AMATH 503: Homework 5
Due May, 28 2019
ID: 1064712

Trent Yarosevich

May 27, 2019

Instructor: Professor Ka-Kit Tung

(1) If we have a Bessel Equation of the form:

$$(xy')' + (\lambda^2 x - \frac{m^2}{x})y = 0$$
$$0 < x < a \tag{1}$$
$$y \text{ bounded at } x = 0, \; , \; y(a) = 0$$

We know from the notes and previous homework that this will be solved by the eigenfunctions $J_m(\lambda x)$. The eigenfunctions are derived with the Frobenius solution, and the eigenvalues are implicitly determined by from the zeros of the eigenfunctions, which are cosine-like. Given this, let the eigenfunctions be $J_m(\lambda_{mn}x)$ and the eigenvalues $(\lambda_{mn} = \frac{z_{mn}}{a})$ where z_{mn} are the zeros of the eigenfunction. We use the equation above and observe that this is a Sturm-Liouville system with:

$$p(x) = x$$
$$r(x) = x \tag{2}$$
$$q(x) = \frac{p^2}{x}$$

We now consider two pairs of eigenfunctions and eigenvalues, $(J_m(\lambda_{mn}x); \lambda_{mn})$ and $(J_k(\lambda_{kn}x)); \lambda_{kx})$ and plug them into the Bessel's Equation, giving:

$$\begin{aligned} (xJ_m(\lambda_{mn}x)')' + (\lambda_{mn}^2 x - \frac{p^2}{x}) = 0 \\ (xJ_k(\lambda_{mn}x)')' + (\lambda_{kn}^2 x - \frac{p^2}{x}) = 0 \end{aligned} \tag{3}$$

We then follow the logic of the general proof of S-L orthogonality by multiplying the first by $J_k(\lambda_{kn}x)$ and the second by $J_m(\lambda_{mn}x)$ then subtracting one from the other:

$$J_k(\lambda_{kn}x)(xJ_m(\lambda_{mn}x)')' - J_m(\lambda_{mn}x)(xJ_k(\lambda_{kn}x)')' = (\lambda_{mn} - \lambda_{kn})xJ_m(\lambda_{mn}x)J_k(\lambda_{kn}x) \tag{4}$$

The LHS is a derivative, so we rewrite as follows:

$$\frac{d}{dx}\Big[J_k(\lambda_{kn}x)(xJ_m(\lambda_{mn}x))') - J_m(\lambda_{mn}x)(J_k(\lambda_{kn}x)')\Big] = \Big[(\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x)\Big] \tag{5}$$

We then integrate both sides giving:

$$\Big[J_k(\lambda_{kn}x)(xJ_m(\lambda_{mn}x))') - J_m(\lambda_{mn}x)(J_k(\lambda_{kn}x)')\Big]\Big|_0^a = \int_0^a \Big[(\lambda_{mn} - \lambda_{kn})xJ_m(\lambda_{mn}x)J_k(\lambda_{kn}x)\Big]dx \tag{6}$$

We then observe that, since this is a singular S-L system and $p(x) = 0$ at $x = 0$ and $x = a$. In this case, $p(x) = x$ so it' a little confusing, but let's just suppose we have a dummy variable for a moment, and $p(s) = s$. Then in a singular S-L system, $p(s) = x = 0$ when $s = 0, a$. From this we can infer that the LHS must be identically zero. This gives:

$$(\lambda_{mn} - \lambda_{kn}) \int_0^a xJ_m(\lambda_{mn}x)J_k(\lambda_{kn}x)dx = 0 \tag{7}$$

We can now simply observe that, if $\lambda_{mn} = \lambda_{kn}$, the leading constant becomes zero, and the integral becomes:

$$\int_0^a x(J_m(\lambda_{mn}x))^2 dx \tag{8}$$

This integral is a positive constant since $x > 0$ for this Bessel function, and the eigenfunction is squared. The integral is $ax(J_m(a\lambda_{mn}x))^2 > 0$ and therefore a positive constant.

Alternatively, if $\lambda_{mn} \neq \lambda_{kn}$, this integral must be identically zero. The resulting integral is thus:

$$\boxed{\int_0^a xJ_m(\lambda_{mn}x)J_k(\lambda_{kn}x)dx = \begin{cases} 0, & \lambda_{mn} \neq \lambda_{kn} \\ c, & \lambda_{mn} = \lambda_{kn} \end{cases} \tag{9}}$$

Where $c > 0$ is a constant.

(2)

(a) From the prompt we know that, with spherical symmetry, the 3D wave equation becomes:

$$u_{tt} = \frac{c^2}{r}(ru)_{rr} \tag{10}$$

Bringing the r to the LHS, we can note that it is constant variable with respect to t, and we can write the PDE as:

$$(ru)_{tt} = c^2(ru)_{rr} \tag{11}$$

Now let's substitute $v = ru$ and plug the resulting equation into the D'Alembert Solution:

$$\begin{aligned} v_{tt} &= c^2 v_{rr} \\ v(r,t) &= \frac{1}{2}\Big[g(r-ct) + g(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} h(s)ds \end{aligned} \tag{12}$$

To transform g back to its equivalent term in u:

$$v(r,0) = g(r) \tag{13}$$

And the final form of the equation is:

$$u(r,0) = f(r) \tag{14}$$

Thus we have:

$$\begin{aligned} \frac{1}{r}v(r,0) &= u(r,0) \\ \frac{1}{r}g(r) &= f(r) \\ g(r) &= rf(r) \end{aligned} \tag{15}$$

Similarly, we observe that:

$$\begin{aligned} v_t &= \frac{d}{dt}\Big[\frac{1}{2c}\int_{r-ct}^{r+ct} h(s)ds\Big] \\ v_t &= \frac{1}{2c}\frac{d}{dt}\Big[H(r+ct) - H(r-ct)\Big] \\ v_t &= \frac{1}{2c}\Big[h(r+ct)(c) - h(r-ct)(-c)\Big] \\ v_t &= \frac{1}{2}\Big[h(r+ct) + h(r-ct) \\ v_t(r,0) &= h(r) \\ ru_t(r,0) &= h(r) \\ u_t(r,0) &= \frac{1}{r}h(r) \\ &\text{So we define some new function:} \\ u_t(r,0) &= k(r) \\ rk(r) &= h(r) \end{aligned} \tag{16}$$

We then substitute all the transformed terms into the solution and replace the r terms with suitable $r \pm ct$ or s and get:

$$\boxed{\begin{aligned} ru(r,t) &= \frac{1}{2}\Big[(r-ct)f(r-ct) + (r+ct)f(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} sk(s)ds \\ u(r,t) &= \frac{1}{2r}\Big[(r-ct)f(r-ct) + (r+ct)f(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} sk(s)ds \end{aligned} \quad (17)}$$

(3)

(a) In the steady state solution $u_{tt} = 0$ thus:

$$\begin{aligned} u_{xx} &= -1 \\ u_x &= -x + c \\ u &= \frac{-x^2}{2} + cx + d \\ u(0) &= 0 = d \\ u(1) &= 0 = -\frac{1}{2} + c \\ c &= \frac{1}{2} \end{aligned} \tag{18}$$

So the steady state solution is:

$$\boxed{u(x,t) = \frac{-x^2}{2} + \frac{x}{2}} \tag{19}$$

Suppose the transient solution is some $v = u - u_{\text{steady}}$. If we plug this into the PDE we get:

$$\begin{aligned} v_{tt} - 0 = v_{xx} - \frac{d^2}{dx^2}(\frac{-x^2}{2} + \frac{x}{2}) + 1 \\ v_{tt} = v_{xx} \end{aligned} \tag{20}$$

By separation of variables let $v = T(t)X(x)$. As we've seen countless times in the class thus far, this yields the solution:

$$\begin{aligned} v(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(\lambda_n t) + B_n cos(\lambda_n t)\Big] sin(\frac{n\pi x}{L}) \\ L = 1 \end{aligned} \tag{21}$$

Now we apply the BCs noting that $u(x,t) = v(x,t) - \frac{x^2}{2} + \frac{x}{2}$

$$\begin{aligned} u(x,0) = 0 = \sum_{n=1}^{\infty} B_n sin(n\pi x) - \frac{x^2}{2} + \frac{x}{2} \\ \sum_{n=1}^{\infty} B_n sin(n\pi x) = \frac{x^2}{2} - \frac{x}{2} \end{aligned} \tag{22}$$

This is a sine series and the coefficient B_n s given:

$$B_n = \int_0^1 x^2 sin(n\pi x)dx - \int_0^1 xsin(n\pi x)dx \tag{23}$$

Proceeding one integral at a time:

$$\begin{aligned} \int_0^1 x^2 sin(n\pi x)dx = \Big[\frac{-x^2 cos(n\pi x)}{n\pi}\Big]\Big|_0^1 + 2\int_0^1 \frac{xcos(n\pi x)}{n\pi}dx \\ \int_0^1 \frac{xcos(n\pi x)}{n\pi}dx = \Big[\frac{xsin}{(n^2\pi^2}\Big]\Big|_0^1 - \int_0^1 \frac{sin(n\pi x)}{n^2\pi^2}dx \\ \int_0^1 \frac{sin(n\pi x)}{n^2\pi^2}dx = \Big[\frac{cos(n\pi x)}{n^3\pi^3}\Big]\Big|_0^1 \end{aligned} \tag{24}$$

Putting these evaluations together we have:

$$\begin{aligned} \Big[\frac{-x^2 cos(n\pi x)}{n\pi}\Big]\Big|_0^1 + 2\Big[\Big[\frac{xsin}{(n^2\pi^2}\Big]\Big|_0^1 + \Big[\frac{cos(n\pi x)}{n^3\pi^3}\Big]\Big|_0^1\Big] \\ = \frac{-(-1)^n}{n\pi} + 2\Big[\frac{(-1)^n)}{n^3\pi^3} - \frac{1}{n^3\pi^3}\Big] \end{aligned} \tag{25}$$

Now the other integral:

$$\begin{aligned} \int_0^1 xsin(n\pi x)dx = \Big[\frac{-xcos(n\pi x)}{n\pi}\Big]\Big|_0^1 - \int_0^1 \frac{cos(n\pi x)}{n\pi}dx \\ \int_0^1 \frac{cos(n\pi x)}{n\pi}dx = \Big[\frac{sin(n\pi x)}{n^2\pi^2}\Big]\Big|_0^1 = 0 \\ \int_0^1 xsin(n\pi x)dx = \frac{-(-1)^n}{n\pi} \end{aligned} \tag{26}$$

Combining the two terms:

$$\begin{aligned} B_n &= \frac{-(-1)^n n^2\pi^2 + 2(-1)^n - 2 + (-1)^n n^2\pi^2}{n^3\pi^3} \\ B_n &= \frac{2((-1)^n - 1)}{n^3\pi^3} \end{aligned} \tag{27}$$

Now we define $n = 1, 3, 5...$ giving:

$$B_n = \frac{-4}{n^3\pi^3} \tag{28}$$

Applying the other initial condition we take the derivative:

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty}\Big[n\pi A_n cos(n\pi t) - n\pi B_n sin(n\pi t)\Big] sin(n\pi x) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\pi A_n sin(n\pi x) = 0 \\ A_n &= 0 \end{aligned} \tag{29}$$

Thus the transient solution and complete solution are:

$$\boxed{\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty}\frac{-4}{n^3\pi^3}cos(n\pi t)sin(n\pi x) \\ u(x,t) &= \sum_{n=1}^{\infty}\frac{-4}{n^3\pi^3}cos(n\pi t)sin(n\pi x) - \frac{x^2}{2} + \frac{x}{2} \\ n &= 1, 3, 5... \end{aligned} \qquad (30)}$$

(b) First we determine the eigenfunctions and eigenvalues that will satisfy the boundary conditions. They are homogeneous dirchelet, and so the eigenfunction is $sin(n\pi x)$. We thus represent the homogeneous form of the PDE as:

$$u(x,t) = \sum_{n=1}^{\infty} T(t) sin(n\pi x) \tag{31}$$

We then represent the forcing term with the eigenfunction/values:

$$f(x,t) = 1 = \sum_{n=1}^{\infty} f_n sin(n\pi x) \tag{32}$$

Next we plug this into the PDE.

$$\begin{aligned} \sum_{n=1}^{\infty} T_n''(t)X(x) - X_n''(x)T_n(t) &= \sum_{n=1}^{\infty} f_n sin(n\pi x) \\ T_n''(t)sin(n\pi x) + (n\pi)^2 sin(n\pi x)T_n(t) &= f_n sin(n\pi x) \\ T_n''(t) + (n\pi)^2 T_n(t) &= f_n \end{aligned} \tag{33}$$

To proceed we'll need f_n, which is the coefficient for the sine series for $f(x) = 1$:

$$\begin{aligned} A_n = 2\int_0^1 sin(n\pi x) &= 2\Big[\frac{-cos(n\pi x)}{n\pi}\Big]\Big|_0^1 \\ A_n &= 2\frac{1-(-1)^n}{n\pi} \\ A_n &= 0 \text{ for even } n, \\ A_n &= \frac{4}{n\pi} \text{ for odd } n \end{aligned} \tag{34}$$

We thus have two scenarios:

$$\begin{aligned} n = 1,3,5... \to T_n''(t) + (n\pi)^2 T_n(t) &= \frac{4}{n\pi} \\ n = 2,4,6... \to T_n''(t) + (n\pi)^2 T_n(t) &= 0 \end{aligned} \tag{35}$$

In the even case, we know from major precedent that the solution is sines and cosines:

$$\begin{aligned} n &= 2,4,6... \\ T_n(t) &= A_n sin(n\pi t) + B_n cos(n\pi t) \end{aligned} \tag{36}$$

For the odd n case we need the homogeneous solution plus the particular. We already have the former, and so to find the particular solution we guess a solution in the form of the forcing term, i.e. a constant k and plug it into the PDE:

$$\begin{aligned} T(t)_{\text{particular}} &= k \\ k(n\pi)^2 &= \frac{4}{n\pi} \\ k &= \frac{4}{(n\pi)^3} \end{aligned} \tag{37}$$

Thus the complete solution for $T(t)$ for odd n is:

$$T_n(t) = \sum_{n=1}^{\infty} A_n sin(n\pi t) + B_n cos(n\pi t) + \frac{4}{(n\pi)^3} \tag{38}$$

Now, I'm not sure how to articulate this accurately, but I think the $n = 2, 4, 6..$ case is not a part of the solution, since it actually doesn't satisfy the PDE. I speculate that this is because f_n can be defined with strictly odd n. We will need to solve the homogeneous case, but the general solution is of the form:

$$\boxed{\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(n\pi t) + B_n cos(n\pi t) + \frac{4}{(n\pi)^3}\Big] sin(n\pi x) \\ n = 1, 3, 5... \end{aligned} \quad (39)}$$

Applying the Initial Conditions:

$$\begin{aligned} u(x,0) = \sum_{n=1}^{\infty} \Big[B_n + \frac{4}{(n\pi)^3}\Big] sin(n\pi x) = 0 \\ B_n = -\frac{4}{(n\pi)^3} \\ u_t(x,t) = \sum_{n=1}^{\infty} \Big[n\pi A_n cos(n\pi t) - n\pi B_n sin(n\pi t)\Big] sin(n\pi x) \\ u_t(x,0) = \sum_{n=1}^{\infty} \Big[n\pi A_n\Big] sin(n\pi x) = 0 \\ A_n = 0 \end{aligned} \tag{40}$$

This gives a final solution of:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\frac{4}{(n\pi)^3}(1 - cos(n\pi t))\right]sin(n\pi x) \qquad n = 1,3,5... \qquad (41)}$$

If we note from **(22)** above that

$$\begin{aligned} n &= 1,3,5.. \\ \sum_{n=1}^{\infty}\frac{-4}{(n\pi)^3}sin(n\pi x) &= \frac{x^2}{2} - \frac{x}{2} \\ \sum_{n=1}^{\infty}\frac{4}{(n\pi)^3}sin(n\pi x) &= -\frac{x^2}{2} + \frac{x}{2} \end{aligned} \tag{42}$$

We can see that the extra series term in **(42)** is equivalent to the non-series term in **(30)** and we have the same answer.