AMATH 353: Homework 6
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Part 1.) Since f_0 is an odd function, we have:

$$\begin{aligned} f_0(z) &= -f_0(-z) \\ f_0'(z) &= f_0'(-z) \\ f_0''(z) &= -f_0''(-z) \end{aligned} \tag{1}$$

First we extend the problem to the entire real line and define f_0 and g_0 as follows:

$$f_0(x) = \begin{cases} f(x), x > 0 \\ -f(-x), x < 0 \end{cases}$$

and

$$g_0(x) = \begin{cases} g(x), x > 0 \\ -g(-x), x < 0 \end{cases}$$

Next, note that $\int_{x-ct}^{x+ct} g_0(s)ds = 0$ because it sums two equal but opposite areas mirrored across the x-axis. Now plugging the solution into the wave equation:

$$\frac{\partial}{\partial t}(\frac{\partial}{\partial t}(f_0(x-ct) + f_0(x+ct))) = c^2 \frac{\partial}{\partial x}(\frac{\partial}{\partial x}(f_0(x-ct) + f_0(x+ct))) \tag{2}$$

$$\frac{\partial}{\partial t}(-cf_{0,t}(x-ct) + cf_{0,t}(x+ct)) = c^2 \frac{\partial}{\partial x}(f_{0,x}(x-ct) + f_{0,x}(x+ct)) \tag{3}$$

$$c^2(f_{0,tt}(x-ct) + f_{0,tt}(x+ct)) = c^2(f_{0,xx}(x-ct) + f_{0,xx}(x+ct)) \tag{4}$$

By superposition, this satisfies the PDE.

Now the side conditions:

$$u(x,0) = \frac{1}{2}(f_0(x) + f_0(x)) + 0 = f_0(x) \tag{5}$$

And since solutions are restricted to $x \geq 0$ an there is no t term we have $f_0(x) = f(x)$.

Now consider the following with $G_0(s) = \int g_0(s)ds$:

$$\begin{aligned} u_t(x,t) &= \frac{1}{2}(-cf_{0,t}(x-ct) + cf_{0,t}(x+ct)) + \frac{1}{2c}\frac{\partial}{\partial t}(G_0(x+ct) - G_0(x-ct)) \\ u_t(x,t) &= \frac{1}{2}(-cf_{0,t}(x-ct) + cf_{0,t}(x+ct)) + \frac{1}{2c}(cg_0(x+ct) + cg_0(x-ct)) \\ u_t(x,0) &= \frac{1}{2}(-cf_0'(x) + cf_0'(x)) + \frac{1}{2c}(cg_0(x) + cg_0(x)) \\ u_t(x,0) &= 0 + g_0(x) \end{aligned} \tag{6}$$

And again because of the restricted x and $t = 0$ we get $u_t(x,0) = g_0(x) = g(x)$.

Lastly for the BC we have the following:

$$u(0,t) = \frac{1}{2}(f_0(-ct) + f_0(ct)) + \frac{1}{2c}(G_0(ct) - G_0(-ct)) \tag{7}$$

And because f_0 is an odd function, $f_0(-ct) = -f_0(ct)$, and because G_0 is an even function, $G_0(-ct) = G_0(ct)$ yielding:

$$u(0,t) = \frac{1}{2}(-f_0(ct) + f_0(ct)) + \frac{1}{2c}(G_0(ct) - G_0(ct)) = 0 \tag{8}$$

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