AMATH 353: Homework 7
Due April, 23 2018
ID: 1064712

Trent Yarosevich

April 23, 2018

Instructor: Jeremy Upsal

Part 1.) Since f_0 is an odd function, we have:

$$\begin{aligned} f_0(z) &= -f_0(-z) \\ f_0'(z) &= f_0'(-z) \\ f_0''(z) &= -f_0''(-z) \end{aligned} \tag{1}$$

First we extend the problem to the entire real line and define f_0 and g_0 as follows:

$$f_0(x) = \begin{cases} f(x), x > 0 \\ -f(-x), x < 0 \end{cases}$$

and

$$g_0(x) = \begin{cases} g(x), x > 0 \\ -g(-x), x < 0 \end{cases}$$

This gives

$$u(x,t) = \frac{1}{2}(f_0(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(s)ds \tag{2}$$

Now the side conditions, with the integral of g_0 being zero because it is a definite integral across a zero domain:

$$u(x,0) = \frac{1}{2}(f_0(x) + f_0(x)) + 0 = f_0(x) \tag{3}$$

And since solutions are restricted to $x \geq 0$ and there is no t term we have $f_0(x) = f(x)$.

Now consider the following with $G_0(s) = \int g_0(s)ds$:

$$\begin{aligned} u_t(x,t) &= \frac{1}{2}(-cf_{0,t}(x-ct) + cf_{0,t}(x+ct)) + \frac{1}{2c}\frac{\partial}{\partial t}(G_0(x+ct) - G_0(x-ct)) \\ u_t(x,t) &= \frac{1}{2}(-cf_{0,t}(x-ct) + cf_{0,t}(x+ct)) + \frac{1}{2c}(cg_0(x+ct) + cg_0(x-ct)) \\ u_t(x,0) &= \frac{1}{2}(-cf_0'(x) + cf_0'(x)) + \frac{1}{2c}(cg_0(x) + cg_0(x)) \\ u_t(x,0) &= 0 + g_0(x) \end{aligned} \tag{4}$$

And again because of the restricted x and $t = 0$ we get $u_t(x,0) = g_0(x) = g(x)$.

Lastly for the BC we have the following:

$$u(0,t) = \frac{1}{2}(f_0(-ct) + f_0(ct)) + \frac{1}{2c}(G_0(ct) - G_0(-ct)) \tag{5}$$

And because f_0 is an odd function, $f_0(-ct) = -f_0(ct)$, and because G_0 is the integral of an odd function, I am assuming it must be an even function[1], so $G_0(-ct) = G_0(ct)$ yielding:

$$u(0,t) = \frac{1}{2}(-f_0(ct) + f_0(ct)) + \frac{1}{2c}(G_0(ct) - G_0(ct)) = 0 \tag{6}$$

Part 2

$$f_0(x) = \begin{cases} -\cos(x) + 1, x > 0 \\ \cos(x) - 1, x < 0 \end{cases}$$

and

$$g_0(x) = \begin{cases} xe^{-x^2}, x > 0 \\ xe^{-x^2}, x < 0 \end{cases}$$

Note that xe^{-x^2} is the odd extension of itself and $\int g_0(s)ds = -\frac{1}{2}e^{-(s)^2}$ (constant of integration is zero), and $c = \sqrt{81} = 9$. From this we get the solution:

$$u(x,t) = \frac{1}{2}(f_0(x-9t) + f_0(x+9t)) + \frac{1}{36}(-e^{-(x+9t)^2} + e^{-(x-9t)^2}) \tag{7}$$

[1] I didn't look all that hard for an authoritative proof of this, but there is one here that my admittedly ignorant self finds convincing: https://math.stackexchange.com/questions/2227677/proof-that-integral-of-odd-function-is-even-function