AMATH 353: Homework 10
Due May, 8 2018
ID: 1064712

Trent Yarosevich

May 8, 2018

Instructor: Jeremy Upsal

Part 1 We consider the heat equation and the following IBVP:

$$\begin{aligned} &u_t = 4u_{xx} && 0 < x < 1, && t > 0 \\ &u(0,t) = u_x(1,t) = 0 && && t > 0 \\ &u(x,0) = x(1-x). \end{aligned}$$

I've used separation of variables in the same fashion as I did in HW 8, which is to say I put the k term with the x equation. Using $k = 4$ this results in:

$$\begin{aligned} G'(t) &= \lambda G(t) \\ F''(x) &= \frac{\lambda}{4} F(x) \end{aligned} \tag{1}$$

As in HW 8, the only allowed λ values with this setup were $\lambda < 0$, which resulted in the following using the characteristic polynomial and Euler's Formula, and $\lambda = -r^2, r \in \mathbb{R}^+$:

$$F(x) = C_1 \cos(\frac{r}{2}x) + C_2 \sin(\frac{r}{2}x) \tag{2}$$

We then apply the BCs to this:

$$\begin{aligned} F(0) = C_1 \cos(0) + C_2 \sin(0) &= 0 \\ C_1 &= 0 \\ F'(x) = \frac{r}{2} C_2 \cos(\frac{r}{2}x)& \\ F'(1) = \frac{r}{2} C_2 \cos(\frac{r}{2}) &= 0 \end{aligned} \tag{3}$$

Thus without setting $C_2 = 0$ the BC is only satisfied when $\cos(\frac{r}{2}) = 0$, which means the argument is equal to odd integer multiples of $\frac{\pi}{2}$:

$$\begin{aligned} n &\in \mathbb{Z}^+ \\ \frac{r}{2} &= \frac{\pi(2n-1)}{2} \\ r &= \pi(2n-1) \\ \lambda_n &= -(\pi(2n-1))^2 \\ F_n(x) &= C_2 \sin(\frac{\pi(2n-1)}{2}x) \end{aligned} \tag{4}$$

Turning to the equation $G'(t) = \lambda G(t)$, we can simply solve with separation of variables (ODE 101 version):

$$\begin{aligned} \int \frac{dG}{dt} &= \int \lambda G(t) \\ ln(G(t)) &= \lambda t + C \\ G(t) &= Ae^{\lambda t} \end{aligned} \tag{5}$$

Combining the above with the result in equation (4) and substituting λ_n, we get a solution for u, and note I have consolidated the product of the arbitrary constants in a single new arbitrary constant. Consequently, we can also use superposition to rewrite the equation as a sum of solutions.

$$\boxed{\begin{aligned} u_n(x,t) &= Ae^{\lambda_n t} \sin(\frac{\pi(2n-1)}{2}x) \\ u(x,t) &= \sum_{n=1}^{\infty} A_n e^{\lambda_n t} \sin(\frac{\pi(2n-1)}{2}x) \end{aligned} \qquad (6)}$$

We now have to match this form to the initial condition $u(x,0) = x(1-x)$, which gives us the following equation. Let this IC be $f(x)$:

$$\begin{aligned} u(x,0) = \sum_{n=1}^{\infty} A_n e^{\lambda_n (0)} \sin(\frac{\pi(2n-1)}{2}x) &= f(x) \\ f(x) &= \sum_{n=1}^{\infty} A_n \sin(\frac{\pi(2n-1)}{2}x) \end{aligned} \tag{7}$$

This gives us the Fourier Series form of the initial condition. Because we need to make use of the orthogonality relations, and $f(x)$ is only defined from $x \in [0,1]$, we will use the odd extension of $f(x)$ because the BCs specify a fixed point at the origin. And indeed, using the even extension in this situation just results in a 0 constant anyway.

We define the odd extension as follows:

$$f(x) = x(1-x) \tag{8}$$

$$f_o(x) = \begin{cases} f(x), & x \geq 0 \\ -f(-x), & x < 0 \end{cases}$$

As stated in the homework, $A_0 = 0$ because $\int_0^1 x(1-x) = 0$. For our one constant A_n we then have the following:

$$\begin{gathered} L = 1 \\ A_n = \int_{-1}^{1} f_o(x) \sin(\frac{\pi(2n-1)}{2}x) \end{gathered} \tag{9}$$

Because both $f_o(x)$ and sin are odd, the resulting function is even, and because $f_o(x) = f(x)$ for all $x \geq 0$ we have the following:

$$\begin{gathered} \int_{-1}^{1} f_o(x) \sin(\frac{\pi(2n-1)}{2}x) = 2\int_{0}^{1} f_o(x) \sin(\frac{\pi(2n-1)}{2}x) \\ 2\int_{0}^{1} f_o(x) \sin(\frac{\pi(2n-1)}{2}x) = 2\int_{0}^{1} f(x) \sin(\frac{\pi(2n-1)}{2}x) \end{gathered} \tag{10}$$

We now solve this integral using integrating factors:

$$\begin{aligned} A_n = 2\int_{0}^{1} f(x) \sin(\frac{\pi(2n-1)}{2}x) \\ u = x(1-x), \quad du = (1-2x)dx \\ dv = \sin(\frac{\pi(2n-1)}{2}x) \\ v = \int dv = \frac{-2}{\pi(2n-1)} \cos(\frac{\pi(2n-1)}{2}x) \\ \int udv = \frac{-2x+2x^2}{\pi(2n-1)} \cos(\frac{\pi(2n-1)}{2}x) - \int \frac{-2+4x}{\pi(2n-1)} \cos(\frac{\pi(2n-1)}{2}x)dx \end{aligned} \tag{11}$$

We then continue to solve $\int vdu$ using integrating factors again:

$$\begin{aligned}
\int \frac{-2+4x}{\pi(2n-1)} \cos(\frac{\pi(2n-1)}{2}x)dx \\
u = \frac{-2+4x}{\pi(2n-1)} \\
du = \frac{4}{\pi(2n-1)}dx \\
dv = \cos(\frac{\pi(2n-1)}{2}x) \\
v = \frac{2}{\pi(2n-1)} \sin(\frac{\pi(2n-1)}{2}x) \\
\int udv = \frac{-4+8x}{\pi^2(2n-1)^2} \sin(\frac{\pi(2n-1)}{2}x) - \int \frac{8}{\pi^2(2n-1)^2)} \sin(\frac{\pi(2n-1)}{2}x) \\
= \frac{-4+8x}{\pi^2(2n-1)^2} \sin(\frac{\pi(2n-1)}{2}x) + \frac{16}{\pi^3(2n-1)^3} \cos(\frac{\pi(2n-1)}{2}x)
\end{aligned} \tag{12}$$

Then substituting this result into the original integral, and consolidating the numerators, we get:

$$A_n = \Big[\frac{(-2x+2x^2)(\pi^2(2n-1)^2)-16)\cos(\frac{\pi(2n-1)}{2}x) + (4-8x)\pi(2n-1)\sin(\frac{\pi(2n-1)}{2}x)}{\pi^3(2n-1)^3}\Big]\Big|_0^1 \tag{13}$$

Letting the denominator be some constant c for a moment, we have

$$\begin{aligned}
A_n c = (0)(\pi^2(2n-1)^2) - 16)\cos(\frac{\pi(2n-1)}{2}) + (-4)\pi(2n-1)\sin(\frac{\pi(2n-1)}{2}) \\
- (0)(\pi^2(2n-1)^2) - 16)\cos(0) + (4)\pi(2n-1)\sin(0)
\end{aligned} \tag{14}$$

Simplifying this and moving the denominator back over, we have:

$$A_n = \frac{-16\cos(\frac{\pi(2n-1)}{2}) - 4\pi(2n-1)\sin(\frac{\pi(2n-1)}{2}) + 16}{\pi^3(2n-1)^3} \tag{15}$$

At this point we can make a few observations. For $n \in \mathbb{Z}^+$, we can see that $\cos(\frac{\pi(2n-1)}{2}) = 0$. Further we can observe that $\sin(\frac{\pi(2n-1)}{2}) = (-1)^{n+1} = -(-1)^n$. Substituting these in and multiplying the entire result of the integral we solved by 2, we get the final value for A_n

$$A_n = \frac{8\pi(2n-1)(-1)^n + 16}{\pi^3(2n-1)^3} \tag{16}$$

bla