

# AMATH 353: Homework 9

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ID: 1064712

Trent YAROSEVICH

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Instructor: Jeremy Upsal

## Part 1

**a.)** We are asked to compute the integral  $\int_a^b x \cos(\frac{n\pi x}{2})$ . I will also show my work here for  $\int_a^b (2-x) \cos(\frac{n\pi x}{2})$  as this is how I computed the  $a_n$  term later on. Starting with the first one, using division by parts we have:

$$\begin{aligned}\int x \cos(\frac{n\pi x}{2}) &= \int u dv \\ u &= x, \quad du = dx \\ dv &= \cos(\frac{n\pi x}{2}) dx \\ v &= \int dv = \frac{2}{n\pi} \sin(\frac{n\pi x}{2}) \\ \int u dv &= uv - \int v du = \frac{2x}{n\pi} \sin(\frac{n\pi x}{2}) + \frac{4}{n^2\pi^2} \cos(\frac{n\pi x}{2})\end{aligned}\tag{1}$$

$$\begin{aligned}\int_a^b x \cos(\frac{n\pi x}{2}) &= \\ \frac{2b}{n\pi} \sin(\frac{n\pi b}{2}) + \frac{4}{n^2\pi^2} \cos(\frac{n\pi b}{2}) - \frac{2a}{n\pi} \sin(\frac{n\pi a}{2}) - \frac{4}{n^2\pi^2} \cos(\frac{n\pi a}{2})\end{aligned}\tag{2}$$

And now the same for  $\int_a^b (2-x) \cos(\frac{n\pi x}{2})$ . Note that I did the definite

integral directly here, unlike the previous equation.

$$\begin{aligned}
\int_a^b (2-x) \cos\left(\frac{n\pi x}{2}\right) dx &= \int_a^b u dv \\
u &= 2-x, \quad du = -dx \\
dv &= \cos\left(\frac{n\pi x}{2}\right) dx \\
v &= \int dv = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \\
\int v du &= \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)
\end{aligned} \tag{3}$$

$$\begin{aligned}
uv \Big|_a^b - \int_a^b v du &= \\
(2-b) \frac{2}{n\pi} \sin\left(\frac{n\pi b}{2}\right) - (2-a) \frac{2}{n\pi} \sin\left(\frac{n\pi a}{2}\right) - \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi b}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi a}{2}\right)
\end{aligned} \tag{4}$$

**b.)**

$$b_n = \frac{1}{2} \int_{-2}^2 f_e(x) \sin\left(\frac{n\pi x}{2}\right) dx \tag{5}$$

Because  $f_e(x)$  is defined as the even extension of  $f(x)$ , and because sine is an odd function, their product is odd. Any integral of an odd function across a symmetrical domain including the origin is equal to zero, and thus  $b_n = 0$ .

**c.)** To find  $a_0$  I use the following equation, which we derived in class and which results from the fact that the average value of sin and cos are both 0:

$$\frac{a_0}{2} = A = \frac{1}{4} \int_{-2}^2 f_e(x) dx \tag{6}$$

Because  $f_e(x)$  is defined as the even extension of  $f(x)$ , and because the resulting domain is defined equivalently in both  $f_e(x)$  and  $f(x)$ , we get the following:

$$\frac{a_0}{2} = A = \frac{1}{4} \int_{-2}^2 f_e(x) dx = 2\left(\frac{1}{4}\right) \int_0^2 f_e(x) dx = \frac{1}{2} \int_0^2 f(x) dx \tag{7}$$

This definite integral is then evaluated for the piece-wise function with an integration constant of 0.

$$\begin{aligned}
A &= \frac{1}{2} \left[ \left(\frac{1}{2}x^2\right) \Big|_0^1 + \left(2x - \frac{1}{2}x^2\right) \Big|_1^2 \right] \\
A &= \frac{1}{2}
\end{aligned} \tag{8}$$

$$a_0 = 2A = 1 \quad (9)$$

**d.)** To find  $a_n$  we use the following equation

$$a_n = \frac{1}{2} \int_{-2}^2 f_e(x) \cos\left(\frac{n\pi x}{2}\right) dx \quad (10)$$

taking note that the product of two even functions,  $f_e(x)$  and  $\cos$ , is also an even function, and that the resulting domain is equivalent in both  $f_e(x)$  and  $f(x)$ , we get the following:

$$a_n = \frac{1}{2} \int_{-2}^2 f_e(x) \cos\left(\frac{n\pi x}{2}\right) dx = 2 \frac{1}{2} \int_0^2 f_e(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \quad (11)$$

Because  $f(x)$  is a piecewise function (shown below), the integral in this domain is defined as follows:

$$f(x) = \begin{cases} x, & 0 < x \leq 1, \\ 2 - x, & 1 < x \leq 2 \end{cases}$$

$$\int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^1 x \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2 - x) \cos\left(\frac{n\pi x}{2}\right) dx \quad (12)$$

Then plugging this into the results of part a.), equations (2) and (4) above, we have the following:

$$\begin{aligned} a_n = & \left[ \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - 0 - \frac{4}{n^2\pi^2} \cos(0) \right] \\ & + \left[ 0 - (1) \left( \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \cos(n\pi) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) \right) \right] \quad (13) \end{aligned}$$

By carrying out the addition and subtraction of sine/cosine terms with the same arguments, and using the equivalence  $\cos(n\pi) = (-1)^n$  we get:

$$a_n = \frac{8 \cos\left(\frac{n\pi}{2}\right) - 4(-1)^n - 4}{n^2\pi^2} \quad (14)$$

**e.)** Now that I have my constants, I can plug them in to the following equation to plot. *Please note I have uploaded the wolfram notebook I used to make these plots.*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad (15)$$

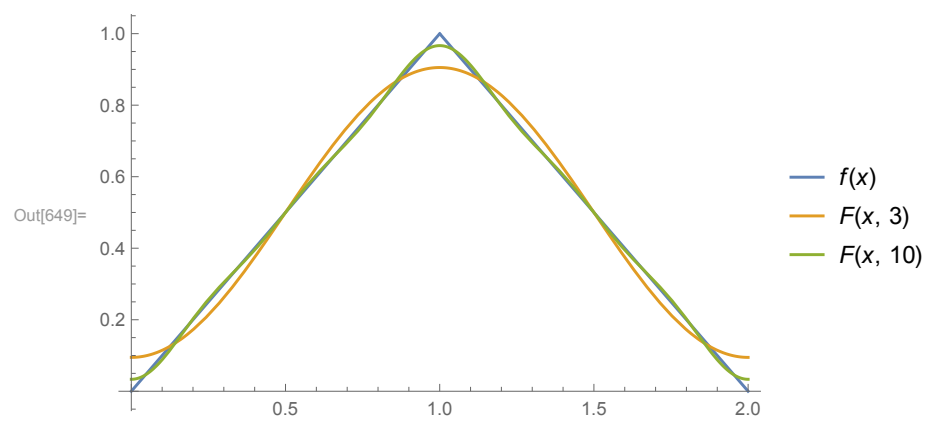


Figure 1:  $f(x)$  and  $n = 3, 10$