AMATH 503: Homework 1
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ID: 1064712

Trent Yarosevich

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Instructor: Ka-Kit Tung

(1) I'll begin by doing separation of variables to find a general solution, assuming it is of the form $u(x,t) = X(x)T(x)$:

$$\begin{aligned} u_x &= T(t)X'(x) \\ u_{xx} &= T(t)X''(x) \\ u_t &= T'(t)X(x) \\ u_{tt} &= T''(t)X(x) \end{aligned} \tag{1}$$

Plugging this into the PDE and isolating terms based on variable we get:

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = K \tag{2}$$

As we've learned in class, we know $K > 0$ and $K = 0$ will give trivial solutions, so let $K = -\lambda^2$, yielding two ODEs. Begining with the x-dependent ODE we have, by ansatz:

$$\begin{aligned} \frac{X''(x)}{X(x)} &= -\lambda^2 \\ X''(x) + \lambda^2 X(x) &= 0 \\ X(x) &= Asin(\lambda x) + Bcos(\lambda x) \end{aligned} \tag{3}$$

Now applying the BCs:

$$\begin{aligned} X(0) = Asin(0) + Bcos(0) &= 0 \\ B &= 0 \\ X(L) = Asin(\lambda L) &= 0 \\ \lambda L &= n\pi \\ \lambda &= \frac{n\pi}{L} \\ X(x) = X_n(x) &= sin(\frac{n\pi x}{L}) \end{aligned} \tag{4}$$

Moving on to the time dependent ODE:

$$\begin{aligned}\frac{T''(t)}{c^2T(t)} &= -\lambda^2 \\ T''(t) + c^2\lambda^2 T(t) &= 0 \\ T(t) &= Asin(\lambda ct) + Bcos(\lambda ct) \\ \text{Let } \omega_n &= c\lambda_n \\ T(t) = T_n(t) &= A_n sin(\omega_n t) + B_n cos(\omega_n t)\end{aligned} \tag{5}$$

Combining the two ODEs, absorbing the A_n from the x-dependent ODE into the A_n and B_n from the t-dependeont one, and using the principle of superposition, we have a sum of possible linear combinations of different solutions:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \big(A_n sin(\omega_n t) + B_n cos(\omega_n t)\big) sin(\frac{n\pi x}{L}) \tag{6}}$$

We now apply the initial conditions, so that we can determine the arbitrary constants, beginning with the IC: $u(x,0) = f(x)$.

$$u(x,0) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L}) = f(x) \tag{7}$$

This is a sine series, so we can use the formula derived in class and in the notes using the orthogonality of the basis to determine B_n:

$$B_n = \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \tag{8}$$

We render this in a piecewise integral given the $f(x)$ provided in the prompt:

$$\begin{aligned}B_n &= \frac{2}{L}\Big[\int_0^L xsin\frac{n\pi x}{L}dx + \int_0^L (L-x)sin\frac{n\pi x}{L}dx\Big] \\ B_n &= \frac{2}{L}\Big[\int_0^L xsin\frac{n\pi x}{L}dx + \int_0^L Lsin\frac{n\pi x}{L}dx - \int_0^L xsin\frac{n\pi x}{L}d\Big]\end{aligned} \tag{9}$$

First I will use integration by parts to determine an indefinite integral of $xsin(\frac{n\pi x}{L})$.

$$\begin{aligned}u &= x, du = 1 \\ dv &= sin(\frac{n\pi x}{L}) \\ v &= -\frac{L}{n\pi}cos(\frac{n\pi x}{L}) \\ uv - \int vdu &= \frac{-Lx}{n\pi}cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}sin\frac{n\pi x}{L}\end{aligned} \tag{10}$$

Substituting this into the formula for B_n we have:

$$\begin{aligned} B_n = \frac{2}{L}\Big[[\frac{-Lx}{n\pi}cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}sin\frac{n\pi x}{L}]\Big|_0^{\frac{L}{2}} - [\frac{cos(\frac{n\pi x}{L})}{n\pi}]\Big|_{\frac{L}{2}}^{L} \\ -[\frac{-Lx}{n\pi}cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}sin\frac{n\pi x}{L}]\Big|_{\frac{L}{2}}^{L} \\ B_n = \frac{2}{L}\Big[\Big(\frac{L^2}{n^2\pi^2}sin(\frac{n\pi}{2})\Big) + \Big(\frac{-L^2(-1)^n}{n\pi}\Big) - \Big((-1)^n(\frac{-L^2}{n\pi}) - \frac{L^2}{n^2\pi^2}sin(\frac{n\pi}{2})\Big)\Big] \\ B_n = \frac{2}{L}\Big[\frac{2L^2}{n^2\pi^2}sin(\frac{n\pi}{2})\Big] \end{aligned} \tag{11}$$

$$\boxed{B_n = \frac{4L}{n^2\pi^2}sin(\frac{n\pi}{2})} \tag{12}$$

To apply the other initial condition we must note that we can move the differential operator inside the summation is a linear combination of well-behaved functions.

$$\begin{aligned} u_t(x,t) = \frac{d}{dt}\Big[\sum_{n=1}^{\infty}\big(A_n sin(\omega_n t) + B_n cos(\omega_n t)\big)sin(\frac{n\pi x}{L})\Big] \\ u_t(x,t) = \sum_{n=1}^{\infty} A_n\omega_n cos(\omega_n t)sin(\frac{n\pi x}{L}) - B_n\omega_n sin(\omega_n t)sin(\frac{n\pi x}{L}) \\ u_t(x,0) = \sum_{n=1}^{\infty} A_n\omega_n sin(\frac{n\pi x}{L}) = 0 \end{aligned} \tag{13}$$

Here we can observe that in order to satisfy the initial condition across all values of x, the arbitrary constant A_n must be equal to zero. Thus we can now plug the values A_n and B_n from **(12)** above and the general solution from **(6)** to obtain a solution that satisfies all the Boundary and Initial Conditions:

$$\boxed{u(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{sin(\frac{n\pi}{2})}{n^2}cos(\omega_n t)sin(\frac{n\pi x}{L})} \tag{14}$$

(2)

(a) If the time derivative is equal to zero, we have:

$$\begin{aligned} \alpha^2 u_{xx} - bu = 0 \\ u_{xx} = \frac{b}{\alpha^2}u \end{aligned} \tag{15}$$

I simply noted by inspection that the solution must be an exponential with a positive or negative coefficient in front of x equal to the inverse of $\sqrt{\frac{b}{\alpha^2}}$, thus when multiplied through twice by taking the derivative, we satisfy the PDE. Any linear combination of such terms works, thus:

$$u(x) = c_1 e^{\frac{\sqrt{b}}{\alpha}x} + c_2 e^{-\frac{\sqrt{b}}{\alpha}x} \tag{16}$$

We then apply the boundary conditions:

$$\begin{aligned} u(0) = c_1 + c_2 &= 0 \\ c_1 &= -c_2 \\ u(L) = c_1 e^{\frac{\sqrt{b}}{\alpha}L} - c_1 e^{-\frac{\sqrt{b}}{\alpha}L} &= 0 \\ c_1 e^{\frac{\sqrt{b}}{\alpha}L} &= c_1 e^{-\frac{\sqrt{b}}{\alpha}L} \end{aligned} \tag{17}$$

Because the two exponentials are different, the only way to satisfy this equation is if $c_1 = c_2 = 0$, which is perfectly sensible since we would expect a rod that is dissipating heat to eventually reach a zero temperature.

(b) We proceed by separation of variables, assuming that the solution is of the form $u(x,t) = X(x)T(t)$. ****Note** that I will reclaim the results from **(1)** and **(3) / (4)** above. Substituting these into the PDE gives:

$$\frac{T''(t)}{T(t)\alpha^2} + \frac{b}{\alpha^2} = \frac{X''(x)}{X(x)} = K \tag{18}$$

Again let $K = -\lambda^2$ since we know other values give trivial solutions. We also already know the solution for $X(x)$ from above, thus we turn to the t-dependent ODE:

$$\begin{aligned} \frac{T''(t)}{T(t)\alpha^2} + \frac{b}{\alpha^2} &= -\lambda^2 \\ T'(t) + bT(t) + \lambda^2\alpha^2 T(t) &= 0 \\ T'(t) + (\lambda^2\alpha^2 + b)T(t) &= 0 \\ T'(t) &= -(\lambda^2\alpha^2 + b)T(t) \end{aligned} \tag{19}$$

Now let $c = -(\lambda^2\alpha^2 + b)$ and we have an easily solved ODE using an exponential:

$$\begin{aligned} T(t) &= \frac{1}{c}e^{ct} \\ T_n(t) &= A_n e^{ct} \\ n &= 1, 2, 3... \end{aligned} \tag{20}$$

Utilizing this $T_n(t)$ and the result from part one for the $X(x)$ term and combining arbitrary constants, we thus have an infinite number of solutions given by $u(x,t) = X_n(x)T_n(t)$:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\alpha^2\lambda^2+b)t} sin(\frac{n\pi x}{L}) \tag{21}$$

As in sections **(3) / (4)** above, this already satisfies the BCs since they are the same in part 2. The initial condition is given by:

$$u(x,0) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L}) \tag{22}$$

This is a sine series and its coefficient can be determined for any $f(x)$ in the domain $0 < x < L$ with the formula derived in class and given in the notes:

$$A_n = \frac{2}{L}\int_0^L f(x) sin(\frac{n\pi x}{L}) \tag{23}$$

Thus our solution satisfies the PDE, IC and BCs. To finally answer the prompt, we take the limit as $t \to \infty$:

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} A_n e^{-(\alpha^2\lambda^2+b)t} sin(\frac{n\pi x}{L}) \tag{24}$$

We can see that the exponential will go to zero, and the steady state solution is zero as $t \to \infty$, which is the same as part **(a)**.