FA18-BSM-044

Assignment 01

Question # 1-1

a) $a_{ii} = a_{11} + a_{22} + a_{33} = 1+4+1 = 6$ (Scalar)

$a_{ij} a_{ij} = 1+1+1+0+16+4+0+1+1 = 25$ (Scalar)

$a_{ij} a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix}$ (M)

$a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ (V)

$a_{ij} b_i b_j = 1+0+2+0+0+0+0+0+4 = 7$ (S)

$b_i b_j = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ (M)

$b_i b_i = 1+0+4 = 5$ (S)

(b) $a_{ii} = 1+2+2 = 5$ (S)

$a_{ij} a_{ij} = 1+4+0+0+4+1+0+16+4 = 30$ (S)

$a_{ij} a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix}$ (M)

$a_{ij} b_j = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ (V)

$b_i b_j = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (M)

$b_i b_i = 4+1+1 = 6$ (S)

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c) $a_{ii} = 1+0+4 = 5 \quad (S)$

$a_{ij}a_{ij} = 1+1+1+1+0+4+0+1+16 = 25 \quad (S)$

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix} \quad M$$

$$a_{ij}b_j = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad (V)$$

$a_{ij}b_ib_j = 1+1+0+1+0+0+0+0 = 3 \quad (S)$

$$b_ib_j = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (M)$$

$b_ib_i = 1+1+0 = 2 \quad (S)$

Q 1-2

(a) $a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ & $a_{[ij]}$ Satisfies appropriate conditions.

(b) $$\frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

$a_{(ij)}$ & $a_{[ij]}$ satisfies conditions

(c) $$\frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$a_{(ij)}$ & $a_{[ij]}$ satisfies :

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Q 1-3

$$a_{ij} b_{ij} = -a_{ji} b_{ji} = -a_{ij} b_{ij} \Rightarrow 2a_{ij} b_{ij} = 0 \Rightarrow a_{ij} b_{ij} = 0$$

From ex 1-2(a): $a_{(ij)} a_{[ij]} = \frac{1}{4} tr \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T \right) = 0$

from 1-2(b): $a_{(ij)} a_{[ij]} = \frac{1}{4} tr \left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T \right) = 0$

from 1-2(c): $a_{(ij)} a_{[ij]} = \frac{1}{4} tr \left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T \right) = 0$

Q 1-4

$$\delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 = \begin{bmatrix} \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij} a_{jk} = \begin{bmatrix} \delta_{11} a_{11} + \delta_{12} a_{21} + \delta_{13} a_{31} & \delta_{11} a_{12} + \delta_{12} a_{22} + \delta_{13} a_{32} & \delta_{11} a_{13} + \delta_{12} a_{23} + \delta_{13} a_{33} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

Q 1-5

$$\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{231} a_{12} a_{23} a_{31} + \varepsilon_{312} a_{13} a_{21} a_{32}$$
$$+ \varepsilon_{321} a_{13} a_{22} a_{31} + \varepsilon_{132} a_{11} a_{23} a_{32} + \varepsilon_{213} a_{12} a_{21} a_{33}$$
$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$
$$- a_{12} a_{21} a_{33} - a_{11} a_{23} a_{32}$$
$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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Q1-6

45° rotation about x_1-axis $\Rightarrow$ $Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

from 1-1 (a): $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$a_{ij}' = Q_{ip} Q_{jp} Q_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$

from ex 1-1 (b): $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$

$a_{ij}' = Q_{ip} Q_{jp} Q_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -.5 \end{bmatrix}$

from 1-1 (c): $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$

$a_{ij}' = Q_{ip} Q_{ip} Q_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$

$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$

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Q1-7

$$Q_{ij} = \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90^\circ - \theta) \\ \cos(90^\circ + \theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b'_i = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \cos\theta + b_2 \sin\theta \\ -b_1 \sin\theta + b_2 \cos\theta \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \begin{bmatrix} a_{11}\cos^2\theta + (a_{12} + a_{21})\sin\theta\cos\theta & a_{12}\cos^2\theta - (a_{11} - a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \\ a_{21}\cos^2\theta + (a_{11} - a_{22})\sin\theta\cos\theta & a_{11}\sin^2\theta - (a_{12} - a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

Q1-8

$$a' \delta'_{ij} = Q_{ip} Q_{jp} Q_{pq} = a\, Q_{ip} Q_{jp} = a\, \delta_{ij}$$

Q1-9

$$\alpha' \delta'_{ij} \delta'_{kl} + \beta' \delta'_{ik} \delta'_{jl} + \gamma' \delta'_{il} \delta'_{jk} = Q_{im} Q_{jn} Q_{kp} Q_{lq} (\alpha \delta_{mn} \delta_{pq} + \beta \delta_{mp} \delta_{nq} + \gamma \delta_{mq} \delta_{np})$$

$$= \alpha Q_{im} Q_{jm} Q_{kp} Q_{lp} + \beta Q_{im} Q_{jm} Q_{km} Q_{in} + \gamma Q_{im} Q_{jn} Q_{km} Q_{in}$$

$$= \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

Q1-10

$$C_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta_{ik} \delta_{jl} + \gamma \delta_{ij} \delta_{jk} = \alpha \delta_{ij} \delta_{kl} + \beta(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$= \alpha \delta_{kl} \delta_{ij} + \beta(\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li}) = C_{klij}$$

Transform Strain Displacement relation from cartesian to cylindrical & spherical coordinates.

(i) Cylindrical Coordinates.

$$u_x = u_r \cos\theta - u_\theta \sin\theta$$

$$u_y = u_r \sin\theta + u_\theta \cos\theta$$

$$u_z = u_z$$

as the derivatives of $x = r\cos\theta$; $y = r\sin\theta$; $z = z$ where $r = \sqrt{x^2+y^2}$; $\theta = \arctan(y/x)$ is given as

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\cdot\frac{\partial}{\partial \theta} = \sin\theta \frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial \theta}$$

It follows

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \left(\cos\theta \sin\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)\right)$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} - \cos\theta\sin\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \left[\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial \theta} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial \theta}\right] + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \sin^2\theta\left[\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] + 2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial \theta}\right)$$

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likewise :-

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \frac{\partial^2}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) - 2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta}\right)$$

we can determine :

$$e_{xx} = \frac{\partial u_x}{\partial x} = \cos\theta \frac{\partial}{\partial r}\left(u_r\cos\theta - u_\theta \sin\theta\right) - \frac{\sin\theta}{r}\left(\frac{\partial}{\partial \theta}\left(u_r\cos\theta - u_\theta\sin\theta\right)\right)$$

$$= \frac{\partial u_r}{\partial r}\cos^2\theta - \frac{\partial u_\theta}{\partial r}\sin\theta\cos\theta - \frac{\partial u_r}{\partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{u_r}{r}\sin^2\theta + \frac{\partial u_\theta}{\partial \theta}\frac{\sin^2\theta}{r}$$

$$+ \frac{u_\theta}{r}\sin\theta\cos\theta$$

$$= \frac{\partial u_r}{\partial r}\cos^2\theta + \left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\sin\theta\cos\theta + \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)\sin^2\theta$$

$$e_{yy} = \frac{\partial u_y}{\partial y} = \sin\theta\frac{\partial}{\partial r}\left(u_r\sin\theta + u_\theta\cos\theta\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(u_r\sin\theta + u_\theta\cos\theta\right)$$

$$e_{xy} = 2\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\right)$$

$$\therefore\ e_{rr} = \frac{\partial u_r}{\partial r}\ ;\ e_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\ ;\ e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

& $e_{zz} = \dfrac{\partial u_z}{\partial z}$.

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Spherical Coordinate :-

$x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$, $z = \rho \cos\phi$

Suppose $R = \rho$

$U_R = R\cos\theta \sin\phi$, $U_\theta = R\sin\theta \sin\phi$, $U_\phi = R\cos\phi$

$$e_R = \frac{\partial U_R}{\partial R} = \cos\theta \sin\theta$$

$$e_\phi = \frac{1}{R}\left(U_R + \frac{\partial U_\phi}{\partial \phi}\right) = \frac{1}{R}\left[R\cos\theta \sin\phi + (-R\sin\phi)\right]$$

$$= \sin\phi(\cos\theta - 1)$$

$$e\theta = \frac{1}{R\sin\phi}\left(\frac{\partial U_\theta}{\partial \theta} + \sin\phi . U_R + \cos\phi \, U_\theta\right)$$

$$= \frac{1}{R\sin\phi}\left[R\cos\theta \sin\phi + \sin\phi R\cos\theta \sin\phi + \cos\phi R\cos\phi\right]$$

$$= \frac{1}{\sin\phi}\left(\cos\theta \sin\phi + \sin^2\phi \cos\theta + \cos^2\phi\right)$$

$$= \cos\theta + \cos\theta \sin\theta + \frac{\cos^2\phi}{\sin\phi}$$

$$= \cos\theta + \cos\theta \sin\phi + \cot\phi \cos\phi .$$

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