Computer Assignment 2: YALMIP/CVXPY

4DM20 Engineering Optimization

Study Questions

- Download and install YALMIP/CVXPY (refer to the installation guide that accompanies this handout).
- Understand how YALMIP/CVXPY works.
- Define the optimization variables, the objective function and the constraints.
- Solve an optimization problem with YALMIP/CVXPY.
- Use different solvers (Mosek, Gurobi, etc.).
- Solve the exercises with YALMIP/CVXPY.

Exercises

Exercise 1: Introduction to YALMIP/CVXPY

One of the intriguing benefits of YALMIP/CVXPY is that we do not have to state our optimization problems in a specific form, since it will parse our problem such that a solver can understand it (among other things). These first exercises are supposed to give you a general understanding of how to utilize the YALMIP/CVXPY syntax.

- Open `Ex1.IntroToYALMIP.mlx` (MATLAB) or `Assignment2_1.ipynb` (Python) and follow the instructions.

I. **Linear Programming**
Consider the following optimization problem
Problem 1

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1 + x_2 \\ \text{s.t.} \quad & 2 \cdot x_1 + x_2 \geq 1 \\ & x_1 + 3 \cdot x_2 \geq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

II. **Quadratic Programming**
Consider the following optimization problem

Problem 2

$$\begin{aligned}\min_{x_1,x_2} \quad & (x_1-5)^2 + (x_2-5)^2 \\ \text{s.t.} \quad & 2\cdot x_1 + x_2 \le 10 \\ & x_1 + x_2 \le 8 \\ & x_1 \le 4\end{aligned}$$

III. **Semidefinite Programming**

Use YALMIP/CVXPY and semidefinite programming to find the ellipse centered at $(0,0)$, i.e., $\mathcal{E} := \{x \in \mathbb{R}^2 | x^\top P^{-1} x \le 1\}$ with $P \succ 0$, with the greatest area that is inscribed in a polytope $\mathcal{H} = \{x \in \mathbb{R}^2 \mid Ax + b \le 0\}$ with $A \in \mathbb{R}^{n\times 2}$ and $b \in \mathbb{R}^n$.

Objective: The volume of $\mathcal{E}$ is proportional to $\det P^{1/2}$. Therefore, to maximize the area, we can instead minimize $-2\log\det P^{1/2} = -\log\det P$.

Constraints: The constraint $\mathcal{E} \subseteq \mathcal{H}$ can be written as

$$\left(\max_{x\in\mathbb{R}^2} \; A_i x \quad \text{s.t. } x^\top P^{-1} x \le 1\right) \le b_i, \quad \forall i \in \{1,\dots,n\}, \tag{1}$$

where A_i denotes the i-th row of A. The optimization problem in the constraints above has a closed-form solution, i.e.,

$$\max_{x\in\mathbb{R}^2} \; A_i x \quad \text{s.t. } x^\top P^{-1} x \le 1 = \sqrt{A_i P A_i^\top}.$$

In a few lectures you should be able to derive this yourself, but do not bother for now. The constraints in (1) can then be written as

$$A_i P A_i^\top - b_i^2 \le 0, \quad \forall i \in \{1,\dots,n\},$$

which are n linear matrix inequalities in P.

The optimization problem becomes:

Problem 3

$$\begin{aligned}\min_{P\succ 0} \quad & -\log\det P \\ \text{s.t.} \quad & A_i P A_i^\top - b_i^2 \le 0, \quad \forall i \in \{1,\dots,n\}\end{aligned}$$

Solve Problem 3 where the polytope $\mathcal{H}$ is defined by

$$\begin{aligned}-2x_1 + x_2 - 2 &\le 0 \\ x_1 + 4x_2 - 5 &\le 0 \\ 2x_1 + 2x_2 - 4 &\le 0 \\ x_1 - 2 &\le 0 \\ -2x_1 + 10x_2 + 8 &\le 0.\end{aligned}$$

IV. **Conic Programming**

Consider the following optimization problem

Problem 4

$$\begin{aligned} \min_{\mathbf{x}} \quad & (\mathbf{x}-\mathbf{c})^\top(\mathbf{x}-\mathbf{c}) \\ \text{s.t.} \quad & \mathbf{x}^\top A\mathbf{x} \le 0 \\ & x_2 \ge 0 \end{aligned}$$

with

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Is this a convex optimization problem?

You will notice that MOSEK does not like this formulation. Reformulate the quadratic constraint as a cone

$$|\mathbf{a}^\top \mathbf{x}| \le \mathbf{b}^\top \mathbf{x}$$

and try again.

Is the second inequality constraint still necessary?

Exercise 2: Linear Programming

A company that produces small toys wants to maximize the production profits. They make two types of toys: Rubik's cubes and Toy cars. Each Rubik's cube takes 4 minutes for each piece to be manufactured in the casting machine, while the Toy Car takes 5 minutes. The casting Machine can only be operated for two hours per day before the plastic runs out. A subsequent phase of production is required for both items: Painting. In this phase, it takes 5 minutes to fully paint a toy car and 3 min for the Cube. Again, the painting machine can only be operated for one hour before the colors run out. The toy car is now ready to be sold but the Cube has to undergo a third phase of Assembling that takes 2 minutes for each piece. The skillful technician who assembles the cubes only works for 30 minutes per day due to a union agreement. The final profit for each cube is 8 Euro and 5 Euro for the car. Consider that product cannot be processed in parallel and the order of the phases is fixed. Also, unfinished items cannot be sold.

Table 1: Production phases and profits.

Toy	Casting (120 min)	Painting (60 min)	Assembling (30 min)	Profit
Toy Car	4 min	5 min	-	5 €
Rubik's Cube	5 min	3 min	2 min	8 €

Being x_1 and x_2, respectively, the number of Rubik's cubes and toy cars to be produced

- Identify the cost function and the constraints for the optimization problem.
- Formulate the problem.
- Plot the constraints and identify the feasible domain.
- Determine the value of the variables that maximize the profits.

Is the problem convex? Give reasons for your answer.

Exercise 3: Quadratic Programming

Analyze the following constrained optimization problem:

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & f(\mathbf{x}) = (x_1 - 4)^2 + (x_2 - 4)^2 \\
\text{s.t.} \quad & g_1(\mathbf{x}) = x_1 + 2x_2 \leq 10 \\
& g_2(\mathbf{x}) = 7x_1 - x_2 \leq 15 \\
& g_3(\mathbf{x}) = x_1 \geq 0 \\
& g_4(\mathbf{x}) = x_2 \geq 0
\end{aligned}
$$

1. Is the problem well constrained? Verify by making a contour plot of the problem and identify in the plot the sets defined by constraints g_1, g_2, g_3, g_4.
2. Classify the types of constraints (active, inactive, redundant constraints).
3. Identify the feasible domain in the contour plot. Is it convex?
4. Can you find local or global minimizers?
5. Use YALMIP/CVXPY to find the minimizer and verify your answer.

Exercise 4: Optimal Racing Strategy

In this exercise we want to find the optimal driving strategy for an electric race car [1]–[3]. Since we know the circuit topography, we choose to optimize for the minimum lap time in the space domain, i.e.,

$$\min\ T = \int_0^S \frac{\mathrm{d}t}{\mathrm{d}s}(s)\,\mathrm{d}s\ ,$$

where S is the length of the circuit and $\frac{\mathrm{d}t}{\mathrm{d}s}(s)$ is the lethargy, essentially the *time per traversed distance*. We can also express the lethargy as the inverse of the velocity, i.e.,

$$\frac{\mathrm{d}t}{\mathrm{d}s}(s) = \frac{1}{v(s)}\ . \tag{2}$$

We simplify the longitudinal dynamics of the vehicle as

$$\frac{\mathrm{d}}{\mathrm{d}s}E_{\mathrm{kin}}(s) = F_{\mathrm{p}}(s) - F_{\mathrm{d}}(s) - F_{\mathrm{br}}(s)\ , \tag{3}$$

where $E_{\mathrm{kin}}(s)$ is the kinetic energy of the vehicle, $F_{\mathrm{p}}(s)$ is the propulsive force, $F_{\mathrm{d}}(s)$ is the drag force, and $F_{\mathrm{br}}(s)$ the braking force. The drag force is expressed as

$$F_{\mathrm{d}}(s) = \frac{\rho_{\mathrm{air}}\cdot c_{\mathrm{d}}\cdot A_{\mathrm{f}}}{2}\cdot v^2(s) = \frac{\rho_{\mathrm{air}}\cdot c_{\mathrm{d}}\cdot A_{\mathrm{f}}}{m}\cdot E_{\mathrm{kin}}(s)\ , \tag{4}$$

where ρ_{air}, c_{d}, and A_{f} are the density of air, the drag coefficient, and the frontal area of the vehicle, respectively, and m is the mass of the vehicle.
The propulsive force is limited by the force and power limits of the electric motor, i.e.,

$$F_{\mathrm{p}}(s) \in [0, F_{\mathrm{p,max}}]\ , \tag{5}$$

$$F_{\mathrm{p}}(s) \leq P_{\mathrm{p,max}}\cdot\frac{\mathrm{d}t}{\mathrm{d}s}(s)\ , \tag{6}$$

where $P_{\mathrm{p,max}}$ is the maximum propulsive power, while the kinetic energy is

$$E_{\mathrm{kin}}(s) = \frac{m}{2}\cdot v(s)^2\ , \tag{7}$$

$$E_{\mathrm{kin}}(s) \leq E_{\mathrm{kin,max}}(s)\ , \tag{8}$$

where $E_{\mathrm{kin,max}}(s)$ is the maximum kinetic energy possible at each spot on the track to not exceed the tire adherence limits. Finally, we constrain the braking force to be non-negative, thus

$$F_{\mathrm{br}}(s) \geq 0\ . \tag{9}$$

Now we can frame the optimization problem as

$$\begin{aligned}\min_{F_{\mathrm{p}},F_{\mathrm{br}}}\quad & T = \int_0^S \frac{\mathrm{d}t}{\mathrm{d}s}(s)\,\mathrm{d}s\\ \text{s.t.}\quad & (2)-(9)\ .\end{aligned}$$

Questions

1. Open `Ex4_minimumLapTime.mlx` (MATLAB) or `Assignment2_4.ipynb` (Python) and follow the instructions.

2. Why do we use MOSEK to solve this problem?

3. Rewrite the constraint for the lower bound of the kinetic energy as a second-order cone (Hint: Check how the lethargy constraint is implemented with the `cone` function).

Exercise 5: Automotive Design Problem

In this exercise we consider a simplified power train for an electric vehicle as shown in Fig. 1 and optimize the fixed-gear transmission ratio in order to reduce the energy consumption for a predefined drive cycle. First, we construct a convex motor model for a given motor using semi-definite programming. Then, we leverage the motor model in a vehicle model and frame an optimization problem to design the gear ratio.

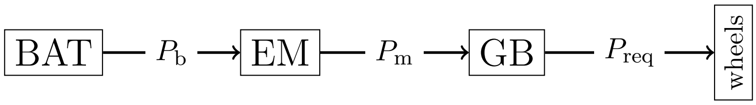

Figure 1: Simple power train layout. Arrows indicate positive direction of power flow.

- Open `Ex5.parameterFitting.mlx` (MATLAB) or `Assignment2_5.ipynb` (Python) and follow the instructions. The text below will give some background information on the exercise.

This exercise is an example for a vehicle design problem that would be performed during the development process of an electric vehicle. Framing it as a convex problem can help achieve fast results for different vehicle layout iterates and give the designer a rough overview of where to steer their design. To emulate "real-world" conditions, designers often leverage predefined standardized drive cycles like the WLTP that is commonly used in CO_2 emissions testing and range estimation.
We use the vehicle model shown in Fig. 1 and use it to obtain the required electric power through backwards simulation starting from the required power at the wheels for the drive cycle, which is obtained through

$$P_\mathrm{req}(t) = m \cdot a(t) \cdot v(t) + \frac{\rho_\mathrm{air} \cdot c_\mathrm{d} \cdot A_\mathrm{f}}{2} \cdot v^3(t) \ , \tag{10}$$

where $v(t)$ and $a(t)$ are the exogenous inputs for the velocity and acceleration throughout the drive cycle, respectively, and m, c_d, and A_f are the vehicle's mass, drag coefficient, and frontal area, while ρ_air is the density of the air.
The required shaft power of the motor is

$$P_\mathrm{m}(t) = \begin{cases} \eta_\mathrm{gb} \cdot P_\mathrm{req}(t) & \text{if } P_\mathrm{req}(t) < 0 \\ \frac{1}{\eta_\mathrm{gb}} \cdot P_\mathrm{req}(t) & \text{if } P_\mathrm{req}(t) \geq 0 \end{cases} , \tag{11}$$

where $P_\mathrm{req}(t)$ is the power required for the drive cycle and η_gb is the gearbox efficiency. The electric power drawn by the motor is obtained through

$$P_\mathrm{b}(t) = P_\mathrm{m}(t) + P_\mathrm{loss}(t) \ , \tag{12}$$

where $P_\mathrm{loss}(t)$ are the losses in the motor.
In our situation we assume that the motor has already been chosen and its performance data is available in the form of a lookup table. However, we would prefer an analytical representation instead, that can then be implemented in our optimization problem. Therefore, the first part of the exercise aims to fit a convex function that best represents the data in the lookup table. For said function we propose the model

$$P_\mathrm{loss}(t) = \mathbf{x}^\top(t) Q \mathbf{x}(t) \ , \tag{13}$$

where $\mathbf{x}(t) = [\,1\,,\,\omega_\mathrm{m}(t)\,,\,P_\mathrm{m}(t)\,]$ and Q is a symmetric and positive semi-definite matrix. Notice that due to the objective of minimizing the energy consumption we can relax (13) in the final design problem as

$$P_\mathrm{loss}(t) \geq \mathbf{x}^\top(t) Q \mathbf{x}(t)\;, \tag{14}$$

which will hold with equality for the optimal solution.
Finally, we introduce the battery dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\mathrm{b}(t) = -P_\mathrm{b}(t)\,. \tag{15}$$

With this information on hand we can state our optimization problems, where we start with the problem of fitting a quadratic model to the motor map:

Problem 5.1: Parameter Fitting Problem

$$\begin{aligned} \min_{Q \succeq 0} \quad & \sum_{i \in \mathcal{M}} \left\| \mathbf{x}_i^\top Q \mathbf{x}_i - P_{\mathrm{loss},i} \right\|_2 \\ \text{s.t.} \quad & \mathbf{x}_i = [\,1\,,\,\omega_i\,,\,P_{\mathrm{m},i}]^\top \quad \forall i \in \mathcal{M} \end{aligned}$$

Then, we discretize our dynamic equations with the Forward Euler Method (e.g., (15) turns into $E_\mathrm{b}[k+1] - E_\mathrm{b}[k] = -P_\mathrm{b}[k]$) and optimize the gear ratio that minimizes the energy expenditure for the given drive cycle.

Problem 5.2: Optimal Gearbox Design

$$\begin{aligned} \min_{\gamma \in \Gamma} \quad & E_{\mathrm{b,init}} - E_{\mathrm{b},N+1} \\ \text{s.t.} \quad & (10) - (12),\ (14) - (15) \\ & \omega_\mathrm{m}(t) \in \Omega \quad \forall t \end{aligned}$$

References

[1] S. Broere and M. Salazar, “Minimum-lap-time control strategies for all-wheel drive electric race cars via convex optimization”, In press. Available online at `https://arxiv.org/abs/2111.04650`, 2022.

[2] J. van Kampen, T. Herrmann, T. Hofman, and M. Salazar, “Optimal endurance race strategies for a fully electric race car under thermal constraints”, 2023, Submitted.

[3] O. Borsboom, C. A. Fahdzyana, T. Hofman, and M. Salazar, “A convex optimization framework for minimum lap time design and control of electric race cars”, vol. 70, no. 9, pp. 8478–8489, 2021.