CSIR UGC NET EXAM (Dec 2018), Q.49

Ojjas Tyagi - MA20BTECH1102/CS20BTECH11060

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Question

Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{F}, P)$ with E(X)= 1 .Let A $\in \mathcal{F}$ be an event with $0 < P(A) < 1$. Which of the following defines another probability measure on $(\Omega, \mathcal{F})$?

1. $Q(B) = P(A \cap B) \quad \forall B \in \mathcal{F}$
2. $Q(B) = P(A \cup B) \quad \forall B \in \mathcal{F}$
3. $Q(B) = E(XI_B) \quad \forall B \in \mathcal{F}$
4. Q(B) $= \begin{cases} P(A|B), & \text{if } P(B) > 0 \\ 0, & \text{if } P(B) = 0 \end{cases}$

Sample space Ω

Definition

Set of all possible outcomes of a random experiment

Event space $\mathcal{F}$

Events

Subsets of Ω which are of interest

σ-algebra

A collection $\mathcal{F}$ of subsets of Ω is called a σ-algebra if:

1. $\phi \in \mathcal{F}$
2. if $A \in \mathcal{F}$ then $A^C \in \mathcal{F}$
3. if $A_1, A_2, ...$ is a countable collection of subsets in $\mathcal{F}$,then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
   Last point also allows us to state that(Using De Morgan's law):
   if $A_1, A_2, ...$ is a countable collection of subsets in $\mathcal{F}$,then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$
   Subsets in $\mathcal{F}$ called $\mathcal{F}$-measurable sets

Measurable space

$(\Omega, \mathcal{F})$ is known as a measurable space

Measure

Definition

A measure is a function $\mu : \mathcal{F} \rightarrow [0, \infty)$ such that:

1. $\mu(\phi)$=0
2. if $A_1, A_2, ...$ is a countable collection of disjoint $\mathcal{F}$-measurable sets, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

Measure space

The triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space.
if $\mu(\Omega) < \infty$ then μ is called a finite measure.

Probability measure

A probability measure $\mathbb{P}$ on$(\Omega, \mathcal{F})$ is a measure with the special property:

$$\mathbb{P}(\Omega) = 1$$

Random variables & Borel sets

Borel sets

A Borel set is any set that can be formed from open sets through the operations of countable union, countable intersection, and complement on some set X.
Alternatively a borel set is a member of the borel σ algebra which is the smallest σ-algebra containing all open sets on some set X and is denoted by $\mathcal{B}(X)$.
i.e if $B \in \mathcal{B}(X)$ then B is a borel set in X.

$\mathcal{F}$-measurable functions

A function $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-measurable if for every $B \in \mathcal{B}(\mathbb{R})$,the pre-image $X^{-1}(B) \in \mathcal{F}$
Where $X^{-1}(B) = \{\omega \in \Omega | X(\omega) \in B\}$

Random variable

A random variable X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$

Solution

Probability measure

Any probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function from $\mathcal{F}$ to [0,1] with the following properties

1. $\mathbb{P}(\phi) = 0$
2. $\mathbb{P}(\Omega) = 1$
3. If $A_1, A_2 ....$ are disjoint $\mathcal{F}$-measurable sets then $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

Conditions

Now we know that $E(XI_B) \geq 0 \; \forall \; B$ as $X \geq 0$
also that for $Q(B) = E(XI_B) \quad \forall B \in \mathcal{F}$ to be valid probability measure

$$E(XI_B) = E(X \times 1) = 1 \text{ when } B = \Omega \tag{1}$$

$$E(XI_B) = E(X \times 0) = 0 \text{ when } B = \phi \tag{2}$$

$$\text{Let } B' = \bigcup_{i=1}^{\infty} A_i \text{ ,where } A_i \text{ are disjoint sets} \in \mathcal{F}$$

$$\implies E(XI_{B'}) = \sum_{i=1}^{\infty} E(XI_{A_i}) \tag{3}$$

Proof

Now to prove that (3) is true

$$E(XI_B) = \sum_{\omega \in \Omega} X(\omega)P(\omega) \times I_B(\omega) \tag{4}$$

Now $I_B(\omega)$ is indicator function which

is 1 if $\omega \in B$ and 0 if $\omega \notin B$ (5)

$$\implies E(XI_B) = \sum_{\omega \in B} X(\omega)P(\omega) \tag{6}$$

$$\implies \sum_{i=1}^{\infty} E(XI_{A_i}) = \sum_{i=1}^{\infty} \sum_{\omega \in A_i} X(\omega)P(\omega) \tag{7}$$

$$\implies \sum_{\omega \in B'} X(\omega)P(\omega) = E(XI'_B) \tag{8}$$

This proves that (3) is true

Similarly to prove (1) and (2)

$$\text{if } B = \phi$$
$$\implies E(XI_B) = \sum_{\omega \in \Omega} X(\omega)P(\omega) \times 0 = 0 \tag{9}$$
$$\text{Also if } B = \Omega$$
$$\implies E(XI_B) = \sum_{\omega \in \Omega} X(\omega)P(\omega) \times 1 = E(X) = 1 \tag{10}$$

This proves that (1) and (2) are true

Answer

Correct answer

Now using (1),(2) and (3) we can state that

$$Q(B) = E(XI_B) \quad \forall B \in \mathcal{F}$$

is a well defined probability measure. $\implies$ option 3 is correct answer.