

# AMATH 503: Homework 5

Due May, 28 2019

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May 26, 2019

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(1) If we have a Bessel Equation of the form:

$$(xy')' + (\lambda^2 x - \frac{m^2}{x})y = 0$$
$$0 < x < a \tag{1}$$

$y$  bounded at  $x = 0$ ,  $y(a) = 0$

We know from the notes and previous homework that this will be solved by the eigenfunctions  $J_m(\lambda x)$ . The eigenfunctions are derived with the Frobenius solution, and the eigenvalues are implicitly determined by from the zeros of the eigenfunctions, which are cosine-like. Given this, let the eigenfunctions be  $J_m(x)$  and the eigenvalues ( $\lambda_{mn} = \frac{z_{mn}}{a}$ ) where  $z_{mn}$  are the zeros of the eigenfunction. We use the equation above and observe that this is a Sturm-Liouville system with:

$$p(x) = x$$
$$r(x) = x \tag{2}$$
$$q(x) = \frac{p^2}{x}$$

We now consider two pairs of eigenfunctions and eigenvalues,  $(J_m(x); \lambda_{mn})$  and  $(J_k(x); \lambda_{kn})$  and plug them into the Bessel's Equation, giving:

$$\begin{aligned} (xJ_m(x))' + (\lambda_{mn}^2 x - \frac{p^2}{x}) &= 0 \\ (xJ_k(x))' + (\lambda_{kn}^2 x - \frac{p^2}{x}) &= 0 \end{aligned} \quad (3)$$

We then follow the logic of the general proof of S-L orthogonality by multiplying the first by  $J_k(x)$  and the second by  $J_m(x)$  then subtracting one from the other:

$$J_k(x)(xJ_m(x))' - J_m(x)(xJ_k(x))' = (\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x) \quad (4)$$

The LHS is a derivative, so we rewrite as follows:

$$\frac{d}{dx} [J_k(x)(xJ_m(x))' - J_m(x)(xJ_k(x))'] = [(\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x)] \quad (5)$$

We then integrate both sides giving:

$$\left[ J_k(x)(xJ_m(x))' - J_m(x)(xJ_k(x))' \right] \Big|_0^a = \int_0^a [(\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x)] dx \quad (6)$$

We then observe that, since this is a singular S-L system and  $p(x) = 0$  at  $x = 0$  and  $x = a$ . In this case,  $p(x) = x$  so it's a little confusing, but let's just suppose we have a dummy variable for a moment, and  $p(s) = s$ . Then in a singular S-L system,  $p(s) = x = 0$  when  $s = 0, a$ . From this we can infer that the LHS must be identically zero. This gives:

$$(\lambda_{mn} - \lambda_{kn}) \int_0^a xJ_m(x)J_k(x)dx = 0 \quad (7)$$

We can now simply observe that, if  $\lambda_{mn} = \lambda_{kn}$ , the leading constant becomes zero, and the integral becomes:

$$\int_0^a x(J_m(x))^2 dx \quad (8)$$

This integral is a positive constant since  $x > 0$  for this Bessel function, and the eigenfunction is squared. The integrand  $ax(J_m(x))^2 > 0$  and therefore the resulting integral will be a positive constant.

Alternatively, if  $\lambda_{mn} \neq \lambda_{kn}$ , this integral must be identically zero. The resulting integral is thus:

$$\int_0^a xJ_m(x)J_k(x)dx = \begin{cases} 0 & \lambda_{mn} \neq \lambda_{kn} \\ c & \lambda_{mn} = \lambda_{kn} \end{cases} \quad (9)$$

Where  $c > 0$  is a constant.

**(2)**

**(a)** From the prompt we know that, with spherical symmetry, the 3D wave equation becomes:

$$u_{tt} = \frac{c^2}{r}(ru)_{rr} \quad (10)$$

Bringing the  $r$  to the LHS, we can note that it is constant variable with respect to  $t$ , and we can write the PDE as:

$$(ru)_{tt} = c^2(ru)_{rr} \quad (11)$$

Now let's substitute  $v = ru$  and plug the resulting equation into the D'Alembert

Solution:

$$v_{tt} = c^2 v_{rr}$$

$$v(r, t) = \frac{1}{2} \left[ g(r - ct) + g(r + ct) + \frac{1}{2c} \int_{r-ct}^{r+ct} h(s) ds \right] \quad (12)$$

To transform  $g$  back to its equivalent term in  $u$ :

$$v(r, 0) = g(r) \quad (13)$$

And the final form of the equation is:

$$u(r, 0) = f(r) \quad (14)$$

Thus we have:

$$\frac{1}{r} v(r, 0) = u(r, 0)$$

$$\frac{1}{r} g(r) = f(r) \quad (15)$$

$$g(r) = rf(r)$$

Similarly, we observe that:

$$\begin{aligned}
v_t &= \frac{d}{dt} \left[ \frac{1}{2c} \int_{r-ct}^{r+ct} h(s) ds \right] \\
v_t &= \frac{1}{2c} \frac{d}{dt} [H(r+ct) - H(r-ct)] \\
v_t &= \frac{1}{2c} [h(r+ct)(c) - h(r-ct)(-c)] \\
v_t &= \frac{1}{2} [h(r+ct) + h(r-ct)] \\
v_t(r, 0) &= h(r) \\
ru_t(r, 0) &= h(r) \\
u_t(r, 0) &= \frac{1}{r} h(r)
\end{aligned} \tag{16}$$

So we define some new function:

$$rk(r) = h(r)$$

We then substitute all the transformed terms into the solution and get:

$$\begin{aligned}
ru(r, t) &= \frac{1}{2} [(r-ct)f(r-ct) + (r+ct)f(r+ct) + \frac{1}{2c} \int_{r-ct}^{r+ct} sk(s) ds] \\
u(r, t) &= \frac{1}{2r} [(r-ct)f(r-ct) + (r+ct)f(r+ct) + \frac{1}{2c} \int_{r-ct}^{r+ct} sk(s) ds]
\end{aligned} \tag{17}$$

**(3)**

(a) In the steady state solution  $u_{tt} = 0$  thus:

$$\begin{aligned}
 u_{xx} &= -1 \\
 u_x &= -x + c \\
 u &= \frac{-x^2}{2} + cx + d \\
 u(0) &= 0 = d \\
 u(1) &= 0 = -\frac{1}{2} + c \\
 c &= \frac{1}{2}
 \end{aligned} \tag{18}$$

So the steady state solution is:

$$u(x, t) = \frac{-x^2}{2} + \frac{x}{2} \tag{19}$$

Suppose the transient solution is some  $v = u - u_{\text{steady}}$ . If we plug this into the PDE we get:

$$\begin{aligned}
 v_{tt} - 0 &= v_{xx} - \frac{d^2}{dx^2} \left( \frac{-x^2}{2} + \frac{x}{2} \right) + 1 \\
 v_{tt} &= v_{xx}
 \end{aligned} \tag{20}$$

By separation of variables let  $v = T(t)X(x)$ . As we've seen countless times in the class thus far, this yields the solution:

$$\begin{aligned}
 v(x, t) &= \sum_{n=1}^{\infty} \left[ A_n \sin(\lambda_n t) + B_n \cos(\lambda_n t) \right] \sin\left(\frac{n\pi x}{L}\right) \\
 L &= 1
 \end{aligned} \tag{21}$$

Now we apply the BCs noting that  $u(x, t) = v(x, t) - \frac{x^2}{2} + \frac{x}{2}$

$$\begin{aligned}
u(x, 0) = 0 &= \sum_{n=1}^{\infty} B_n \sin(n\pi x) - \frac{x^2}{2} + \frac{x}{2} \\
\sum_{n=1}^{\infty} B_n \sin(n\pi x) &= \frac{x^2}{2} - \frac{x}{2}
\end{aligned} \tag{22}$$

This is a sine series and the coefficient  $B_n$  s given:

$$B_n = \int_0^1 x^2 \sin(n\pi x) dx - \int_0^1 x \sin(n\pi x) dx \tag{23}$$

Proceeding one integral at a time:

$$\begin{aligned}
\int_0^1 x^2 \sin(n\pi x) dx &= \left[ \frac{-x^2 \cos(n\pi x)}{n\pi} \right]_0^1 + 2 \int_0^1 \frac{x \cos(n\pi x)}{n\pi} dx \\
\int_0^1 \frac{x \cos(n\pi x)}{n\pi} dx &= \left[ \frac{x \sin(n\pi x)}{n^2 \pi^2} \right]_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n^2 \pi^2} dx \\
&= \left[ \frac{\cos(n\pi x)}{n^3 \pi^3} \right]_0^1
\end{aligned} \tag{24}$$

Putting these evaluations together we have:

$$\begin{aligned}
&\left[ \frac{-x^2 \cos(n\pi x)}{n\pi} \right]_0^1 + 2 \left[ \left[ \frac{x \sin(n\pi x)}{n^2 \pi^2} \right]_0^1 + \left[ \frac{\cos(n\pi x)}{n^3 \pi^3} \right]_0^1 \right] \\
&= \frac{-(-1)^n}{n\pi} + 2 \left[ \frac{(-1)^n}{n^3 \pi^3} - \frac{1}{n^3 \pi^3} \right]
\end{aligned} \tag{25}$$

Now the other integral:

$$\begin{aligned}
\int_0^1 x \sin(n\pi x) dx &= \left[ \frac{-x \cos(n\pi x)}{n\pi} \right]_0^1 - \int_0^1 \frac{\cos(n\pi x)}{n\pi} dx \\
\int_0^1 \frac{\cos(n\pi x)}{n\pi} dx &= \left[ \frac{\sin(n\pi x)}{n^2 \pi^2} \right]_0^1 = 0 \\
\int_0^1 x \sin(n\pi x) dx &= \frac{-(-1)^n}{n\pi}
\end{aligned} \tag{26}$$

Combining the two terms:

$$B_n = \frac{-(-1)^n n^2 \pi^2 + 2(-1)^n - 2 + (-1)^n n^2 \pi^2}{n^3 \pi^3} \quad (27)$$

$$B_n = \frac{2((-1)^n - 1)}{n^3 \pi^3}$$

Now we define  $n = 1, 3, 5 \dots$  giving:

$$B_n = \frac{-4}{n^3 \pi^3} \quad (28)$$

Applying the other initial condition we take the derivative:

$$u_t(x, t) = \sum_{n=1}^{\infty} \left[ n\pi A_n \cos(n\pi t) - n\pi B_n \sin(n\pi t) \right] \sin(n\pi x)$$

$$u_t(x, 0) = \sum_{n=1}^{\infty} n\pi A_n \sin(n\pi x) = 0 \quad (29)$$

$$A_n = 0$$

Thus the transient solution and complete solution are:

$$v(x, t) = \sum_{n=1}^{\infty} \frac{-4}{n^3 \pi^3} \cos(n\pi t) \sin(n\pi x)$$

$$u(x, t) = \sum_{n=1}^{\infty} \frac{-4}{n^3 \pi^3} \cos(n\pi t) \sin(n\pi x) - \frac{x^2}{2} + \frac{x}{2} \quad (30)$$

$$n = 1, 3, 5 \dots$$