Opti Mid2

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1 Dual feasibility

$$\mathrm{A} \in \mathbb{R}^{m \times n}, \mathrm{x} \in \mathbb{R}^n, \mathrm{y} \in \mathbb{R}^m, \mathrm{c} \in \mathbb{R}^n, \mathrm{b} \in \mathbb{R}^m$$

1.1 Canonical form

$$LP = \min \mathrm{c}^T \mathrm{y} \quad \text{s.t.} \quad \mathrm{Ax} \geq \mathrm{b}, \mathrm{x} \geq 0$$
$$DP = \max \mathrm{b}^T \mathrm{y} \quad \text{s.t.} \quad \mathrm{A}^T \mathrm{y} \leq \mathrm{c}, \mathrm{y} \geq 0$$

1.2 Standard form

$$LP = \min \mathrm{c}^T \mathrm{y} \quad \text{s.t.} \quad \mathrm{Ax} = \mathrm{b}, \mathrm{x} \geq 0$$
$$DP = \max \mathrm{b}^T \mathrm{y} \quad \text{s.t.} \quad \mathrm{A}^T \mathrm{y} \leq c$$

1.3 Dual of dual is primal

- LP: $\min \mathrm{c}^T \mathrm{x}$ s.t. $\mathrm{Ax} \geq \mathrm{b}, \mathrm{x} \geq 0$
- DP(LP): $\max \mathrm{b}^T \mathrm{y}$ s.t. $-\mathrm{A}^T \mathrm{y} \geq -c, \mathrm{y} \geq 0$
- DP(DP(LP)): $\max -\mathrm{c}^T \mathrm{z}$ s.t. $(-\mathrm{A}^T)^T \mathrm{z} \geq -(-\mathrm{b})^T, \mathrm{z} \geq 0$
- $\min \mathrm{c}^T \mathrm{z}$ s.t. $\mathrm{Az} \geq \mathrm{b}, \mathrm{z} \geq 0$

1.4 Converting between forms

1.4.1 LP canonical to DP canonical through standard

LP canonical:

$$\min \mathrm{c}^T \mathrm{x} \quad \text{s.t. } \mathrm{Ax} \geq b, \mathrm{x} \geq 0$$

LP standard:

$$\min \begin{bmatrix} \mathrm{c} \\ 0 \end{bmatrix}^T \begin{bmatrix} \mathrm{x} \\ \mathrm{z} \end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix} \mathrm{A} & -I \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{z} \end{bmatrix} = b, \mathrm{x}, \mathrm{z} \geq 0$$

DP standard:

$$\max \mathrm{b}^T \mathrm{y} \quad \text{s.t.} \quad \begin{bmatrix} A^T \\ -I \end{bmatrix} \mathrm{y} \leq \begin{bmatrix} \mathrm{c} \\ 0 \end{bmatrix}$$

$$\max \mathrm{b}^T \mathrm{y} \quad \text{s.t. } \mathrm{A}^T \mathrm{y} \leq \mathrm{c}, -\mathrm{y} \leq 0$$

DP canonical:

$$\max \mathrm{b}^T \mathrm{y} \quad \text{s.t. } \mathrm{A}^T \mathrm{y} \leq \mathrm{c}, \mathrm{y} \geq 0$$

1.4.2 LP standard to DP standard through canonical

LP standard:

$$\min \mathrm{c}^T\mathrm{x} \ \text{ s.t. } \ \mathrm{Ax} = \mathrm{b}, \mathrm{x} \geq 0$$

LP canonical:

$$\min \mathrm{c}^T\mathrm{x} \ \text{ s.t. } \ \begin{bmatrix} \mathrm{A} \\ -\mathrm{A} \end{bmatrix} \mathrm{x} \geq \begin{bmatrix} \mathrm{b} \\ -\mathrm{b} \end{bmatrix}, \mathrm{x} \geq 0$$

DP canonical:

$$\max \begin{bmatrix} \mathrm{b} \\ -\mathrm{b} \end{bmatrix}^T \begin{bmatrix} \mathrm{u} \\ \mathrm{v} \end{bmatrix} \ \text{ s.t. } \ \begin{bmatrix} \mathrm{A} \\ -\mathrm{A} \end{bmatrix}^T \begin{bmatrix} \mathrm{u} \\ \mathrm{v} \end{bmatrix} \leq \mathrm{c}, \mathrm{u}, \mathrm{v} \geq 0$$

DP standard: Let $\mathrm{y} = \mathrm{u} - \mathrm{v}$

$$\max \mathrm{b}^T(\mathrm{u} - \mathrm{v}) \ \text{ s.t. } \ \mathrm{A}^T(\mathrm{u} - \mathrm{v}) \leq \mathrm{c}$$
$$\max \mathrm{b}^T\mathrm{y} \ \text{ s.t. } \ \mathrm{A}^T\mathrm{y} \leq \mathrm{c}$$

2 Optimality checklist

1. Basic: $\mathrm{Ax} = \mathrm{b}$
2. Feasibility: $\mathrm{x} \geq 0$
3. Dual: $\mathrm{b}^T\mathrm{y} \leq \mathrm{c}$
4. Supervisor: $\mathrm{b}^T\mathrm{y} = \mathrm{c}^T\mathrm{x}$

If we have all these conditions, then we have optimality by supervisor principle.

2.1 Weak duality

Given (LP), (DP). Suppose x is feasible in (LP) and y is feasible in (DP). Then,

$$ofv_{DP} \leq ofv_{LP}$$

Proof standard: Since x, y feasible in (LP), (DP), respectively, $\mathrm{Ax} = \mathrm{b}, \mathrm{x} \geq 0$ and $\mathrm{A}^T\mathrm{y} \leq \mathrm{c}$, respectively. Thus,

$$\begin{aligned} \mathrm{b}^T\mathrm{y} &= (\mathrm{Ax})^T\mathrm{y} && \text{Since (1) } \mathrm{Ax} = \mathrm{b} \\ &= \mathrm{x}^T\mathrm{A}^T\mathrm{y} \\ &\leq \mathrm{x}^T\mathrm{c} && \text{Since (3) } \mathrm{A}^T\mathrm{y} \leq c \\ &= \mathrm{c}^T\mathrm{x} \end{aligned}$$

Proof symmetric: Since x, y feasible in (LP), (DP), respectively, $\mathrm{Ax} \leq \mathrm{b}, \mathrm{x} \geq 0$ and $\mathrm{A}^T\mathrm{y} \leq \mathrm{c}$, respectively. Thus,

$$\begin{aligned} \mathrm{b}^T\mathrm{y} &\leq (\mathrm{Ax})^T\mathrm{y} && \text{Since (1) } \mathrm{Ax} \leq \mathrm{b} \\ &= \mathrm{x}^T\mathrm{A}^T\mathrm{y} \\ &\leq \mathrm{x}^T\mathrm{c} && \text{Since (3) } \mathrm{A}^T\mathrm{y} \leq c \\ &= \mathrm{c}^T\mathrm{x} \end{aligned}$$

2.2 Supervisor Principle

Given x feasible in (LP), y feasible in (DP). If $ofv(x) = ofv(y)$ ($\mathrm{b}^T\mathrm{y} = \mathrm{c}^T\mathrm{x}$), then x is optimal in (LP) and y is optimal in (DP).

$$\begin{aligned} \mathrm{c}^T\mathrm{x} &= \mathrm{c}_B^T\mathrm{x}_B + \mathrm{c}_N^T\mathrm{x}_N \\ &= \mathrm{c}_B^T\mathrm{x}_B \\ &= \mathrm{c}_B^T(\mathrm{B}^{-1}\mathrm{b}) \\ &= (\mathrm{c}_B^T\mathrm{B}^{-1})\mathrm{b} \\ &= \mathrm{y}^T\mathrm{b} \end{aligned}$$

2.2.1 Corollary

If (LP) is unbounded, then (DP) is infeasible.

2.3 Strong duality

If (LP) is feasible and objective function is bounded below, then (LP) has an optimal solution and (DP) has an optimal solution. Furthermore, their oofv are equal ($\mathrm{b}^T\mathrm{y} = \mathrm{c}^T\mathrm{x}$).
Proof: At optimality of standard LP,

$$\begin{aligned} -\mathrm{r}_N^T &\leq 0 \\ r_N^T &\geq 0 \\ \mathrm{c}_\mathrm{B}^T\mathrm{B}^{-1}\mathrm{A} - \mathrm{c}^T &\geq 0 \\ \mathrm{c}_\mathrm{B}^T\mathrm{B}^{-1}\mathrm{A} &\geq \mathrm{c}^T \end{aligned}$$

Choose $\mathrm{y}^{*T} = \mathrm{c}_B^T\mathrm{B}^{-1}$ as potential solution for (DP).

1. We know that y^* is dual feasible (3) because of how we chose it,

$$\begin{aligned} \mathrm{c}_\mathrm{B}^T\mathrm{B}^{-1}\mathrm{A} &\geq \mathrm{c}^T \\ \mathrm{y}^{*T}\mathrm{A} &\geq \mathrm{c}^T \end{aligned}$$

2. We know that y^* is optimal by the supervisor principle.

$$\begin{aligned} \mathrm{x} &= \begin{bmatrix} \mathrm{B}^{-1}\mathrm{b} \\ 0 \end{bmatrix} \\ \mathrm{c}^T\mathrm{x} &= \mathrm{c}^T\mathrm{B}^{-1}\mathrm{b} \\ \mathrm{c}^T\mathrm{x} &= \mathrm{y}^T\mathrm{b} \\ \mathrm{c}^T\mathrm{x} &= \mathrm{b}^T\mathrm{y} \end{aligned}$$

2.4 Complementary slackness for standard

Suppose x is feasible in (LP) and y feasible in (DP). Then x is optimal in (LP) and y is optimal in (DP) iff dual slack is complementary to x. That is,

$$(c - \mathrm{A}^T\mathrm{y}) \cdot \mathrm{x} = 0$$

Proof: Recall from weak duality,

$$\begin{aligned} \mathrm{b}^T\mathrm{y} &= (\mathrm{Ax})^T\mathrm{y} \\ &= \mathrm{x}^T\mathrm{A}^T\mathrm{y} \\ &\leq \mathrm{x}^T\mathrm{c} \\ &= \mathrm{c}^T\mathrm{x} \end{aligned}$$

Note that x, y are respectively optimal in (LP), (DP) iff $\mathrm{b}^T\mathrm{y} = \mathrm{c}^T\mathrm{x}$ because of strong duality in the forward direction, and supervisor in the reverse direction. In order for this to be true from weak duality, we need for $\mathrm{x}^T\mathrm{A}^T\mathrm{y} \leq \mathrm{x}^T\mathrm{c}$ to be true. Putting them both on one side and factoring out x, we see that this is true iff $0 = \mathrm{x}^T(\mathrm{c} - \mathrm{A}^T\mathrm{y})$

2.5 Complementary slackness for symmetric

Suppose x is feasible in (LP) and y feasible in (DP). Then x is optimal in (LP) and y is optimal in (DP) iff dual slack is complementary to x. That is,

$$(c - \mathrm{A}^T\mathrm{y}) \cdot \mathrm{x} = 0 \text{ and } (A\mathrm{x} - b) \cdot \mathrm{y} = 0$$

Proof: Recall from weak duality,

$$\begin{aligned} \mathrm{b}^T\mathrm{y} &\leq (\mathrm{Ax})^T \\ &= \mathrm{x}^T\mathrm{A}^T\mathrm{y} \\ &\leq \mathrm{x}^T\mathrm{c} \\ &= \mathrm{c}^T\mathrm{x} \end{aligned}$$

Note that x, y are respectively optimal in (LP), (DP) iff $\mathrm{b}^T\mathrm{y} = \mathrm{c}^T\mathrm{x}$ because of strong duality in the forward direction, and supervisor in the reverse direction. In order for this to be true from weak duality, we need for $\mathrm{x}^T\mathrm{A}^T\mathrm{y} \leq \mathrm{x}^T\mathrm{c}$ and $b^T\mathrm{y} \leq (\mathrm{Ax})^T\mathrm{y}$ to be true. Putting them both on one side and factoring out x, we see that this is true iff $0 = \mathrm{x}^T(\mathrm{c} - \mathrm{A}^T\mathrm{y})$ and $0 = \mathrm{y}^T(\mathrm{Ax} - \mathrm{b})$

3 Sensitivity

Consider optimal tableau for LP. Then, adjust by small enough Δb.

- Assume nondegeneracy (otherwise, would need to use dual simplex).
- If $\mathrm{B}^{-1}(\mathrm{b} + \Delta\mathrm{b}) \geq 0$, bfs will remain feasible and optimal.

Each unit of change in b changes oofv by y for each element:

$$\Delta oofv = \mathrm{y}^{*T}\Delta\mathrm{b}$$

$$\partial oofv/\partial b_i = \partial y_i$$

4 Dual simplex

4.1 Tableau

Primal linear program:

$$\min \mathrm{c}^T\mathrm{x}, \mathrm{Ax} \geq \mathrm{b}, \mathrm{x} \geq 0$$

Convert min to max, add slack:

$$\max -\mathrm{c}^T\mathrm{x}, \mathrm{Ax} - \mathrm{Iz} = \mathrm{b}, \mathrm{x} \geq 0$$

Covnvert slack to identity for starting columns:

$$\max -\mathrm{c}^T\mathrm{x}, -\mathrm{Ax} + \mathrm{Iz} = -\mathrm{b}, \mathrm{x} \geq 0$$

1	-c	0	0
0	-A	I	-b

4.2 Algorithm

1. If the top is ≤ 0, and the right is ≥ 0, we are done. While this is not true, keep looping.
2. Take the most negative (smallest) entry in the rightmost column.
3. Considering negative entries in that row, choose the smallest ratio of top row to that row. If no entries are negative, then DP infeasible. If ratio is 0, then it is stalling.
4. Row reduction.
5. Repeat.
6. To write out solution, 0 if not basis. follow 1 to right hand side if basis. Can either include slack or not, depends on whether you are doing canonical or standard form.

Note: Reasoning behind step 3. Say that minimum ratio is α_*/β_*. α_0, β_0 old values, and α_1, β_1 new values for the current column. For each column,

$$\alpha_1 = -\beta_0 * \alpha_*/\beta_* + \alpha_0$$

Starting with the fact that we know that this is the smallest ratio,

$$\frac{\alpha_*}{\beta_*} \leq \frac{\alpha_0}{\beta_0}$$

$$\frac{\alpha_*\beta_0}{\beta_*} \leq \alpha_0$$

$$0 \geq \alpha_0 - \frac{\alpha_*\beta_0}{\beta_*}$$

Therefore, the top row will remain nonpositive.

4.3 Optimality

Dual slack:

$$c - A^T y$$

At optimality:

$$\begin{aligned}
A^T y &\le c \\
0 &\le c - A^T y \\
0 &\le c^T - y^T A \\
&= c^T - y^T A \\
&= \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} - (c_B^T B^{-1}) \begin{bmatrix} B & N \end{bmatrix} \\
&= \begin{bmatrix} 0 & c_N^T - c_B^T B^{-1} N \end{bmatrix} \\
&= \begin{bmatrix} 0 & r_N^T \end{bmatrix}
\end{aligned}$$

5 Comparing primal/dual

5.1 Choosing algorithms

- If right is positive and top is negative, optimal!
- If right is positive and top mixed or positive, then primal simplex.
- If top is negative, and right is negative or mixed, then dual simplex.
- If both mixed, use 2 phase or big M to try to find basis.

5.2 Maintenance

- In both, (1) and (4) are given.
- In primal simplex, maintain (2) $x \ge 0$ by pivoting, work towards (3) $A^T y \le c$ by going from basis to basis. In other words, in primal simplex, we maintain right as positive, and terminate when top is negative.
- In dual simplex, maintain (3) $A^T y \le c$ by pivoting, work towards (2) $x \ge 0$ by going from basis to basis. In other words, in dual simplex, we maintain the top as negative, and terminate when the right is positive.

5.3 Special conditions

- LP, DP both bounded.
- If LP is unbounded, then DP is infeasible.
- If DP is unbounded, then LP is infeasible.
- LP, DP infeasible.

Note that infeasible does not imply unbounded, since both can be infeasible.

- DP unbounded if pivot row is all positive.
- LP unbounded if pivot column is all negative.