AMATH 503: Homework 2
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(a) **A few preliminaries:** I will frequently refer back to the results in section **(a)** as the work will be repeated throughout. I will also be using the ansatz solutions for ODEs from section 1 of the notes. Also please note that I will often use the fact that $cos(n\pi) = (-1)^n$.

Assuming a solution of the form $u(x,t) = X(x)T(t)$ we have the following, with K being an arbitrary constant.

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = K \tag{1}$$

If we suppose $K = 0$ we quickly get a trivial soution:

$$\begin{aligned} X''(x) &= 0 \\ X(x) &= Ax + B \\ X(0) &= A(0) + B = 0 \\ B = 0X(\pi) &= A(\pi) = 0 \\ A &= 0 \end{aligned} \tag{2}$$

Similarly, if $K > 0$ we have solutions of the form:

$$\begin{aligned} X(x) &= Ae^{\sqrt{x}} + Be^{-\sqrt{x}} \\ X(0) &= A + B = 0 \\ B &= -A \\ X(\pi) &= Ae^{\sqrt{\pi}} - Ae^{-\sqrt{\pi}} = 0 \\ A &= 0 \end{aligned} \tag{3}$$

These trivial solutions have been covered quite a bit, but I include them here for later reference. We now proceed with $-\lambda^2 = K < 0$.

$$\begin{aligned} X(x) &= Asin(\lambda x) + Bcos(\lambda x) \\ X(0) &= B = 0 \\ X(\pi) &= Asin(\lambda\pi) \\ \lambda\pi &= \pi n \\ \lambda &= n \\ X_n(x) &= A_n sin(nx) \end{aligned} \tag{4}$$

We combine this with the T equation to get a general solution, consolidating the arbitrary constant from the X equation, and then applying the ICs:

$$\begin{aligned} T_n(t) &= A_n sin(nt) + B_n cos(nt) \\ \sum_{n=1}^{\infty} \Big[A_n sin(nt) &+ B_n cos(nt) \Big] sin(nx) \\ u(x,0) &= \sum_{n=1}^{\infty} B_n sin(nx) = 1 \\ B_n = \frac{2}{\pi} \int_0^{\pi} sin(nx)dx &= \frac{2}{\pi} \Big[\frac{-cos(nx)}{n} \Big] \Big|_0^{\pi} \\ B_n &= \frac{2}{\pi} (\frac{-cos(n\pi) + 1)}{n}) \\ B_n &= \frac{2 - 2(-1)^n}{\pi n} \end{aligned} \tag{5}$$

To apply the other IC we need the time derivative:

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} \Big[A_n n cos(nt) - B_n n sin(nt) \Big] sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} A_n sin(nx) = 0 \end{aligned} \tag{6}$$

It has been asserted in class and office hours without proof that for a series of this form to be equal to zero, the coefficient must be zero, thus we have $A_n = 0$. Combining these we have the complete solution:

$$\boxed{u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} cos(nt)(sin(nx) \tag{7}}$$

(b) We can observe from sections **(2)** and **(3)** above that the addition of the coefficient of 2 in this problem will not change the triviality of solutions with $K = 0$ and $K > 0$, therefore we move directly to solutions with $K < 0$. Given the formula,

$$\frac{T''(t)}{2T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \tag{8}$$

we can see that the $X(x)$ equation will be identical to **(4)**. We therefore move on to the $T(t)$ equation, again using the ansatz from the notes and the fact that $\lambda = n$:

$$T_n(t) = A_n sin(\sqrt{2}nt) + B_n cos(\sqrt{2}nt) \tag{9}$$

The resulting general solution before applying ICs is thus:

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(\sqrt{2}nt) + B_n cos(\sqrt{2}nt)\Big] sin(nx) \tag{10}$$

Now applying the first IC:

$$u(x,0) = \sum_{n=1}^{\infty} B_n sin(nx) = 0 \tag{11}$$

For the reasons given near **(6)** above $B_n = 0$, we now take the time derivative and apply the other IC:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n sin(\sqrt{2}nt) sin(nx) \\ u_t(x,t) &= \sum_{n=1}^{\infty} n\sqrt{2}A_n cos(\sqrt{2}nt) sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\sqrt{2}A_n sin(nx) = 1 \\ n\sqrt{2}A_n &= \frac{2}{\pi}\int_0^{\pi} sin(nx)dx \end{aligned} \tag{12}$$

We can recover the result of this integral from **(5)** above, giving:

$$\begin{aligned} n\sqrt{2}A_n &= \frac{2-2(-1)^n}{\pi n} \\ A_n &= \frac{2-2(-1)^n}{\sqrt{2}\pi n^2} \end{aligned} \tag{13}$$

Thus we have the full solution:

$$u(x,t) = \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\Big[\frac{1-(-1)^n}{n^2}sin(\sqrt{2}nt)\Big]sin(nx) \tag{14}$$

(c) For the same reasons stated above in **(8)** and **(9)** we can move directly to the general solution, simply substituting $\sqrt{3}$ in place of $\sqrt{2}$:

$$u(x,t) = \sum_{n=1}^{\infty}\Big[A_n sin(\sqrt{3}nt) + B_n cos(\sqrt{3}nt)\Big]sin(nx) \tag{15}$$

We will first take the time derivative in order to apply an IC that is likely to eliminate a coefficient:

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty}\Big[n\sqrt{3}A_n cos(\sqrt{3}nt) - n\sqrt{3}B_n sin(\sqrt{3}nt)\Big]sin(nx) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\sqrt{3}A_n sin(nx) = 0 \end{aligned} \tag{16}$$

Again, for the reasons state in **(6)** $A_n = 0$. Making this substitution, we then apply the other IC:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n cos(\sqrt{3}nt)sin(nx) \\ u(x,0) &= \sum_{n=1}^{\infty} B_n sin(nx) = sin^3(x) \end{aligned} \tag{17}$$

Rather than using an integral to determine B_n here, we can simply observe that the objective is to obtain a sine series for $sin^3(x)$. By the triple-angle formula we can see that $sin^3x = \frac{3}{4}sinx - \frac{1}{4}sin(3x)$. It is immediately clear that this *is* a sine series with only two terms at $n = 1$ and $n = 3$ with corresponding coefficients $B_1 = \frac{3}{4}$ and $B_3 = -\frac{1}{4}$. Since all other terms are zero, the resulting function is quite simple, since the only values remaining from the sum are n=1 and n=3.

$$\boxed{u(x,t) = \frac{3}{4}cos(\sqrt{3}t)sin(x) - \frac{1}{4}cos(3\sqrt{3}t)sin(3x) \qquad (18)}$$

(d) As in **(15)** above, we can move directly to the following, substituting $\sqrt{4} = 2$ for $\sqrt{3}$:

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(2nt) + B_n cos(2nt)\Big] sin(nx) \tag{19}$$

Applying the first IC we have:

$$\begin{aligned} u(x,0) = \sum_{n=1}^{\infty} B_n sin(nx) &= x \\ B_n = \frac{2}{\pi}\int_0^{\pi} xsin(x)dx \\ u = x \ , \ du &= 1 \\ dv &= sin(nx) \\ v &= \frac{-1}{n}cos(nx) \\ uv - \int vdu &= \frac{-xcos(nx)}{n} + \frac{sin(nx)}{n^2} \\ \Big[\frac{-xcos(nx)}{n} + \frac{sin(nx)}{n^2}\Big]\Big|_0^{\pi} &= (\frac{-\pi cos(n\pi)}{n}) \\ B_n = \frac{2}{\pi}(\frac{-\pi(-1)^n}{n}) &= \frac{-2(-1)^n}{n} \end{aligned} \tag{20}$$

To apply the other IC we take the time derivative (note I will retain the arbitrary coefficients for brevity):

$$\begin{aligned} u_t(x,t) = \sum_{n=1}^{\infty}\Big[2nA_n cos(2nt) - 2nB_n csin(2nt)\Big]sin(nx) \\ u_t(x,0) = \sum_{n=1}^{\infty} 2nA_n sin(nx) = -x \\ \text{Recoving the integral above:} \\ -2nA_n = \frac{2}{\pi}\int_0^{\pi} xsin(x)dx \\ -2nA_n = \frac{-2(-1)^n}{n} \\ A_n = \frac{(-1)^n}{n^2} \end{aligned} \tag{21}$$

Substituting our coefficients, we arrive at the full solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\Big[\frac{(-1)^n}{n^2}sin(2nt) - \frac{2(-1)^n}{n}cos(2nt)\Big]sin(nx) \qquad (22)}$$

(e) The TA asserted in Recitation on Friday 4/12 that with mixed BCs one still need only check $K < 0$. However, it is easy to see in **(2)** above that if $K = 0$ we would be applying a BC to get $X'(\pi) = C = 0$, and then similarly for $X(0) = D = 0$. $K > 0$ similarly still leads to a situation in which two exponentials with different powers were equal, once again requiring the trivial solution.

$$\begin{aligned} X(0) = A + B = 0 \\ B = -AX'(x) = \frac{Ae^{\sqrt{x}}}{2\sqrt{x}} + \frac{Ae^{-\sqrt{x}}}{2\sqrt{x}} \end{aligned} \tag{23}$$

We can immediately see that X'(0) is not even defined, and so obviously cannot match any boundary conditions. We thus move forward with $K = -\lambda^2 < 0$. By ansatz, we then take the derivative and apply the BCs:

$$\begin{aligned}X(x) &= Asin(\lambda x) + Bcos(\lambda x)\\ X(0) &= B_n = 0\\ X'(x) &= \lambda Acos(\lambda x)\\ X'(\pi) &= \lambda Acos(\lambda \pi) = 0\\ \lambda\pi &= (n+\frac{1}{2})\pi\\ \lambda &= (n+\frac{1}{2})\end{aligned} \tag{24}$$

Combining with the general solution for $T(t)$ that has been used repeatedly above we have:

$$u(x,t) = \sum_{n=1}^{\infty}\Big[A_n sin((n+\frac{1}{2})t) + B_n cos((n+\frac{1}{2})t)\Big] sin((n+\frac{1}{2})x) \tag{25}$$

First I'll take the time derivative in order to eliminate a coefficient:

$$\begin{aligned}u_t(x,t) = \sum_{n=1}^{\infty}\Big[(n+\frac{1}{2})A_n cos((n+\frac{1}{2})t) - (n+\frac{1}{2})B_n sin((n+\frac{1}{2})t)\Big] sin((n+\frac{1}{2})x)\\ u_t(x,0) = \sum_{n=1}^{\infty}(n+\frac{1}{2})A_n sin((n+\frac{1}{2})x) = 0\end{aligned} \tag{26}$$

As in **(6)** above, $A_n = 0$. We now apply the other IC:

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\Big[B_n cos((n+\frac{1}{2})t)\Big] sin((n+\frac{1}{2})x)\\ u(x,0) &= \sum_{n=1}^{\infty} B_n sin((n+\frac{1}{2})x)\end{aligned} \tag{27}$$

This gives modes that we are not used to, so in accordance with what we learned in Recitation, I'll extend the domain to $0 < x < 2\pi$ and take the even extension of the initial condition, $f(x) = 1$, which is easy since it's the same function with the expanded domain. We then let $n + \frac{1}{2} = \frac{j}{2}$ with $j = 1, 3, 5, 7...$ and calculate the coefficient as follows:

$$\begin{aligned} C_j = \frac{1}{\pi}\int_0^{2\pi} sin(\frac{jx}{2})dx = \frac{1}{\pi}\Big[\frac{-2cos\frac{jx}{2}}{j}\Big]\Big|_0^{2\pi} \\ C_j = \frac{1}{\pi}\Big[\frac{-2cos\pi j + 2}{j}\Big] \end{aligned} \tag{28}$$

If j were even, C_j would be equal to zero, but we've defined it as odd only. So if j is only odd and we substitute n back in by observing that $j = 2n + 1$ we have:

$$\begin{aligned} C_j = \frac{4}{\pi j} \\ B_n = \frac{4}{\pi(2n+1)} \end{aligned} \tag{29}$$

Substituting this back into the general solution we have the final solution satisfying the PDE and IC/BCs:

$$\boxed{u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{cos((n+\frac{1}{2})t)sin((n+\frac{1}{2})x)}{(2n+1)\pi}} \tag{30}$$

(f) Because this problem has entirely different BCs from the previous ones, namely Neumann Boundary Conditions, I started by checking $K = 0$. Recovering the equation from **(2)** above we have:

$$\begin{aligned} X''(x) = 0 \\ X'(x) = C \end{aligned} \tag{31}$$

We here we can apply either BC, and we should have:

$$\begin{aligned} X'(0) = X'(2\pi) = C = 0 \\ C = 0 \\ X'(x) = 0 \\ X(x) = D \end{aligned} \tag{32}$$

Thus $X(x)$ is an arbitrary constant. Let $D = 1$ and we'll move on to the $T(t)$ equation. From **(1)** we have the following equation, to which we'll apply the ICs:

$$\begin{aligned} \frac{T''(t)}{2T(t)} = 0 \\ T''(t) = 0 \\ T'(t) = C \\ T'(0) = C = 1 \\ C = 1 \\ T(t) = t + D \\ T(0) = 0 + D = -1 \\ D = -1 \end{aligned} \tag{33}$$

Thus we have already have a solution that satisfies the PDE and the Initial and Boundary Conditions. This worked out so easily with $K = 0$ because the ICs were simple linear functions. ****Please Note** at the Thursday 4/18 Office Hours, Alex said that if we find a satisfactory solution with $K = 0$, there is no need to continue on to check the other conditions for K. I would speculate that the others would either give trivial solutions, or the same solution? (I would check but I'm really busy...). The final answer, while simple, still satisfies the PDE, ICs and BCs:

$$\boxed{u(x,t) = t - 1 \qquad (34)}$$

(g) Again to be thorough, I started by checking the conditions for K, starting with $K = 0$. Recovering the results of **(31)** and **(32)** we have $X(x) = 1$ and:

$$\begin{aligned} T(t) &= C \\ T(0) = C &= x(x-1) \\ C &\neq x(x-1) \end{aligned} \tag{35}$$

So clearly $K = 0$ will not let us satisfy the ICs. If $K > 0$ we have the same situation as **(23)** and the function is not defined at the boundary conditions. Thus we move on again to $K = -\lambda^2 < 0$

$$\begin{aligned} X(x) &= Asin(\lambda x) + Bcos(\lambda x) \\ X'(x) &= \lambda Acos(\lambda x) - B\lambda sin(\lambda x) \\ X'(0) &= \lambda A = 0 \\ \lambda &= 0 \text{ is trivial, thus:} \\ A &= 0 \\ X'(1) &= -B\lambda sin(\lambda) = 0 \\ \lambda &= n\pi \\ X_n(x) &= \sum_{n=1}^{\infty} B_n cos(n\pi x) \end{aligned} \tag{36}$$

Consolidating arbitrary constants we have the general solution:

$$\sum_{n=1}^{\infty} \Big[A_n sin(n\pi t) + B_n cos(n\pi t)\Big] sin(n\pi x) \tag{37}$$

First I'll take the derivative to use one of the ICs to eliminate a coefficient:

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} \Big[\pi n A_n cos(n\pi t) - \pi n B_n sin(n\pi t)\Big] sin(n\pi x) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\pi A_n cos(n\pi x) = 0 \\ A_n &= 0 \end{aligned} \tag{38}$$

Thus we have:

$$u(x,t) = \sum_{n=1}^{\infty} B_n cos(n\pi t) cos(n\pi x) \tag{39}$$

Here we have something different, a cosine series! Using the derivation of the formula from the notes, which makes use of the orthogonality of the basis, we have:

$$\begin{aligned} B_0 &= \int_0^1 x(1-x)dx = \int_0^1 xdx - \int_0^1 x^2 dx \\ B_0 &= \Big[\frac{x^2}{2}\Big]\Big|_0^1 - \Big[\frac{x^3}{3}\Big]\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \end{aligned} \tag{40}$$

And for B_n we have the following very long integral!

$$B_n = 2\int_0^1 x(1-x)cos(n\pi x)dx = 2\int_0^1 xcos(n\pi x)dx - 2\int_0^1 x^2 cos(n\pi x)dx \tag{41}$$

Starting with the first integral:

$$\begin{aligned} u &= x \ , \ du = 1 \\ dv &= cos(n\pi x) \\ v &= \frac{sin(n\pi x)}{n\pi} \\ \int xcos(n\pi x)dx &= \frac{xsin(n\pi x)}{n\pi} - \int \frac{sin(n\pi x)}{n\pi} dx \\ \int xcos(n\pi x)dx &= \frac{xsin(n\pi x)}{n\pi} + \frac{cos(n\pi x)}{n^2\pi^2} \\ \Big[\frac{xsin(n\pi x)}{n\pi} + \frac{cos(n\pi x)}{n^2\pi^2}\Big]\Big|_0^1 &= \frac{cos(n\pi) - 1}{n^2\pi^2} \end{aligned} \tag{42}$$

Moving on to the second integral we have:

$$\begin{aligned} u = x^2 \ , \ du = 2x \\ dv = cos(n\pi x) \\ v = \frac{sin(n\pi x)}{n\pi} \\ \int x^2 cos(n\pi x)dx = \frac{x^2 sin(n\pi x)}{n\pi} - \frac{2}{n\pi}\int xsin(n\pi x)dx \end{aligned} \tag{43}$$

We now pause to solve this integral:

$$\begin{aligned} u = x \ , \ du = 1 \\ dv = sin(n\pi x) \\ v = -\frac{cos(n\pi x)}{n\pi} \\ \int xsin(n\pi x)dx = -\frac{xcos(n\pi x)}{n\pi} + \int \frac{cos(n\pi x)}{n\pi} \\ \int xsin(n\pi x)dx = -\frac{xcos(n\pi x)}{n\pi} + \frac{sin(n\pi x)}{n^2\pi^2} \end{aligned} \tag{44}$$

Plugging this back into the ongoing integral we get:

$$\begin{aligned} \int_0^1 x^2 cos(n\pi x)dx = \Big[\frac{x^2 sin(n\pi x)}{n\pi} - \frac{2}{n\pi}\Big[\frac{-xcos(n\pi x)}{n\pi} + \frac{sin(n\pi x)}{n^2\pi^2}\Big]\Big]\Big|_0^1 \\ \int_0^1 x^2 cos(n\pi x)dx = \frac{2cos(n\pi)}{n^2\pi^2} \end{aligned} \tag{45}$$

Now combining this with the results from **(42)** we have:

$$\begin{aligned} B_n = 2\Big[\frac{cos(n\pi) - 1}{n^2\pi^2} - \frac{2cos(n\pi)}{n^2\pi^2}\Big] \\ B_n = \frac{-2 - 2cos(n\pi)}{n^2\pi^2} \end{aligned} \tag{46}$$

Putting all this together, at last we have:

$$\boxed{u(x,t) = \frac{1}{6} - 2\sum_{n=1}^{\infty}\frac{1+cos(n\pi)}{n^2\pi^2}cos(n\pi x)cos(n\pi t) \qquad (47)}$$