MAE 5803 - Homework #2 Problem #2

Table of Contents

Tim Coon: 9, February 2017

```
clear; close all; clc;
```

Set default figure properties

```
set(0,'defaultlinelinewidth',2.5)
set(0,'defaultaxeslinewidth',2.5)
set(0,'defaultpatchlinewidth',2.5)
set(0,'defaulttextfontsize',14)
set(0,'defaultaxesfontsize',14)
set(0,'defaultTextInterpreter','latex')
```

Equilibrium Points and Stability

Fo the following systems, find the equilibrium points and determine their stability. Indicate if the stability is asymptotic and if it is global.

$$\dot{x} = -x^3 + \sin^4(x)$$

$$\dot{x} = (5 - x)^5$$

$$\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8(x) \cos^2(3x)$$

$$\ddot{x} + (x-1)^4 \dot{x}^7 + x^5 = x^3 \sin^3(x)$$

$$\ddot{x} + (x-1)^2 \dot{x}^7 + x = \sin \frac{\pi x}{2}$$

Eqn #1:

First-order state equation

$\dot{x} = -x^3 + \sin^4(x)$

This system has an equilibrium point at $x = 0$. Apply Theorem 3.3 to find it is a globally asymptotically stable. $\dot{V}(x)$ is negative definite because $x^4 > x\sin^4(x)$

1. $V(\mathbf{x}) > 0 \quad \forall \quad \mathbf{x} \neq \mathbf{0}$

2. $\dot{V}(\mathbf{x}) < 0 \quad \forall \quad \mathbf{x} \neq \mathbf{0}$

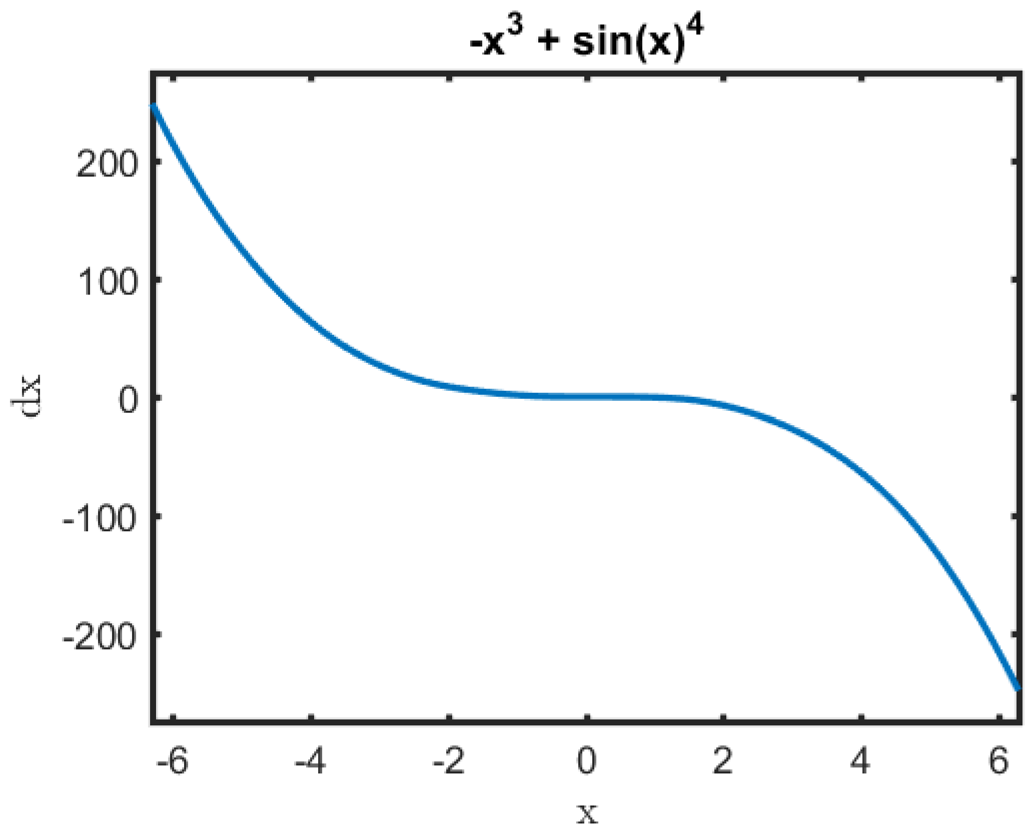
3. $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$

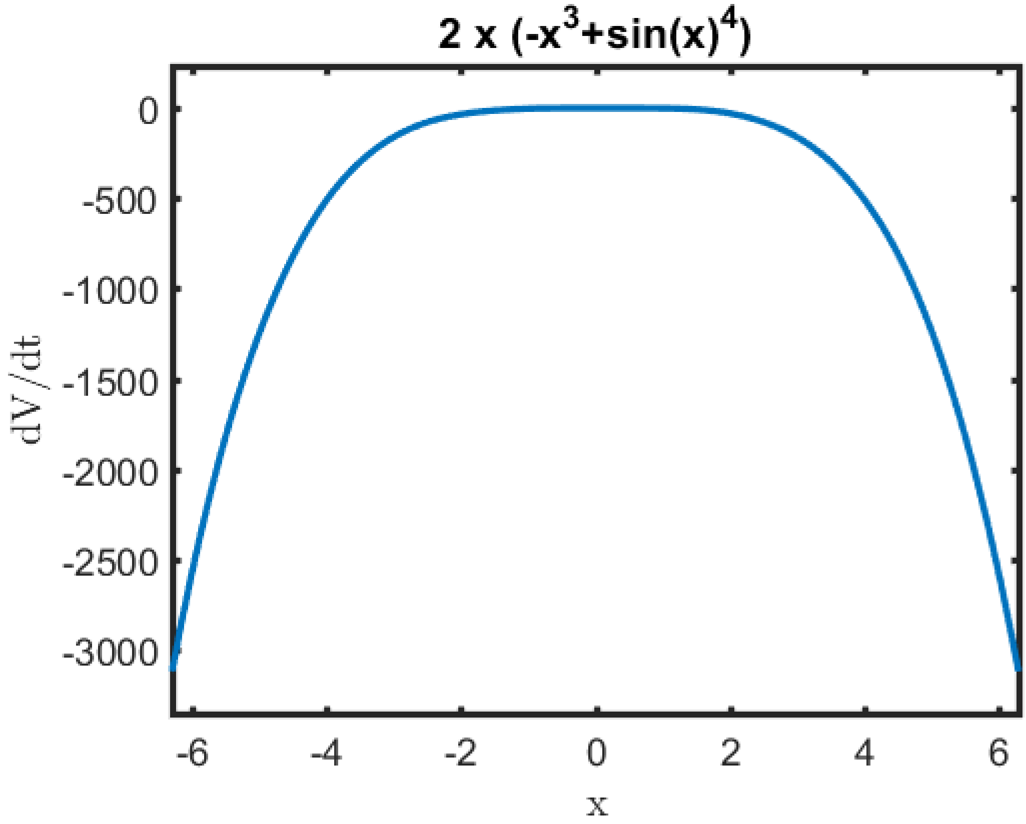
The Lyapunov function is:

$V = x^2$

$\dot{V}(x) = 2x\dot{x} = 2x(-x^3 + \sin^4(x)) \leq 0$

```
figure()
ezplot('-x^3 + sin(x)^4')
xlabel('x'); ylabel('dx');
figure()
ezplot('2*x*(-x^3+sin(x)^4)')
xlabel('x'); ylabel('dV/dt');
```





Eqn #2:

First-order state equation

$\dot{x} = (5 - x)^5$

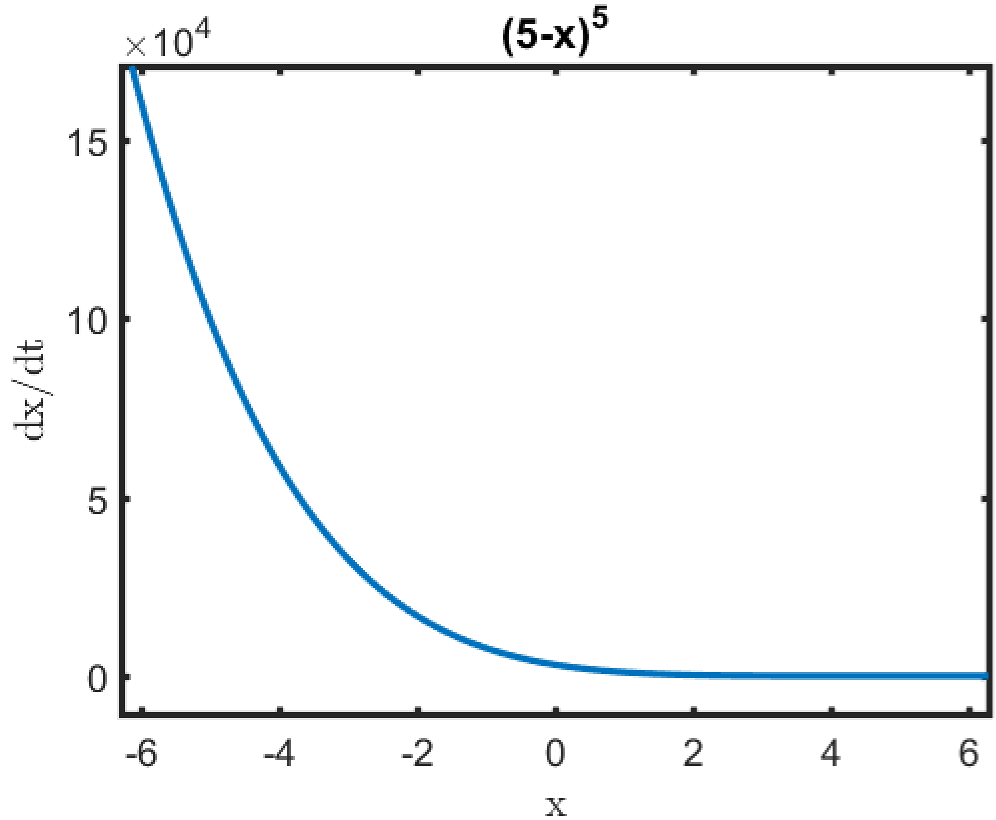
This system has an equilibrium point at $x = 5$. It is an unstable node as evidenced by using the candidate Lyapunov function to show violation of Theorem 3.2. Clearly, $\dot{V}(x)$ is not negative semidefinite.

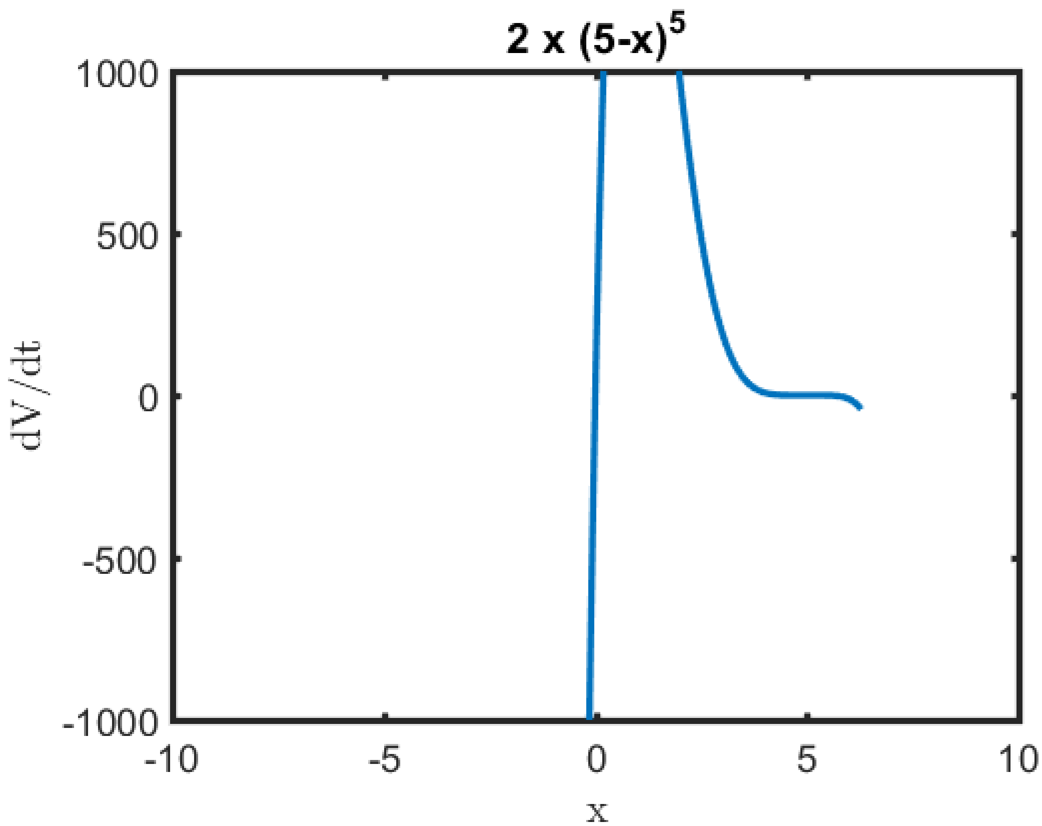
The Lyapunov function is:

$V = x^2$

$\dot{V}(x) = 2x\dot{x} = 2x(5 - x)^5$

```
figure()
ezplot('(5-x)^5')
xlabel('x'); ylabel('dx/dt');
figure()
ezplot('2*x*(5-x)^5')
axis([-10 10 -1000 1000])
xlabel('x'); ylabel('dV/dt');
```


(5-x)^5
×10^4
15
10
5
0
dx/dt
-6
-4
-2
0
2
4
6
x


2 x (5-x)^5
1000
500
0
-500
-1000
dV/dt
-10
-5
0
5
10
x

Eqn #3:

$$\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8(x) \cos^2(3x)$$

$$k(x) = x^7 - x^2 \sin^8(x) \cos^2(3x)$$

$$\ddot{x} + \dot{x}^5 + k(x) = 0$$

The Lyapunov function is:

$$V(\mathbf{x}) = \frac{1}{2}\dot{x} + \int_0^x k(\xi) d\xi$$

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{x}\ddot{x} + k(x)\dot{x} \\ &= \dot{x}\ddot{x} + k(x)\dot{x} \\ &= \dot{x}(-\dot{x}^5 - k) + k(x)\dot{x} \\ &= -\dot{x}^6 \end{aligned}$$

From Theorem 3.3:

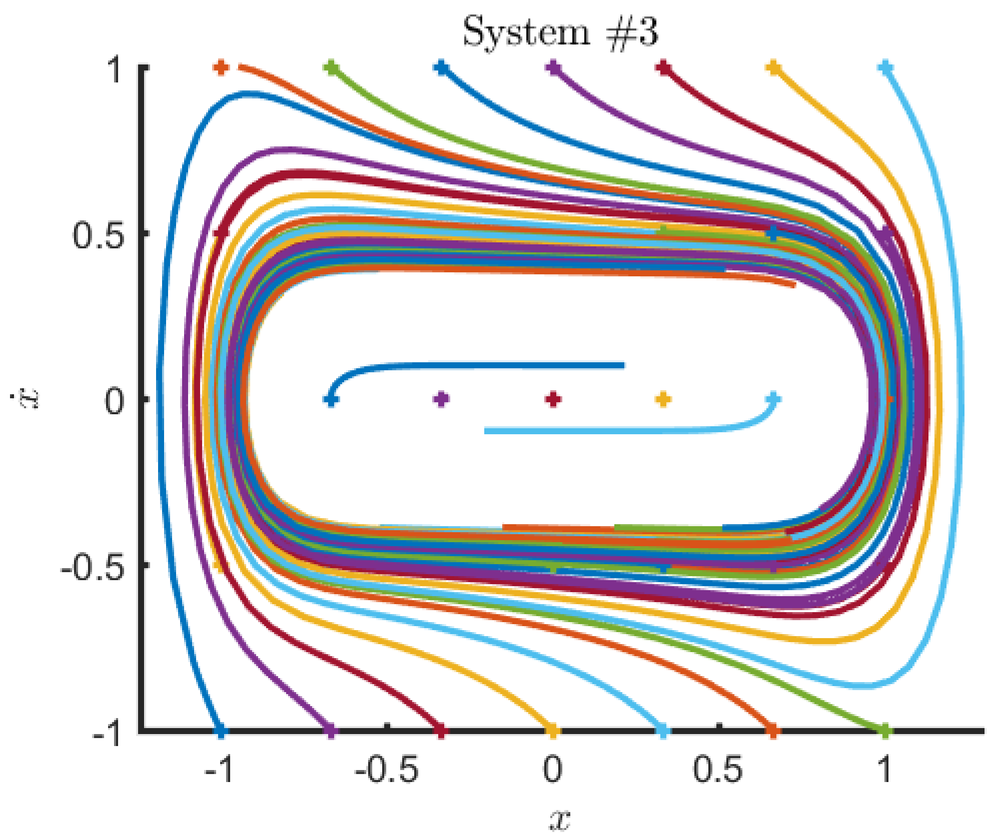
1. $V(\mathbf{x})$ is positive definite. The first term is squared and the second is positive by analogy of stored energy in a displaced spring no matter what direction the displacement.

2. $\dot{V}(\mathbf{x})$ is negative semidefinite as evidenced by the above simplification.

3. $V(\mathbf{x})$ is radially unbounded. More speed means more energy and more spring displacement means more energy.

The origin is a globally asymptotically stable equilibrium point and is the only equilibrium point.

```
HW2P2_plotPhasePortrait(3,[0 10],[-1 1],[-1 1],'System \#3')
```



Eqn #4:

$$\ddot{x} + (x-1)^4\dot{x}^7 + x^5 = x^3\sin^3(x)$$

$$k(x) = x^5 - x^3\sin^3(x)$$

$$\ddot{x} + (x-1)^4\dot{x}^7 + k(x) = 0$$

The Lyapunov function is:

$$V(\mathbf{x}) = \frac{1}{2}\dot{x} + \int_0^x k(\xi)d\xi$$

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \dot{x}\ddot{x} + k(x)\dot{x} \\ &= \dot{x}(-(x-1)^4\dot{x}^7 - k) + k(x)\dot{x} \\ &= -(x-1)^4\dot{x}^8\end{aligned}$$

From Theorem 3.3:

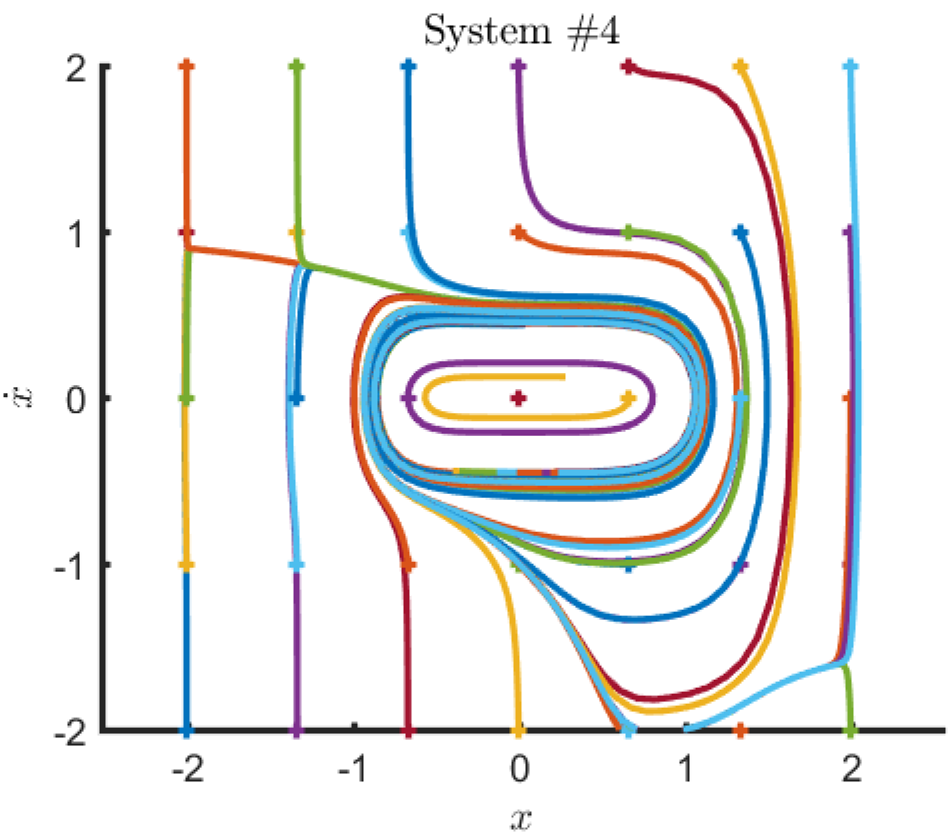
1. $V(\mathbf{x})$ is positive definite. The first term is squared and the second is positive by analogy of stored energy in a displaced spring no matter what direction the displacement.

2. $\dot{V}(\mathbf{x})$ is negative semidefinite as evidenced by the above simplification.

3. $V(\mathbf{x})$ is radially unbounded. More speed means more energy and more spring displacement means more energy.

The origin is a globally asymptotically stable equilibrium point and is the only equilibrium point.

```
HW2P2_plotPhasePortrait(4,[0 20],[-2 2],[-2 2],'System \#4')
```



Eqn #5:

$$\ddot{x} + (x-1)^2\dot{x}^7 + x = \sin(\frac{\pi x}{2})$$

$$k(x) = x - \sin(\frac{\pi x}{2})$$

$$\ddot{x} + (x-1)^2\dot{x}^7 + k(x) = 0$$

The Lyapunov function is:

$$V(\mathbf{x}) = \frac{1}{2}\dot{x} + \int_0^x k(\xi)d\xi$$

$$\begin{aligned}\dot{V}(\mathbf{x}) \quad &= \dot{x}\ddot{x} + k(x)\dot{x} \\ &= \dot{x}(-(x-1)^2\dot{x}^7 - k(x)) + k(x)\dot{x} \\ &= -(x-1)^2\dot{x}^8\end{aligned}$$

From Theorem 3.4:

1. $V(\mathbf{x}) \to \infty$ as $||x|| \to \infty$ This is typical of energy functions.

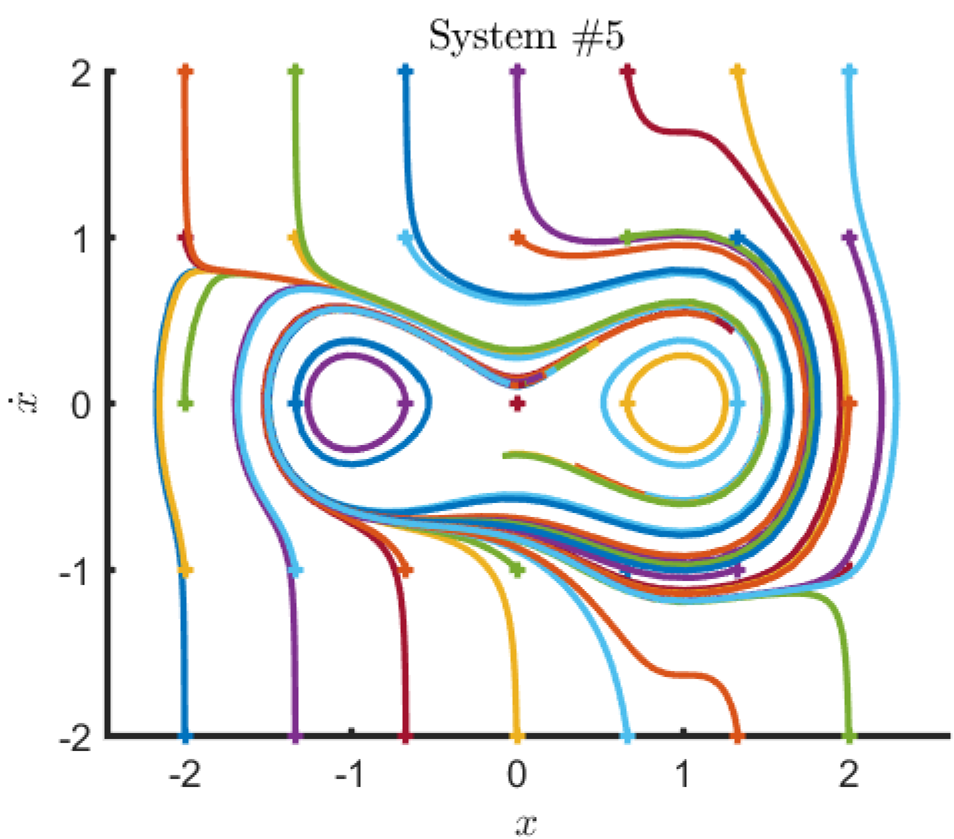
2. $\dot{V}(\mathbf{x})$ is negative semidefinite over the entire state space as evidenced by the preceeding simplification.

3. R composes all points satisfying $\dot{V}(\bar{x}) = 0$ which is the union of (1,u) and (v,0) for all real values u,v.

R = {\bar{x} : \dot{V}(\bar{x})=0} = (1,u) \bigcup (v,0) \forall u,v \in \R$

Though the origin fits this definition, it does not fall within the region Ω_ℓ defined in Theorem 3.4. It seems it should not fit here, but I can't figure out why. The origin is an unstable equilibrium point. The other two equilibrium points, (1,0) and (-1,0) meet the criteria above as well as those for Theorem 3.3. These are globally asymptotically stable equilibrium points. I tried to show this by finding the points where energy is zero, but I could only get the points (0,0), (-1.4483,0), and (1.4483,0).

```
HW2P2_plotPhasePortrait(5,[0 10],[-2 2],[-2 2],'System \#5')
```



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