Assignment 5

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Download latex-tikz codes from

https://github.com/tyagio/AI1103/tree/main/assignment5/assignment5.tex

1 Problem

Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{F}, P)$ with E(X)= 1 .Let A $\in \mathcal{F}$ be an event with $0 < P(A) < 1$. Which of the following defines another probability measure on $(\Omega, \mathcal{F})$?

1) $Q(B) = P(A \cap B) \quad \forall B \in \mathcal{F}$
2) $Q(B) = P(A \cup B) \quad \forall B \in \mathcal{F}$
3) $Q(B) = E(XI_B) \quad \forall B \in \mathcal{F}$
4) Q(B) $=\begin{cases} P(A|B), & \text{if } P(B) > 0 \\ 0, & \text{if } P(B) = 0 \end{cases}$

2 Solution

Any probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function from $\mathcal{F}$ to [0,1] with the following properties

1) $\mathbb{P}(\phi) = 0$
2) $\mathbb{P}(\Omega) = 1$
3) If A_1, A_2.... are disjoint $\mathcal{F}$-measurable sets then $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

Now we know that $E(XI_B) \geq 0 \,\forall\, B$ as $X \geq 0$
also that for $Q(B) = E(XI_B) \quad \forall B \in \mathcal{F}$ to be valid probability measure

$$E(XI_B) = E(X \times 1) = 1 \text{ when } B = \Omega \tag{2.0.1}$$

$$E(XI_B) = E(X \times 0) = 0 \text{ when } B = \phi \tag{2.0.2}$$

$$\text{Let } B' = \bigcup_{i=1}^{\infty} A_i \text{ ,where } A_i \text{ are disjoint sets} \in \mathcal{F}$$

$$\implies E(XI_{B'}) = \sum_{i=1}^{\infty} E(XI_{A_i}) \tag{2.0.3}$$

Now to prove that (2.0.3) is true

$$E(XI_B) = \sum_{\omega \in \Omega} \frac{X(\omega) \times I_B(\omega)}{|\Omega|} \tag{2.0.4}$$

Now $I_B(\omega)$ is indicator function which is 1 if $\omega \in B$ and 0 if $\omega \notin B$ (2.0.5)

$$\implies E(XI_B) = \sum_{\omega \in B} \frac{X(\omega)}{|\Omega|} \tag{2.0.6}$$

$$\implies \sum_{i=1}^{\infty} E(XI_{A_i}) = \sum_{i=1}^{\infty} \sum_{\omega \in A_i} \frac{X(\omega)}{|\Omega|} \tag{2.0.7}$$

$$\implies \sum_{\omega \in B'} \frac{X(\omega)}{|\Omega|} = E(XI'_B) \tag{2.0.8}$$

This proves that (2.0.3) is true
Similarly to prove (2.0.1) and (2.0.2)

if $B = \phi$

$$\implies E(XI_B) = \sum_{\omega \in \Omega} \frac{X(\omega) \times 0}{|\Omega|} = 0 \tag{2.0.9}$$

Also if $B = \Omega$

$$\implies E(XI_B) = \sum_{\omega \in \Omega} \frac{X(\omega) \times 1}{|\Omega|} = E(X) = 1 \tag{2.0.10}$$

This proves that (2.0.1) and (2.0.2) are true Now we can state that

$$Q(B) = E(XI_B) \quad \forall B \in \mathcal{F}$$

is a well defined probability measure.
$\implies$ option 3 is correct answer.