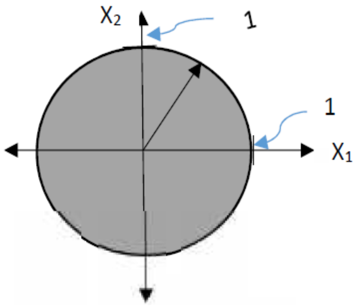
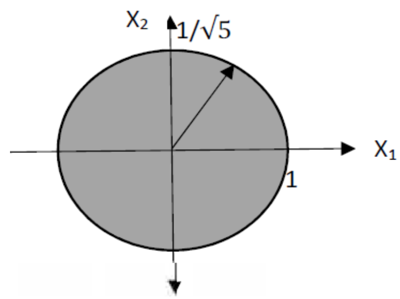
MAE 5803 Nonlinear Control Systems **Assigned: Jan 31, 2017**
Homework #2 (solutions) **Due: Feb 9, 2017**

1. *Courtesy of Yash Shah*

$\|x\|^2 = x_1^2 + x_2^2 \le 1$ (Euclidian Norm)

$$\|x\|^2 = x_1^2 + 5x_2^2 \le 1$$
$$= \frac{x_1^2}{5} + x_2^2 \le \frac{1}{5}$$



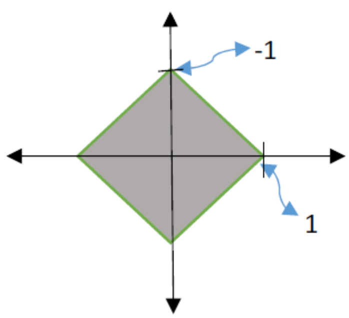


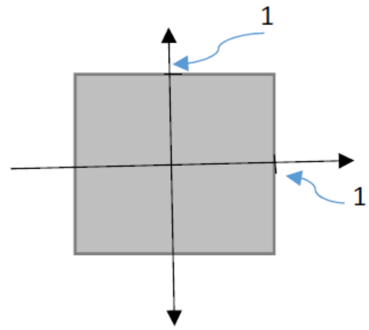
$-1 < x_1 > 1$

$-1/\sqrt{5} \le x_2 \ge 1/\sqrt{5}$

$\|x\| = |x_1| + |x_2| \le 1$

$\|x\| = \sup(|x_1|, |x_2|) \le 1$





$\text{Max}(|x_1|, |x_2|) \le 1$

2. *Courtesy of Nasir Hariri*

(a) 1st order system:

$$\dot{x} = -x^3 + \sin^4 x \tag{1}$$

Equilibrium point:

$$-x^3 + \sin^4 x = 0 \quad \rightarrow \quad x = 0 \tag{2}$$

Define a scalar function:

$$V(x) = x^2 \quad \rightarrow \quad > 0 \quad \text{Positive definite, radially unbounded} \tag{3}$$

Thus:

$$\dot{V}(x) = 2 * x * \dot{x} \tag{4}$$

$$\dot{V}(x) = -2(x^4 - x * \sin^4 x) \ \text{negative definite} \tag{5}$$

since

$$x^4 - x sin^4 x \ge x^4 - x sin^3 x = x(x^3 - sin^3 x) = x(x - sinx)(x^2 + xsinx + sin^2 x) \ge 0$$

This indicates that the equilibrium point at the origin is globally asymptotic stable (G.A.S).

(b) 1^{st} order system:

$$\dot{x} = (5 - x)^5 \tag{6}$$

Equilibrium point:

$$(5 - x)^5 = 0 \quad \rightarrow \quad x = 5 \tag{7}$$

Change of variable:

$$y = 5 - x$$
$$=> \dot{y} + y^5 = 0$$

Consider the Lyapunov function candidate:

$$V(y) = y^2$$

which is positive definite and $V(y) \to \infty \; as \; \|y\| \to \infty$.

$$\dot{V}(y) = 2y\dot{y} = -2y^6 < 0 \;\; for \;\; y \neq 0$$

This indicates that the equilibrium point (5,0) is globally asymptotically stable (G.A.S).

(c) 2^{nd} order system:

$$\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8 x \cos^2 3x \tag{8}$$

Singular/equilibrium point:

$$x^7 - x^2 \sin^8 x \cos^2 3x = 0 \quad \rightarrow \quad (x, \dot{x}) = (0, 0) \tag{9}$$

Define and test a candidate Lyapunov function:

$$V(x) = \frac{1}{2}\dot{x}^2 + \int_0^x (y^7 - y^2 \sin^8 y \cos^2 3y) \; dy \quad \rightarrow$$

> 0 Positive definite and radially unbounded (need to prove this)

Thus:

$$\dot{V}(x) = \dot{x}\ddot{x} + x^7 - x^2 \, sin^8 \, x \, cos^2 \, 3x \tag{10}$$
$$\dot{V}(x) = -\dot{x}^6 \quad \rightarrow \text{ negative semi-definite}$$

Using invariant set theorem

$$R: \dot{V}(x) = 0 \;\; => \;\; \dot{x} = 0 \;\; => \;\; \overline{x} = 0 \;\; => \;\; M \equiv R = \{\overline{x} = \overline{0}\}$$

which means all trajectories tend to origin.

So the equilibrium point is **globally asymptotically stable**.

(d) 2^{nd} order system:

$$\ddot{x} + (x - 1)^4 \dot{x}^7 + x^5 = x^3 \sin^3 x \tag{11}$$

Equilibrium point:

$$x^3 \sin^3 x - x^5 = 0 \quad \rightarrow \quad (x, \dot{x}) = (0, 0) \tag{12}$$

Candidate of Lyapunov function:

$$V(x) = \frac{1}{2}\dot{x}^2 + \int_0^x (y^5 - y^3 \sin^3 y) \; dy$$

> 0 Positive definite and radially unbounded (need to prove)

Thus:

$$\dot{V}(x) = \dot{x}\ddot{x} + (x^5 - x^3 \sin^3 x)\dot{x} \tag{13}$$
$$\dot{V}(x) = -(x - 1)^4 \dot{x}^8 \quad \rightarrow \quad \leq 0 \text{ negative semi-definite} \tag{14}$$

Invariant set theorem:

$$R: \quad \dot{V} = 0 \tag{15}$$

$x = 1$ $\qquad$ $x = 0$

$$\text{if} \quad x = 1 \quad \rightarrow \quad \ddot{x} = 0.6 \quad \rightarrow \quad \neq 0 \quad \text{“Not an invariant set”} \tag{16}$$

$$\text{if} \quad \dot{x} = 0 \quad \rightarrow \quad \ddot{x} = 0 \quad \rightarrow \quad \mathrm{M} = \{\mathbf{x} = 0\} \quad \text{“an invariant set: Equilibrium point”} \tag{17}$$

By the invariant set theorem, all motion trajectories converge to the equilibrium point (0,0). Hence, the origin is globally asymptotic stable (G.A.S).

(e) 2nd order system:

$$\ddot{x} + (x-1)^2\dot{x}^7 + x = \sin\frac{\pi x}{2} \tag{18}$$

Equilibrium points:

$$\sin\frac{\pi x}{2} - x = 0 \tag{19}$$

Hence there are three equilibrium points: (-1,0), (0,0), and (1,0)

Define and test a candidate Lyapunov function:

$$V(x) = \frac{1}{2}\dot{x}^2 + \int_0^x \left(y - \sin\frac{\pi y}{2}\right) dy \quad \rightarrow \quad > 0 \quad \text{Positive definite and radially unbounded (prove)} \tag{20}$$

Thus:

$$\dot{V}(x) = \dot{x}\ddot{x} + \left(x - \sin\frac{\pi x}{2}\right)\dot{x} \tag{21}$$

$$\dot{V}(x) = -(x-1)^2\dot{x}^8 \quad \rightarrow \quad \leq 0 \ \text{ negative semi-definite} \tag{22}$$

Invariant set theorem: possible solutions for ($\dot{V} = 0$):

$$R: \quad \dot{V} = 0 \tag{23}$$

$x = 1$	$\dot{x} = 0$

$$\text{if} \quad x = 1 \quad \rightarrow \quad \ddot{x} = 0 \quad \rightarrow \quad \text{“an invariant set"} \tag{24}$$

$$\text{if} \quad \dot{x} = 0 \quad \rightarrow \quad \ddot{x} = 0 \quad \rightarrow \quad \mathrm{M} = \{\mathbf{x} = (-1{,}0), (0{,}0), (1{,}0)\} \quad \text{“an invariant set"} \tag{25}$$

According to invariant set theorem, the system converges globally to $(x, \dot{x}) = (1, 0)$ or $(x, \dot{x}) = (-1, 0)$ or $(x, \dot{x}) = (0, 0)$. The first two are stable since they correspond to local minima of *V*. But the equilibrium point $(x, \dot{x}) = (0, 0)$ is unstable because it is a local maximum of *V*.

So $(x, \dot{x}) = (\pm 1, 0)$ are **locally asymptotically stable** and $(x, \dot{x}) = (0, 0)$ is **unstable**.

3. *Courtesy of Rakhayai Mangsatabam*

$\dot{v} + 2a|v|v + bv = 0$; $a > 0$, $b > 0$

Equilibrium point,

$2a|v|v + bv = 0$.

$\therefore v(2a|v| + b) = 0$.

$\therefore v = 0$, $|v| = \frac{-b}{2a}$ (magnitude cannot be negative)

$\therefore v = 0$ //

Let $V(v) = \frac{1}{2}v^2 \quad > 0$ (+ve def) (radially unbounded)

$$\dot{V}(v) = v\dot{v}$$
$$= v(-2a|v|v - bv)$$
$$= -2a|v|v^2 - bv^2 \quad < 0 \text{ (-ve def).}$$

always true

$\therefore$ Global Asymptotic Stable.

4. *Courtesy of Casey Clark*

4

$$L(\dot{q})\ddot{q} + R(\dot{q}) + C(q) = 0$$

$\dot{q}R(\dot{q}) > 0 \; ; \; \dot{q} \neq 0$

$qC(q) > 0 \; ; \; q \neq 0$

$L(\dot{q}) > 0$

$R(0) = C(0) = 0$

$$\vec{x} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \qquad \ddot{q} = \dot{q} = 0$$
$$C(q) = 0$$
$$C(0) = 0$$
$$q = 0$$
$$\vec{x} = \vec{0} \quad \text{Eq. Pt.}$$

$$V(\vec{x}) = \int_0^{\dot{q}} L(w)\,w\,dw + \int_0^{q} C(y)\,dy > 0 \quad \text{(radially unbounded) (Pos. Def)}$$

$$\dot{V}(\vec{x}) = L(\dot{q})\,\dot{q}\,\ddot{q} + C(q)\dot{q}$$

$$\dot{V}(\vec{x}) = L(\dot{q})\dot{q}\left(\frac{-R(\dot{q}) - C(q)}{L(\dot{q})}\right) + C(q)\dot{q}$$

$$\dot{V}(\vec{x}) = -\dot{q}R(\dot{q})$$ ➔ negative semi-definite

Then use invariant set theorem to conclude that all trajectories will converge to the origin (asymptotically stable).

Furthermore, if

$$\int_0^{\infty} C(y)\,dy \rightarrow \infty \Rightarrow \text{radially Unbounded}$$

$$\int_0^{\infty} L(w)\,w\,dw \rightarrow \infty \Rightarrow \text{radially unbounded}$$

➔ Global asymptotic stability

5. *Courtesy of Nasir Hariri*

$$\dot{x} = 4\,x^2 y - f_1(x)[x^2 + 2\,y^2 - 4] \qquad (26)$$

$$\dot{y} = -2\,x^3 - f_2(y)[x^2 + 2\,y^2 - 4] \qquad (27)$$

Use the following Lyapunov function candidate:

$$V = [x^2 + 2\,y^2 - 4]^2 \;\;\rightarrow\;\; > 0 \quad \text{Positive definite} \qquad (28)$$

Thus:

$$\dot{V} = 2\,[x^2 + 2\,y^2 - 4] * (2x\dot{x} + 4y\dot{y}) \qquad (29)$$

$$\dot{V} = -4\,[x^2 + 2\,y^2 - 4]^2 * \big(x\,f_1(x) + 2y\,f_2(y)\big) \qquad (30)$$

Since the continuous functions f_1 and f_2 have the same sign as their argument, thus:

$$x.f_1(x) \;\; > 0; \qquad x \neq 0 \qquad (31)$$

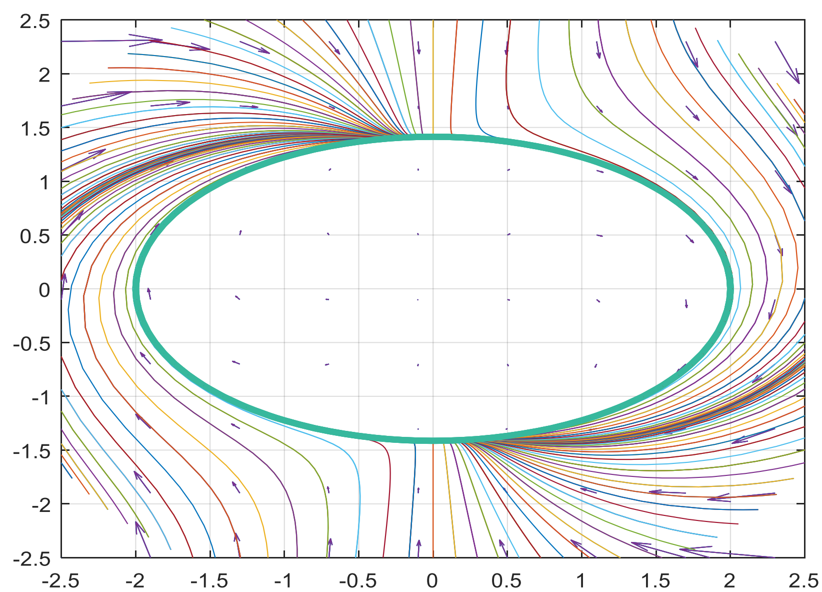
$$y.f_2(y) \;\; > 0; \qquad y \neq 0 \qquad (32)$$

Therefore, the $\dot{V}$ is a negative semi-definite function. Invariant set theorem:

$$R: \qquad \dot{V} = 0 \qquad (33)$$

$x\,f_1(x) + 2y\,f_2(y) = 0$ $x, y = 0$ "Equilibrium point"	$x^2 + 2\,y^2 = 4$ Closed curve: "Limit cycle"

Also, by plotting the phase portrait plot as shown in the figure (below), it can be seen that this is a stable clockwise limit cycle.



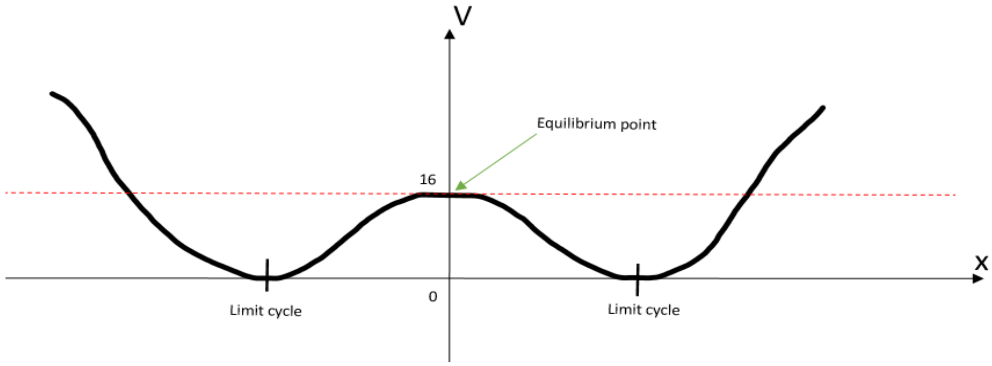
At equilibrium point, when (x = y = 0):

$$V = 16 \qquad (34)$$

Where at limit cycle:

$$V = 0 \qquad (35)$$

Which can be represented as the following sketch (y=0):



In order to trap only the limit cycle:

$$\text{If} \quad l = 16 \;\Rightarrow\; \Omega_{16}: \quad V < 16; \quad \dot{V} \leq 0 \;\Rightarrow\; M: limit\ cycle \qquad (36)$$

By the invariant set theorem, all motion trajectories converge to the limit cycle, which indicates that the limit cycle is stable and the system tends toward the limit cycle independent of the explicit values of f_1 and f_2. Since all trajectories tend to go to the limit cycle including trajectories inside the limit cycle, therefore, equilibrium point isn't stable as also shown in the portrait plot.