AMATH 353: Homework 13
Due May, 23 2018
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Part 1

(a) Because the PDE in question is homogeneous, we have $x = c(u_o(x_0))t + x_0$. For this equation, $c(u(x,t)) = u(x,t)$. From this we get the following:

$$\begin{aligned} \frac{du}{dt} &= 0 \\ u(x(t),t) &= A \\ A = u(x(t),0) &= u_0(x_0) = e^{-x_0^2} \end{aligned} \tag{1}$$

And the characteristic lines:

$$\boxed{\begin{aligned} x &= x_0 + te^{-x_0^2} \\ t &= (x - x_0)e^{x_0^2} \end{aligned}} \tag{2}$$

(b) On the following page is a plot of the characteristics:

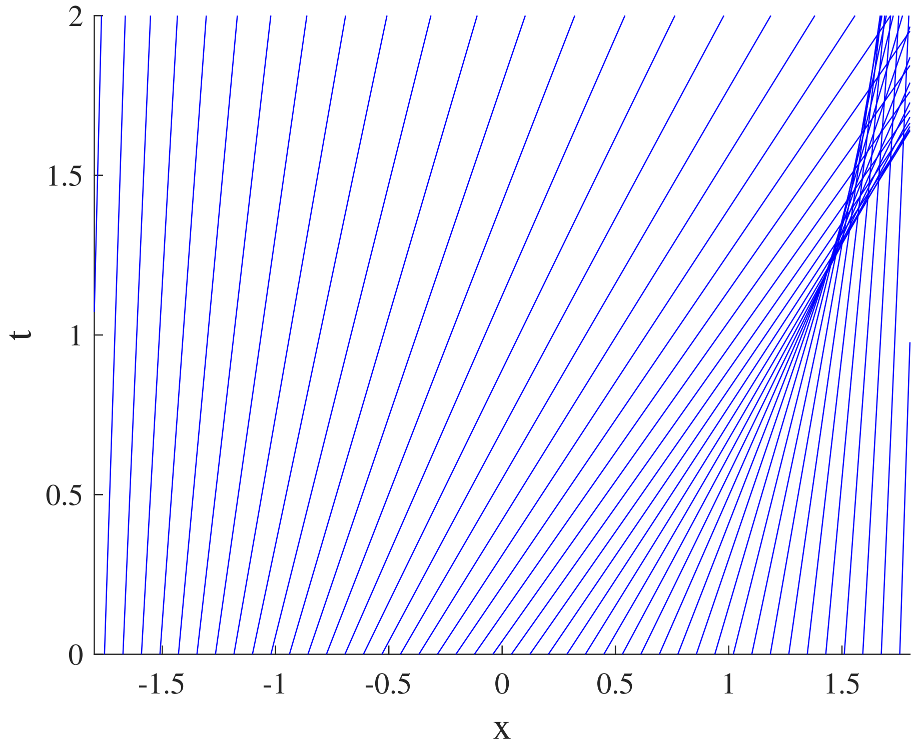

Figure 1: $x \in [-1.8, 1.8]$

(c) I wrote a script in matlab to find the roots for $f(x_0) = x_0 - x + te^{-x^2}$, which is included in my HW submission.

(d) and (e) Below are the plots, using x_0 values from my rootfinder, for $t = 0, 0.6, 0.9, 1.18$ and $x \in [-1.8, 1.8]$ with a mesh of values. From the first figure, it is clear that at $t = 0$ the rootfinder is returning the correct values, yielding the profile of $u_0(x_0) = e^{-x^2}$. As the figures below show, as t increases, the top of the profile has a faster rate of change than the bottom, causing a 'crashing wave' to form. This is to be expected, since the time velocity of the PDE is depending on both the spatial velocity and the value of u itself, thus the larger u values have larger time velocity.

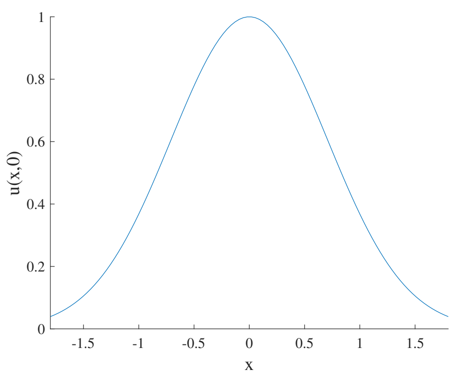

Figure 2: $t = 0$

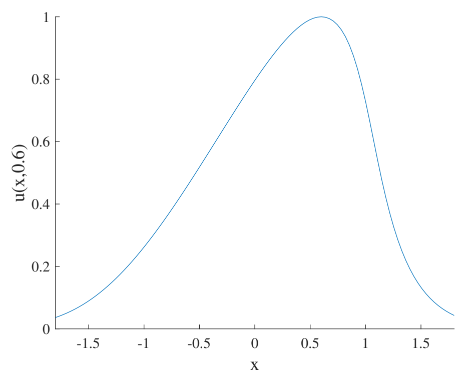

Figure 3: $t = 0.6$

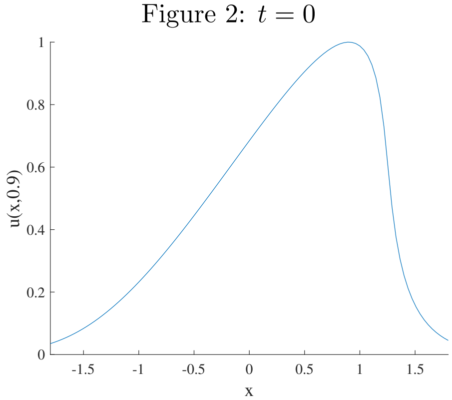

Figure 4: $t = 0.9$

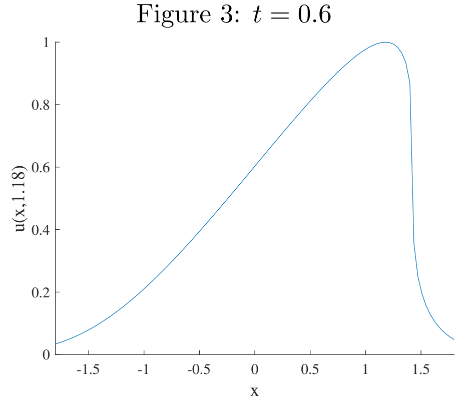

Figure 5: $t = 1.18$

(f) When my rootfinder is used to plot the values at $t = 1.5$, it appears to get 'stuck' at the point of the wave crashing and falling over. This occurs at the point that the characteristics start to cross, which by closely examining (zooming in) on Figure 1, would seem to happen around $t = 1.2$. Below are two plots using my rootfinder, of $t = 1.5$ and $t = 2.5$. In both, we see an abrupt change in the value of u, caused by crossing characteristic lines which result in the value of u 'jumping' between two x_0 values produced by my rootfinder. The plot at $t = 2.5$ shows what the profile looks like as even more of the profile (non-zero u values) fall inside the range where the characteristics begin to cross begin to cross.

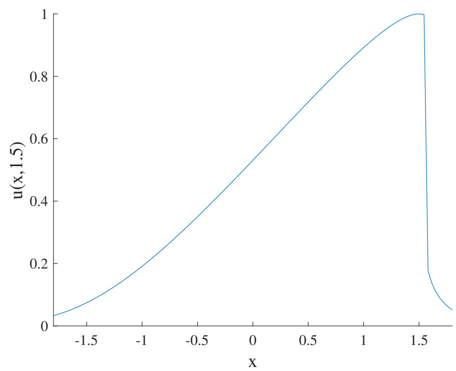

Figure 6: $t = 1.5$

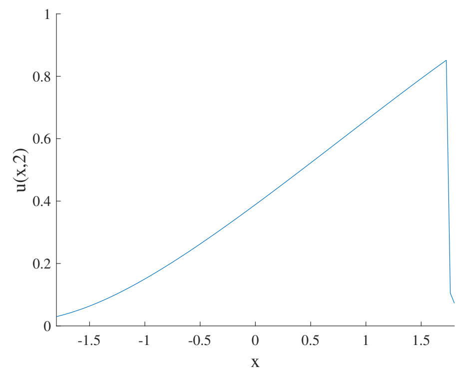

Figure 7: $t = 2.5$

Part 2

(a) Given the provided initial condition, and the fact that the PDE is homogeneous, we have the following:

$$\begin{aligned} u(x(t),t) = u(x(0),0) = u_0(x_0) &= \frac{1}{1+x_0^2} \\ \frac{du}{dt} &= 0 \\ u(x(t),t) = A &= \frac{1}{1+x_0^2} \end{aligned} \tag{3}$$

And then for the characteristic lines:

$$\begin{aligned} c(u(x,t)) &= u(x,t)^2 \\ u(x,t)^2 = u_0(x_0)^2 &= \frac{1}{(1+x_0^2)^2} \end{aligned} \tag{4}$$

We then plug this into the formula $x = x_0 + tc(u(x,t))$ to arrive at the equation for the characteristics:

$$\boxed{x = x_0 + \frac{t}{(1+x_0^2)^2}} \tag{5}$$

Here is a plot of these characteristic curves generally, as well as a plot that scales up the area where the first characteristic lines appear to cross.

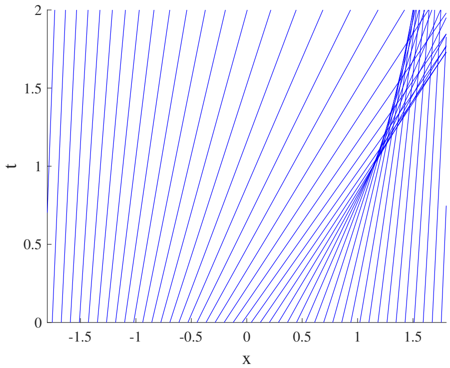

Figure 8: $x \in [-1.8, 1.8]$

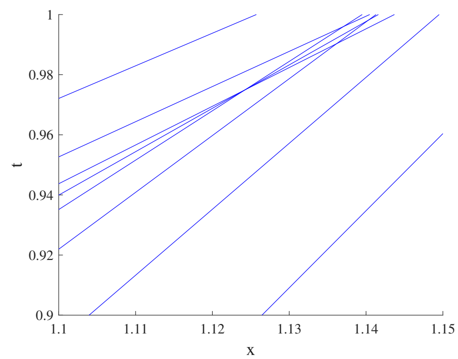

Figure 9: $x \in [1.1, 1.15]$

Judging from the zoomed in figure, I would estimate $t_b \approx .97$.

(c) - (d) First, we need to take the implicit derivative of equation (5) above in order to find $\frac{dx_0}{dx}$, so that we can examine at what point in time it blows up to ∞:

$$\begin{aligned} \frac{d}{dx}x &= \frac{d}{dx}\Big[x_0 + \frac{t}{(1+x_0^2)^2}\Big] \\ 1 &= \frac{dx_0}{dx} + t\frac{d}{dx}\Big[(1+x_0^2)^{-2}\Big] \\ 1 &= \frac{dx_0}{dx} + t\Big[t(-2(1+x_0^2)^{-3}(2x_0)(\frac{dx_0}{dx})\Big] \\ 1 &= \frac{dx_0}{dx}\Big[\frac{-4tx_0}{(1+x_0^2)^3}\Big] \end{aligned} \tag{6}$$

$$\frac{dx_0}{dx} = \frac{1}{\frac{-4tx_0}{(1+x_0^2)^3} + 1} \tag{7}$$

Examining this equation, it is clear that it can only go to ∞ when the denominator approaches zero, and so we have:

$$\frac{-4tx_0}{(1+x_0^2)^3} + 1 = 0 \tag{8}$$

$$t = \frac{(1+x_0^2)^3}{4x_0} \tag{9}$$

We are interested in the lowest time t at which this equation holds, so we will use (9) to make a function of x_0, take the derivative of that function, and see at what x_0 when this derivative is equal to zero.

$$\begin{aligned}
F(x_0) &= \frac{(1+x_0^2)^3}{4x_0} \\
F'(x_0) &= 3(1+x_0^2)^2(2x_0)(4x_0)^{-1} + (1+x_0^2)^3(-\frac{1}{4})(x_0)^{-2} \\
F'(x_0) &= \frac{3}{2}(1+x_0^2)^2 - \frac{(1+x_0^2)^3}{4x_0^2} = 0 \\
6x_0^2(1+x_0^2)^2 &= (1+x_0^2)^3 \\
6x_0^2 + 12x_0^4 + 6x_0^6 &= 1 + 3x_0^2 + 3x_0^4 + x_0^6 \\
3x_0^2 + 9x_0^4 + 5x_0^6 - 1 &= 0
\end{aligned} \tag{10}$$

Using Mathematica to find the roots of this polynomial, we get:

$$x_0 = i, i, -i, -i, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \tag{11}$$

Since we are only interested in real roots, we simply have the $\pm\frac{1}{\sqrt{5}}$. However, $-\frac{1}{\sqrt{5}}$ yields $t < 0$, so the only remaining x_0 value is $x_0 = \frac{1}{\sqrt{5}}$. Plugging this into equation describing t as the denominator of $\frac{dx_0}{dx} \to 0$ we get the following breaking time, which is indeed quite close to the answer I got above in (a) from (Figure 9).

$$\boxed{\begin{aligned}
t_b &= \frac{(1+\frac{1}{\sqrt{5}}^2)^3}{\frac{4}{\sqrt{5}}} \\
t_b &= \frac{54}{25\sqrt{5}} \approx .965981
\end{aligned} \qquad (12)}$$