AMATH 353: Homework 15
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Part 1 First, let's find the solutions a long the characteristic lines using the method of characteristics.

$$\begin{aligned} \frac{du}{dt} &= 0 \\ u(x(t),t) &= A \\ u_0(x_0) &= A \\ u_0(x_0) &= \begin{cases} u_1, & x_0 \leq 0 \\ 0, & x_0 > 0 \end{cases} \end{aligned} \tag{1}$$

The characteristic lines are then given as follows:

$$\begin{aligned} c(u) &= v_1(1 - 2\frac{2u}{u_1}) \\ c(u_0(x_0)) &= \begin{cases} v_1(1 - 2\frac{2u_1}{u_1}) = -v_1, & x_0 \leq 0 \\ v_1(1 - 2\frac{0}{u_1}) = v_1, & x_0 > 0 \end{cases} \\ x(t) &= \begin{cases} x_0 - v_1 t, & x_0 \leq 0 \\ x_0 + v_1 t, & x_0 > 0 \end{cases} \end{aligned} \tag{2}$$

From this we get the following values for x_0:

$$x_0 = \begin{cases} x + v_1 t & x_0 \leq 0 \\ x - v_1 t, & x_0 > 0 \end{cases} \tag{3}$$

We can then plug these into the result of (1) above to get the solution to u along the characteristic lines only:

$$u(x,t) = \begin{cases} u_1, & x \leq -v_1 t \\ 0, & x > v_1 t \end{cases} \tag{4}$$

This leaves the rarefaction wave. We know that, taking from the notes and Knobel, the solution along the rarefaction wave will be of the form $u(x,t) = g(\frac{x}{t})$, with the rarefaction lines taking the form $x = ct$. Plugging this into the traffic problem PDE simplifies to

$$\frac{1}{t}g'(x/t)\Big(g(x/t) - \frac{x}{t}\Big) = 0 \tag{5}$$

We know that g is not constant (otherwise we'd still have a discontinuity), so we can examine $g(x/t) - \frac{x}{t} = 0$. This means that $g(x/t) = \frac{x}{t}$. Making use of this, and the fact that from our PDE we know $c = v_1(1 - \frac{2u}{u_1})$, we can solve for u in the following manner:

$$\begin{aligned}
c &= \frac{x}{t} \\
g(x/t) = g(c) = c &= v_1(1 - \frac{2u}{u_1}) \\
v_1(1 - \frac{2u}{u_1}) &= \frac{x}{t} \\
v_1 - \frac{2uv_1}{u_1} &= \frac{x}{t} \\
\frac{2uv_1}{u_1} &= -\frac{x}{t} + v_1 \\
2uv_1 &= u_1(v_1 - \frac{x}{t} \\
u &= \frac{u_1}{2}(1 - \frac{x}{v_1 t})
\end{aligned} \tag{6}$$

We can then insert this solution into the region that the characteristics in (2) leave undefined, and substitute the x_0 values into the piecewise solution to get the final answer:

$$\boxed{u(x,t) = \begin{cases} u_1, & x \leq -v_1 t \\ \frac{u_1}{2}(1 - \frac{x}{v_1 t}), & -v_1 t < x < v_1 t \\ 0, & x > v_1 t \end{cases} \qquad (7)}$$