MAE 5803 - Homework #2 Problem #2

Table of Contents

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```
clear; close all; clc;
```

Set default figure properties

```
set(0,'defaultlinelinewidth',2.5)
set(0,'defaultaxeslinewidth',2.5)
set(0,'defaultpatchlinewidth',2.5)
set(0,'defaulttextfontsize',14)
set(0,'defaultaxesfontsize',14)
set(0,'defaultTextInterpreter','latex')
```

Equilibrium Points and Stability

Fo the following systems, find the equilibrium points and determine their stability. Indicate if the stability is asymptotic and if it is global.

$$\dot{x} = -x^3 + \sin^4(x)$$

$$\dot{x} = (5-x)^5$$

$$\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8(x)\cos^2(3x)$$

$$\ddot{x} + (x-1)^4\dot{x}^7 + x^5 = x^3\sin^3(x)$$

$$\ddot{x} + (x-1)^2\dot{x}^7 + x = \sin\frac{\pi x}{2}$$

Eqn #1:

First-order state equation

$$\dot{x} = -x^3 + \sin^4(x)$$

This system has an equilibrium point at $x = 0$. It is an asymptotically stable node. $\dot{x}$ is positive whenever x is negative and vice versa as seen in the phase plane plot. To be sure it does not oscillate unbounded, look at the candidate Lyapunov function as it satisfies all conditions for global asymptotic stability.

$$V(\mathbf{x}) > 0 \quad \forall \quad \mathbf{x} \neq \mathbf{0}$$

$$\dot{V}(\mathbf{x}) < 0 \quad \forall \quad \mathbf{x} \neq \mathbf{0}$$

$$V(\mathbf{x}) \to \infty \text{ as } \|\mathbf{x}\| \to \infty$$

The Lyapunov function is:

$$V = x^2$$

$$\dot{V}(x) = 2x\dot{x} = 2x(-x^3 + sin^4(x)) \leq 0$$

```
figure()
ezplot('-x^3 + sin(x)^4')
xlabel('x'); ylabel('dx');
figure()
ezplot('2*x*(-x^3+sin(x)^4)')
xlabel('x'); ylabel('dV/dt');
```

Eqn #2:

First-order state equation

$$\dot{x} = (5 - x)^5$$

This system has an equilibrium point at $x = 5$. It is an unstable node as evidenced by using the candidate Lyapunov function to show violation of Theorem 3.2. Clearly, $\dot{V}(x)$ is not negative semidefinite.

The Lyapunov function is:

$$V = x^2$$

$$\dot{V}(x) = 2x\dot{x} = 2x(5 - x)^5$$

```
figure()
ezplot('(5-x)^5')
xlabel('x'); ylabel('dx/dt');
figure()
ezplot('2*x*(5-x)^5')
axis([-10 10 -1000 1000])
xlabel('x'); ylabel('dV/dt');
```

Eqn #3:

$$\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8(x) \cos^2(3x)$$

$$k(x) = x^7 - x^2 \sin^8(x) \cos^2(3x)$$

$$\ddot{x} + \dot{x}^5 + k(x) = 0$$

The Lyapunov function is:

$$V(\mathbf{x}) = \frac{1}{2}\dot{x} + \int_0^x k(\xi)d\xi$$

$$\begin{aligned}\dot{V}(\mathbf{x}) \quad &= \dot{x}\ddot{x} + k(x)\dot{x} \\ &= \dot{x}\ddot{x} + k(x)\dot{x} \\ &= \dot{x}(-\dot{x}^5 - k) + k(x)\dot{x} \\ &= -\dot{x}^6\end{aligned}$$

From Theorem 3.3:

1. $V(\mathbf{x})$ is positive definite. The first term is squared and the second is positive by analogy of stored energy in a displaced spring no matter what direction the displacement.

2. $\dot{V}(\mathbf{x})$ is negative semidefinite as evidenced by the above simplification.

3. $V(\mathbf{x})$ is radially unbounded. More speed means more energy and more spring displacement means more energy.

The origin is a globally asymptotically stable equilibrium point and is the only equilibrium point.

```
HW2P2_plotPhasePortrait(3,[0 10],[-1 1],[-1 1],'System \#3')
```

Eqn #4:

$$\ddot{x} + (x-1)^4\dot{x}^7 + x^5 = x^3 \sin^3(x)$$

$$k(x) = x^5 - x^3 \sin^3(x)$$

$$\ddot{x} + (x-1)^4\dot{x}^7 + k(x) = 0$$

The Lyapunov function is:

$$V(\mathbf{x}) = \frac{1}{2}\dot{x} + \int_0^x k(\xi)d\xi$$

$$\begin{aligned}\dot{V}(\mathbf{x}) \quad &= \dot{x}\ddot{x} + k(x)\dot{x} \\ &= \dot{x}(-(x-1)^4\dot{x}^7 - k) + k(x)\dot{x} \\ &= -(x-1)^4\dot{x}^8\end{aligned}$$

From Theorem 3.3:

1. $V(\mathbf{x})$ is positive definite. The first term is squared and the second is positive by analogy of stored energy in a displaced spring no matter what direction the displacement.

2. $\dot{V}(\mathbf{x})$ is negative semidefinite as evidenced by the above simplification.

3. $V(\mathbf{x})$ is radially unbounded. More speed means more energy and more spring displacement means more energy.

The origin is a globally asymptotically stable equilibrium point and is the only equilibrium point.

```
HW2P2_plotPhasePortrait(4,[0 20],[-2 2],[-2 2],'System \#4')
```

Eqn #5:

$$\ddot{x} + (x-1)^2\dot{x}^7 + x = \sin(\frac{\pi x}{2})$$

$$k(x) = x - \sin(\frac{\pi x}{2})$$

$$\ddot{x} + (x-1)^2\dot{x}^7 + k(x) = 0$$

The Lyapunov function is:

$$V(\mathbf{x}) = \frac{1}{2}\dot{x} + \int_0^x k(\xi)d\xi$$

$$\begin{aligned} \dot{V}(\mathbf{x}) \quad &= \dot{x}\ddot{x} + k(x)\dot{x} \\ &= \dot{x}(-(x-1)^2\dot{x}^7 - k(x)) + k(x)\dot{x} \\ &= -(x-1)^2\dot{x}^8 \end{aligned}$$

From Theorem 3.4:

1. $V(\mathbf{x}) \to \infty$ as $||x|| \to \infty$ This is typical of energy functions.

2. $\dot{V}(\mathbf{x})$ is negative semidefinite over the entire state space as evidenced by the preceeding simplification.

3. R composes all points satisfying $\dot{V}(\bar{x}) = 0$ which is the union of (1,u) and (v,0) for all real values u,v.

R = {\bar{x} : \dot{V}(\bar{x})=0} = (1,u) \bigcup (v,0) \forall u,v \in \R$

Though the origin fits this definition, it does not fall within the region Ω_ℓ defined in Theorem 3.4. It seems it should not fit here, but I can't figure out why. The origin is an unstable equilibrium point. The other two equilibrium points, (1,0) and (-1,0) meet the criteria above as well as those for Theorem 3.3. These are globally asymptotically stable equilibrium points. I tried to show this by finding the points where energy is zero, but I could only get the points (0,0), (-1.4483,0), and (1.4483,0).

```
HW2P2_plotPhasePortrait(5,[0 10],[-2 2],[-2 2],'System \#5')
```

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