PROBLEM #1 (30pts)

$$\dddot{x} + a_1 \dot{x}^2 \cos(x) + a_2 \sin(2x) = u$$

a_1 & a_2 ARE UNKNOWN CONSTANTS, u IS CONTROL IN

a) DESIGN AN ADAPTIVE CONTROLLER FOR THE SYSTEM TO TRACK A DISIRED TRAJECTORY, $x_d(t)$

WE WANT TO CONTROL THE DYNAMICS OF $\tilde{x}$ WITH A SIMPLE FIRST-ORDER SYSTEM OF S

$$s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x} \qquad n = 3$$

$$= \ddot{x} - \ddot{x}_d + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x}$$

$$= \ddot{x} - \ddot{x}_r \qquad \ddot{x}_r = \ddot{x}_d - 2\lambda\dot{\tilde{x}} - \lambda^2\tilde{x}$$

$$\dot{s} = \dddot{x} - \dddot{x}_r$$

$$= (-a_1 \dot{x}^2 \cos(x) - a_2 \sin(2x) + u) - \dddot{x}_r$$

SIMPLIFY EXPRESSION USING γ AND $\bar{a}$

$$\dot{s} = u - \gamma\bar{a} - \dddot{x}_r$$

$$\hookrightarrow \gamma = [\dot{x}^2\cos(x) \;\; \sin(2x)] \qquad \bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

EMPLOY LYAPUNOV THEORY TO ESTABLISH A CONTROL LAW CAPABLE OF DRIVING $s \to 0$ AND, THEREFORE, $\tilde{x} \to 0$

CANDIDATE LYAPUNOV FUNCTION

$$V = \frac{1}{2} \cancel{\gamma_0^{-1}} s^2$$

WE WANT $\dot{V} < 0$ TO HELP SATISFY BARBALAT'S LEMMA ASSURING $s \to 0$

$$\dot{V} = s\dot{s}$$
$$= s(u - \gamma\bar{a} - \dddot{x}_r)$$

ASSUME $\bar{a}$ IS KNOWN, CHOOSE $u = \gamma\bar{a} - ks + \dddot{x}_r$ $\quad k > 0$
THIS WILL MAKE $\dot{V}$ = SOMETHING VERY SIMPLE

$$\dot{V} = -ks^2$$

HOW DO WE ASSURE $s \to 0$? LET'S USE BARBALAT'S LEMMA

IF ① V IS LOWER-BOUNDED
$\hookrightarrow s^2 \geq 0$ b/c $s = \ddot{x} - \ddot{x}_r$ ✓

AND ② $\dot{V}$ IS NEGATIVE SEMI-DEFINITE
$\hookrightarrow k > 0$ SO $-ks^2 \leq 0$ ✓

AND ③ $\dot{V}$ IS UNIFORMLY CONTINUOUS
$\hookrightarrow \ddot{V}(s)$ IS FINITE B/C s^2 IS BOUNDED ✓

THEN $\dot{V} \to 0$ SO $s \to 0 \Rightarrow \begin{matrix} \tilde{x} \to 0 \\ \dot{\tilde{x}} \to 0 \end{matrix}$

PROBLEM #1

HOWEVER, WE DO NOT KNOW $\bar{a}$. THE BEST WE CAN DO IS GUESS AT VALUES FOR OUR CONTROL LAW

CONTROL LAW

$$\boxed{u = \gamma \hat{a} - ks + \dddot{x}_r} \qquad k > 0$$

NOW,

$$\dot{V} = s(\gamma \hat{a} - ks + \dddot{x}_r - \gamma \bar{a} - \dddot{x}_r)$$

$\tilde{a}$ = PARAM. EST. ERROR

$$= -ks^2 + s\gamma\tilde{a} \qquad \tilde{a} = \hat{a} - \bar{a}$$

I WANT TO GET RID OF THIS SO I CAN USE BARBALAT'S

TRY ADDING A TERM TO LYAPUNOV

$$V = \frac{1}{2}s^2 + \frac{1}{2}\tilde{a}^T P^{-1}\tilde{a} \qquad \dot{\hat{a}} \neq 0$$

★ TO TAKE DERIVATIVE AND KEEP SOMETHING FROM THE ADDED TERM, $\tilde{a}$ MUST BE TIME VARYING ($\hat{a}(t)$)

$$\dot{V} = s\dot{s} + \dot{\hat{a}}^T P^{-1}\tilde{a}$$

$$= -ks^2 + s\gamma\tilde{a} + \dot{\hat{a}}^T P^{-1}\tilde{a}$$

$$= -ks^2 + (s\gamma + \dot{\hat{a}}^T P^{-1})\tilde{a}$$

TO ELIMINATE EFFECT OF UNKNOWN PARAMETERS $(s\gamma + \dot{\hat{a}}^T P^{-1}) = 0$

ADAPTATION LAW

$$\boxed{\dot{\hat{a}} = -P\gamma^T s}$$

b) SIMULATE THE RESPONSES OF THE SYSTEM USING THE ADAPTIVE CONTROLLER DESIGNED IN (a) TO TRACK $X_d(t)$. CHOOSE VALUES FOR P, λ, AND k TO ACHIEVE GOALS

$X_d = \sin(0.8t) \qquad P \leq 5I \qquad \lambda \leq 4 \qquad k \leq 10$

FOR GOOD TRACKING BEHAVIOR,

$P = I \qquad \lambda = 2 \qquad k = 1$

c) SIMULATE THE FOLLOWING CASES AND COMMENT ON EACH PARAMETER'S IMPACT ON TRACKING PERFORMANCE, PARAMETER CONVERGENCE, AND MAGNITUDE OF CONTROL INPUT

	TRACKING PERFORMANCE	PARAMETER CONVERGENCE	MAGNITUDE OF CONTROL INPUT
INCREASE P	CLOSER	FASTER	GREATER
INCREASE λ	CLOSER	FASTER	GREATER
INCREASE k	CLOSER	SLOWER	GREATER

PROBLEM #2

$$\ddot{x} + a_1 \dot{x}^2 \cos(x) + a_2 \sin(2x) = u + d(t)$$

WHERE $d(t) \leq D = \text{const}$

a) DESIGN AN ADAPTIVE TRACKING CONTROLLER ROBUST TO THE SPECIFIED DISTURBANCE

$$S = \left(\tfrac{d}{dt} + \lambda\right)^{n-1} \tilde{x} \qquad n=3$$

$$= \ddot{x} - \ddot{x}_r \qquad \ddot{x}_r = \ddot{x}_d - 2\lambda\dot{\tilde{x}} - \lambda^2 \tilde{x}$$

$$\dot{S} = \dddot{x} - \dddot{x}_r$$

$$= \left(-a_1 \dot{x}^2 \cos(x) - a_2 \sin(2x) + u + d\right) - \dddot{x}_r$$

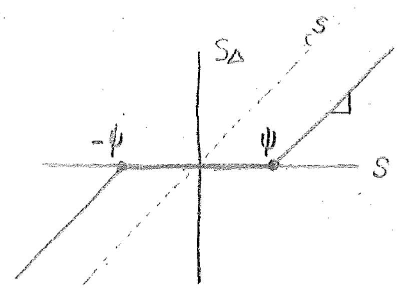
$$= u + d - \gamma \bar{a} - \dddot{x}_r$$

$$\hookrightarrow \gamma = [\dot{x}^2\cos(x) \quad \sin(2x)] \qquad \bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

WE WOULD LIKE TO $S \to 0$, BUT THE UNKNOWN DISTURBANCE WILL PREVENT THIS FROM HAPPENING. WE WOULD LIKE TO ESTABLISH A BOUNDARY LAYER ABOUT $S=0$ WHERE WE BELIEVE THE TRACKING ERROR IS DOMINATED BY THE UNKNOWN DISTURBANCE BECAUSE WE HAVE FOUND VALUES FOR $\gamma \hat{S}$ WHICH GIVE THE BEST ESTIMATE POSSIBLE. CHOOSE, THEN

$$S_\Delta = S - \psi \, \text{sat}\left(\tfrac{S}{\psi}\right)$$

SO THE CONTROL AND ADAPTATIONS NOW HAVE A "DEAD BAND"



$$S_\Delta = \begin{cases} 0, & |s| \leq \psi \\ s - \psi, & s > \psi \\ s + \psi, & s < \psi \end{cases}$$

EMPLOY LYAPUNOV TO DEVELOP A CONTROL LAW TO DRIVE $S_\Delta \to 0$ AND, THEREFORE, $\tilde{x} < \varepsilon$.

CANDIDATE LYAPUNOV: $V = \frac{1}{2}\cancel{\alpha_0} S_\Delta^2 + \frac{1}{2}\tilde{a}^T P^{-1} \tilde{a}$

USE ADDITIONAL TERM FROM ADAPTATION EFFORT IN THE LAST PROBLEM

$$\dot{V} = S_\Delta \dot{S} + \dot{\hat{a}}^T P^{-1} \tilde{a}$$

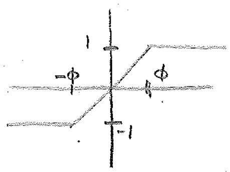
SUB $\dot{S}$

$$= S_\Delta \left(u + d - \gamma \bar{a} - \dddot{x}_r\right) + \dot{\hat{a}}^T P^{-1} \tilde{a}$$

CHOOSE A CONTROL LAW, DON'T USE $-kS_\Delta$ B/C, I THINK, THIS WILL MAKE THE BL BIGGER. INSTEAD, USE $-k\,\text{sat}\left(\frac{S}{\psi}\right)$ AS WE HAVE BEFORE

PROBLEM #2

$$\mathrm{sat}\left(\frac{s}{\phi}\right) = \begin{cases} 1, & s \geq \phi \\ -1, & s \leq -\phi \\ s/\phi, & |s| < \phi \end{cases}$$



$\tilde{a} = \hat{a} - \bar{a}$

$k = D + \eta$

SO, THE CONTROL LAW IS CHOSEN TO BE

$$\boxed{u = -k\,\mathrm{sat}\left(\tfrac{s}{\phi}\right) - \hat{d} + \gamma \hat{a} + \dddot{x}_r} \qquad k > 0$$

$\hat{d} = 0$ B/C WE DON'T KNOW $d(t)$ AT ALL

SUCH THAT

$$\dot{V} = -S_\Delta k\,\mathrm{sat}\left(\tfrac{s}{\phi}\right) + S_\Delta(d - \hat{d}_0) + S_\Delta \gamma \tilde{a} + \dot{\hat{a}}^T P^{-1} \tilde{a}$$

$\rightarrow$ IF $\phi = \psi$, THEN $-S_\Delta k\,\mathrm{sat}\left(\frac{s}{\phi}\right) = -k|S_\Delta|$

$\rightarrow$ IF CHOOSE $k = D + \eta$, THEN $-k|S_\Delta| + S_\Delta(d - \hat{d}_0) \leq 0$

$$\dot{V} = -k|S_\Delta| + S_\Delta(d - \hat{d}_0) + \underbrace{\left(S_\Delta \gamma + \dot{\hat{a}}^T P^{-1}\right)}_{\text{MAKE THIS } = 0} \tilde{a}$$

ADAPTATION LAW

$$\boxed{\dot{\hat{a}}^T = -P \gamma^T S}$$

$$\dot{V} = -(D + \eta)|S_\Delta| + S_\Delta d$$

$$\hookrightarrow \dot{V} \leq -\eta |S_\Delta| \leq 0$$

USE BARBALAT'S LEMMA TO SHOW $\dot{V} \rightarrow 0$ SIMILAR TO PREVIOUS

BARBALAT OR INV SET THM?

NOTICE, B/C $\dot{V} \rightarrow 0$, S_Δ IS "SQUEEZED" TO ZERO

---

b) SIMULATE RESPONSE TO TRACKING $x_d = \sin(0.8t)$

$a_1 = 2 \quad a_2 = 5 \quad P = 5I \quad \lambda = 3 \quad \eta = 1$

$D = [0.1 \;\; 0.5 \;\; 1.0]$

DESCRIBE EFFECT OF CHANGING D

INCREASE D

OBSERVATIONS

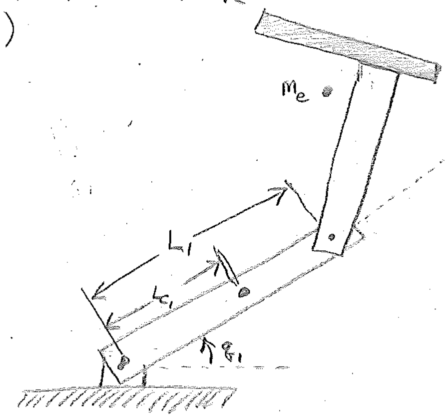
- CLOSER TRACKING
- VARIED EFFECT ON PARAMETER ESTIMATION. STEADY-STATE ERROR IS NOT MONOTONIC.
- APPEARS TO INCREASE CONTROL INPUT AT TIME START, BUT LESS AS TIME PROGRESSES

EXPLANATIONS

(D) DIRECTLY AFFECTS (k) ACTING AS A GAIN ON THE CONTROL OUTSIDE OF THE BL. MORE GAIN MEANS FASTER REACHING TIME, BUT THE POSSIBLE OVERSHOOT IS OBVIATED BY THE BL CONDITION. IT AFFECTS PARAMETER ESTIMATION IN A COMPLEX MANNER BECAUSE IT IMPACTS THE TIME THE STATE IS OUTSIDE THE BL AND, THEREFORE, THE TIME & CONDITIONS ADAPTATION LAW IS APPLIED. HIGHER GAIN MEANS MORE CONTROL EFFORT, CAUSING FASTER CONVERGENCE, TO WHERE LESS CONTROL IS REQUIRED.

PROBLEM # 3

TWO-LINK MANIPULATOR
(EXAMPLE 9.1)



$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h(\dot{q}_1+\dot{q}_2) \\ h\dot{q}_1 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$H_{11} = a_1 + 2a_3\cos(q_2) + 2a_4\sin(q_2)$
$H_{12} = H_{21} = a_2 + a_1\cos(q_2) + a_4\sin(q_2)$
$H_{22} = a_2$
$h = a_3\sin(q_2) - a_4\cos(q_2)$
$a_1 = I_1 + m_1 l_{c_1}^2 + I_e + m_e l_{ce}^2 + m_e l_1^2$
$a_2 = I_e + m_e l_{ce}^2$
$a_3 = m_e l_1 l_{ce}\cos(\delta_e)$
$a_4 = m_e l_1 l_{ce}\sin(\delta_e)$

$m = Kg$
$I = Kg\cdot m^2$

$m_1 = 1 \quad l_1 = 1 \quad m_e = 2 \quad \delta_e = 30^\circ \quad I_1 = 0.12 \quad l_{c_1} = 0.5 \quad I_e = 0.25 \quad l_{ce} = 0.6$

@ $t=0$, $q_1 = q_2 = \dot{q}_1 = \dot{q}_2 = 0$ $\quad q_{d_1} = 60^\circ \quad q_{d_2} = 90^\circ$

$K_D = 100\,I \qquad K_p = 20K_D$

$\bar{\tau} = -K_p\tilde{q} - K_D\dot{q}$

$x_1 = q_1 \quad x_2 = q_2 \quad x_3 = \dot{q}_1 \quad x_4 = \dot{q}_2$

$\dot{x}_1 = \dot{q}_1 = x_3$
$\dot{x}_2 = \dot{q}_2 = x_4$ $\rightarrow$ $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$

$\dot{x}_3 = \ddot{q}_1 =$
$\dot{x}_4 = \ddot{q}_2 =$ $\rightarrow$ $\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = H^{-1}\left[\bar{\tau} - C\dot{q}\right]$

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ & H^{-1} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \tau_1 \\ \tau_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ 0 & -H^{-1}C \end{bmatrix}\bar{x}$$

PROBLEM #3a

SIMULATE THE RESPONSE OF THE MANIPULATOR $(q_1(t)\ q_2(t))$ USING A PD-CONTROLLER STARTING FROM ZERO ICs AND SEEKING THE DESIRED FINAL POSITIONS, $\bar{q}_d$

$m_1 = 1 \quad l_1 = 1 \quad m_e = 2 \quad \delta_e = 30^\circ \quad I_1 = 0.12 \quad l_{c1} = 0.5$

$I_e = 0.25 \quad l_{ce} = 0.6 \quad K_D = 100I \quad K_P = 20K_D \quad \bar{q}_d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

IN DESIGNING A PD-CONTROLLER, WE MAY TAKE FOR GRANTED THE CONTROL WILL HAVE THE FORM

$$\bar{\tau} = K_P \tilde{q} - K_D \dot{q}$$

NOTE, THERE IS NOT YET UNCERTAINTY IN THE PROBLEM

AS SUCH, THE SET-UP FOR SIMULATION IS TRIVIAL AFTER TRANSFORMING TO A SYSTEM OF FIRST-ORDER ODEs AS DONE ON THE PREVIOUS PAGE

ANALYSIS

THE CONTROL TORQUE IS LARGE, INITIALLY. THIS IS A CONSEQUENCE OF A LARGE PROPORTIONAL FEEDBACK GAIN

THE RESULTS OF SIMULATION ARE AS EXPECTED.

PROBLEM #3b

DESIGN AN ADAPTIVE CONTROLLER FOR THE MANIPULATOR ASSUMING NO INITIAL KNOWLEDGE OF PARAMETERS, $\bar{a}$. EXAMPLE 9.3 GIVES VALUES FOR MATRIX [Y]. SIMULATE THE RESPONSE TO DESIRED TRAJECTORY

$$\begin{cases} \bar{q}_{d_1} = [1-e^{-t} \quad 2(1-e^{-t})]^T \\ \bar{q}_{d_2} = [1-\cos(2\pi t) \quad 2(1-\cos(2\pi t))]^T \end{cases}$$

$$\begin{cases} P_1 = \text{diag}([0.6 \; 0.1 \; 0.1 \; 0.06]) \\ P_2 = 200\, P_1 \\ P_3 = 0.1\, P_1 \end{cases}$$

$$\Lambda = 20I \qquad K_D = 100I$$

DISCUSS TRACKING PERFORMANCE AND PARAMETER CONVERGENCE FOR EACH CASE.

FIND DEFINITIONS FOR CONTROL LAW AND ADAPTATION LAW USING LYAPUNOV THEORY. FIRST SIMPLIFY TO FIRST-ORDER SYSTEM BY INTRODUCING INTERMEDIATE PARAMETER.

$$\bar{s} = \left(\frac{d}{dt} + \Lambda\right)^{n-1} \tilde{q} \qquad n=2$$

$$= \dot{\tilde{q}} + \Lambda \tilde{q}$$

$$= \dot{q} - \dot{q}_r \qquad (\dot{q}_r = \dot{q}_d - \Lambda \tilde{q})$$

$$\dot{\bar{s}} = \ddot{q} - \ddot{q}_r \quad \longleftarrow$$

CANDIDATE LYAPUNOV FUNCTION

$$V = \frac{1}{2}\bar{s}^T H \bar{s} + \frac{1}{2}\tilde{a}^T P^{-1} \tilde{a}$$

$$\dot{V} = \bar{s}^T H \dot{\bar{s}} + \frac{1}{2}\bar{s}^T \dot{H} \bar{s} + \dot{\hat{a}}^T P^{-1} \tilde{a}$$

$$= \bar{s}^T (H\ddot{q} - H\ddot{q}_r) + \frac{1}{2}\bar{s}^T \dot{H} \bar{s} + \dot{\hat{a}}^T P^{-1} \tilde{a}$$

$$= \bar{s}^T (\tau - C\dot{q} - H\ddot{q}_r) + \frac{1}{2}\bar{s}^T \dot{H} \bar{s} + \dot{\hat{a}}^T P^{-1} \tilde{a}$$

$$= \bar{s}^T (\tau - H\ddot{q}_r - C\bar{s} - C\dot{q}_r) + \frac{1}{2}\bar{s}^T \dot{H} \bar{s} + \dot{\hat{a}}^T P^{-1} \tilde{a}$$

$$= \bar{s}^T (\tau - H\ddot{q}_r - C\dot{q}_r) + \frac{1}{2}\bar{s}^T \cancel{(\dot{H} - 2C)}^{\,0} \bar{s} + \dot{\hat{a}}^T P^{-1} \tilde{a}$$

$$= \bar{s}^T (\tau - Y\bar{a}) + \dot{\hat{a}}^T P \tilde{a}$$

★ ASSUMING C IS "PROPER" ($C\ddot{q}$ IS UNIQUE)

COULD FIND Y USING $Y\bar{a} = H\ddot{q}_r - C\dot{q}_r$ IF IT WAS NOT GIVEN

PROBLEM #3b

CHOOSE $\bar{\tau}$ TO MAKE CONTROL LAW (WE WANT $\dot{V} < 0$)

$$\boxed{\bar{\tau} = Y\hat{a} - K_D\bar{s}} \qquad K_D = K_D^T > 0$$

$\tilde{a} = \hat{a} - \bar{a}$

SO $\quad \dot{V} = -\bar{s}^T K_D \bar{s} + \bar{s} Y \tilde{a} + \dot{\hat{a}}^T P^{-1} \tilde{a}$

I WANT $\dot{V} < 0$, SO GET RID OF BY ADDING TO LYAP. FUNC. * AND CHOOSING $\dot{\hat{a}}$, YIELDING ADAPTATION LAW

$$\boxed{\dot{\hat{a}} = -PY^T\bar{s}}$$

NOW, $\dot{V} = -\bar{s}^T K_D \bar{s} < 0$ SO, USING BARBALAT'S LEMMA,

$$\dot{V} \to 0 \Rightarrow \bar{s} \to 0 \Rightarrow \begin{matrix} \tilde{X} \to 0 \\ \dot{\tilde{X}} \to 0 \end{matrix}$$

SIMULATION IS ACCOMPLISHED IN A SIMILAR MANNER AS BEFORE, IN PART 1 & 2 WHERE $\tilde{a}$ IS INTEGRATED ALONG WITH THE DYNAMICS AS THOUGH THEY ARE ADDITIONAL STATES.

DISCUSS TRACKING PERFORMANCE AND PARAMETER CONVERGENCE

DESCRIBE EACH DESIRED TRAJECTORY AND EFFECTS

$\bar{q}d_1$ HAS ONLY EXPONENTIAL TERMS. AS SUCH, THE ACTUATORS MUST WORK HARD TO ADJUST TO FAST POSITION CHANGE, BUT LATER DOES NOT HAVE TO MOVE MUCH.

$\bar{q}d_2$ HAS ONLY SINUSOIDAL TERMS, AS SUCH, THE DESIRED CHANGE SPEED IS PERSISTENTLY CYCLIC, SO EXPECT CONTROL EFFORT TO BE CYCLIC

DESCRIBE EFFECT OF MAGNITUDE OF [P]

[P] CONTAINS THE GAINS ACTING ON THE DYNAMICS OF THE PARAMETER ESTIMATION PROGRESSION HIGHER GAINS CAUSE SHORTER REACHING TIME W/ POSSIBILITY FOR OVERSHOOT (ON PARAM ESTIMATION)

OBSERVATIONS OF SIMULATION RESULTS

- EACH CASE WITH $\bar{q}d_1$ SETTLES TO ZERO POSITION ERROR
- EACH CASE WITH $\bar{q}d_2$ HAS PERSISTENT, CYCLIC ERROR & TORQUE
- FOR $\bar{q}d_1$, EXPONENTIAL: FOR [P] AT SMALLER VALUES, THE POSITION ERROR IS CONTROLLED MORE QUICKLY AND ACCURATELY. THIS IS BECAUSE THE $(\tilde{a})$-VALUE PREDICTION DOES NOT OVERSHOOT SO EXTREMELY AS SEEN IN PLOTS OF $(\tilde{a})$ FOR $[P_2]$.
- FOR $\bar{q}d_2$, SINUSOIDAL: FOR [P] AT LARGER VALUES, THE SINUSOIDAL $\bar{q}d_2$ IS CONTROLLED MORE QUICKLY AND THE RESULTING REDUCTION IN POSITION ERROR CAUSES THE PROPORTIONAL-GAIN CONTROL TORQUE TO BE LOWER OVER TIME. HOWEVER, THE INITIAL RESPONSE CHATTERS FROM LARGE OVERSHOOT.