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MAE 5803
NONLINEAR CONTROL SYSTEMS

Nonlinear Systems Analysis and Stability

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Stability Concepts for Autonomous Systems

Equilibrium Points

- *Definition*: **x*** is an equilibrium point (or state) of system if once $\mathbf{x}(t) = \mathbf{x}^*$, it remains equal to **x*** for all future time
- Mathematically: for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

  $$\Rightarrow \text{ equilibrium point: } \dot{\mathbf{x}} = \mathbf{0} \Rightarrow \mathbf{f}(\mathbf{x}^*) = \mathbf{0}$$

- Nonlinear system may have several (or infinitely many) isolated equilibrium points
  - *Note*: in linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with **A** nonsingular, there is only a single isolated equilibrium point at $\mathbf{x} = \mathbf{0}$
- Transformation can often be done so that the origin ($\mathbf{x} = \mathbf{0}$) becomes one of the equilibrium points of interest

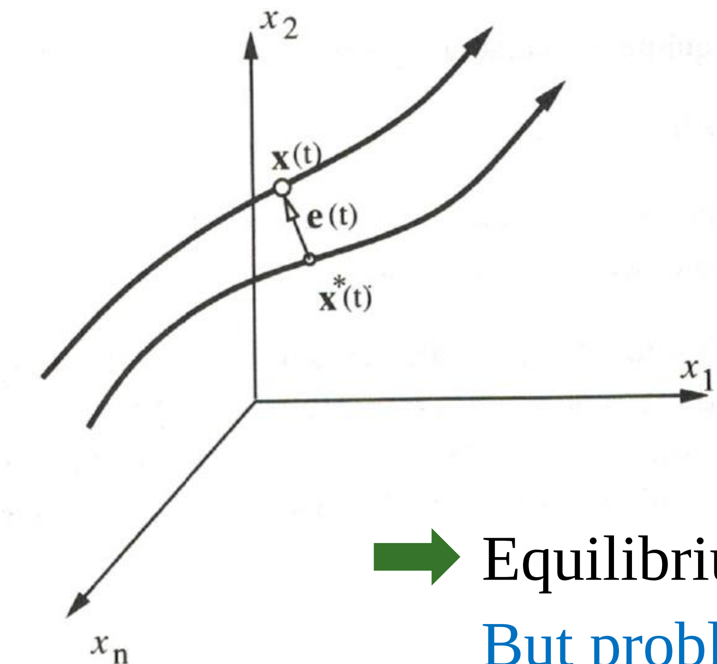
$$\mathbf{y} = \mathbf{x} - \mathbf{x}^* \Rightarrow \mathbf{x} = \mathbf{y} + \mathbf{x}^* \quad \dot{\mathbf{y}} = \dot{\mathbf{x}}$$

$$\Rightarrow \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y} + \mathbf{x}^*)$$

Equilibrium point: $\mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{x} = \mathbf{x}^*$

Nominal Motion

- *Nominal motion*: solution $\mathbf{x}^*(t)$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ corresponding to initial condition $\mathbf{x}^*(0) = \mathbf{x}_0$
  - In practical problems, this solution often represents nominal or reference motion trajectory
  - Concept of nominal motion can be made equivalent to equilibrium point by looking at error variation about $\mathbf{x}^*(t)$



$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$$

$$\dot{\mathbf{x}}^* = \mathbf{f}(\mathbf{x}^*) \quad ; \quad \mathbf{x}^*(0) = \mathbf{x}_0$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad ; \quad \mathbf{x}(0) = \mathbf{x}_0 + \delta\mathbf{x}_0$$

$$\Rightarrow \quad \dot{\mathbf{e}} = \mathbf{f}(\mathbf{x}^* + \mathbf{e}, t) - \mathbf{f}(\mathbf{x}^*, t)$$

$$\Rightarrow \quad \dot{\mathbf{e}} = \mathbf{g}(\mathbf{e}, t) \quad ; \quad \mathbf{e}(0) = \delta\mathbf{x}_0$$

$\Rightarrow$ Equilibrium point: $\mathbf{e} = \mathbf{0}$ $\Leftrightarrow$ $\mathbf{x}(t) = \mathbf{x}^*(t)$

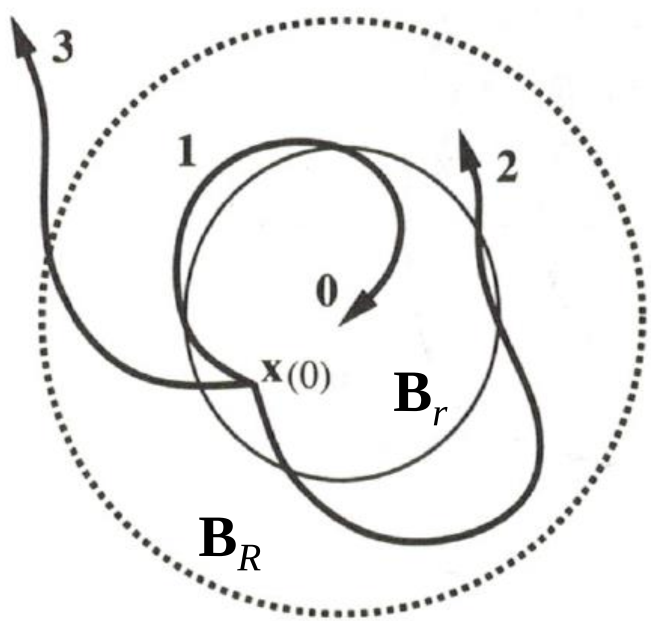
But problem becomes non-autonomous

Stability Concepts (1)

- *Definition*: The equilibrium $\mathbf{x} = \mathbf{0}$ is (Lyapunov) stable if

$$\forall R > 0,\ \exists r > 0, \underbrace{\| \mathbf{x}(0) \| < r}_{\mathbf{x}(0) \in \mathbf{B}_r} \quad \Rightarrow \quad \forall t \geq 0,\ \underbrace{\| \mathbf{x}(t) \| < R}_{\mathbf{x}(t) \in \mathbf{B}_R}$$

- *Definition*: The equilibrium $\mathbf{x} = \mathbf{0}$ is asymptotically stable if it is stable and $\exists r > 0, \| \mathbf{x}(0) \| < r \quad \Rightarrow \quad \mathbf{x}(t) \to \mathbf{0},\ t \to \infty$



1 – asymptotically stable
2 – marginally stable (Lyapunov stable but not asymptotically stable)
3 – unstable

For asymptotic stability, $\mathbf{B}_r$ is *domain of attraction* of the equilibrium point

Stability Concepts (2)

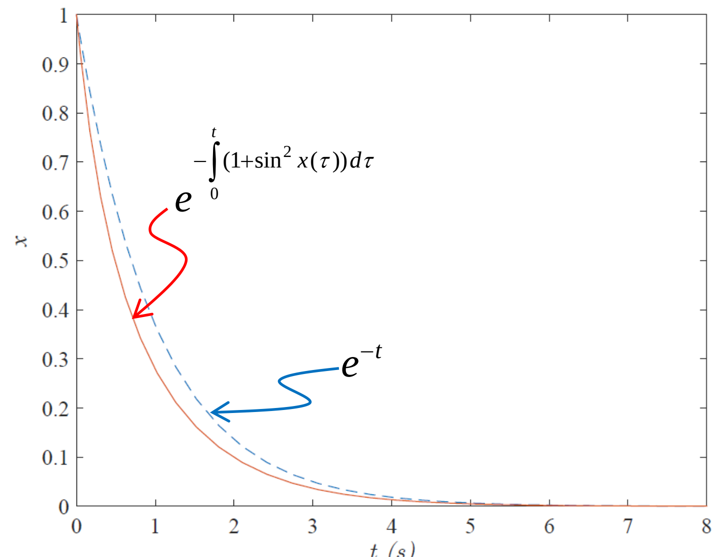
- *Definition*: The equilibrium $\mathbf{x} = \mathbf{0}$ is exponentially stable if inside $\mathbf{B}_r$, $\exists \alpha > 0, \exists \lambda > 0 \quad \Rightarrow \quad \forall t \geq 0, \|\mathbf{x}(t)\| \leq \alpha \|\mathbf{x}(0)\| e^{-\lambda t}$

  ➡ λ is similar to *time constant* in linear system, indicating rate of exponential convergence

Example: $\dot{x} = -(1+\sin^2 x)x$

Solution: $x = x(0)e^{-\int_0^t (1+\sin^2 x(\tau))d\tau}$

➡ $|x(t)| \leq |x(0)| e^{-t}$ ➡ exponentially stable with a rate of at least 1



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Stability Concepts (3)

- Exponential stability implies asymptotic stability, but asymptotic stability does not imply exponential stability
- In the previous asymptotic or exponential stability, if it holds for any initial states ($r \to \infty$ or $\mathbf{B}_r$ has infinite radius)

  ➡ asymptotic or exponential stability *in the large*, or *global asymptotic or exponential stability*
- Discussion on stability of system is only relevant if it involves global asymptotic or exponential stability
- Global asymptotic or exponential stability is only relevant if there is only one equilibrium point
- For linear systems: all these stability definitions collapse into one ➡ no differentiation needed

Lyapunov's Indirect Method (1)

- Local stability of nonlinear system can be determined based on its linearization about equilibrium point of interest
  - Nonlinear system should behave similarly to its linearized approximation in the small range of motions

  ➡ Lyapunov's indirect method or linearization method

- Linearization:

  Nonlinear autonomous system: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

  Taylor series expansion about $\mathbf{x} = \mathbf{0}$: $\dot{\mathbf{x}} = \left(\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} + h.o.t.(\mathbf{x})$

  Linearized approximation about $\mathbf{x} = \mathbf{0}$:

  $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ $\quad$ $\mathbf{A} = \left(\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}}$ ➡ Jacobian matrix of $\mathbf{f}$ at $\mathbf{x} = \mathbf{0}$, eigenvalues: $\lambda_i(\mathbf{A})$, $i = 1,\ldots,n$

Lyapunov's Indirect Method (2)

- In a system with control: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$

  Taylor series expansion about $\mathbf{x} = \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$:

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\substack{\mathbf{x}=\mathbf{0},\\ \mathbf{u}=\mathbf{0}}} \mathbf{x} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{\substack{\mathbf{x}=\mathbf{0},\\ \mathbf{u}=\mathbf{0}}} \mathbf{u} + h.o.t.(\mathbf{x}, \mathbf{u})$$

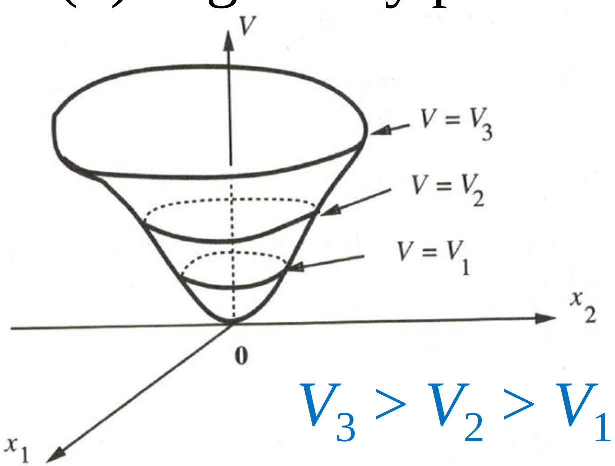
  Linearized approximation about $\mathbf{x} = \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{A} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\substack{\mathbf{x}=\mathbf{0},\\ \mathbf{u}=\mathbf{0}}} \qquad \mathbf{B} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{\substack{\mathbf{x}=\mathbf{0},\\ \mathbf{u}=\mathbf{0}}}$$

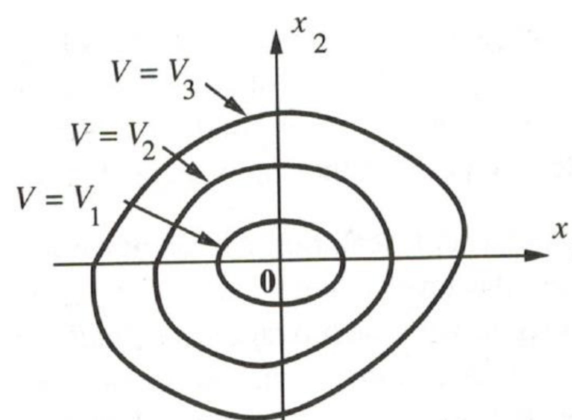
- Lyapunov's indirect method:
  - If $\mathrm{Re}\{\lambda_i(\mathbf{A})\} < 0$ for all i, $\mathbf{x} = \mathbf{0}$ is locally asymptotically stable
  - If $\mathrm{Re}\{\lambda_i(\mathbf{A})\} > 0$ for at least one i, $\mathbf{x} = \mathbf{0}$ is locally unstable
  - If $\mathrm{Re}\{\lambda_i(\mathbf{A})\} = 0$ for at least one i, the local stability of $\mathbf{x} = \mathbf{0}$ cannot be concluded

Lyapunov’s Direct Method (1)

- Lyapunov’s direct method is generalization of concept of *energy* of system
  - Basic procedure: formulation of *scalar energy-like function* for the system and evaluation of its *time variation*
    - System with energy dissipation: stable
    - System with energy growth: unstable
- *Definition*: Scalar continuous function $V(\mathbf{x})$ is *locally positive definite* if $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}, \|\mathbf{x}\| < R$
  - $V(\mathbf{x})$ is globally positive definite if $R \rightarrow \infty$



$V_3 > V_2 > V_1$



Lyapunov's Direct Method (2)

- For $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, time derivative of V:

$$\dot{V} = \frac{dV(\mathbf{x})}{dt} = \frac{\partial V}{\partial \mathbf{x}}\dot{\mathbf{x}} = \frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})$$

- *Local stability theorem*:

The equilibrium $\mathbf{x} = \mathbf{0}$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is *stable* if $\exists V(\mathbf{x})$ such that in $\|\mathbf{x}\| < R$:

  - $V(\mathbf{x})$ is positive definite
  - $\dot{V}(\mathbf{x})$ negative semi-definite for any solution $\mathbf{x}(t)$

- If in the theorem above $\dot{V}(\mathbf{x})$ is negative definite in $\|\mathbf{x}\| < R$, then equilibrium $\mathbf{x} = \mathbf{0}$ is *asymptotically stable*

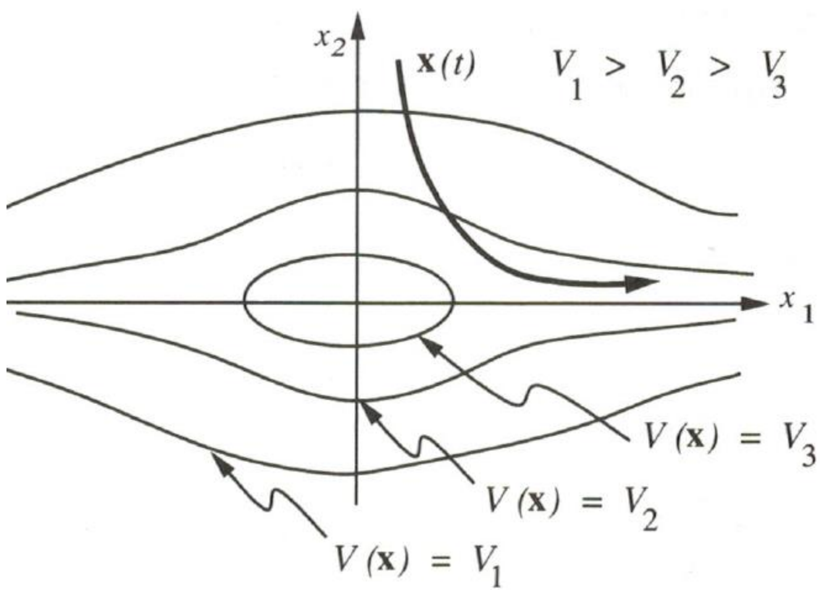
Lyapunov's Direct Method (3)

- *Global stability theorem*:

  The equilibrium $\mathbf{x} = \mathbf{0}$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is *globally asymptotically stable* if $\exists V(\mathbf{x})$ such that:

  - $V(\mathbf{x})$ is positive definite
  - $\dot{V}(\mathbf{x})$ is negative definite for any solution $\mathbf{x}(t)$
  - $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$ (radial unboundedness)

- Motivation for the radial unboundedness:



Radial unboundedness guarantees that V = *constant* corresponds to closed curves

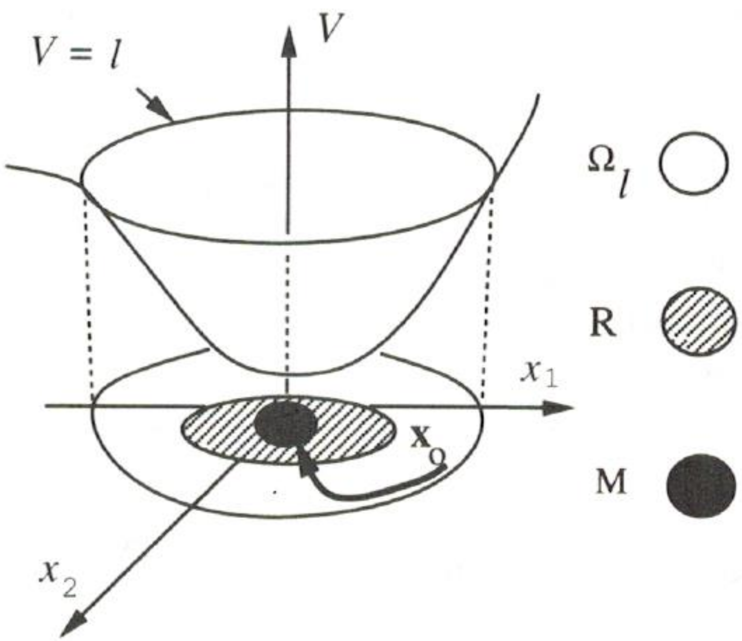
Invariant Set

- *Definition*: A set **G** is invariant if once in it, the trajectory stays in it
- Examples of invariant set:
  - Equilibrium point
  - Domain of attraction of equilibrium point
  - Any trajectory of autonomous system
  - Limit cycles
- Invariant set idea can often be used to describe convergence to dynamic behaviors other than equilibrium points, e.g. convergence to limit cycle

Local Invariant Set

- *Local invariant set theorem*: Let $V(\mathbf{x})$ be a scalar function of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with the following properties:
  - For some $l > 0$, the region $\mathbf{\Omega}_l$ defined by $V(\mathbf{x}) < l$ is bounded
  - $\dot{V}(\mathbf{x}) \leq 0$ in $\mathbf{\Omega}_l$

  Let $\mathbf{R} \subset \mathbf{\Omega}_l$ where $\dot{V}(\mathbf{x}) = 0$, and $\mathbf{M}$ be the largest invariant set in $\mathbf{R}$, then all trajectories $\mathbf{x}(t)$ starting in $\mathbf{\Omega}_l$ tends to $\mathbf{M}$



Note: Lyapunov local asymptotic stability theorem is a special case of local invariant set theorem, where $\mathbf{M}$ consists only of the origin

Global Invariant Set

- *Global invariant set theorem*: Let $V(\mathbf{x})$ be a scalar function of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with the following properties:
  - $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$
  - $\dot{V}(\mathbf{x}) \le 0$ over the whole state space

  Let **R**: $\dot{V}(\mathbf{x}) = 0$, and **M** be the largest invariant set in **R**, then all trajectories $\mathbf{x}(t)$ globally converge to **M**
- Note: Lyapunov global asymptotic stability theorem is a special case of global invariant set theorem, where **M** is the origin

Analysis Based on Lyapunov's Direct Method

- Lyapunov stability analysis is applicable to all systems: linear or nonlinear
  - Lyapunov functions can be considered as common language between linear and nonlinear systems
- Key in analysis based on Lyapunov's direct method: finding Lyapunov function
  - Key question: how to find Lyapunov function for a specific problem
  - No general way of finding Lyapunov functions for nonlinear systems
  - For linear systems, Lyapunov functions can be found systematically

Symmetric Matrices and Positive Definiteness

- Square matrix $\mathbf{M}$ is *symmetric* if $\mathbf{M}^T = \mathbf{M}$
- Square matrix $\mathbf{M}$ is *skew-symmetric* if $\mathbf{M}^T = -\mathbf{M}$

  ➡ $\forall \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x}^T \mathbf{M} \mathbf{x} = 0$

- For any square matrix $\mathbf{M}$:

$$\mathbf{M} = \underbrace{\frac{\mathbf{M} + \mathbf{M}^T}{2}}_{\text{symmetric}} + \underbrace{\frac{\mathbf{M} - \mathbf{M}^T}{2}}_{\text{skew-symmetric}}$$

➡ $$\forall \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x}^T \mathbf{M} \mathbf{x} = \mathbf{x}^T \underbrace{\left(\frac{\mathbf{M} + \mathbf{M}^T}{2} \right)}_{\text{symmetric}} \mathbf{x}$$

general

- $\mathbf{M}$ is positive definite ($\mathbf{M} > \mathbf{0}$) if $\forall \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x}^T \mathbf{M} \mathbf{x} > 0$
- $\mathbf{M}$ is positive semi-definite ($\mathbf{M} > \mathbf{0}$) if $\forall \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0$

  In considering positive-definiteness, without loss of generality, $\mathbf{M}$ can always be assumed symmetric

Lyapunov Functions for LTI Systems

- LTI system: $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$
- Candidate Lyapunov function: $V = \mathbf{x}^T\mathbf{P}\mathbf{x} \quad ; \quad \mathbf{P} = \mathbf{P}^T > \mathbf{0}$

$$\Rightarrow \dot{V} = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} = \mathbf{x}^T\underbrace{\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)}_{-\mathbf{Q}}\mathbf{x}$$

$\Rightarrow$ $\mathbf{Q} > \mathbf{0}$ for asymptotically stable system

- For determining the Lyapunov function:
  - Start with $\mathbf{Q} > \mathbf{0}$
  - Solve $\mathbf{P}$ from Lyapunov equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$
  - Check whether $\mathbf{P} > \mathbf{0}$

    If $\mathbf{P} > \mathbf{0}$, then the LTI system is globally asymptotically stable (necessary and sufficient condition)

Lyapunov Functions for Nonlinear Systems (1)

- There are mathematically-motivated techniques of construction Lyapunov functions, e.g.:
  - Krasovskii's method
  - Variable gradient method

  but their applicability to physical systems is often limited
- Elegant and powerful Lyapunov analysis, even for very complex systems, can often be done by properly exploiting system's physical properties and engineering insight

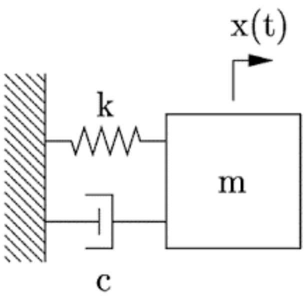
  ➡ Physically motivated Lyapunov functions
  - Concepts of energy is often useful for Lyapunov analysis
  - For mechanical systems: total mechanical energy (sum of kinetic and potential energy) is often a good Lyapunov function candidate

Lyapunov Functions for Nonlinear Systems (2)

- Example: 2nd-order scalar systems

  Analogy with linear mass-spring-damper systems is often used in Lyapunov analysis of 2nd-order nonlinear systems

  Linear mass-spring-damper system:



EOM: $m\ddot{x} + c\dot{x} + kx = 0$

Lyapunov function candidate based on total mechanical energy:

$$V = \underbrace{\tfrac{1}{2} m \dot{x}^2}_{\text{KE}} + \underbrace{\int_0^x k\,y\,dy}_{\text{PE}}$$

2nd-order nonlinear system: $m\ddot{x} + b(\dot{x}) + c(x) = 0$

Often-used Lyapunov function: $V = \tfrac{1}{2} m \dot{x}^2 + \int_0^x c(y)\,dy$

Lyapunov Functions for Nonlinear Systems (3)

- Example: A class of multivariable systems

Linear dynamical system: $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$

Energy-based Lyapunov function: $V = \underbrace{\tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}}_{\text{KE}} + \underbrace{\tfrac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q}}_{\text{PE}}$

In similar fashion, for nonlinear dynamical systems:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{K}(\mathbf{q}) = \mathbf{0}$$

Good Lyapunov function candidate:

$$V = \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \tfrac{1}{2}\mathbf{K}^T(\mathbf{q})\mathbf{K}(\mathbf{q})$$

Control Design Using Lyapunov’s Direct Method

- Two ways of using Lyapunov’s direct method for designing a stable control system:
  - Hypothesize a control law, then find a Lyapunov function to justify it
    - The control law is stabilizing if Lyapunov function can be found
  - Hypothesize a Lyapunov function candidate, then find a control law to make this candidate a real Lyapunov function
- Performance is not clearly addressed

Stability Concepts for Non-Autonomous Systems

Equilibrium Points and Invariant Sets

- Many of the stability concepts for non-autonomous systems are similar to those of the autonomous systems
  - Main difference: explicit dependence on initial time t_0

  ➡ Uniformity concept

- *Definition*: **x*** is an equilibrium point (or state) of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ if $\mathbf{f}(\mathbf{x}, t) \equiv \mathbf{0} \quad ; \quad \forall t \geq t_0$
- Definition of invariant sets is the same as autonomous systems
  - For non-autonomous systems: trajectory $\mathbf{x}(t)$ is not an invariant set

Extension of Previous Stability Concepts

- *Definition*: The equilibrium $\mathbf{x} = \mathbf{0}$ is (Lyapunov) stable at t_0 if $\forall R > 0,\ \exists r(R, t_0) > 0, \underbrace{\| \mathbf{x}(t_0) \| < r}_{\mathbf{x}(t_0) \in \mathbf{B}_r} \quad \Rightarrow \quad \forall t \geq t_0,\ \underbrace{\| \mathbf{x}(t) \| < R}_{\mathbf{x}(t) \in \mathbf{B}_R}$
- *Definition*: Equilibrium $\mathbf{x} = \mathbf{0}$ is asymptotically stable at t_0 if it is stable and $\exists r(t_0) > 0, \| \mathbf{x}(t_0) \| < r(t_0) \quad \Rightarrow \quad \mathbf{x}(t) \to \mathbf{0},\ t \to \infty$
  - Domain of attraction $\mathbf{B}_r$ is dependent on t_0
- *Definition*: The equilibrium $\mathbf{x} = \mathbf{0}$ is exponentially stable if inside $\mathbf{B}_r$, $\exists \alpha > 0, \exists \lambda > 0 \Rightarrow \forall t \geq t_0, \| \mathbf{x}(t) \| \leq \alpha \| \mathbf{x}(t_0) \| e^{-\lambda(t - t_0)}$
- *Definition*: Equilibrium $\mathbf{x} = \mathbf{0}$ is globally asymptotically stable if $\forall \mathbf{x}(t_0) \quad \Rightarrow \quad \mathbf{x}(t) \to \mathbf{0},\ t \to \infty$

Uniform Stability Concepts

- *Definition*: Equilibrium $\mathbf{x} = \mathbf{0}$ is locally *uniformly stable* if it is stable with r independent of t_0, i.e. $r = r(R)$
- *Definition*: Equilibrium $\mathbf{x} = \mathbf{0}$ is locally uniformly asymptotically stable if it is uniformly stable and the domain of attraction $\mathbf{B}_r$ is independent of t_0, such that

$$\forall \mathbf{x}(t_0) \subset \mathbf{B}_r \quad \Rightarrow \quad \mathbf{x}(t) \to \mathbf{0}, \; t \to \infty$$

  - If in the definition above $\mathbf{B}_r$ includes the whole state space, then the equilibrium is globally uniformly asymptotically stable

Uniform asymptotic stability ➡ asymptotic stability

Positive Definite and Decrescent Functions

- *Definition*: Scalar continuous function $V(\mathbf{x}, t)$ is *locally positive definite* if $V(\mathbf{0},t)=0$ and $\exists\ V_0(\mathbf{x})>0$ such that $\forall t \geq t_0,\ V(\mathbf{x},t) \geq V_0(\mathbf{x})$
  - *Time-variant function* is locally positive definite if it dominates a positive-definite *time-invariant function*
- *Definition*: Scalar continuous function $V(\mathbf{x}, t)$ is *decrescent* if $V(\mathbf{0},t)=0$ and $\exists\ V_l(\mathbf{x})>0$ such that $\forall t \geq t_0,\ V(\mathbf{x},t) \leq V_l(\mathbf{x})$
  - *Decrescent function* is dominated by a positive-definite *time-invariant function*

Lyapunov Theorems for Non-Autonomous Systems

- *Local stability theorem*:

  The equilibrium $\mathbf{x} = \mathbf{0}$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ is *stable* if $\exists V(\mathbf{x}, t)$ such that in $\| \mathbf{x} \| < R$:

  - $V(\mathbf{x}, t)$ is positive definite
  - $\dot{V}(\mathbf{x}, t)$ is negative semi-definite

- If in the definition above $\dot{V}(\mathbf{x}, t)$ is negative definite, $\mathbf{x} = \mathbf{0}$ is asymptotically stable
- If furthermore $V(\mathbf{x}, t)$ is decrescent, then $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable
- If furthermore $V(\mathbf{x}, t)$ is radially unbounded, then $\mathbf{x} = \mathbf{0}$ is globally uniformly asymptotically stable

Lyapunov Analysis for LTV Systems

- LTV system: $\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x}$
- Sufficient condition for stability:

  $\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x}$ is asymptotically stable if

  $\exists \lambda > 0,\ \forall i,\ \forall t \geq 0,\ \lambda_i(\mathbf{A}(t) + \mathbf{A}^T(t)) \leq -\lambda$

- Using Lyapunov function: $V = \mathbf{x}^T\mathbf{x}$

  ➡ $\dot{V} = \dot{\mathbf{x}}^T\mathbf{x} + \mathbf{x}^T\dot{\mathbf{x}} = \mathbf{x}^T(\mathbf{A}(t) + \mathbf{A}^T(t))\mathbf{x}$

  ➡ If $\lambda_i(\mathbf{A}(t) + \mathbf{A}^T(t)) \leq -\lambda$, then

  $$\dot{V} = \mathbf{x}^T(\mathbf{A}(t) + \mathbf{A}^T(t))\mathbf{x} \leq -\lambda\mathbf{x}^T\mathbf{x} = -\lambda V$$

  ➡ $\forall t \geq 0, \quad 0 \leq V(t) \leq V(0)e^{-\lambda t}$

  ➡ $\mathbf{x}(t) \rightarrow \mathbf{0}$ exponentially

Barbalat's Lemma

- Some facts about differentiable function *f*(*t*):
  - $\dot{f} \to 0 \quad \not\Rightarrow \quad f$ converges

    Example: $f(t) = \sin(\log t)$ ➡ $\dot{f}(t) = \dfrac{\cos(\log t)}{t} \to 0$ as $t \to \infty$

    continuously oscillating
  - f converges $\not\Rightarrow \dot{f} \to 0$

    Example: $f(t) = e^{-t}\sin(e^{2t}) \to 0$ ➡ $\dot{f}(t) \to \infty$ as $t \to \infty$
  - If *f* is lower bounded and decreasing ($\dot{f} \le 0$), then it is converges to a limit
    - $\dot{f}$ may not go to zero
- *Barbalat's lemma*: If $f \longrightarrow$ finite limit as $t \longrightarrow \infty$ and $\dot{f}$ is continuous ($\ddot{f}$ is bounded), then $\dot{f}(t) \to 0$ as $t \to \infty$

Lyapunov-Like Analysis Using Barbalat's Lemma

- *Lyapunov-like lemma*: If $V(\mathbf{x}, t)$ satisfies the following:
  - $V(\mathbf{x}, t)$ is lower bounded $(V(\mathbf{x}, t) \geq 0)$
  - $\dot{V}(\mathbf{x}, t) \leq 0$
  - $\dot{V}(\mathbf{x}, t)$ is uniformly continuous ($\ddot{V}(\mathbf{x}, t)$ is bounded)

  then $\dot{V}(\mathbf{x}, t) \to 0$ as $t \to \infty$
- The same challenge as in Lyapunov analysis: proper choice of V
- Application in control: choose $\mathbf{u}$ to shape V to make $\dot{V}$ useful
  - Variables to converge should be contained in $\dot{V}$