AMATH 503: Homework 2
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(a)

A few preliminaries: I will frequently refer back to the results in section **(a)** as the work will be repeated throughout. I will also be using the ansatz solutions for ODEs from section 1 of the notes. Also please note that I will often use the fact that $cos(n\pi) = (-1)^n$.

Assuming a solution of the form $u(x,t) = X(x)T(t)$ we have the following, with K being an arbitrary constant.

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = K \tag{1}$$

If we suppose $K = 0$ we quickly get a trivial soution:

$$\begin{aligned} X''(x) &= 0 \\ X(x) &= Ax + B \\ X(0) &= A(0) + B = 0 \\ B &= 0X(\pi) = A(\pi) = 0 \\ A &= 0 \end{aligned} \tag{2}$$

Similarly, if $K > 0$ we have solutions of the form:

$$\begin{aligned} X(x) &= Ae^{\sqrt{x}} + Be^{-\sqrt{x}} \\ X(0) &= A + B = 0 \\ B &= -A \\ X(\pi) &= Ae^{\sqrt{\pi}} - Ae^{-\sqrt{\pi}} = 0 \\ A &= 0 \end{aligned} \tag{3}$$

These trivial solutions have been covered quite a bit, but I include them here for later reference. We now proceed with $-\lambda^2 = K < 0$.

$$\begin{aligned} X(x) &= Asin(\lambda x) + Bcos(\lambda x) \\ X(0) &= B = 0 \\ X(\pi) &= Asin(\lambda \pi) \\ \lambda \pi &= \pi n \\ \lambda &= n \\ X_n(x) &= A_n sin(nx) \end{aligned} \tag{4}$$

We combine this with the T equation to get a general solution, consolidating the arbitrary constant from the X equation, and then applying the ICs:

$$\begin{aligned} T_n(t) &= A_n sin(nt) + B_n cos(nt) \\ \sum_{n=1}^{\infty} \Big[A_n sin(nt) &+ B_n cos(nt)\Big] sin(nx) \\ u(x,0) &= \sum_{n=1}^{\infty} B_n sin(nx) = 1 \\ B_n = \frac{2}{\pi}\int_0^{\pi} sin(nx)dx &= \frac{2}{\pi}\Big[\frac{-cos(nx)}{n}\Big]\Big|_0^{\pi} \\ B_n &= \frac{2}{\pi}(\frac{-cos(n\pi)+1}{n}) \\ B_n &= \frac{2-2(-1)^n}{\pi n} \end{aligned} \tag{5}$$

To apply the other IC we need the time derivative:

$$\begin{aligned} u_t(x,t) = \sum_{n=1}^{\infty} \Big[A_n n cos(nt) - B_n n sin(nt)\Big] sin(nx) \\ u_t(x,0) = \sum_{n=1}^{\infty} A_n sin(nx) = 0 \end{aligned} \tag{6}$$

It has been asserted in class and office hours without proof that for a series of this form to be equal to zero, the coefficient must be zero, thus we have $A_n = 0$. Combining these we have the complete solution:

$$\boxed{\begin{gathered} u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} cos(nt)(sin(nx) \\ \text{Or perhaps more efficiently, let } j = 2n+1\text{:} \\ u(x,t) = \frac{2}{\pi}\sum_{j=0}^{\infty} \frac{1-(-1)^j}{j} cos(jt)(sin(jx) \end{gathered} \qquad (7)}$$

(b) We can observe from sections **(2)** and **(3)** above that the addition of the coefficient of 2 in this problem will not change the triviality of solutions with

$K = 0$ and $K > 0$, therefore we move directly to solutions with $K < 0$. Given the formula,

$$\frac{T''(t)}{2T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \tag{8}$$

we can see that the $X(x)$ equation will be identical to **(4)**. We therefore move on to the $T(t)$ equation, again using the ansatz from the notes and the fact that $\lambda = n$:

$$T_n(t) = A_n sin(\sqrt{2}nt) + B_n cos(\sqrt{2}nt) \tag{9}$$

The resulting general solution before applying ICs is thus:

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(\sqrt{2}nt) + B_n cos(\sqrt{2}nt)\Big] sin(nx) \tag{10}$$

Now applying the first IC:

$$u(x,0) = \sum_{n=1}^{\infty} B_n sin(nx) = 0 \tag{11}$$

For the reasons given near **(6)** above $B_n = 0$, we now take the time derivative and apply the other IC:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n sin(\sqrt{2}nt) sin(nx) \\ u_t(x,t) &= \sum_{n=1}^{\infty} n\sqrt{2} A_n cos(\sqrt{2}nt) sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\sqrt{2} A_n sin(nx) = 1 \\ n\sqrt{2} A_n &= \frac{2}{\pi} \int_0^{\pi} sin(nx) dx \end{aligned} \tag{12}$$

We can recover the result of this integral from **(5)** above, giving:

$$\begin{aligned} n\sqrt{2}A_n &= \frac{2-2(-1)^n}{\pi n} \\ A_n &= \frac{2-2(-1)^n}{\sqrt{2}\pi n^2} \end{aligned} \tag{13}$$

Thus we have the full solution:

$$\boxed{\begin{gathered} u(x,t) = \frac{2}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\Big[\frac{1-(-1)^n}{n^2}sin(\sqrt{2}nt)\Big]sin(nx) \\ \text{Or perhaps more efficiently let } j = 2n+1\text{:} \\ u(x,t) = \frac{2}{\pi\sqrt{2}}\sum_{j=1}^{\infty}\Big[\frac{1-(-1)^j}{j^2}sin(\sqrt{2}jt)\Big]sin(jx) \end{gathered} \quad (14)}$$

(c) For the same reasons stated above in **(8)** and **(9)** we can move directly to the general solution, simply substituting $\sqrt{3}$ in place of $\sqrt{2}$:

$$u(x,t) = \sum_{n=1}^{\infty}\Big[A_n sin(\sqrt{3}nt) + B_n cos(\sqrt{3}nt)\Big]sin(nx) \tag{15}$$

We will first take the time derivative in order to apply an IC that is likely to eliminate a coefficient:

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty}\Big[n\sqrt{3}A_n cos(\sqrt{3}nt) - n\sqrt{3}B_n sin(\sqrt{3}nt)\Big]sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\sqrt{3}A_n sin(nx) = 0 \end{aligned} \tag{16}$$

Again, for the reasons state in **(6)** $A_n = 0$. Making this substitution, we then apply the other IC:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n cos(\sqrt{3}nt) sin(nx) \\ u(x,0) &= \sum_{n=1}^{\infty} B_n sin(nx) = sin^3(x) \end{aligned} \tag{17}$$

Rather than using an integral to determine B_n here, we can simply observe that the objective is to obtain a sine series for $sin^3(x)$. By the triple-angle formula we can see that $sin^3 x = \frac{3}{4} sinx - \frac{1}{4} sin(3x)$. It is immediately clear that this *is* a sine series with only two terms at $n = 1$ and $n = 3$ with corresponding coefficients $B_1 = \frac{3}{4}$ and $B_3 = -\frac{1}{4}$. Since all other terms are zero, the resulting function is quite simple, since the only values remaining from the sum are n=1 and n=3.

$$\boxed{u(x,t) = \frac{3}{4} cos(\sqrt{3}t) sin(x) - \frac{1}{4} cos(3\sqrt{3}t) sin(3x) \qquad (18)}$$

(d)

As in **(15)** above, we can move directly to the following, substituting $\sqrt{4} = 2$ for $\sqrt{3}$:

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(2nt) + B_n cos(2nt) \Big] sin(nx) \tag{19}$$

Applying the first IC we have:

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} B_n sin(nx) = x \\ B_n &= \frac{2}{\pi}\int_0^{\pi} xsin(x)dx \\ u &= x \ , \ du = 1 \\ dv &= sin(nx) \\ v &= \frac{-1}{n}cos(nx) \\ uv - \int vdu &= \frac{-xcos(nx)}{n} + \frac{sin(nx)}{n^2} \\ \Big[\frac{-xcos(nx)}{n} + \frac{sin(nx)}{n^2}\Big]\Big|_0^{\pi} &= (\frac{-\pi cos(n\pi)}{n}) \\ B_n &= \frac{2}{\pi}(\frac{-\pi(-1)^n}{n}) = \frac{-2(-1)^n}{n} \end{aligned} \tag{20}$$

To apply the other IC we take the time derivative (note I will retain the arbitrary coefficients for brevity):

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty}\Big[2nA_n cos(2nt) - 2nB_n csin(2nt)\Big]sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} 2nA_n sin(nx) = -x \\ &\text{Recoving the integral above:} \\ -2nA_n &= \frac{2}{\pi}\int_0^{\pi} xsin(x)dx \\ -2nA_n &= \frac{-2(-1)^n}{n} \\ A_n &= \frac{(-1)^n}{n^2} \end{aligned} \tag{21}$$

Substituting our coefficients, we arrive at the full solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\Big[\frac{(-1)^n}{n^2}sin(2nt) - \frac{2(-1)^n}{n}cos(2nt)\Big]sin(nx) \qquad (22)}$$

(e) The TA asserted in Recitation on Friday 4/12 that with mixed BCs one still need only check $K < 0$. However, it is easy to see in **(2)** above that if $K = 0$ we would be applying a BC to get $X'(\pi) = C = 0$, and then similarly for $X(0) = D = 0$. $K > 0$ similarly still leads to a situation in which two exponentials with different powers were equal, once again requiring the trivial solution.

$$\begin{aligned} X(0) = A + B = 0 \\ B = -AX'(x) = \frac{Ae^{\sqrt{x}}}{2\sqrt{x}} + \frac{Ae^{-\sqrt{x}}}{2\sqrt{x}} \end{aligned} \tag{23}$$

We can immediately see that X'(0) is not even defined, and so obviously cannot match any boundary conditions. We thus move forward with $K = -\lambda^2 < 0$. By ansatz, we then take the derivative and apply the BCs:

$$\begin{aligned} X(x) = Asin(\lambda x) + Bcos(\lambda x) \\ X(0) = B_n = 0 \\ X'(x) = \lambda Acos(\lambda x) \\ X'(\pi) = \lambda Acos(\lambda \pi) = 0 \\ \lambda\pi = (n + \frac{1}{2})\pi \\ \lambda = (n + \frac{1}{2}) \end{aligned} \tag{24}$$

Combining with the general solution for $T(t)$ that has been used repeatedly above we have:

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin((n + \frac{1}{2})t) + B_n cos((n + \frac{1}{2})t)\Big] sin((n + \frac{1}{2})x) \tag{25}$$

First I'll take the time derivative in order to eliminate a coefficient:

$$u_t(x,t) = \sum_{n=1}^{\infty} \Big[(n+\frac{1}{2})A_n cos((n+\frac{1}{2})t) - (n+\frac{1}{2})B_n sin((n+\frac{1}{2})t)\Big] sin((n+\frac{1}{2})x)$$
$$u_t(x,0) = \sum_{n=1}^{\infty}(n+\frac{1}{2})A_n sin((n+\frac{1}{2})x) = 0 \tag{26}$$

As in **(6)** above, $A_n = 0$. We now apply the other IC:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \Big[B_n cos((n+\frac{1}{2})t)\Big] sin((n+\frac{1}{2})x) \\ u(x,0) &= \sum_{n=1}^{\infty} B_n sin((n+\frac{1}{2})x) \end{aligned} \tag{27}$$

This gives modes that we are not used to, so in accordance with what we learned in Recitation, I'll extend the domain to $0 < x < 2\pi$ and take the even extension of the initial condition, $f(x) = 1$, which is easy since it's the same function with the expanded domain. We then let $n + \frac{1}{2} = \frac{j}{2}$ with $J = 1, 3, 5, 7...$ and calculate the coefficient as follows:

$$\begin{aligned} u(x,0) &= \sum_{j=1}^{\infty} C_j sin(\frac{jx}{2}) \\ C_j = \frac{1}{\pi}\int_0^{2\pi} sin(\frac{jx}{2})dx &= \frac{1}{\pi}\Big[\frac{-2cos\frac{jx}{2}}{j}\Big]\Big|_0^{2\pi} \\ C_j &= \frac{1}{\pi}\Big[\frac{-2cos\pi j + 2}{j}\Big] \end{aligned} \tag{28}$$

If J were even, C_j would be equal to zero, but we've defined it as odd only. So if j is only odd and we substitute n back in by observing that $j = 2n+1$ we have:

$$\begin{aligned} C_j &= \frac{4}{\pi j} \\ B_n &= \frac{4}{\pi(2n+1)} \end{aligned} \tag{29}$$

Substituting this back into the general solution we have the final solution satisfying the PDE and IC/BCs:

$$\boxed{u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{cos((n+\frac{1}{2})t)sin((n+\frac{1}{2})x)}{(2n+1)\pi} \qquad (30)}$$

(f) Because this problem has entirely different BCs from the previous ones, namely Neumann Boundary Conditions, I started by checking $K = 0$. Recovering the equation from **(2)** above we have:

$$\begin{aligned} X''(x) &= 0 \\ X'(x) &= C \end{aligned} \tag{31}$$

We here we can apply either BC, and we should have:

$$\begin{aligned} X'(0) = X'(2\pi) = C &= 0 \\ C &= 0 \\ X'(x) &= 0 \\ X(x) &= D \end{aligned} \tag{32}$$

Thus $X(x)$ is an arbitrary constant. Let $D = 1$ and we'll move on to the $T(t)$ equation. From **(1)** we have the following equation, to which we'll apply the

ICs:

$$\begin{aligned}
\frac{T''(t)}{2T(t)} &= 0 \\
T''(t) &= 0 \\
T'(t) &= C \\
T'(0) = C &= 1 \\
C &= 1 \\
T(t) &= t + D \\
T(0) = 0 + D &= -1 \\
D &= -1
\end{aligned} \tag{33}$$

Thus we have already have a solution that satisfies the PDE and the Initial and Boundary Conditions. This worked out so easily with $K = 0$ because the ICs were simple linear functions. ****Please Note** at the Thursday 4/18 Office Hours, Alex said that if we find a satisfactory solution with $K = 0$, there is no need to continue on to check the other conditions for K. I would speculate that the others would either give trivial solutions, or the same solution? (I would check but I'm really busy...)

$$\boxed{u(x,t) = t - 1 \tag{34}}$$

(g) Again to be thorough, I started by checking the conditions for K, starting with $K = 0$. Recovering the results of **(31)** and **(32)** we have $X(x) = 1$ and:

$$\begin{aligned}
T(t) &= C \\
T(0) = C &= x(x-1) \\
C &\neq x(x-1)
\end{aligned} \tag{35}$$

So clearly $K = 0$ will not let us satisfy the ICs. If $K > 0$ we have the same situation as **(23)** and the function is not defined at the boundary conditions. Thus we move on again to $K = -\lambda^2 < 0$

$$\begin{aligned}
X(x) &= Asin(\lambda x) + Bcos(\lambda x) \\
X'(x) &= \lambda Acos(\lambda x) - B\lambda sin(\lambda x) \\
X'(0) &= \lambda A = 0 \\
\lambda = 0 &\text{ is trivial, thus:} \\
A &= 0 \\
X'(1) &= -B\lambda sin(\lambda) = 0 \\
\lambda &= n\pi \\
X_n(x) &= \sum_{n=1}^{\infty} B_n cos(n\pi x)
\end{aligned} \tag{36}$$

Consolidating arbitrary constants we have the general solution:

$$\sum_{n=1}^{\infty} \Big[A_n sin(n\pi t) + B_n cos(n\pi t)\Big] sin(n\pi x) \tag{37}$$

First I'll take the derivative to use one of the ICs to eliminate a coefficient:

$$\begin{aligned}
u_t(x,t) &= \sum_{n=1}^{\infty} \Big[\pi n A_n cos(n\pi t) - \pi n B_n sin(n\pi t)\Big] sin(n\pi x) \\
u_t(x,0) &= \sum_{n=1}^{\infty} n\pi A_n cos(n\pi x) = 0 \\
A_n &= 0
\end{aligned} \tag{38}$$

Thus we have:

$$u(x,t) = \sum_{n=1}^{\infty} B_n cos(n\pi t) cos(n\pi x) \tag{39}$$

Here we have something different, a cosine series! Using the derivation of the formula from the notes, which makes use of the orthogonality of the basis, we have:

$$\begin{aligned} B_0 &= \int_0^1 x(1-x)dx = \int_0^1 xdx - \int_0^1 x^2dx \\ B_0 &= \left[\frac{x^2}{2}\right]\Big|_0^1 - \left[\frac{x^3}{3}\right]\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \end{aligned} \tag{40}$$

And for B_n we have the following very long integral!

$$B_n = 2\int_0^1 x(1-x)cos(n\pi x)dx = \int_0^1 xcos(n\pi x)dx - \int_0^1 x^2cos(n\pi x)dx \tag{41}$$

Starting with the first integral:

$$\begin{aligned} u &= x \text{ , } du = 1 \\ dv &= cos(n\pi x) \\ v &= \frac{sin(n\pi x)}{n\pi} \\ \int xcos(n\pi x)dx &= \frac{xsin(n\pi x)}{n\pi} - \int \frac{sin(n\pi x)}{n\pi}dx \\ \int xcos(n\pi x)dx &= \frac{xsin(n\pi x)}{n\pi} + \frac{cos(n\pi x)}{n^2\pi^2} \\ \left[\frac{xsin(n\pi x)}{n\pi} + \frac{cos(n\pi x)}{n^2\pi^2}\right]\Big|_0^1 &= \frac{cos(n\pi) - 1}{n^2\pi^2} \end{aligned} \tag{42}$$

Moving on to the second integral we have:

$$\begin{aligned} u &= x^2 \text{ , } du = 2x \\ dv &= cos(n\pi x) \\ v &= \frac{sin(n\pi x)}{n\pi} \\ \int x^2cos(n\pi x)dx &= \frac{x^2sin(n\pi x)}{n\pi} - \frac{2}{n\pi}\int xsin(n\pi x)dx \end{aligned} \tag{43}$$

We now pause to solve this integral:

$$
\begin{aligned}
u = x \ , \ du = 1 \\
dv = sin(n\pi x) \\
v = -\frac{cos(n\pi x)}{n\pi} \\
\int xsin(n\pi x)dx = -\frac{xcos(n\pi x)}{n\pi} + \int \frac{cos(n\pi x)}{n\pi} \\
\int xsin(n\pi x)dx = -\frac{xcos(n\pi x)}{n\pi} + \frac{sin(n\pi x)}{n^2\pi^2}
\end{aligned}
\tag{44}
$$

Plugging this back into the ongoing integral we get:

$$
\begin{aligned}
\int_0^1 x^2cos(n\pi x)dx = \Big[\frac{x^2sin(n\pi x)}{n\pi} - \frac{2}{n\pi}\Big[\frac{-xcos(n\pi x)}{n\pi} + \frac{sin(n\pi x)}{n^2\pi^2}\Big]\Big]\Big|_0^1 \\
\int_0^1 x^2cos(n\pi x)dx = \frac{cos(n\pi)}{n^2\pi^2}
\end{aligned}
\tag{45}
$$

Now combining this with the results from **(42)** we have:

$$
\begin{aligned}
B_n = 2\Big[\frac{cos(n\pi) - 1}{n^2\pi^2} - \frac{cos(n\pi)}{n^2\pi^2}\Big] \\
B_n = \frac{-2 - 2cos(n\pi)}{n^2\pi^2}
\end{aligned}
\tag{46}
$$

Putting all this together, at last we have:

$$
\boxed{u(x,t) = \frac{1}{6} - 2\sum_{n=1}^{\infty} \frac{1 + cos(n\pi)}{n^2\pi^2} cos(n\pi x)cos(n\pi t) \qquad (47)}
$$