MAE 5803 Nonlinear Control Systems **Assigned: Feb. 14, 2017**
Homework #3 (solutions) **Due: Feb 23, 2017**

1. *Courtesy of Nasir Hariri*

(a) Show that origin (0,0) is a unique equilibrium point for the system:

The 2nd order nonlinear equation of the system is:

$$\boldsymbol{A}_1\ddot{\boldsymbol{y}} + \boldsymbol{A}_2\dot{\boldsymbol{y}} + \boldsymbol{A}_3\boldsymbol{y} = \boldsymbol{0} \tag{1}$$

Singular point can be found by making $\dot{\boldsymbol{y}} = 0$, which leads to the following:

$$\boldsymbol{A}_3\boldsymbol{y} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{y} = \boldsymbol{0} \tag{2}$$

Thus, there is a single equilibrium point of the system at $\boldsymbol{y}_0 = \begin{Bmatrix} \boldsymbol{y} \\ \dot{\boldsymbol{y}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix}$

(b) Show that the system is globally asymptotically stable (G.A.S):

Define and test a candidate Lyapunov function for the 2nd order system:

$$V = \frac{1}{2}\dot{\boldsymbol{y}}^T\boldsymbol{A}_1\dot{\boldsymbol{y}} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{A}_3\boldsymbol{y} \tag{3}$$

Therefore:

$$\dot{V}(\boldsymbol{y}) = \frac{1}{2}\ddot{\boldsymbol{y}}^T\boldsymbol{A}_1\dot{\boldsymbol{y}} + \frac{1}{2}\dot{\boldsymbol{y}}^T\boldsymbol{A}_1\ddot{\boldsymbol{y}} + \frac{1}{2}\dot{\boldsymbol{y}}^T\boldsymbol{A}_3\boldsymbol{y} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{A}_3\dot{\boldsymbol{y}} \tag{4}$$

$$\dot{V}(\boldsymbol{y}) = \frac{1}{2}\ddot{\boldsymbol{y}}^T\boldsymbol{A}_1\dot{\boldsymbol{y}} + \frac{1}{2}\dot{\boldsymbol{y}}^T\boldsymbol{A}_1(-\boldsymbol{A}_2\boldsymbol{A}_1^{-1}\dot{\boldsymbol{y}} - \boldsymbol{A}_3\boldsymbol{A}_1^{-1}\boldsymbol{y}) + \frac{1}{2}\dot{\boldsymbol{y}}^T\boldsymbol{A}_3\boldsymbol{y} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{A}_3\dot{\boldsymbol{y}} \tag{5}$$

$$\dot{V}(\boldsymbol{y}) = \frac{1}{2}\ddot{\boldsymbol{y}}^T\boldsymbol{A}_1\dot{\boldsymbol{y}} - \frac{1}{2}\dot{\boldsymbol{y}}^T\boldsymbol{A}_2\dot{\boldsymbol{y}} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{A}_3\dot{\boldsymbol{y}} \tag{6}$$

$$\dot{V}(\boldsymbol{y}) = \frac{1}{2}(-\boldsymbol{A}_2\boldsymbol{A}_1^{-1}\dot{\boldsymbol{y}}^T - \boldsymbol{A}_3\boldsymbol{A}_1^{-1}\boldsymbol{y}^T)\,\boldsymbol{A}_1\dot{\boldsymbol{y}} - \frac{1}{2}\dot{\boldsymbol{y}}^T\boldsymbol{A}_2\dot{\boldsymbol{y}} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{A}_3\dot{\boldsymbol{y}} \tag{7}$$

$$\dot{V}(\boldsymbol{y}) = -\dot{\boldsymbol{y}}^T\boldsymbol{A}_2\dot{\boldsymbol{y}} \qquad \leq 0 \quad \text{“negative semi-definite”} \tag{8}$$

Since **A**$_j$ are all symmetric positive definite matrices, it can be seen that $\dot{V}(\boldsymbol{y})$ is a negative semi-definite function, while $V(\boldsymbol{y})$ is positive definite and radially unbounded; therefore, it can be stated that the system is locally stable about the equilibrium point (origin). Global stability can be investigated by applying the invariant set theorem:

Let: R: $\dot{V}(\boldsymbol{y}) = 0$ (9)

$$-\boldsymbol{A}_2\boldsymbol{A}_1^{-1}\,\dot{\boldsymbol{y}}^2 = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{y}} = \boldsymbol{0} \tag{10}$$

If: $\dot{\boldsymbol{y}} = \boldsymbol{0} \Rightarrow \ddot{\boldsymbol{y}} = \boldsymbol{0} \Rightarrow M = \{\boldsymbol{y} = \boldsymbol{0}\}$ “an invariant set: equilibrium point” (11)

By the invariant set theorem, all motion trajectories converge to the equilibrium point at the origin. As a result, the unique equilibrium point is globally asymptotically stable (G.A.S).

2. *Courtesy of Nasir Hariri*

The equation of the linear time invariant (LTI) system is:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \boldsymbol{x} \tag{12}$$

By considering a quadratic Lyapunov function candidate:

$$V = \boldsymbol{x}^T \boldsymbol{P}\, \boldsymbol{x} \tag{13}$$

Where **P** is a symmetric positive definite matrix. Thus:

$$\dot{V} = \dot{\boldsymbol{x}}^T \boldsymbol{P}\, \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{P}\, \dot{\boldsymbol{x}} = -\boldsymbol{x}^T\, \boldsymbol{Q}\, \boldsymbol{x} \tag{14}$$

The system is (G.A.S) if:

$$\boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q} \tag{15}$$

Let's start with the following value of **Q**:

$$\boldsymbol{Q} = \boldsymbol{I} \tag{16}$$

Thus:

$$\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{17}$$

Thus:

$$-2P_{11} = -1 \quad \Rightarrow \quad P_{11} = \frac{1}{2} \tag{18}$$

$$2P_{11} - 2P_{12} = 0 \quad \Rightarrow \quad P_{12} = \frac{1}{2} \tag{19}$$

$$4P_{12} - 2P_{22} = -1 \quad \Rightarrow \quad P_{22} = \frac{3}{2} \tag{20}$$

Therefore:

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \tag{21}$$

P is positive definite, confirmed by checking the eigenvalues of **P**:

$$\det[\lambda_i \boldsymbol{I} - \boldsymbol{P}] = 0 \tag{22}$$

$$\det\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}\right] = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + \frac{1}{2} = 0 \tag{23}$$

(24)

$$\lambda_1 = 0.2929$$
$$\lambda_2 = 1.7071 \tag{25}$$

The Lyapunov function can be written as follows:

$$V = \boldsymbol{x}^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \boldsymbol{x} \tag{26}$$

3. *Courtesy of Casey Clark*

Part a

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 2\sin t \\ 0 & -(t+1) \end{bmatrix} \mathbf{x}$$

Lyapunov function,

$$V = \mathbf{x}^{\mathbf{T}}\mathbf{x}$$

$$\dot{V} = \dot{\mathbf{x}}^{\mathbf{T}}\mathbf{x} + \mathbf{x}^{\mathbf{T}}\dot{\mathbf{x}}$$

$$\mathbf{A}(t) + \mathbf{A}^T(t) = \begin{bmatrix} -2 & -2\sin t \\ -2\sin t & -2(t+1) \end{bmatrix}$$

$$det(\lambda\mathbf{I} - (\mathbf{A}(t) + \mathbf{A}^T(t))) = \begin{vmatrix} \lambda + 2 & 2\sin t \\ 2\sin t & \lambda + 2(t+1) \end{vmatrix} = (\lambda+2)(\lambda+2t+2) - 4\sin^2 t = 0$$

$$0 = \lambda^2 + 2t\lambda + 2\lambda + 2\lambda + 4t + 4 - 4\sin^2 t = \lambda^2 + (2t+4)\lambda + 4t + 4 - 4\sin^2 t$$

$$0 = \lambda^2 + (2t+4)\lambda + 4t + 4(1 - \sin^2 t) = \lambda^2 + (2t+4)\lambda + 4t + 4\cos^2 t$$

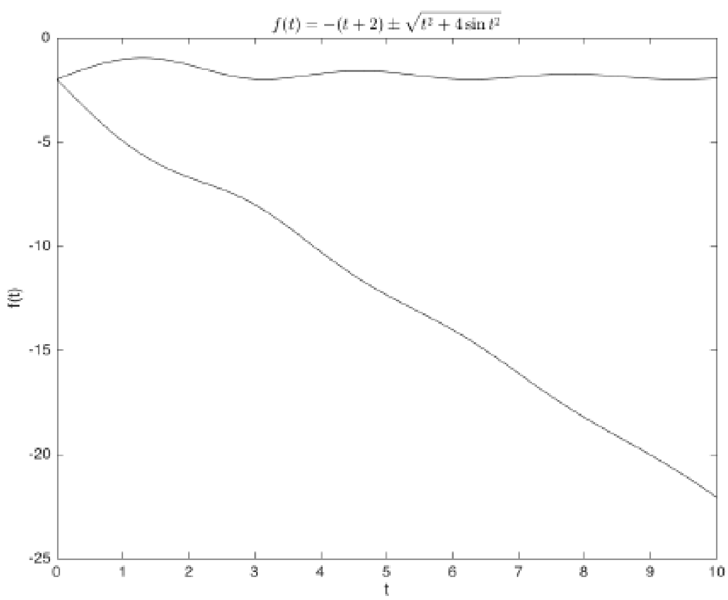
$$0 = \lambda^2 + (2t+4)\lambda + 4(t + \cos^2 t)$$

$$\lambda_{1,2} = \frac{-(2t+4) \pm \sqrt{(2t+4)^2 - 16(t + \cos^2 t)}}{2} = -(t+2) \pm \frac{\sqrt{4t^2 + 16t + 16 - 16t - 16\cos^2 t}}{2}$$

$$\lambda_{1,2} = -(t+2) \pm \frac{\sqrt{4(t^2 + 4 - 4\cos^2 t}}{2} = -(t+2) \pm \sqrt{t^2 + 4(1 - \cos^2 t)}$$

$$\lambda_{1,2} = -(t+2) \pm \sqrt{t^2 + 4\sin^2 t}$$

Figure 1: Eigenvalues of $A(t) + A^T(t)$



We can observe in Figure 1

$$\lambda_{1,2} \leq -0.9754$$

The LTV system is globally asymptotically stable as:

$$\text{V is positive definite}$$

$$\text{V is radially unbounded}$$

$$\dot{V} \text{ is negative definite}$$

$$\lambda_i(\mathbf{A}(t) + \mathbf{A}^T(t)) < -\lambda$$

Part b

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -2 \end{bmatrix} \mathbf{x}$$

$$\mathbf{A}(t) + \mathbf{A}^T(t) = \begin{bmatrix} -2 & e^{2t} \\ e^{2t} & -4 \end{bmatrix}$$

$$det(\lambda \mathbf{I} - (\mathbf{A}(t) + \mathbf{A}^T(t))) = \begin{vmatrix} \lambda + 2 & -e^{2t} \\ -e^{2t} & \lambda + 4 \end{vmatrix} = (\lambda + 2)(\lambda + 4) - e^{4t} = 0$$

$$0 = \lambda^2 + 6\lambda + 8 - e^{4t}$$

$$\lambda_{1,2} = \frac{-6 \pm \sqrt{36 - 4(8 - e^{4t})}}{2} = -3 \pm \frac{\sqrt{4(1 + e^{4t})}}{2}$$

$$\lambda_{1,2} = -3 \pm \sqrt{1 + e^{4t}}$$

$$3 \not> \sqrt{1 + e^{4t}} \rightarrow \lambda_{1,2} \not\leq -\lambda$$

Stability cannot be verified using this method.

Expand state-space

$$\dot{x_1} = -x_1 + e^{2t}x_2$$

$$\dot{x_2} = -2x_2$$

Apply integrating factor method for linear first order ODE of the form

$$\frac{dy}{dx} + a(x)y = b(x)$$

Yielding

$$x_2 = c_1 e^{-2t} \text{ (asymptotically stable)}$$

Substituting

$$\dot{x_1} = -x_1 + e^{2t}(c_1 e^{-2t})$$

$$\dot{x_1} = -x_1 + c_1$$

Applying integrating factor method again

$$x_1 = c_2 e^{-t} + c_1$$

$$x_1 \text{ converges, but not to } 0 \quad \forall \quad c_1 \neq 0 \text{ (stable, but not asymptotically)}$$

Although, x_1 is asymptotically stable and x_2 converges, x_2 is not asymptotically stable implying that the system is marginally or Lyapunov stable.

4. *Courtesy of Casey Clark*

There are no local maximum or minimum values of a function if it is solely decreasing or solely increasing.

Keeping this in mind when starting with

$$\forall \quad t \geq 0 \quad : \quad \dot{V}(t) = 0$$

When $\dot{V}(t) = 0$ there are 3 possibilities:

Local minimum

Local maximum

Inflection point

The only possibility for $\dot{V}(t) = 0$ is the presence of an inflection point as V(t) has no local min or max.

An inflection point only occurs when $\ddot{V}(t) = 0$

$$\dot{V}(t) = 0 \rightarrow \ddot{V}(t) = 0$$

5. *Courtesy of Casey Clark*

$$\ddot{x} - \omega_0^2 x \sin x = u$$

$$0 < \omega_0^2 \leq 2$$

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Part a

We know from Lyapunov's direct method that a dynamic equation in the forr

$$\ddot{x} + b(\dot{x}) + c(x) = 0$$

has a globally asymptotically stable equilibrium point at the origin if

$$\dot{x}b(\dot{x}) > 0 \quad ; \quad x \neq 0$$

$$xc(x) > 0 \quad ; \quad x \neq 0$$

Choose a control law

$$u = u_1(\dot{x}) + u_2(x)$$

Rearrange in form $\ddot{x} + b(\dot{x}) + c(x) = 0$

$$\ddot{x} - u_1(\dot{x}) - (\omega_0^2 x \sin x + u_2(x)) = 0$$

To come up with proper u_1 & u_2

$$\dot{x}(u_1(\dot{x})) < 0 \quad ; \quad \dot{x} \neq 0$$

$$x(\omega_0^2 x \sin x + u_2(x)) < 0 \quad ; \quad x \neq 0$$

Choose

$$u_1(\dot{x}) = -\dot{x}$$

$$u_2(x) = -2x|\sin x| - x$$

Yielding

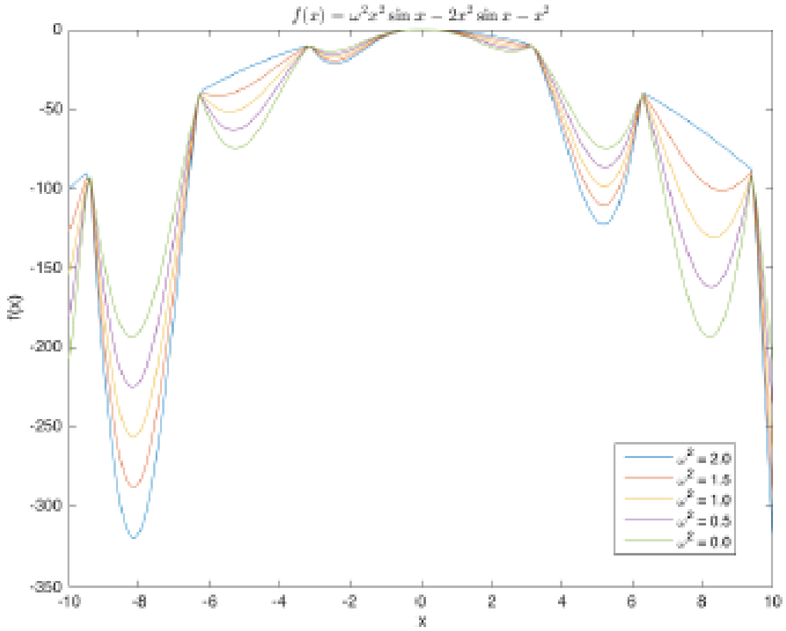
$$-\dot{x}^2 < 0 \quad ; \quad \dot{x} \neq 0 \text{ (Trivial)}$$

$$\omega_0^2 x^2 \sin x - 2x^2|\sin x| - x^2 < 0 \quad ; \quad x \neq 0 \text{ (shown in Figure 2)}$$

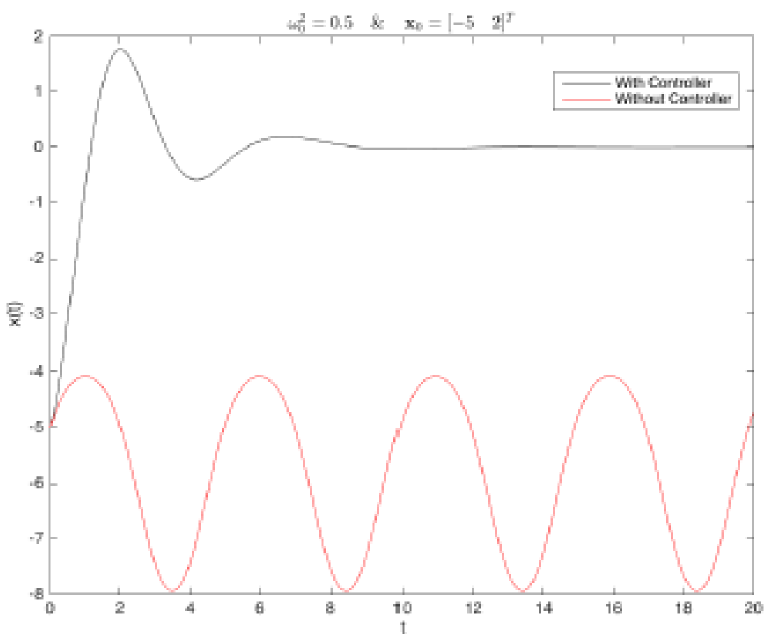
Use the following controller

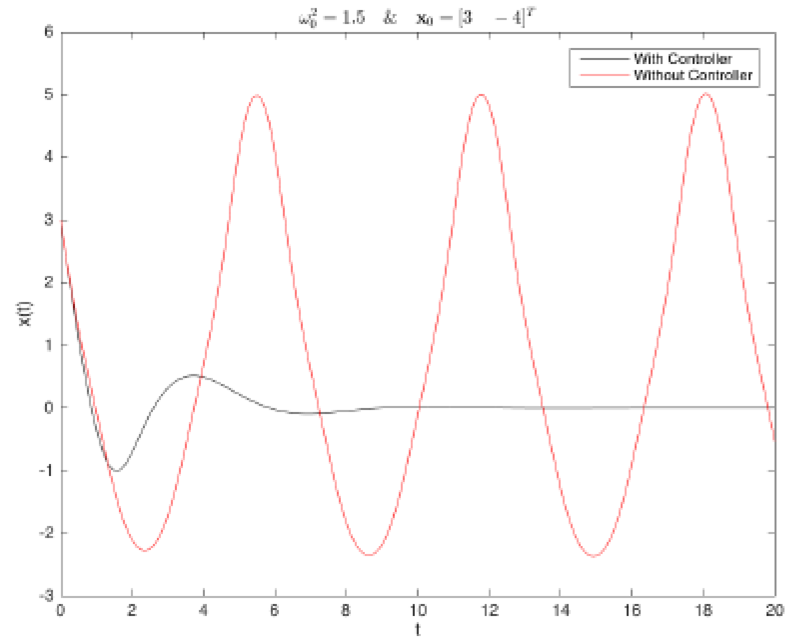
$$u = -\dot{x} - 2x|\sin x| - x$$

to make the equilibrium point at the origin globally asymptotically stable

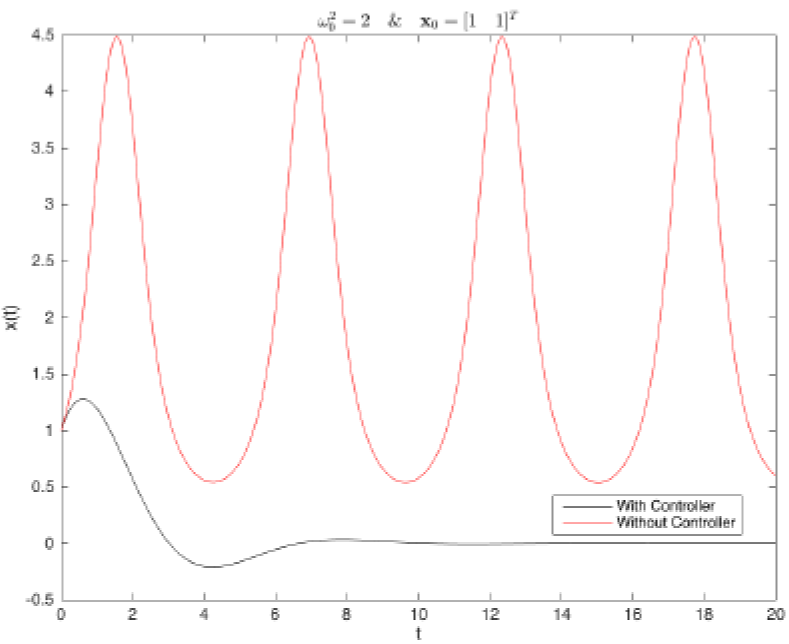

$f(x) = \omega^2 x^2 \sin x - 2x^2 \sin x - x^2$
f(x)
x
ω^2 = 2.0
ω^2 = 1.5
ω^2 = 1.0
ω^2 = 0.5
ω^2 = 0.0

Part b


$\omega_0^2 = 0.5 \quad \& \quad \mathbf{x}_0 = [-5 \quad 2]^T$
With Controller
Without Controller
x(t)
t



$\omega_0^2 = 1.5 \quad \& \quad \mathbf{x}_0 = [3 \quad -4]^T$
With Controller
Without Controller
x(t)
t



$\omega_0^2 = 2 \quad \& \quad \mathbf{x}_0 = [1 \quad 1]^T$
With Controller
Without Controller
x(t)
t