Florida Institute of Technology
High Tech with a Human Touch™

MAE 5803
NONLINEAR CONTROL SYSTEMS

Phase Plane Analysis

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About Phase Plane Analysis

- *Phase plane analysis*: graphical method for studying second-order systems
  - Basic idea: generation of motion trajectories to various initial conditions and plot them in state space
  - Useful for visualization of dynamics
  - Reveals stability and motion patterns
- Phase plane analysis is applicable to both linear and nonlinear systems

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

or $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

State space: space spanned by x_1 and x_2

➡ also called phase plane

- Graphical visualization is limited to second-order systems, although concept can be extended to higher-order systems

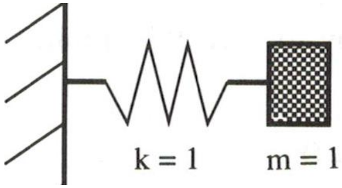
Phase Portraits

- *Phase-plane trajectory*: geometrical plot of the solution $\mathbf{x}(t)$
- *Singular points*: equilibrium points in phase plane

$$f_1(x_1, x_2) = 0 \quad f_2(x_1, x_2) = 0$$

- *Phase portrait*: family of phase-plane trajectories

Example:



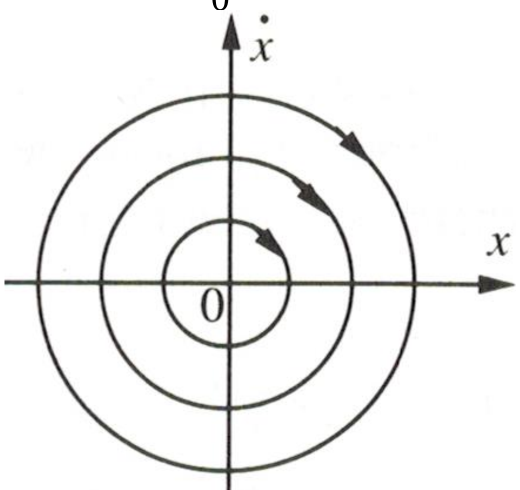
EOM: $\ddot{x} + x = 0$

Singular point: $(x, \dot{x}) = (0, 0)$

Solution: $x(t) = x_0 \cos t \Rightarrow \dot{x}(t) = -x_0 \sin t$

Eliminating t: $x^2 + \dot{x}^2 = x_0^2 \Rightarrow$ circle in phase plane

Phase portrait:



Phase Plane Analysis of Linear Systems

- General form of linear second-order system:

$$\dot{x}_1 = a x_1 + b x_2 \qquad \text{or} \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \text{with} \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\dot{x}_2 = c x_1 + d x_2$$

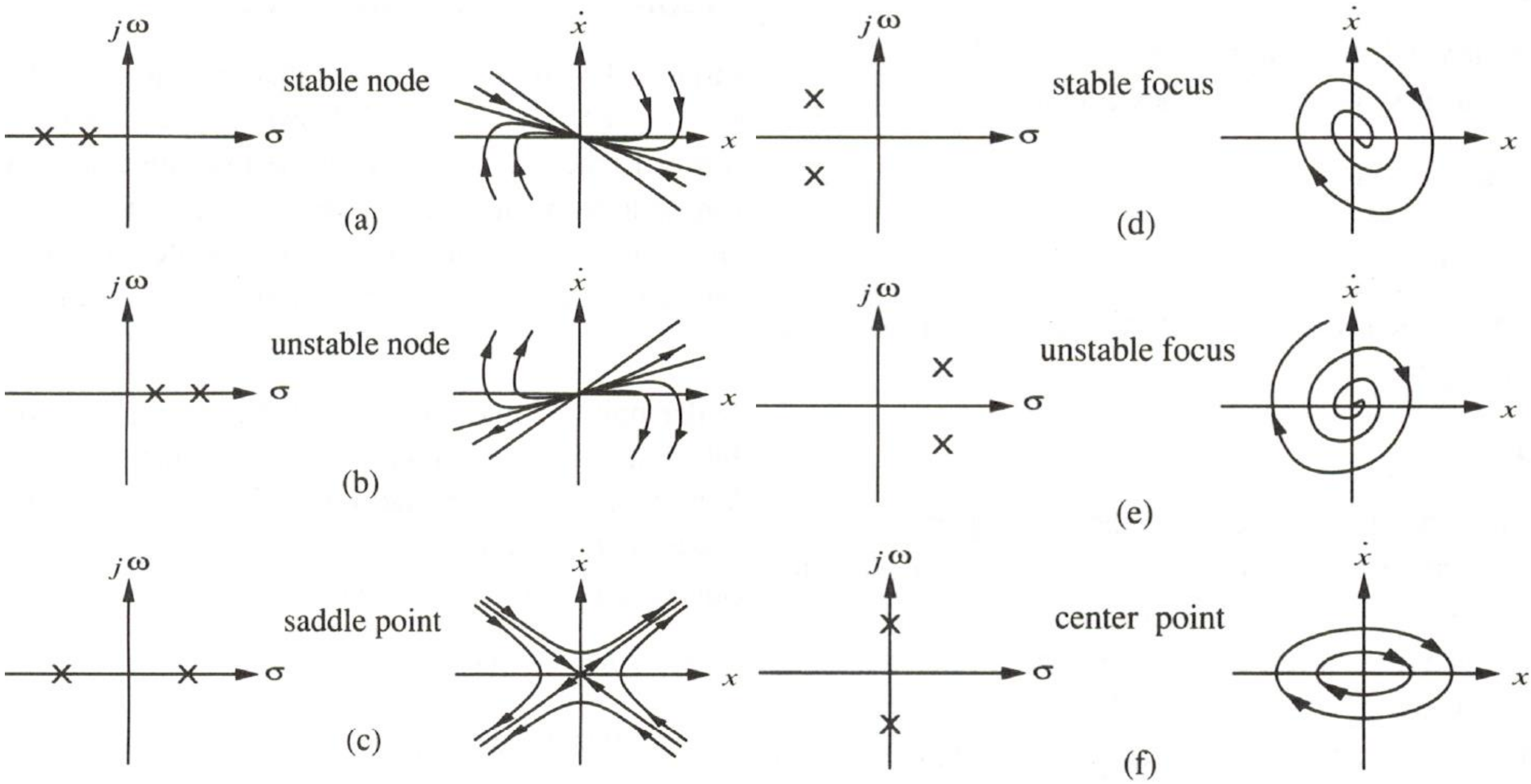
➡ Only one singular point: origin

➡ Eigenvalues/characteristic roots: λ_1 and λ_2

From linear system theory: system is stable if Re(λ_1) and Re(λ_2) are both negative

Phase-plane analysis can reveal dynamic characteristics of the system

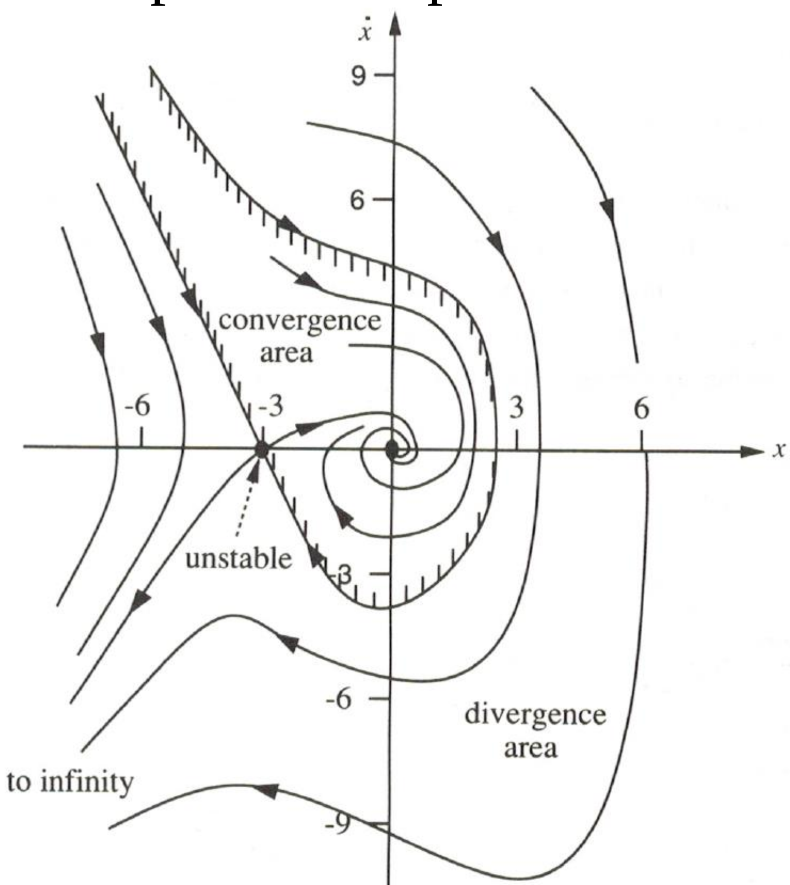
Phase Portraits of Linear Systems



Phase Plane Analysis of Nonlinear Systems

- Nonlinear system can display much more complicated phase portraits
  - There can be multiple singular points and limit cycles

Example: Phase portrait of $\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$



Phase portraits in the vicinity of the singular points similar to those of linear systems

Local Behavior or Nonlinear Systems

- Linearization about a singular point:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

$$\Rightarrow \quad \dot{x}_1 = a x_1 + b x_2 + g_1(x_1, x_2)$$
$$\dot{x}_2 = c x_1 + d x_2 + g_2(x_1, x_2)$$

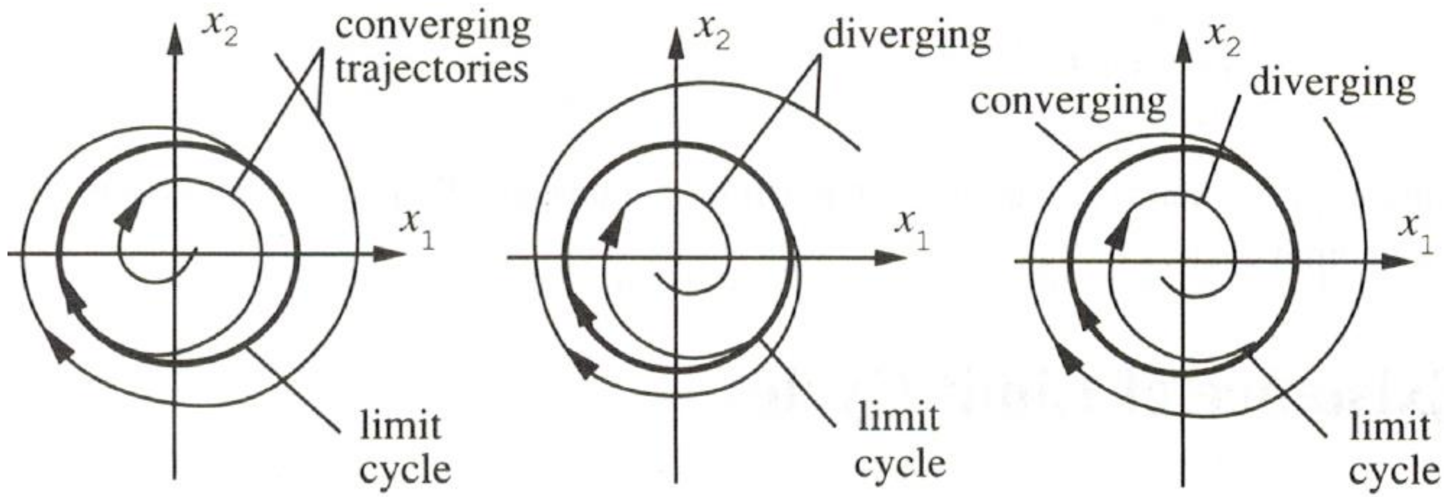
higher order terms, can be neglected near a singular point

$$\Rightarrow \quad \dot{x}_1 = a x_1 + b x_2$$
$$\dot{x}_2 = c x_1 + d x_2$$

$\Rightarrow$ Local behavior can be approximated by the linearized system's behavior

Limit Cycles

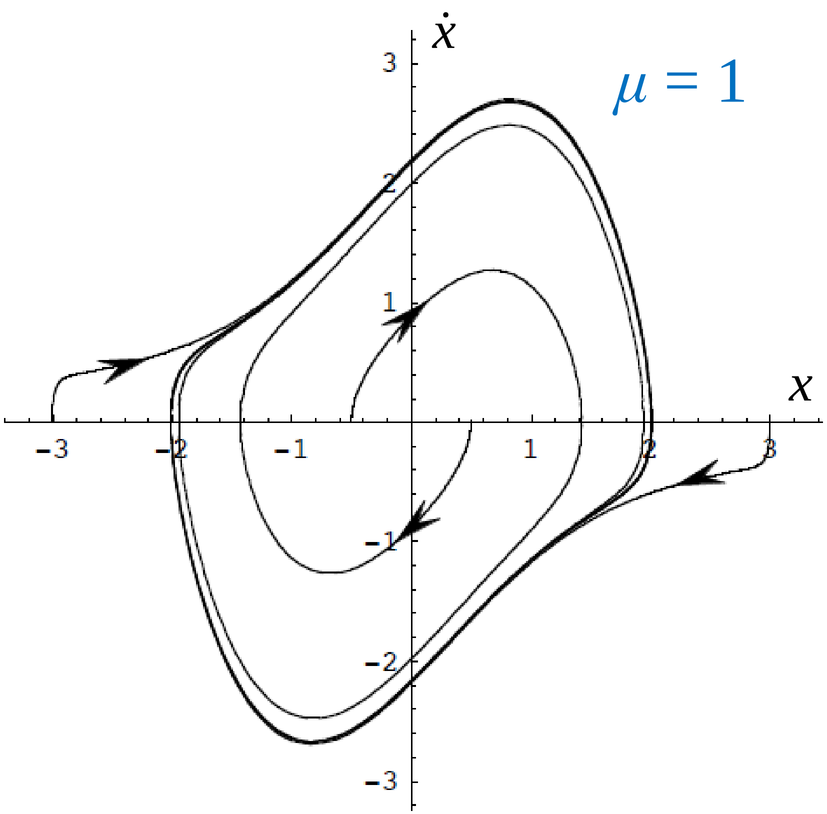
- *Limit cycle*: isolated closed curve in the phase plane
  - ➡ *closed curve* indicates periodic motion
  - ➡ *isolated* indicates limiting nature of the motion (trajectories in the vicinity of the limit cycle converge or diverge from it)
- Three kinds of limit cycles:

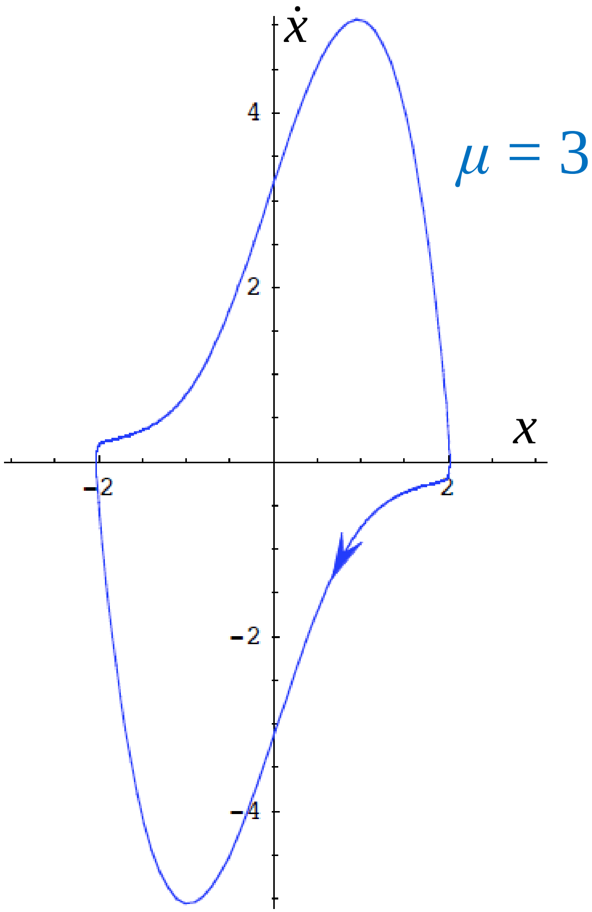


Stable Unstable Semi-stable

Example: Van der Pol Equation

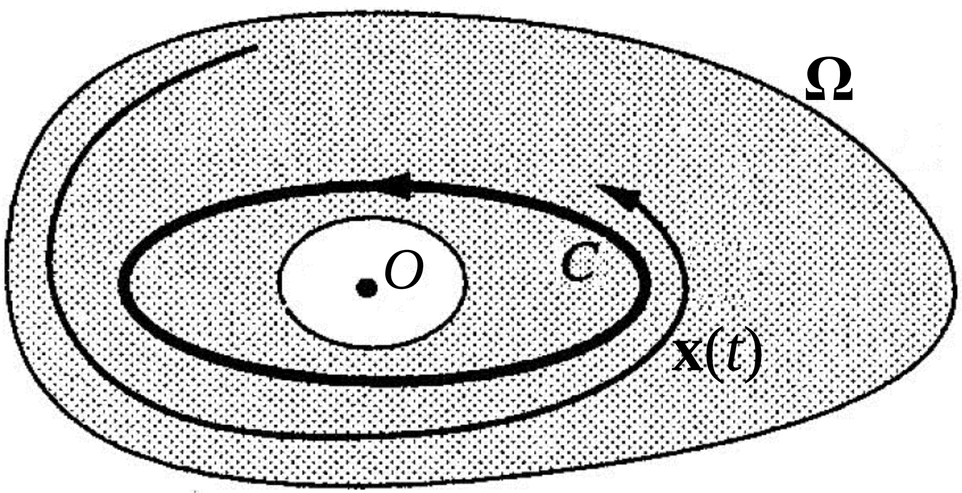
- Van der Pol equation: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$





Limit Cycle Theorem

- *Poincare-Bendixson theorem*: If a trajectory of a second-order autonomous system remains in a finite region $\mathbf{\Omega}$, then one of the following is true:
  - the trajectory goes to an equilibrium point
  - the trajectory tends to a stable limit cycle
  - the trajectory itself is a limit cycle



If $\mathbf{x}(t) \subset \mathbf{\Omega}$ for $t > t_0$:

- $\mathbf{x}(t) \to O$ as $t \to \infty$, or
- $\mathbf{x}(t) \to C$ (stable limit cycle)