Florida Institute of Technology
High Tech with a Human Touch™

MAE 5803
NONLINEAR CONTROL SYSTEMS

Course Introduction

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Mechanical & Aerospace Engineering

Course Contents

- Nonlinear system fundamentals
  - Peculiar dynamics
- Nonlinear system analysis
  - Dynamics behavior
  - Stability concept and analysis
- Nonlinear control design
  - Lyapunov-based control methods
  - Gain scheduling
  - Feedback linearization
  - Sliding control
  - Adaptive control

➡ Background knowledge: linear control system analysis and design

Course Administration (1)

- Course materials:
  - Lecture notes:
    - Lecture slides: posted in Canvas
    - Class notes written on the board
    - Supplementary materials: posted in Canvas or distributed in class
  - Homework/assignments: posted in Canvas
- Textbook: Slotine & Li, *Applied Nonlinear Control*, Prentice-Hall, 1991
- Other references:
  - Khalil, *Nonlinear Control*, Pearson, 2015
  - Astrom & Wittenmark, *Adaptive Control*, Addison-Wesley, 1995

Course Administration (2)

- Grading:
  - Homeworks/assignments: 30%
  - Midterm exam: 30%
  - Term paper: 40%
    - The topic of the term paper is open, but it has to be related to nonlinear control design or analysis
    - Students are encouraged to select a topic relevant to their area of interest or research
    - Topic selection has to be finalized before spring break
    - Term paper is due one-week before the last day of semester
    - Presentation of the term paper will be held on the last week of the semester

Goals of Control

- Regardless of the type of the system (linear or nonlinear), goals of control are the same:
  - To *stabilize* an unstable system
  - To improve stability of a system
    - To have better *relative stability*
    - Equivalent to improving *transient response* of the system
  - To improve *tracking performance* (*command-following characteristics*)
    - Reduce/eliminate *steady-state errors* to certain type of inputs
  - To maintain adequate performance in the presence of *disturbances* and *uncertainties*
    - Good *disturbance-rejection* and *robustness* characteristics

Linear Control

- Linear control has been widely used and studied:
  - Well-developed theory $\Rightarrow$ clear system structure
    - Unique equilibrium

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

$\mathbf{x}(t) \in \mathbb{R}^n$: state vector

$\mathbf{u}(t) \in \mathbb{R}^m$: input vector

Equilibrium: $\dot{\mathbf{x}}(t) = \mathbf{0}$ $\Rightarrow$ unique $(\mathbf{x}_0, \mathbf{u}_0)$ solution

    - Principle of superposition

$$\mathbf{u} = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 \longrightarrow \boxed{\text{Linear System}} \longrightarrow \mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$$

α, β: arbitrary constants

  - Various powerful methods for analysis and design
    - Classical SISO (root locus, frequency response) and modern MIMO methods (eigenstructure assignment, LQR, LQG, H_∞, etc.)
  - Successful applications

Limitations of Linear Control

- *Fact*: all physical systems are inherently *nonlinear*
- Linear control is based on *linear model of a system*, which is usually obtained by linearization about certain operational condition (equilibrium)
  - OK for some applications
  - *But*, linear model may have very limited range of validity
  - Also, linearization does not make sense for some classes of nonlinear systems
    - Some nonlinear phenomena *cannot be captured* by the linearized model

Motivation for Nonlinear Control

- Improvement over existing linear control systems
  - Not limited by the validity of the linear assumption
  - Potentially better performance as design considers nonlinear effects
- Presence of *hard nonlinearities* (discontinuity)
  - Linear approximation cannot be applied
- Handling of model uncertainties
  - Certain classes of nonlinear controllers can tolerate model uncertainties better than linear controllers
- Design simplicity
  - Nonlinear control designs may be simpler and more intuitive than their linear counterparts

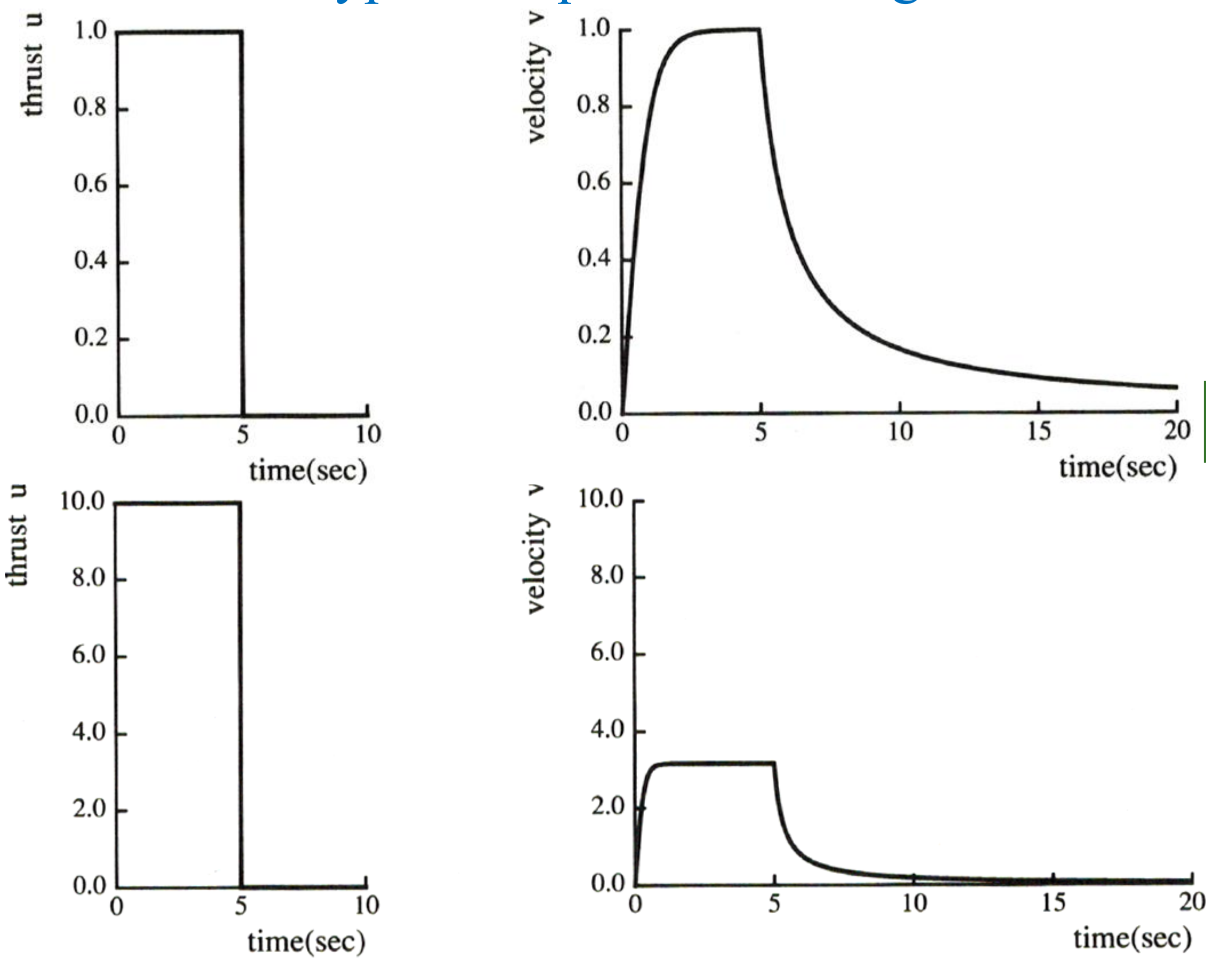
Example of Nonlinear Behavior

- Consider simplified model of underwater vehicle motion:

$$\dot{v} + \underbrace{|v|v}_{\text{Typical ``square-law'' drag}} = u$$

v: speed

u: thrust from propeller



Different response speed for positive and negative inputs

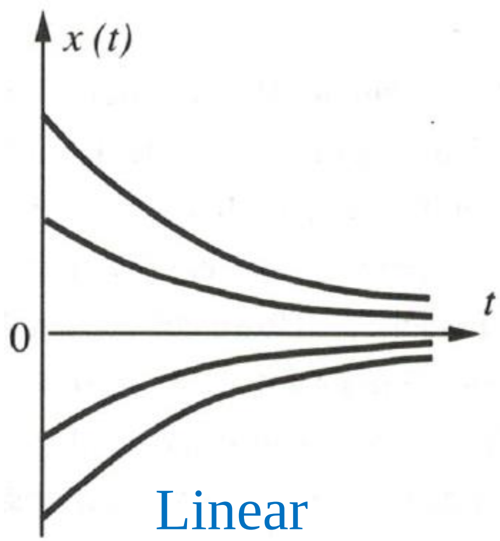
Response does not scale proportionally with the input

Nonlinear Phenomena (1)

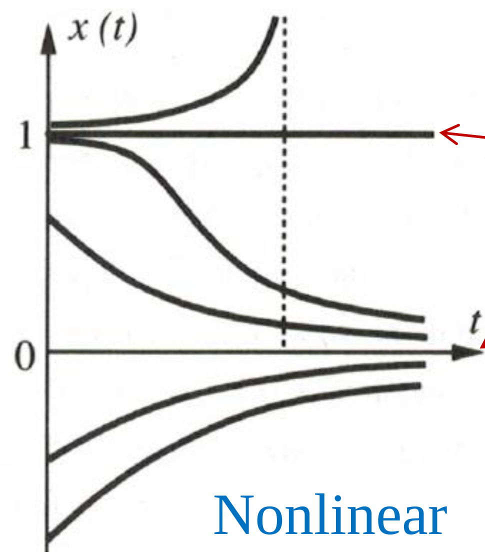
- Some common nonlinear phenomena:
  - Multiple isolated equilibria
    - System may settle at one of these equilibria depending on the initial condition

Example: $\dot{x} = -x + x^2$ ➡ Linearization: $\dot{x} = -x$

Response comparison:



Linear



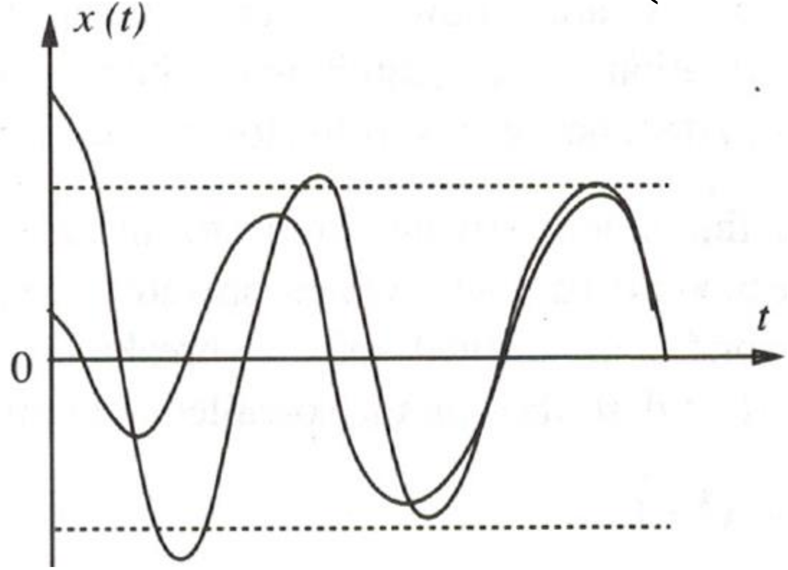
Nonlinear

Nonlinear Phenomena (2)

- Limit cycles
  - Sometimes called *self-excited oscillations*: oscillations with constant amplitude and frequency without external excitation

Famous example: Van der Pol equation

$$m\ddot{x} + 2c(x^2 - 1)\dot{x} + kx = 0$$

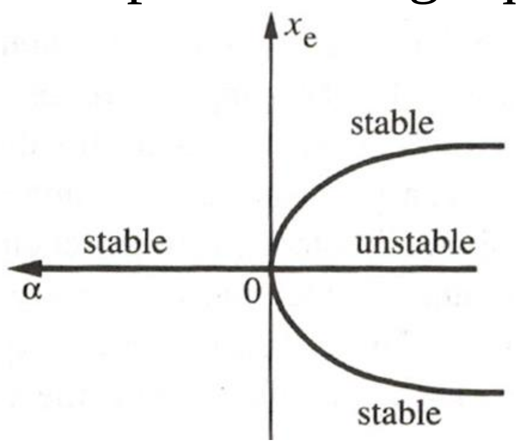


Amplitude of oscillations independent of initial conditions

Nonlinear Phenomena (3)

- Bifurcations
  - Change in system parameters results in change the number of equilibrium points and their stability characteristics

Example: Duffing equation $\ddot{x} + \alpha x + x^3 = 0$

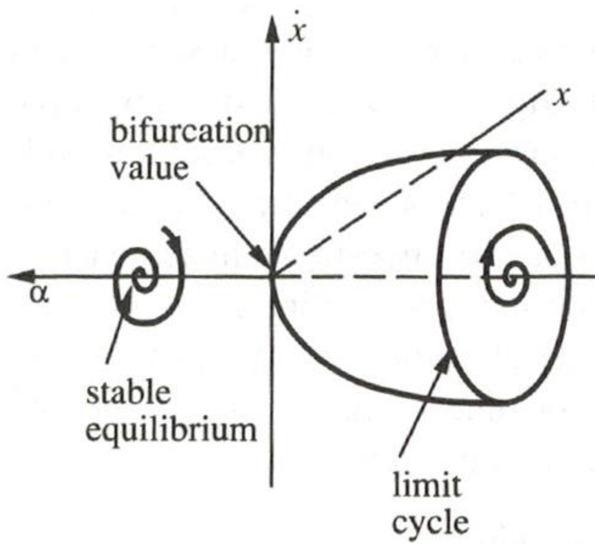


α = 0: *critical* or *bifurcation* value

➡ Pitchfork bifurcation

Another example: Hopf bifurcation

➡ Emergence/disappearance of limit cycles as parameter is varied across its bifurcation value



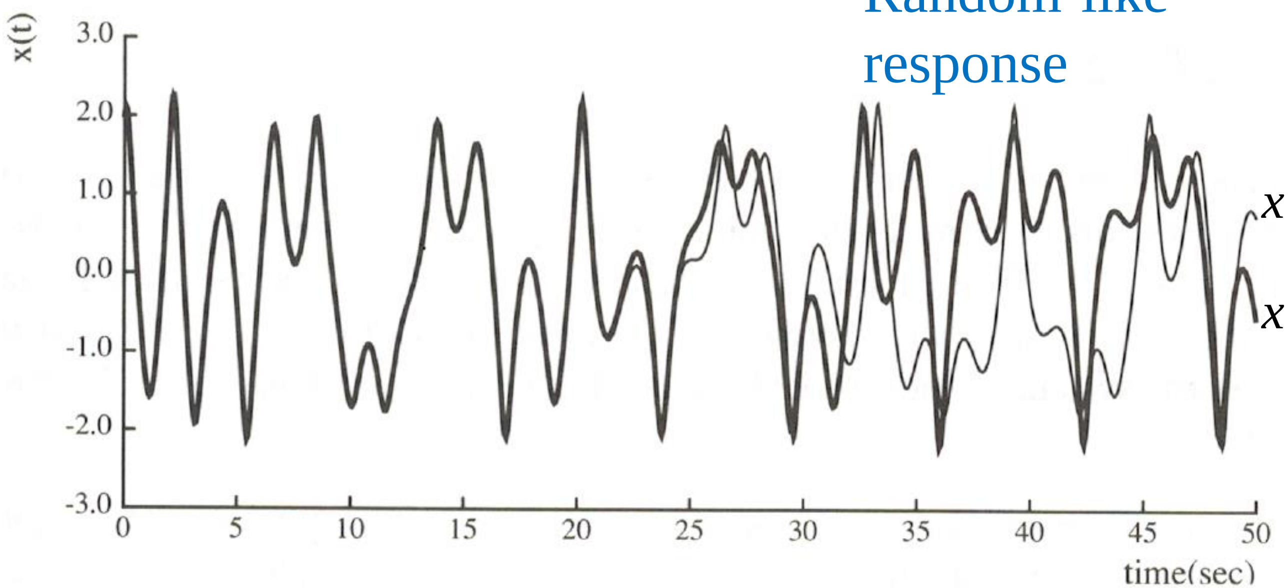
Nonlinear Phenomena (4)

- Chaos
  - Unpredictability of system response

Example: $\ddot{x} + 0.1\dot{x} + x^5 = 6\sin t$ (deterministic equation)

Response comparison:

Random-like response



$x(0) = 2.01, \dot{x}(0) = 3.01$

$x(0) = 2, \dot{x}(0) = 3$

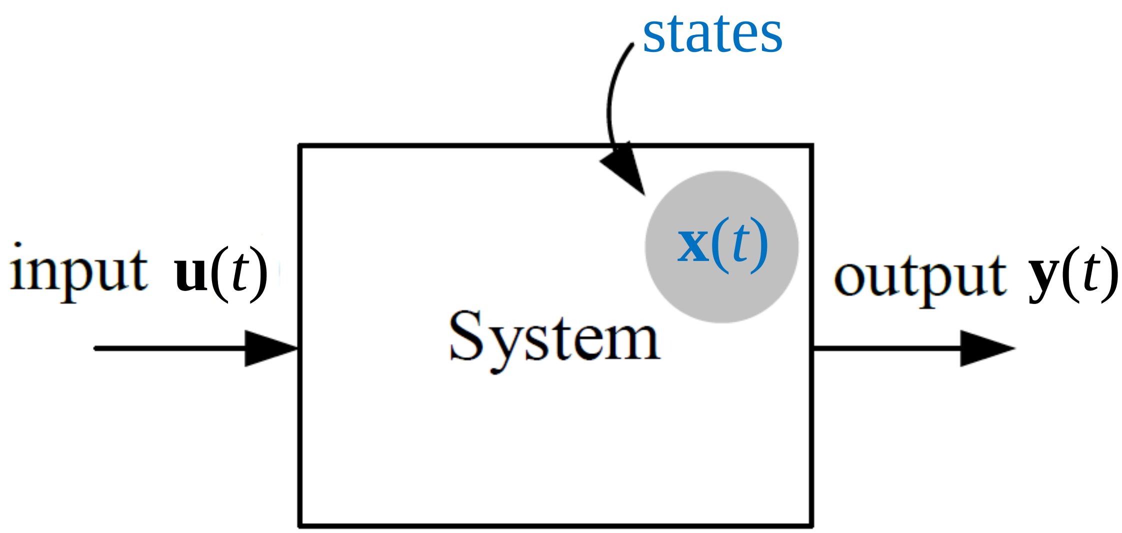
➡ Very sensitive to initial conditions

Nonlinear Phenomena (5)

- Other nonlinear phenomena:
  - *Jump resonance*: sudden jump in the amplitude and frequency of oscillations
  - *Finite escape time*: response goes to infinity in finite time
  - *Subharmonic or harmonic oscillations*: Under periodic excitation, nonlinear system response may oscillate with frequencies that are submultiples or multiples of input frequency

Nonlinear Dynamical System Model (1)

- Model used to express nonlinear dynamical system:



Note: output may comprise some system states or combination of states

- Scalar differential equations
  - Sometimes describe direct input-output relationship
- State-space models
  - Capture internal system dynamics (*state dynamics*)

Nonlinear Dynamical System Model (2)

- General state-space model:

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t)\\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t),\mathbf{u}(t),t)\end{aligned}}$$

➡ state dynamics

➡ output equation

$\mathbf{x}(t) \in \mathbb{R}^n$: state vector, describing the "state" of the system

$\mathbf{u}(t) \in \mathbb{R}^m$: input vector, containing input variables

$\mathbf{y}(t) \in \mathbb{R}^p$: output vector, containing output variables

$\mathbf{f}$ and $\mathbf{g}$ are vector-valued functions

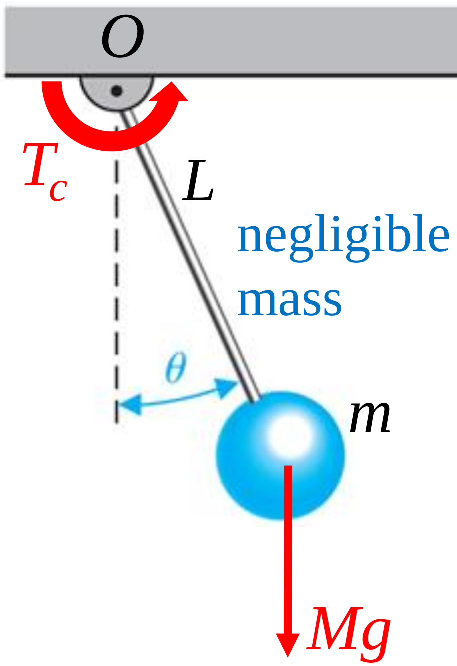
- Choice of state variables is not unique, but often desirable to choose *physically meaningful* state variables
  - Often associated with the energy storage elements in the system
    - For example mechanical-system state variables:

      positions ➡ potential energy

      velocities ➡ kinetic energy

Example: Simple Pendulum (1)

- Differential equation representation of simple pendulum motion:



$$\sum \mathbf{T} = J\boldsymbol{\alpha} \Rightarrow \sum T_O = J_O\,\alpha$$

$$\left.\begin{aligned} \sum T_O &= T_c - mgL\sin\theta \\ J_O &= mL^2 \\ \alpha &= \ddot{\theta} \end{aligned}\right\} mL^2\,\ddot{\theta} = T_c - mgL\sin\theta$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{T_c}{mL^2}}$$

$\Rightarrow$ nonlinear

Example: Simple Pendulum (2)

- State-space model of the simple pendulum motion with the displacement θ as the output of interest:

State variables: $x_1 = \theta$ ➡ potential energy

$x_2 = \dot{\theta}$ ➡ kinetic energy

Input variable: $u = T_c$ ➡ external torque

Output variable: $y = \theta$

State-space model:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(g/L)\sin x_1 + (1/mL^2)u \\ y &= x_1 \end{aligned}\right\} \quad \begin{aligned} &\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} x_2 \\ -(g/L)\sin x_1 + (1/mL^2)u \end{bmatrix} \\ &y = x_1 \end{aligned}$$

Special System Models

- System with no explicit presence of **u**:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

  - Either no external input or the input has been specified as a function of time and/or state variables
  - Two cases:
    - $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ ➡ Autonomous/time invariant
    - $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ ➡ Non-autonomous/time varying
- In control, system model is often expressed as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

  - $\mathbf{u} = \mathbf{u}(t)$ ➡ Non-autonomous/time varying
  - $\mathbf{u} = \mathbf{u}(\mathbf{x})$ ➡ Autonomous feedback control
    - typically in stabilization
  - $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ ➡ Non-autonomous feedback control
    - typically in tracking

Nonlinear Control Problems (1)

- *Stabilization problems*: to stabilize the states of the system around an equilibrium condition
  - *Asymptotic stabilization problem*:

    Given $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ with equilibrium $\mathbf{x} = \mathbf{0}$,
    find $\mathbf{u}$ such that starting from anywhere in a region Ω,
    $\mathbf{x} \to \mathbf{0}$ as $t \to \infty$
- *Tracking problems*: to track certain output trajectory
  - *Asymptotic tracking problem*:

    Given $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$, $\mathbf{y} = \mathbf{g}(\mathbf{x})$ and desired output trajectory $\mathbf{y}_d$,
    find $\mathbf{u}$ such that starting from anywhere in a region Ω,
    $\mathbf{y} \to \mathbf{y}_d$ as $t \to \infty$, while $\mathbf{x}$ remains bounded

Nonlinear Control Problems (2)

- Relations between stabilization and tracking problems:
  - Tracking problems can often be treated as stabilization problems

    $$\mathbf{e} = \mathbf{y} - \mathbf{y}_d \quad \Rightarrow \quad \dot{\mathbf{e}} = \mathbf{h}(\mathbf{e}, \mathbf{u}, t)$$

    Tracking problem: find $\mathbf{u}$ to make $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$

    ⇒ stabilization problem
  - Stabilization problem can often be regarded as special case of tracking problems ⇒ $\mathbf{y}_d$ = constant

Evaluation of Control Characteristics

- Desired control behaviors for nonlinear systems need to be examined in the operating region of interest
- Relevant control characteristics:
  - *Stability*: guarantee of stability in local or global sense, region of stability, convergence
  - *Accuracy and speed of response*: tolerable accuracy and consistent tracking for some typical motions in the region of operation
  - *Robustness*: degree of sensitivity to effects not considered in the nominal design, e.g. disturbances, measurement noise, unmodeled dynamics, etc.
  - *Cost*: requirement on number and type of actuators, sensors, and controller complexity

Nonlinear Control Methods

- *Trial-and-error*: apply linear compensation techniques (lead and/or lag) based on knowledge of system behaviors
- *Gain-scheduling*: schedule gains (based on operating conditions) from applying linear control methodologies to linearization of nonlinear system about several points
- *Compensate for nonlinearities*: nominal nonlinearities can be compensated e.g. feedback linearization
- *Robust control*: dominate nonlinearities by including model uncertainties, e.g. sliding control
- *Adaptive control*: allow controller to adapt to unknown or changing system parameters
- *Neural-network approach*: allow controller to predict the system behavior and compensate for it