Calc Final

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Surfaces

Surface area

Let T_u, T_v be tangents. Surface area is the sum of parallelogram areas.

$$SA(S) = \iint_S dS = \iint_D ||T_u \times T_v|| dudv$$

Surface area reparameterization

$$\iint_S f(\mathrm{x}) dS = \iint_D f(\Phi(u, v)) ||T_u \times T_v|| dudv$$

Surface orientation

If it exists, a surface orientation of $S \in \mathbb{R}^n$ is a choice of unit normal vector at each point of the surface so that the vectors vary continuously. At any point, although there are many options for normal vectors, there are only options for unit vectors (inside versus outside, or above versus below). E.g. a mobius strip doesn't have an orientation.

Flux (vector surface integral)

Measures vector field flow across surface. For $F : \mathbb{R}^3 \to \mathbb{R}^3$, $S = \Phi(D)$ where $\Phi : D \to \mathbb{R}^3$ a surface with a parameterization.

$$\iint_S F \cdot dS = \iint_D F(\Phi(u, v)) \cdot (T_u \times T_v) dudv$$

Green's theorem

$$\int_{\partial D} F \cdot \mathrm{ds} = \int_{\partial D} P(x, y)dx + Q(x, y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Where ∂D represents the boundary. Note that this is the scalar curl of F. Note that the boundary can be piecewise, in which case you just split the integral into pieces and sum them.

Area of a bounded region

$$A(D) = \frac{1}{2} \oint xdy - ydx$$

Circulation

$$\int_c F \cdot \mathrm{ds}$$

Stokes Theorem

Let S be a bounded piecewise smooth oriented surface in $\mathbb{R}^3$ where ∂S is a finite set of piecewise c1-closed curves each oriented consistently wrt S. For F a c1 vector field on S,

$$\iint_S (\nabla \times F) \cdot d\mathrm{S} = \oint_{\partial S} F \cdot \mathrm{ds}$$

If a surface has no boundary, the circulation is 0.

Equivalence

For F a c1 vector field on a simply connected domain such as $\mathbb{R}^3$ the following are equivalent:

- F is conservative (iff $F = \nabla f$)
- For c a simple closed curve, $\int_c F \cdot d\mathrm{s} = 0$
- For any 2 oriented simple curves with the same endpoints, $\int_{c_1} F \cdot d\mathrm{s} = \int_{c_2} F \cdot d\mathrm{s}$
- $\nabla \times F = 0$

Application

- Use LHS.
- Use RHS.
- Change to a different surface with the same boundary.

Gauss Theorem

Let D be a bounded solid region in $\mathbb{R}^3$ whose boundary consists of a finite number of piecewise smooth, closed orientable surfaces. Oriented so that the normals point away from D. Then, the flux across the boundary:

$$\iint_{\partial D} F \cdot d\mathrm{S} = \iiint_D (\nabla \cdot F) dV$$

Extrema

Definitions

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, $\mathrm{x}_0 \in U$.

- x_0 is local min of f iff
$$\exists\ \mathcal{N}(\mathrm{x}_0) \text{ s.t. } \forall\ \mathrm{x} \in \mathcal{N}(\mathrm{x}_0),\ f(\mathrm{x}) \geq f(\mathrm{x}_0)$$
- x_0 is a strict local min of f iff
$$\exists\ \mathcal{N}(\mathrm{x}_0) \text{ s.t. } \forall\ \mathrm{x} \in \mathcal{N}(\mathrm{x}_0),\ \mathrm{x} \neq \mathrm{x}_0\ f(\mathrm{x}) > f(\mathrm{x}_0)$$
- x_0 is an absolute min iff
$$\forall\ \mathrm{x} \in U,\ f(\mathrm{x}) \leq f(\mathrm{x}_0)$$
- Critical point - Either $Df(\mathrm{x})$ not defined or $Df(\mathrm{x}_0) = 0$.
- Extrema - Local mins or maxes.
- Saddle point - Critical point that is not an extremum.

A critical point is not necessarily extreme, but an extreme point is a critical

- If a point is critical, it may not be an extrema. E.g. saddle point.
- If f is differentiable at $\mathrm{x}_0 \in U$ and x_0 is extreme, then $Df(\mathrm{x}_0) = 0$ and x_0 is critical.

Extreme Value Theorem

- Bounded - set U is bounded iff
$$\forall\ \mathrm{x} \in U,\ \exists\ M \in \mathbb{R}^+, \|\mathrm{x}\| < M$$
In other words, we can draw a ball around it.
- Closed - set U is closed iff it contains interior points and boundary points.
- Level sets are closed and bounded.

Let U be closed and bounded.

$$f : U \subset \mathbb{R}^n \to \mathbb{R} \text{ is continuous} \implies f \text{ has global min and max on } U$$

Hessian

Let $f : \mathbb{R}^n \to \mathbb{R}$ be c2 at x_0. Then, the Hessian matrix of f at x_0 is the symmetric matrix,

$$Hf(\mathrm{x}_0) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right] \in \mathbb{R}^{n \times n}$$

The Hessian as a function of $\mathbb{R}^n$,

$$Hf(\mathrm{x}_0)(\mathrm{h}) = \frac{1}{2}\mathrm{h}^T Hf(\mathrm{x}_0)\mathrm{h}$$

Hessian is positive definite iff

- Hessian function returns positive.
$$\forall\ \mathrm{h} \in \mathbb{R}^n, \begin{cases} H(f)(\mathrm{x})(\mathrm{h}) \geq 0 & \mathrm{h} \neq 0 \\ Hf(\mathrm{x})(\mathrm{h}) = 0 & \mathrm{h} = 0 \end{cases}$$
- Or, determinant is positive and all of its diagonal sub-determinants are positive.

Positive definite hessian implies local min

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be c2 function at critical point $\mathrm{x}_0 \in D$.

$$Hf(\mathrm{x}_0) \text{ is PD} \implies \mathrm{x}_0 \text{ is a local min}$$

Lagrange Multipliers

Looks for constrained extrema by finding where gradient and level set are perpendicular. Recall that level sets are closed and bounded.

Single constraint

- Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, $g : U \subset \mathbb{R}^n \to \mathbb{R}$ be c1 functions.
- Let S_c be c-level set of g, and $\mathrm{x}_0 \in U$ and $g(\mathrm{x}_0) = c$.
- Let $f|S_c$ be f restricted to S_c.

Assume $\nabla g(\mathrm{x}_0) \neq 0$ (smooth),

$$\mathrm{x}_0 \text{ is local extrema on } f|S_c \iff \exists\ \lambda \text{ s.t. } \nabla f(\mathrm{x}_0) = \lambda \nabla g(\mathrm{x}_0)$$

Multiple constraints

Use multiple lambdas $\lambda_{1..k}$ for smooth constraints $g_{1..k}$.

$$\mathrm{x}_0 \text{ is local extrema on } f|S_c \iff \exists\ \lambda_1, \ldots, \lambda_k \text{ s.t. } \nabla f(\mathrm{x}_0) = \lambda_1 \nabla g_1(\mathrm{x}_0) + \cdots + \lambda_k \nabla g_k(\mathrm{x}_0)$$

Finding extrema

1. Check continuity
2. Find critical points on open interior (No constraints, $\nabla f(x) = 0$
3. Find all constrained critical points on boundary using Lagrange multipliers (Constraints, $\nabla f(x) = \lambda_i \nabla g_i(\mathrm{x})$)
4. Evaluate function at critical points and compare.

Parameterizations

A differential is an instantaneous change in a variable's value. E.g. for $t \in \mathbb{R}$, dt is instantaneous linear change in its value.

Physics

Let $c : [a, b] \to \mathbb{R}^n$ smooth enough s.t. derivatives exist,

- Velocity: $\mathrm{c}'(t)$
- Speed: $||\mathrm{c}'(t)||$
- Acceleration: $\mathrm{c}''(t)$
- Vector displacement: $d\mathrm{s} = [dx_i]dt \in \mathbb{R}^n$
- Scalar displacement: $ds = ||d\mathrm{s}|| \in \mathbb{R}$
- Arclength: $s(t) = \int_a^t ||c'(u)||du$
- Length w/ c: $L(\mathrm{c}) = \int_a^b ||\mathrm{c}'(t)||dt = \int_{\mathrm{c}} ds$
- Length w/ arclength: $L(\mathrm{c}) = \int_a^b s'(t)dt = s(b) - s(a) = s(b)$

If c is not c1 at a finite number of points, length can still be computed by breaking interval at nondifferentiable points, and integrate over differentiable intervals.

Vector fields

A vector field is a map $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$. In other words, assigns each $\mathrm{x} \in U$, a vector $F(\mathrm{x})$ based at $\mathrm{x} \in \mathbb{R}^n$.

Conservative

F is conservative iff $\exists\ f : \mathbb{R}^n \to \mathbb{R}$ a c1 function, where

$$F(\mathrm{x}) = \nabla f(\mathrm{x})$$

This is because $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ can be viewed as a vector field, aka a gradient field. Mixed partials must be equal (function can't be discontinuous).

Flow line

Curve $\mathrm{c} : \mathbb{R} \to \mathbb{R}^n$ is a flow line of F iff

$$\forall t \in \mathbb{R},\ \mathrm{c}'(t) = F(\mathrm{c}(t))$$

In other words, the velocity of a particle on the curve must be same as the vector field at that point.

Del operator

View ∇ as a function. $\nabla : (f : \mathbb{R}^n \to \mathbb{R}) \to (F : \mathbb{R}^n \to \mathbb{R}^n)$.

$$\nabla = \left[\frac{\partial}{\partial \mathrm{x}_1}, \ldots, \frac{\partial}{\partial \mathrm{x}_n}\right]^T$$

Divergence

Measures rate of expansion of volume at each point in a vector field. Results in scalar field, since it maps to $\mathbb{R}$.

$$div(F) = \nabla \cdot F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

If $= 0$ stays same size, if > 0 expanding, and if < 0 decreasing in size

Curl

Measures sense of rotation. Results in vector field, since it maps to $\mathbb{R}^n$.

$$curl(F) = \nabla \times F$$

Irrotational

Vector field F is irrotational iff

$$curl(F) = \nabla \times F = 0$$

- Gradient vector fields have zero curl and are thus irrotational ($\nabla \times (\nabla f) = 0$). Contrapositive, too.

Divergence of curl

$$div(curl(F)) = \nabla \cdot (\nabla \times F) = 0$$

- Curl of a vector field has no divergence.
- Nonzero divergence means not a curl field.

Multiple integration

Cavalier's principle

Take a solid $S \subset \mathbb{R}^n$, volume is the summing of areas of slices along one direction. Then, we can integrate again for the area.

$$V(S) = \int_a^b A(x)dx = \int_a^b \int_c^d f(x,y)dydx$$

Elementary region

Domain $D \in \mathbb{R}^2$ is elementary region if either,

- y-simple:
$$D = \{(x,y) \in \mathbb{R} \mid a < x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$$
- x-simple:
$$D = \{(x,y) \in \mathbb{R} \mid c < y \leq d, \phi_1(y) \leq x \leq \phi_2(y)\}$$

If both true, simple region.

Fubini's and double integrals

If simple region,

$$\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

If y-simple,

$$\int_R f(x,y)dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dydx$$

If x-simple,

$$\int_R f(x,y)dA = \int_c^d \int_{\phi_1(y)}^{\phi_2(y)} f(x,y)dxdy$$

When swapping from x-simple to y-simple, need to swap bounds as well. Sometimes, might become piecewise.

Change of variables

Map properties

Let T be a map $T : D^* \to D$,

- T is one-to-one (injective) iff
$$T(u_1, v_1) = T(u_2, v_2) \implies u_1 = u_2, v_1 = v_2$$
In other words, no two distinct points in domain map to the same point in image.

- T is onto (surjective) iff
$$T(D^*) = D$$
In other words, for every point in the image, there exist a point in the domain that map to that value.

Nonsingular linear transformations are bijective (one-to-one and onto). Let $\mathrm{A}_{2\times 2}$ be a nonsingular ($det(\mathrm{A}) \neq 0$), and $T : \mathbb{R}^2 \to \mathbb{R}$, $T(\mathrm{x}) = \mathrm{Ax}$.

Nice region to desired region

Idea: Able to map a nicer region D^* into the desired region D.
Let D^*, D be elementary regions in $\mathbb{R}^2$ and let $T : D^* \to D$ be a c1 transformation which is bijective at least on the interior of D^*, D. Then, for $f : D \to \mathbb{R}$,

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v), y(u,v))) \Big| \frac{\partial(x,y)}{\partial(u,v)} \Big| dudv$$

Where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ is the absolute value of the jacobian determinant (the determinant of the derivative matrix).
E.g. For polar coordinates, D^* is r, θ, and D is x, y. Jacobian determinant is $[cos(\theta) * rcos(\theta)] - [\sin(\theta) * -r\sin(\theta) = r$

Desired region to nice region

Idea: Able to map desired region D into a nicer region D^*. Instead of using inverse map, write f in terms of new mapping, and do the inverse Jacobian determinant.

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v), y(u,v)) \Big| \frac{\partial(x,y)}{\partial(u,v)} \Big|^{-1} dudv$$

Line integrals

Scalar line integral

Scalar line integral of $f : \mathbb{R}^n \to \mathbb{R}^n$ along $\mathrm{c} : [a,b] \to \mathbb{R}^n$

$$\int_{\mathrm{c}} f ds = \int_{\mathrm{c}} f(\mathrm{c}(t)) ||\mathrm{c}'(t)|| dt$$

Where ds is essentially arclength. In other words, area under a curve along the line, like the structure underneath a rollercoaster.

Vector line integral

Let F be a c0 vector field defined along a c1 curve $\mathrm{c} : [a,b] \to \mathbb{R}^n$. Then, the vector line interval of F along c is

$$\int_{\mathrm{c}} \mathrm{F} \cdot d\mathrm{s} = \int_a^b \mathrm{F}(\mathrm{c}(t)) \cdot \mathrm{c}'(t) dt$$

Vector line integral is scalar line integral

For a c1 path, $\mathrm{c} : [a, b] \to \mathbb{R}^n$,

$$\begin{aligned}
\int_{\mathrm{c}} \mathrm{F} \cdot d\mathrm{s} &= \int_a^b \mathrm{F}(\mathrm{c}(t)) \cdot \mathrm{c}'(t) dt \\
&= \int_a^b \mathrm{F}(\mathrm{c}(t)) \cdot \frac{\mathrm{c}'(t)}{||c'(t)||} * ||c'(t)|| dt \\
&= \int_a^b [\mathrm{F}(\mathrm{c}(t)) \cdot \mathrm{T}(t)] ||c'(t)|| dt \\
&= \int_a^b (\mathrm{F} \cdot \mathrm{T}) d\mathrm{s}
\end{aligned}$$

Applying FTC

If conservative, then $F = \nabla f$, then you can apply fundamental theorem of calculus

$$\int_{\mathrm{c}} \mathrm{F} \cdot d\mathrm{s} = f(c(b)) - f(c(a))$$

Reparameterization

Line reparameterization

Let $h : I \to J$ be a c1-real valued function that is one-to-one on $I = [a, b] \subset \mathbb{R}$ and onto $J = [c, d]$. For $\mathrm{c} : J \to \mathbb{R}^n$ a piecewise c1 curve, the composition $\mathrm{p} = \mathrm{c} \circ h : I \to \mathbb{R}^n$ is a reparameterization of c.

- Orientation preserving: $h(a) = c,\ h(b) = d$
- Orientation reversing: $h(a) = d,\ h(b) = c$

Integral of reparameterization

Let F be a c0 vector field and f a c0 function on a domain that contains a piecewise c1 curve $\mathrm{c} : [a, b] \to \mathbb{R}^n$. Let $\mathrm{p} : [c, d] \to \mathbb{R}^n$ be any reparameterization. Then,

$$\int_{\mathrm{c}} f d\mathrm{s} = \int_{\mathrm{p}} f d\mathrm{s}$$

$$\int_{\mathrm{c}} F \cdot d\mathrm{s} = \int_{\mathrm{p}} F \cdot d\mathrm{s} \text{ if p is orientation preserving}$$

$$\int_{\mathrm{c}} F \cdot d\mathrm{s} = -\int_{\mathrm{p}} F \cdot d\mathrm{s} \text{ if p is orientation reversing}$$

Simple, closed, one-to-one, onto

- One-to-one - no two points in domain map to the same point in image.

$$f(\mathrm{x}_1) = \mathrm{y}, f(\mathrm{x}_2) = \mathrm{y} \implies \mathrm{x}_1 = \mathrm{x}_2$$

- Onto - every point in image is mapped by domain.

$$\forall \ \mathrm{y} \in Im(f), \exists \ \mathrm{x} \text{ s.t. } f(\mathrm{x}) = \mathrm{y}$$

- Simple - $\mathrm{c} : [a, b] \to \mathbb{R}^n$ c0 curve is simple iff does not cross same point in $Im(\mathrm{c})$ more than once.
- Closed - $\mathrm{c} : [a, b] \to \mathbb{R}^n$ c0 curve is closed iff $\mathrm{c}(a) = \mathrm{c}(b)$.
- Simple and closed - closed, and simple on $[a, b)$.

Misc

Coordinates to remember

- Polar: $(r\cos\theta, r\sin\theta)$, r
- Cylindrical: $(r\cos\theta, r\sin\theta, z)$, r
- Spherical: $(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$, $r^2\sin\phi$