AMATH 503: Homework 2
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(a) Assuming a solution of the form $u(x,t) = X(x)T(t)$ we have the following, with K being an arbitrary constant.

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = K \tag{1}$$

If we suppose $K = 0$ we quickly get a trivial soution:

$$\begin{aligned} X''(x) &= 0 \\ X(x) &= Ax + B \\ X(0) &= A(0) + B = 0 \\ B &= 0X(\pi) = A(\pi) = 0 \\ A &= 0 \end{aligned} \tag{2}$$

Similarly, if $K > 0$ we have solutions of the form:

$$\begin{aligned} X(x) &= Ae^{\sqrt{x}} + Be^{-\sqrt{x}} \\ X(0) &= A + B = 0 \\ B &= -A \\ X(\pi) &= Ae^{\sqrt{\pi}} - Ae^{-\sqrt{\pi}} = 0 \\ A &= 0 \end{aligned} \tag{3}$$

These trivial solutions have been covered quite a bit, but I include them here for later reference. We now proceed with $-\lambda^2 = K < 0$.

$$\begin{aligned} X(x) &= Asin(\lambda x) + Bcos(\lambda x) \\ X(0) &= B = 0 \\ X(\pi) &= Asin(\lambda\pi) \\ \lambda\pi &= \pi n \\ \lambda &= n \\ X_n(x) &= A_n sin(nx) \end{aligned} \tag{4}$$

We combine this with the T equation to get a general solution, consolidating the arbitrary constant from the X equation, and then applying the ICs:

$$\begin{aligned} T_n(t) &= A_n sin(nt) + B_n cos(nt) \\ \sum_{n=1}^{\infty} &\Big[A_n sin(nt) + B_n cos(nt)\Big] sin(nx) \\ u(x,0) &= \sum_{n=1}^{\infty} B_n sin(nx) = 1 \\ B_n = \frac{2}{\pi}\int_0^{\pi} sin(nx)dx &= \frac{2}{\pi}\Big[\frac{-cos(nx)}{n}\Big]\Big|_0^{\pi} \\ B_n &= \frac{2}{\pi}(\frac{-cos(n\pi)+1}{n}) \\ B_n &= \frac{2-2(-1)^n}{\pi n} \end{aligned} \tag{5}$$

To apply the other IC we need the time derivative:

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} \Big[A_n ncos(nt) - B_n nsin(nt)\Big] sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} A_n sin(nx) = 0 \end{aligned} \tag{6}$$

It has been asserted in class and office hours without proof that for a series of this form to be equal to zero, the coefficient must be zero, thus we have $A_n = 0$. Combining these we have the complete solution:

$$\boxed{u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} cos(nt)(sin(nx) \qquad (7)}$$

(b) We can observe from sections **(2)** and **(3)** above that the addition of the coefficient of 2 in this problem will not change the triviality of solutions with $K = 0$ and $K > 0$, therefore we move directly to solutions with $K < 0$. Given the formula,

$$\frac{T''(t)}{2T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \tag{8}$$

we can see that the $X(x)$ equation will be identical to **(4)**. We therefore move on to the $T(t)$ equation, again using the ansatz from the notes and the fact that $\lambda = n$:

$$T_n(t) = A_n sin(\sqrt{2}nt) + B_n cos(\sqrt{2}nt) \tag{9}$$

The resulting general solution before applying ICs is thus:

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(\sqrt{2}nt) + B_n cos(\sqrt{2}nt)\Big] sin(nx) \tag{10}$$

Now applying the first IC:

$$u(x,0) = \sum_{n=1}^{\infty} B_n sin(nx) = 0 \tag{11}$$

For the reasons given near **(6)** above $B_n = 0$, we now take the time derivative and apply the other IC:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n sin(\sqrt{2}nt) sin(nx) \\ u_t(x,t) &= \sum_{n=1}^{\infty} n\sqrt{2}A_n cos(\sqrt{2}nt) sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\sqrt{2}A_n sin(nx) = 1 \\ n\sqrt{2}A_n &= \frac{2}{\pi}\int_0^{\pi} sin(nx)dx \end{aligned} \tag{12}$$

We can recover the result of this integral from **(5)** above, giving:

$$\begin{aligned} n\sqrt{2}A_n &= \frac{2-2(-1)^n}{\pi n} \\ A_n &= \frac{2-2(-1)^n}{\sqrt{2}\pi n^2} \end{aligned} \tag{13}$$

Thus we have the full solution:

$$\boxed{u(x,t) = \frac{2}{\pi\sqrt{2}} \sum_{n=1}^{\infty} \Big[\frac{1-(-1)^n}{n^2} sin(\sqrt{2}nt) \Big] sin(nx) \qquad (14)}$$