

AMATH 353: Homework 8

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Part 1

a.) We are asked to compute the integral $\int_a^b x \cos(\frac{n\pi x}{2})$. I will also show my work here for $\int_a^b (2-x) \cos(\frac{n\pi x}{2})$ as this is how I computed the a_n term later on. Starting with the first one, using division by parts we have:

$$\begin{aligned}\int x \cos(\frac{n\pi x}{2}) &= \int u dv \\ u &= x, \quad du = dx \\ dv &= \cos(\frac{n\pi x}{2}) dx \\ v &= \int dv = \frac{2}{n\pi} \sin(\frac{n\pi x}{2}) \\ \int u dv &= uv - \int v du = \frac{2x}{n\pi} \sin(\frac{n\pi x}{2}) + \frac{4}{n^2\pi^2} \cos(\frac{n\pi x}{2})\end{aligned}\tag{1}$$

$$\begin{aligned}\int_a^b x \cos(\frac{n\pi x}{2}) &= \\ \frac{2b}{n\pi} \sin(\frac{n\pi b}{2}) + \frac{4}{n^2\pi^2} \cos(\frac{n\pi b}{2}) - \frac{2a}{n\pi} \sin(\frac{n\pi a}{2}) - \frac{4}{n^2\pi^2} \cos(\frac{n\pi a}{2})\end{aligned}\tag{2}$$

And now the same for $\int_a^b (2-x) \cos(\frac{n\pi x}{2})$. Note that I did the definite integral directly here, unlike the previous equation.

$$\begin{aligned}
\int_a^b (2-x) \cos(\frac{n\pi x}{2}) &= \int_a^b u dv \\
u &= 2-x, \quad du = -dx \\
dv &= \cos(\frac{n\pi x}{2}) dx \\
v &= \int dv = \frac{2}{n\pi} \sin(\frac{n\pi x}{2}) \\
\int v du &= \frac{4}{n^2 \pi^2} \cos(\frac{n\pi x}{2})
\end{aligned} \tag{3}$$

$$\begin{aligned}
uv \Big|_a^b - \int_a^b v du &= \\
(2-b) \frac{2}{n\pi} \sin(\frac{n\pi b}{2}) - (2-a) \frac{2}{n\pi} \sin(\frac{n\pi a}{2}) - \frac{4}{n^2 \pi^2} \cos(\frac{n\pi b}{2}) + \frac{4}{n^2 \pi^2} \cos(\frac{n\pi a}{2})
\end{aligned} \tag{4}$$

b.)

$$b_n = \frac{1}{2} \int_{-2}^2 f_e(x) \sin(\frac{n\pi x}{2}) dx \tag{5}$$

Because $f_e(x)$ is defined as the even extension of $f(x)$, and because sine is an odd function, their product is odd. Any integral of an odd function across a symmetrical domain including the origin is equal to zero, and thus $b_n = 0$.

c.) To find a_0 I use the following equation:

$$\frac{a_0}{2} = A = \frac{1}{4} \int_{-2}^2 f_e(x) x \tag{6}$$

Because $f_e(x)$ is defined as the even extension of $f(x)$, and because the resulting domain is defined equivalently in both $f_e(x)$ and $f(x)$, we get the following:

$$\frac{a_0}{2} = A = \frac{1}{4} \int_{-2}^2 f_e(x) x = 2 \left(\frac{1}{4} \right) \int_0^2 f_e(x) dx = \frac{1}{2} \int_0^2 f(x) dx \tag{7}$$

This definite integral is then evaluated for the piece-wise function with an integration constant of 0.

$$\begin{aligned}
A &= \frac{1}{2} \left[\left(\frac{1}{2} x^2 \right) \Big|_0^1 + \left(2x - \frac{1}{2} x^2 \right) \Big|_1^2 \right] \\
A &= \frac{1}{2} \\
a_0 &= 2A = 1
\end{aligned} \tag{8}$$

d.) To find a_n we use the following equation:

$$a_n = \frac{1}{2} \int_{-2}^2 f_e(x) \cos\left(\frac{n\pi x}{2}\right) dx \quad (9)$$

Taking note that the product of two even functions, $f_e(x)$ and *cosine*, is also an even function, and that the resulting domain is equivalent in both $f_e(x)$ and $f(x)$ we get the following:

$$a_n = \frac{1}{2} \int_{-2}^2 f_e(x) \cos\left(\frac{n\pi x}{2}\right) dx = 2 \frac{1}{2} \int_0^2 f_e(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \quad (10)$$

Because $f(x)$ is a piecewise function (shown below), the integral in this domain is defined as follows:

$$f(x) = \begin{cases} x, & 0 < x \leq 1, \\ 2 - x, & 1 < x \leq 2 \end{cases}$$

$$\int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^1 x \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2 - x) \cos\left(\frac{n\pi x}{2}\right) dx \quad (11)$$

Then plugging this into the results of part a.), equations (2) and (4) above, we have the following:

$$\begin{aligned} a_n = & \left[\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - 0 - \frac{4}{n^2\pi^2} \cos(0) \right] \\ & + \left[0 - (1) \left(\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \cos(n\pi) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) \right) \right] \quad (12) \end{aligned}$$

By canceling the terms that simply subtract out, adding the cosine terms, and using the equivalence $\cos(n\pi) = (-1)^n$ we get:

$$a_n = \frac{8 \cos\left(\frac{n\pi}{2}\right) - 4(-1)^n - 4}{n^2\pi^2} \quad (13)$$

bla