AMATH 353: Homework 14
Due May, 30 2018
ID: 1064712

Trent Yarosevich

May 30, 2018

Instructor: Jeremy Upsal

Part 1

(a) Given the following initial condition, we get:

$$u(x,0) = \begin{cases} \frac{\pi}{2}, & x < 1 \\ \frac{\pi}{4}, & x \geq 1 \end{cases} \tag{1}$$

$$\begin{aligned} \frac{du}{dt} &= 0 \\ u(x(t),t) &= A \\ A = u_0(x_0) &= u(x,0) \end{aligned} \tag{2}$$

We can then make use of $x(t) = x_0 + c(u_0(x_0)t$: and determine the equations for the characteristics:

$$\begin{aligned} c(u(x(t),t)) &= \sin(u) \\ c(u_0(x_0) &= \sin(u_0(x_0)) \end{aligned} \tag{3}$$

$$c(u_0(x_0) \begin{cases} 1, & x_0 < 1 \\ \frac{\sqrt{2}}{2}, & x_0 \geq 1 \end{cases} \tag{4}$$

This in turn gives the following characteristics curves:

$$x(t) = \begin{cases} x_0 + t, & x_0 < 1 \\ x_0 + \frac{\sqrt{2}}{2}t, & x_0 \geq 1 \end{cases} \tag{5}$$

$$t = \begin{cases} x - x_0, & x_0 < 1 \\ \sqrt{2}(x - x_0) & x_0 \geq 1 \end{cases} \tag{6}$$

Additionally, here are the functions of x_0 in terms of x and t, which will be useful later:

$$x_0 = \begin{cases} x - t, & x_0 < 1 \\ x - \frac{\sqrt{2}}{2}t & x_0 \geq 1 \end{cases} \tag{7}$$

(b) Because $u(x(t), t) = u_0(x_0)$ we can find the breaking time by finding where its derivative goes to ∞.

$$u_x = u_0'(x_0)\frac{dx_0}{dx} \tag{8}$$

It is clear from the initial condition given above that we cannot take the derivative $u_0'(x_0)$ because $u_0(x_0)$ has a discontinuity when $x_0 = 1$. Plugging this into equation (7) above at $t = 0$ we see that this occurs at $x = 1$. Last, we need to confirm that when this discontinuity occurs, that the characteristic lines are actually crossing. Plugging in the value for x_0 we have:

$$t = \begin{cases} x - 1, & x_0 < 1 \\ \sqrt{2}(x-1) & x_0 \geq 1 \end{cases} \tag{9}$$

The slope of the characteristic line for $x_0 < 1$ is 1, and the slope for $x_0 \geq 1$ is $\sqrt{(2)}$. Both lines pass through $x = 1, t = 0$, so the lines will indeed cross at $t = 0$. In the class we seem comfortable saying this, but technically, don't the characteristics cross at an infinitely small $t > 0$?

(c) Since $\phi(x, t) = \sin(u)u_x$ we can integrate to get the following (note that I am ignoring the constant of integration, since it drops out in the Rankine-Hugoniot relation anyway):

$$\phi(x, t) = -\cos(u) \tag{10}$$

We know from equation (2) above that $u^- = \frac{\pi}{2}$ and $u^+ = \frac{\pi}{4}$ giving:

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ \frac{[\phi]}{[u]} &= \frac{-\cos(u^+) + \cos(u^-)}{u^+ - u^-} \\ \frac{dx_s}{dt} &= \frac{-\frac{\sqrt{2}}{2}}{-\frac{\pi}{4}} \end{aligned} \tag{11}$$

Which simplifies to:

$$\frac{dx_s}{dt} = \frac{2\sqrt{2}}{\pi} \tag{12}$$

We then integrate and apply the initial condition found in (b):

$$
\boxed{\begin{aligned} x_s(t) &= \frac{2\sqrt{2}}{\pi}t + C \\ x_s(0) &= 1 = C \\ x_s(t) &= \frac{2\sqrt{2}}{\pi}t + 1 \end{aligned}} \tag{13}
$$

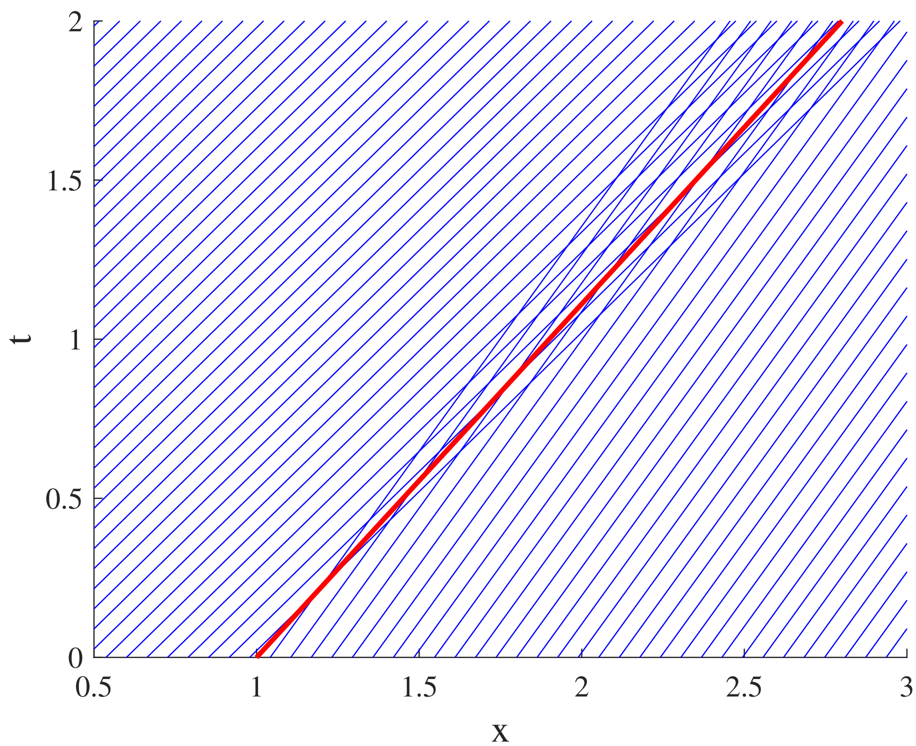

Figure 1: Characteristics and x_s

(d) The solution to the left of the shockline is a line at $u = \frac{\pi}{2}$, and to the right, a line at $u = \frac{\pi}{4}$. This is defined as:

$$
\boxed{u(x,t) = \begin{cases} \frac{\pi}{2}, & x < x_s \\ \frac{\pi}{4} & x \geq x_s \end{cases}} \tag{14}
$$

(e) If we change the characteristics in this way, the speed in the u^+ region is actually greater than it is in the u^-. As a result, the space between the solutions for u actually increases in time, giving a growing region in which u is not defined.

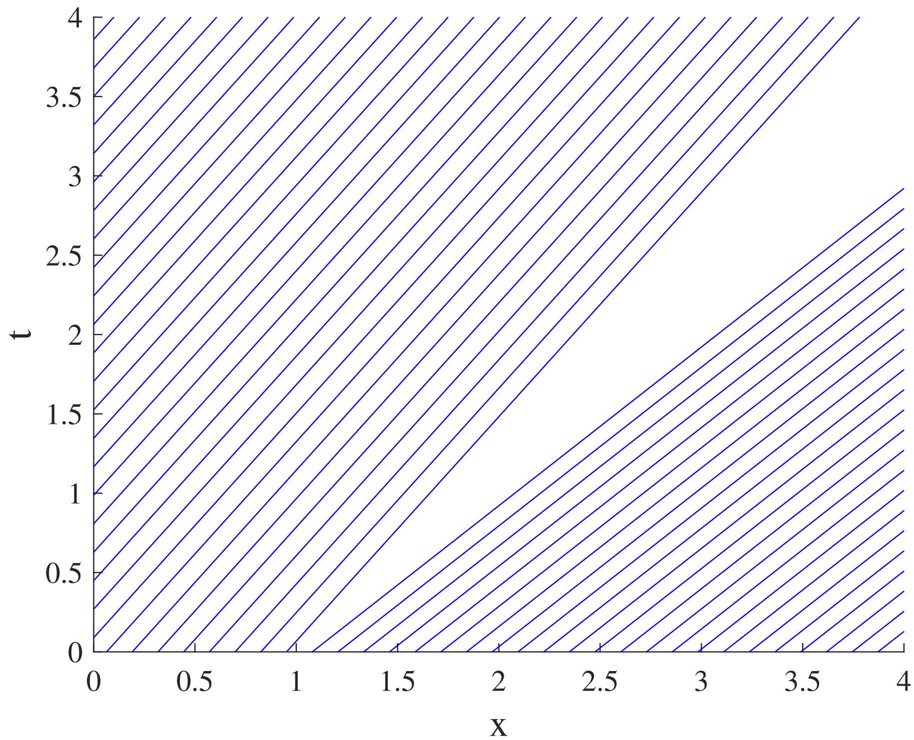

Figure 2: $u^- = \frac{\pi}{4}$ and $u^+ = \frac{\pi}{2}$

Part 2

(a)

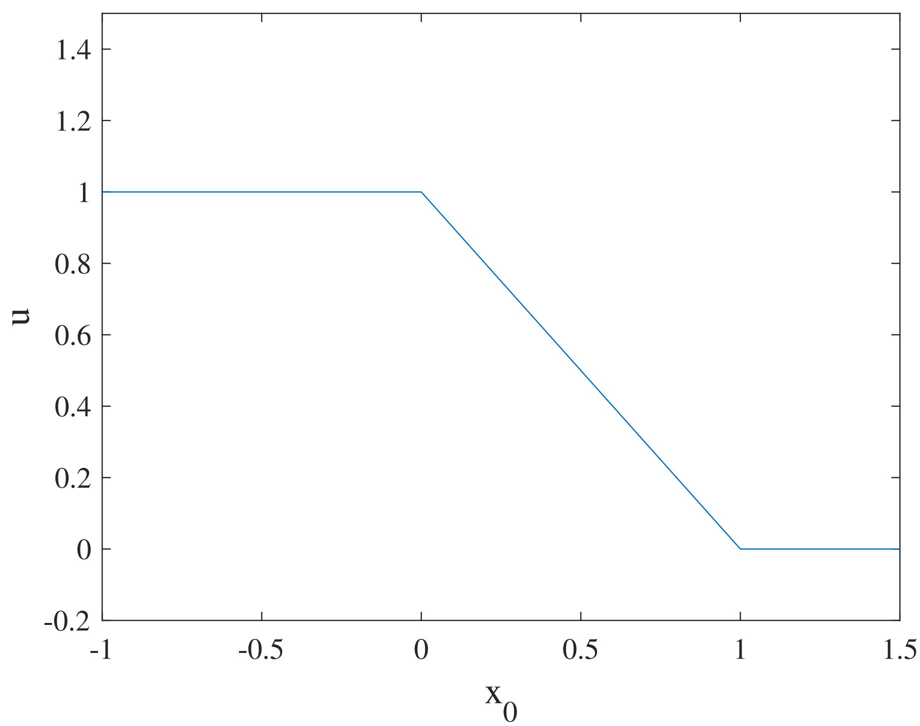

Figure 3: $u(x, 0)$

(b) First, we obtain the solution to u along the characteristics:

$$\begin{aligned} \frac{du}{dt} &= 0 \\ u(x(t),t) &= A \\ A = u(x,0) &= u_0(x_0) \end{aligned} \tag{15}$$

$$u(x(t),t) = \begin{cases} 1, & x_0 \leq 0 \\ 1 - x_0, & 0 < x_0 < 1 \\ 0, & x_0 \geq 1 \end{cases} \tag{16}$$

Using the values for $u_0(x_0)$ and $x(t) = x_0 + c(u_0(x_0))t$, we then obtain the characteristic curves:

$$\boxed{x(t) = \begin{cases} x_0 + t, & x_0 \leq 0 \\ x_0 + t(1 - x_0), & 0 < x_0 < 1 \\ x_0, & x_0 \geq 1 \end{cases} \tag{17}}$$

(c) We know from Knobel's that a solution for $u(x,t)$ when a shock forms must have continuous first derivatives in the regions to the left and right of the shock line. When we take the derivative $u_0'(x_0)$ from equaton (15) we have the following:

$$u_0'(x_0) = \begin{cases} 0, & x_0 \leq 0 \\ -1, & 0 < x_0 < 1 \\ 0, & x_0 \geq 1 \end{cases} \tag{18}$$

This derivative is discontinuous at $x_0 = 0, 1$ and so no two regions exist such that there would be continuous derivatives on each side. However, we can deduce a few things looking at the speed of the initial profile, which is equal to u. The value of u when $x_0 < 0$ is 1, so it is moving to the right, and the value when $x \geq 1$ is 0, so it is not moving in time. The middle region's value from $0 < x < 1$ is $1 - x_0$ so the top of the profile is moving faster than the bottom, so over time it will form a vertical line as x_0 approaches 1. This means that if these characteristic lines at $x_0 = 0, 1$ intersect, there will be smooth first derivatives on each side, and furthermore there will be an jump in the solutions, and thus a shockwave. Plugging these values of x_0 into equation (16) we get:

$$\boxed{\begin{aligned} x_0 = 0 &= x - t \\ x_0 = 1 &= x \\ 0 &= 1 - t \\ t_b &= 1 \end{aligned} \tag{19}}$$

(d) Using equation (16) above, we can solve for x_0 for each of the 3 ranges. $x_0 \leq 0$ and $x_0 \geq 0$ are fairly straightforward, so I won't show my work for those. For $0 < x_0 < 1$ however, we have the following:

$$\begin{aligned} x(t) &= x_0 + t(1 - x_0) \\ \frac{x}{x_0} &= 1 - t + \frac{t}{x_0} \\ \frac{x}{x_0} - \frac{t}{x_0} &= 1 - t \\ \frac{1}{x_0}(x - t) &= 1 - t \\ x_0 &= \frac{x - t}{1 - t} \end{aligned} \tag{20}$$

We then have:

$$x_0 = \begin{cases} x - t, & x_0 \leq 0 \\ \frac{x-t}{1-t}, & 0 < x_0 < 1 \\ x, & x_0 \geq 1 \end{cases} \tag{21}$$

We can then substitute these values into equation (16) to obtain the solution for $t < t_b$. Again, the substitutions are fairly straightforward, but I will show the work for $0 < x_0 < 1$.

$$\begin{aligned} 0 < \frac{x - t}{1 - t} &< 1 \\ 0 < x - t &< 1 - t \\ t < x &< 1 \end{aligned} \tag{22}$$

And finally substituting all this into (15) gives the solution for $0 \leq t < t_b$. We can see in this solution that the middle constraint solutions only exist in a range between $t = 0$, $x = 1$ and $t = x$, as described in part (c).

$$\boxed{u(x,t) = \begin{cases} 1, & x \leq t \\ 1 - \frac{x-t}{1-t}, & t < x < 1 \\ 0, & x \geq 1 \end{cases} \qquad (23)}$$

(e) We can see from the stated problem that $\phi_x = uu_x$. Integrating this gives

$$\phi = \frac{u^2}{2} \tag{24}$$

Considering equation (22) above, we can see that at the breaking time $t_b = 1$, the solution is defined in two regions as the second constraint $t < x < 1$ is not satisfied for any $t > t_b = 1$. This leaves the other two constraints:

$$\begin{aligned} u^- &= 1, \ x < t \\ u^+ &= 0, \ x \geq 1 \end{aligned} \tag{25}$$

With this information we can then use the Rankine-Hugoniot relation:

$$\begin{aligned}
\frac{dx_0}{dt} &= \frac{\frac{0^2}{2} - \frac{1^2}{2}}{0-1} \\
\frac{dx_0}{dt} &= \frac{1}{2} \\
x_s(t) &= \frac{1}{2}t + C \\
x_s(1) = 1 &= \frac{1}{2}(1) + C \\
x_s(t) &= \frac{1}{2}t + \frac{1}{2}
\end{aligned} \tag{26}$$

(f) Making use equations (22), (24), and (25) above:

$$u(x,t) = \begin{cases} \begin{cases} 1, & x \leq t \\ 1 - \frac{x-t}{1-t}, & t < x < 1 \\ 0, & x \geq 1 \end{cases} & 0 < t < 1 \\ \begin{cases} 1, & x < x_s(t) \\ 0, & x > x_s(t) \end{cases} & t \geq 1 \end{cases} \tag{27}$$

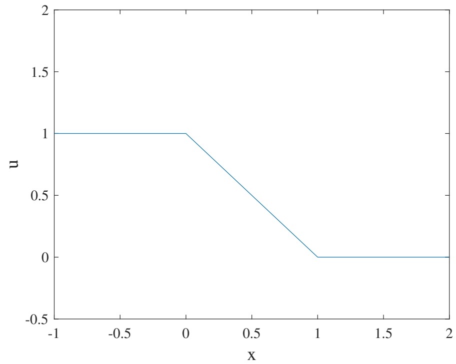

Figure 4: $t = 0$

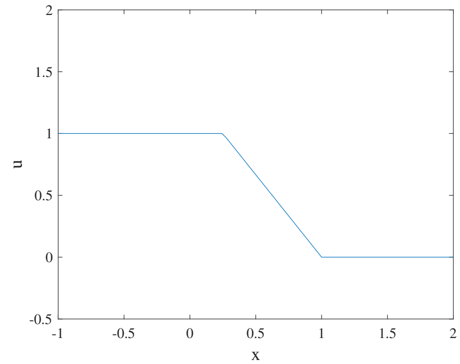

Figure 5: $t = 0.25$

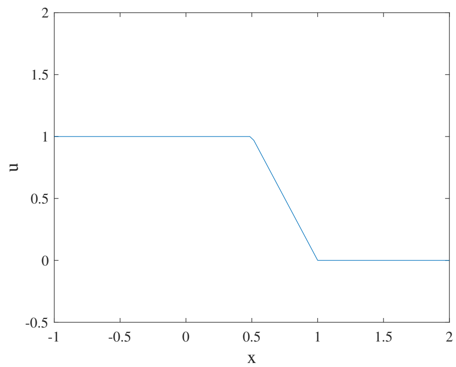

Figure 6: $t = 0.5$

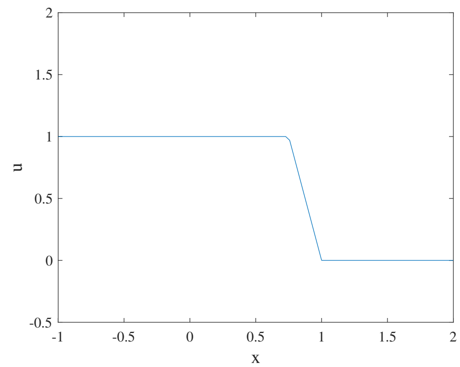

Figure 7: $t = .75$

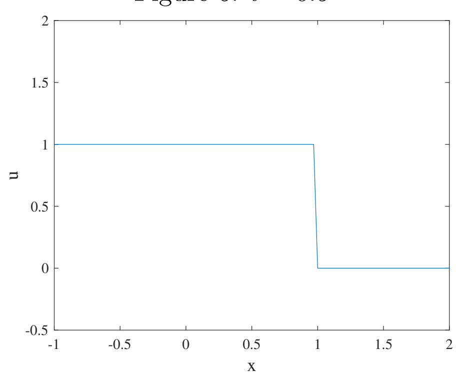

Figure 8: $t = 1$

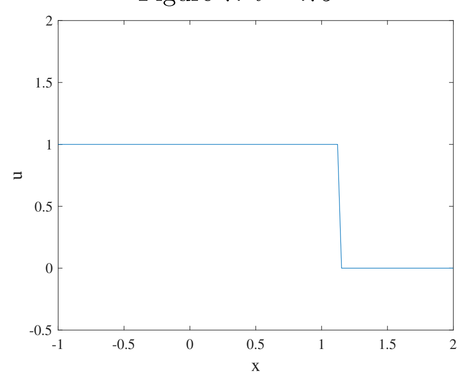

Figure 9: $t = 1.25$

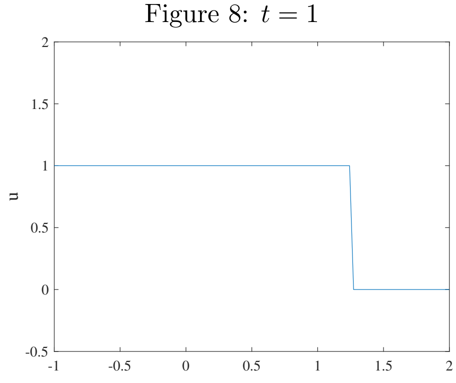

Figure 10: $t = 1.5$

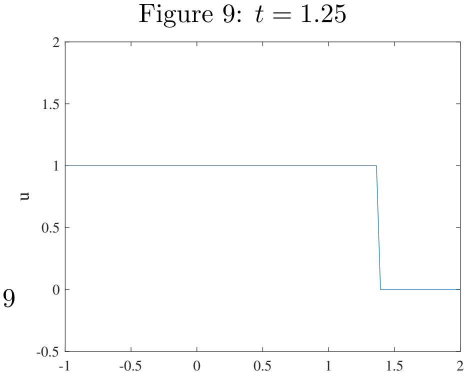

Figure 11: $t = 1.75$

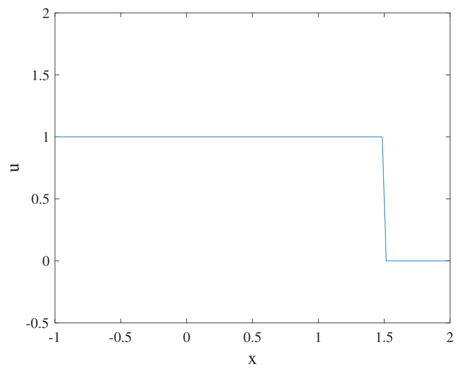

Figure 12: $t = 2$

(g)

Part 3

(a) For these values, there is no change in density of cars at any point, thus no shock/traffic jam forms. Traffic travels forward at a uniform speed of $v = v_1(1 - \frac{\frac{u_1}{3}}{u_1}) = \frac{2}{3}v_1$.

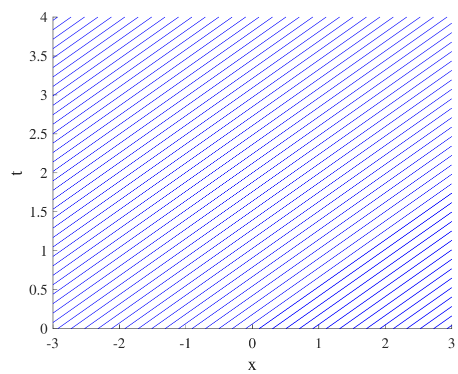

Figure 13

(b) For these values a traffic jam forms immediately at $t = 0$. If you zoom in on the graph, it's clear that information is traveling forward, which is also clear since the u^+ value is less than u_1. Overall this means traffic hits the traffic jam and then immediately slows to a much lower speed. Information is moving forward i.e. traffic is moving slowly forward, at the following rate:

$$\begin{aligned}
\phi &= v_1(u - \frac{u^2}{u_1}) \\
\phi^+ &= v_1(\frac{2u_1}{3} - \frac{(\frac{2u_1}{3})^2}{u_1} \\
\phi^+ &= v_1(\frac{2u_1}{3} - \frac{4u_1}{9}) = \frac{2}{9}v_1u_1 \\
\phi^- &= v_1(\frac{u_1}{6} - \frac{(\frac{u_1}{6})^2}{u_1} \\
\phi^- &= v_1(\frac{u_1}{6} - \frac{u_1}{36}) = \frac{5}{36}v_1u_1
\end{aligned} \tag{28}$$

Then using this to calculate the slope of the shock line we have:

$$\begin{aligned}
\frac{dx_s}{dt} &= \frac{\frac{2u_1v_1}{9} - \frac{5u_1v_1}{36}}{\frac{2u_1}{3} - \frac{u1}{6}} \\
\frac{dx_s}{dt} &= 2\frac{1}{u_1}(\frac{2u_1v_1}{9} - \frac{5u_1v_1}{36}) \\
\frac{dx_s}{dt} &= \frac{4}{9}v_1 - \frac{10}{36}v_1
\end{aligned} \tag{29}$$

$$\boxed{\frac{dx_s}{dt} = \frac{v_1}{6}} \tag{30}$$

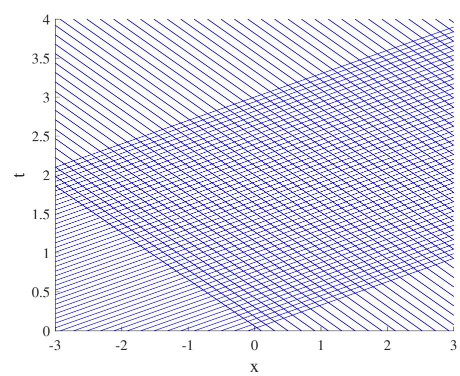

Figure 14

(c) For these values, traffic is moving more slowly on the left and more quickly on the right, and thus a region forms in which there are no characteristics, and thus no solution to u. Intuitively, this would seem to model an area in which there are no cars because two 'packs' of cars are getting further and further apart. As to why this would happen, I'm not sure. I suppose it could model an expanding work crew is diverting traffic to a side street.

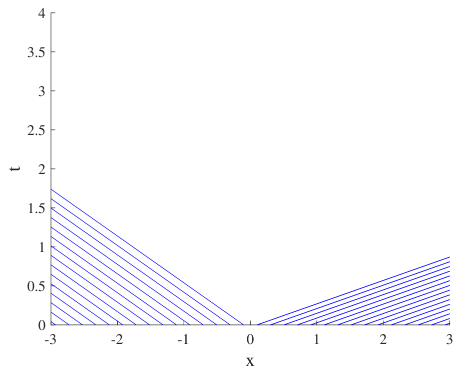

Figure 15

(d) The slope of the characteristics in the traffic problem are give by

$$\frac{1}{v_1(1-\frac{2u^{\pm}}{u_1})} \tag{31}$$

We can see that when the slope of the characteristics in the left region are greater than the ones in the right, they will cross (regardless of whether the slopes are positive or negative). We then have:

$$\frac{1}{v_1(1-\frac{2u^{-}}{u_1})} > \frac{1}{v_1(1-\frac{2u^{+}}{u_1})} \tag{32}$$

This then very quickly simplifies down to $u^+ \geq u^-$, which is to say if the initial density is higher in the right region than it is on the left region, a shock will form. This also correctly deals with the scenario in which the density on the left is higher than the right, in which the characteristics are going away from each other since in that situation, the characteristics in the left region have lower slope (opposite sign) than those in the right.