AMATH 353: Homework 9
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Part 1

a.) We are asked to compute the integral $\int_a^b x\cos(\frac{n\pi x}{2})$. I will also show my work here for $\int_a^b (2-x)\cos(\frac{n\pi x}{2})$ as this is how I computed the a_n term later on. Starting with the first one, using division by parts we have:

$$\begin{aligned} \int x\cos(\frac{n\pi x}{2}) &= \int udv \\ u = x\ ,\ du &= dx \\ dv &= \cos(\frac{n\pi x}{2})dx \\ v = \int dv &= \frac{2}{n\pi}\sin(\frac{n\pi x}{2}) \\ \int udv = uv - \int vdu &= \frac{2x}{n\pi}\sin(\frac{n\pi x}{2}) + \frac{4}{n^2\pi^2}\cos(\frac{n\pi x}{2}) \end{aligned} \tag{1}$$

$$\boxed{\begin{aligned} &\int_a^b x\cos(\frac{n\pi x}{2}) = \\ &\frac{2b}{n\pi}\sin(\frac{n\pi b}{2}) + \frac{4}{n^2\pi^2}\cos(\frac{n\pi b}{2}) - \frac{2a}{n\pi}\sin(\frac{n\pi a}{2}) - \frac{4}{n^2\pi^2}\cos(\frac{n\pi a}{2}) \quad (2) \end{aligned}}$$

And now the same for $\int_a^b (2-x)\cos(\frac{n\pi x}{2})$. Note that I did the definite integral directly here, unlike the previous equation.

$$\begin{aligned}\int_a^b (2-x)cos(\frac{n\pi x}{2}) &= \int_a^b udv \\ u = 2-x\ ,\ du &= -dx \\ dv &= cos(\frac{n\pi x}{2})dx \\ v = \int dv &= \frac{2}{n\pi}\sin(\frac{n\pi x}{2}) \\ \int vdu &= \frac{4}{n^2\pi^2}\cos(\frac{n\pi x}{2})\end{aligned} \tag{3}$$

$$\boxed{\begin{aligned}&uv\Big|_a^b - \int_a^b vdu = \\ &(2-b)\frac{2}{n\pi}\sin(\frac{n\pi b}{2})-(2-a)\frac{2}{n\pi}\sin(\frac{n\pi a}{2})-\frac{4}{n^2\pi^2}\cos(\frac{n\pi b}{2})+\frac{4}{n^2\pi^2}\cos(\frac{n\pi a}{2})\end{aligned} \tag{4}}$$

b.)

$$b_n = \frac{1}{2}\int_{-2}^{2} f_e(x)\sin(\frac{n\pi x}{2})dx \tag{5}$$

Because $f_e(x)$ is defined as the even extension of $f(x)$, and because sine is an odd function, their product is odd. Any integral of an odd function across a symmetrical domain including the origin is equal to zero, and thus $b_n = 0$.

c.) To find a_0 I use the following equation, which we derived in class and which results from the fact that the average value of sin and cos are both 0:

$$\frac{a_0}{2} = A = \frac{1}{4}\int_{-2}^{2} f_e(x)dx \tag{6}$$

Because $f_e(x)$ is defined as the even extension of $f(x)$, and because the resulting domain is defined equivalently in both $f_e(x)$ and $f(x)$, we get the following:

$$\frac{a_0}{2} = A = \frac{1}{4}\int_{-2}^{2} f_e(x)dx = 2(\frac{1}{4})\int_0^2 f_e(x)dx = \frac{1}{2}\int_0^2 f(x)dx \tag{7}$$

This definite integral is then evaluated for the piece-wise function with an integration constant of 0.

$$\begin{aligned}A = \frac{1}{2}\Big[(\frac{1}{2}x^2)\Big|_0^1 + (2x - \frac{1}{2}x^2)\Big|_1^2\Big] \\ A = \frac{1}{2}\end{aligned} \tag{8}$$

$$\boxed{a_0 = 2A = 1 \qquad (9)}$$

d.) To find a_n we use the following equation

$$a_n = \frac{1}{2}\int_{-2}^{2} f_e(x)\cos(\frac{n\pi x}{2})dx \tag{10}$$

taking note that the product of two even functions, $f_e(x)$ and cos, is also an even function, and that the resulting domain is equivalent in both $f_e(x)$ and $f(x)$, we get the following:

$$a_n = \frac{1}{2}\int_{-2}^{2} f_e(x)\cos(\frac{n\pi x}{2})dx = 2\frac{1}{2}\int_{0}^{2} f_e(x)\cos(\frac{n\pi x}{2})dx = \int_{0}^{2} f(x)\cos(\frac{n\pi x}{2})dx \tag{11}$$

Because $f(x)$ is a piecewise function (shown below), the integral in this domain is defined as follows:

$$f(x) = \begin{cases} x, & 0 < x \leq 1, \\ 2 - x, & 1 < x \leq 2 \end{cases}$$

$$\int_{0}^{2} f(x)\cos(\frac{n\pi x}{2})dx = \int_{0}^{1} x\cos(\frac{n\pi x}{2})dx + \int_{1}^{2}(2-x)cos(\frac{n\pi x}{2})dx \tag{12}$$

Then plugging this into the results of part a.), equations (2) and (4) above, we have the following:

$$\begin{aligned} a_n = & \\ & \Big[\frac{2}{n\pi}\sin(\frac{n\pi}{2}) + \frac{4}{n^2\pi^2}\cos(\frac{n\pi}{2}) - 0 - \frac{4}{n^2\pi^2}\cos(0)\Big] \\ & + \Big[0 - (1)(\frac{2}{n\pi}\sin(\frac{n\pi}{2}) - \frac{4}{n^2\pi^2}\cos(n\pi) + \frac{4}{n^2\pi^2}\cos(\frac{n\pi}{2})\Big] \end{aligned} \tag{13}$$

By carrying out the addition and subtraction of sine/cosine terms with the same arguments, and using the equivalence $cos(n\pi) = (-1)^2$ we get:

$$\boxed{a_n = \frac{8\cos(\frac{n\pi}{2}) - 4(-1)^n - 4}{n^2\pi^2} \qquad (14)}$$

e.) Now that I have my constants, I can plug them in to the following equation to plot. *Please note I have uploaded the wolfram notebook I used to make these plots.* As the graphs show, f_3 and f_{10} superficially resemble $f(x)$. As N increases, we see that the fourier series is beginning to converge. In particular, comparing f_3 and f_{10}, we see the larger number of oscillations as more waves are added, but they are flattening out and more closely resembling a line. In addition, it's clear looking at the bounds that f_{10} is much closer to the proper bounds of $f(0) = 0$ and $f(2) = 0$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{2}) \tag{15}$$

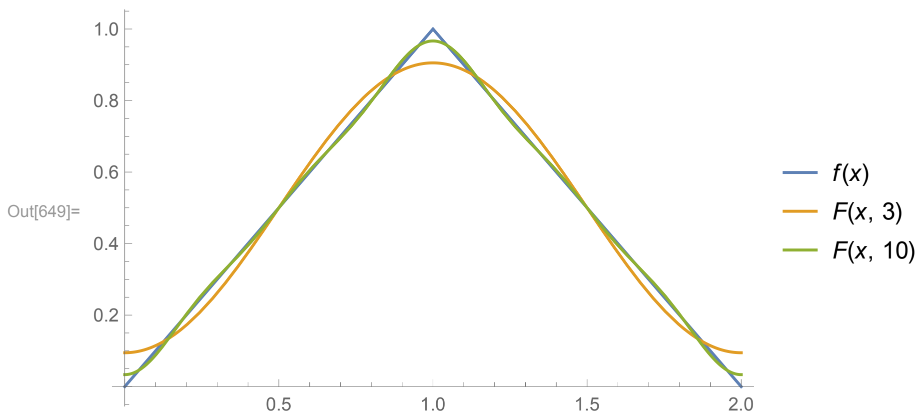

Figure 1: $f(x)$ and $n = 3, 10$