AMATH 503: Homework 3
Due April, 29 2019
ID: 1064712

Trent Yarosevich

April 29, 2019

Instructor: Ka-Kit Tung

(1)

(a) Using the formula for a complex Fourier series in the notes we have the following in a domain of $-2 < x < 2$ with $n = 0, \pm 1, \pm 2, \pm 3...$

$$|x| = \sum_{-\infty}^{\infty} c_n e^{\frac{-in\pi x}{2}} \tag{1}$$

Similarly, we obtain the coefficient c_n with the following:

$$c_n = \frac{1}{4} \int_{-2}^{2} |x| e^{\frac{in\pi x}{2}} dx \tag{2}$$

To solve this, we split the integral into two domains and integration by parts:

$$\begin{aligned}
c_n &= \frac{1}{4}\Big[\int_0^2 xe^{\frac{in\pi x}{2}}dx - \int_{-2}^0 xe^{\frac{in\pi x}{2}}dx\Big] \\
u &= x \ , \ du = 1 \\
dv &= e^{\frac{in\pi x}{2}} \\
v &= \frac{2}{in\pi}e^{\frac{in\pi x}{2}} \\
uv = \int vdu &= \frac{2x}{in\pi}e^{\frac{in\pi x}{2}} - \frac{4}{n^2\pi^2}e^{\frac{in\pi x}{2}} \\
c_n = \frac{1}{4}\Bigg[\Big[\frac{2xe^{\frac{in\pi x}{2}}}{in\pi} + \frac{4e^{\frac{in\pi x}{2}}}{n^2\pi^2}\Big]\Big|_0^2 &- \Big[\frac{2xe^{\frac{in\pi x}{2}}}{in\pi} + \frac{4e^{\frac{in\pi x}{2}}}{n^2\pi^2}\Big]\Big|_{-2}^0\Bigg] \\
c_n = \frac{1}{4}\Bigg[\Big[\frac{4e^{in\pi}}{in\pi} + \frac{4e^{in\pi}}{n^2\pi^2} - \frac{4}{n^2\pi^2}\Big] - \Big[\frac{4}{n^2\pi^2} &+ \frac{4e^{-in\pi}}{in\pi} + \frac{4e^{-in\pi}}{n^2\pi^2}\Big]\Bigg] \\
c_n &= \frac{-in\pi e^{in\pi} + in\pi e^{-in\pi} + e^{in\pi} + e^{-in\pi} - 2}{n^2\pi^2}
\end{aligned} \tag{3}$$

Now we observe that:

$$\begin{aligned}
e^{in\pi} &= cos(n\pi) + isin(n\pi) = cos(n\pi) = (-1)^n \\
e^{-in\pi} &= cos(n\pi) - isin(n\pi) = cos(n\pi) = (-1)^n \\
e^{in\pi} &= e^{-in\pi}
\end{aligned} \tag{4}$$

This allows for the following simplification:

$$c_n = \frac{2(-1)^n - 2}{n^2\pi^2} \tag{5}$$

This is obviously not defined for $n = 0$ however, and so we calculate this independently:

$$\begin{aligned}
c_0 &= \frac{1}{4}\int_{-2}^2 |x|dx \\
c_0 &= \frac{1}{4}\Big[[\frac{x^2}{2}]\Big|_0^2 - [\frac{x^2}{2}]\Big|_{-2}^0\Big] = 1
\end{aligned} \tag{6}$$

Putting all this togther, we arrive at the following Fourier series, noting that the even modes are zero (except c_0), and that $n = \pm 1, \pm 2, \pm 3...$

$$\boxed{|x| = 1 + \frac{2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n - 1}{n^2} e^{-\frac{in\pi x}{2}}} \tag{7}$$

(b) Using Euler's Formula we can rewrite the exponential above in the following way:

$$e^{-\frac{in\pi x}{2}} = cos(\frac{n\pi x}{2}) - isin(\frac{n\pi x}{2}) \tag{8}$$

If we observe that sin is an odd function, it follows that $isin(\frac{-n\pi x}{2}) = -isin(\frac{n\pi x}{2})$. This means that when $n < 0$, all the sin values in the sum will cancel with their corresponding values from $n > 0$, for example when $n = 1$ we have $-isin(\frac{\pi x}{2})$ and when $n = -1$ we have $isin(\frac{n\pi x}{2})$.

We can also observe that the even modes are zero, substituting $n = 2k - 1$ and define our sum with $k = 1, 2, 3....$ This means $c_n = \frac{-4}{n^2\pi^2}$ as well. The resulting cosine series is:

$$\boxed{|x| = 1 - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{cos(\frac{(2k-1)\pi x}{2})}{(2k-1)^2}} \tag{9}$$

(c)

$$\frac{d}{dx}[1 - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{cos(\frac{(2k-1)\pi x}{2})}{(2k-1)^2}] = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{sin(\frac{n\pi x}{2})}{2k-1} \tag{10}$$

This is the sine series for $f(x) = 1$ in the class notes, section (6.4.1). This is actually to be expected, since we know the derivative of $|x|$ is -1 from $-\infty < x < 0$ and 1 from $0 < x < \infty$. Furthermore, we know that a fourier sine series for a non-periodic function will only be accurate in the domain $0 < x < L$, and because it's an odd function, will be reflected accordingly outside that domain. Thus we would expect the Fourier series for the derivative of $|x|$ to be accurate from $-L < x < L$ and be $2L$ periodic outside this. The resulting function would be the sine series of $f(x) = 1$ and that is precisely what we got.

(2) We first define the fourier transforms in the different dimensions:

$$\begin{aligned} \mathcal{F}_1[u(x,y,z,t)] &= \int_{-\infty}^{\infty} u(x,y,z,t,)e^{i\omega_1 x}dx \\ \mathcal{F}_2[u(x,y,z,t)] &= \int_{-\infty}^{\infty} u(x,y,z,t,)e^{i\omega_2 y}dy \\ \mathcal{F}_3[u(x,y,z,t)] &= \int_{-\infty}^{\infty} u(x,y,z,t,)e^{i\omega_3 z}dz \end{aligned} \tag{11}$$

We can then fully define u in fourier space:

$$\mathcal{F}_1[\mathcal{F}_2[\mathcal{F}_3[u(x,y,z,t)]]] = U(\omega_1,\omega_2,\omega_3,t) \tag{12}$$

Now defining the terms in the PDE, in particular using the definition of a derivative in Fourier space from the notes:

$$\begin{aligned} \mathcal{F}_1\mathcal{F}_2\mathcal{F}_3[u_t] &= \frac{d}{dt}U(\omega_1,\omega_2,\omega_3,t) \\ \mathcal{F}_1\mathcal{F}_2\mathcal{F}_3[u_x x] &= (i\omega_1)^2U(\omega_1,\omega_2,\omega_3,t) \\ \mathcal{F}_1\mathcal{F}_2\mathcal{F}_3[u_y y] &= (i\omega_2)^2U(\omega_1,\omega_2,\omega_3,t) \\ \mathcal{F}_1\mathcal{F}_2\mathcal{F}_3[u_z z] &= (i\omega_3)^2U(\omega_1,\omega_2,\omega_3,t) \end{aligned} \tag{13}$$

We can now write out the full PDE in Fourier space:

$$\frac{d}{dt}U = -D(\omega_1^2+\omega_2^2+\omega_3^2)U \tag{14}$$

We then separate variables and integrate with respect to t, letting $D = \alpha^2$:

$$\begin{aligned} \frac{dU}{U} &= -\alpha^2(\omega_1^2 + \omega_2^2 + \omega_3^2)dt \\ lnU &= -\alpha^2(\omega_1^2 + \omega_2^2 + \omega_3^2)t + C \\ U &= Ae^{-\alpha^2(\omega_1^2+\omega_2^2+\omega_3^2)t} \end{aligned} \tag{15}$$

Applying the initial condition, we have:

$$\begin{aligned} A &= U(\omega_1, \omega_2, \omega_3, 0) \\ U(\omega_1, \omega_2, \omega_3, 0) &= \mathcal{F}_1\mathcal{F}_2\mathcal{F}_3\delta(x)\delta(y)\delta(z) \end{aligned} \tag{16}$$

As we learned in class, the integral across a delta function is equal to 1 because it is summing the area under an infinitely small Riemann rectangle. Thus we have:

$$U = e^{-\alpha^2(\omega_1^2+\omega_2^2+\omega_3^2)t} \tag{17}$$

To apply the reverse transform, we can split U up into three exponentials and apply the reverse transform to each one:

$$\begin{aligned} U &= e^{-\alpha^2\omega_1^2 t}e^{-\alpha^2\omega_2^2 t}e^{-\alpha^2\omega_3^2 t} \\ u(x, y, z, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\omega_1^2 t}e^{i\omega_1 x}d\omega_1 \\ &\quad * \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\omega_2^2 t}e^{i\omega_1 y}d\omega_2 \\ &\quad * \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\omega_3^2 t}e^{i\omega_1 z}d\omega_3 \end{aligned} \tag{18}$$

These integrals can be re-written in the form:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha^2\omega_1^2 t + i\omega_1 x}d\omega_1 \tag{19}$$

Using the general formula for Euler's Integral, taken from the class notes section (8.2), and noting that t is being treated as a constant here since we are integrating in respect to the omegas. The formula is as follows:

$$\int_{-\infty}^{\infty} e^{p^2\omega^2 \pm q\omega} d\omega = \frac{\sqrt{\pi}}{p} e^{\frac{q^2}{4p^2}} \tag{20}$$

Applying this to each integral, we get the answer:

$$\boxed{\begin{aligned} u(x,y,z,t) &= (\frac{1}{2\pi})^3 (\frac{\sqrt{\pi}}{\alpha\sqrt{t}})^3 e^{\frac{-x^2}{4\alpha^2 t}} e^{\frac{-y^2}{4\alpha^2 t}} e^{\frac{-z^2}{4\alpha^2 t}} \\ u(x,y,z,t) &= \frac{1}{8\pi^3})(\frac{\pi^{\frac{3}{2}}}{\alpha^3 t^{\frac{3}{2}}}) e^{-\frac{(x^2+y^2+z^2)}{4\alpha^2 t}} \\ u(x,y,z,t) &= \frac{1}{8\alpha^3 t^{\frac{3}{2}} \pi^{\frac{3}{2}}} e^{-\frac{(x^2+y^2+z^2)}{4\alpha^2 t}} \end{aligned} \quad (21)}$$