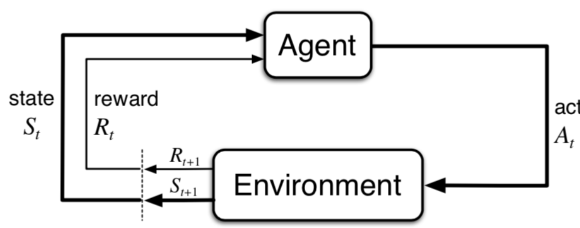
# Reinforcement Learning (RL)

## Symbols

- $a$ action, arm (multi-armed bandit problem)
- $A$ advantage
- $\alpha$ learning rate, step size
- $b$ bandit, baseline estimate, bias
- $c$ cost, context (= state)
- $d$ difference error
- $\Delta$ difference
- $D$ replay buffer
- $E$ eligibility trace
- $\epsilon$ exploration rate (0 1)
- $\phi$ state transition function
- $g$ gain (reward over time)
- $\gamma$ discount factor
- $\nabla$ gradient (spacial derivative)
- $H$ horizon
- $J$ expected return (see loss)
- $L$ expected (squared) loss (see return)
- $\lambda$ trace decay (0 1)
- $m$ momentum
- $Pr$ transition distribution
- $\pi$ policy (*state* $\rightarrow$ *action*)
- $q$ quality
- $r$ reward
- $R$ return
- $\rho$ regret
- $s$ state
- $t$ time
- $\tau$ trajectory
- $\theta$ parameter weight (learning target)
- $v$ value
- $vm$ variance momentum
- $w$ weight, winning probability
- $y$ target / prediction
- $*$ optimal
- $'$ next / derivative
- $\hat{}$ estimation
- $\|\|$ vector norm

## Agent-Environment Interface




The Agent at each step $t$ receives a representation of the environment's *state*, $S_t \in S$ and it selects an action $A_t \in A(s)$. From its action the agent receives a *reward*, $R_{t+1} \in R \in \mathbb{R}$.

## Bandits

- solve multi-armed bandit problem
- no state
- reduces RL to exploration vs exploitation

They estimate $\hat{r}_a \approx \mathbb{E}[r|a]$

### Greedy

$$\begin{aligned} n_a &= \textstyle\sum_t^a 1 \\ \hat{r}_a &= \textstyle\sum_t^a \frac{r_t}{n_a} \\ a_t &= \operatorname*{argmax}_{a \in A} \hat{r}_a \end{aligned} \tag{1}$$

### Optimistic-Greedy

large initial $\hat{r}_a$

### $\epsilon$-Greedy

for probability $\epsilon$, pick randomly

### Upper Confidence Bound (UCB)

try all for k rounds, then

$$a_t = \operatorname*{argmax}_{a \in A} \hat{r}_a + \sqrt{\frac{2 \log t}{n_a}} \tag{2}$$

### Contextual Bandit

### Linear UCB (LinUCB)

$$\begin{aligned} x_{t,a} &= \Phi(s_{t,a}) \\ \mathbb{E}[r_{t,a}|x_{t,a}] &= \theta_a \cdot x_{t,a} \\ \hat{\theta}_a &= A^{-1} b \end{aligned} \tag{3}$$

### Posterior / Thompson sampling

Pick actions by probability they maximize expected reward

### Greedy in the Limit with Infinite Exploration (GLIE)

- infinitely explores
- converges on greedy

## Concepts

### Reward

$$G_t = \sum_{k=0}^{H} \gamma^k r_{t+k+1} \tag{4}$$

for *discount factor* $\gamma$ and (infinite?) *horizon* $H$.

### Policy

A *policy* maps a state to an action

$$\pi_t(a|s) \tag{5}$$

probability to pick an action $A_t = a$ if $S_t = s$.

### Markov Decision Process (MDP)

a 5-tuple $(S, A, P, R, \gamma)$

$$\begin{aligned} &\text{state transition probabilities:} \\ &p(s'|s,a) = Pr\{S_{t+1} = s' | S_t = s, A_t = a\} \\ &\text{expected reward for state-action-nexstate:} \\ &r(s', s, a) = \mathbb{E}[R_{t+1} | S_{t+1} = s', S_t = s, A_t = a] \end{aligned} \tag{6}$$

### Value Function

Value (expected return, total discounted reward) of a specific state $s$ under policy $\pi$ for a MDP:

$$\begin{aligned} v_\pi(s) &= \mathbb{E}_\pi[G_t | S_t = s] \\ &= \sum_a \pi(a|s) \sum_{s'} \sum_r p(s', r|s, a)[r + \gamma v_\pi(s')] \\ v_*(s) &= \max_\pi v_\pi(s) \end{aligned} \tag{7}$$

### Action-Value (Q) Function

expected reward for state-action pairs:

$$\begin{aligned} q_\pi(s,a) &= \mathbb{E}_\pi[G_t | S_t = s, A_t = a] \\ &= \sum_{s', r} p(s', r|s, a)[r + \gamma V_\pi(s')] \\ q_*(s,a) &= \max_\pi q_\pi(s,a) \end{aligned} \tag{8}$$

rewriting $v_*$ for $q_*(s,a)$:

$$v_*(s) = \max_{a \in A(s)} q_{\pi_*}(s,a) \tag{9}$$

i.e. state value under optimal policy = expected return from its best action.

### Bellman Equation

Recursive property $Value$ 7 / $Q$ 8 functions

### Contraction Mapping

For metric space $(X, d)$ and $f : X \rightarrow X$, $f$ is a *contraction* given a *Lipschitz coefficient* $k \in [0, 1)$ where for all $x$ / $y$ in $X$:

$$d(f(x), f(y)) \leq k d(x, y) \tag{10}$$

### Contraction Mapping theorem

For complete metric space $(X, d)$ and contraction $f : X \rightarrow X$, there is only 1 fixed point $x^*$ where $f(x^*) = x^*$.
For point $x$ in $X$, and $f^n(x)$ inductively defined by $f^n(x) = f(f^{n1}(x))$, $f^n(x) \rightarrow x^*$ as $n \rightarrow \infty$, yielding a unique optimal solution for DP.

## Model-based Methods (known MDP)

### Exhaustive Search

brute force, usually computationally unviable.

### Dynamic Programming (DP)

+ bootstrap (learn mid-episode)

Find $\pi_*$ for $V$ / $Q$:

### Policy Iteration

Initialize $V(s) \in \mathbb{R}$e.g. 0, $\Delta \leftarrow 0$, $\pi(s) \in A$ for all $s \in S$
1. Policy Evaluation
**while** $\Delta < \theta$ *(e.g. 0.001)* **do**
  **foreach** $s \in S$ **do**
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)[r + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
  **end**
**end**
2. Policy Improvement
*policy-stable* $\leftarrow$ *true*
**while** *not policy-stable* **do**
  **foreach** $s \in S$ **do**
    *old-action* $\leftarrow \pi(s)$
    $\pi(s) \leftarrow \operatorname*{argmax}_a \sum_{s',r} p(s', r|s, a)[r + \gamma V(s')]$
    *policy-stable* $\leftarrow$ *old-action* $\neq \pi(s)$
  **end**
**end**

Policy iteration methods:

- gradient-based (policy gradient methods): gradient ascent
- gradient-free: simulated annealing, cross-entropy search or methods of evolutionary computation
- value search/iteration: stop after 1 state sweep
- async DP: update iteratively, no full sweeps
- generated policy iteration (GPI)

### Value Iteration

ditch $V(s)$ convergence for policy improvement and truncated policy eval step in 1 operation:

Initialize $V(s) \in \mathbb{R}$e.g. 0, $\Delta \leftarrow 0$
**while** $\Delta < \theta$ *(e.g. 0.001)* **do**
  **foreach** $s \in S$ **do**
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
  **end**
**end**
**output:** deterministic policy $\pi \approx \pi_*$ where
$\pi(s) = \text{argmax}_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$

# Model-free methods

## Monte Carlo (MC) Methods

- uses **averaging sample returns** per state-action pair
- episodic

\+ sampling

Initialize for all $s \in S, a \in A(s)$ :
  $Q(s,a) \leftarrow$ arbitrary
  $\pi(s) \leftarrow$ arbitrary
  $Returns(s,a) \leftarrow$ empty list
**while** *forever* **do**
  Pick $S_0 \in S$ and $A_0 \in A(S_0)$, all $p(s,a) > 0$
  Generate an episode starting at $S_0, A_0$ following $\pi$
  **foreach** *pair* $s, a$ *in the episode* **do**
    $G \leftarrow$ return for first occurrence of $s, a$
    Append $G$ to $Returns(s,a)$)
    $Q(s,a) \leftarrow average(Returns(s,a))$
  **end**
  **foreach** $s$ *in the episode* **do**
    $\pi(s) \leftarrow \text{argmax}_a Q(s,a)$
  **end**
**end**

estimate for non-stationary problems:

$$V(S_t) \leftarrow V(S_t) + \alpha[G_t - V(S_t)] \tag{11}$$

for learning rate $\alpha$, how much we want to forget about past experiences.

### Temporal Difference (TD)

\+ DP's bootstrap

\+ MC's sampling

- substitutes expected discounted reward $G_t$ from the episode with an estimation:

$$V(S_t) \leftarrow V(S_t) + \alpha[R_{t+1} + \gamma V(S_{t+1} - V(S_t)] \tag{12}$$

### State-action-reward-state-action (SARSA)

- on-policy TD control
- can use priors

Initialize $Q(s,a)$ arbitrarily and
 $Q(terminal - state, ) = 0$
**foreach** *episode* $\in$ *episodes* **do**
  Pick $a$ from $s$ by policy from $Q$ (e.g. $\epsilon$-greedy)
  **while** $s$ *is not terminal* **do**
    Take action $a$, observe $r, s'$
    Pick $a'$ from $s'$ by policy from $Q$ (e.g., $\epsilon$-greedy)
    $y \leftarrow r + \gamma Q(s', a')$
    $Q(s,a) \leftarrow Q(s,a) + \alpha[y - Q(s,a)]$
    $s \leftarrow s'$
    $a \leftarrow a'$
  **end**
**end**

### $n$-step Sarsa ($n$-step TD)

for $n$-step Q-Return:

$$\begin{aligned} q_t^{(n)} &= \gamma^n Q(S_{t+n}) + \textstyle\sum_{i=1}^n \gamma^{i-1} R_{t+i} \\ Q(s_t, a_t) &\leftarrow Q(s_t, a_t) + \alpha \left[ q_t^{(n)} - Q(s_t, a_t) \right] \end{aligned} \tag{13}$$

### Forward View Sarsa($\lambda$) (/ TD($\lambda$))

$$\begin{aligned} q_t^{\lambda} &= (1-\lambda) \textstyle\sum_{n=1}^{\infty} \lambda^{n-1} q_t^{(n)} \\ Q(s_t, a_t) &\leftarrow Q(s_t, a_t) + \alpha \left[ q_t^{\lambda} - Q(s_t, a_t) \right] \end{aligned} \tag{14}$$

### Backward View Sarsa($\lambda$) (/ TD($\lambda$))

\+ more efficient

\+ can update at every time-step

- eligibility traces : find cause in frequency vs recency

$$\begin{aligned} E_0(s,a) &= 0 \\ E_t(s,a) &= \gamma\lambda E_{t-1}(s,a) + \mathbf{1}(S_t = t, A_t = a) \\ \delta_t &= R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t) \\ Q(s,a) &\leftarrow Q(s,a) + \alpha \delta_t E_t(s,a) \end{aligned} \tag{15}$$

### Linear Function Approximation

\+ efficient

\+ generalize

update temporal difference error, minimize squared loss:

$$\begin{aligned} \delta &\leftarrow r_t + \gamma \theta^T \phi(s_{t+1}) - \theta^T \phi(s_t) \\ \theta &\leftarrow \theta + \alpha \delta \phi(s_t) \\ J(\theta) &= ||\delta||^2 \end{aligned} \tag{16}$$

### Q Learning

$$\begin{aligned} \delta &\leftarrow r_t + \gamma \underset{a \in A}{\text{argmax}}\, Q(\phi(s_{t+1}, a); \theta_i^-) - Q(\phi(s_t, a); \theta_i) \\ Q(s,a) &\leftarrow Q(s,a) + \alpha[r + \gamma \max_{a'} Q(s', a') - Q(s,a)] \end{aligned} \tag{17}$$

### Replay Memory

update $\theta$ by SGD

$$\nabla_{\theta_i} J(\theta_i) = \mathbb{E}_{(s_t, a_t, r_t, s_{t+1}) \sim D} \delta \nabla_{\theta_i} Q(\phi(s_t, a_t); \theta) \tag{18}$$

### Deep Q Learning (DQL)

Made by *DeepMind*, uses a deep neural net (*Q-network*) for the $Q$ function. Keeps $N$ observations in a *memory* to train on.

$$\begin{aligned} y &= r_t + \gamma \max_{a \in A} Q(\phi(s_{t+1}, a); \theta_i^-) \\ \nabla_i J(\theta_i) &= \mathbb{E}_{(s,a,r,s') \sim U(D)} [(y - \underbrace{Q(s, a; \theta_i)}_{\text{prediction}})^2] \end{aligned} \tag{19}$$

for network weights $\theta$ and experience replay history $U(D)$.

Initialize replay memory $D$ with capacity $N$
Initialize $Q(s,a)$ arbitrarily
**foreach** *episode* $\in$ *episodes* **do**
  Pick $a$ from $s$ by policy from $Q$ (e.g. $\epsilon$-greedy)
  **while** $s$ *is not terminal* **do**
    Take action $a$, observer $r, s'$
    Store transition $(s, a, r, s')$ in $D$
    Sample random transitions from $D$
    $y_i \leftarrow \begin{cases} r_j & \text{for terminal } s'_j \\ r_j + \gamma \max_a Q(s', a'; \theta) & \text{otherwise} \end{cases}$
    Perform gradient descent step on
     $(y_j - Q(s_j, a_j; \Theta))^2$
    $s \leftarrow s'$
  **end**
**end**

### Prioritized Replay Memory

learn esp. from high loss (traumas)

$$p(s_t, a_t, r_t, s_{t+1}) \propto r_t + \gamma \max_{a \in A} Q(\Phi(s_{t+1}, a); \theta_i^-) \tag{20}$$

### Double DQN

solve bias from reused $\theta$, faster

$$y = r_t + \gamma Q(\Phi(s_{t+1}, \underset{a \in A}{\text{argmax}}\, Q(\Phi(s_{t+1}, a); \theta_i)); \theta_i^-) \tag{21}$$

### Direct Policy Search

learn $\pi$.

### Policy Gradient

\+ simpler than $Q$ or $V$

\+ allows stochastic policy (rock-paper-scissors)

− local optimum

− less sample efficient

$$L = \mu(log p(a|s) \cdot R(s)) \tag{22}$$

## Likelihood Ratio

return state-action trajectory:

$$R(\tau) = \sum_{t=0}^{T} R(s_t, a_t) \tag{23}$$

expected return:

$$\begin{aligned} J(\theta) &= \mathbb{E}[\textstyle\sum_{t=0}^{T} R(s_t, a_t); \pi_\theta] \\ &= \textstyle\sum_{\tau}^{T} R(s_t, a_t); \pi_\theta \end{aligned} \tag{24}$$

find $\theta$ to max:

$$J(\theta) = \sum_{\tau} P(\tau;\theta) R(\tau) \tag{25}$$

yielding:

$$\max_\theta J(\theta) = \max_\theta \sum_{\tau} P(\tau;\theta) R(\tau) \tag{26}$$

$$\nabla_\theta J(\theta) = \sum_{\tau} P(\tau;\theta) \nabla_\theta \log P(\tau;\theta) R(\tau) \tag{27}$$

sampled over m trajectories:

$$\nabla_\theta J(\theta) = \frac{1}{m} \sum_{i=1}^{m} P(\tau;\theta) \nabla_\theta \log P(\tau^i;\theta) R(\tau^i) \tag{28}$$

gradient chases reward.

## REINFORCE

- Get info on states/actions/rewards for policy during episode
- Calc episode return for collected rewards
- Update model params toward policy gradient

$$\nabla_\theta J(\theta) = \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t) R \tag{29}$$

desired loss function:

$$\frac{1}{m} \sum_{t=1}^{m} \nabla_\theta \log \pi_\theta(a_t|s_t) R \tag{30}$$

## Baselined REINFORCE

use baselined rewards

$$\nabla_\theta J(\theta) = \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t)(R - V_{\phi(s_t)}) \tag{31}$$

## Actor-critic

+ less variance than Baselined REINFORCE
- actor (makes policy): policy gradient
- critic: policy iteration

advantage:

$$A_\pi(s, a) = Q(s, a) - V(s) \tag{32}$$

## Asynchronous Advantage Actor Critic (A3C)

- 5-step Q-Value estimation
- shares params between actor/critic

+ run parallel, one policy
+ no more need for DQN's replay policy

github.com/tycho01/Reinforcement-Learning-Cheat-Sheet