AMATH 503: Homework 2
Due April, 22 2019
ID: 1064712

Trent Yarosevich

April 21, 2019

Instructor: Ka-Kit Tung

(a)

A few preliminaries: I will frequently refer back to the results in section **(a)** as the work will be repeated throughout. I will also be using the ansatz solutions for ODEs from section 1 of the notes. Also please note that I will often use the fact that $cos(n\pi) = (-1)^n$.

Assuming a solution of the form $u(x,t) = X(x)T(t)$ we have the following, with K being an arbitrary constant.

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = K \tag{1}$$

If we suppose $K = 0$ we quickly get a trivial soution:

$$\begin{aligned} X''(x) &= 0 \\ X(x) &= Ax + B \\ X(0) &= A(0) + B = 0 \\ B &= 0X(\pi) = A(\pi) = 0 \\ A &= 0 \end{aligned} \tag{2}$$

Similarly, if $K > 0$ we have solutions of the form:

$$\begin{aligned} X(x) &= Ae^{\sqrt{x}} + Be^{-\sqrt{x}} \\ X(0) &= A + B = 0 \\ B &= -A \\ X(\pi) &= Ae^{\sqrt{\pi}} - Ae^{-\sqrt{\pi}} = 0 \\ A &= 0 \end{aligned} \tag{3}$$

These trivial solutions have been covered quite a bit, but I include them here for later reference. We now proceed with $-\lambda^2 = K < 0$.

$$\begin{aligned} X(x) &= Asin(\lambda x) + Bcos(\lambda x) \\ X(0) &= B = 0 \\ X(\pi) &= Asin(\lambda\pi) \\ \lambda\pi &= \pi n \\ \lambda &= n \\ X_n(x) &= A_n sin(nx) \end{aligned} \tag{4}$$

We combine this with the T equation to get a general solution, consolidating the arbitrary constant from the X equation, and then applying the ICs:

$$\begin{aligned} T_n(t) &= A_n sin(nt) + B_n cos(nt) \\ \sum_{n=1}^{\infty} &\Big[A_n sin(nt) + B_n cos(nt)\Big] sin(nx) \\ u(x,0) &= \sum_{n=1}^{\infty} B_n sin(nx) = 1 \\ B_n = \frac{2}{\pi}\int_0^{\pi} sin(nx)dx &= \frac{2}{\pi}\Big[\frac{-cos(nx)}{n}\Big]\Big|_0^{\pi} \\ B_n &= \frac{2}{\pi}(\frac{-cos(n\pi)+1}{n}) \\ B_n &= \frac{2-2(-1)^n}{\pi n} \end{aligned} \tag{5}$$

To apply the other IC we need the time derivative:

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} \Big[A_n n cos(nt) - B_n n sin(nt)\Big] sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} A_n sin(nx) = 0 \end{aligned} \tag{6}$$

It has been asserted in class and office hours without proof that for a series of this form to be equal to zero, the coefficient must be zero, thus we have $A_n = 0$. Combining these we have the complete solution:

$$\boxed{\begin{gathered} u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n} cos(nt)(sin(nx) \\ \text{Or perhaps more efficiently, let } j = 2n+1\text{:} \\ u(x,t) = \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{1-(-1)^j}{j} cos(jt)(sin(jx) \end{gathered} \qquad (7)}$$

(b) We can observe from sections **(2)** and **(3)** above that the addition of the coefficient of 2 in this problem will not change the triviality of solutions with

$K = 0$ and $K > 0$, therefore we move directly to solutions with $K < 0$. Given the formula,

$$\frac{T''(t)}{2T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \tag{8}$$

we can see that the $X(x)$ equation will be identical to **(4)**. We therefore move on to the $T(t)$ equation, again using the ansatz from the notes and the fact that $\lambda = n$:

$$T_n(t) = A_n sin(\sqrt{2}nt) + B_n cos(\sqrt{2}nt) \tag{9}$$

The resulting general solution before applying ICs is thus:

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(\sqrt{2}nt) + B_n cos(\sqrt{2}nt)\Big] sin(nx) \tag{10}$$

Now applying the first IC:

$$u(x,0) = \sum_{n=1}^{\infty} B_n sin(nx) = 0 \tag{11}$$

For the reasons given near **(6)** above $B_n = 0$, we now take the time derivative and apply the other IC:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n sin(\sqrt{2}nt) sin(nx) \\ u_t(x,t) &= \sum_{n=1}^{\infty} n\sqrt{2}A_n cos(\sqrt{2}nt) sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\sqrt{2}A_n sin(nx) = 1 \\ n\sqrt{2}A_n &= \frac{2}{\pi}\int_0^{\pi} sin(nx)dx \end{aligned} \tag{12}$$

We can recover the result of this integral from **(5)** above, giving:

$$\begin{aligned} n\sqrt{2}A_n &= \frac{2-2(-1)^n}{\pi n} \\ A_n &= \frac{2-2(-1)^n}{\sqrt{2}\pi n^2} \end{aligned} \tag{13}$$

Thus we have the full solution:

$$\boxed{\begin{gathered} u(x,t) = \frac{2}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\Big[\frac{1-(-1)^n}{n^2}sin(\sqrt{2}nt)\Big]sin(nx) \\ \text{Or perhaps more efficiently let } j = 2n+1\text{:} \\ u(x,t) = \frac{2}{\pi\sqrt{2}}\sum_{j=1}^{\infty}\Big[\frac{1-(-1)^j}{j^2}sin(\sqrt{2}jt)\Big]sin(jx) \end{gathered}} \tag{14}$$

(c) For the same reasons stated above in **(8)** and **(9)** we can move directly to the general solution, simply substituting $\sqrt{3}$ in place of $\sqrt{2}$:

$$u(x,t) = \sum_{n=1}^{\infty}\Big[A_n sin(\sqrt{3}nt) + B_n cos(\sqrt{3}nt)\Big]sin(nx) \tag{15}$$

We will first take the time derivative in order to apply an IC that is likely to eliminate a coefficient:

$$\begin{aligned} u_t(x,t) = \sum_{n=1}^{\infty}\Big[n\sqrt{3}A_n cos(\sqrt{3}nt) - n\sqrt{3}B_n sin(\sqrt{3}nt)\Big]sin(nx) \\ u_t(x,0) = \sum_{n=1}^{\infty} n\sqrt{3}A_n sin(nx) = 0 \end{aligned} \tag{16}$$

Again, for the reasons state in **(6)** $A_n = 0$. Making this substitution, we then apply the other IC:

$$u(x,t) = \sum_{n=1}^{\infty} B_n cos(\sqrt{3}nt) sin(nx)$$
$$u(x,0) = \sum_{n=1}^{\infty} B_n sin(nx) = sin^3(x) \tag{17}$$

Rather than using an integral to determine B_n here, we can simply observe that the objective is to obtain a sine series for $sin^3(x)$. By the triple-angle formula we can see that $sin^3 x = \frac{3}{4} sinx - \frac{1}{4} sin(3x)$. It is immediately clear that this *is* a sine series with only two terms at $n = 1$ and $n = 3$ with corresponding coefficients $B_1 = \frac{3}{4}$ and $B_3 = -\frac{1}{4}$. Since all other terms are zero, the resulting function is quite simple, since the only values remaining from the sum are n=1 and n=3.

$$\boxed{u(x,t) = \frac{3}{4} cos(\sqrt{3}t) sin(x) - \frac{1}{4} cos(3\sqrt{3}t) sin(3x) \tag{18}}$$

(d) As in **(15)** above, we can move directly to the following, substituting $\sqrt{4} = 2$ for $\sqrt{3}$:

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(2nt) + B_n cos(2nt) \Big] sin(nx) \tag{19}$$

Applying the first IC we have:

$$\begin{aligned}
u(x,0) = \sum_{n=1}^{\infty} B_n sin(nx) &= x \\
B_n = \frac{2}{\pi}\int_0^{\pi} xsin(x)dx \\
u = x \ , \ du = 1 \\
dv = sin(nx) \\
v = \frac{-1}{n}cos(nx) \\
uv - \int vdu = \frac{-xcos(nx)}{n} + \frac{sin(nx)}{n^2} \\
\Big[\frac{-xcos(nx)}{n} + \frac{sin(nx)}{n^2}\Big]\Big|_0^{\pi} = (\frac{-\pi cos(n\pi)}{n}) \\
B_n = \frac{2}{\pi}(\frac{-\pi(-1)^n}{n}) = \frac{-2(-1)^n}{n}
\end{aligned} \tag{20}$$

To apply the other IC we take the time derivative (note I will retain the arbitrary coefficients for brevity):

$$\begin{aligned}
u_t(x,t) = \sum_{n=1}^{\infty}\Big[2nA_n cos(2nt) - 2nB_n csin(2nt)\Big]sin(nx) \\
u_t(x,0) = \sum_{n=1}^{\infty} 2nA_n sin(nx) = -x \\
\text{Recoving the integral above:} \\
-2nA_n = \frac{2}{\pi}\int_0^{\pi} xsin(x)dx \\
-2nA_n = \frac{-2(-1)^n}{n} \\
A_n = \frac{(-1)^n}{n^2}
\end{aligned} \tag{21}$$

Substituting our coefficients, we arrive at the full solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\Big[\frac{(-1)^n}{n^2}sin(2nt) - \frac{2(-1)^n}{n}cos(2nt)\Big]sin(nx) \qquad (22)}$$