AMATH 503: Homework 5
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$(\mathbf{1})$ If we have a Bessel Equation of the form:

$$(xy')' + (\lambda^2 x - \frac{m^2}{x})y = 0$$

$$0 < x < a \tag{1}$$

$$y \text{ bounded at } x = 0, \ , \ y(a) = 0$$

We know from the notes and previous homework that this will be solved by the eigenfunctions $J_m(\lambda x)$. The eigenfunctions are derived with the Frobenius solution, and the eigenvalues are implicitly determined by from the zeros of the eigenfunctions, which are cosine-like. Given this, let the eigenfunctions be $J_m(x)$ and the eigenvalues $(\lambda_{mn} = \frac{z_{mn}}{a})$ where z_{mn} are the zeros of the eigenfunction. We use the equation above and observe that this is a Sturm-Liouville system with:

$$p(x) = x$$

$$r(x) = x \tag{2}$$

$$q(x) = \frac{p^2}{x}$$

We now consider two pairs of eigenfunctions and eigenvalues, $(J_m(x); \lambda_{mn})$ and $(J_k(x)); \lambda_{kx})$ and plug them into the Bessel's Equation, giving:

$$(xJ_m(x)')' + (\lambda_{mn}^2 x - \frac{p^2}{x}) = 0$$
$$(xJ_k(x)')' + (\lambda_{kn}^2 x - \frac{p^2}{x}) = 0$$

(3)

We then follow the logic of the general proof of S-L orthogonality by multiplying the first by $J_k(x)$ and the second by $J_m(x)$ then subtracting one from the other:

$$J_k(x)(xJ_m(x)')' - J_m(x)(xJ_k(x)')' = (\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x)$$

(4)

The LHS is a derivative, so we rewrite as follows:

$$\frac{d}{dx}\Big[J_k(x)(xJ_m(x))')' - J_m(x)(J_k(x)')\Big] = \Big[(\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x)\Big]$$

(5)

We then integrate both sides giving:

$$\Big[J_k(x)(xJ_m(x))')' - J_m(x)(J_k(x)')\Big]\Big|_0^a = \int_0^a \Big[(\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x)\Big]dx$$

(6)

We then observe that, since this is a singular S-L system and $p(x) = 0$ at $x = 0$ and $x = a$. In this case, $p(x) = x$ so it' a little confusing, but let's just suppose we have a dummy variable for a moment, and $p(s) = s$. Then in a singular S-L system, $p(s) = x = 0$ when $s = 0, a$. From this we can infer that the LHS must be identically zero. This gives:

$$(\lambda_{mn} - \lambda_{kn}) \int_0^a xJ_m(x)J_k(x)dx = 0$$

(7)

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We can now simply observe that, if $\lambda_{mn} = \lambda_{kn}$, the leading constant becomes zero, and the integral becomes:

$$\int_0^a x(J_m(x))^2 dx \tag{8}$$

This integral is a positive constant since $x > 0$ for this Bessel function, and the eigenfunction is squared. The integrand $ax(J_m(x))^2 > 0$ and therefore the resulting integral will be a positive constant.

Alternatively, if $\lambda_{mn} \neq \lambda_{kn}$, this integral must be identically zero. The resulting integral is thus:

$$\int_0^a x J_m(x) J_k(x) dx = \begin{cases} 0 & \lambda_{mn} \neq \lambda_{kn} \\ \\ c & \lambda_{mn} = \lambda_{kn} \end{cases} \tag{9}$$

Where $c > 0$ is a constant.

(2)

(a) From the prompt we know that, with spherical symmetry, the 3D wave equation becomes:

$$u_{tt} = \frac{c^2 r}{(} r u)_{rr} \tag{10}$$

Bringing the r to the LHS, we can note that it is constant variable with respect to t, and we can write the PDE as:

$$(ru)_{tt} = c^2 (ru)_{rr} \tag{11}$$

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Now let's substitute $v = ru$ and plug the resulting equation into the D'Alembert Solution:

$$v_{tt} = c^2 v_{rr}$$

$$v(r,t) = \frac{1}{2}\Big[g(r - ct) + g(r + ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} h(s)ds\Big] \tag{12}$$

To transform g back to its equivalent term in u:

$$v(r,0) = g(r) \tag{13}$$

And the final form of the equation is:

$$u(r,0) = f(r) \tag{14}$$

Thus we have:

$$\frac{1}{r}v(r,0) = u(r,0)$$

$$\frac{1}{r}g(r) = f(r) \tag{15}$$

$$g(r) = rf(r)$$

Similarly, we observe that:

$$v_t = \frac{d}{dt}\left[\frac{1}{2c}\int_{r-ct}^{r+ct} h(s)ds\right]$$

$$v_t = \frac{1}{2c}\frac{d}{dt}\left[H(r+ct) - H(r-ct)\right]$$

$$v_t = \frac{1}{2c}\left[h(r+ct)(c) - h(r-ct)(-c)\right]$$

$$v_t = \frac{1}{2}\left[h(r+ct) + h(r-ct)\right.$$

$$v_t(r,0) = h(r)$$

$$ru_t(r,0) = h(r)$$

$$u_t(r,0) = \frac{1}{r}h(r)$$

(16)

So we define some new function:

$$rk(r) = h(r)$$

We then substitute all the transformed terms into the solution and get:

$$ru(r,t) = \frac{1}{2}\left[(r-ct)f(r-ct) + (r+ct)f(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} sk(s)ds\right]$$

$$u(r,t) = \frac{1}{2r}\left[(r-ct)f(r-ct) + (r+ct)f(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} sk(s)ds\right]$$

(17)