AMATH 503: Homework 3
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(1)

(a) Using the formula for a complex Fourier series in the notes we have the following in a domain of $-2 < x < 2$ with $n = 0, \pm 1, \pm 2, \pm 3...$

$$|x| = \sum_{-\infty}^{\infty} c_n e^{\frac{-in\pi x}{2}} \tag{1}$$

Similarly, we obtain the coefficient c_n with the following:

$$c_n = \frac{1}{4}\int_{-2}^{2} |x| e^{\frac{in\pi x}{2}} dx \tag{2}$$

To solve this, we split the integral into two domains and integration by parts:

$$
\begin{aligned}
c_n &= \frac{1}{4}\Big[\int_0^2 xe^{\frac{in\pi x}{2}}dx - \int_{-2}^0 xe^{\frac{in\pi x}{2}}dx\Big] \\
u &= x\ ,\ du = 1 \\
dv &= e^{\frac{in\pi x}{2}} \\
v &= \frac{2}{in\pi}e^{\frac{in\pi x}{2}} \\
uv = \int vdu &= \frac{2x}{in\pi}e^{\frac{in\pi x}{2}} - \frac{4}{n^2\pi^2}e^{\frac{in\pi x}{2}} \\
c_n = \frac{1}{4}\Bigg[\Big[\frac{2xe^{\frac{in\pi x}{2}}}{in\pi} &+ \frac{4e^{\frac{in\pi x}{2}}}{n^2\pi^2}\Big]\Big|_0^2 - \Big[\frac{2xe^{\frac{in\pi x}{2}}}{in\pi} + \frac{4e^{\frac{in\pi x}{2}}}{n^2\pi^2}\Big]\Big|_{-2}^0\Bigg] \\
c_n = \frac{1}{4}\Bigg[\Big[\frac{4e^{in\pi}}{in\pi} + \frac{4e^{in\pi}}{n^2\pi^2} - \frac{4}{n^2\pi^2}\Big] &- \Big[\frac{4}{n^2\pi^2} + \frac{4e^{-in\pi}}{in\pi} + \frac{4e^{-in\pi}}{n^2\pi^2}\Big]\Bigg] \\
c_n &= \frac{-in\pi e^{in\pi} + in\pi e^{-in\pi} + e^{in\pi} + e^{-in\pi} - 2}{n^2\pi^2}
\end{aligned}
\tag{3}
$$

Now we observe that:

$$
\begin{aligned}
e^{in\pi} &= cos(n\pi) + isin(n\pi) = cos(n\pi) = (-1)^n \\
e^{-in\pi} &= cos(n\pi) - isin(n\pi) = cos(n\pi) = (-1)^n \\
e^{in\pi} &= e^{-in\pi}
\end{aligned}
\tag{4}
$$

This allows for the following simplification:

$$
c_n = \frac{2(-1)^n - 2}{n^2\pi^2} \tag{5}
$$

This is obviously not defined for $n = 0$ however, and so we calculate this independently:

$$
\begin{aligned}
c_0 &= \frac{1}{4}\int_{-2}^2 |x|dx \\
c_0 &= \frac{1}{4}\Big[[\frac{x^2}{2}]\Big|_0^2 - [\frac{x^2}{2}]\Big|_{-2}^0\Big] = 1
\end{aligned}
\tag{6}
$$

Putting all this togther, we arrive at the following Fourier series, noting that the even modes are zero (except c_0), and that $n = \pm 1, \pm 2, \pm 3...$

$$\boxed{|x| = 1 + \frac{2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n - 1}{n^2} e^{-\frac{in\pi x}{2}}} \tag{7}$$

(b) Using Euler's Formula we can rewrite the exponential above in the following way:

$$e^{-\frac{in\pi x}{2}} = cos(\frac{n\pi x}{2}) - isin(\frac{n\pi x}{2}) \tag{8}$$

If we observe that sin is an odd function, it follows that $isin(\frac{-n\pi x}{2}) = -isin(\frac{n\pi x}{2})$. This means that when $n < 0$, all the sin values in the sum will cancel with their corresponding values from $n > 0$, for example when $n = 1$ we have $-isin(\frac{\pi x}{2})$ and when $n = -1$ we have $isin(\frac{n\pi x}{2})$.

We can also observe that the even modes are zero, substituting $n = 2k - 1$ and define our sum with $k = 1, 2, 3....$ This means $c_n = \frac{-4}{n^2\pi^2}$ as well. The resulting cosine series is:

$$\boxed{|x| = 1 - \frac{4}{\pi^2} \sum_{k=1}^{\infty} cos(\frac{(2k-1)\pi x}{2})} \tag{9}$$

(c)

$$\frac{d}{dx}[1 + \sum_{n=-\infty}^{\infty} c_n cos(\frac{n\pi x}{2})] = -\sum_{n=-\infty}^{\infty} c_n sin(\frac{n\pi x}{2})(\frac{n\pi}{2}) \tag{10}$$

Plugging c_n back in we get:

$$\frac{d}{dx}|x| = -\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n - 1}{n} sin(\frac{n\pi x}{2}) \tag{11}$$