AMATH 503: Homework 5
Due May, 28 2019
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June 3, 2019

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(1)

(a) Given the solution to the Green's Function for the Heat Equation in 1-D stated on page 228 of the notes, we can define the semi-infinite domain as $0 < x < \infty$, and then subtract the same solution defined at the location $x = -\xi$ to satisfy the boundary condition $G = 0$ at $x = 0$. This gives the Green's Function:

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi D(t-\tau)}}\left[e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4D(t-\tau)}}\right] \tag{1}$$

To show this satisfies the BC, we can simply observe that if $x = 0$, the two exponential terms are equivalent since the only difference is the $\pm\xi$ term, which is squared anyway. The other boundary condition is still satisfied since both of these exponentials are raised to a strictly negative power increasing with x, and will go to zero as $x \to \infty$.

(b) To satisfy a boundary condition of $\frac{\partial}{\partial x}G$ at $x = 0$, we can observe that if we add a solution mirrored across the G axis, the two resulting normal distributions will have equal slope, but opposite in sign, at $x = 0$:

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi D(t-\tau)}}\left[e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} + e^{-\frac{(x+\xi)^2}{4D(t-\tau)}}\right] \tag{2}$$

We can show this satisfies the BC at $x = 0$ by taking the derivatives of the exponentials:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{-2(x-\xi)}{4D(t-\tau)}e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \\ \frac{\partial}{\partial x} &= \frac{-2(x+\xi)}{4D(t-\tau)}e^{-\frac{(x+\xi)^2}{4D(t-\tau)}} \end{aligned} \tag{3}$$

When $x = 0$, again the two terms cancel since the sign on the $-\xi$ carries out, and the $\pm\xi$ in the exponents are equivalent because they are squared:

$$\frac{2(\xi)}{4D(t-\tau)}e^{-\frac{(-\xi)^2}{4D(t-\tau)}} - \frac{2(\xi)}{4D(t-\tau)}e^{-\frac{(\xi)^2}{4D(t-\tau)}} = 0 \tag{4}$$

Again, the BC of $G = 0$ as $x \to \infty$ is still satisfied since both normal distributions are raised to strictly negative exponent increasing with x.

(2)

(a) We want to show that

$$\begin{aligned} (\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2})u = \delta(x-\xi)\delta(t-\tau) \\ u = 0\ ,\ \frac{\partial}{\partial t}u = 0 \text{ at } t = 0 \end{aligned} \tag{5}$$

is equivalent to:

$$\begin{aligned} (\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2})u = 0 \\ u = 0 \text{ and } \frac{\partial}{\partial t}u = \delta(x-\xi) \text{ at } t = \tau \end{aligned} \tag{6}$$

To do so, we make some observations:

(1): On account of the time delta function, by definition the RHS of the non-homogeneous equation is zero at all t except for $t = \tau$. Also, because the initial condition is $u = 0$ and $\frac{\partial}{\partial t}u = 0$ at $= 0$, we know that nothing happens until $t = \tau$.

(2): We also know that when $t > \tau$, the RHS is again zero because of the delta function. So the original equation is actually homogeneous everywhere except at $t = \tau$.

(3): So the question is, what happens at $t = \tau$ in the non-homogeneous PDE? We need this information to define our second initial condition as well as to ascertain whether there is a jump in u at $t = \tau$, which would make this task impossible. To do this we integrate the non-homogeneous PDE with respect to time across an arbitrarily small domain around $t = \tau$:

$$\begin{aligned}\int_{\tau-}^{\tau+} \Big[(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2})u\Big]dt &= \int_{\tau-}^{\tau+} \delta(x-\xi)\delta(t-\tau)dt \\ \int_{\tau-}^{\tau+} \frac{\partial^2}{\partial t^2}udt - \int_{\tau-}^{\tau+} c^2\frac{\partial^2}{\partial x^2}udt &= \delta(x-\xi)\int_{\tau-}^{\tau+} \delta(t-\tau)dt\end{aligned} \tag{7}$$

Now a few observations. We know that when $t < \tau$ that $u = 0$, so the first integral is simply equivalent to $\frac{\partial}{\partial t}u$ evaluated at $\tau+$. Integrating the spatial second derivative across an arbitrarily small time span must be zero. If this weren't the case, u itself would be a delta function in time, and the second derivative of u in the original PDE doesn't indicate any huge singularity like that on the RHS. Lastly, the RHS is equal to $\delta(x-\xi)$ since the derivative across the zero of a delta function is by definition equal to 1. This means that one of the initial conditions of the homogeneous equation must take the following into account at $t = \tau$:

$$\frac{\partial}{\partial t}\Big|_{\tau+} = \delta(x - \xi) \tag{8}$$

Lastly, we can observe that if there was a jump at $t = \tau$, $\frac{\partial}{\partial t}u$ at that precise point would be very large, i.e. it would itself be equal to some kind of t delta function. This isn't the case, and so we know there is no jump there and thus the other initial condition must be $u = 0$ at $t = \tau$, and we have achieved our objective, as these are the "initial conditions" the prompt gives for the homogeneous, fundamental problem.

(b) In accordance with the D'Alembert solution we assume a solution with the form of two travelling waves moving with speed c in relation to time $t - \tau$.

$$u = R(x - c(t - \tau)) + L(x + c(t - \tau)) \tag{9}$$

Applying the first initial condition $u = 0$ at $t = \tau$ we have:

$$\begin{aligned} u = R(x) + L(x) = 0 \\ L(x) = -R(x) \\ u = R(x + c(t - \tau) - R(x - c(t - \tau)) \end{aligned} \tag{10}$$

To plug in the second initial condition, we need to take the derivative of u in time and then plug in the initial condition of the derivative at $t = \tau$:

$$\begin{aligned} u_t = -cR'(x + c(t - \tau)) - cR'(x - c(t - \tau)) \\ u_t(t = \tau) = -2cR'(x) = \delta(x - \xi) \end{aligned} \tag{11}$$

To find R we integrate, introducing a dummy variable. Note that we are integrating from $-\infty$ to x since when $x > \xi$ we won't be picking up anything because we are integrating a delta function. Thus we have:

$$\begin{aligned}\int R' &= -\frac{1}{2c}\int \delta(x-\xi)dx \\ R &= -\frac{1}{2c}\int_{-\infty}^{x} \delta(\bar{x}-\xi)d\bar{x}\end{aligned} \tag{12}$$

The integral of a delta function is the Heavyside step function, thus we have the following for R:

$$R = -\frac{1}{2c}H(x-\xi) = \begin{cases} 0, & x < \xi \\ \frac{-1}{2c} & x > \xi \end{cases} \tag{13}$$

Thus the solution to the PDE with both initial conditions applied is as follows:

$$\boxed{u(x,t) = \frac{1}{2c}\Big[H(x-\xi+c(t-\tau)) - H(x-\xi-c(t-\tau))\Big] \qquad (14)}$$

(c) To reconstruct a solution we integrate the fundamental problem multiplied by the forcing term evaluated at τ and ξ across space and time to get the particular solution, noting that the evaulation of τ stops at t because when $\tau > t$ the Green's Function is equal to zero anyway (I think?).

$$u_p = \frac{1}{2c}\int_0^t d\tau \int_{-\infty}^{\infty} \Big[H(x-\xi+c(t-\tau)) - H(x-\xi-c(t-\tau))\Big]Q(\xi,\tau)d\xi \tag{15}$$

Because the Heavyside functions describe an expanding box starting at $x = \xi$ and then expanding to the left and right as time increases, it is easy to simplify the integral to simply evaluate 1 across the domain, thus giving the area. Since we've pulled out the $\frac{1}{2c}$, the height of the box is one and it's width extends from $x-\xi-c(t-\tau)$ to $x-\xi+c(t-\tau)$. The ξ can be dropped from the bounds since it would fall out in the evaluation of the integral anyway. We are left with:

$$\boxed{u_p(x,t) = \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} Q(\xi,\tau)d\xi} \tag{16}$$

Lastly, we need the homogeneous solution to the PDE, though it seems obvious to me it must be trivial since we have zero initial condition, zero initial velocity and a zero boundary condition. Using d'Alembert's solution I'll use the variable v to distinguish it from the forced PDE: $v_{tt} - c^2 v_{xx} = 0$ with $v(x,0) = 0$ and $\frac{d}{dt}v(x,0) = 0$.

$$\begin{aligned} v(x,t) &= R(x-ct) + L(x+ct) \\ v(x,0) &= 0 \\ v(x,t) &= R(x-ct) - R(x+ct) \\ v_t(x,t) &= -cR'(x-ct) - cR'(x+ct) \\ v_t(x,0) &= 0 = -2cR'(x) \end{aligned} \tag{17}$$

Integrating:

$$\begin{aligned} -2cR(x) &= K \\ R(x) &= K \\ v(x,t) &= K - K = 0 \end{aligned} \tag{18}$$

Thus the solution should just be the above particular solution by itself.