AMATH 503: Homework 3
Due April, 29 2019
ID: 1064712

Trent Yarosevich

April 29, 2019

Instructor: Ka-Kit Tung

(1)

(a) Using the formula for a complex Fourier series in the notes we have the following in a domain of $-2 < x < 2$ with $n = 0, \pm 1, \pm 2, \pm 3...$

$$|x| = \sum_{-\infty}^{\infty} c_n e^{\frac{-in\pi x}{2}} \tag{1}$$

Similarly, we obtain the coefficient c_n with the following:

$$c_n = \frac{1}{4}\int_{-2}^{2} |x| e^{\frac{in\pi x}{2}} dx \tag{2}$$

To solve this, we split the integral into two domains and integration by parts:

$$\begin{aligned} c_n &= \frac{1}{4}\Big[\int_0^2 xe^{\frac{in\pi x}{2}}dx - \int_{-2}^0 xe^{\frac{in\pi x}{2}}dx\Big] \\ u &= x\ ,\ du = 1 \\ dv &= e^{\frac{in\pi x}{2}} \\ v &= \frac{2}{in\pi}e^{\frac{in\pi x}{2}} \\ uv = \int vdu &= \frac{2x}{in\pi}e^{\frac{in\pi x}{2}} - \frac{4}{n^2\pi^2}e^{\frac{in\pi x}{2}} \\ c_n &= \frac{1}{4}\left[\Big[\frac{2xe^{\frac{in\pi x}{2}}}{in\pi} + \frac{4e^{\frac{in\pi x}{2}}}{n^2\pi^2}\Big]\Big|_0^2 - \Big[\frac{2xe^{\frac{in\pi x}{2}}}{in\pi} + \frac{4e^{\frac{in\pi x}{2}}}{n^2\pi^2}\Big]\Big|_{-2}^0\right] \\ c_n &= \frac{1}{4}\left[\Big[\frac{4e^{in\pi}}{in\pi} + \frac{4e^{in\pi}}{n^2\pi^2} - \frac{4}{n^2\pi^2}\Big] - \Big[\frac{4}{n^2\pi^2} + \frac{4e^{-in\pi}}{in\pi} + \frac{4e^{-in\pi}}{n^2\pi^2}\Big]\right] \\ c_n &= \frac{-in\pi e^{in\pi} + in\pi e^{-in\pi} + e^{in\pi} + e^{-in\pi} - 2}{n^2\pi^2} \end{aligned} \tag{3}$$

Now we observe that:

$$\begin{aligned} e^{in\pi} &= cos(n\pi) + isin(n\pi) = cos(n\pi) = (-1)^n \\ e^{-in\pi} &= cos(n\pi) - isin(n\pi) = cos(n\pi) = (-1)^n \\ e^{in\pi} &= e^{-in\pi} \end{aligned} \tag{4}$$

This allows for the following simplification:

$$c_n = \frac{2(-1)^n - 2}{n^2\pi^2} \tag{5}$$

This is obviously not defined for $n = 0$ however, and so we calculate this independently:

$$\begin{aligned} c_0 &= \frac{1}{4}\int_{-2}^2 |x|dx \\ c_0 &= \frac{1}{4}\Big[[\frac{x^2}{2}]\Big|_0^2 - [\frac{x^2}{2}]\Big|_{-2}^0\Big] = 1 \end{aligned} \tag{6}$$

Putting all this togther, we arrive at the following Fourier series, noting that the even modes are zero (except c_0), and that $n = \pm 1, \pm 2, \pm 3...$

$$\boxed{|x| = 1 + \frac{2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n - 1}{n^2} e^{-\frac{in\pi x}{2}}} \tag{7}$$

(b) Using Euler's Formula we can rewrite the exponential above in the following way:

$$e^{-\frac{in\pi x}{2}} = cos(\frac{n\pi x}{2}) - isin(\frac{n\pi x}{2}) \tag{8}$$

If we observe that sin is an odd function, it follows that $isin(\frac{-n\pi x}{2}) = -isin(\frac{n\pi x}{2})$. Additionally, the n term in the divisor is squared, so it is no different for $\pm n$. This means that when $n < 0$, all the sin values in the sum will cancel with their corresponding values from $n > 0$, for example when $n = 1$ we have $-isin(\frac{\pi x}{2})$ and when $n = -1$ we have $isin(\frac{n\pi x}{2})$.
Similarly, we can observe that cosine is an even function and $cos(nx) = cos(-nx)$. Thus we can change our sum to be from $0 < n < \infty$ and multiply the sum by 2. Lastly, we can also observe that the even modes are zero, substituting $n = 2k-1$ and define our sum with $k = 1, 2, 3....$ This means $c_n = \frac{-4}{n^2\pi^2}$ as well. The resulting cosine series after discarding the sines, changing the range of n, eliminating even modes, and multiplying by 2, is:

$$\boxed{|x| = 1 - \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{cos(\frac{(2k-1)\pi x}{2})}{(2k-1)^2}} \tag{9}$$

(c)

$$\frac{d}{dx}[1 - \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{cos(\frac{(2k-1)\pi x}{2})}{(2k-1)^2}] = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{sin(\frac{n\pi x}{2})}{2k-1} \tag{10}$$

This is the sine series for $f(x) = 1$ in a domain of $0 < x < 2$. To show this, we simply observe that if $f(x) = 1$, our sine series coefficient is:

$$\begin{aligned}
a_n &= \int_0^2 sin(\frac{n\pi x}{2}) \\
a_n &= \Big[-\frac{2}{n\pi} cos(\frac{n\pi x}{2}) \Big] \Big|_0^2 \\
a_n &= \frac{2 - 2(-1)^n}{n\pi} \\
&\text{Even modes are zero, thus:} \\
f(x) &= \sum_{n=1}^{\infty} \frac{4}{n\pi} sin(\frac{n\pi x}{2}) \\
f(x) &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{sin(\frac{(2k-1)\pi x}{2})}{2k-1}
\end{aligned} \tag{11}$$

This is the same as the result of taking our derivative, which makes sense since we know the derivative of $|x|$ is -1 from $-\infty < x < 0$ and 1 from $0 < x < \infty$. Furthermore, we know that a fourier sine series for a non-periodic function will only be accurate in the domain $0 < x < L$, and because it's an odd function, will be reflected accordingly outside that domain. Thus we would expect the Fourier series for the derivative of $|x|$ to be accurate from $-L < x < L$ and be $2L$ periodic outside this. The resulting function would be the sine series of $f(x) = 1$ and that is precisely what we got.

(2) We first define the fourier transforms in the different dimensions:

$$\begin{aligned}
\mathcal{F}_1[u(x,y,z,t)] &= \int_{-\infty}^{\infty} u(x,y,z,t,)e^{i\omega_1 x} dx \\
\mathcal{F}_2[u(x,y,z,t)] &= \int_{-\infty}^{\infty} u(x,y,z,t,)e^{i\omega_2 y} dy \\
\mathcal{F}_3[u(x,y,z,t)] &= \int_{-\infty}^{\infty} u(x,y,z,t,)e^{i\omega_3 z} dz
\end{aligned} \tag{12}$$

We can then fully define u in fourier space:

$$\mathcal{F}_1[\mathcal{F}_2[\mathcal{F}_3[u(x,y,z,t)]]] = U(\omega_1,\omega_2,\omega_3,t) \tag{13}$$

Now defining the terms in the PDE, in particular using the definition of a derivative in Fourier space from the notes:

$$\begin{aligned}
\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3[u_t] &= \frac{d}{dt}U(\omega_1,\omega_2,\omega_3,t) \\
\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3[u_x x] &= (i\omega_1)^2 U(\omega_1,\omega_2,\omega_3,t) \\
\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3[u_y y] &= (i\omega_2)^2 U(\omega_1,\omega_2,\omega_3,t) \\
\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3[u_z z] &= (i\omega_3)^2 U(\omega_1,\omega_2,\omega_3,t)
\end{aligned} \tag{14}$$

We can now write out the full PDE in Fourier space:

$$\frac{d}{dt}U = -D(\omega_1^2+\omega_2^2+\omega_3^2)U \tag{15}$$

We then separate variables and integrate with respect to t, letting $D=\alpha^2$:

$$\begin{aligned}
\frac{dU}{U} &= -\alpha^2(\omega_1^2+\omega_2^2+\omega_3^2)dt \\
lnU &= -\alpha^2(\omega_1^2+\omega_2^2+\omega_3^2)t + C \\
U &= Ae^{-\alpha^2(\omega_1^2+\omega_2^2+\omega_3^2)t}
\end{aligned} \tag{16}$$

Applying the initial condition, we have:

$$\begin{aligned}
A &= U(\omega_1,\omega_2,\omega_3,0) \\
U(\omega_1,\omega_2,\omega_3,0) &= \mathcal{F}_1\mathcal{F}_2\mathcal{F}_3\delta(x)\delta(y)\delta(z)
\end{aligned} \tag{17}$$

As we learned in class, the integral across a delta function is equal to 1 because it is summing the area under an infinitely small Riemann rectangle. Thus we have:

$$U = e^{-\alpha^2(\omega_1^2+\omega_2^2+\omega_3^2)t} \tag{18}$$

To apply the reverse transform, we can split U up into three exponentials and apply the reverse transform to each one:

$$U = e^{-\alpha^2\omega_1^2 t}e^{-\alpha^2\omega_2^2 t}e^{-\alpha^2\omega_3^2 t}$$

$$\begin{aligned} u(x,y,z,t) = & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\omega_1^2 t}e^{i\omega_1 x}d\omega_1 \\ & * \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\omega_2^2 t}e^{i\omega_1 y}d\omega_2 \\ & * \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\omega_3^2 t}e^{i\omega_1 z}d\omega_3 \end{aligned} \tag{19}$$

These integrals can be re-written in the form:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\omega_1^2 t + i\omega_1 x}d\omega_1 \tag{20}$$

Using the general formula for Euler's Integral, taken from the class notes section (8.2), and noting that t is being treated as a constant here since we are integrating in respect to the omegas. The formula is as follows:

$$\int_{-\infty}^{\infty} e^{p^2\omega^2 \pm q\omega}d\omega = \frac{\sqrt{\pi}}{p}e^{\frac{q^2}{4p^2}} \tag{21}$$

Applying this to each integral, we get the answer:

$$\boxed{\begin{aligned} u(x,y,z,t) &= (\frac{1}{2\pi})^3(\frac{\sqrt{\pi}}{\alpha\sqrt{t}})^3 e^{\frac{-x^2}{4\alpha^2 t}}e^{\frac{-y^2}{4\alpha^2 t}}e^{\frac{-z^2}{4\alpha^2 t}} \\ u(x,y,z,t) &= \frac{1}{8\pi^3})(\frac{\pi^{\frac{3}{2}}}{\alpha^3 t^{\frac{3}{2}}})e^{-\frac{(x^2+y^2+z^2)}{4\alpha^2 t}} \\ u(x,y,z,t) &= \frac{1}{8\alpha^3 t^{\frac{3}{2}}\pi^{\frac{3}{2}}}e^{-\frac{(x^2+y^2+z^2)}{4\alpha^2 t}} \end{aligned} \quad (22)}$$