

AMATH 353: Homework 8

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Part 1

a.) Assuming solutions of the form $u(x, t) = G(t)F(x)$ and the PDE $u_{tt} + u_{xx} = 0$ we get the following:

$$\begin{aligned} G''(t)F(x) &= -F''(x)G(t) \\ \frac{G''(t)}{G(t)} &= -\frac{F''(x)}{F(x)} = \lambda \end{aligned} \tag{1}$$

The two equations are equal to the constant λ because neither the RHS or LHS is in terms of its independent variable. This results in the two ODEs:

$$\begin{aligned} G''(t) &= \lambda G(t) \\ F''(x) &= -\lambda F(x) \end{aligned} \tag{2}$$

b.) Assuming solutions of the form $u(x, t) = G(t)F(x)$ and the PDE $u_t = \kappa u_{xx}$ we get the following:

$$\begin{aligned} G'(t)F(x) &= \kappa F''(x)G(t) \\ \frac{G'(t)}{G(t)} &= \kappa \frac{F''(x)}{F(x)} = \lambda \end{aligned} \tag{3}$$

The two equations are equal to the constant λ as above, yielding the two ODEs:

$$\begin{aligned} G'(t) &= \lambda G(t) \\ F''(x) &= \frac{\lambda}{\kappa} F(x) \end{aligned} \tag{4}$$

Part 2

a.) Starting with the equation $u_{tt} + u_{xx} = 0$ we have the following x dependent ODE and three cases for λ :

$$F''(x) = -\lambda F(x) \quad (5)$$

$$\lambda = 0$$

In this case we have $F''(x) = -0 \cdot F(x)$. Integrating twice gives $F(x) = A + Bx$. With the BC $u(0, t) = 0$ we get $F(0) = A + B(0) = 0$ and thus $A = 0$. With the other BC $u_x(L, t) = 0$ and $F'(L) = B$ we must conclude that $B = 0$ as well, yielding only trivial solutions, so $\lambda = 0$ is not an allowed value.

$$\lambda > 0$$

In this case we have $F''(x) = -\lambda F(x)$ where λ is a positive real number. Let $\lambda = r^2$, with $r \in \mathbb{R}_{>0}$. Then by characteristic polynomial we get following, per Euler's Formula:

$$\begin{aligned} F(x) &= C_1 e^{irx} + C_2 e^{-irx} \\ F(x) &= C_1 \cos(rx) + C_2 \sin(rx) \end{aligned} \quad (6)$$

We then apply the BC $u(0, t) = 0$:

$$\begin{aligned} F(0) &= C_1 \cos(0) + C_2 \sin(0) = 0 \\ C_1 &= 0 \\ F(x) &= C_2 \sin(rx) \end{aligned} \quad (7)$$

And then the second BC:

$$\begin{aligned} F'(x) &= C_2 r \cos(rx) \\ F'(L) &= C_2 r \cos(rL) = 0 \end{aligned} \quad (8)$$

Because $\cos(x) = 0$ at odd multiples of $\frac{\pi}{2}$ it follows that to satisfy the second BC without setting $C_2 = 0$ that we must have the argument of cosine, $rL = \frac{\pi(2n-1)}{2}$ where $n > 0$ is a positive integer. Thus with positive eigenvalues λ we have the eigenfunctions:

$$F_n(x) = C_n \sin\left(\frac{\pi(2n-1)x}{2L}\right), \quad n \in \mathbb{Z}^+ \quad (9)$$

$$\lambda < 0$$

In the same fashion as above, but now let $\lambda = -r^2$, $r \in \mathbb{R}_{>0}$. From characteristic polynomial we then get:

$$F(x) = C_1 e^{rx} + C_2 e^{-rx} \quad (10)$$

Applying the first BC we get:

$$\begin{aligned}
F(0) &= C_1 e^0 + C_2 e^{-0} = 0 \\
C_1 + C_2 &= 0 \\
C_2 &= -C_1 \\
F(x) &= C(e^{rx} - e^{-rx})
\end{aligned} \tag{11}$$

Then applying this to the second BC:

$$\begin{aligned}
F'(x) &= C(re^{rx} + re^{-rx}) \\
F'(L) &= C(re^{rL} + re^{-rL}) = 0 \\
e^{rL} &= -e^{-rL}
\end{aligned} \tag{12}$$

Since there is no possibility of $e^{rL} = -e^{-rL}$, this means the only way to satisfy the BCs with $\lambda < 0$ is to have $C = 0$, yielding a trivial solution. Thus $\lambda < 0$ is not an allowed value.

b.) We now consider the same possible values for λ for the second equation from above, and the ODE for x :

$$\begin{aligned}
u_t &= \kappa u_{xx} \\
F''(x) &= \frac{\lambda}{\kappa} F(x)
\end{aligned} \tag{13}$$

$$\lambda = 0$$

Similarly to the previous equation, setting $\lambda = 0$ and integrating twice gives us

$$F(x) = A + Bx \tag{14}$$

which gives only trivial solutions in a manner identical to above, Part 2, subsection a.).

$$\lambda > 0$$

As above we'll let $\lambda = r^2$ with $r \in \mathbb{Z}^+$ giving:

$$\begin{aligned}
F''(x) &= \frac{r^2}{\kappa} F(x) \\
F''(x) &= \left(\frac{r}{\sqrt{\kappa}}\right)^2 F(x)
\end{aligned} \tag{15}$$

As above, by ansatz this results in

$$F(x) = C_1 e^{\frac{r}{\sqrt{\kappa}}x} + C_2 e^{-\frac{r}{\sqrt{\kappa}}x} \tag{16}$$

This lambda yields trivial solutions for the same reasons as $\lambda < 0$ in the previous equation, as shown in equations (11) and (12). The only difference are the coefficients, which don't affect the validity of this particular lambda.

$$\lambda < 0$$

As in the case above for the previous equation, equations (6), (7), and (8) we have an oscillatory solution when $\lambda < 0$.

$$F(x) = C_1 \cos\left(\frac{r}{\sqrt{\kappa}}x\right) + C_2 \sin\left(\frac{r}{\sqrt{\kappa}}x\right) \quad (17)$$

As before, the cosine is dropped to satisfy the first BC. The second is very slightly different since the $\sqrt{\kappa}$ cancels:

$$\begin{aligned} F'(x) &= C \frac{r}{\sqrt{\kappa}} \cos\left(\frac{r}{\sqrt{\kappa}}x\right) \\ F'(L) &= C \frac{r}{\sqrt{\kappa}} \cos\left(\frac{r}{\sqrt{\kappa}}L\right) = 0 \\ \frac{r}{\sqrt{\kappa}}L &= \frac{\pi(2n-1)}{2} \\ r &= \frac{\pi\sqrt{\kappa}(2n-1)}{2L} \end{aligned} \quad (18)$$

When we plug this r value back into $F(x)$, the κ value cancels giving:

$$F(x)_n = C_n \sin\left(\frac{\pi(2n-1)}{2L}x\right) \quad (19)$$

That the $\sqrt{\kappa}$ should cancel makes sense, since it is a diffusion term affecting the time velocity, and it does not cancel if we solve the t equation, instead appearing as a damping term (if $\kappa > 0$ which it is).

So to summarize the only allowable value for lambda is $\lambda < 0$, yielding the eigenfunctions in (19).