AMATH 503: Homework 5
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(1) If we use the eigenfunction and eigenvalue from the prompt, assuming we started with a a λ^2 in the function and not λ, we have:

$$\begin{aligned} \phi_p &= J_p(\lambda r) \\ \lambda_{pj} &= \frac{z_{jp}}{a} \end{aligned} \tag{1}$$

Where z_{jp} is the p'th zero of J. We know from the notes that for a cylindrical Bessel's function like this, we have the following values for p, r, q when $b = 0$ as it does here.

$$\begin{aligned} p(r) &= \lambda r \\ r(r) &= \lambda r \\ q(r) &= \frac{p^2}{\lambda r} \end{aligned} \tag{2}$$

We now consider two pairs of eigenfunctions and eigenvalues:

$$\phi_k = J_k(\lambda r); \lambda_k = \frac{z_{pk}}{a}$$
$$\phi_j = J_j(\lambda r); \lambda_j = \frac{z_{pj}}{a} \tag{3}$$

Plugging these into the Bessel function:

$$\begin{aligned}(p\phi_k')' + (\lambda_k x - \frac{p^2}{x})\phi_k = 0 \\ (p\phi_j')' + (\lambda_j x - \frac{p^2}{x})\phi_j = 0\end{aligned} \tag{4}$$

We then follow the logic of the general proof of S-L orthogonality by multiplying the first by ϕ_j and the second by ϕ_k then subtracting one from the other:

$$\phi_j(p\phi_k')' - \phi_k(p\phi_j)' = (\lambda_j - \lambda_k)r\phi_j\phi_k \tag{5}$$

The LHS is a derivative, so we rewrite as follows:

$$\frac{d}{dr}\Big[\phi_j(p\phi_k')' - \phi_k(p\phi_j')\Big] = (\lambda_j - \lambda_k)r(r)\phi_j\phi_k \tag{6}$$

We then integrate both sides giving:

$$\Big[\phi_j(p\phi_k')' - \phi_k(p\phi_j')\Big]\Big|_0^a = (\lambda_j - \lambda_k)\int_0^a r(r)\phi_j\phi_k dr \tag{7}$$

We can observe here that the LHS goes to zero because (i) p(0) = 0, and (ii) because at the boundary a the eigenfunctions will be $\phi_{j/k} = J_{j/k}(z_{mn}(\frac{a}{a})$ which goes to zero because the a's cancel, leaving only the zero of the eigenfunction.

This results in:

$$(\lambda_j - \lambda_k) \int_0^a r(r)\phi_j\phi_k dr = 0 \tag{8}$$

We can now simply observe that, if $\lambda_j = \lambda_k$, the leading constant becomes zero, and the integral becomes:

$$\int_0^a r(r)(\phi_j)^2 dr \tag{9}$$

This integral is a positive constant since the Bessel function defines $r(r) > 0$, and the eigenfunction is squared.

Alternatively, if $\lambda_j \neq \lambda_k$, this integral must be identically zero. The resulting integral is thus:

$$\int_0^a r(r)\phi_k\phi_j dr = \begin{cases} 0, & \lambda_k \neq \lambda_j \\ c, & \lambda_k = \lambda_j \end{cases} \tag{10}$$

Where $c > 0$ is a constant.

(2)

(a) It is extremely unclear to me what this problem is asking, and so I'm going to follow the derivation of the D'Alembert solution in the context of the problem.

From the prompt we know that, with spherical symmetry, the 3D wave equation becomes:

$$u_{tt} = \frac{c^2}{r}(ru)_{rr} \tag{11}$$

Bringing the r to the LHS, we can note that it is constant variable with respect to t, and we can write the PDE as:

$$(ru)_{tt} = c^2(ru)_{rr} \tag{12}$$

Let $z = ru$, and then we proceed to a D'Alembert solution, starting with a variable transform:

$$\begin{gathered} k = r + ct, s = r - ct \\ r = \frac{1}{2}(r+s), t = \frac{1}{2c}(r-s) \end{gathered} \tag{13}$$

The differential operators are now:

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial}{\partial t}\frac{\partial t}{\partial k} + \frac{\partial}{\partial r}\frac{\partial r}{\partial k} = \frac{1}{2c}\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r} = \frac{1}{2c}(\frac{\partial}{\partial t} + c\frac{\partial}{\partial r}) \\ \frac{\partial}{\partial s} &= \frac{\partial}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial}{\partial r}\frac{\partial r}{\partial s} = -\frac{1}{2c}\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r} = -\frac{1}{2c}(\frac{\partial}{\partial t} + c\frac{\partial}{\partial r}) \end{aligned} \tag{14}$$

Now if:

$$\begin{aligned} \frac{\partial^2}{\partial t^2}z - c^2\frac{\partial^2}{\partial r^2}z = 0 \\ (\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial r^2})z = 0 \\ (\frac{\partial}{\partial t} - c^2\frac{\partial}{\partial r})(\frac{\partial}{\partial t} - c^2\frac{\partial}{\partial r})z = 0 \\ \text{using our differential operators above:} \\ (2c\frac{\partial}{\partial k})(-2c\frac{\partial}{\partial s})z = 0 \\ (-4c^2)\frac{\partial}{\partial k}\frac{\partial}{\partial s}z = 0 \\ \frac{\partial}{\partial k}\frac{\partial}{\partial s}z = 0 \end{aligned} \tag{15}$$

Solving this we get:

$$\begin{aligned} \frac{\partial z}{\partial s} = f(s) \\ \text{where } f(s) \text{ is some function of } s. \\ z(s,k) = \int f(s) + G(k) \\ \text{where } G(k) \text{ is some function of } k. \\ z(s,k) = F(s) + G(k) \\ \text{Where } F \text{ is the antiderivative of } f\text{:} \end{aligned} \tag{16}$$

Substituting the original functions of k/s:

$$z(r,t) = F(r-ct) + G(r+ct) \tag{17}$$

Fitting this to general initial conditions:

$$\begin{aligned} z(r,0) = F(r) + G(r) = z_0 z_t(r,t) = -cF'(r-ct) + cG'(r-ct) \\ z_t(r,0) = -cF'(r) + cG'(r) = v_0 \end{aligned} \tag{18}$$

Integrating in order to get two equations with the same form of F and G:

$$-cF(r) + cG(r) = \int_0^r v_0(\bar{x})d\bar{x} \tag{19}$$

Now we multiply $z(r,0)$ above by c and add it to the second equation:

$$\begin{aligned} 2cG(r) = \int_0^r v_0(\bar{x})d\bar{x} + cz_0(r) \\ G(x) = \frac{1}{2c}(\int_0^r v_0(\bar{x})d\bar{x}) + \frac{1}{2}z_0(r) \end{aligned} \tag{20}$$

Next we multiply $z(r,0)$ by $-c$ and add to the second equation:

$$\begin{aligned} -2cF(r) = -cz_0(r) + \int_0^r v_0(\bar{x})d\bar{x} \\ F(r) = \frac{1}{2}z_0(r) - \frac{1}{2c}(\int_0^r v_0(\bar{x})d\bar{x}) \end{aligned} \tag{21}$$

To get our solution we simply plug these into our original function above, and combine the integrals by changing the sign of the second one and swapping its bounds of integration.

$$z(r,t) = \frac{1}{2}(z_0(r-ct) + z_0(r+ct)) + \frac{1}{2c}\int_{r-ct}^{r+ct} v_0(\bar{x})d\bar{x} \tag{22}$$

Allowing that $u = \frac{z}{r}$ the final solution is:

$$\boxed{u(r,t) = \frac{1}{2r}(z_0(r-ct) + z_0(r+ct)) + \frac{1}{2rc}\int_{r-ct}^{r+ct} v_0(\bar{x})d\bar{x}} \tag{23}$$

(b) The outgoing wave is the one determined by $z_0(r-ct)$, since this is the compression wave corresponding with an increasing r as t increases in order to track the wave's movement.

(3)

(a) In the steady state solution $u_{tt} = 0$ thus:

$$\begin{aligned}
u_{xx} &= -1 \\
u_x &= -x + c \\
u &= \frac{-x^2}{2} + cx + d \\
u(0) = 0 &= d \\
u(1) = 0 &= -\frac{1}{2} + c \\
c &= \frac{1}{2}
\end{aligned} \tag{24}$$

So the steady state solution is:

$$\boxed{u(x,t) = \frac{-x^2}{2} + \frac{x}{2}} \tag{25}$$

Suppose the transient solution is some $v = u - u_{\text{steady}}$. If we plug this into the PDE we get:

$$\begin{aligned}
v_{tt} - 0 = v_{xx} - \frac{d^2}{dx^2}\left(\frac{-x^2}{2} + \frac{x}{2}\right) + 1 \\
v_{tt} = v_{xx}
\end{aligned} \tag{26}$$

By separation of variables let $v = T(t)X(x)$. As we've seen countless times in the class thus far, this yields the solution:

$$\begin{gathered} v(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(\lambda_n t) + B_n cos(\lambda_n t)\Big] sin(\frac{n\pi x}{L}) \\ L = 1 \end{gathered} \tag{27}$$

Now we apply the BCs noting that $u(x,t) = v(x,t) - \frac{x^2}{2} + \frac{x}{2}$

$$\begin{aligned} u(x,0) = 0 = \sum_{n=1}^{\infty} B_n sin(n\pi x) - \frac{x^2}{2} + \frac{x}{2} \\ \sum_{n=1}^{\infty} B_n sin(n\pi x) = \frac{x^2}{2} - \frac{x}{2} \end{aligned} \tag{28}$$

This is a sine series and the coefficient B_n s given:

$$B_n = \int_0^1 x^2 sin(n\pi x)dx - \int_0^1 x sin(n\pi x)dx \tag{29}$$

Proceeding one integral at a time:

$$\begin{aligned} \int_0^1 x^2 sin(n\pi x)dx = \Big[\frac{-x^2 cos(n\pi x)}{n\pi}\Big]\Big|_0^1 + 2\int_0^1 \frac{x cos(n\pi x)}{n\pi}dx \\ \int_0^1 \frac{x cos(n\pi x)}{n\pi}dx = \Big[\frac{x sin}{(n^2\pi^2}\Big]\Big|_0^1 - \int_0^1 \frac{sin(n\pi x)}{n^2\pi^2}dx \\ \int_0^1 \frac{sin(n\pi x)}{n^2\pi^2}dx = \Big[\frac{cos(n\pi x)}{n^3\pi^3}\Big]\Big|_0^1 \end{aligned} \tag{30}$$

Putting these evaluations together we have:

$$\begin{aligned} \Big[\frac{-x^2 cos(n\pi x)}{n\pi}\Big]\Big|_0^1 + 2\Big[\Big[\frac{x sin}{(n^2\pi^2}\Big]\Big|_0^1 + \Big[\frac{cos(n\pi x)}{n^3\pi^3}\Big]\Big|_0^1\Big] \\ = \frac{-(-1)^n}{n\pi} + 2\Big[\frac{(-1)^n)}{n^3\pi^3} - \frac{1}{n^3\pi^3}\Big] \end{aligned} \tag{31}$$

Now the other integral:

$$\int_0^1 xsin(n\pi x)dx = \left[\frac{-xcos(n\pi x)}{n\pi}\right]\Big|_0^1 - \int_0^1 \frac{cos(n\pi x)}{n\pi}dx$$
$$\int_0^1 \frac{cos(n\pi x)}{n\pi}dx = \left[\frac{sin(n\pi x)}{n^2\pi^2}\right]\Big|_0^1 = 0 \tag{32}$$
$$\int_0^1 xsin(n\pi x)dx = \frac{-(-1)^n}{n\pi}$$

Combining the two terms:

$$B_n = \frac{-(-1)^n n^2\pi^2 + 2(-1)^n - 2 + (-1)^n n^2\pi^2}{n^3\pi^3}$$
$$B_n = \frac{2((-1)^n - 1)}{n^3\pi^3} \tag{33}$$

Now we define $n = 1, 3, 5...$ giving:

$$B_n = \frac{-4}{n^3\pi^3} \tag{34}$$

Applying the other initial condition we take the derivative:

$$u_t(x,t) = \sum_{n=1}^{\infty}\left[n\pi A_n cos(n\pi t) - n\pi B_n sin(n\pi t)\right]sin(n\pi x)$$
$$u_t(x,0) = \sum_{n=1}^{\infty} n\pi A_n sin(n\pi x) = 0 \tag{35}$$
$$A_n = 0$$

Thus the transient solution and complete solution are:

$$\boxed{\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty}\frac{-4}{n^3\pi^3}cos(n\pi t)sin(n\pi x) \\ u(x,t) &= \sum_{n=1}^{\infty}\frac{-4}{n^3\pi^3}cos(n\pi t)sin(n\pi x) - \frac{x^2}{2} + \frac{x}{2} \qquad (36) \\ n &= 1, 3, 5... \end{aligned}}$$

(b) First we determine the eigenfunctions and eigenvalues that will satisfy the boundary conditions. They are homogeneous dirchelet, and so the eigenfunction is $sin(n\pi x)$. We thus represent the homogeneous form of the PDE as:

$$u(x,t) = \sum_{n=1}^{\infty} T(t) sin(n\pi x) \tag{37}$$

We then represent the forcing term with the eigenfunction/values:

$$f(x,t) = 1 = \sum_{n=1}^{\infty} f_n sin(n\pi x) \tag{38}$$

Next we plug this into the PDE.

$$\begin{aligned} \sum_{n=1}^{\infty} T_n''(t)X(x) - X_n''(x)T_n(t) &= \sum_{n=1}^{\infty} f_n sin(n\pi x) \\ T_n''(t) sin(n\pi x) + (n\pi)^2 sin(n\pi x) T_n(t) &= f_n sin(n\pi x) \\ T_n''(t) + (n\pi)^2 T_n(t) &= f_n \end{aligned} \tag{39}$$

To proceed we'll need f_n, which is the coefficient for the sine series for $f(x) = 1$:

$$\begin{aligned} A_n = 2\int_0^1 sin(n\pi x) = 2\Big[\frac{-cos(n\pi x)}{n\pi}\Big]\Big|_0^1 \\ A_n = 2\frac{1-(-1)^n}{n\pi} \\ A_n = 0 \text{ for even } n, \\ A_n = \frac{4}{n\pi} \text{ for odd } n \end{aligned} \tag{40}$$

We thus have two scenarios:

$$\begin{aligned} n = 1, 3, 5... &\rightarrow T_n''(t) + (n\pi)^2 T_n(t) = \frac{4}{n\pi} \\ n = 2, 4, 6... &\rightarrow T_n''(t) + (n\pi)^2 T_n(t) = 0 \end{aligned} \tag{41}$$

In the even case, we know from major precedent that the solution is sines and cosines:

$$\begin{gathered} n = 2, 4, 6... \\ T_n(t) = A_n sin(n\pi t) + B_n cos(n\pi t) \end{gathered} \tag{42}$$

For the odd n case we need the homogeneous solution plus the particular. We already have the former, and so to find the particular solution we guess a solution in the form of the forcing term, i.e. a constant k and plug it into the PDE:

$$\begin{aligned} T(t)_{\text{particular}} &= k \\ k(n\pi)^2 &= \frac{4}{n\pi} \\ k &= \frac{4}{(n\pi)^3} \end{aligned} \tag{43}$$

Thus the complete solution for $T(t)$ for odd n is:

$$T_n(t) = \sum_{n=1}^{\infty} A_n sin(n\pi t) + B_n cos(n\pi t) + \frac{4}{(n\pi)^3} \tag{44}$$

Now, I'm not sure how to articulate this accurately, but I think the $n = 2, 4, 6..$ case is not a part of the solution, since it actually doesn't satisfy the PDE. I speculate that this is because f_n can be defined with strictly odd n. We will need to solve the homogeneous case, but the general solution is of the form:

$$\boxed{\begin{gathered} u(x,t) = \sum_{n=1}^{\infty} \left[A_n sin(n\pi t) + B_n cos(n\pi t) + \frac{4}{(n\pi)^3}\right] sin(n\pi x) \\ n = 1, 3, 5... \end{gathered} \tag{45}}$$

Applying the Initial Conditions:

$$u(x,0) = \sum_{n=1}^{\infty} \left[B_n + \frac{4}{(n\pi)^3} \right] sin(n\pi x) = 0$$

$$B_n = -\frac{4}{(n\pi)^3}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[n\pi A_n cos(n\pi t) - n\pi B_n sin(n\pi t) \right] sin(n\pi x) \tag{46}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left[n\pi A_n \right] sin(n\pi x) = 0$$

$$A_n = 0$$

This gives a final solution of:

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left[\frac{4}{(n\pi)^3}(1 - cos(n\pi t)) \right] sin(n\pi x) \\ n &= 1, 3, 5... \end{aligned} \qquad (47)}$$

If we note from **(22)** above that

$$n = 1, 3, 5..$$

$$\sum_{n=1}^{\infty} \frac{-4}{(n\pi)^3} sin(n\pi x) = \frac{x^2}{2} - \frac{x}{2} \tag{48}$$

$$\sum_{n=1}^{\infty} \frac{4}{(n\pi)^3} sin(n\pi x) = -\frac{x^2}{2} + \frac{x}{2}$$

We can see that the extra series term in **(42)** is equivalent to the non-series term in **(30)** and we have the same answer.