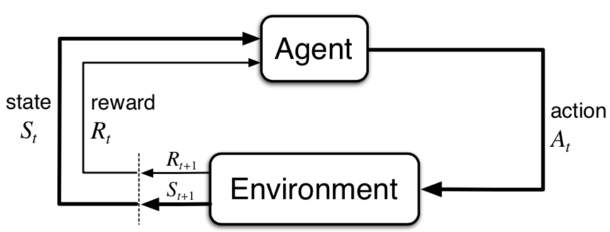
Reinforcement Learning (RL)

Symbols

a	action	Ω	observations
A	advantage	Pr	transition distribution
α	learning rate, step size	π	policy (*state* $\rightarrow$ *action*)
b	bandit, baseline estimate	q	quality
c	cost, context (= state)	r	reward
d	difference error	R	return
Δ	difference	s	state
D	replay buffer	t	time
E	eligibility trace	T	conditional transition probabilities between states
ϵ	exploration rate ($0 \sim 1$)	τ	trajectory
Φ	state transition function	θ	parameter weight
g	gain	v	value
γ	discount factor	w	weight
∇	gradient	y	target / prediction
H	horizon	$_*$	optimal
J	expected return	$'$	next / derivative
L	loss / regret	$\hat{}$	estimation
λ	trace decay ($0 \sim 1$)	$\|\|\|\|$	vector norm
O	conditional observation probabilities		

Agent-Environment Interface



General model

1. the *agent* at each time step t receives a representation of the environment's *state* $S_t \in S$
2. it selects an action $A_t \in A(s)$
3. from its action the agent receives a *reward* $R_{t+1} \in R \in \mathbb{R}$

General challenges

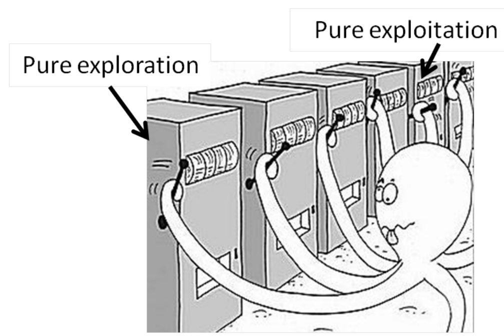
- choosing the right *Representation* of the problem / state
- *Generalization* beyond the cases trained on
- *Temporal Credit Assignment*: what caused this outcome?
- balance *Exploitation* vs *Exploration* (short vs long term)

Concepts (no state)

$$\begin{aligned}
\underbrace{a}_{\text{state}} &\in \underbrace{A}_{\text{finite set of actions}} \\
\underbrace{R}_{\text{return}} &= \underbrace{\sum_{t=0}^{\overbrace{H}^{\text{horizon } (\infty?)}} r_t}_{\text{total } reward \text{ over time}} \\
\underbrace{G_t}_{\text{gain}} &= \underbrace{\sum_{k=0}^{H} \overbrace{\gamma^k}^{\text{discount factor}} r_{t+k+1}}_{\text{total discounted reward over time}} \\
\underbrace{L}_{\text{regret}} &= \underbrace{\overbrace{G^*}^{\text{optimal gain}} - \overbrace{G_\pi}^{\text{gain under policy}}}_{\text{gain missed under policy}}
\end{aligned} \tag{1}$$

Bandit algorithms

- *multi-armed bandit problem*: which slot arm gives most?
- no notion of state!
- this leaves just *exploration vs exploitation*



Bandits = heuristics. Image from research.microsoft.com

$$\begin{aligned}
\underbrace{n_a}_{\text{action's use count}} &= \underbrace{\sum_{t=0}^{T} \begin{cases} 1 & a_t = a \\ 0 & \text{otherwise} \end{cases}}_{\text{tries for action so far}} \\
\underbrace{\hat{r}_a}_{\text{action's expected reward}} &= \underbrace{\sum_{t}^{a} \frac{r_t}{n_a}}_{\text{average action reward so far}}
\end{aligned} \tag{2}$$

Greedy

- 100% exploitation
- wants to pick best arm but won't know enough

– if 1st arm ok won't try others (stuck in local optimum)

$$\begin{aligned}
\underbrace{a_t}_{\text{action at time } t} &= \underbrace{\operatorname*{argmax}_{a \in A} \hat{r}_a}_{\text{action with max expected reward}} \\
\underbrace{L_T}_{\text{total regret}} &\propto \underbrace{O(n)}_{\text{order of magnitude } linear \text{ to the number of actions}}
\end{aligned} \tag{3}$$

Optimistic-Greedy

- like greedy but large initial $\hat{r}_a$

\+ tries all

– no 2nd chance for arms unlucky on first try

ϵ-Greedy

- at probability ϵ pick randomly

\+ not fooled by unlucky first try

– linear regret from constant exploration

Upper Confidence Bound (UCB)

try all for k rounds, then

$$a_t = \operatorname*{argmax}_{a \in A}[\underbrace{\hat{r}_a}_{\text{reward}} + \underbrace{\sqrt{\frac{2 \log t}{n_a}}}_{\text{under-explored bonus}}] \tag{4}$$

- explore then gradually exploit

\+ logarithmic regret

Contextual Bandit

Linear UCB (LinUCB)

$$\begin{aligned}
x_{t,a} &= \Phi(s_{t,a}) \\
\mathbb{E}[r_{t,a}|x_{t,a}] &= \theta_a \cdot x_{t,a} \\
\hat{\theta}_a &= A^{-1} b
\end{aligned} \tag{5}$$

Posterior / Thompson sampling

Pick actions by probability they maximize expected reward

Greedy in the Limit with Infinite Exploration (GLIE)

- infinitely explores
- converges on greedy

Markov Decision Process (MDP)

a 5-tuple (S, A, P, R, γ)

partially observable Markov decision process (POMDP)

a 7-tuple $(S, A, T, R, \Omega, O, \gamma)$

$$
\begin{aligned}
\underbrace{s}_{\text{state}} &\in \underbrace{S}_{\text{finite set of states}} \\
\underbrace{\pi_t(a|s)}_{\text{policy}} &: \text{a mapping from a state to an action} \\
p(s'|s,a) &= \underbrace{Pr\{S_{t+1} = s'|S_t = s, A_t = a\}}_{\text{state transition probabilities}} \\
r(s',s,a) &= \underbrace{\mathbb{E}[R_{t+1}|S_{t+1} = s', S_t = s, A_t = a]}_{\text{expected return for state-action-nexstate}} \\
\underbrace{v_\pi(s)}_{\text{value function}} &= \underbrace{\mathbb{E}_\pi[G_t|S_t = s]}_{\text{expected gain for policy } \pi \text{ in state } s} \\
&= \underbrace{\sum_a \pi(a|s) \sum_{s'} \sum_r p(s',r|s,a)[r + \gamma v_\pi(s')]}_{\text{recursive definition (Bellman equation)}} \\
\underbrace{q_\pi(s,a)}_{\text{action-value (Q) function}} &= \underbrace{\mathbb{E}_\pi[G_t|S_t = s, A_t = a]}_{\text{expected reward for policy } \pi \text{ for a state-action pair}} \\
&= \underbrace{\sum_{s',r} p(s',r|s,a)[r + \gamma V_\pi(s')]}_{\text{recursive definition (Bellman equation)}} \\
\underbrace{q_*(s,a)}_{\text{optimal Q function}} &= \max_\pi q_\pi(s,a) \\
\underbrace{v_*(s)}_{\text{state value for optimal policy}} &= \max_\pi v_\pi(s) \\
&= \underbrace{\max_{a \in A(s)} q_{\pi_*}(s,a)}_{\text{expected return from its best action}}
\end{aligned} \tag{6}
$$

Contraction Mapping

For metric space (X, d) and $f : X \to X$, f is a *contraction* given a *Lipschitz coefficient* $k \in [0, 1)$ where for all x / y in X:

$$d(f(x), f(y)) \leq kd(x,y) \tag{7}$$

Contraction Mapping theorem

For complete metric space (X, d) and contraction $f : X \to X$, there is only 1 fixed point x^* where $f(x^*) = x^*$.
For point x in X, and $f^n(x)$ inductively defined by $f^n(x) = f(f^{n1}(x))$, $f^n(x) \to x^*$ as $n \to \infty$, yielding a unique optimal solution for DP.

Model-based Methods (known MDP)

Unlike bandit algorithms, exploitation here thinks beyond the first next step.

Exhaustive Search

brute force, usually computationally unviable.

Dynamic Programming (DP)

+ bootstrap (learn mid-episode)

Find π_* for V / Q:

Policy Iteration

Initialize $V(s) \in \mathbb{R}$e.g. 0, $\Delta \leftarrow 0$, $\pi(s) \in A$ for all $s \in S$
1. Policy Evaluation
while $\Delta < \theta$ *(e.g. 0.001)* **do**
  foreach $s \in S$ **do**
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
  end
end
2. Policy Improvement
policy-stable $\leftarrow$ *true*
while *not policy-stable* **do**
  foreach $s \in S$ **do**
    old-action $\leftarrow \pi(s)$
    $\pi(s) \leftarrow \operatorname{argmax}_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$
    policy-stable $\leftarrow$ *old-action* $\neq \pi(s)$
  end
end

Policy iteration methods:

- gradient-based (policy gradient methods): gradient ascent
- gradient-free: simulated annealing, cross-entropy search or methods of evolutionary computation
- value search/iteration: stop after 1 state sweep
- async DP: update iteratively, no full sweeps
- generated policy iteration (GPI)

Value Iteration

ditch $V(s)$ convergence for policy improvement and truncated policy eval step in 1 operation:

Initialize $V(s) \in \mathbb{R}$e.g. 0, $\Delta \leftarrow 0$
while $\Delta < \theta$ *(e.g. 0.001)* **do**
  foreach $s \in S$ **do**
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
  end
end
output: deterministic policy $\pi \approx \pi_*$ where
$\pi(s) = \operatorname{argmax}_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$

Model-free methods

Monte Carlo (MC) Methods

- uses **averaging sample returns** per state-action pair
- episodic

+ sampling

Initialize for all $s \in S, a \in A(s)$:
  $Q(s,a) \leftarrow$ arbitrary
  $\pi(s) \leftarrow$ arbitrary
  $Returns(s,a) \leftarrow$ empty list
while *forever* **do**
  Pick $S_0 \in S$ and $A_0 \in A(S_0)$, all $p(s,a) > 0$
  Generate an episode starting at S_0, A_0 following π
  foreach *pair* s, a *in the episode* **do**
    $G \leftarrow$ return for first occurrence of s, a
    Append G to $Returns(s,a)$)
    $Q(s,a) \leftarrow average(Returns(s,a))$
  end
  foreach s *in the episode* **do**
    $\pi(s) \leftarrow \operatorname{argmax}_a Q(s,a)$
  end
end

estimate for non-stationary problems:

$$V(S_t) += \alpha[G_t - V(S_t)] \tag{8}$$

for learning rate α, how much we want to forget about past experiences.

Temporal Difference (TD)

+ DP's bootstrap
+ MC's sampling

- substitutes expected discounted reward G_t from the episode with an estimation:

$$V(S_t) += \alpha[R_{t+1} + \gamma V(S_{t+1} - V(S_t)] \tag{9}$$

State-action-reward-state-action (SARSA)

- on-policy TD control
- can use priors

Initialize $Q(s,a)$ arbitrarily and
 $Q(terminal - state,) = 0$
foreach *episode* $\in$ *episodes* **do**
  Pick a from s by policy from Q (e.g. ϵ-greedy)
  while s *is not terminal* **do**
    Take action a, observe r, s'
    Pick a' from s' by policy from Q (e.g., ϵ-greedy)
    $y \leftarrow r + \gamma Q(s',a')$
    $Q(s,a) += \alpha[y - Q(s,a)]$
    $s \leftarrow s'$
    $a \leftarrow a'$
  end
end

n-step Sarsa (n-step TD)

for n-step Q-Return:

$$
\begin{aligned}
q_t^{(n)} &= \gamma^n Q(S_{t+n}) + \textstyle\sum_{i=1}^n \gamma^{i-1} R_{t+i} \\
Q(s_t, a_t) &+= \alpha \left[q_t^{(n)} - Q(s_t, a_t) \right]
\end{aligned} \tag{10}
$$

Forward View Sarsa(λ) (/ TD(λ))

$$
\begin{aligned}
q_t^\lambda &= (1 - \lambda) \textstyle\sum_{n=1}^\infty \lambda^{n-1} q_t^{(n)} \\
Q(s_t, a_t) &+= \alpha \left[q_t^\lambda - Q(s_t, a_t) \right]
\end{aligned} \tag{11}
$$

Backward View Sarsa(λ) (/ TD(λ))

+ more efficient
+ can update at every time-step
- eligibility traces : find cause in frequency vs recency

$$\begin{aligned} E_0(s,a) &= 0 \\ E_t(s,a) &= \gamma\lambda E_{t-1}(s,a) + \mathbf{1}(S_t = t, A_t = a) \\ \delta_t &= R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t) \\ Q(s,a) &+= \alpha\delta_t E_t(s,a) \end{aligned} \tag{12}$$

Linear Function Approximation

+ efficient
+ generalize

update temporal difference error, minimize squared loss:

$$\begin{aligned} \delta &\leftarrow r_t + \gamma\theta^T\Phi(s_{t+1}) - \theta^T\Phi(s_t) \\ \theta &+= \alpha\delta\Phi(s_t) \\ J(\theta) &= ||\delta||^2 \end{aligned} \tag{13}$$

Q Learning

$$\begin{aligned} \delta &\leftarrow r_t + \gamma \operatorname*{argmax}_{a\in A} Q(\Phi(s_{t+1},a);\theta_i^-) - Q(\Phi(s_t,a);\theta_i) \\ Q(s,a) &+= \alpha[r + \gamma \max_{a'} Q(s',a') - Q(s,a)] \end{aligned} \tag{14}$$

Replay Memory

update θ by SGD

$$\nabla_{\theta_i} J(\theta_i) = \mathbb{E}_{(s_t,a_t,r_t,s_{t+1})\sim D}\delta\nabla_{\theta_i} Q(\Phi(s_t,a_t);\theta) \tag{15}$$

Deep Q Learning (DQL)

Made by *DeepMind*, uses a deep neural net (*Q-network*) for the Q function. Keeps N observations in a *memory* to train on.

$$\begin{aligned} y &= r_t + \gamma \max_{a\in A} Q(\Phi(s_{t+1},a);\theta_i^-) \\ \underbrace{\nabla_i J(\theta_i)}_{\text{loss function}} &= \mathbb{E}_{(s,a,r,s')\sim \underbrace{U(D)}_{\text{memory}}}[(\underbrace{y}_{\text{target}} - \underbrace{Q(s,a;\theta_i)}_{\text{prediction}})^2] \end{aligned} \tag{16}$$

for network weights θ and experience replay history $U(D)$.

```
Initialize replay memory D with capacity N
Initialize Q(s,a) arbitrarily
foreach episode ∈ episodes do
    Pick a from s by policy from Q (e.g. ε-greedy)
    while s is not terminal do
        Take action a, observer r, s'
        Store transition (s, a, r, s') in D
        Sample random transitions from D
        y_i ← { r_j                          for terminal s'_j
              { r_j + γ max_a Q(s', a'; θ)   otherwise
        Perform gradient descent step on
          (y_j − Q(s_j, a_j; Θ))^2
        s ← s'
    end
end
```

Prioritized Replay Memory / Prioritized Experience Replay

learn esp. from high loss (traumas)

$$p(s_t,a_t,r_t,s_{t+1}) \propto r_t + \gamma \max_{a\in A} Q(\Phi(s_{t+1},a);\theta_i^-) \tag{17}$$

Double DQN

solve bias from reused θ, faster

$$y = r_t + \gamma Q(\Phi(s_{t+1}, \operatorname*{argmax}_{a\in A} Q(\Phi(s_{t+1},a);\theta_i));\theta_i^-) \tag{18}$$

Direct Policy Search

learn π.

Policy Gradient

+ simpler than Q or V
+ allows stochastic policy (rock-paper-scissors)
− local optimum
− less sample efficient

$$L = \mu(logp(a|s)\cdot R(s)) \tag{19}$$

Likelihood Ratio

$$\begin{aligned} & \text{return state-action trajectory:} \\ R(\tau) &= \textstyle\sum_{t=0}^T R(s_t,a_t) \\ & \text{expected return:} \\ J(\theta) &= \mathbb{E}[\textstyle\sum_{t=0}^T R(s_t,a_t);\pi_\theta] \\ &= \textstyle\sum_\tau^T R(s_t,a_t);\pi_\theta \\ & \text{find } \theta \text{ to max:} \\ &= \textstyle\sum_\tau P(\tau;\theta)R(\tau) \\ & \text{yielding:} \\ \max_\theta J(\theta) &= \max_\theta \textstyle\sum_\tau P(\tau;\theta)R(\tau) \\ & \text{gradient chasing reward:} \\ \nabla_\theta J(\theta) &= \textstyle\sum_\tau P(\tau;\theta)\nabla_\theta \log P(\tau;\theta)R(\tau) \\ & \text{sampled over m trajectories:} \\ &= \tfrac{1}{m}\textstyle\sum_{i=1}^m P(\tau;\theta)\nabla_\theta \log P(\tau_i;\theta)R(\tau_i) \\ & \text{combined:} \\ &= \tfrac{1}{m}\textstyle\sum_{i=1}^m \sum_{t=0}^T \nabla_\theta \log \pi_\theta(a_t|s_t)R(\tau_i) \end{aligned} \tag{20}$$

REINFORCE

- check policy/episode's states/actions/rewards
- calc episode return for collected rewards
- update model params toward policy gradient

$$\nabla_\theta J(\theta) = \sum_{t=0}^T \nabla_\theta \log \pi_\theta(a_t|s_t)R \tag{21}$$

desired loss function:

$$\frac{1}{m}\sum_{t=1}^m \nabla_\theta \log \pi_\theta(a_t|s_t)R \tag{22}$$

Baselined REINFORCE

$$\nabla_\theta J(\theta) = \sum_{t=0}^T \nabla_\theta \log \pi_\theta(a_t|s_t) \underbrace{(R - V_{\Phi(s_t)})}_{\text{baselined reward}} \tag{23}$$

Actor-critic

+ less variance than Baselined REINFORCE
- actor (makes policy): policy gradient
- critic: policy iteration

$$\underbrace{A_\pi(s,a)}_{\text{advantage}} = Q(s,a) - V(s) \tag{24}$$

Asynchronous Advantage Actor Critic (A3C)

- 5-step Q-Value estimation
- shares params between actor/critic
+ run parallel, one policy
+ no more need for DQN's replay policy

Actor-Critic with Experience Replay (ACER)

- *Retrace* rewritten from truncated importance sampling with bias correction
- stochastic dueling network architectures
- new TRPO method

Advantage Actor Critic (A2C)

synchronous, deterministic variant of A3C.

Actor Critic using Kronecker-Factored Trust Region (ACKTR)

more sample-efficient than TRPO/A2C.

Deterministic Policy Gradient (DPG)

based on actor-critic

Deep Deterministic Policy Gradient (DDPG)

DPG variant that can operate over continuous action spaces

NoisyNet

A3C/DQN/dueling, but add parametric noise to weights to aid exploration

Trust Region Policy Optimization (TRPO)

natural policy gradient method, iterative procedure for optimizing policies, with guaranteed monotonic improvement.

Proximal Policy Optimization (PPO)

policy gradient method, new objective function enabling multiple epochs of minibatch updates. beats TRPO.

Categorical Distributional RL (CDRL)

value-based, model distribution of returns i/o expected values.

C51

51-tuple returns.

Rainbow

combines:

1. DQN (Deep Q-Network)
2. DDQN (Double Deep Q-Network)
3. N-Step Q-Learning
4. Prioritized Experience Replay (PER)
5. Dueling Q-Network
6. Distributional RL: C51
7. Noisy Network