AMATH 353: Homework 12
Due May, 18 2018
ID: 1064712

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Part 1 Assuming $u(x,t) = u(x(t),t)$, by the chain rule we have $\frac{d}{dt}u = u_t + u_x \frac{dx}{dt}$. Given that we are solving $u_t + 2u_x = 0$, if we assume $\frac{dx}{dt} = 2$ we get the following:

$$\frac{d}{dt}(u(x(t),t)) = u_t + 2u_x = 0 \tag{1}$$

This gives us the two ODEs:

$$\boxed{\begin{aligned} \frac{dx}{dt} &= 2 \\ \frac{du}{dt} &= 0 \end{aligned}} \tag{2}$$

Solving the first ODE by separation of variables, we get the following equation for the characteristic curves, which are shown in the plot below:

$$x(t) = 2t + x_0 \tag{3}$$

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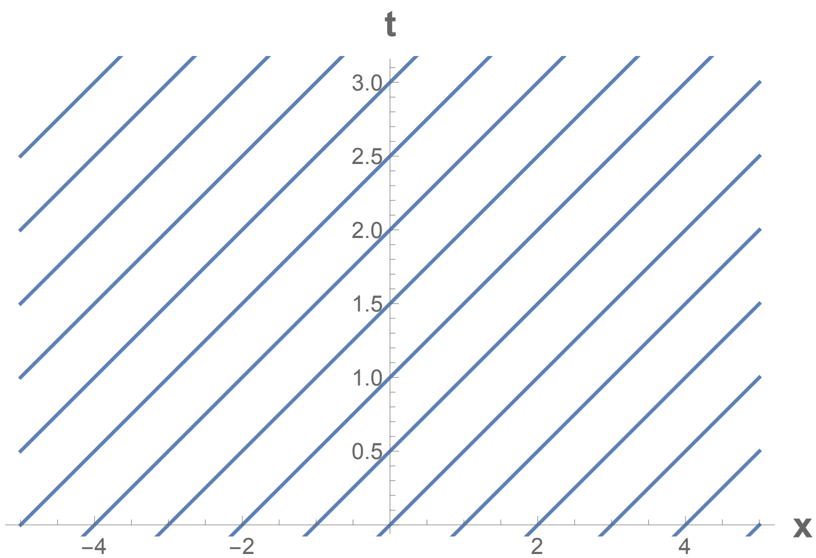

Figure 1: $t = \frac{x - x_0}{2}$

Solving the second ODE we simply get a constant:

$$\begin{aligned} \int \frac{du}{dt} &= \int 0 dt \\ u(x(t), t) &= A \end{aligned} \tag{4}$$

Making use of the initial condition, $u(x, 0) = e^{-x^2}$ we have the following:

$$\begin{aligned} u(x(t), 0) = u(x_0, 0) = u_0(x_0) &= A \\ u_0(x_0) &= e^{-x_0^2} \\ A &= e^{-x_0^2} \end{aligned} \tag{5}$$

$$\boxed{u(x(t), t) = e^{-x_0^2} \qquad (6)}$$

This means that along any given characteristic line $u(x, t) = u(x(t), t)$ we have u constant at a value determined by the initial value of that particular characteristic.

Now using an example of a point $(3, 4)$ we plug it into the characteristic curve and determine it's x_0 value, then determine the value of u_0 at that point, and thus u along that entire characteristic curve:

$$\begin{aligned} x_0 &= x - 2t \\ x_0 &= 3 - 4(4) = -5 \end{aligned} \tag{7}$$

$$\boxed{u_0(-5) = e^{-(-5)^2} = e^{-25}} \tag{8}$$

We can arrive at the same value generally by plugging the x_0 equation into the equation derived above for $u(x(t), t)$, then plugging $(3, 4)$ into that:

$$\begin{aligned} x_0 &= x - 2t \\ u(x(t), t) &= e^{-x_0^2} \\ u(x, t) &= e^{-(x-2t)^2} \end{aligned} \tag{9}$$

$$\boxed{e^{-(3-2(4)^2} = e^{-25}} \tag{10}$$

Note the results of (8) and (10) are the same.

Part 2

a.) The characteristic curves and ODEs for this question are arrived at in the same manner as equations (1) - (4) above, albeit restricted to $x \geq 0$:

$$\boxed{\begin{aligned} \frac{dx}{dt} &= 2 \\ x(t) &= 2t + x_0 \\ \frac{du}{dt} &= 0 \\ u(x(t), t) &= A \end{aligned}} \tag{11}$$

Note that $x_0 = x - 2t$, and so with the restriction that $x > 2t$, we can assume that x_0 will be positive, and thus will be defined by the initial value $u(x, 0) = 0$. We can then use this to solve the constant in $u(x(t), t)$:

$$\begin{aligned} u(x(t), t) &= A \\ u(x(t), 0) = u_0(x_0) &= A \\ u_0(x_0) &= 0 \\ A &= 0 \end{aligned} \tag{12}$$

From this it follows that when $x > 2t$:

$$\boxed{u(x, t) = 0} \tag{13}$$

b.) To solve for u when $x < 2t$, I took the same approach, but integrated $\frac{dx}{dt}$ differently in order to get t in terms of x and some t_0:

$$\begin{aligned} \frac{dx}{dt} &= 2 \\ dt &= \frac{2}{dx} \\ \int dt &= \int \frac{2}{dx} \\ t &= \frac{1}{2}x + t_0 \end{aligned} \tag{14}$$

From this equation we can see that $t_0 = t - \frac{1}{2}x$ and so when $x < 2t$, the value of t_0 will be greater than 0 and thus defined by the initial condition $u_0 = u(0,t) = \frac{t}{1+t^2}$. From this, we follow the same procedure as above to solve for u:

$$\begin{aligned} u(x(t),t) &= A \\ u(0,t) &= u_0(t_0) \\ u_0(t_0) = \frac{t_0}{1+t_0^2} &= A \end{aligned} \tag{15}$$

Then plugging in the value above for t_0 we arrive at the solution for the value of $u(x,t)$ when $x < 2t$:

$$\boxed{u(x,t) = \frac{t - \frac{1}{2}x}{1 + (t - \frac{1}{2}x)^2}} \tag{16}$$

Here are the profiles of the solution at $t = 0, 1, 2, 3$.

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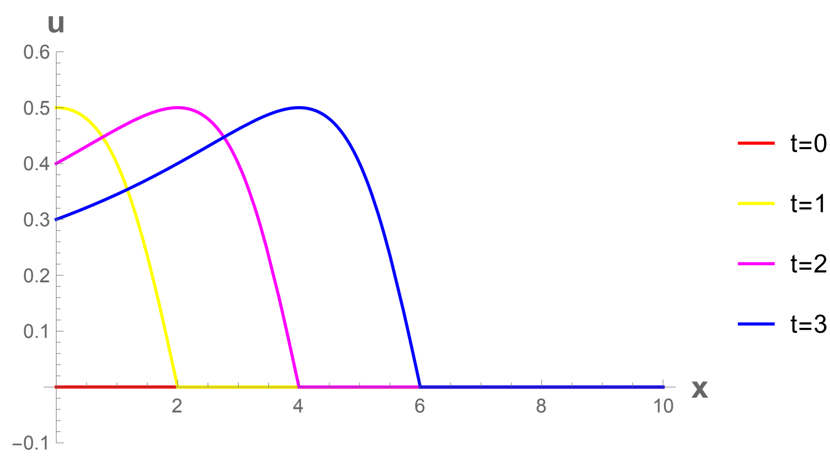

Figure 2: $t = 0, 1, 2, 3$

Part 3 This problem yields the same characteristic lines as in Part 1, namely:

$$\begin{aligned} \frac{dx}{dt} &= 2 \\ x &= 2t + x_0 \end{aligned} \tag{17}$$

However, the second ODE is now:

$$\frac{du}{dt} = -u(x(t), t) \tag{18}$$

This equation describes the rate of change of $u(x(t), t)$ along the characteristic lines, and is a function entirely of t, and it can be solved as follows:

$$\begin{aligned} \int \frac{du}{u} &= \int -dt \\ ln(u) &= -t + C \\ u(x(t), t) = e^{-t+C} = e^C e^{-t} &= Ae^{-t} \end{aligned} \tag{19}$$

We now use the same method as above to solve for the constant A:

$$\begin{aligned} u(x, 0) = u(x(0), 0) = u_0(x_0) &= e^{-x_0^2} \\ u(x(t), 0) &= Ae^0 \\ A &= e^{-x_0^2} \end{aligned} \tag{20}$$

So we see that along the characteristic lines we have some constant defined by the initial condition x_0, and that over the characteristic line this value is damped over time. While this could be simplified to a single exponent, I prefer to leave them separate so as to elucidate the damping term.

$$u(x(t), t) = e^{-t} e^{-x_0^2} \tag{21}$$

Finally, we can plug in the value for x_0 in terms of (x, t) and arrive at a general solution:

$$\boxed{u(x, t) = e^{-t} e^{-(x-2t)^2} \qquad (22)}$$

To compare this with Part 1, let us calculate (x, t):

$$u(3, 4) = e^{-4} e^{-(3-2(4))^2} == e^{-4} e^{-25} \tag{23}$$

$$\boxed{u(3, 4) = e^{-29} \qquad (24)}$$

As expected, effect of the $-u$ term has been to dampen the solution over time relative to the solution in Part 1.

The request that we draw the characteristic lines in the (t, x) plane seems a little odd to me, and perhaps it is a typo? The characteristic lines for this problem are identical to those in Part 1, and (Figure 1) above displays them. If, however, it is not a typo, we can re-orient the lines in Figure 1 by determining t in terms of x and t_0 as was done in equation (14) above, then plot them:

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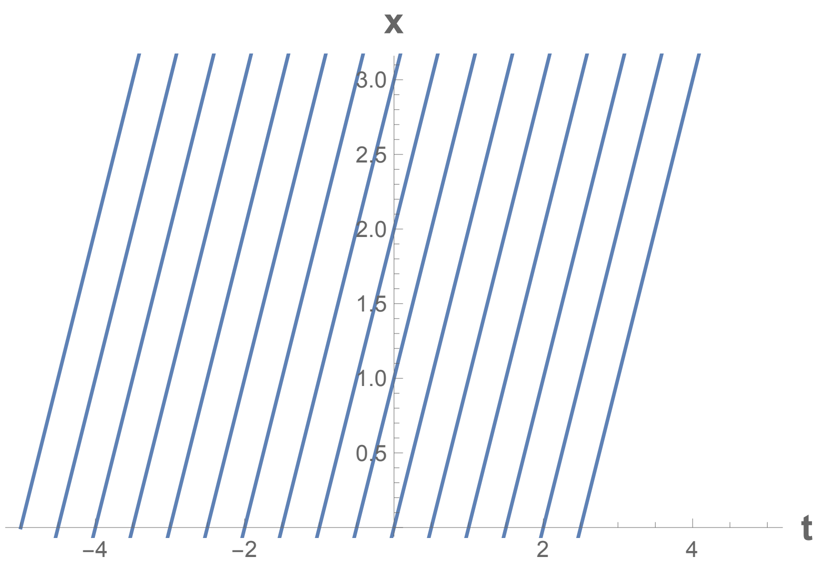

Figure 3: $t = 0, 1, 2, 3$