AMATH 503: Homework 5
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(1)

(a) Given the solution to the Green's Function for the Heat Equation in 1-D stated on page 228 of the notes, we can define the semi-infinite domain as $0 < x < \infty$, and then subtract the same solution defined at the location $x = -\xi$ to satisfy the boundary condition $G = 0$ at $x = 0$. This gives the Green's Function:

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi D(t-\tau)}}\left[e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4D(t-\tau)}}\right] \tag{1}$$

To show this satisfies the BC, we can simply observe that if $x = 0$, the two exponential terms are equivalent since the only difference is the $\pm\xi$ term, which is squared anyway. The other boundary condition is still satisfied since both of these exponentials have negative terms, and will go to zero as $x \to \infty$.

(b) To satisfy a boundary condition of $\frac{\partial}{\partial x}G$ at $x = 0$, we can observe that if we add a solution mirrored across the G axis, the two resulting normal distributions will have equal slope, but opposite in sign, at $x = 0$:

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi D(t-\tau)}}\Big[e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} + e^{-\frac{(x+\xi)^2}{4D(t-\tau)}}\Big] \tag{2}$$

We can show this satisfies the BC at $x = 0$ by taking the derivatives of the exponentials:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{-2(x-\xi)}{4D(t-\tau)}e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \\ \frac{\partial}{\partial x} &= \frac{-2(x+\xi)}{4D(t-\tau)}e^{-\frac{(x+\xi)^2}{4D(t-\tau)}} \end{aligned} \tag{3}$$

When $x = 0$, again the two terms cancel since the sign on the $-\xi$ carries out, and the $\pm\xi$ in the exponents are equivalent because they are squared:

$$\frac{2(\xi)}{4D(t-\tau)}e^{-\frac{(-\xi)^2}{4D(t-\tau)}} - \frac{2(\xi)}{4D(t-\tau)}e^{-\frac{(\xi)^2}{4D(t-\tau)}} = 0 \tag{4}$$

Again, the BC of $G = 0$ as $x \to \infty$ is still satisfied since both normal distributions are raised to strictly negative exponent dependent on x.

(2)

(a) We want to show that

$$\begin{aligned} (\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2})u &= \delta(x-\xi)\delta(t-\tau) \\ u = 0 \ , \ \frac{\partial}{\partial t}u &= 0 \text{ at } t = 0 \end{aligned} \tag{5}$$

is equivalent to:

$$\begin{aligned} (\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2})u &= 0 \\ u = 0 \text{ and } \frac{\partial}{\partial t}u &= \delta(x-\xi) \text{ at } t = \tau \end{aligned} \tag{6}$$

To do so, we make some observations:

(1): by definition, the RHS of the non-homogeneous equation is zero at all t except for $t = \tau$. Thus, because the initial condition is $u = 0$ and $\frac{\partial}{\partial t}u = 0$ at $= 0$, we know that nothing happens until $t = \tau$.

(2): We also know that when $t > \tau$, the RHS is again zero because of the delta function. So the original equation is actually homogeneous everywhere except at $t = \tau$.

(3): So the question is, what happens at $t = \tau$ in the non-homogeneous PDE? We need this information to define our second initial condition. To do this we integrate the non-homogeneous PDE with respect to time across an arbitrarily small domain around $t = \tau$:

$$\begin{aligned} \int_{\tau-}^{\tau+} \Big[(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2})u\Big]dt &= \int_{\tau-}^{\tau+} \delta(x-\xi)\delta(t-\tau)dt \\ \int_{\tau-}^{\tau+} \frac{\partial^2}{\partial t^2}udt - \int_{\tau-}^{\tau+} c^2\frac{\partial^2}{\partial x^2}udt &= \delta(x-\xi)\int_{\tau-}^{\tau+} \delta(t-\tau)dt \end{aligned} \tag{7}$$

Now a few observations. We know that when $t < \tau$ that $u = 0$, so the first integral is simply equivalent to $\frac{\partial}{\partial t}u$ evaluated at $\tau+$. Integrating the spatial second derivative across an arbitrarily small time span must be zero. If this weren't the case, u itself would be a delta function in time, which isn't indicated by the PDE in any way. Lastly, the RHS is equal to $\delta(x-\xi)$ since the derivative across the zero of a delta function is by definition equal to 1. This means the second BC is:

$$\frac{\partial}{\partial t}\Big|_{\tau+} = \delta(x-\xi) \tag{8}$$

Which is equivalent to the objective.