification Factor} = \lim_{(\Delta u, \Delta v) \to (0,0)} \frac{\text{Area}(T(A))}{\text{Area}(A)}$$

where $\text{Area}(A) = \Delta u \Delta v$. Now what is the area of $T(A)$? Consider the rectangle A in S, with sides of vectors $\langle 0, \Delta v\rangle$ and $\langle \Delta u, 0\rangle$. After applying T to this, the new rectangle $T(A)$ will be approximately a parallelogram. The sides of this parallelogram is $\left\langle \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u \right\rangle$, which can be thought of as since you're moving right a certain Δu, then by the definition of the partial derivative you're moving a certain $\frac{\partial x}{\partial u}\Delta u$ of x to the right. The rest can be done the same way, with the other side being the vector $\left\langle \frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v \right\rangle$.

Now, we want to find the area of this parallelogram. As the area is abs val the det of the edge vectors, we have

$$\begin{aligned}
\text{Area}(T(A)) &= |\det(\text{edge vectors})| \\
&= \left| \begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} \right| \\
&= \left| \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \right| \Delta u \Delta v
\end{aligned}$$

Now that we have this, our magnfication factor will just be (by the previous equation)

$$\begin{aligned}
\text{Magnification Factor} &= \lim_{(\Delta u, \Delta v) \to (0,0)} \frac{\text{Area}(T(A))}{\text{Area}(A)} \\
&= \lim_{(\Delta u, \Delta v) \to (0,0)} \frac{\left| \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \right| \Delta u \Delta v}{\Delta u \Delta v} \\
&= \left| \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \right|
\end{aligned}$$

There is a simpler way to write this, as simply the determinant of a matrix, called the **Jacobian**.

Jacobian of Two and Three Variables —

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

This leads to the theorem

Theorem 3.4 (Change of Variables in Double Integrals)

$$\iint\limits_R f\,dA = \iint\limits_S (f\circ T)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

where T is a differentiable bijection from S (in the (u,v) plane) to R (in the (x,y) plane).

Example — An example of this is taking the change of cartesian to polar coordinates.

As we have the functions

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$$

the jacobian would be

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} &= \begin{vmatrix}\cos(\theta) & -3\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{vmatrix}\\ &= r\cos^2(\theta) - \left(-r\sin^2(\theta)\right)\\ &= r\end{aligned}$$

Plugging this into the change of variables formula gives

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f(r,\theta) r\,dA$$

which is the formula for the double integral of a polar curve.

Change of Variables for Triple Integrals

Say we have a region R in the (x,y,z) plane and a region S in the (u,v,w) plane, with

$$x = g(u,v,w) \qquad\qquad y = h(u,v,w) \qquad\qquad z = k(u,v,w)$$

and a transformation $T : S \to R$ being a differentiable bijection between these two regions. Then, we have

Theorem 3.5 (Change of Varables in 3 Dimentions)

$$\iiint\limits_R f\,dV = \iiint\limits_S (f\circ T)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV$$

where $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ is the Jacobian in 3 variables.

Example — Find the magnification factor when convereting to spheerical coordinates.

Solution. The magnification factor is the absolute value of the Jacobian of 3 variables. As we have the formulas for the translation as

$$x = \rho\sin(\phi)\cos(\theta) \qquad\qquad y = \rho\sin(\phi)\sin(\theta) \qquad\qquad z = \rho\cos(\phi)$$

which gives the jacobian as

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} \\
&= \begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{vmatrix} \\
&= -\rho^2\sin^3(\phi)\cos^2(\theta) - \rho^2\sin(\phi)\cos^2(\phi)\sin^2(\theta) - \rho^2\sin(\phi)\cos^2(\phi)\cos^2(\theta) - \rho^2\sin^3(\theta)\sin^2(\theta) \\
&= -\rho^2\sin^3(\phi) - \rho^2\sin(\phi)\cos^2(\phi) = \rho^2\sin(\phi)
\end{aligned}$$

As the magnification factor is the absolute value of this jacobian, and $\rho^2\sin(\phi)$ is positive since $0 \le \phi \le \pi$, then this gives the magnification factor as

$$\rho^2\sin(\phi)$$

□

In general, to evaluate an integral by change of variables,

1. Choose a transformation which makes things nicer
2. Understand the change in geometry of the region
3. Compute hte magnification factor
4. Evaluate the integral

Example — Calculate

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$$

Where R is the quadrilateral between the points $(0,1)$, $(0,2)$, $(1,0)$, $(2,0)$.

Solution. This looks like a hard integral, so we want to make cos() simpler. Notice that if we change variabels such that

$$u = y + x \qquad v = y - x$$

This results in a much easier integral to evaluate. However, what would this region look like? We have to solve for x and y in terms of u and v.

Solving gives us $x = \frac{u-v}{2}$ and $y = \frac{u+v}{2}$, which is the transformation we wanted.

Now, how do we graph out this transformation into a new region? Notice that our old region, R, is bounded by the lines $x + y = 2$ and $x + y = 1$, so in the new graph, its bounded by $u = 1$ and $u = 2$, and as the old region R is bounded by $y = 0$ and $x = 0$, the new graph is also bounded by $u + v = 0$ and $u - v = 0$. This gives the region S enclosed by $(1,1)$, $(2,2)$, $(-1,-1)$, $(-2,-2)$ in a trapezoid shape.

Now, the magnficiation factor is the absolute value of the jacobian, which is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$
$$= \frac{1}{2}$$

Now, we can solve the integral. We have that

$$\begin{aligned}
\iint_R \cos\left(\frac{y-x}{y+x}\right) dA &= \iint_S \left(\cos\left(\frac{v}{u}\right)\right)\left(\frac{1}{2}\right) dA \\
&= \int_1^2 \int_{-u}^{u} \frac{1}{2}\cos\left(\frac{v}{u}\right) dv\, du \\
&= \int_1^2 \frac{1}{2} u \sin\left(\frac{v}{u}\right)\Bigg|_{v=-u}^{v=u} du \\
&= \int_1^2 \frac{1}{2}\left(u\sin(1) - u\sin(-1)\right) du \\
&= \int_1^2 u\sin(1)\, du \\
&= \frac{\sin(1)}{2} u^2 \Bigg|_{u=1}^{u=2} \\
&= \boxed{\frac{3}{2}\sin(1)}
\end{aligned}$$

□

Example — Calculate

$$\iint_R \frac{1}{\sqrt{xy}}\, dA$$

where R is the quadrilateral with corners $(3,1)$, $(1,3)$, $(\frac{1}{4},\frac{3}{4})$, and $(\frac{3}{4},\frac{1}{4})$.

Solution. It is bounded by the lines $y = \frac{x}{3}$, $y = 3x$, $x+y=4$, and $x+y=1$. Note that this can be done over x nad y coordinates, but it wouldn't be clean, as we would have to divide it into 3 parts to do 3 seperate integrals.

However, we can do a change in region to evaluate it easier. Note that applying a change with

$$x = u^2 \qquad y = v^2$$

would make this much easier, as not only will the integrated function be simpler, the integrated region becomes a simple pie slice, a much simpler region too.

Calculating out the new region S gives us its bound as $\frac{v}{u} = \sqrt{3}$ and $\frac{v}{u} = \frac{1}{\sqrt{3}}$, which are two lines with an angle of $\frac{pi}{6}$ between them. Inside these two lines its bounded by the two circles $u^2+v^2=1$ and $u^2+v^2=4$. This is indeed a bijection, which we can check.

The Jacobian/magnification factor is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix} = 4uv$$

This gives the integral as

$$\iint\limits_R \frac{1}{\sqrt{xy}}\, dA = \iint\limits_S \frac{1}{uv}\,(4uv)\, dA = 4 \iint\limits_S dA$$

which is simply 4 times the area of the region S. The area of the region S is jsut the area of the pi slice, or $\left(\frac{\frac{\pi}{6}}{2\pi}\right)\pi(4\pi - 1\pi)$, giving answer as four times of this, or

$$\boxed{\pi}$$

□

Example — Let R be a region in hte plane between the circles defined by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the first quadrant. Find a one-to-one (non-linear) differentiable transformation T which transforms $[0,1] \times [0,1]$ into R.

Solution. This region is circular, so we consider the transformation

$$x = (v)\cos(u) \qquad\qquad y = (v)\sin(u)$$

which, when $0 \le v \le 1$ and $0 \le u \le 1$, maps onto the xy region similar to this. However, when $v = 0$, it maps onto $x^2 + y^2 = 0$ and when $v = 1$ it maps onto $x^2 + y^2 = 1$, over an angle of 0 to 1. We want the angle from 0 to $\frac{pi}{2}$ so it draws in the first quadrant, and from $x^2 + y^2 = 1$ to $x^2 + y^2 = 4$, so by translating this transformation we get our final transformation of

$$\boxed{x = (v+1)\cos\left(\frac{\pi}{2}u\right) \qquad\qquad y = (v+1)\cos\left(\frac{\pi}{2}u\right)}$$

□

§3.6 More Applications of Double and Triple Integrals

These are more applications of integrals that aren't included in the lecture, but are in the textbook. Most of them are relating to physics. The applications that were included in the lecture are here.

Applications of Double Integrals

Once again, given a density function

$$\rho(x,y) = \lim \frac{\Delta m}{\Delta A}$$

where mass over area is density. This gives

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

approximating the mass of the whole object, and taking the limit as the nubmer of these sub-rectangles goes to infinity gives us

$$m = \iint\limits_D \rho(x,y)\, dA$$

a formula in the notes before.

Now, the moments are defined as

Moments — The moment of the entire lamina about the x-axis is

$$M_x = \iint_D y\rho(x,y)\,dA$$

and about the y-axis is

$$M_y = \iint_D x\rho(x,y)\,dA$$

These moments help us calculate the center of mass, which is given by

Center of Mass — a point $(\overline{x}, \overline{y})$ such that

$$\overline{x} = \frac{M_x}{m} \qquad \overline{y} = \frac{M_y}{m}$$

The moments of inertia (also called the second moment) are calculated as

Moments of Inertia — The moment of inertia about the x-axis is

$$I_x = \iint_D y^2\rho(x,y)\,dA$$

and about the y-axis is

$$I_y = \iint_D x^2\rho(x,y)\,dA$$

and about the origin (whihc is also called the polar moment of inertia) is

$$I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA$$

Applications of Triple Integrals

Just like before, and similarly to double integrals, we can show the mass is

$$m = \iint_E \rho(x,y,z)\,dV$$

where $\rho(x,y,z)$ is the density function for a region E.

Moments — The moments about the coordinate planes are

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV \qquad M_{xz} = \iiint_E y\rho(x,y,z)\,dV$$

$$M_{xy} = \iiint_E z\rho(x,y,z)\,dV$$

Center of Mass — The center of mass is a point $(\overline{x}, \overline{y}, \overline{z})$ such that

$$\overline{x} = \frac{M_{yz}}{m} \qquad y = \frac{M_{xz}}{m} \qquad \frac{M_{xy}}{m}$$

The moments of inertia are also

Moments of Inertia — About the three coordinate axes, the moments of inertia are

$$I_x = \iiint\limits_E (y^2 + z^2)\rho(x, y, z)\, dV \qquad I_y = \iiint\limits_E (x^2 + z^2)\rho(x, y, z)\, dV$$

$$I_z = \iiint\limits_E (x^2 + y^2)\rho(x, y, z)\, dV$$

§4 Module 4 (Vector Calculus)

§4.1 Vector Fields and Line Integrals

Definition — A vector field on $\mathbb{R}^2$ is a function $\vec{\mathbf{F}}$ associating to each point $(x, y) \in \mathbb{R}^2$ a vector

$$\vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y)\rangle$$

A vector field on $\mathbb{R}^3$ is the same; for every point $(x, y, z) \in \mathbb{R}^3$, there is a vector such that

$$\vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

A physical example of this is wind velocity, as at each point, the wind is moving in some vector at a certain speed, so you can represent it with a vector field. They can also describe force fields, such as the gravitational field, an electric field, and so on.

Example — Gravitational Field

Lets say at the origin, theres a point with mass M. The gravitational attraction from a point r away from the origin with mass m has gravitational attraction of

$$G\frac{Mm}{r^2} = \left(\frac{GM}{r^2}\right) m$$

The gravitational field is a vector pointing towards the origin with magnitude $\frac{GM}{r^2}$. Now, the unit vector pointing to the origin is $-\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}$ (negative so its pointing to the origin), and now this gives us the function as

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \left(\frac{GM}{|\vec{\mathbf{r}}|^2}\right)\left(-\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}\right) = \boxed{-\frac{GM\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3}}$$

which is defined for all points except the origin, where the magnitude of the vector would be infinite.

Conservative Vector Fields

If f is a real-valued function on $\mathbb{R}^2$ or $\mathbb{R}^3$, (or any number of dimensions), then ∇f is a (gradient) vector field.

On $\mathbb{R}^2$, $\nabla f = \langle f_x, f_y\rangle$.

On $\mathbb{R}^3$, $\nabla f = \langle f_x, f_y, f_z\rangle$.

Definition — A vector field $\vec{\mathbf{F}}$ is called conservative if

$$\vec{\mathbf{F}} = \nabla f$$

for some function f.

The function f is sometimes called a potential.

We can guess out functions of f given the gradient, to see if its conservative. For example, the vector field

$$\vec{\mathbf{F}} = \left\langle 3x^2 \sin(y), x^3 \cos(y)\right\rangle$$

is conservative, as it is the gradient of f, where

$$f = x^3 \sin(y)$$

However, the vector field

$$\vec{\mathbf{F}} = \left\langle x^3 \sin(y), y^2 \right\rangle$$

is not conservative.

Now, how do we tell if something is conservative or not without guessing and checking?

(One method learned later is here)

Fact. If f is a differentiable function on $\mathbb{R}^2$ then

$$f(x,y) = f(0,y) + \int_0^x f_x(t,y)\,dt$$

This is true because for a fixed y, define $g(t) = f(t,y)$, and the rest follows from the fundamental theorem of calculus, where

$$\frac{\mathrm{d}g}{\mathrm{d}t}(t) = f_x(t,y)$$

Example — Find f given $f_x = 3x^2 \sin(y)$ and $f_y = x^3 \cos(y)$.

Solution. Using the fact above, if we have $f_x = 3x^2 \sin(y)$ and $f_y = x^3 \cos(y)$, then f would be

$$\begin{aligned} f(x,y) &= g(y) + x^3 \sin(y) \\ \implies f_y &= g'(y) + x^3 \cos(y) \end{aligned}$$

Now, since we have $f_y = x^3 \cos(y)$, this means $g'(y) = 0$ or $g(y) = C$, leading to our final equation for f being

$$\boxed{f(x,y) = x^3 \sin(y) + c}$$

□

Example — Try to find f given $f_x = x^3 \sin(y)$ and $f_y = y^2$.

Solution. We can integrate out x, getting

$$f = \frac{x^4}{4} \sin(y) + g(y)$$

Now, differentiating on y gives

$$f_y = \frac{x^4}{4} \cos(y) + g'(y)$$

$$\implies g'(y) = y^2 - \frac{x^4}{4} \cos(y)$$

This contradicts itself though, as $g(y)$ is supposed to be a function depending only on y, but it has x terms in it. This means that f does not exist, and the vector field

$$\vec{\mathbf{F}} = \left\langle x^3 \sin(y), y^2 \right\rangle$$

is not conservative. □

Example — Is the gravitational field

$$\vec{\mathbf{F}} = \frac{-GM\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3}$$

represented by the function $f = \frac{GM}{|\vec{\mathbf{r}}|}$?

Solution. Given $f = \frac{GM}{|\vec{\mathbf{r}}|}$, we can write out f as $f = GM(x^2 + y^2 + z^2)^{-1/2}$. This gives

$$f_x = \frac{-GMx}{|\vec{\mathbf{r}}|^3}$$

$$f_y = \frac{-GMy}{|\vec{\mathbf{r}}|^3}$$

$$f_z = \frac{-GMz}{|\vec{\mathbf{r}}|^3}$$

and so

$$\nabla f = \frac{-GM}{|\vec{\mathbf{r}}|^3}\langle x, y, z\rangle = \frac{-GM}{|\vec{\mathbf{r}}|^3}$$

which was the answer we were looking for. □

Line Integrals in $\mathbb{R}^2$

Let C be a parametrized curve, which means theres an interval from a to b, which is being mapped to the xy plane as a curve from $\vec{\mathbf{r}}(a)$ to $\vec{\mathbf{r}}(b)$, with a point on it beign $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle$.

There are three different kinds of "line integrals" over C.

1. $\int\limits_C f\,ds$ where $f : C \to \mathbb{R}$ (integration wrt arc length)
2. $\int\limits_C f\,dx$, $\int\limits_C f\,dy$ (integration wrt to x and y)
3. $\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}}$ is a vector field

These are all different, with different properties, definitions, and purposes.

Integration WRT Arc Length

Divide up the interval from a to b into tiny pieces, with

$$a = t_0 < t_1 < \cdots < t_n = b \qquad \Delta t = t_i - t_{i-1} = \frac{b-a}{n}$$

This will also divide the curve up into tiny points, with

$$\vec{\mathbf{r}}\,(t_i) = \langle x(t_i), y(t_i)\rangle$$
$$\vec{\mathbf{r}}\,(t_{i-1}) = \langle x(t_{i-1}), y(t_{i-1})\rangle$$

Now, between these two points on the curve, theres a vector pointing from $\vec{\mathbf{r}}(t_{i-1})$ to $\vec{\mathbf{r}}(t_i)$, represented as $\langle \Delta x_i, \Delta y_i\rangle$. As the lenght of the curve is the sum of all these vectors' lengths, call the length of this vector as

$$\Delta s = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = |\vec{\mathbf{r}}(t_i) - \vec{\mathbf{r}}(t_{i-1})|$$

Now, we can define the integral as

Integration wrt Arc Length —

$$\int_C f\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\Delta s_i$$

$$\int_C f\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\left|\vec{\mathbf{r}}\,'(t)\right|dt$$

An example of this is $\int_C 1\,ds$ which gives the length of the curve C.

A more general example would be looking at a function, where if $f > 0$ on the curve C, then

$$\int_C f\,ds = \text{area under the graph of } f \text{ over } C$$

This can also be seen through the limit definition of the integral, wehre

$$\int_C f\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\Delta s_i$$

where we're approxiamting teh region by rectangular strips and adding up the areas.

An application of this is in physics.

Applications — Suppose that the curve C is a wire with mass desnity ρ. Then the mass of the wire is

$$M = \int_C \rho\,ds$$

The center of mass of the wire is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{1}{M}\int x\rho\,ds$$

$$\overline{y} = \frac{1}{M}\int y\rho\,ds$$

The integral

$$\int_C f\,ds$$

does not depend on the parametrization as long as you don't "backtrack".

Proof. Why is this true? Consider the interval a to b, with a parameter t between it. It's being mapped by $\vec{\mathbf{r}}$ to a curve C, which starts at $\vec{\mathbf{r}}(a)$ and ends at $\vec{\mathbf{r}}(b)$.

Consider another parametrization of this curve C, which goes from c to d, with a parameter u between it. Now, the parameter t is going to be a function of u, or $t = g(u)$. There are two ways now to think of this curve; t mapping to $\vec{\mathbf{r}}(t)$, and u mapping to $\vec{\mathbf{r}}(t(u))$.

Assume that g is a differentiable bijection, so there won't be "backtracking". Now, there are two cases.

Case 1: $g(c) = a$, $g(d) = b$, and $g' \geq 0$.

Now, the integral wrt arc length of this is

$$\begin{aligned}
&\int_c^d f(\vec{\mathbf{r}}(t(u)))\left|\frac{\mathrm{d}}{\mathrm{d}u}\vec{\mathbf{r}}(t(u))\right| du \\
&= \int_c^d f(\vec{\mathbf{r}}(t(u)))\left|\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbf{r}}(t)}_{\text{vector}}\ \underbrace{\frac{\mathrm{d}t}{\mathrm{d}u}}_{\substack{\text{scalar}\\\text{nonnegative}}}\right| du \\
&= \int_c^d f(\vec{\mathbf{r}}(t(u)))\left|\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbf{r}}(t)\right|\frac{\mathrm{d}t}{\mathrm{d}u}\, du \\
&= \int_a^b f(\vec{\mathbf{r}}(t))\left|\vec{\mathbf{r}}'(t)\right| dt
\end{aligned}$$

and so we've proved that, for this case, these two integrals are equal.

Case 2: $g(c) = b$, $g(d) = a$, $g' \leq 0$, or the curve going opposite from before.

This results in basically the same thing, giving

$$\begin{aligned}
&\int_c^d f(\vec{\mathbf{r}}(t(u)))\left|\frac{\mathrm{d}}{\mathrm{d}u}\vec{\mathbf{r}}(t(u))\right| du \\
&= \int_c^d f(\vec{\mathbf{r}}(t(u)))\left|\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbf{r}}(t)}_{\text{vector}}\ \underbrace{\frac{\mathrm{d}t}{\mathrm{d}u}}_{\substack{\text{scalar}\\\text{nonpositive}}}\right| du \\
&= -\int_c^d f(\vec{\mathbf{r}}(t(u)))\left|\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbf{r}}(t)\right|\frac{\mathrm{d}t}{\mathrm{d}u}\, du \\
&= -\int_b^a f(\vec{\mathbf{r}}(t))\left|\vec{\mathbf{t}}'\right| dt \\
&= \int_a^b f(\vec{\mathbf{r}}(t))\left|\vec{\mathbf{r}}'(t)\right| dt
\end{aligned}$$

So we've proven that, for both our cases, the integral wrt arc length doesn't depend on the parametrization of the curve. $\square$

Line Integrals WRT x Or y

Definition — Let C be a parametrized curve, meaning for a point $t \in [a, b]$ then $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle$. Let $f : C \to \mathbb{R}$

$$\int_C f\, dx = \int_a^b f(x(t), y(t))x'(t)\, dt$$

$$\int_C f\, dy = \int_a^b f(x(t), y(t))y'(t)\, dt$$

The geometric interpretation of this line integral is the image mapped onto the x or y plane of the curve, and the area under that curve.

We can compare this with the arc length, which is

$$\int_C f\,ds = \int_a^b f(x(t),y(t))\underbrace{\sqrt{x'(t)^2+y'(t)^2}}_{\text{length of velocity vector}}$$

$$\int_C f\,dx = \int_a^b f(x(t),y(t))\underbrace{\sqrt{x'(t)^2+y'(t)^2}}_{x\text{ component of velocity}}$$

$$\int_C f\,dy = \int_a^b f(x(t),y(t))\underbrace{\sqrt{x'(t)^2+y'(t)^2}}_{y\text{ component of velocity}}$$

Another property to compare is is

$$\int_C 1\,ds = \text{length}(C)$$

$$\int_C 1\,dx = x(b) - x(a) \qquad \int_C 1\,dy = y(b) - y(a)$$

The integrals wrt x or y

$$\int_C f\,dx \qquad \int_C f\,dy$$

do not depend on the parametrization (even allowing backtracking), except that if you switch the endpoints/go the opposite direction, then you have to switch the sign (multiply by -1). Another way to write this is, a curve with a chosen direction is called an <u>oriented curve</u>. Then $-C = C$ (opposite way), and

$$\int_{-C} f\,dx = -\int_C f\,dx$$

$$\int_{-C} f\,dy = -\int_C f\,dy$$

This means these integrals are well defined over oriented curves.

The geometric interpretation also gives intuition for this fact; as the area under a curve being "projected" onto a plane is the same no matter how the curve is drawn, the only thing that matters is the direction of integration; if it's positive integrating from "right" to "left", then it'll be negative the other way.

Example — Compute

$$\int_C (x+y)\,dx$$

where C is the portion of the parabola $y = x^2$ from $(-2,4)$ to $(1,1)$.

Solution. To compute this, we first have to choose a parametrization. we can choose

$$x = t \qquad y = t^2 \qquad -2 \le t \le 1$$

Then, we have

$$\begin{aligned}
\int_C (x+y)\ dx &= \int_{-2}^{1} (x(t)+y(t))\, x'(t)\, dt \\
&= \int_{-2}^{1} \left(t+t^2\right)(1)\ dt \\
&= \left(\frac{t^2}{2}+\frac{t^3}{3}\right)\Bigg|_{t=-2}^{t=1} \\
&= \left(\frac{1}{2}+\frac{1}{3}\right) - \left(2-\frac{8}{3}\right) = \boxed{\frac{3}{2}}
\end{aligned}$$

As it doesn't depend on the parametrization, we can also try another one to make sure its the same. We can use the parametrization

$$x = t^3 \qquad y = t^6 \qquad (-2)^{\frac{1}{3}} \le t \le 1$$

Now, we have

$$\begin{aligned}
\int_C (x+y)\ dx &= \int_{-(2)^{\frac{1}{3}}}^{1} \left(t^3+t^6\right)\left(3t^2\right)\ dt \\
&= \int_{-(2)^{\frac{1}{3}}}^{1} \left(3t^5+3t^8\right)\ dt \\
&= \left(\frac{t^6}{2}+\frac{t^9}{3}\right)\Bigg|_{t=2^{\frac{1}{3}}}^{t=1} \\
&= \boxed{\frac{3}{2}}
\end{aligned}$$

Trying a third paramtrization,

$$x = -t \qquad y =^2 \qquad -1 \le t \le 2$$

Using this, we get

$$\begin{aligned}
\int_C (x+y)\ dx &= \int_{-1}^{2} (x(t)+y(t))\, x'(t)\, dt \\
&= \int_{-1}^{2} \left(-t+t^2\right)(-1)\ dt \\
&= \left(\frac{t^2}{2}+\frac{t^3}{3}\right)\Bigg|_{t=-1}^{t=2} \\
&= \left(2-\frac{8}{3}\right) - \left(\frac{1}{2}+\frac{1}{3}\right) = \boxed{-\frac{3}{2}}
\end{aligned}$$

This makes sense, as we've switched the orientation, so the final answer's sign has switched. □

Line Integral of a Vector Field

Let C be a parametrized curve in $\mathbb{R}^3$. Let the parameter $t \in [a,b]$ be defined with a $\langle x(t), y(t)\rangle$ on the curve C. Let $\vec{\mathbf{F}} = \langle P(x,y), Q(x,y)\rangle$ be a vector field defined in a domain D in $\mathbb{R}^2$ containing C.

Definition — Now, we can define the Line Integral of a Vector Field as

$$\begin{aligned}\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b \vec{\mathbf{F}}\left(\vec{\mathbf{r}}(t)\right) \cdot \vec{\mathbf{r}}\,'(t)\,dt \\ &= \int_a^b \left\langle P\left(\vec{\mathbf{r}}(t)\right), Q\left(\vec{\mathbf{r}}(t)\right)\right\rangle \cdot \left\langle x'(t), y'(t)\right\rangle\, dt \\ &= \int_a^b \big(P\left(\vec{\mathbf{r}}(t)\right) x'(t) + Q\left(\vec{\mathbf{r}}(t)\right) y'(t)\big)\, dt = \int_a^b \left(P\,dx + Q\,dy\right)\end{aligned}$$

What does this mean? The velocity vector of the curve, when perpendicular to the vector field $\vec{\mathbf{F}}$, has very little contribution to the integral, and when pointing opposite $\vec{\mathbf{F}}$, negative contribution. The most contribution is when it is "going with the flow" of the vector field.

The physical interpretation is that if $\vec{\mathbf{F}}$ is a sort of force field, then the integral

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

is the work done as you go along the curve.

The Line Integral also depends on the orientation of the curve, as it depends on teh integrals wrt x and y. Specifically, this means that

$$\int\limits_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

All of this is the same in $\mathbb{R}^3$.

Integrals in 3D — Say we have a curve C with $t \mapsto \langle x(t), y(t), z(t)\rangle = \vec{\mathbf{r}}(t)$, with $a \leq t \leq b$, and a function $f : C \to R$, with $F = \langle P, Q, R\rangle$ being a vector field, we have the same three integrals

$$\int\limits_C f\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt = \int_a^b f\left(\vec{\mathbf{r}}(t)\right)\left|\vec{\mathbf{r}}\,'(t)\right| dt \tag{1}$$

$$\begin{gathered}\int\limits_C f\,dx = \int_a^b f\left(\vec{\mathbf{r}}(t)\right) x'(t)\,dt \qquad \int\limits_C f\,dy = \int_a^b f\left(\vec{\mathbf{r}}(t)\right) y'(t)\,dt \\ \int\limits_C f\,dz = \int_a^b f\left(\vec{\mathbf{r}}(t)\right) z'(t)\,dt\end{gathered} \tag{2}$$

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}\left(\vec{\mathbf{r}}(t)\right) \cdot \vec{\mathbf{r}}\,'(t)\,dt = \int\limits_C \left(P\,dx + Q\,dy + R\,dz\right) \tag{3}$$

Just like in the 2D case, equation (1) don't depend on parametrizations as long as no "backtrack", equation (2) and equation (3) depend on the orientation, and are multiplied by -1 if the orientation is switched.

Example — Calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}} = \langle z, xy, x+z \rangle$ and C consists fo three line segments, from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.

Solution. We can split these three line segments into three curves, C_1, C_2, C_3, and write out different integrals for all of them.

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} \right) \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Now, we can caculate them individually.

For curve 1, we can parametrize into $x = t$, $y = 0$, $z = 0$, $0 \le t \le 1$.

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \langle 0, 0, t \rangle \cdot \langle 1, 0, 0 \rangle \; dt = \int_0^1 0 \, dt = 0$$

For curve 2, we can parametrize into $x = 1$, $y = t$, $z = 0$, $0 \le t \le 1$.

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \langle 0, t, 1 \rangle \cdot \langle 0, 1, 0 \rangle = \int_0^1 t \, dt = \frac{1}{2}$$

For curve 3, we can parametrize into $x = 1$, $y = 1$, $z = t$, $0 \le t \le 1$.

$$\int_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \langle t, 1, 1+t \rangle \cdot \langle 0, 0, 1 \rangle \; dt = \int_0^1 (1+t) \; dt = \frac{3}{2}$$

This gives the final answer as

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} \right) \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0 + \frac{1}{2} + \frac{3}{2} = \boxed{2}$$

□

Example — Calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}} = \langle z, xy, x+z \rangle$ and $\widetilde{C}$ is the line segment from $(0,0,0)$ to $(1,1,1)$.

Solution. This is the same starting and ending points, but with a different curve going traveling from the start to the end. We can parametrize this curve $\widetilde{C}$ as $x = t$, $y = t$, $z = t$ with $0 \le t \le 1$, and write the integral as

$$\begin{aligned} \int_{\widetilde{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \left\langle t, t^2, 2t \right\rangle \cdot \langle 1, 1, 1 \rangle \; dt \\ &= \int_0^1 \left(3t + t^2 \right) \; dt \\ &= \left(\frac{3t^2}{2} + \frac{t^3}{3} \right) \Bigg|_{t=0}^{t=1} \\ &= \frac{3}{2} + \frac{1}{3} = \boxed{\frac{11}{6}} \end{aligned}$$

□

Note how this isn't the same answer as we got below, and this tells us that the work done by $\vec{\mathbf{F}}$, the vector field, is more in the first example than in the second example, and depends on the path you take between these two different paths.

§4.2 Fundamental Theorem of Line Integrals

Theorem (Fundamental Theorem of Line Integrals)
If $\vec{\mathbf{F}} = \boldsymbol{\nabla} f$ ($\vec{\mathbf{F}}$ is conservative) and C is a parametrized curve from A to B, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(B) - f(A)$$

Proof. Choose a parametrization of C as $\vec{\mathbf{r}}(t)$ where $a \leq t \leq b$, $\vec{\mathbf{r}}(a) = A$, $\vec{\mathbf{r}}(b) = B$. Now, the chain rule says

$$\frac{\mathrm{d}}{\mathrm{d}t} f\left(\vec{\mathbf{r}}(t)\right) = \boldsymbol{\nabla} f\left(\vec{\mathbf{r}}(t)\right) \cdot \vec{\mathbf{r}}\,'(t)$$

Now, by the fundamental theorem of calculus,

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b \underbrace{\vec{\mathbf{F}}\left(\vec{\mathbf{r}}(t)\right)}_{\boldsymbol{\nabla} f(\vec{\mathbf{r}}(t))} \cdot \vec{\mathbf{r}}\,'(t)\, dt \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f\left(\vec{\mathbf{r}}(t)\right)\, dt = f\left(\vec{\mathbf{r}}(b)\right) - f\left(\vec{\mathbf{r}}(a)\right) \\
&= \boxed{f(B) - f(A)}
\end{aligned}$$

□

This is very similar to the fundamental theorem of calculus, which says that

$$\int_a^b \frac{\mathrm{d}g}{\mathrm{d}x}\, dx = g(b) - g(a)$$

Corollary
If $\vec{\mathbf{F}}$ is conservative then $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ depends only on A, B and not on C.

Example — Calculate

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

where $\vec{\mathbf{F}} = \left\langle 2xyz + e^{x+z}, x^2z, x^2y + e^{x+z} \right\rangle$ and C is a curve from $(1, 2, 0)$ to $(1, 0, 5)$.

Solution. As $\vec{\mathbf{F}}$ is conservative (the only way this problem is solvable, because we can't solve unless knowing C if otherwise), we hvae to find the f such that $\vec{\mathbf{F}} = \boldsymbol{\nabla} f$. Looking at the terms, we can guess that

$$f = x^2yz + e^{x+z}$$

which satisfies the partial derivatives given in $\vec{\mathbf{F}}$. By the fundamental theorem of line integrals (FTLI),

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= f(1,0,5) - f(1,2,0) \\ &= \left(0 + e^6\right) - \left(0 + e^{1+0}\right) \\ &= \boxed{e^6 - e}\end{aligned}$$

□

Definition

1. A curve C from A to B is closed if $A = B$.
2. A curve C is simple if it doesn't cross itself.
3. A region D is open if it doesn't contain any of its boundary points.
4. A region D is connected if we can connect any two points in the region with a curve that lies completely in D.
5. A region D is simply connected if it is connected and contains no holes.

Closed Curves and Conservative Vector Fields — If $\vec{\mathbf{F}}$ is conservative and C is closed, then by FTLI

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(B) - f(A) = 0$$

This means that the work done around a closed curve is 0.

An example of this is the gravitational field, where if it were conservative, moving in a closed curve in it would result in infinite nonstopping motion, which is not possible in real life, so the real life version of the gravitational field is not conservative.

Theorem

Let $\vec{\mathbf{F}}$ be a vector field in a domain D. Then

$$\vec{\mathbf{F}} \text{ is conservative } \iff \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

for every closed curve C in D.

Proof. To prove this, we have to prove both sides of the theorem.

To prove $\implies$ we need to show that

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

for every closed curve in D. However, we know this, as we've proved that before, with FTLI.

To prove $\Longleftarrow$ we need to show that if

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

for every closed curve in D, then there exists a function f in D such that $\vec{\mathbf{F}} = \boldsymbol{\nabla} f$. Suppose for this proof that $D = \mathbb{R}^2$, where the general case's proof is similar.

Define $f(x,y) = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where C is a curve from $(0,0)$ to (x,y). This doesn't depend on the choice C, as if C' is another curve from $(0,0)$ to (x,y),

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C-C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

as the curve $C - C'$ is a closed curve, and this was our assumption.

However, we still need to show that $\boldsymbol{\nabla} f = \vec{\mathbf{F}}$, or $\vec{\mathbf{F}}(x,y) = \boldsymbol{\nabla} f(x,y)$. Suppose $x, y > 0$ (general case is similar, once again). If $\vec{\mathbf{F}} = \langle P, Q \rangle$, we just have to show that

$$\frac{\partial f}{\partial x}(x,y) = P(x,y)$$
$$\frac{\partial f}{\partial y}(x,y) = Q(x,y)$$

To prove the first case, think about fixing y, and varying x. We can then travel first from $(0,0)$ to $(0,y)$ (call it C_1), then from $(0,y)$ to (x,y) (call it C_2). Then, we have

$$f(x,y) = \underbrace{\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}}_{\text{0 after d/dx}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \qquad \text{where} \qquad \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^x P(t,y)\,dt$$

Now, taking the partial derivative wrt x of $f(x,y)$, we have (by the fundamental theorem of calculus)

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x P(t,y)\,dt \\ &= P(x,y) \end{aligned}$$

To prove the second case, we can fix x, and vary y. Similarly, by drawing the curve C_1 as $(0,0)$ to $(x,0)$, and C_2 as $(x,0)$ to (x,y), and similarly, we have that

$$f(x,y) = \underbrace{\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}}_{\text{0 after d/dy}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \qquad \text{where} \qquad \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^y Q(x,t)\,dt$$

which gives

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y Q(x,t)\,dt \\ &= Q(x,y) \end{aligned}$$

thus concluding our proof. $\square$

To summarize this proof, we prove both sides of the $\Longleftrightarrow$ equation, where $\Longrightarrow$ side is already proven, and to prove the $\Longleftarrow$ side, we first prove that the condition doesn't depend on C, then sepereate C into two different curves, one regarding x as a constant and the other regarding y as a constant, and finally, we take the partial derivatives and show that the gradient is equal to the vector field.

Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a diffrentiable vector field on a domain $D \in \mathbb{R}^2$. If $\vec{\mathbf{F}}$ is conservative then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

This is true, as if $\vec{\mathbf{F}} = \nabla f$ then $P = f_x$ and $Q = f_y$. This means

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} \\ \implies f_{xy} &= f_{yx} \end{aligned}$$

which is true by Clairaut's Theorem.

Example — Is $\vec{\mathbf{F}} = \left\langle x^2 \cos(y), x + y^3 \right\rangle$ conservative?

Solution. If it's conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

We have that

$$\begin{aligned} \frac{\partial P}{\partial y} &= -x^2 \sin(y) \\ \frac{\partial Q}{\partial x} &= 1 \end{aligned}$$

and since these are unequal, that means $\vec{\mathbf{F}}$ is $\boxed{\text{not conservative}}$. ☐

However, the converse is no necessarily true. The converse leads to the theorem below, where

Theorem 4.1

If D is simply connected and $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field on D such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

then $\vec{\mathbf{F}}$ is conservative.

The sketch of the proof of this theorem is shown here.

§4.3 Green's Theorem

Theorem 4.2 (Jordan Curve Theorem)

A simple closed curve $C \in \mathbb{R}^2$ is the boundary of a (unique) simply connected region.

Definition — Let C be a simple closed curve in $\mathbb{R}^2$ and let D be the region that it bounds. We say C is positively oriented if "D is to the left as you walk along C".

The notation of this is, if C is a simple closed curve and P and Q are functions, then

$$\oint_C (P\,dx + Q\,dy)$$

is the line integral

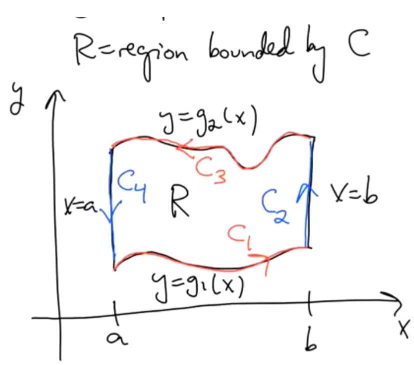
$$\int_C (P\,dx + Q\,dy)$$

Theorem (Green's Theorem)

Let C be a simple closed curve in $\mathbb{R}^2$. Let D be the region that it bounds. Let P, Q be differentiable functions on D. Then

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Proof. Case 1: R is a region bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, and $Q = 0$. Call, from bottom going counterclockwise, the 4 different sides of this curvy quadrilateral shape as C_1, C_2, C_3, and C_4, as shown in the image below.



Let the parametrization of C_1 be $x = t$, $y = g_1(x)$, $a \le t \le b$. Then,

$$\int_{C_1} P\,dx = \int_a^b P(t, g_1(t))\,dt = \int_a^b P(x, g_1(x)\,dx$$

Let the parametrization of $-C_3$ be $x = t$, $y = g_2(x)$, $a \le t \le b$. Then,

$$\int_{C_3} P\,dx = -\int_a^b P(x, g_2(x))\,dx$$

For the curves C_2 and C_4, as x doesn't change, this results in the integrals being 0.

With this, we have that

$$\begin{aligned}\oint_C P\,dx &= \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx \\ &= \int_a^b \left(P(x, g_1(x)) - P(x, g_2(x))\right)\,dx\end{aligned}$$

Now, the other side of green's theorem is

$$\iint_R -\frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} -\frac{\partial P}{\partial y}(x,y)\, dy\, dx$$
$$= \int_a^b \left(-P(x, g_2(x)) + P(x, g_1(x))\right) dx$$

We can see why these last two lines are equal with the fundamental theorem of calculus, and can make more clear by defining $g(x) = P(x,y)$ such that $\frac{\partial P}{\partial y}(x,y) = \frac{\mathrm{d}g}{\mathrm{d}y}(y)$.

As these two expressions are equal, we have proved that this is correct for this special case.

Case 2: R is defined by $g_1(x) \le x \le g_2(x)$ and $a \le y \le b$, and $P = 0$. The proof of this is very similar

Step 3: R is both of these types of regions (very general), and P, Q are anything.

Looking at the integral

$$\oint_C (P\, dx + Q\, dy) = \oint_C P\, dx + \oint_C Q\, dy$$

where

$$\oint_C P\, dx = -\iint_R \frac{\partial P}{\partial y}\, dA \qquad \oint_C Q\, dy = \iint_R \frac{\partial Q}{\partial x}\, dA$$

which we proved in the other two cases. This gives

$$\oint_C (P\, dx + Q\, dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Finalize The General case is dividing R into subregions, each of which is both of the types of regions described in case 1 and case 2.

Then, if Green's theorem holds for all subregions, then green's theorem holds for the whole region.

We can see this last sentence is correct as all "overlapping boundaries" curves cancel out. If R_1 and R_2 are regions which hold wiht Green's Theorem, and $R_1 \cup R_2 = R$ then basically this says that

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
$$= \oint_{C_1} (P\, dx + Q\, dy) + \oint_{C_2} (P\, dx + Q\, dy)$$
$$= \oint_C (P\, dx + Q\, dy)$$

This is kind of the basic idea of the proof. □

Example — Calculate

$$\oint_C \left(-x^2 y\, dx + x y^2\, dy\right)$$

where $C =$ unit circle.

Solution. We'll do two ways, one by green's theorem and one by direct calculation.

Solution 1: Parameterize C: $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$.

$$\begin{aligned}
\oint_C \left(-x^2y\,dx + xy^2\,dy\right) &= \int_0^{2\pi} \left(-x^2(t)y(t)x'(t) + x(t)y(t)^2y'(t)\right)\,dt \\
&= \int_0^{2\pi} \left(\cos^2(t)\sin^2(t) + \cos^2(t)\sin^2(t)\right)\,dt \\
&= 2\int_0^{2\pi} \cos^2(t)\sin^2(t)\,dt \\
&= \frac{1}{2}\int_0^{2\pi} \sin^2(2t)\,dt \\
&= \frac{1}{4}\int_0^{2\pi} \left(1 - \cos(4t)\right)\,dt \\
&= \boxed{\frac{\pi}{2}}
\end{aligned}$$

Solution 2:

$$\begin{aligned}
\oint_C \left(\underbrace{-x^2y}_{P}\,dx + \underbrace{xy^2}_{Q}\,dy\right) &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \iint_D \left(y^2 - (-x^2)\right)\,dA \\
&= \iint_D \left(x^2 + y^2\right)\,dA \\
&= \int_0^{2\pi}\int_0^1 r^2 r\,dr\,d\theta \\
&= \int_0^{2\pi} \left(\frac{r^4}{4}\right)\Big|_{r=0}^{r=1}\,d\theta \\
&= \int_0^{2\pi} \frac{1}{4}\,d\theta \\
&= \boxed{\frac{\pi}{2}}
\end{aligned}$$

□

If C is a simple closed curve, nad D is the region it bounds, then

$$\begin{aligned}
\text{Area}(D) &= \oint_C x\,dy \\
&= \oint_C -y\,dx \\
&= \oint_C \frac{1}{2}\left(x\,dy - y\,dx\right)
\end{aligned}$$

Proof. By greene's theorem,

$$\oint_C x\,dy = \iint_D 1\,dA = \text{Area}(D)$$

$$\oint_C -y\,dx = \iint_D 1\,dA = \text{Area}(D)$$

The third one is just half the area plus half the area is the whole area. □

We can test this with an example. Let C be the unit circle, with $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$.

$$\begin{aligned}
\text{Area}(C) &= \oint_C \frac{1}{2}(x\,dy - y\,dx) \\
&= \int_0^{2\pi} \frac{1}{2}\big(x(t)y'(t) - y(t)x'(t)\big)\ dt \\
&= \frac{1}{2}\int_0^{2\pi} (\cos(t)\cos(t) - \sin(t)\,(-\sin(t)))\ dt \\
&= \frac{1}{2}\int_0^{2\pi} \big(\cos^2(t) + \sin^2(t)\big)\ dt \\
&= \frac{1}{2}\int_0^{2\pi} 1\,dt \\
&= \pi
\end{aligned}$$

which is obviously correct.

Now, we will show a sketch of the proof of the theorem we stated before here. It states that If $\vec{\mathbf{F}} = \langle P, Q\rangle$ is a differnetiable vector field defined on a simply connected domain D, then

$$\vec{\mathbf{F}} \text{ is conservative } \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Proof. We already know $\implies$ by Clairuat's Theorem.

Proving $\impliedby$. Suppose $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. We need to show that $\vec{\mathbf{F}}$ is conservative. This means it is enough to show that

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

for every closed curve C in D (as vector field is conservative), and it is enough to do this when C is simple and closed, as if the curve is not simple, you can divide it up into simple curves (similar to the proof of the equation above, where we seperated into simple curves and proved).

If C is simple then C bounds a region R. Since D is simply connected, $R \subset D$, which is true as D doesn't have any holes.

Now, by green's theorem,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C (P\,dx + Q\,dy) = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0 \text{ by assumption}} dA = 0$$

□

Interesting Non-Conservative Vector Field

Let there be a vector field such that

$$\vec{\mathbf{F}} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

In this example,

$$P_y = Q_x = \frac{-x^2+y^2}{\left(x^2+y^2\right)^2}$$

but $\vec{\mathbf{F}}$ is not conservative, which is a contradiction to what we just proved. Now why does our theorem not hold? This is as the domain of this vector field is not simply connected, and not defined at the origin.

This is not conservative because if C is the unit circle, with $x = \cos(t)$, $y = \sin(t)$, $0 \leq t \leq 2\pi$, then

$$\begin{aligned}
\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \frac{-y(t)}{x(t)^2+y(t)^2} x'(t)\,dt + \int_0^{2\pi} \frac{x(t)}{x(t)^2+y(t)^2} y'(t)\,dt \\
&= \int_0^{2\pi} \frac{\sin^2(t)+\cos^2(t)}{\cos^2(t)+\sin^2(t)}\,dt \\
&= \int_0^{2\pi} 1\,dt \\
&= 2\pi
\end{aligned}$$

This is just a note that just because $P_y = Q_x$, it doesn't immediately follow that $\vec{\mathbf{F}} = \langle P, Q \rangle$ is conservative.

This curve is special for other reasons too.

Let C be a closed curve in $\mathbb{R}^2$ not going through $(0,0)$, with

$$\vec{\mathbf{F}} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2\pi \cdot (\text{Winding number of } C \text{ around } (0,0))$$

where that's the amount of times it goes around the origin (always an integer).

This is true, as we parametrize C with domain $[a,b]$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle dt = \int_a^b \underbrace{\frac{-x'y+xy'}{x^2+y^2}}_{\text{"}\frac{\mathrm{d}\theta}{\mathrm{d}t}\text{"}}\,dt$$

How does the last part make sense? If we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$, then

$$\begin{aligned}
x' &= r'\cos(\theta) - r\sin(\theta)\theta' \\
y' &= r'\sin(\theta) + r\cos(\theta)\theta'
\end{aligned}$$

giving

$$\left.\begin{aligned}
-yx' &= -rr'\cos(\theta)\sin(\theta) + y^2\theta' \\
xy' &= rr'\cos(\theta)\sin(\theta) + x^2\theta'
\end{aligned}\right\} \implies \theta' = \frac{xy'-yx'}{x^2+y^2}$$

This means that the net change in θ is what we want, so if it goes around the origin twice, the change is $\theta = 2\pi \cdot (2)$.

Review More here and here.

Review — Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a vector field in two dimensions.

1. $\vec{\mathbf{F}}$ is conservative $\iff \displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for all closed curves C.
2. If $\vec{\mathbf{F}}$ is conservative then $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.
3. Green's Theorem If C is a simple closed curve in $\mathbb{R}^2$ and R is the region it bounds, then
$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$
4. If $\vec{\mathbf{F}}$ is defined on a simply connected domain and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, then $\vec{\mathbf{F}}$ is conservative.

The interpretation of $\displaystyle\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is the work of $\vec{\mathbf{F}}$ to go around C such that R the region is on the left of the parameterization of C. We can also think of it as the "circulation of $\vec{\mathbf{F}}$ around C".

The interpretation of $\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right)$ in $\displaystyle\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$ is the "local rotation" of $\vec{\mathbf{F}}$.

An example of this is suppose $\vec{\mathbf{F}} = \langle -y, x \rangle$, so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$. Over a square, on the right side its going upwards, on the top its going to the left, on the left its going down, and on the bottom its going right. This is kind of like a rotation, and the "total rotation" is kind of the difference of the partial derivativees.

§4.4 Curl and Divergence

Curl

Definition — Suppose $\vec{\mathbf{F}} = \langle P, Q, R \rangle$, a vector field on some domain in $\mathbb{R}^3$. Define a new vector field
$$\text{curl } \vec{\mathbf{F}} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Curl — The way to remember this long formula is consider the upside down triangle as
$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \qquad \vec{\mathbf{F}} = \langle P, Q, R \rangle$$

Then,
$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

The interpretation of this, is that the curl of $\vec{\mathbf{F}}$ measures "local rotation" of $\vec{\mathbf{F}}$, where the vector points in the "local axis of rotation", and the magnitude of the curl is "amount of local rotation". (More here).

Example — Suppose $\vec{\mathbf{F}} = \langle -y, x, 0\rangle$. Find the curl of $\vec{\mathbf{F}}$.

Solution.

$$\begin{aligned}\text{curl } \vec{\mathbf{F}} &= \nabla \times \vec{\mathbf{F}} \\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle -y, x, 0\rangle \\ &= \boxed{\langle 0, 0, 2\rangle}\end{aligned}$$

This has magnitude 2 and points in the z direction. □

Theorem

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is conservative (and partial derivs of P, Q, R defined, continuous) then $\nabla \times \vec{\mathbf{F}} = 0$.

Proof. Suppose $\vec{\mathbf{F}}$ is conservative. Then $\vec{\mathbf{F}} = \boldsymbol{\nabla} f$ for some function f. Then

$$\begin{aligned}\nabla \times \vec{\mathbf{F}} &= \nabla \times \boldsymbol{\nabla} f \\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\ &= \left\langle \underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial z}}_{0 \text{ by Clairaut's Theorem}} \right\rangle = 0\end{aligned}$$

□

Example — Is $\vec{\mathbf{F}} = \langle xz, \cos(y), z\rangle$ conservative?

Solution.

$$\begin{aligned}\nabla \times \vec{\mathbf{F}} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle xz, \cos(y), z\rangle \\ &= \langle 0, x, 0\rangle \\ &\neq 0\end{aligned} \tag{4}$$

which means $\vec{\mathbf{F}}$ is not conservaitve. □

Theorem

If curl $\vec{\mathbf{F}} = 0$ and $\vec{\mathbf{F}}$ is defined on a simply connected region, then $\vec{\mathbf{F}}$ is conservative.

Simply connected in $\mathbb{R}^3$ is a bit different, where $\mathbb{R}^3 \setminus \{(0,0,0)\}$ or $\mathbb{R}^3$ with the origin removed is also simply connected, but a donut shape is not.

Example — $\vec{\mathbf{F}} = \langle 2xyz, x^2z + z, x^2y + y\rangle$. Is $\vec{\mathbf{F}}$ conservative?

Solution.

$$\begin{aligned}\nabla \times \vec{\mathbf{F}} &= \left\langle \frac{\partial}{\partial y}(x^2y+y) - \frac{\partial}{\partial z}(x^2z+z), \frac{\partial}{\partial z}(2xyz) - \frac{\partial}{\partial x}(x^2y+y), \frac{\partial}{\partial x}\left(x^2z+z\right) - \frac{\partial}{\partial y}\left(2xyz\right)\right\rangle \\ &= \left\langle x^2+1-x^2+1, 2xy-2xy, 2xz-2xz\right\rangle \\ &= 0\end{aligned}$$

This means that $\vec{\mathbf{F}}$ is conservativie.

Find f with $\boldsymbol{\nabla} f = \vec{\mathbf{F}}$. As we have $f_x = 2xyz$, we can integrate out x to get $f = x^2yz + \underbrace{g(y,z)}_{f(0,y,z)}$. When we differentiate this equation wrt to y, we get that $f_y = x^2z + g_y = x^2z + z$, giving $g_y = z$ or $g = yz + h(z)$. Finally, this means $f = x^2yz + yz + h(z)$, which when differentiating, we have $f_z = x^2y + y + h'(z) = x^2y + y$, giving $h'(z) = 0$ or $h(z) = c$. This gives

$$\boxed{f = x^2yz + yz + c}$$

□

Example — $\vec{\mathbf{F}} = \langle y, z, x\rangle$. Is $\vec{\mathbf{F}}$ conservative?

Solution.

$$\nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}, \frac{\partial y}{\partial z} - \frac{\partial x}{\partial x}, \frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\right\rangle = \langle -1, -1, -1\rangle \neq 0$$

which means that $\vec{\mathbf{F}}$ is not conservative.

Lets try anyway to find a f such that $\boldsymbol{\nabla} f = \vec{\mathbf{F}}$.

$$f_x = y \implies f = xy + g(y,z)$$

$$f_y = z = x + g_y \implies g_y = z - x$$

which has a contradiction as g_y should only depend on y and z, but it has an x term in it. □

Divergence

Definition — Suppose $\vec{\mathbf{F}} = \langle P, Q, R\rangle$, a vector field on some domain in $\mathbb{R}^3$. Define a function

$$\operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle P, Q, R\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

By this definition,

$$\operatorname{div}\langle x, y, z\rangle = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\operatorname{div}\langle -x, -y, -z\rangle = -\operatorname{div}\langle x, y, z\rangle = 3$$

$$\operatorname{div}\langle -y, x, 0\rangle = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) = 0$$

Now, what do these vector fields look like?

The first vector field looks like a vector field pointing away from the orign, the further the greater the magnitude.

The second example looks exactly same vectors, but opposite direction, with the further the amgnitude.

The third vector field is a circular cylindaric shape kinda.

Curl and Divergence Intuition

Curl measures "rotation", divergence measures "expansion" and "compression". What does this exactly mean?

<ins>Curl</ins>

Well, to understand curl, let's first look at a simpler case of curl in 2d vector fields. In 2d, suppose that there is a curl there if you imagine the flow of a point along the vector field turning in a circle-like shape. This curl is positive if there's counterclockwise motion, and negative if clockwise motion. If there's no net rotation, then the curl is 0.

This leads to the the next case. First, we want to pretend like the output of the curl of a 2d case is a 3d vector field. Now, what does this vector field look like? It involves the right-hand rule, where if we imagine "curling" our right hand in the direction of the rotation in 2d, then the thumb points to the direction of the vector at that point. The more "curly" it is, the more magnitude, and vice versa. Note that this only describes the vector in the z direction, as your thumb will only point up and down.

Finally, we have a 3d vector field, where we want to find the curl of. To do this, we notice that the curl is, just as in the previous case, essentially describing a "rotation" of particles as they travel along the vector field.

If we take any point, and describe the rotation among that point, along with the right hand rule, then that gives the vector for the curl at that point, and the curl vector field is describing that rotation at every given point in space.

<ins>Divergence</ins>

Divergence, unlike curl, is describing the movement of particles around a neighborhood of a point. Imagine a neighborhood around a point, and as particles move around the vector field, if the amount of particles inward is the same outward, then the divergence would be 0.

The divergence would be positive if the points are "diverging" around that neighborhood, such as if the points are "moving outward" of that neighborhood, then the divergence is positive. If the points are "going inward", then the divergence is negative. Real life examples of this include the divergence around a "bomb exploding" would be positive, whereas an "implosion" would be negative.

Note that divergence takes in a vector field as an input, and outputs a scalar valued function, as it's taking in a point, and its describing whether a fluid tends to diverge away or to it, and by how much.

Relations Between Grad, Curl, and Div

Theorem

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field on a domain $D \in \mathbb{R}^3$ then $\text{div}(\text{curl}(\vec{\mathbf{F}})) = 0$.

Proof. As we have that $\text{curl}(\vec{\mathbf{F}}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$,

$$\begin{aligned}\text{div}(\text{curl}(\vec{\mathbf{F}})) &= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_x \\ &= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \\ &= 0\end{aligned}$$

□

Fact. If $\vec{\mathbf{F}}$ is a vector field on $\mathbb{R}^3$ and div $\vec{\mathbf{F}} = 0$ then $\vec{\mathbf{F}} =$ curl (some other vector field).

This can all be summed up in the following diagram.

$$\begin{tikzcd}
\text{Function} \arrow[d, "\text{gradient}"] \arrow[dd, bend right=60, "0"'] \\
\text{Vector field} \arrow[d, "\text{Curl}"] \arrow[dd, bend left=60, "0"] \\
\text{Vector field} \arrow[d, "\text{Divergence}"] \\
\text{Function}
\end{tikzcd}$$

Laplacian — If f is a function on $\mathbb{R}^3$, define the <u>Laplacian</u> $\Delta f = \nabla^2 f$ by

$$\nabla^2 f = \text{div}(\text{grad } f) = \nabla \cdot \boldsymbol{\nabla} f$$

$$\boldsymbol{\nabla} f = \langle f_x, f_y, f_z \rangle \qquad\qquad \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\implies \nabla \cdot \boldsymbol{\nabla} f = f_{xx} + f_{yy} + f_{zz}$$

A cool fact is if f is the electric potential then $\nabla^2 f = 0$ where there are no charges.

Alternate Statement of Green's Theorem — Let a surface R have a boundary curve C. Imagine $\vec{\mathbf{F}} = \langle P, Q, 0 \rangle$, so it's a 3d vector field. We can look at the vector $\vec{\mathbf{n}} = \langle 0, 0, 1 \rangle$, which is perpendicular to the surface R. The alternate statement of Green's Theorem is then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R \left(\text{curl } \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} \right) dA$$

To show that this is the same as before, we just have to show that

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \text{curl } \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}$$

which is true as

$$\text{curl } \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, 0 \rangle = \left\langle \dots, \dots, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Our goal, knowing this, is to generalize to other surfaces in 3d.

§4.5 Parametrized Surfaces and Surface Integrals

A surface needs two parameters, which we can call u and v. On the uv plane, the domain D maps, by $\vec{\mathbf{r}}$, to a surface S in 3d space. A point (u, v) in D gets mapped to a $\vec{\mathbf{r}}(u, v)$ in S. The horizontal and vertical lines also get mapped onto S.

> **Example** — What is the surface described by $x = r\sin(u)\cos(v)$, $y = r\sin(u)\sin(v)$, $z = r\cos(u)$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$ with r as a constant?

Solution. We can draw the rectangle on uv with sides π and 2π. We can eliminate the parameters with trig identities, so

$$\begin{aligned} x^2 + y^2 &= r^2\sin^2(u)\left(\cos^2(v) + \sin^2(v)\right) = r^2\sin^2(u) \\ z^2 &= r^2\cos^2(u) \end{aligned}$$

$$\implies x^2 + y^2 + z^2 = r^2$$

which means that this is a sphere of radius r centered at the origin.

The horizontal lines in the uv plane where $z =$ constant, those are the lines which go from top of sphere down to bottom of sphere, and the vertical lines where u is a constant correspond to the vertical lines around the circumpherence of the planes that intersects sphere horizontally. □

> **Example** — Same equations as before, except $0 \le u \le \frac{\pi}{2}$ and $0 \le u \le \pi$.

Solution. We're still going to get a sphere of radius r, but as we bounded it more, we won't get the whole surface, and instead just the upper hemisphere as $z \ge 0$. We have to pay attention to not only teh surface but the parameters. □

> **Example** — What's the surface described by $x = u\cos(v)$, $y = u\sin(v)$, $z = u$?

Solution. As there are no restrictions on the parameters, they can be all real numbers. We can eliminate parameters to find the surface, with

$$x^2 + y^2 = u^2\cos^2(v) + u^2\sin^2(v) = u^2 = z^2$$

which is the equation of a cone. We can draw the lines as $u = c$, which as v goes up by 2π, this'll give $z = c$ all the way around the cone.

If we look at lines where v is fixed, we see that this corresponds to lines that pass through the origin and are inside the cone. □

> **Example** — Parametrize the surface of revolution of $y = f(x)$, $a \le x \le b$, around the x-axis.

Solution. We can use $x = u$ as one of the parameters. Then, we have that for a fixed u, a circle with radius $f(u)$ is drawn in the plane $x = u$ centered at the origin. This gives motivation to write

$$\begin{aligned} y &= f(u)\cos(v) \\ z &= f(u)\sin(v) \end{aligned}$$

and, since we want to cover this exactly once, then $a \leq u \leq b$ (as we just subbed x for u) and $0 \leq v \leq 2\pi$ to draw out the whole circle.

The vertical lines in the uv planes correspond to the circumference of the circles drawn at the plane $x = u$, and the horizontal plane corresponds to values of the graph rotated around the surface. □

Tangent Plane To a Parametrized Surface at a Point

Let the surface be $\vec{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, and fixing (u, v), how do we find the tangent plane at $\vec{\mathbf{r}}(u, v)$?

There are two tangent vectors:

$$\vec{\mathbf{r}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$
$$\vec{\mathbf{r}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

These are tangent, as these are just the vectors that are tangent to the lines at v is a constant and u is a constant on the curve (partial derivatives property).

This parametrized surface is smooth at (u, v) if $\vec{\mathbf{r}}_u$, $\vec{\mathbf{r}}_v$ are linearly independent (one is not a scalar multiple of another), aka $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v \neq 0$.

Then, this means that the normal vector to the tangent plane is $\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$, and the equation for the thangent plane is $\vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$ where $\vec{\mathbf{r}}_0 = \vec{\mathbf{r}}(u, v)$.

Tangent Plane — With a vector on a surface $\vec{\mathbf{r}}$ as $\vec{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, and a normal

$$\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v \neq 0$$

with

$$\vec{\mathbf{r}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$
$$\vec{\mathbf{r}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

then the equation for the tangent plane is

$$\vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$$

Example — Find the tangent plaen to the surface $x = u^2$, $y = u - v^2$, $z = v^2$ with u, $v \geq 0$ at the point $(1, 0, 1)$.

Solution. Step 1: The point $(x, y, z) = (1, 0, 1)$ corresponds to $(u, v) = (1, 1)$.

Step 2: The tangent vectors $\vec{\mathbf{r}}_u$ and $\vec{\mathbf{r}}_v$ at $(u, v) = (1, 1)$ are

$$\vec{\mathbf{r}}_u = \langle 2u, 1, 0\rangle = \langle 2, 1, 0\rangle$$

$$\vec{\mathbf{r}}_v = \langle 0, -2v, 2v\rangle = \langle 0, -2, 2\rangle$$

This gives the normal vector as

$$\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 2, 1, 0\rangle \times \langle 0, -2, 2\rangle = \langle 2, -4, -4\rangle$$

As we started on the point $\vec{\mathbf{r}}_0 = \langle 1, 0, 1\rangle$, with $\vec{\mathbf{r}} = \langle x, y, z\rangle$, we have the equation as

$$\begin{aligned}\vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) &= \langle 2, -4, -4\rangle \cdot \langle x-1, y, z-1\rangle \\ &= \boxed{2(x-1) - 4y - 4(z-1) = 0}\end{aligned}$$

□

Example — Let's look at the cone $x^2 + y^2 = z^2$, which can be parametrized as $x = u\cos(v)$, $y = u\sin(v)$, and $z = u$. Let's calculate the normal vector.

We have that

$$\vec{\mathbf{r}}_u = \langle \cos(v), \sin(v), 1\rangle$$

$$\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), 0\rangle$$

Now, the cross product gives

$$\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -u\cos(v), -u\sin(v), u\rangle$$

When is this vector 0? This is only 0 when $u = 0$, which is when $z = 0$, so it has well defined tangent planes everywhere except at the origin, which makes sense as there's a singularity at the origin of a cone centered at the origin.

In general, you might choose a bad parameterization which leads the cross product to be 0, but if its well defined at that point, then you can always choose another parameterization. However, if it's a singularity, it will always be 0 for any parameterization.

Area of Parametrized Surface

To review, a parameterized surface has a domain D in the uv plane, which is mapped by $\vec{\mathbf{r}}$ to a surface in the xyz space. The idea for calculating the area is to divide the domain up into tiny rectangles, and look at their image in the surface, and add those tiny areas up.

If we look at a small rectangle, with side lengths Δu and Δv, this will map to approximately a parallelo gram in the xyz space (assuming its differentiable). As we want to find the area of this parallelogram, we need the edge vectors first.

One of the edge vectors is $(\Delta u)\vec{\mathbf{r}}_u$, and the other is $(\Delta v)\vec{\mathbf{r}}_v$. This gives the area of the parllelogram as

$$\text{Area } = |(\Delta u)\,\vec{\mathbf{r}}_u \times (\Delta v)\,\vec{\mathbf{r}}_v| = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\Delta u \Delta v$$

The area is now the limit as these rectangles become smaller, or

Area of Surface —

$$\text{Area } = \lim_{(\Delta u, \Delta v)\to(0,0)} \sum_{\text{Rectangles}} \frac{\text{Area (parlelogram)}}{|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\Delta u \Delta v} = \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA$$

Example — Let $S =$ unit sphere. What is the surface area?

Solution. We can parameterize this as $x = \sin(u)\cos(v)$, $y = \sin(u)\sin(v)$, $z = \cos(u)$, with $0 \le u \le \pi$, $0 \le v \le 2\pi$.

Then, we have

$$\vec{\mathbf{r}}_u = \langle \cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)\rangle$$

$$\vec{\mathbf{r}}_v = \langle -\sin(u)\sin(v), \sin(u)\cos(v), 0\rangle$$

which gives

$$\begin{aligned}\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v &= \left\langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u)\right\rangle \\ &= \sin(u)\langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\rangle \\ &= \sin(u)\langle x, y, z\rangle\end{aligned}$$

This gives

$$\begin{aligned}\text{Area}(S) &= \int_0^{\pi}\int_0^{2\pi} |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dv\, du \\ &= \int_0^{\pi}\int_0^{2\pi} \sin(u)\, dv\, du \\ &= 2\pi\int_0^{\pi} \sin(u)\, du \\ &= 2\pi\left(-\cos(u)\right)\Big|_{u=0}^{u=\pi} \\ &= 2\pi\left(-(-1)-(-1)\right) \\ &= \boxed{4\pi}\end{aligned}$$

□

Example — Let S = graph of $z = f(x, y)$ where $f : D \to \mathbb{R}$ (defined by $D \in \mathbb{R}^2$ in $\mathbb{R}$).

Solution. Let the parameterization be $x = u$, $y = v$, and $z = f(u, v)$ with $(u, v) \in D$.
Then, we have that

$$\begin{aligned}\vec{\mathbf{r}}_u &= \langle 1, 0, f_x\rangle \\ \vec{\mathbf{r}}_v &= \langle 0, 1, f_y\rangle\end{aligned}$$

which gives

$$\begin{aligned}\text{Area} &= \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA \\ &= \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA\end{aligned}$$

which is the same formula we got before. □

Integrals over Surfaces

Let S = parametrized surface in $\mathbb{R}^3$. This is described by the function $\vec{\mathbf{r}} : D \to \mathbb{R}^3$, where D is a domain in the uv plane, and a function $\vec{\mathbf{r}}$ takes each point in the uv plane to a surface S in the xyz space. There are two integrals to introduce,

Integration WRT Surface Area —

$$\iint_S f\, ds$$

where $f : S \to \mathbb{R}$ (f is a real valued function on surface).

This may be confusing, as the S below the integral is referring to the surface S, yet the dS is referring to integration wrt surface area, related to the lower case s in "integration wrt arc length", with some books calling this instead dA (notation varies).

Flux Integral —

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

where $\vec{\mathbf{F}}$ is a 3d vector field defined on S.

Let's first define Integration wrt surface area.

Definition —

$$\iint_S f\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))\underbrace{|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|}_{\substack{\text{magnification}\\ \text{factor}}}\,dA$$

This has the property that $\iint_S 1\,dS = \text{area}(S)$.

Mass — The geometric definition is if S is a sheet with mass density $\rho : S \to \mathbb{R}$, then $M = \iint_S \rho\,dS$ is the total mass of the sheet. The center of mass is just $\frac{1}{M}$ of this, analagous to before.

Example — Calculate

$$\iint_S yz\,dS$$

where S is given by $x = uv$, $y = u + v$, $z = u - v$, $u^2 + v^2 \leq 1$ (the unit disk).

Solution. Lets first calculate the magnification factor.

$$\vec{\mathbf{r}} = \langle uv, u+v, u-v\rangle$$

$$\vec{\mathbf{r}}_u = \langle v, 1, 1\rangle$$

$$\vec{\mathbf{r}}_v = \langle u, 1, -1\rangle$$

which gives

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -2, u+v, v-u\rangle$$

$$\implies |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| = \sqrt{4 + (u+v)^2 + (v-u)^2} = \sqrt{4 + 2(u^2+v^2)}$$

Now, this gives our integral as

$$\iint_S yz\,dS = \iint_D \underbrace{(u+v)}_{y}\underbrace{(u-v)}_{z}\underbrace{\sqrt{4+2(u^2+v^2)}}_{\substack{\text{magnification}\\ \text{factor}}}\,dA$$

$$= \iint_D u^2\sqrt{4+2(u^2+v^2)}\,dA - \iint_D v^2\sqrt{4+2(u^2+v^2)}\,dA$$

We can notice now that since its symmetric, we can do a change of variables of

$$u \to v \qquad\qquad v \to u$$

which will have a jacobian of

$$\frac{\partial(v,u)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

(so a magnification factor of 1), with the domain D staying the same. This converts one integral into the other, which means these two integrals are equal, giving

$$\iint_D u^2\sqrt{4+2(u^2+v^2)}\,dA - \iint_D v^2\sqrt{4+2(u^2+v^2)}\,dA = \boxed{0}$$

□

In general, this is a neat trick to use with a change of variables, where we noticed both integrals were symmetric so we can use change of variabels to prove that they are equal.

<u>Orientations of Surfaces</u>

Recall that $\displaystyle\int_C f\,ds$ doesn't depned on parametrization of curve, as long as there's no bactracking allowed.

However, $\displaystyle\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ depends on an <u>orientation</u>.

The integral wrt surface area $\displaystyle\int_S f\,dS$ does not depend on parametrization (as long as tehre's no backtracking).

However, the integral of a vector field on a surface $\displaystyle\int_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}}$ depends on an orientation of S.

Definition — An <u>orientation</u> of a surface S is a choice of a <u>unit normal vector</u> at each point, which varies continuously as you move along S.

This basically means, at any point on a surface, we have a tangent plane (assume that we're talking about a smooth surface). The unit normal vector is perpendicular to the tangent plane, and has length 1.

There are two choices, one going "up", and the other "down", that you can choose, and it should stay consistent throughout the surface.

Example — A graph of $z = f(x,y)$ has a distinguished "upward" orientation.

This means, the unit normal vector is pointing "upward" throughout the whole surface. This means, with the tangent vectors of the tangent planes being

$$\left\langle 1, 0, \frac{\partial z}{\partial x}\right\rangle \qquad\qquad \left\langle 0, 1, \frac{\partial z}{\partial y}\right\rangle$$

giving the unit normal vector as

$$\vec{\mathbf{n}} = \frac{\left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right\rangle}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}$$

Example — If S is the boundary of a solid region E, then S has a distinguished "outward" orientation.

This means, on the surface S, we ant to pick the unit normal vector that goes "outward".

Theorem

If S is <u>connected</u> (one piece), then S has either 2 or 0 orientations.

This is obvious, as you can switch the normal to go the other way. However, some don't have an orientation, such as the Mobius strip, so its called no orientable. This is as this mobius strip only has one side.

With orientation covered, now we can move on to integration of vector fields over a surface.

Integration of a Vector Field

Let S be an oriented surface, which means it has a chosen unit normal vector at each point. Let $\vec{\mathbf{F}}$ be a vector field, which is defined on the domain that includes S.

Definition — Define

$$\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iint\limits_S \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right) dS$$

This basically starts with a vector field, and ends with a double integral of a function with the element of surface area.

The physics interpretation is

$$\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \underline{\text{flux}} \text{ of } \vec{\mathbf{F}} \text{ through } S$$

A flux kinda means how much a vector field is flowing through S. For example, suppose there was a river, and the vector field will tell the velocity of the water at any point. Take a surface that's perpendicular to the river, and the flux will tell us at what rate the water will pass through the surface will be the flux.

At each point in the surface, the normal vector is dot producted with the velocity of the water, which is telling how much teh water is moving through, and this is integrated over the whole surface to give the total.

This integral of a vector field over a surface depends on the orientation, as the opposite orientation would be $\vec{\mathbf{n}} \to -\vec{\mathbf{n}}$, which would lead to

$$\iint\limits_S \left(\vec{\mathbf{F}} \cdot -\vec{\mathbf{n}}\right) = -\iint\limits_S \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right) dS$$

or could also be represented as

$$\iint\limits_{-S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = -\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$

How do we compute this? If $\vec{\mathbf{r}}$ has a smooth parameterization ($\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v \neq 0$), then the orientation is determined by

$$\vec{\mathbf{n}} = \frac{\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v}{|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|}$$

With this orientation,

$$\int_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iint_S \vec{\mathbf{F}} \cdot \left(\frac{\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v}{|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|} \right) dS$$
$$= \iint_D \vec{\mathbf{F}} \cdot \left(\frac{\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v}{|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|} \right) |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA$$

which cancels out to result in our final equation

Computing Integral of Vector Field over a Surface

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$$

with $\vec{\mathbf{F}}$ being the vector field, $\vec{\mathbf{r}}_u$, $\vec{\mathbf{r}}_v$ being the vector partial derivatives of the parameterization $\vec{\mathbf{r}}$.

To sum up, the three steps to compute this integral is

1. Choose a smooth parametrization
2. Check that the orientation is correct
3. Evaluate the integral

Example — Let S = triangle with vertices at $(0,0,0)$, $(1,0,1)$, $(1,1,2)$, oriented upward. Calculate

$$\iint_S \langle 3,4,5\rangle \cdot d\vec{\mathbf{s}}$$

Solution. We first need a parametrization of the surface. It's part of a plane, so we can write out the equation for the plane. S is the graph of funciton $g(x,y) = ax + by + c = x + y$ (by plugging in points and solving for a, b, and c). It is the graph of this over D where D is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. As this is a smooth surface, we have $\vec{\mathbf{r}}(x,y) = \langle x, y, x+y\rangle$.

Now, we have to check if this is the correct orientation.

$$\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y = \langle 1,0,1\rangle \times \langle 0,1,1\rangle = \langle -1,-1,1\rangle$$

and since the z term is positive, it is oriented upward (correct).

Finally, we can compute the double integral

$$\iint_S \langle 3,4,5\rangle \cdot d\vec{\mathbf{s}} = \iint_D \langle 3,4,5\rangle \cdot \langle -1,-1,1\rangle\ dA$$
$$= \iint_D (-2)\, dA$$
$$= -2\,(\text{Area}(D))$$
$$= \boxed{-1}$$

It's fine that this is negative, it just means the vector field is generally pointing downward wrt the surface. □

Example — More generally, calculate

$$\iint_S \langle P, Q, R\rangle \cdot d\vec{\mathbf{s}}$$

where S is the graph of $z = g(x,y)$ over D (with upward orientation, generally is unless otherwise specified).

Solution. We can do this as follows:

Parameterization: $\vec{\mathbf{r}}(x,y) = \langle x, y, g(x,y)\rangle$ with $(x,y) \in D$. This generally gives the correct orientation as

$$\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y = \langle 1, 0, g_x\rangle \times \langle 0, 1, g_y\rangle = \langle -g_x, -g_y, 1\rangle$$

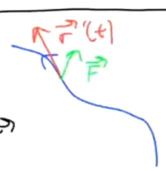
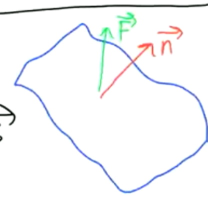
and as z is positive, it is upwardly oriented.

Then, we have the double integral as

$$\begin{aligned}\iint_S \langle P, Q, R\rangle \; d\vec{\mathbf{s}} &= \iint_D \langle P, Q, R\rangle \cdot \langle -g_x, -g_y, 1\rangle \; dA \\ &= \boxed{\iint_D (-Pg_x - Qg_y + R) \; dA}\end{aligned}$$

□

Summary More here and here.

Integrals over curves	Integrals over Surfaces	Properties
$\int_C f\,ds$ integration w.r.t. arc length	$\iint_S f\,dS$ integration w.r.t. surface area	does not depend on (bijective) parametrization
$\int_C f\,dx$, $\int_C f\,dy$	$\iint_S f\,dx\,dy$, $\iint_S f\,dx\,dz$, $\iint_S f\,dy\,dz$	depends on orientation, otherwise does not depend on (bijective) parametrization. Switching the orientation multiplies the integral by -1
$\int_C \vec F \cdot d\vec r$  measures circulation of $\vec F$ along C; use tangent vector to C	$\iint_S \vec F \cdot d\vec S$  measures flux across S; use normal vector to S	

§4.6 Stokes' Theorem

Let S be an oriented surface with boundary curve C. If C is oriented such that S is to the left of the parameterization C, then C is positively oriented (the unit normal vector to S is pointing upward). Can be seen with right hand rule.

Theorem (Stokes' Theorem)
Let S be a surface bounded by a positively oriented curve C. Let $\vec{\mathbf{F}}$ be a differentiable vector field defined on S. Then,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot d\vec{\mathbf{s}}$$

Proof. Let's start with a special case.

Let S be the graph of $z = g(x, y)$ over a domain D int eh xy plane (upward orientation) with $\vec{\mathbf{F}} = \langle P, Q, R\rangle$. Show that

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot d\vec{\mathbf{s}}$$

To do this, lets first evaluate the curl, which is

$$\nabla \times \vec{\mathbf{F}} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$$

The parameterization of S is $\vec{\mathbf{r}}(x, y) = \langle x, y, g(x, y)\rangle$ with $(x, y) \in D$. this means $\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y = \langle -g_x, -g_y, 1\rangle$. By green's,

$$\begin{aligned}\iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot d\vec{\mathbf{s}} &= \iint_D \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \langle -g_x, -g_y, 1\rangle \; dA \\ &= \iint_D \left(g_x\left(Q_z - R_y\right) + g_y\left(R_x - P_z\right) + \left(Q_x - P_y\right)\right) dA\end{aligned}$$

Now, evaluating the left side, we first need a parametrization for C.We can choose a paramtrization for $C_0 = \langle x(t), y(t)\rangle$ surrounding the domain where $a \le t \le b$, which gives a parametrizationf or C (which we do by taking every point above that. THis gives

$$C = \langle x(t), y(t), g(x(t), y(t))\rangle \qquad a \le t \le b$$

This gives

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b \langle P, Q, R\rangle \cdot \left\langle x', y', g_x x' + g_y y'\right\rangle \, dt \\ &= \int_a^b \left(Px' + Qy' + Rg_x x' + Rg_y y'\right) dt\end{aligned}$$

Turnign back to C_0 we can write $x'\,dt \to dx$ and $y'\,dt \to dy$ which gives

$$\begin{aligned}&= \int_{C_0} \left(\left(P + Rg_x\right) dx + \left(Q + Rg_y\right) dy\right) \\ &= \iint_D \left(\left(Q + Rg_y\right)_x - \left(P + Rg_x\right)_y\right) dA\end{aligned}$$

now, using the chain rule,

$$\begin{aligned}(Q+Rg_y)_x &= \frac{\partial}{\partial x}\left(Q(x,y,g(x,y))+R(x,y,g(x,y))g_y(x,y)\right)\\ &= Q_x+Q_zg_x+\left(R_x+R_zg_x\right)g_y+Rg_{yx}\end{aligned}$$

with the other part being

$$(P+Rg_x)_y = P_y+P_zg_y+\left(R_y+R_zg_y\right)g_x+Rg_{xy}$$

Now, subtracting (as does in the integrand) cancels, giving

$$\begin{aligned}&= \iint\limits_D \left((Q+Rg_y)_x-(P+Rg_x)_y\right)\,dA\\ &= \iint\limits_D \left(\left(Q_z-R_y\right)g_x+\left(R_x-P_z\right)g_y+\left(Q_x-P_y\right)\right)\,dA\end{aligned}$$

which agrees with the expression on teh other page, so these are equal in this special case.

Step 2 of the proof is

This also works if S is a graph with $y=g(x,z)$ or $x=g(y,z)$.

Step 3 is

For a general surface S, divide it into simpler surfaces S_i, each of which is a graph in step 1 or step 2. Now, we have

$$\begin{aligned}\iint\limits_S \left(\nabla\times\vec{\mathbf{F}}\right)\cdot d\vec{\mathbf{s}} &= \sum_i \iint\limits_{S_i}\left(\nabla\times\vec{\mathbf{F}}\right)\cdot d\vec{\mathbf{s}}\\ &= \sum_i \int\limits_{C_i}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\\ &= \int\limits_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\end{aligned}$$

as the parts of the curves that overlap cancel each other out. □

The physical interpretation of htis, is imagining $\vec{\mathbf{F}}$ as a velocity vector of a fluid, then the lhs is the ciculation of $\vec{\mathbf{F}}$ around the curve C. The rhs measures rotation of $\vec{\mathbf{F}}$, which is a vector pointing in the direction that depends on the rotation of $\vec{\mathbf{F}}$. As we're dotting it, we're only taking the normal to it, so it basically says if we take the rotation that is tangential to the surface, and integrate over the whole surface, then we get the circulation.

If the boundary of S has several curves, then Stokes' Theorem is true where on the lhs, you add up the integrals over all the boudnary curves.

Example — Given $\vec{\mathbf{F}}=\langle P,Q,0\rangle$ where P and Q depends only on x, y. Lets suppose $S=$ domain in the x, y plane with upward orientation ($\vec{\mathbf{n}}=\langle 0,0,1\rangle$). What does stokes' Theorem say?

Solution. We have

$$\nabla\times\vec{\mathbf{F}}=\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\times\langle P,Q,0\rangle=\langle\ldots,\ldots,Q_x-P_y\rangle$$

Stokes' Theorem says that

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint\limits_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{n}}\, dS$$
$$= \boxed{\iint\limits_S \left(Q_x - P_y\right)\, dA} \qquad \square$$

We can notice that this is green's theorem, if the surface is a flat surface. Stokes' theorem just generalizess this.

Example — What does stokes' theorem say if $\vec{\mathbf{F}}$ is consrevative?

Solution. Stokes' theroem says

$$\iint\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint\limits_S \left(\nabla \times \boldsymbol{\nabla} f\right) \cdot d\vec{\mathbf{S}}$$

and this is true as the lhs is 0 by FTLI, and the right side is 0 as teh curl of a conservtaive vector field is 0. $\square$

Theorem

If $\vec{\mathbf{F}}$ is defined on $\mathbb{R}^3$ and $\nabla \times \vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is conservative.

Proof. It's enough to show that

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

for all closed curves C. We can restrict our attention to the case where C is simple and the boundary of an oriented surface S. Then if we give C the positive orientation, by stokes' theorem,

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint\limits_S \underbrace{\nabla \times \vec{\mathbf{F}}}_{0} \cdot d\vec{\mathbf{s}} = 0$$

(only proved for 2d, but is true for all dimensions). $\square$

Example — Calculate $\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $C = \left(\vec{\mathbf{r}}(t) = \langle \cos(t), 0, \sin(t)\rangle, 0 \le t \le 2\pi\right)$ with $\vec{\mathbf{F}} = \left\langle \sin\left(x^3\right) + z^3, \sin\left(y^3\right), \sin\left(z^3\right) - x^3 \right\rangle$.

Solution. We know that $C =$ unit circle in the xz plane, so C is the boundary of S where $S =$ unit disk in teh xz plane. As C can be thought of as travelinging clockwise, that means S is oriented to the right for positive orientation.

This gives the normal vector as

$$\vec{\mathbf{n}} = \langle 0, -1, 0 \rangle$$

With this, by stokes' theorem

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iint\limits_{x^2+z^2\le 1} \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \langle 0, -1, 0 \rangle\, dA$$

$$= \iint\limits_{x^2+z^2\leq 1} -\left(\frac{\partial}{\partial z}\left(\sin\left(x^3\right)+z^3\right)-\frac{\partial}{\partial x}\left(\sin\left(z^3\right)-x^3\right)\right)\,dA$$
$$= \iint\limits_{x^2+z^2\leq 1} -3x^2-3z^2\,dA$$
$$= \int_0^{2\pi}\int_0^1 \left(-3r^2\right) r\,dr\,d\theta$$
$$= 2\pi\left(-\frac{3}{4}\right) = \boxed{\frac{-3\pi}{2}}$$

□

Example — Calculate $\int\limits_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}} = \left\langle \sin(\sin(z)), z^3, -y^3\right\rangle$ and C is the curve $x^2+y^2=1$, $z=x$, oriented counterclockwise when viewed from above.

Solution. This curve is an ellipse, , in the plane $z=x$. Let $S =$ the part of the plane $z=x$ bounded by C. The parameterization of this surface is $x=u$, $y=v$, and $z=u$, with $u^2+v^2\leq 1$. Now, $\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v = \langle 1,0,1\rangle\times\langle 0,1,0\rangle = \langle -1,0,1\rangle$, which points upwards so its giving us the correct orientation.

The curl of $\vec{\mathbf{F}}$ is

$$\nabla\times\vec{\mathbf{F}} = \left\langle -3y^2-3z^2,\ldots,0\right\rangle = \left\langle -3v^2-3u^2,\ldots,0\right\rangle$$

Now, by stokes theorem,

$$\int\limits_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \iint\limits_{u^2+v^2\leq 1}\left(\nabla\times\vec{\mathbf{F}}\right)\cdot\left(\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v\right)\,dA$$
$$= \iint\limits_{u^2+v^2\leq 1} 3(u^2+v^2)\,dA$$
$$= \int_0^{2\pi}\int_0^1 3r^2 r\,dr\,d\theta$$
$$= 2\pi\left(\frac{3}{4}\right) = \boxed{\frac{3\pi}{2}}$$

An alternate parametrization of S is $x=u\cos(v)$, $y=u\sin(v)$, with $0\leq u\leq 1$, $0\leq v\leq 2\pi$, and $z=x=u\cos(v)$. We can calulate the final answer to be teh same thing.

We can calculate $\vec{\mathbf{r}}_u = \langle\cos(v),\sin(v),\cos(v)\rangle$, and $\vec{\mathbf{r}}_v = \langle -u\sin(v), u\cos(v), -u\sin(v)\rangle$ giving $\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v = \langle -u,0,u\rangle$, which gives correct orientaiton.

Then, we have that

$$\int\limits_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \iint\limits_D \left\langle -3u^2,\ldots,0\right\rangle\cdot\langle -u,0,u\rangle$$

which leads to the same thing, so this is just another way to solve it. □

§4.7 The Divergence Theorem

Theorem (Divergence Theorem)

Let E be a bounded solid region in $\mathbb{R}^3$.

Let S be the boundary surface of E, with the outward orientation (unit normal vector points outward).

Let $\vec{\mathbf{F}}$ be a differentiable vector fiel defined on E. Then

$$\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iiint\limits_E \left(\text{div } \vec{\mathbf{F}}\right) dV$$

This is pretty intuitive, as the left hand side measures the net flux of $\vec{\mathbf{F}}$ out of S, and the divergence is positive if $\vec{\mathbf{F}}$ is expanding, and opposite if not. This is exactly the same, as if the divergence is positive, that means the flux is positive too (more stuff is going out).

Proof. Special Case A special case is $\vec{\mathbf{F}} = \langle 0, 0, R\rangle$, and E is the type of region where $E = \{(x, y, z) \mid (x, y) \in D, g_1(x, y) \le z \le g_2(x, y)\}$.

We can divide this E into three parts, the top, the bottom, and the side. Let the top be oriented up, side be to the side outward, and the bottom downward. This means

$$\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \left(\iint\limits_{\text{top}} - \iint\limits_{\text{bottom}} + \iint\limits_{\text{side}}\right) \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$

Now we have that

$$\begin{aligned}\iint\limits_{\text{top}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} &= \iint\limits_D \left\langle -\frac{\partial g_2}{\partial x}, \frac{\partial g_2}{\partial y}, 1\right\rangle \cdot \langle 0, 0, R\rangle \, dA \\ &= \iint\limits_D R(x, y, g_2(x, y))\, dA\end{aligned}$$

The bottom is the same, except its going to have a $g_1(x, y)$ in the final answer. The double integral over the side is just 0 as $\langle 0, 0, 1\rangle$ is tangent to the side, so it has no z component.

This means that

$$\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iint\limits_D R(x, y, g_2(x, y))\, dA - \iint\limits_D R(x, y, g_1(x, y))\, dA$$

Now, looking at the other side of the divergence formula, we have that (simplyfing as $\vec{\mathbf{F}}$ has only z component)

$$\begin{aligned}\iiint\limits_E \left(\text{div } \vec{\mathbf{F}}\right) dV &= \iiint\limits_E \frac{\partial R}{\partial z}\, dV \\ &= \iint\limits_D \int_{g_1(x,y)}^{g_2(x,y)} \frac{\partial R}{\partial z}(x, y, z)\; dz\, dA \\ &= \iint\limits_D R(x, y, g_2(x, y)) - R(x, y, g_1(x, y))\end{aligned}$$

by the FTC, which proves that they're equal. The other special cases are that E is between the graphs of two functions y of x, z, and $\vec{\mathbf{F}} = \langle 0, Q, 0\rangle$ and that E is between graphs on x of y, z, with $\vec{\mathbf{F}} = \langle P, 0, 0\rangle$.

The general case is proved is for general region E, divide it into "simple" regions, which are types of the previous special cases. Then, we know the divergence for each of teh tiny regions, E_i. For the total case, we have that

$$\iiint_E \left(\text{div } \vec{\mathbf{F}}\right) \, dV = \sum_i \iiint_{E_i} \left(\text{div } \vec{\mathbf{F}}\right) \, dV = \sum_i \iint_{S_i} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$

and since the interior boundary fluxes cancel out this is equal to

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$

□

This is similar to the proof of stokes theorem.

Example — Suppose that $\vec{\mathbf{F}} = \text{curl } \vec{\mathbf{G}}$. What does divergence theorem say?

Solution. The divergence theorem says that

$$\iint_S \left(\text{curl } \vec{\mathbf{G}}\right) \cdot d\vec{\mathbf{s}} = \iiint_E \text{div}(\text{curl } \vec{\mathbf{G}}) \, dV$$
$$\implies 0 = 0$$

The left hand side is 0, by stokes theorem, because S has no boundary, and the right hand side is 0 as divergence of curl is 0. □

Example — Suppose $\vec{\mathbf{F}}$ is a vector field on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ with div $\vec{\mathbf{F}} = \sqrt{x^2+y^2+z^2}$. Let S_1 be the sphere $x^2+y^2+z^2 = 1$, oriented outward. Let S_2 be the sphere $x^2+y^2+z^2 = 4$, oriented outward. Suppose

$$\iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 2\pi$$

Calculate

$$\iint_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$

Solution. The idea is to relate the flux over these two surfaces by finding the flux in between the larger sphere S_2 and the smaller one S_1. Call this region in between as $E = \{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 4\}$. Divergence theorem says that

$$\iiint_E \left(\text{div } \vec{\mathbf{F}}\right) \, dV = \iint_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} - \iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$

This is subtraction, as boht S_1 and S_2 are oriented out, so S_1 is technically oriented inward into E, yet divergence thm needs S_2 to be oriented outward, so it's negative.

To get the answer, we have

$$\iint_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 2\pi + \iiint_E \left(\text{div } \vec{\mathbf{F}}\right) \, dV$$

$$
\begin{aligned}
&= 2\pi + \iiint\limits_E \left(\sqrt{x^2+y^2+z^2}\right)\, dV \\
&= 2\pi + \int_0^\pi \int_0^{2\pi} \int_1^2 \rho\left(\rho^2 \sin(\phi)\right)\, d\rho\, d\theta\, d\phi \\
&= 2\pi + \int_0^\pi \int_0^{2\pi} \left(\frac{\rho^4}{4}\sin(\phi)\right)\Bigg|_{\rho=1}^{\rho=2} d\theta\, d\phi \\
&= 2\pi + \frac{15}{4}\int_0^\pi \int_0^{2\pi} \sin(\phi)\, d\theta\, d\phi \\
&= 2\pi + \frac{15\pi}{2}\int_0^\pi \sin(\phi)\, d\phi \\
&= 2\pi + 15\pi = \boxed{17\pi}
\end{aligned}
$$

□

The wrong approach to this problem is to apply divergence theorem to the ball $x^2+y^2+z^2 \leq 4$. If we do it this way, we have

$$
\iint\limits_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iiint\limits_B \left(\operatorname{div} \vec{\mathbf{F}}\right)\, dV
$$

which is similar to before, but now $0 \leq \rho \leq 1$, which leads to the wrong answer, and this is wrong as the vector field is defined on $\vec{\mathbf{F}}$ is not defined on the origin, yet this ball is including the origin.

In this example, we used divergence thm to relate the flux of two different surfaces. **PAY ATTENTION TO ORIENTATION; MUST BE OUTWARD!!!!**

Example — Calculate

$$
\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}
$$

where $\vec{\mathbf{F}} = \langle z, y, x\rangle$ and S is the upper hemisphere of the unit sphere $\left(x^2+y^2+z^2 = 1, z \geq 0\right)$, oriented upward.

Solution. First solution is direct calculation.

S = graph of $g(x,y) = \sqrt{1-x^2-y^2}$ over the unit disk. This gives

$$
\begin{aligned}
\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} &= \iint\limits_D \langle z, y, x\rangle \cdot \langle -g_x, -g_y, 1\rangle\, dA \\
&= \iint\limits_D \left\langle \sqrt{1-x^2-y^2}, y, x\right\rangle \cdot \left\langle \frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right\rangle dA \\
&= \iint\limits_D \left(x + \frac{y^2}{\sqrt{1-x^2-y^2}} + x\right) dA \\
&= \iint\limits_D \left(\frac{y^2}{\sqrt{1-x^2-y^2}}\right) dA \\
&= \int_0^{2\pi} \int_0^1 \frac{r^2 \sin^2(\theta)}{\sqrt{1-r^2}} r\, dr\, d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left(-3r^2\sqrt{1-r^2} - \frac{2}{3}\left(1-r^2\right)^{\frac{3}{2}} \sin^2(\theta) \right) \Big|_{r=0}^{r=1} d\theta \\
&= \frac{2}{3}\int_0^{2\pi} \sin^2(\theta)\, d\theta \\
&= \boxed{\frac{2\pi}{3}}
\end{aligned}
$$

Another solution is using divergence theorem.

If we call the upper hemisphere S and the shape it encloses with the xy plane E. What if we look at a certain S', which is the flat disk below on hte domain of S? Then, we have (as $z = 0$)

$$
\begin{aligned}
\iint_{S'} \langle z, y, x\rangle \cdot d\vec{\mathbf{s}} &= \iint_{D^2} \langle 0, y, x\rangle \cdot \langle 0, 0, 1\rangle \; dA \\
&= \iint_{D^2} x\, dA = 0
\end{aligned}
$$

Now, what is the actual thing we want to calculate? We want to find the double itnegral over S, which we notice is the difference between the whole surface of the whole object and S', so we can write

$$
\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} &= \left(\iint_{\text{surface of } E} + \underbrace{\iint_{S'}}_{0} \right) \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} \\
&= \iiint_E \operatorname{div} \langle y, z, x\rangle \; dV \\
&= \iiint_E 1\, dV \\
&= \text{volume of half ball} \\
&= \boxed{\frac{2\pi}{3}}
\end{aligned}
$$

□

In conclusion, if you want to evaluate a surface integral over a surface without boundary (boundary of a solid region), can use divergence theorem to integratee over interior. If has boundary, can replace inner by anotheer surface and fidn difference.

Two-Dimensional version of the Divergence Theorem

Theorem

Suppose that we have a region R surrounded by C, positively oriented. Suppose that we have $\vec{\mathbf{F}} = \langle P, Q\rangle$ (ddifferentiable) defined on all of R, so that div $\vec{\mathbf{F}} = P_x + Q_y$.

$$
\iint_R \left(\operatorname{div} \vec{\mathbf{F}} \right) dA = \int_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}} \right) ds
$$

where $\vec{\mathbf{n}}$ is an outwrad unit normal vector to C.

Proof. Parametrize C as $(x(t), y(t))$ with $\alpha \leq t \leq \beta$. A tangent velocity vector would be $\langle x'(t), y'(t)\rangle$, and a perpendicular vector would be $\langle y'(t), -x'(t)\rangle$ as the dot product is 0. This means that

$$\vec{\mathbf{n}} = \frac{\langle y', -x'\rangle}{\sqrt{(x')^2 + (y')^2}}$$

Then,

$$\begin{aligned}
\int\limits_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right) ds &= \int_\alpha^\beta \langle P, Q\rangle \cdot \frac{\langle y'(t), -x'(t)\rangle}{\sqrt{x'(t)^2 + y'(t)^2}} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\
&= \int_\alpha^\beta \left(Py' - Qx'\right) dt \\
&= \int\limits_C (P\, dy - Q\, dx) \\
&= \iint\limits_R (P_x - (-Q_y)) \\
&= \iint\limits_R \left(\operatorname{div} \vec{\mathbf{F}}\right) dA
\end{aligned}$$

$\square$

Comparing this in green's theorem,

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int\limits_C (P\, dx + Q\, dy) = \iint\limits_R (Q_x - P_y)\, dA$$

$$\int\limits_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right) ds = \int\limits_C (P\, dy - Q\, dx) = \iint\limits_R \left(P_x + Q_y\right) dA$$

§4.8 Review

Other review sections here and here.

FTLI —

$$\int\limits_C \nabla f \cdot d\vec{\mathbf{r}} = f(B) - f(A)$$

where C is a curve starting from A to B.

Green's Theorem —

$$\int\limits_C (P\, dx + Q\, dy) = \iint\limits_R (Q_x - P_y)\, dA$$

with R is bounded by C positively oriented.

Stokes' Theorem —

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint\limits_S \left(\text{curl } \vec{\mathbf{F}}\right) \cdot d\vec{\mathbf{s}}$$

where S is a curved surface in 3d orietned positively surrounded by C positively oriented.

Divergence Theorem —

$$\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iiint\limits_E \left(\text{div } \vec{\mathbf{F}}\right) dV$$

where E is a 3d surrounded by S oriented positively.

These all have the form of

$$\text{Integral over boundary } = \text{ Integral of derivatives over interior}$$

Theres a generalization called generalized stokes' therorem, which generalizes all of these over higher dimensions too. Now, what can we do with these?

Stokes' Theorem:

Find $\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ by choosing an oriented surface S with boundary C (closed) and integrating the curl $\vec{\mathbf{F}}$ over S.

can also do if C is not a closed curve, which we can replace with another more convenient curl, C', which is a surface, so we have

$$\int\limits_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint\limits_S \text{curl } \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} + \int\limits_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$

Divergence Theorem:

If S is a closed surface whichis the boundary of E, then

$$\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \pm \iiint\limits_E (\text{div } \vec{\mathbf{F}})\, dV$$

It's positve if S is oriented positive, negative otherwise.

If S isn't closed, we can look at S' which ahs the same boundary C, so we now have that (as it now covers a region E)

$$\iint\limits_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \iiint\limits_E (\text{div } \vec{\mathbf{F}})\, dV + \iint\limits_{S'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$