# AMATH 353: Homework 14
## Due May, 30 2018
## ID: 1064712

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## Part 1

**(a)** Given the following initial condition, we get:

$$u(x, 0) = \begin{cases} \frac{\pi}{2}, & x < 1 \\ \frac{\pi}{4}, & x \geq 1 \end{cases} \tag{1}$$

$$\frac{du}{dt} = 0$$
$$u(x(t), t) = A \tag{2}$$
$$A = u_0(x_0) = u(x, 0)$$

We can then make use of $x(t) = x_0 + c(u_0(x_0)t$: and determine the equations for the characteristics:

$$c(u(x(t), t)) = \sin(u)$$
$$c(u_0(x_0) = \sin(u_0(x_0)) \tag{3}$$

$$c(u_0(x_0) \begin{cases} 1, & x_0 < 1 \\ \frac{\sqrt{2}}{2}, & x_0 \geq 1 \end{cases} \tag{4}$$

This in turn gives the following characteristics curves:

$$x(t) = \begin{cases} x_0 + t, & x_0 < 1 \\ x_0 + \frac{\sqrt{2}}{2}t, & x_0 \geq 1 \end{cases} \tag{5}$$

$$t = \begin{cases} x - x_0, & x_0 < 1 \\ \sqrt{2}(x - x_0) & x_0 \geq 1 \end{cases} \tag{6}$$

Additionally, here are the functions of $x_0$ in terms of $x$ and $t$, which will be useful later:

$$x_0 = \begin{cases} x - t, & x_0 < 1 \\ x - \frac{\sqrt{2}}{2}t & x_0 \geq 1 \end{cases} \tag{7}$$

**(b)** Because $u(x(t), t) = u_0(x_0)$ we can find the breaking time by finding where its derivative goes to $\infty$.

$$u_x = u_0'(x_0)\frac{dx_0}{dx} \tag{8}$$

It is clear from the initial condition given above that we cannot take the derivative $u_0'(x_0)$ because $u_0(x_0)$ has a discontinuity when $x_0 = 1$. Plugging this into equation (7) above at $t = 0$ we see that this occurs at $x = 1$.

**(c)** Since $\phi(x, t) = \sin(u)u_x$ we can integrate to get the following (note that I am ignoring the constant of integration, since it drops out in the Rankine-Hugoniot relation anyway):

$$\phi(x, t) = -\cos(u) \tag{9}$$

We know from equation (2) above that $u^- = \frac{\pi}{2}$ and $u^+ = \frac{\pi}{4}$ giving:

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}$$
$$\frac{[\phi]}{[u]} = \frac{-\cos(u^+) + \cos(u^-)}{u^+ - u^-} \tag{10}$$
$$\frac{dx_s}{dt} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\pi}{4}}$$

Which simplifies to:

$$\frac{dx_s}{dt} = \frac{2\sqrt{2}}{\pi} \tag{11}$$

We then integrate and apply the initial condition found in (b):

$$x_s(t) = \frac{2\sqrt{2}}{\pi}t + C$$
$$x_s(0) = 1 = C \tag{12}$$
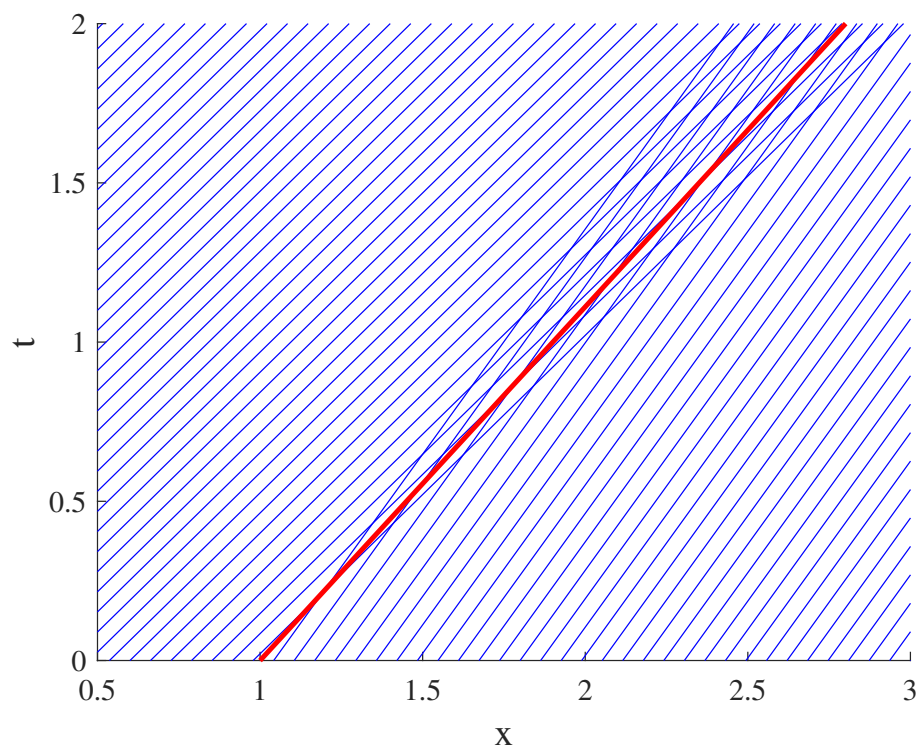$$x_s(t) = \frac{2\sqrt{2}}{\pi}t + 1$$

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Figure 1: Characteristics and $x_s$

**(d)** The solution to the left of the shockline is a line at $u = \frac{\pi}{2}$, and to the right, a line at $u = \frac{\pi}{4}$. This is defined as:

$$u(x,t) = \begin{cases} \frac{\pi}{2}, & x < x_s \\ \frac{\pi}{4} & x \geq x_s \end{cases} \tag{13}$$

**(e)** If we change the characteristics in this way,the density increases at the discontinuity, so the speed for $u^+$ is actually greater than $u^-$. As a result, the space between the solutions for $u$ actually increases in time, giving a growing region in which $u$ is not defined.
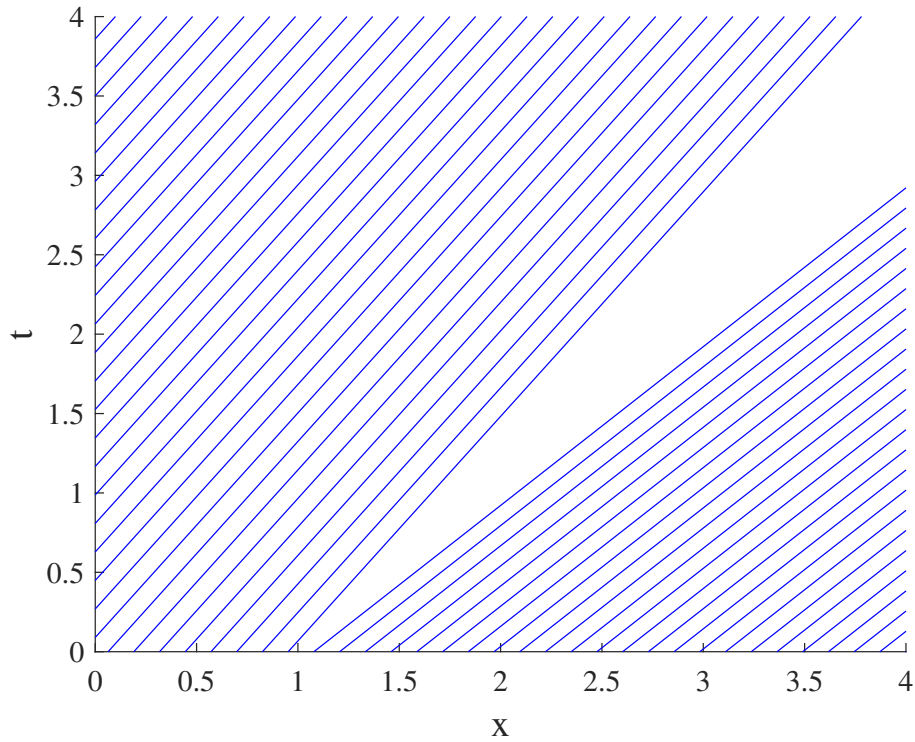


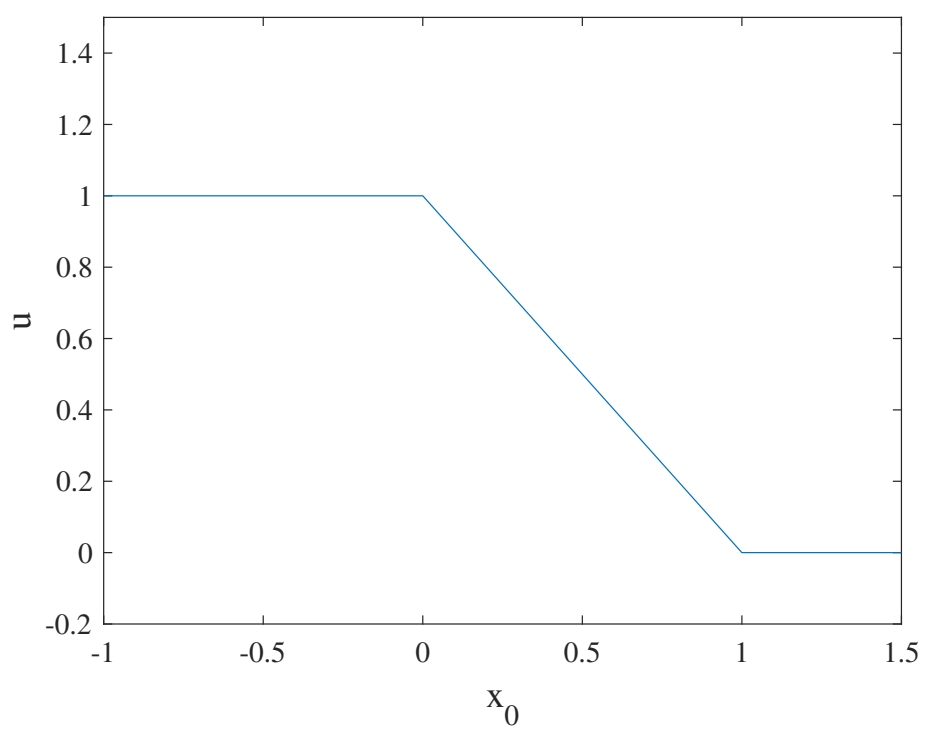Figure 2: $u^- = \frac{\pi}{4}$ and $u^+ = \frac{\pi}{2}$

## Part 2

**(a)**

Figure 3: $u(x, 0)$

**(b)** First, we obtain the solution to $u$ along the characteristics:

$$\frac{du}{dt} = 0$$
$$u(x(t), t) = A \tag{14}$$
$$A = u(x, 0) = u_0(x_0)$$

$$u(x(t), t) = \begin{cases} 1, & x_0 \leq 0 \\ 1 - x_0, & 0 < x_0 < 1 \\ 0, & x_0 \geq 1 \end{cases} \tag{15}$$

Using the values for $u_0(x_0)$ and $x(t) = x_0 + c(u_0(x_0))t$, we then obtain the characteristic curves:

$$x(t) = \begin{cases} x_0 + t, & x_0 \leq 0 \\ x_0 + t(1 - x_0), & 0 < x_0 < 1 \\ x_0, & x_0 \geq 1 \end{cases} \tag{16}$$

**(c)** We know from Knobel's that a solution for $u(x, t)$ when a shock forms must have continuous first derivatives in the regions to the left and right of the shock line. When we take the derivative $u_0'(x_0)$ from equaton (15) we have the following:

$$u_0'(x_0) = \begin{cases} 0, & x_0 \leq 0 \\ -1, & 0 < x_0 < 1 \\ 0, & x_0 \geq 1 \end{cases} \tag{17}$$

This derivative is discontinuous at $x_0 = 0, 1$ and so no two regions exist such that there would be continuous derivatives on each side. However, we can deduce a few things looking at the speed of the initial profile, which is equal to $u$. The value of $u$ when $x_0 < 0$ is 1, so it is moving to the right, and the value when $x \geq 1$ is 0, so it is not moving in time. The middle region's value from $0 < x < 1$ is $1 - x_0$ so the top of the profile is moving faster than the bottom, so over time it will form a vertical line as $x_0$ approaches 1. This means that if these characteristic lines at $x_0 = 0, 1$ intersect, there will be smooth first derivatives on each side, and furthermore there will be an jump in the solutions, and thus a shockwave. Plugging these values of $x_0$ into equation (16) we get:

$$x_0 = 0 = x - t$$
$$x_0 = 1 = x$$
$$0 = 1 - t \tag{18}$$
$$t_b = 1$$

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**(d)** Using equation (16) above, we can solve for $x_0$ for each of the 3 ranges. $x_0 \leq 0$ and $x_0 \geq 0$ are fairly straightforward, so I won't show my work for those. For $0 < x_0 < 1$ however, we have the following:

$$x(t) = x_0 + t(1 - x_0)$$

$$\frac{x}{x_0} = 1 - t + \frac{t}{x_0}$$

$$\frac{x}{x_0} - \frac{t}{x_0} = 1 - t \tag{19}$$

$$\frac{1}{x_0}(x - t) = 1 - t$$

$$x_0 = \frac{x - t}{1 - t}$$

We then have:

$$x_0 = \begin{cases} x - t, & x_0 \leq 0 \\ \frac{x-t}{1-t}, & 0 < x_0 < 1 \\ x, & x_0 \geq 1 \end{cases} \tag{20}$$

We can then substitute these values into equation (16) to obtain the solution for $t < t_b$. Again, the substitutions are fairly straightforward, but I will show the work for $0 < x_0 < 1$.

$$0 < \frac{x - t}{1 - t} < 1$$

$$0 < x - t < 1 - t \tag{21}$$

$$t < x < 1$$

And finally substituting all this into (15) gives the solution for $0 \leq t < t_b$. We can see in this solution that the middle constraint solutions only exist in a range between $t = 0$, $x = 1$ and $t = x$, as described in part (c).

$$u(x, t) = \begin{cases} 1, & x \leq t \\ 1 - \frac{x-t}{1-t}, & t < x < 1 \\ 0, & x \geq 1 \end{cases} \tag{22}$$

**(e)** We can see from the stated problem that $\phi_x = u u_x$. Integrating this gives

$$\phi = \frac{u^2}{2} \tag{23}$$

Considering equation (22) above, we can see that at the breaking time $t_b = 1$, the solution is defined in two regions as the second constraint $t < x < 1$ is not satisfied for any $t > t_b = 1$. This leaves the other two constraints:

$$u^- = 1, \; x < t$$

$$u^+ = 0, \; x \geq 1 \tag{24}$$

With this information we can then use the Rankine-Hugoniot relation:

$$\frac{dx_0}{dt} = \frac{\frac{0^2}{2} - \frac{1^2}{2}}{0 - 1}$$

$$\frac{dx_0}{dt} = \frac{1}{2}$$

$$x_s(t) = \frac{1}{2}t + C \tag{25}$$

$$x_s(1) = 1 = \frac{1}{2}(1) + C$$

$$x_s(t) = \frac{1}{2}t + \frac{1}{2}$$

(**f**)  Making use equations (22), (24), and (25) above:

$$u(x,t) = \begin{cases} \begin{cases} 1, & x \leq t \\ 1 - \frac{x-t}{1-t}, & t < x < 1 \quad 0 < t < 1 \\ 0, & x \geq 1 \end{cases} \\ \begin{cases} 1, & x < x_s(t) \\ 0, & x > x_s(t) \end{cases} \quad t \geq 1 \end{cases} \tag{26}$$
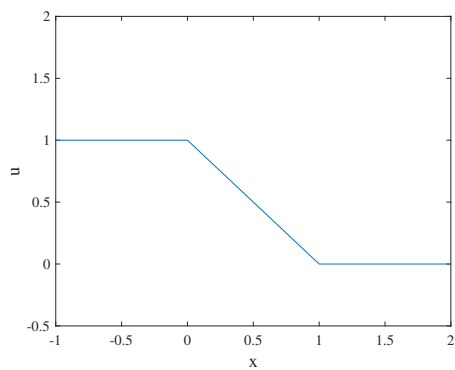
Figure 4: $t = 0$
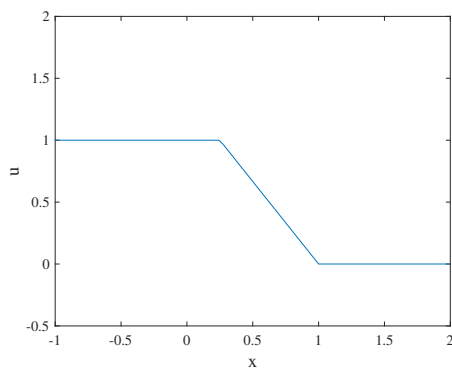


Figure 5: $t = 0.25$
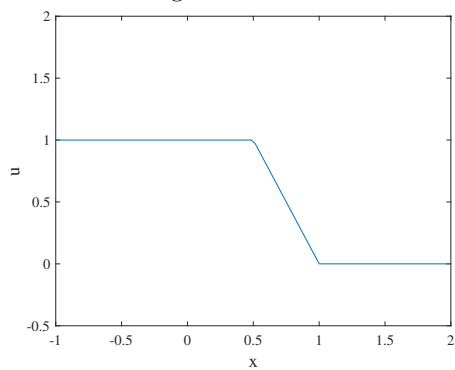


Figure 6: $t = 0.5$
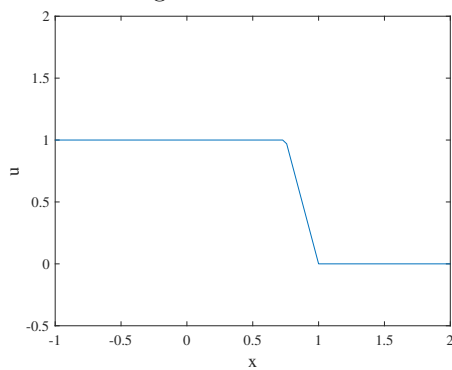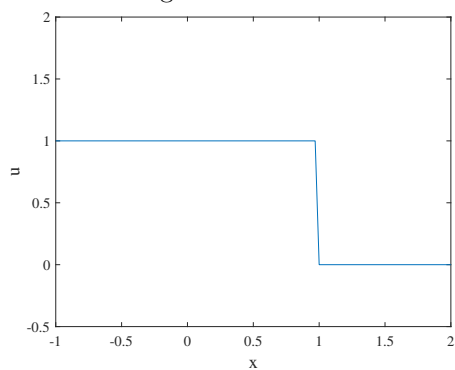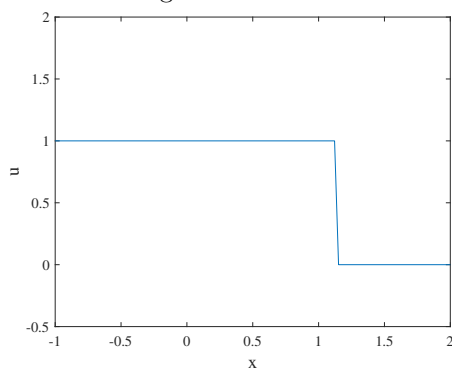


Figure 7: $t = .75$
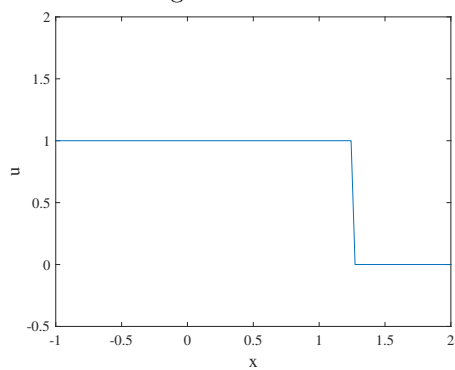


Figure 8: $t = 1$


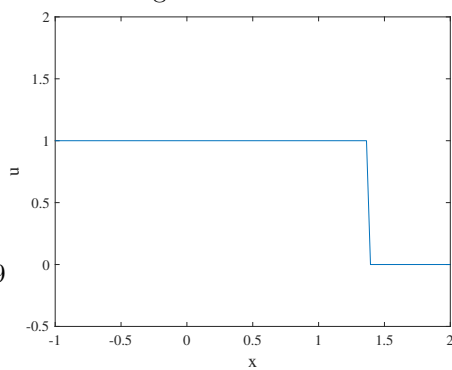
Figure 9: $t = 1.25$
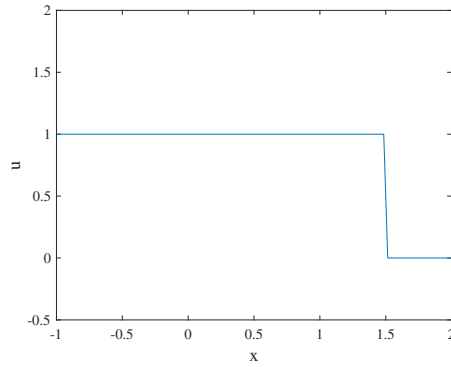


Figure 10: $t = 1.5$



Figure 11: $t = 1.75$

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Figure 12: $t = 2$

(g)

# Part 3

(a) For these values, there is no change in density of cars at any point, thus no shock/traffic jam forms. Information travels forward at a uniform speed of $v = v_1(1 - \frac{\frac{u_1}{3}}{u_1}) = \frac{2}{3}v_1$.
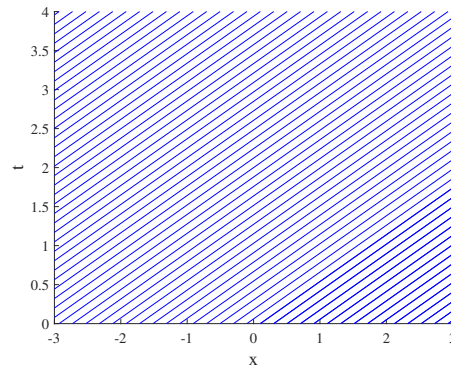


Figure 13

(b) For these values a traffic jam forms immediately at $t = 0$. If you zoom in on the graph, it's clear that information is traveling forward, which is also clear since the $u^+$ value is less than $u_1$. Overall this means traffic hits the traffic jam and then immediately slows to a much lower speed.
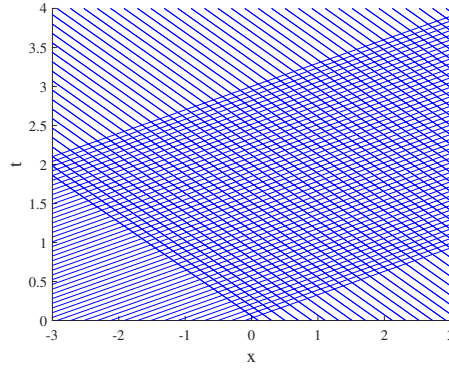
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Figure 14

(c) For these values, traffic is moving more slowly on the left and more quickly on the right, and thus a region forms in which there are no characteristics, and thus no solution to $u$. Intuitively, this would seem to model an area in which there are no cars because two 'packs' of cars are getting further and further apart. As to why this would happen, I can think of a few scenarios, but some, like a change in speed limit, mess with our model. Perhaps an area in which a single lane opens up to two lanes, but this wouldn't explain the widening region in which $u$ is undefined.
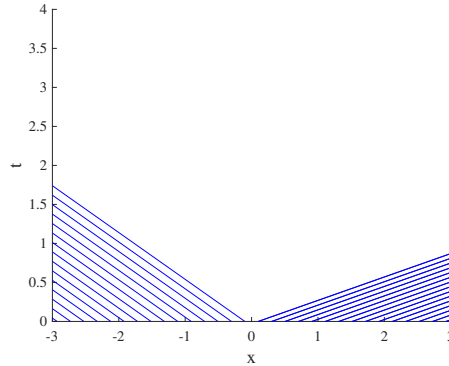


Figure 15

(d) The slope of the characteristics in the traffic problem are give by

$$\frac{1}{v_1(1 - \frac{2u^{\pm}}{u_1})} \tag{27}$$

We can see that when the slope of the characteristics in the left region are greater than the ones in the right, they will cross (regardless of whether the

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slopes are positive or negative). We then have:

$$\frac{1}{v_1(1 - \frac{2u^-}{u_1})} \geq \frac{1}{v_1(1 - \frac{2u^+}{u_1})} \tag{28}$$

This then very quickly simplifies down to $u^- \geq u^+$, which is to say if the initial density is higher on the left than it is on the right, no shock will form.