AMATH 503: Homework 5
Due May, 28 2019
ID: 1064712

Trent Yarosevich

May 27, 2019

Instructor: Professor Ka-Kit Tung

(1) If we have a Bessel Equation of the form:

$$
\begin{aligned}
(xy')' + (\lambda^2 x - \frac{m^2}{x})y = 0 \\
0 < x < a \qquad (1) \\
y \text{ bounded at } x = 0, \ , y(a) = 0
\end{aligned}
$$

We know from the notes and previous homework that this will be solved by the eigenfunctions $J_m(\lambda x)$. The eigenfunctions are derived with the Frobenius solution, and the eigenvalues are implicitly determined by from the zeros of the eigenfunctions, which are cosine-like. Given this, let the eigenfunctions be $J_m(x)$ and the eigenvalues $(\lambda_{mn} = \frac{z_{mn}}{a})$ where z_{mn} are the zeros of the eigenfunction. We use the equation above and observe that this is a Sturm-Liouville system with:

$$
\begin{aligned}
p(x) = x \\
r(x) = x \qquad (2) \\
q(x) = \frac{p^2}{x}
\end{aligned}
$$

We now consider two pairs of eigenfunctions and eigenvalues, $(J_m(x); \lambda_{mn})$ and $(J_k(x)); \lambda_{kx})$ and plug them into the Bessel's Equation, giving:

$$\begin{aligned} (xJ_m(x)')' + (\lambda_{mn}^2 x - \frac{p^2}{x}) = 0 \\ (xJ_k(x)')' + (\lambda_{kn}^2 x - \frac{p^2}{x}) = 0 \end{aligned} \tag{3}$$

We then follow the logic of the general proof of S-L orthogonality by multiplying the first by $J_k(x)$ and the second by $J_m(x)$ then subtracting one from the other:

$$J_k(x)(xJ_m(x)')' - J_m(x)(xJ_k(x)')' = (\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x) \tag{4}$$

The LHS is a derivative, so we rewrite as follows:

$$\frac{d}{dx}\Big[J_k(x)(xJ_m(x))') - J_m(x)(J_k(x)')\Big] = \Big[(\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x)\Big] \tag{5}$$

We then integrate both sides giving:

$$\Big[J_k(x)(xJ_m(x))') - J_m(x)(J_k(x)')\Big]\Big|_0^a = \int_0^a \Big[(\lambda_{mn} - \lambda_{kn})xJ_m(x)J_k(x)\Big]dx \tag{6}$$

We then observe that, since this is a singular S-L system and $p(x) = 0$ at $x = 0$ and $x = a$. In this case, $p(x) = x$ so it' a little confusing, but let's just suppose we have a dummy variable for a moment, and $p(s) = s$. Then in a singular S-L system, $p(s) = x = 0$ when $s = 0, a$. From this we can infer that the LHS must be identically zero. This gives:

$$(\lambda_{mn} - \lambda_{kn}) \int_0^a xJ_m(x)J_k(x)dx = 0 \tag{7}$$

We can now simply observe that, if $\lambda_{mn} = \lambda_{kn}$, the leading constant becomes zero, and the integral becomes:

$$\int_0^a x(J_m(x))^2 dx \tag{8}$$

This integral is a positive constant since $x > 0$ for this Bessel function, and the eigenfunction is squared. The integrand $ax(J_m(x))^2 > 0$ and therefore the resulting integral will be a positive constant.

Alternatively, if $\lambda_{mn} \neq \lambda_{kn}$, this integral must be identically zero. The resulting integral is thus:

$$\boxed{\int_0^a xJ_m(x)J_k(x)dx = \begin{cases} 0 & \lambda_{mn} \neq \lambda_{kn} \\ c & \lambda_{mn} = \lambda_{kn} \end{cases} \quad (9)}$$

Where $c > 0$ is a constant.

(2)

(a) From the prompt we know that, with spherical symmetry, the 3D wave equation becomes:

$$u_{tt} = \frac{c^2}{r}(ru)_{rr} \tag{10}$$

Bringing the r to the LHS, we can note that it is constant variable with respect to t, and we can write the PDE as:

$$(ru)_{tt} = c^2(ru)_{rr} \tag{11}$$

Now let's substitute $v = ru$ and plug the resulting equation into the D'Alembert Solution:

$$v_{tt} = c^2 v_{rr}$$
$$v(r,t) = \frac{1}{2}\Big[g(r-ct) + g(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} h(s)ds \tag{12}$$

To transform g back to its equivalent term in u:

$$v(r,0) = g(r) \tag{13}$$

And the final form of the equation is:

$$u(r,0) = f(r) \tag{14}$$

Thus we have:

$$\begin{aligned} \frac{1}{r}v(r,0) &= u(r,0) \\ \frac{1}{r}g(r) &= f(r) \\ g(r) &= rf(r) \end{aligned} \tag{15}$$

Similarly, we observe that:

$$
\begin{aligned}
v_t &= \frac{d}{dt}\Big[\frac{1}{2c}\int_{r-ct}^{r+ct} h(s)ds\Big] \\
v_t &= \frac{1}{2c}\frac{d}{dt}\Big[H(r+ct) - H(r-ct)\Big] \\
v_t &= \frac{1}{2c}\Big[h(r+ct)(c) - h(r-ct)(-c)\Big] \\
v_t &= \frac{1}{2}\Big[h(r+ct) + h(r-ct) \\
v_t(r,0) &= h(r) \\
ru_t(r,0) &= h(r) \\
u_t(r,0) &= \frac{1}{r}h(r) \\
\text{So we define some new function:} \\
rk(r) &= h(r)
\end{aligned}
\tag{16}
$$

We then substitute all the transformed terms into the solution and get:

$$
\boxed{
\begin{aligned}
ru(r,t) &= \frac{1}{2}\Big[(r-ct)f(r-ct) + (r+ct)f(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} sk(s)ds \\
u(r,t) &= \frac{1}{2r}\Big[(r-ct)f(r-ct) + (r+ct)f(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} sk(s)ds
\end{aligned}
\quad (17)}
$$

(3)

(a) In the steady state solution $u_{tt} = 0$ thus:

$$\begin{aligned} u_{xx} &= -1 \\ u_x &= -x + c \\ u &= \frac{-x^2}{2} + cx + d \\ u(0) &= 0 = d \\ u(1) &= 0 = -\frac{1}{2} + c \\ c &= \frac{1}{2} \end{aligned} \tag{18}$$

So the steady state solution is:

$$\boxed{u(x,t) = \frac{-x^2}{2} + \frac{x}{2}} \tag{19}$$

Suppose the transient solution is some $v = u - u_{\text{steady}}$. If we plug this into the PDE we get:

$$\begin{aligned} v_{tt} - 0 = v_{xx} - \frac{d^2}{dx^2}(\frac{-x^2}{2} + \frac{x}{2}) + 1 \\ v_{tt} = v_{xx} \end{aligned} \tag{20}$$

By separation of variables let $v = T(t)X(x)$. As we've seen countless times in the class thus far, this yields the solution:

$$\begin{aligned} v(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(\lambda_n t) + B_n cos(\lambda_n t)\Big] sin(\frac{n\pi x}{L}) \\ L = 1 \end{aligned} \tag{21}$$

Now we apply the BCs noting that $u(x,t) = v(x,t) - \frac{x^2}{2} + \frac{x}{2}$

$$\begin{aligned} u(x,0) = 0 = \sum_{n=1}^{\infty} B_n sin(n\pi x) - \frac{x^2}{2} + \frac{x}{2} \\ \sum_{n=1}^{\infty} B_n sin(n\pi x) = \frac{x^2}{2} - \frac{x}{2} \end{aligned} \tag{22}$$

This is a sine series and the coefficient B_n s given:

$$B_n = \int_0^1 x^2 sin(n\pi x)dx - \int_0^1 xsin(n\pi x)dx \tag{23}$$

Proceeding one integral at a time:

$$\begin{aligned} \int_0^1 x^2 sin(n\pi x)dx = \Big[\frac{-x^2 cos(n\pi x)}{n\pi}\Big]\Big|_0^1 + 2\int_0^1 \frac{xcos(n\pi x)}{n\pi}dx \\ \int_0^1 \frac{xcos(n\pi x)}{n\pi}dx = \Big[\frac{xsin}{(n^2\pi^2}\Big]\Big|_0^1 - \int_0^1 \frac{sin(n\pi x)}{n^2\pi^2}dx \\ \int_0^1 \frac{sin(n\pi x)}{n^2\pi^2}dx = \Big[\frac{cos(n\pi x)}{n^3\pi^3}\Big]\Big|_0^1 \end{aligned} \tag{24}$$

Putting these evaluations together we have:

$$\begin{aligned} \Big[\frac{-x^2 cos(n\pi x)}{n\pi}\Big]\Big|_0^1 + 2\Big[\Big[\frac{xsin}{(n^2\pi^2}\Big]\Big|_0^1 + \Big[\frac{cos(n\pi x)}{n^3\pi^3}\Big]\Big|_0^1\Big] \\ = \frac{-(-1)^n}{n\pi} + 2\Big[\frac{(-1)^n)}{n^3\pi^3} - \frac{1}{n^3\pi^3}\Big] \end{aligned} \tag{25}$$

Now the other integral:

$$\begin{aligned} \int_0^1 xsin(n\pi x)dx = \Big[\frac{-xcos(n\pi x)}{n\pi}\Big]\Big|_0^1 - \int_0^1 \frac{cos(n\pi x)}{n\pi}dx \\ \int_0^1 \frac{cos(n\pi x)}{n\pi}dx = \Big[\frac{sin(n\pi x)}{n^2\pi^2}\Big]\Big|_0^1 = 0 \\ \int_0^1 xsin(n\pi x)dx = \frac{-(-1)^n}{n\pi} \end{aligned} \tag{26}$$

Combining the two terms:

$$\begin{aligned} B_n &= \frac{-(-1)^n n^2\pi^2 + 2(-1)^n - 2 + (-1)^n n^2\pi^2}{n^3\pi^3} \\ B_n &= \frac{2((-1)^n - 1)}{n^3\pi^3} \end{aligned} \tag{27}$$

Now we define $n = 1, 3, 5...$ giving:

$$B_n = \frac{-4}{n^3\pi^3} \tag{28}$$

Applying the other initial condition we take the derivative:

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} \Big[n\pi A_n cos(n\pi t) - n\pi B_n sin(n\pi t) \Big] sin(n\pi x) \\ u_t(x,0) &= \sum_{n=1}^{\infty} n\pi A_n sin(n\pi x) = 0 \\ A_n &= 0 \end{aligned} \tag{29}$$

Thus the transient solution and complete solution are:

$$\boxed{\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} \frac{-4}{n^3\pi^3} cos(n\pi t) sin(n\pi x) \\ u(x,t) &= \sum_{n=1}^{\infty} \frac{-4}{n^3\pi^3} cos(n\pi t) sin(n\pi x) - \frac{x^2}{2} + \frac{x}{2} \\ n &= 1, 3, 5... \end{aligned} \qquad (30)}$$

(b) First we determine the eigenfunctions and eigenvalues that will satisfy the boundary conditions. They are homogeneous dirchelet, and so the eigenfunction is $sin(n\pi x)$. We thus represent the homogeneous form of the PDE as:

$$u(x,t) = \sum_{n=1}^{\infty} T(t) sin(n\pi x) \tag{31}$$

We then represent the forcing term with the eigenfunction/values:

$$f(x,t) = 1 = \sum_{n=1}^{\infty} f_n sin(n\pi x) \tag{32}$$

Next we plug this into the PDE.

$$\begin{aligned} \sum_{n=1}^{\infty} T_n''(t)X(x) - X_n''(x)T_n(t) &= \sum_{n=1}^{\infty} f_n sin(n\pi x) \\ T_n''(t)sin(n\pi x) + (n\pi)^2 sin(n\pi x)T_n(t) &= f_n sin(n\pi x) \\ T_n''(t) + (n\pi)^2 T_n(t) &= f_n \end{aligned} \tag{33}$$

To proceed we'll need f_n, which is the coefficient for the sine series for $f(x) = 1$:

$$\begin{aligned} A_n = 2\int_0^1 sin(n\pi x) &= 2\Big[\frac{-cos(n\pi x)}{n\pi}\Big]\Big|_0^1 \\ A_n &= 2\frac{1-(-1)^n}{n\pi} \\ A_n &= 0 \text{ for even } n, \\ A_n &= \frac{4}{n\pi} \text{ for odd } n \end{aligned} \tag{34}$$

We thus have two scenarios:

$$\begin{aligned} n = 1, 3, 5... \rightarrow T_n''(t) + (n\pi)^2 T_n(t) &= \frac{4}{n\pi} \\ n = 2, 4, 6... \rightarrow T_n''(t) + (n\pi)^2 T_n(t) &= 0 \end{aligned} \tag{35}$$

In the even case, we know from major precedent that the solution is sines and cosines:

$$\begin{aligned} n &= 2, 4, 6... \\ T_n(t) = A_n sin(n\pi t) + B_n cos(n\pi t) \end{aligned} \tag{36}$$

For the odd n case we need the homogeneous solution plus the particular. We already have the former, and so to find the particular solution we guess a solution in the form of the forcing term, i.e. a constant k and plug it into the PDE:

$$\begin{aligned} T(t)_{\text{particular}} &= k \\ k(n\pi)^2 &= \frac{4}{n\pi} \\ k &= \frac{4}{(n\pi)^3} \end{aligned} \tag{37}$$

Thus the complete solution for $T(t)$ for odd n is:

$$T_n(t) = A_n sin(n\pi t) + B_n cos(n\pi t) + \frac{4}{(n\pi)^3} \tag{38}$$

And the complete general solutions are:

$$\begin{aligned} n = 2,4,6 \rightarrow u(x,t) &= \sum_{n=1}^{\infty} \Big[A_n sin(n\pi t) + B_n cos(n\pi t)\Big] sin(n\pi x) \\ n = 1,3,5... \rightarrow u(x,t) &= \sum_{n=1}^{\infty} \Big[A_n sin(n\pi t) + B_n cos(n\pi t) + \frac{4}{(n\pi)^3}\Big] sin(n\pi x) \end{aligned} \tag{39}$$

However, and I'm not fully clear on the explanation for this, the solution when n is even will simply go to zero if one applies the boundary conditions. I suppose there is a sense in which this is to be expected, since the forcing term itself doesn't include even modes in the sine series used to represent it. I suppose, then, that those cases were irrelevant to begin with (though we did need to find the homogeneous solution as part of the process for solving the odd n). In any case the full general solution is thus:

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \Big[A_n sin(n\pi t) + B_n cos(n\pi t) + \frac{4}{(n\pi)^3}\Big] sin(n\pi x) \\ n = 1,3,5... \end{aligned} \tag{40}$$

Applying the Initial Conditions:

$$u(x,0) = \sum_{n=1}^{\infty} \Big[B_n + \frac{4}{(n\pi)^3}\Big] sin(n\pi x) = 0$$

$$B_n = -\frac{4}{(n\pi)^3}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \Big[n\pi A_n cos(n\pi t) - n\pi B_n sin(n\pi t)\Big] sin(n\pi x) \tag{41}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \Big[n\pi A_n\Big] sin(n\pi x) = 0$$

$$A_n = 0$$

This gives a final solution of:

$$\boxed{\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \Big[\frac{4}{(n\pi)^3}(1 - cos(n\pi t))\Big] sin(n\pi x) \\ n = 1, 3, 5... \end{aligned}} \tag{42}$$

If we note from **(22)** above that

$$\begin{aligned} n = 1, 3, 5.. \sum_{n=1}^{\infty} \frac{-4}{(n\pi)^3} sin(n\pi x) = \frac{x^2}{2} - \frac{x}{2} \\ \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3} sin(n\pi x) = -\frac{x^2}{2} + \frac{x}{2} \end{aligned} \tag{43}$$

We can see that the extra series term in **(42)** is equivalent to the non-series term in **(30)** and we have the same answer.