AMATH 353: Homework 12
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ID: 1064712

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Part 1   Assuming $u(x,t) = u(x(t), t)$, by the chain rule we have $\frac{d}{dt}u = u_t + u_x \frac{dx}{dt}$. Given that we are solving $u_t + 2u_x = 0$, if we assume $\frac{dx}{dt} = 2$ we get the following:

$$\frac{d}{dt}(u(x(t), t)) = u_t + 2u_x = 0 \tag{1}$$

This gives us the two ODEs:

$$\begin{aligned} \frac{dx}{dt} &= 2 \\ \frac{du}{dt} &= 0 \end{aligned} \tag{2}$$

Solving the first ODE by separation of variables, we get the following equation for the characteristic curves, which are shown in the plot below:
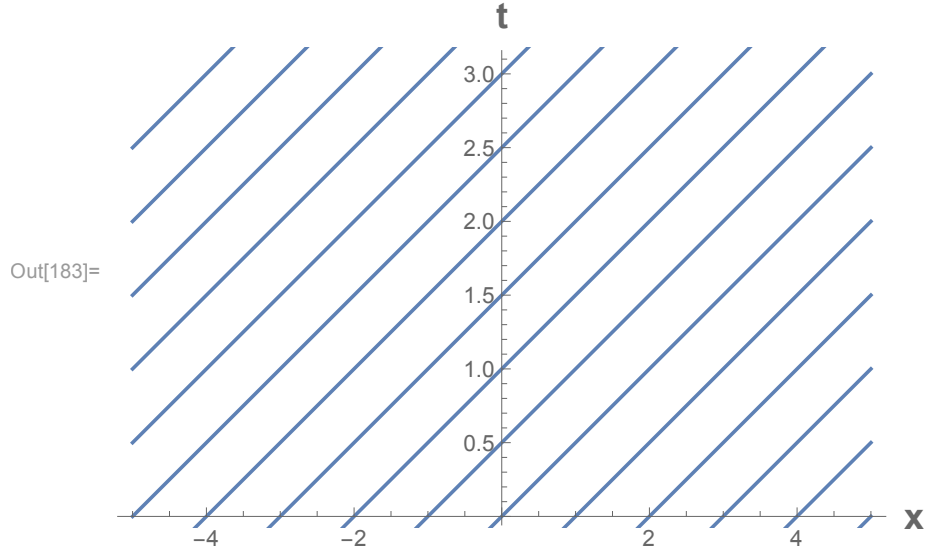
$$x(t) = 2t + x_0 \tag{3}$$

Figure 1: $t = \frac{x - x_0}{2}$

Solving the second ODE we simply get a constant:

$$\int \frac{du}{dt} = \int 0 dt$$
$$u(x(t), t) = A \tag{4}$$

Making use of the initial condition, $u(x, 0) = e^{-x^2}$ we have the following:

$$u(x(t), 0) = u(x_0, 0) = u_0(x_0) = A$$
$$u_0(x_0) = e^{-x_0^2} \tag{5}$$
$$A = e^{-x_0^2}$$

$$u(x(t), t) = e^{-x_0^2} \tag{6}$$

This means that along any given characteristic line $u(x, t) = u(x(t), t)$ we have u constant at a value determined by the initial value of that particular characteristic.

Now using an example of a point $(3, 4)$ we plug it into the characteristic curve and determine it's x_0 value, then determine the value of u_0 at that point, and thus u along that entire characteristic curve:

$$x_0 = x - 2t$$
$$x_0 = 3 - 4(4) = -5 \tag{7}$$

2

$$u_0(-5) = e^{-(-5)^2} = e^{-25} \tag{8}$$

We can arrive at the same value generally by plugging the x_0 equation into the equation derived above for $u(x(t), t)$, then plugging $(3, 4)$ into that:

$$x_0 = x - 2t$$
$$u(x(t), t) = e^{-x_0^2} \tag{9}$$
$$u(x, t) = e^{-(x-2t)^2}$$

$$e^{-(3-2(4))^2} = e^{-25} \tag{10}$$

Note the results of (8) and (10) are the same.

Part 2

a.) The characteristic curves and ODEs for this question are arrived at in the same manner as equations (1) - (4) above, albeit restricted to $x \geq 0$:

$$\frac{dx}{dt} = 2$$
$$x(t) = 2t + x_0$$
$$\frac{du}{dt} = 0 \tag{11}$$
$$u(x(t), t) = A$$

Note that $x_0 = x - 2t$, and so with the restriction that $x > 2t$, we can assume that x_0 will be positive, and thus will be defined by the initial value $u(x, 0) = 0$. We can then use this to solve the constant in $u(x(t), t)$:

$$u(x(t), t) = A$$
$$u(x(t), 0) = u_0(x_0) = A$$
$$u_0(x_0) = 0 \tag{12}$$
$$A = 0$$

From this it follows that when $x > 2t$:

$$u(x, t) = 0 \tag{13}$$

b.) To solve for u when $x < 2t$, I took the same approach, but integrated $\frac{dx}{dt}$ differently in order to get t in terms of x and some t_0:

$$\frac{dx}{dt} = 2$$
$$dt = \frac{2}{dx}$$
$$\int dt = \int \frac{2}{dx}$$
$$t = \frac{1}{2}x + t_0$$

(14)

From this equation we can see that $t_0 = t - \frac{1}{2}x$ and so when $x < 2t$, the value of t_0 will be greater than 0 and thus defined by the initial condition $u_0 = u(0, t) = \frac{t}{1+t^2}$. From this, we follow the same procedure as above to solve for u:

$$u(x(t), t) = A$$
$$u(0, t) = u_0(t_0)$$
$$u_0(t_0) = \frac{t_0}{1 + t_0^2} = A$$

(15)

Then plugging in the value above for t_0 we arrive at the solution for the value of $u(x, t)$ when $x < 2t$:

$$u(x, t) = \frac{t - \frac{1}{2}x}{1 + (t - \frac{1}{2}x)^2}$$

(16)

bla