P4.

Step 1

$$\int_a^b g''(x)h''(x)\,dx = g''(x)h'(x)\Big|_a^b - \int_a^b g'''(x)h'(x)\,dx$$

$g''(a) = g''(b) = 0$ and $g'''(x)$ is piecewise constant on (a,b)

and so $$\int_a^b g''(x)h''(x)\,dx = 0 - \int_a^b g'''(x)h'(x)\,dx$$

$$= 0 - g'''(x_1^+)\int_{x_1}^{x_2} h'(x)\,dx$$

$$+ \cdots$$

$$= 0 - \sum_{j=1}^{n-1} g'''(x_j^+)\int_{x_j}^{x_{j+1}} h'(x)\,dx$$

$$= 0 - \sum_{j=1}^{n-1} g'''(x_j^+)\left[h(x_{j+1}) - h(x_j)\right]$$

This is 0 because $h(x_j) = 0$ for all j

as $\tilde{g}(x_j) = g(x_j)$ (they both interpolate all points)

Step 2.

$$\int_a^b (\tilde{g}''(t))^2 dt = \int_a^b (h''(t) - g''(t))^2 dt$$

$$= \int_a^b (h''(t))^2 dt + \int_a^b (g''(t))^2 - 2 \cdot \int_a^b h''(t) g''(t) dt$$

$$-2 \int_a^b h''(t) g''(t) dt = 2 \sum_{j=1}^{n-1} g'''(x_j^+) [h(x_{j+1}) - h(x_j)]$$

$$= \int_a^b (h''(t))^2 dt + \int_a^b (g''(t))^2 dt - 2 \times 0$$

$$\geq \int_a^b (g''(t))^2 dt$$

and equality holds when $\int_a^b (h''(t))^2 dt = 0$

This means that h is either a constant on $[a,b]$ or h is a linear function on $[a,b]$, but since $h(a) = 0 = h(b)$, we must have that h's slope is 0 and so $h = 0$ on $[a,b]$

Step 3

I think this should be easy to argue. Let's say any f that is the minimizer isn't a natural cubic spline. Note that f doesn't need to interpolate all the existing points, and in either case, we can always set up a natural cubic spline to interpolate all $(x_1, f(x_1))$, $(x_2, f(x_2))$ .... $(x_n, f(x_n))$ and so its squares term would be equal to that of f and its penalty term $\leq$ that of f.

Thus, doesn't hold, and f must be a natural cubic spline to be the minimizer. This completes the proof.