AMATH 353: Homework 8
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Part 1

a.) Assuming solutions of the form $u(x,t) = G(t)F(x)$ and the PDE $u_{tt} + u_{xx} = 0$ we get the following:

$$\begin{aligned} G''(t)F(x) &= -F''(x)G(t) \\ \frac{G''(t)}{G(t)} &= -\frac{F''(x)}{F(x)} = \lambda \end{aligned} \tag{1}$$

The two equations are equal to the constant λ because neither the RHS or LHS is in terms of its independent variable. This results in the two ODEs:

$$\begin{aligned} G''(t) &= \lambda G(t) \\ F''(x) &= -\lambda F(x) \end{aligned} \tag{2}$$

b.) Assuming solutions of the form $u(x,t) = G(t)F(x)$ and the PDE $u_t = \kappa u_{xx}$ we get the following:

$$\begin{aligned} G'(t)F(x) &= \kappa F''(x)G(t) \\ \frac{G'(t)}{G(t)} &= \kappa\frac{F''(x)}{F(x)} = \lambda \end{aligned} \tag{3}$$

The two equations are equal to the constant λ as above, yielding the two ODEs:

$$\begin{aligned} G'(t) &= \lambda G(t) \\ F''(x) &= \frac{\lambda}{\kappa}F(x) \end{aligned} \tag{4}$$

Part 2

a.) Starting with the equation $u_{tt} + u_{xx} = 0$ we have the following x dependent ODE and three cases for λ:

$$F''(x) = -\lambda F(x) \tag{5}$$

$lambda = 0$

In this case we have $F''(x) = -0 * F(x)$. Integrating twice gives $F(x) = A + Bx$. With the BC $u(0,t) = 0$ we get $F(0) = A + B(0) = 0$ and thus $A = 0$. With the other BC $u_x(L,t) = 0$ and $F'(L) = B$ we must conclude that $B = 0$ as well, yielding only trivial solutions, so $\underline{\lambda = 0 \text{ is not an allowed value.}}$

$\lambda > 0$

In this case we have $F''(x) = -\lambda F(x)$ where λ is a positive real number. Let $\lambda = r^2$, with $r \in \mathbb{R}_{>0}$. Then by characteristic polynomial we get following, per Euler's Formula:

$$\begin{aligned} F(x) &= C_1 e^{irx} + C_2 e^{-irx} \\ F(x) &= C_1 \cos(rx) + C_2 sin(rx) \end{aligned} \tag{6}$$

We then apply the BC $u(0,t) = 0$:

$$\begin{aligned} F(0) = C_1 \cos(0) + C_2 \sin(0) &= 0 \\ C_1 &= 0 \\ F(x) &= C_2 \sin(rx) \end{aligned} \tag{7}$$

And then the second BC:

$$\begin{aligned} F'(x) &= C_2 r \cos(rx) \\ F'(L) &= C_2 r \cos(rL) = 0 \end{aligned} \tag{8}$$

Because $\cos(x) = 0$ at odd multiples of $\frac{\pi}{2}$ it follows that to satisfy the second BC without setting $C_2 = 0$ that we must have the argument of cosine, $rL = \frac{\pi(2n-1)}{2}$ where $n > 0$ is a positive integer. Thus with positive eigenvalues λ we have the eigenfunctions:

$$F_n(x) = C_n \sin(\frac{\pi(2n-1)x}{2L} \text{ , } n \in \mathbb{Z}^+ \tag{9}$$

$\lambda < 0$

In the same fashion as above, but now let $\lambda = -r^2$, $r \in \mathbb{R}_{>0}$. From characteristic polynomial we then get:

$$F(x) = C_1 e^{rx} + C_2 e^{-rx} \tag{10}$$

Applying the first BC we get:

$$
\begin{aligned}
F(0) = C_1 e^0 + C_2 e^{-0} &= 0 \\
C_1 + C_2 &= 0 \\
C_2 &= -C_1 \\
F(x) &= C(e^{rx} - e^{-rx})
\end{aligned}
\tag{11}
$$

Then applying this to the second BC:

$$
\begin{aligned}
F'(x) &= C(re^{rx} + re^{-rx}) \\
F'(L) &= C(re^{rL} + re^{-rL}) = 0 \\
e^{rL} &= -e^{-rL}
\end{aligned}
\tag{12}
$$

Since there is no possibility of $e^{rL} = -e^{-rL}$, this means the only way to satisfy the BCs with $\lambda < 0$ is to have $C = 0$, yielding a trivial solution. Thus $\underline{\lambda < 0 \text{ is not an allowed value.}}$

b.) We now consider the same possible values for λ for the second equation from above, and the ODE for x:

$$
\begin{aligned}
u_t &= \kappa u_{xx} \\
F''(x) &= \frac{\lambda}{\kappa} F(x)
\end{aligned}
\tag{13}
$$

$\lambda = 0$

Similarly to the previous equation, setting $\lambda = 0$ and integrating twice gives us

$$
F(x) = A + Bx \tag{14}
$$

which gives only trivial solutions in a manner identical to above, Part 2, subsection a.).