AMATH 353: Homework 14
Due May, 30 2018
ID: 1064712

Trent Yarosevich

May 28, 2018

Instructor: Jeremy Upsal

Part 1

(a) Given the following initial condition, we get:

$$u(x,0) = \begin{cases} \frac{\pi}{2}, & x < 1 \\ \frac{\pi}{4}, & x \geq 1 \end{cases} \tag{1}$$

$$\begin{gathered} \frac{du}{dt} = 0 \\ u(x(t),t) = A \\ A = u_0(x_0) = u(x,0) \end{gathered} \tag{2}$$

We can then make use of $x(t) = x_0 + c(u_0(x_0)t$: and determine the equations for the characteristics:

$$\begin{gathered} c(u(x(t),t)) = \sin(u) \\ c(u_0(x_0) = \sin(u_0(x_0)) \end{gathered} \tag{3}$$

$$c(u_0(x_0) \begin{cases} 1, & x_0 < 1 \\ \frac{\sqrt{2}}{2}, & x_0 \geq 1 \end{cases} \tag{4}$$

This in turn gives the following characteristics curves:

$$x(t) = \begin{cases} x_0 + t, & x_0 < 1 \\ x_0 + \frac{\sqrt{2}}{2}t, & x_0 \geq 1 \end{cases} \tag{5}$$

$$t = \begin{cases} x - x_0, & x_0 < 1 \\ \sqrt{2}(x - x_0) & x_0 \geq 1 \end{cases} \tag{6}$$

Additionally, here are the functions of x_0 in terms of x and t, which will be useful later:

$$x_0 = \begin{cases} x - t, & x_0 < 1 \\ x - \frac{\sqrt{2}}{2}t & x_0 \geq 1 \end{cases} \tag{7}$$

(b) Because $u(x(t), t) = u_0(x_0)$ we can find the breaking time by finding where its derivative goes to ∞.

$$u_x = u_0'(x_0)\frac{dx_0}{dx} \tag{8}$$

It is clear from the initial condition given above that we cannot take the derivative $u_0'(x_0)$ because $u_0(x_0)$ has a discontinuity when $x_0 = 1$. Plugging this into equation (7) above at $t = 0$ we see that this occurs at $x = 1$.

(c) Since $\phi(x, t) = \sin(u)u_x$ we can integrate to get the following (note that I am ignoring the constant of integration, since it drops out in the Rankine-Hugoniot relation anyway):

$$\phi(x, t) = -\cos(u) \tag{9}$$

We know from equation (2) above that $u^- = \frac{\pi}{2}$ and $u^+ = \frac{\pi}{4}$ giving:

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ \frac{[\phi]}{[u]} &= \frac{-\cos(u^+) + \cos(u^-)}{u^+ - u^-} \\ \frac{dx_s}{dt} &= \frac{-\frac{\sqrt{2}}{2}}{-\frac{\pi}{4}} \end{aligned} \tag{10}$$

Which simplifies to:

$$\frac{dx_s}{dt} = \frac{2\sqrt{2}}{\pi} \tag{11}$$

We then integrate and apply the initial condition found in (b):

$$\boxed{\begin{aligned} x_s(t) &= \frac{2\sqrt{2}}{\pi}t + C \\ x_s(0) &= 1 = C \\ x_s(t) &= \frac{2\sqrt{2}}{\pi}t + 1 \end{aligned} \tag{12}}$$

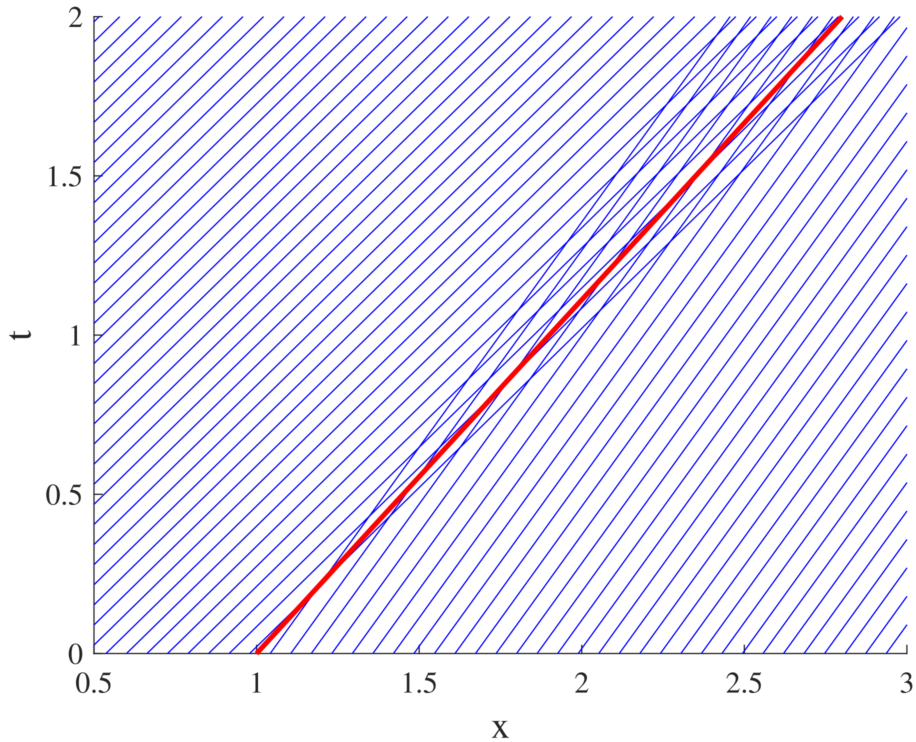

Figure 1: Characteristics and x_s

(d) The solution to the left of the shockline is a line at $u = \frac{\pi}{2}$, and to the right, a line at $u = \frac{\pi}{4}$. This is defined as:

$$\boxed{u(x,t) = \begin{cases} \frac{\pi}{2}, & x < x_s \\ \frac{\pi}{4} & x \geq x_s \end{cases} \qquad (13)}$$

(e) If we change the characteristics in this way,the density increases at the discontinuity, so the speed for u^+ is actually greater than u^-. As a result, the space between the solutions for u actually increases in time, giving a growing region in which u is not defined.

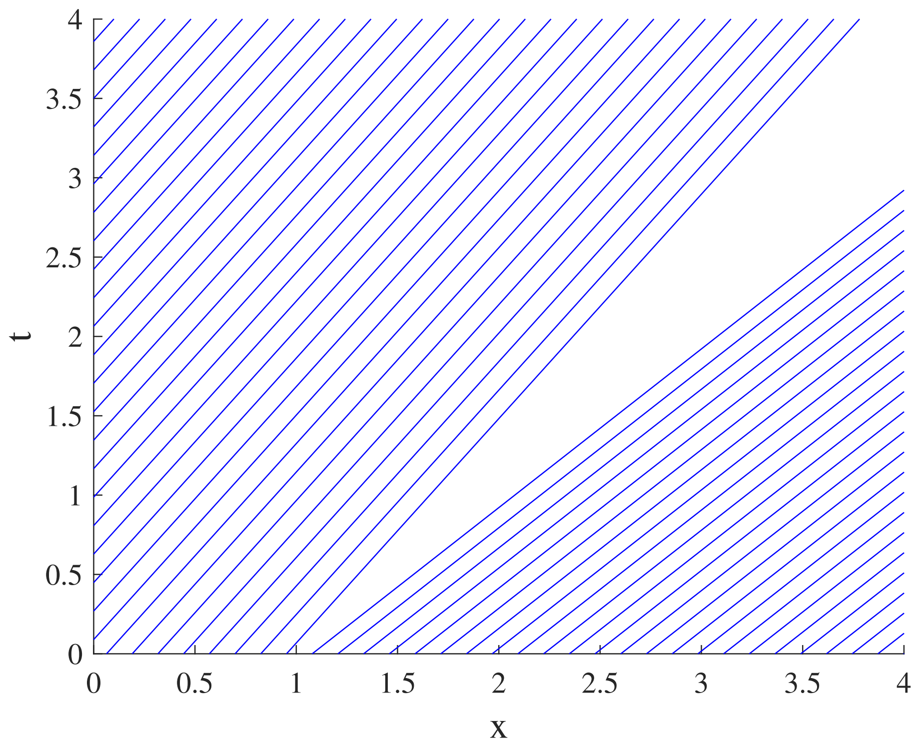

Figure 2: $u^- = \frac{\pi}{4}$ and $u^+ = \frac{\pi}{2}$

Part 2

(a)

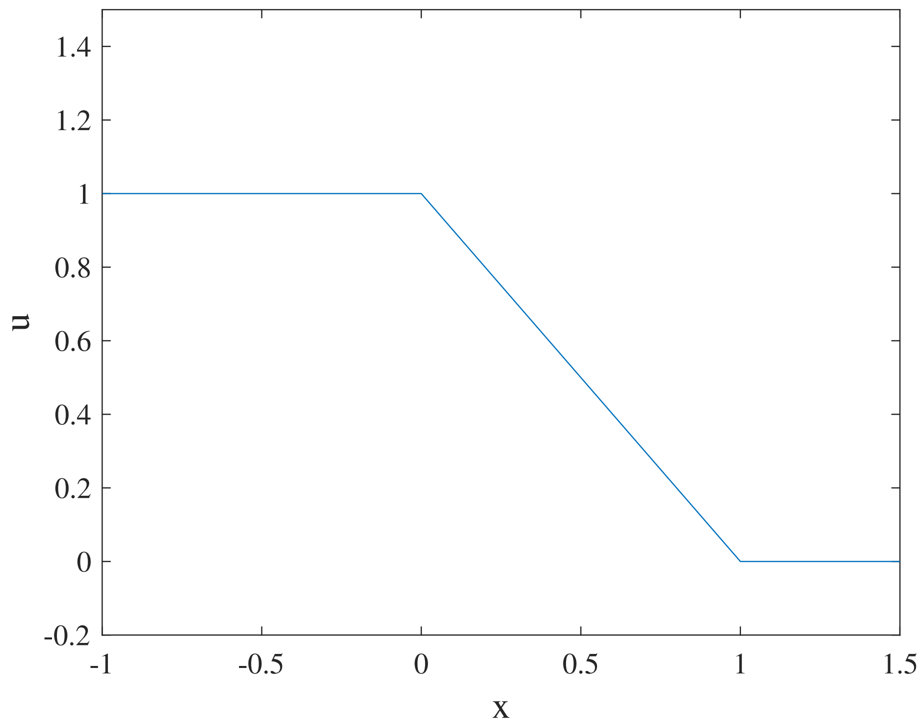

Figure 3: $u(x, 0)$

(b) First, we obtain the solution to u along the characteristics:

$$\begin{aligned} \frac{du}{dt} &= 0 \\ u(x(t),t) &= A \\ A = u(x,0) &= u_0(x_0) \end{aligned} \tag{14}$$

$$u(x(t),t) = \begin{cases} 1, & x_0 \leq 0 \\ 1 - x_0, & 0 < x_0 < 1 \\ 0, & x_0 \geq 1 \end{cases} \tag{15}$$

Using the values for $u_0(x_0)$ and $x(t) = x_0 + c(u_0(x_0))t$, we then obtain the characteristic curves:

$$\boxed{x(t) = \begin{cases} x_0 + t, & x_0 \leq 0 \\ x_0 + t(1 - x_0), & 0 < x_0 < 1 \\ x_0, & x_0 \geq 1 \end{cases} \tag{16}}$$

d