MAE 5803 Nonlinear Control Systems **Assigned: Feb 28, 2017**
Homework #4 (Solutions) **Due: Mar 21, 2017**

a. Courtesy of Casey Clark

A single-degree of freedom underwater vehicle motion under viscous drag can be modeled as:

$$m\ddot{x} + (\alpha_1 + \alpha_2 \cos^2 x)|\dot{x}|\,\dot{x} = u + d \tag{1}$$

with bounds

$$4 \le \alpha_1 \le 6$$
$$1 \le \alpha_2 \le 2$$
$$|d| \le 1$$

Part a

For $m = 1$ equation (1) simplifies

$$\ddot{x} = -(\alpha_1 + \alpha_2 \cos^2 x)|\dot{x}|\,\dot{x} + d + u$$

A switching controller can be designed for an equation in the form $\ddot{x} = f + u$ using the control law

$$u = \hat{u} - ksgn(s)$$
$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}}$$
$$k = F + \eta$$

where the sliding line is defined as

$$s = \dot{\tilde{x}} + \lambda\tilde{x}$$

with tracking error $\tilde{x}$

$$\tilde{x} = x - x_d$$
$$\dot{\tilde{x}} = \dot{x} - \dot{x}_d$$

F is bounded by

$$\left|f - \hat{f}\right| \le F$$

with

$$f = -(\alpha_1 + \alpha_2 \cos^2 x)|\dot{x}|\,\dot{x} + d$$

An appropriate selection of parameters for $\hat{f}$ can be made by substituting the mean of each constant into f

$$\alpha_1 = 4 + \frac{6-4}{2} = 5$$
$$\alpha_2 = 1 + \frac{2-1}{2} = 1.5$$
$$d = -1 + \frac{1-(-1)}{2} = 0$$

Yielding

$$\hat{f} = -(5 + 1.5\cos^2 x)|\dot{x}|\,\dot{x}$$

The values of the parameters for F can be chosen to be the difference between the max and mean bound

$$\alpha_1 = 6 - 5 = 1$$
$$\alpha_2 = 2 - 1.5 = 0.5$$
$$d = 1 - 0 = 1$$

Yielding

$$F = (1 + 0.5\cos^2 x)\dot{x}^2 + 1$$

Substituting F into k yields

$$k = (1 + 0.5\cos^2 x)\dot{x}^2 + 1 + \eta$$

Substituting $\hat{f}$ into $\hat{u}$

$$\hat{u} = (5 + 1.5\cos^2 x)|\dot{x}|\,\dot{x} + \ddot{x}_d - \lambda\dot{\tilde{x}}$$

Finally substitute $\hat{u}$ into u

$$u = (5 + 1.5\cos^2 x)|\dot{x}|\,\dot{x} + \ddot{x}_d - \lambda\dot{\tilde{x}} - ksgn(s)$$

Part b

The system response was simulated to track $x_d(t) = 2\sin t$ using the above controller and gain with

$$\alpha_1(t) = 5 + \cos t$$

$$\alpha_2(t) = 1 + |\sin t|$$

$$d(t) = \cos 1.3t$$

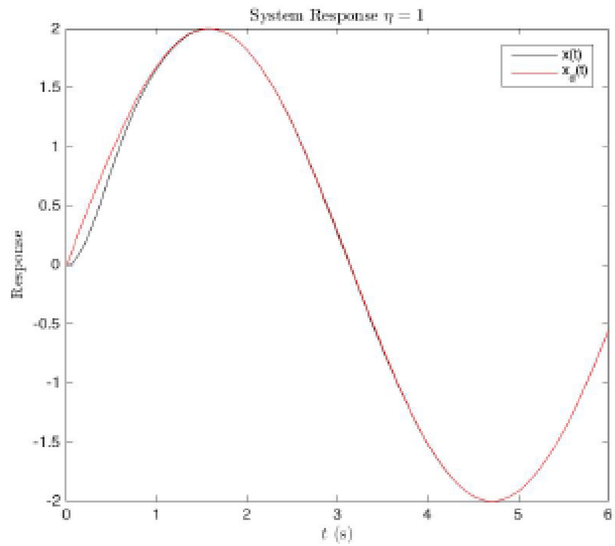
$$\lambda = 4$$

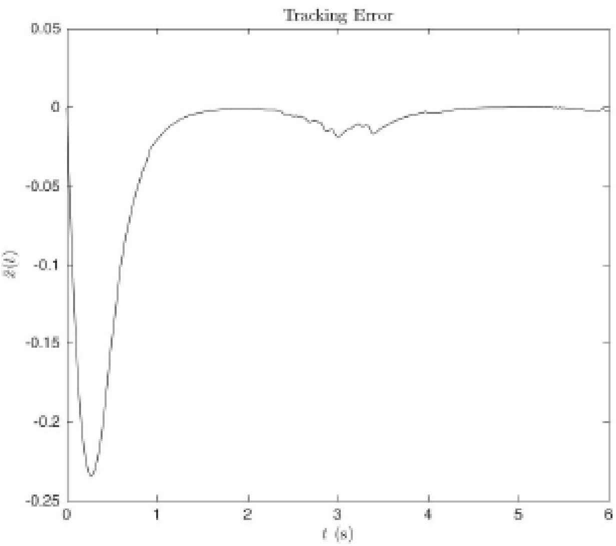
Note that

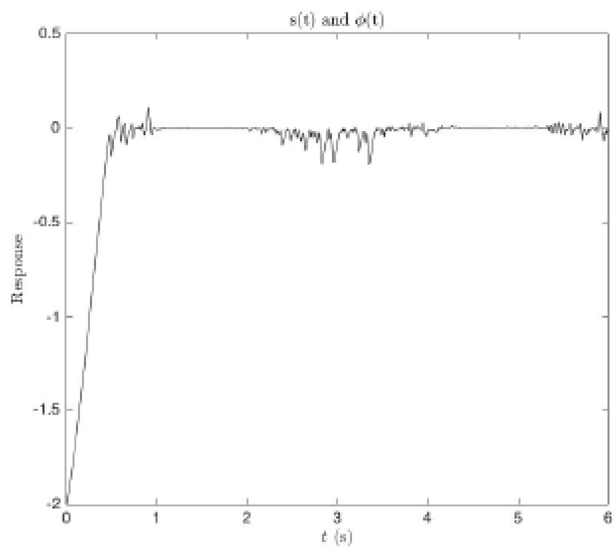
$$\dot{x}_d(t) = 2\cos t$$

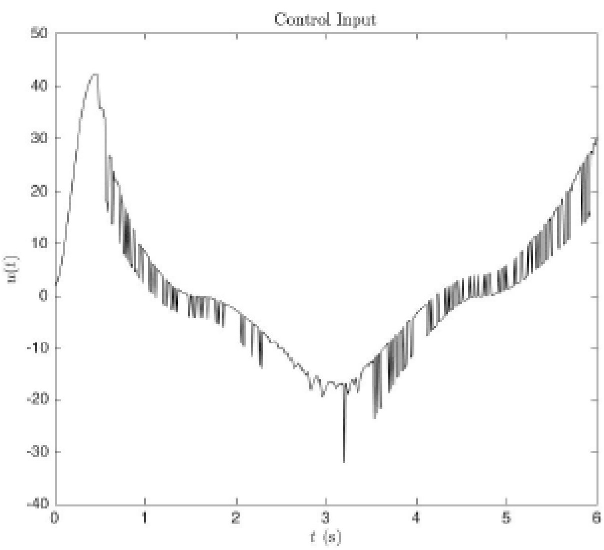
$$\ddot{x}_d(t) = -2\sin t$$

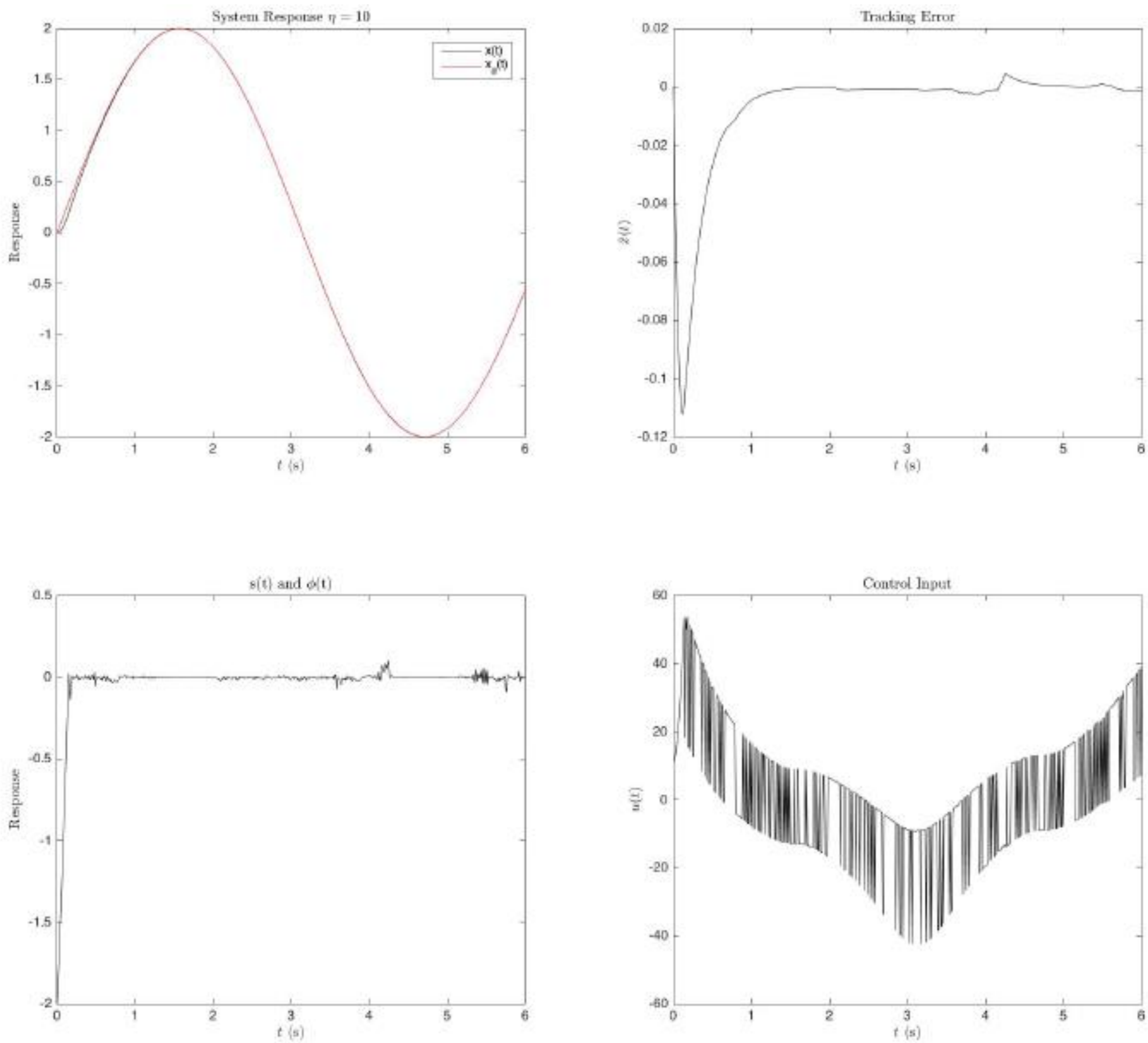
There is noticeable chattering in the control responses for both values of η. An increase in η appears to decrease transient response time of $x(t)$ as well decrease the tracking error.


System Response $\eta = 1$
x(t)
x_d(t)
Response
t (s)


Tracking Error
$\tilde{x}(t)$
t (s)


s(t) and ϕ(t)
Response
t (s)


Control Input
u(t)
t (s)


System Response $\eta = 10$
x(t)
$x_d(t)$
Response
t (s)
Tracking Error
$\tilde{x}(t)$
t (s)
s(t) and ϕ(t)
Response
t (s)
Control Input
$u(t)$
t (s)

Part c

A sliding controller can be designed for an equation in the form $\ddot{x} = f + u$ using the control law

$$u = \hat{u} - (k - \dot{\phi}) sat\left(\frac{s}{\phi}\right)$$

$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}}$$

$$\dot{\phi} + \lambda\phi = k(x_d)$$

$$k(x_d) = F(x_d) + \eta$$

Use the same $\hat{u}$, $\hat{f}$, and k used in the switching controller

$$\hat{u} = (5 + 1.5\cos^2 x)|\dot{x}|\,\dot{x} + \ddot{x}_d - \lambda\dot{\tilde{x}}$$

$$\hat{f} = -(5 + 1.5\cos^2 x)|\dot{x}|\,\dot{x}$$

$$k = F + \eta$$

with tracking error $\tilde{x}$

$$\tilde{x} = x - x_d$$

$$\dot{\tilde{x}} = \dot{x} - \dot{x}_d$$

and the same F used in the switching controller

$$F = (1 + 0.5\cos^2 x)\dot{x}^2 + 1$$

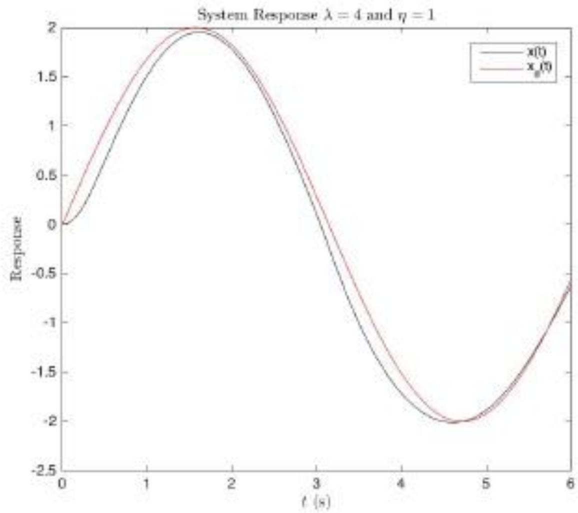
Now $k(x_d)$ can be evaluated as

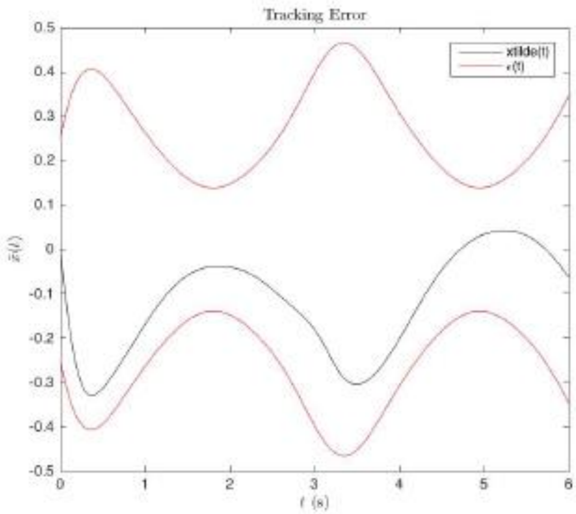
$$k(x_d) = (1 + 0.5\cos^2 x_d)\dot{x_d}^2 + 1 + \eta$$

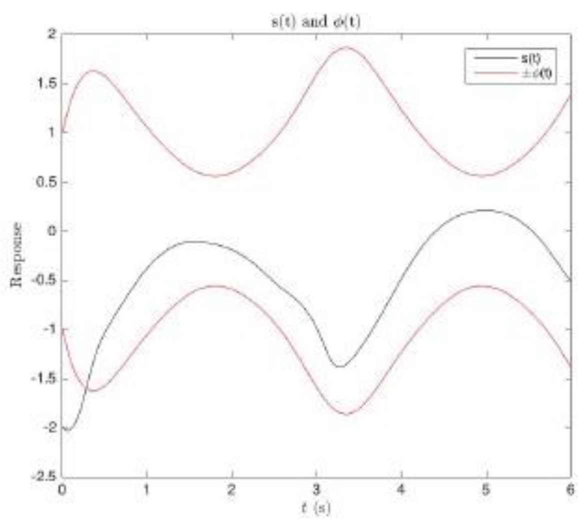
Yielding the controller

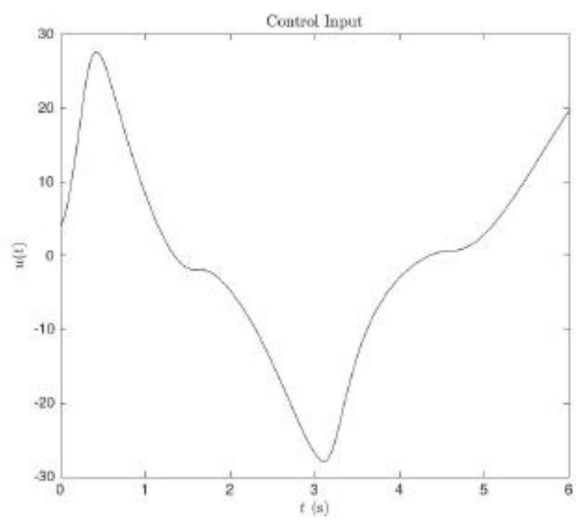
$$u = (5 + 1.5\cos^2 x)|\dot{x}|\,\dot{x} + \ddot{x}_d - \lambda\dot{\tilde{x}} - \left[(1 + 0.5\cos^2 x)\dot{x}^2 - (1 + 0.5\cos^2 x_d)\dot{x_d}^2 + \lambda\phi\right] sat\left(\frac{s}{\phi}\right)$$

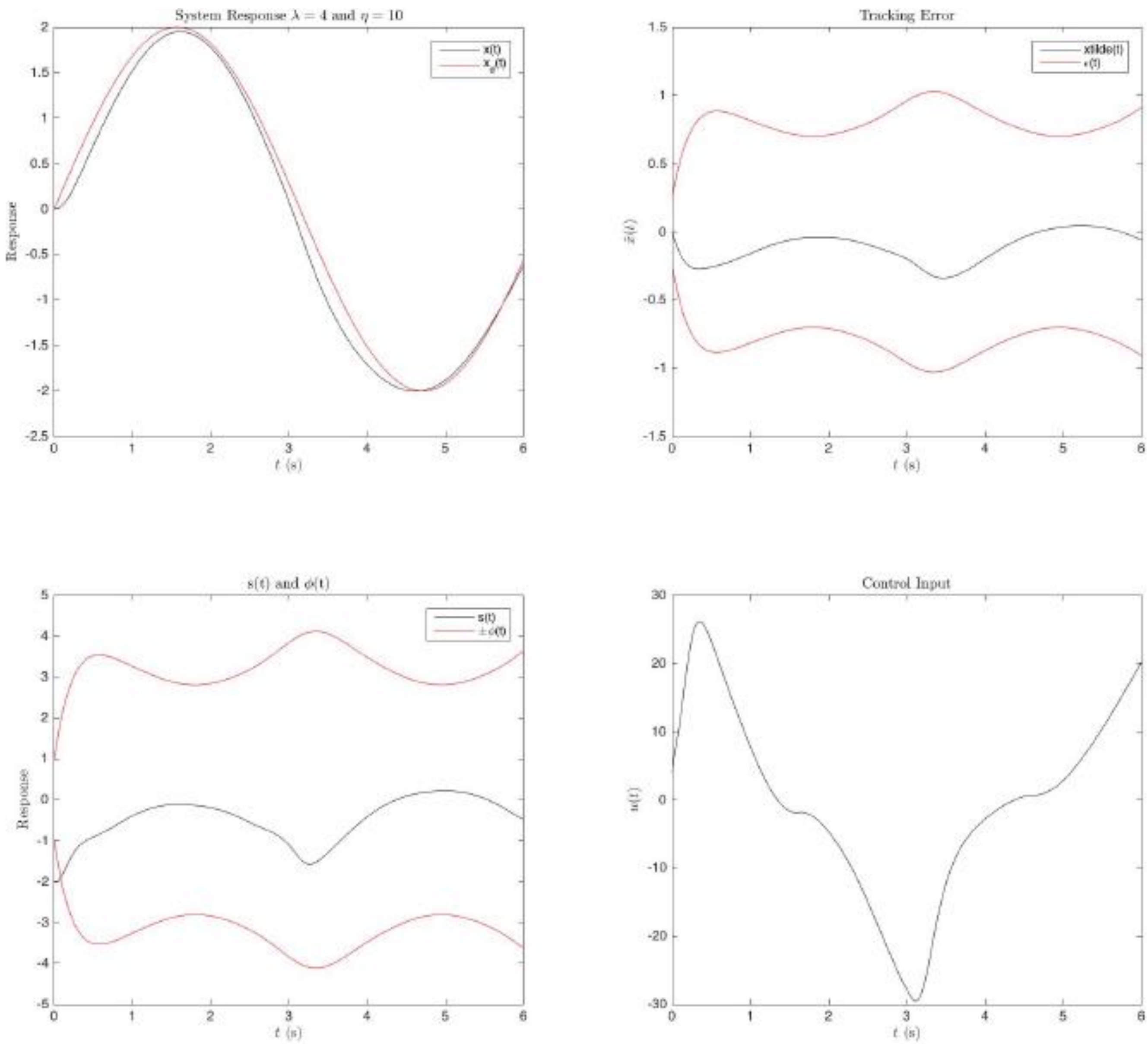
Part d


System Response λ = 4 and η = 1
x(t)
x_d(t)
Response
t (s)


Tracking Error
xtilde(t)
ε(t)
x̃(t)
t (s)


s(t) and φ(t)
s(t)
±φ(t)
Response
t (s)


Control Input
u(t)
t (s)



As λ increases, the tracking error becomes less and less, also the width of boundary layer becomes smaller which shows that sliding variable for large λ stays closer to s=0 than smaller λ.

As η increases the controller acts faster in bringing the sliding variable to s=0 line, hence the tracking is done in shorter time.

Based on theory $|\tilde{x}| \leq \frac{\phi}{\lambda^{n-1}} \rightarrow \; |\tilde{x}| \leq \frac{\phi}{\lambda^{1}}$

Part e

A modification to the switching controller can be made to account for a time-varying m bounded by

$$1 \leq m \leq 2$$

Now

$$\ddot{x} = -\frac{1}{m}(\alpha_1 + \alpha_2 \cos^2 x)|\dot{x}|\,\dot{x} + \frac{1}{m}d + \frac{1}{m}u$$

with

$$f = -\frac{1}{m}(\alpha_1 + \alpha_2 \cos^2 x)|\dot{x}|\,\dot{x}$$

$$b = \frac{1}{m}$$

To account for varying mass define β as

$$\beta = \sqrt{\frac{b_{max}}{b_{min}}} = \sqrt{2}$$

$$\beta^{-1} \leq \frac{b}{\hat{b}} \leq \beta$$

where $\hat{b}$ is the estimate of b defined as the geometric mean of b

$$\hat{b} = \sqrt{b_{min} b_{max}} = \frac{\sqrt{2}}{2}$$

$$\hat{m} = \frac{1}{\hat{b}} = \sqrt{2}$$

A sliding controller can be designed for an equation in the form $\ddot{x} = f + bu$ using the control law

$$u = \hat{u} - (k - \dot{\phi})sat\left(\frac{s}{\phi}\right)$$

$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}}$$

$$\dot{\phi} + \lambda\phi = k(x_d)$$

$$k(x_d) = F(x_d) + \eta$$

with tracking error $\tilde{x}$

$$\tilde{x} = x - x_d$$

$$\dot{\tilde{x}} = \dot{x} - \dot{x}_d$$

Modifying $\hat{f}$ and $\hat{u}$ to account for varying mass

$$\hat{f} = -\frac{1}{1.4}(5 + 1.5\cos^2 x)|\dot{x}|\,\dot{x}$$

$$\hat{u} = \frac{1}{1.4}(5 + 1.5\cos^2 x)|\dot{x}|\,\dot{x} + \ddot{x}_d - \lambda\dot{\tilde{x}}$$

The gain k is modified to account for the varying mass by redefining it as

$$k = \beta(F + \eta) + (\beta - 1)\left|\ddot{x}_d - \lambda\dot{\tilde{x}}\right|$$

Modify F to account for the varying mass

$$F = \frac{1}{1.4}\left((1 + 0.5\cos^2 x)\dot{x}^2 + 1\right)$$

Part f

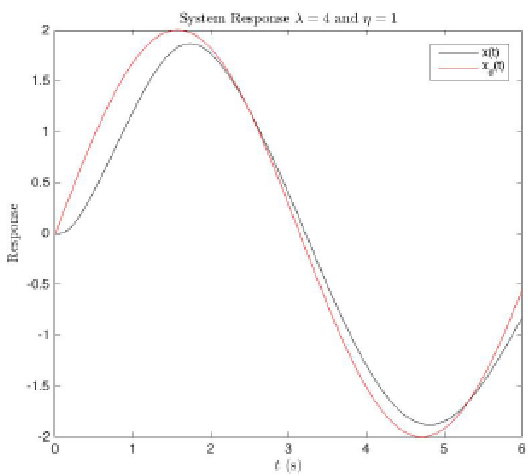
The response was simulated for

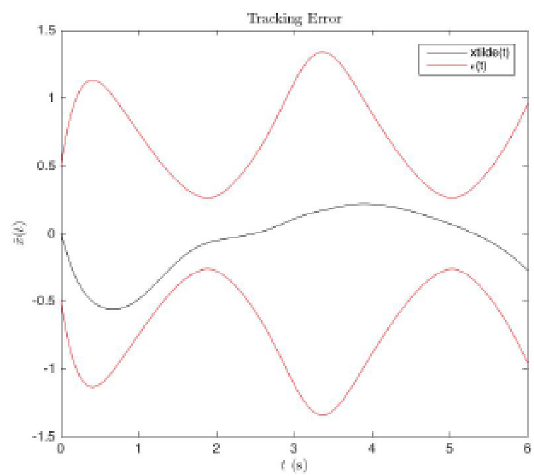
$$\alpha_1(t) = 5 + \cos t$$

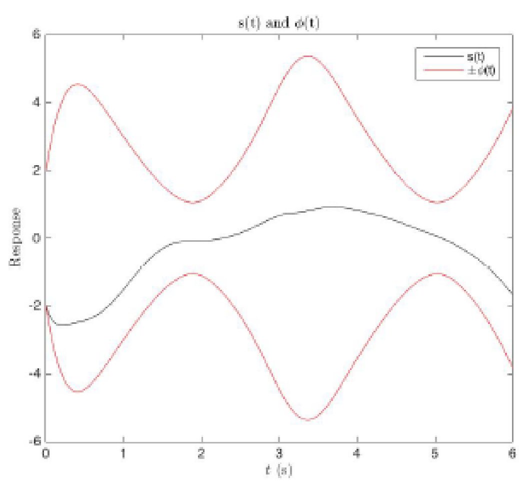
$$\alpha_2(t) = 1 + |\sin t|$$

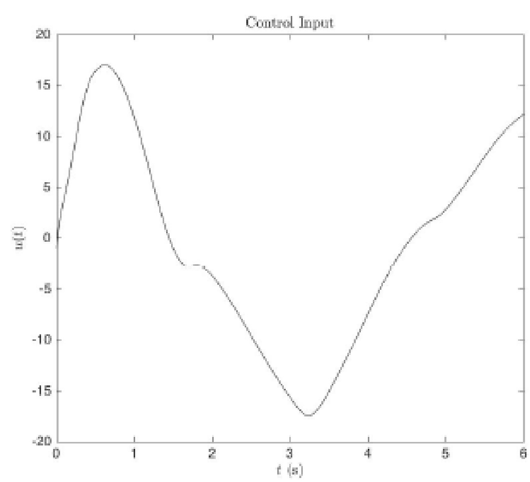
$$d(t) = \cos 1.3t$$

$$m(t) = 2 - |\cos 1.5t|$$









2. *Courtesy of Shingo Kunito*

2) $\ddot{x} + \alpha_1(t)|x|\dot{x}^2 + \alpha_2(t)x^3\cos 2x = 5\dot{u} + u$

$|\alpha_1(t)| \le 1$, $-1 \le \alpha_2 \le 5$

a) let $v = 5\dot{u} + u$

$\tilde{x} = x - x_d$ (tracking error)

let $s = \dot{\tilde{x}} + \lambda\tilde{x}$

$\dot{s} = \ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}}$

where $\ddot{x} = v \underbrace{- \alpha_1(t)|x|\dot{x}^2 - \alpha_2(t)x^3\cos 2x}_{f}$

let $f = -\alpha_1(t)|x|\dot{x}^2 - \alpha_2(t)x^3\cos 2x$

$\hat{f} = -\hat{\alpha}_1|x|\dot{x}^2 - \hat{\alpha}_2 x^3\cos 2x$

$v = \hat{v} - k\,\mathrm{sgn}(s)$

$\hat{v} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}}$

Assuming $\hat{\alpha}_1 = 0$, $\hat{\alpha}_2 = 2$

$\Rightarrow \hat{v} = 2x^3\cos 2x + \ddot{x}_d + \lambda\dot{\tilde{x}}$

$v = 2x^3\cos 2x + \ddot{x}_d + \lambda\dot{\tilde{x}} - k\,\mathrm{sgn}(s)$

$\Rightarrow \dot{s} = 2x^3\cos 2x - \alpha_1|x|\dot{x}^2 - \alpha_2 x^3\cos 2x - k\,\mathrm{sgn}(s)$

$F = |f - \hat{f}|$

$= |\hat{\alpha}_1|x|\dot{x}^2 + \hat{\alpha}_2 x^3\cos 2x - \alpha_1|x|\dot{x}^2 - \alpha_2 x^3\cos 2x|$

$= |2x^3\cos 2x - \alpha_1|x|\dot{x}^2 - \alpha_2 x^3\cos 2x|$

$F = |x|\dot{x}^2 + 3x^2|x|\,|\cos 2x|$

$k = F + \eta$

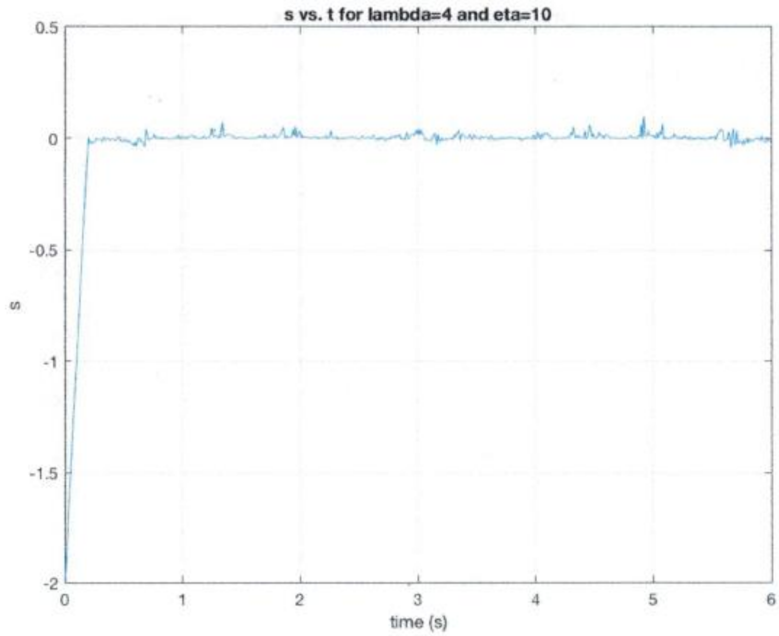
b) Assuming

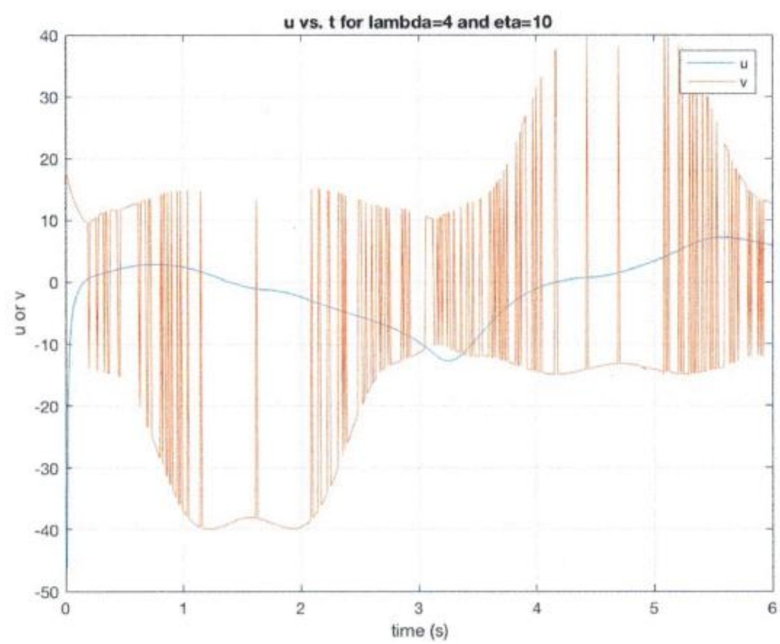
$\alpha_1 = \cos(t)$ ⇒ selected based on its boundary $|\alpha_1| \leq 1$

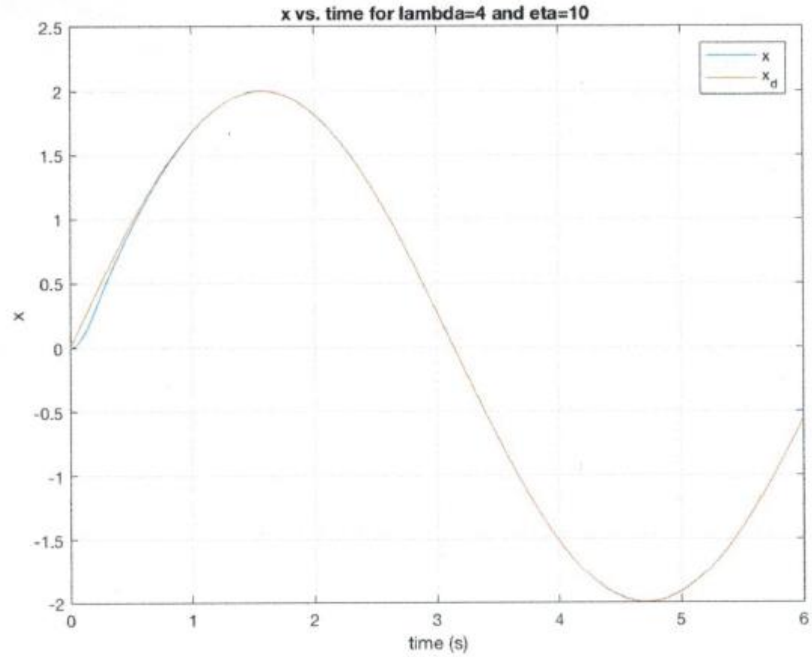
$\alpha_2 = 3\sin t + 2$ ⇒ 〃 〃 $-1 \leq \alpha_2 \leq 5$

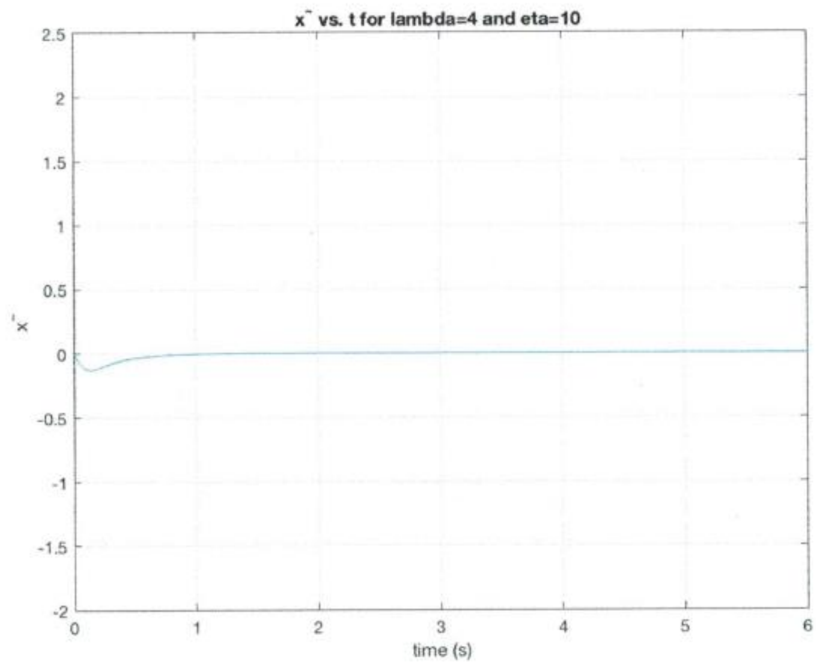
$x_d = 2\sin t$

$\dot{u} = \frac{v - u}{5}$






x vs. time for lambda=4 and eta=10
x
x_d
x
time (s)


x~ vs. t for lambda=4 and eta=10
x~
time (s)