Assignment 5

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Download latex-tikz codes from

https://github.com/tyagio/AI1103/tree/main/assignment5/assignment5.tex

1 Problem

Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{F}, P)$ with E(X)= 1 .Let A $\in \mathcal{F}$ be an event with $0 < P(A) < 1$. Which of the following defines another probability measure on $(\Omega, \mathcal{F})$?

1) $Q(B) = P(A \cap B) \quad \forall B \in \mathcal{F}$
2) $Q(B) = P(A \cup B) \quad \forall B \in \mathcal{F}$
3) $Q(B) = E(XI_B) \quad\quad \forall B \in \mathcal{F}$
4) Q(B) $= \begin{cases} P(A|B), & \text{if } P(B) > 0 \\ 0, & \text{if } P(B) = 0 \end{cases}$

2 Solution

Any probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function from $\mathcal{F}$ to [0,1] with the following properties

$$\mathbb{P}(\phi) = 0 \tag{2.0.1}$$

$$\mathbb{P}(\Omega) = 1 \tag{2.0.2}$$

If $A_1, A_2....$ are disjoint $\mathcal{F}$-measurable sets

$$\implies \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) \tag{2.0.3}$$

Now we know that $E(XI_B) \geq 0 \,\forall\, B$ as $X \geq 0$
also

$$E(XI_B) = E(X \times 1) = 1 \text{ when } B = \Omega \tag{2.0.4}$$

$$E(XI_B) = E(X \times 0) = 0 \text{ when } B = \phi \tag{2.0.5}$$

Let $B = \bigcup_{i=1}^{\infty} A_i$,where A_i are disjoint sets $\in \mathcal{F}$

$$\implies E(XI_B) = \sum_{i=1}^{\infty} E(XI_{A_i}) \tag{2.0.6}$$

(2.0.6) is true because Indicator function I_B is 1 when element within B and 0 otherwise due to this the summation in the RHS is equivalent to $E(XI_B)$ but calculated for smaller intervals where X is set to 0 when it is not in disjoint subinterval of B and then summed up over all subintervals to get same value as LHS.
Now we can state that

$$Q(B) = E(XI_B) \quad \forall B \in \mathcal{F}$$

is a well defined probability measure.
$\implies$ option 3 is correct answer.