MAE 5803 Nonlinear Control Systems **Assigned: Jan 19, 2017**
Homework #1 (solutions) **Due: Jan 26, 2017**

1. *Courtesy of Nasir Hariri*

(a) Draw the phase portrait plot, identify the singular points and their types:

The differential equation of the 2nd order nonlinear system is:

$$\ddot{\theta}(t) + 0.6\,\dot{\theta}(t) + 3\,\theta(t) + \theta^2(t) = 0 \quad (1)$$

Defining the following state variables for the system:

$$x_1 = \theta(t) \quad (2)$$
$$x_2 = \dot{\theta}(t) \quad (3)$$

The state-space dynamics of the nonlinear system can be expressed as following:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} x_2 \\ -0.6\,x_2 - 3\,x_1 - x_1^2 \end{bmatrix} \quad (4)$$

Therefore, the singular points for the nonlinear system are: $(x_1, x_2) = (0,0)$ and $(-3,0)$

The phase portrait is as shown:

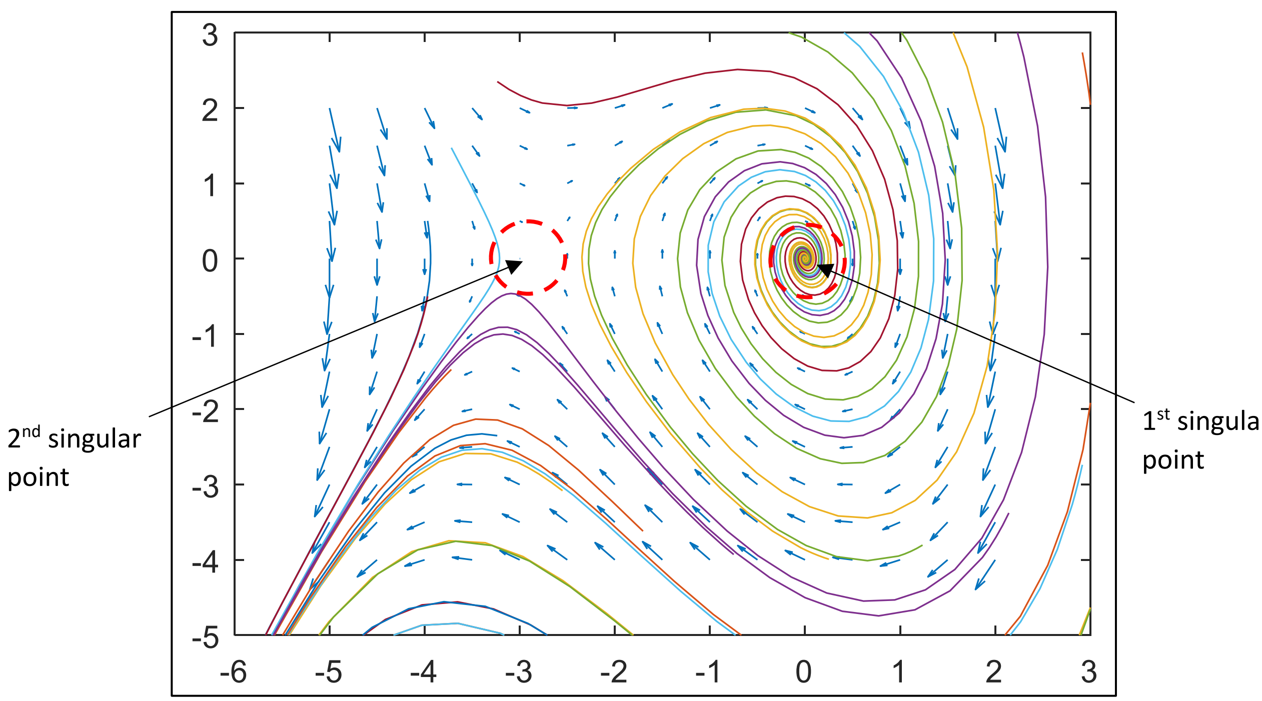


Figure 1: Phase portrait plot

The singular point located at the origin (0,0) is stable focus and the singular point at (-3,0) is a saddle point.

(b) Linearized equations about singular points, eigenvalues, and stability of singular points:

For linearization about the first singular "equilibrium" point (0,0), the system "Jacobian" matrix **A** is:

$$\boldsymbol{A} = \left.\frac{\partial f}{\partial x}\right|_{0,0} = \begin{bmatrix} 0 & 1 \\ -3 - 2 * x_1 & -0.6 \end{bmatrix}_{0,0} = \begin{bmatrix} 0 & 1 \\ -3 & -0.6 \end{bmatrix} \quad (5)$$

The eigenvalues of the system at the 1st singular point (0,0) are complex conjugate values with a negative real part, which indicates that the system is stable and considered as stable focus system about the 1st singular point:

```
A1 =

         0    1.0000
   -3.0000   -0.6000
```

```
eigenvalue_A_1_equili =

  -0.3000 + 1.7059i
  -0.3000 - 1.7059i
```

For linearization about the second singular "equilibrium" point (-3,0), the system matrix **A**:

$$A = \frac{\partial f}{\partial x}\Big|_{-3,0} = \begin{bmatrix} 0 & 1 \\ -3-2*x_1 & -0.6 \end{bmatrix}_{-3,0} = \begin{bmatrix} 0 & 1 \\ 3 & -0.6 \end{bmatrix} \quad (6)$$

The eigenvalues of the system at the 2nd singular point (-3,0) consist of a positive and negative real values, which indicates that the system is not stable (real$\{\lambda\} > 0$) and considered as saddle point system about the 2nd singular point:

```
A2 =

         0    1.0000
    3.0000   -0.6000
```

```
eigenvalue_A_2_equili =

    1.4578
   -2.0578
```

(c) Phase portraits for the linearized equations:

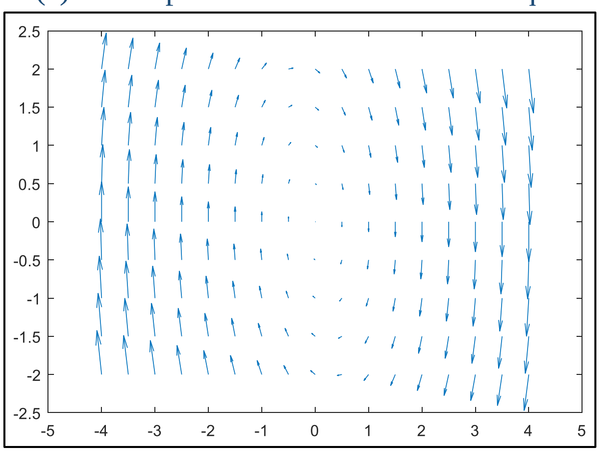


Figure 2: Phase portrait of linearized equation (1st singular point)

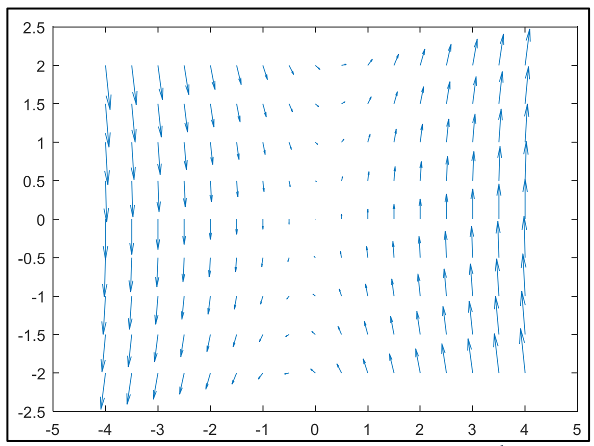


Figure 3: Phase portrait of linearized equation (2nd singular point)

The phase portraits of the linearized and nonlinear equations in the neighborhood of the singular points compare very well.

2. *Courtesy of Tim Coon*

$$\dot{x}_1 = \mu - x_1^2$$

$$\dot{x}_2 = -x_2$$

a) Identify Singular points

For $\mu = 1$:

First Singular Point

The first singular point is a *stable node* at (1,0). Use the Jacobian to linearize about this point. Both eigenvalues have negative real parts, supporting the ID as a stable focus.

$$A_1 = \left.\frac{\partial \bar{f}}{\partial \bar{x}}\right|_{\bar{x}=(1,0)}$$

$$\frac{\partial f_1}{\partial x_1} = -2x_1 \quad \frac{\partial f_1}{\partial x_2} = 0 \quad \frac{\partial f_2}{\partial x_1} = 0 \quad \frac{\partial f_2}{\partial x_2} = -1$$

$$A_1 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

```
eValue1 = eig([-2 0; 0 -1])
```

```
eValue1 =

  -2
  -1
```

Second Singular Point

The second singular point is a *saddle point* at (-1,0). Use the Jacobian to linearize about this point. Both eigenvalues have negative real parts, supporting the ID as a stable focus.

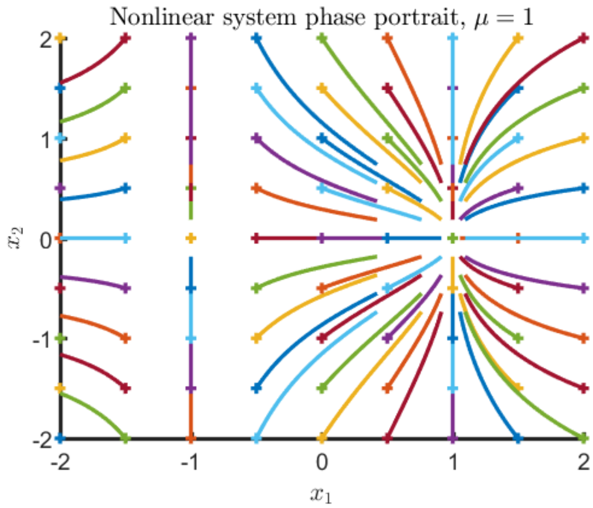
$$A_2 = \left.\frac{\partial \bar{f}}{\partial \bar{x}}\right|_{\bar{x}=(-1,0)}$$

$$A_2 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

```
eValue2 = eig([2 0; 0 -1])
```

```
eValue2 =

  -1
   2
```



b) For $\mu = 0$:

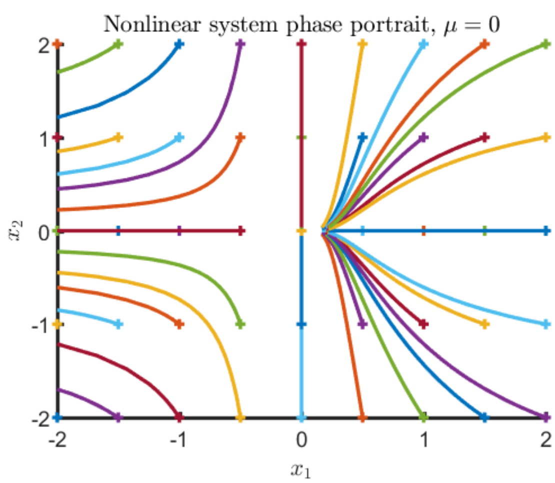
Singular Point, $\mu = 0$

$$A_2 = \left.\frac{\partial \bar{f}}{\partial \bar{x}}\right|_{\bar{x}=(0,0)}$$

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

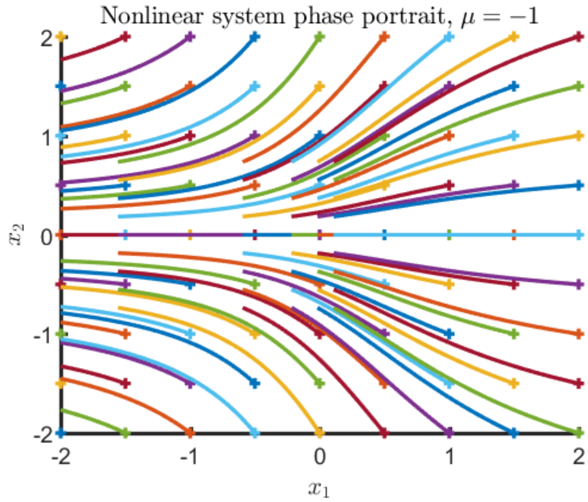
```
eValue1 = eig([0 0; 0 -1])
eValue1 =
   -1
    0
```

One eigenvalue at the origin of the complex plane with no negative eigenvalues means the stability of the system cannot be determined by the eigenvalues alone. From the phase portrait, it is clear any state in the right-half plane tends toward the origin. This would indicate stability were it mirrored by the left-half plane. However, any state in the left-hand plane escapes along the negative x_2 axis, so the node is unstable.



c) Let $\mu = -1$

No Singular Points for $\mu = -1$



d) Comments

The number of singular points change as μ is varied from positive to negative. This phenomenon is called bifurcation.

3. *Courtesy of Shingo Kunito*

a)

$$f_1(x_1, x_2) = x_2$$

$$f_2(x_1, x_2) = -x_1 + (\mu - x_1^2)x_2$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \Bigg|_{\binom{0}{0}}$$

$$= \begin{bmatrix} 0 & 1 \\ (-1 - 2x_1x_2) & \mu - x_1^2 \end{bmatrix} \Bigg|_{\binom{0}{0}}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

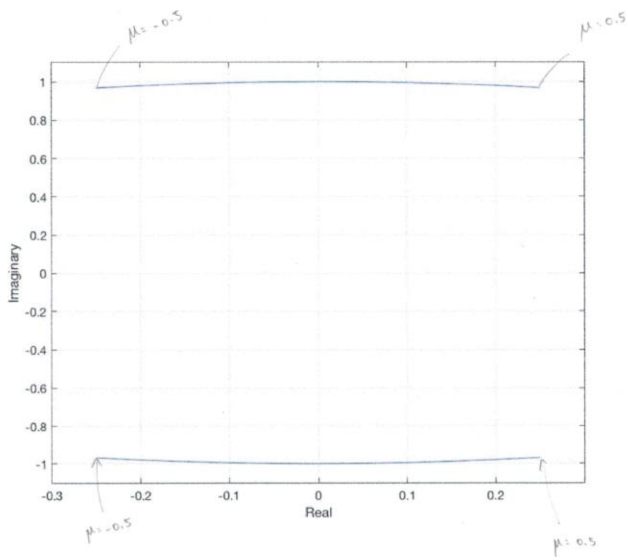
$$\det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda-\mu \end{bmatrix} = 0$$

$$\lambda(\lambda-\mu) + 1 = 0$$

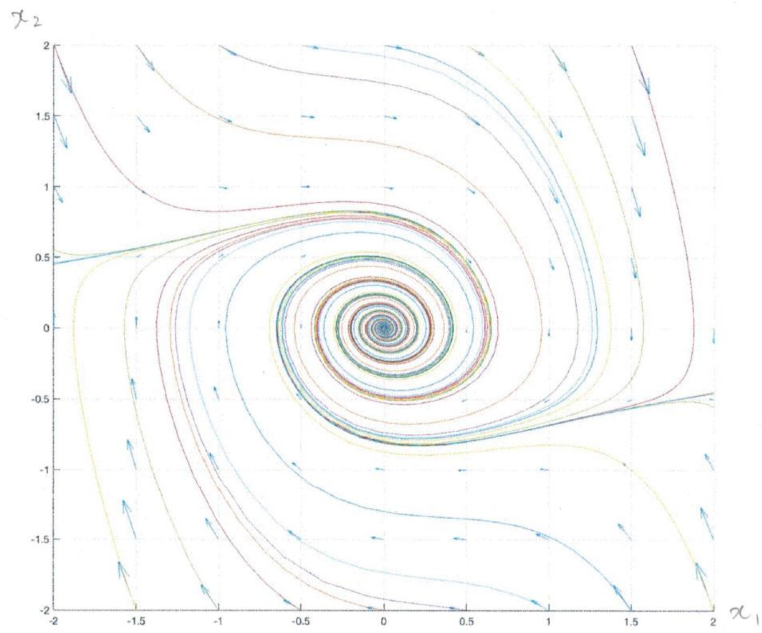
$$\lambda^2 - \lambda\mu + 1 = 0$$

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

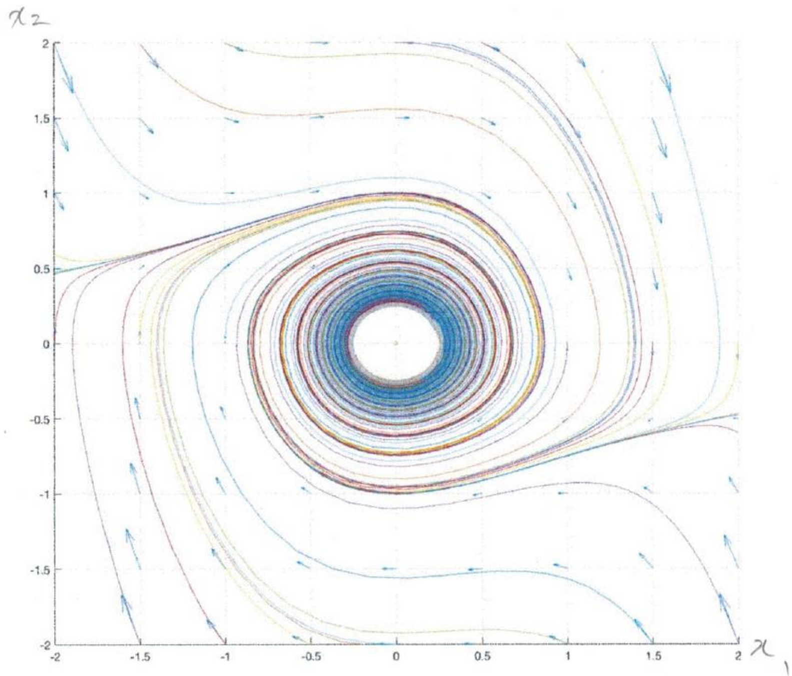
$$= \frac{\mu + \sqrt{\mu^2-4}}{2}, \quad \frac{\mu - \sqrt{\mu^2-4}}{2}$$



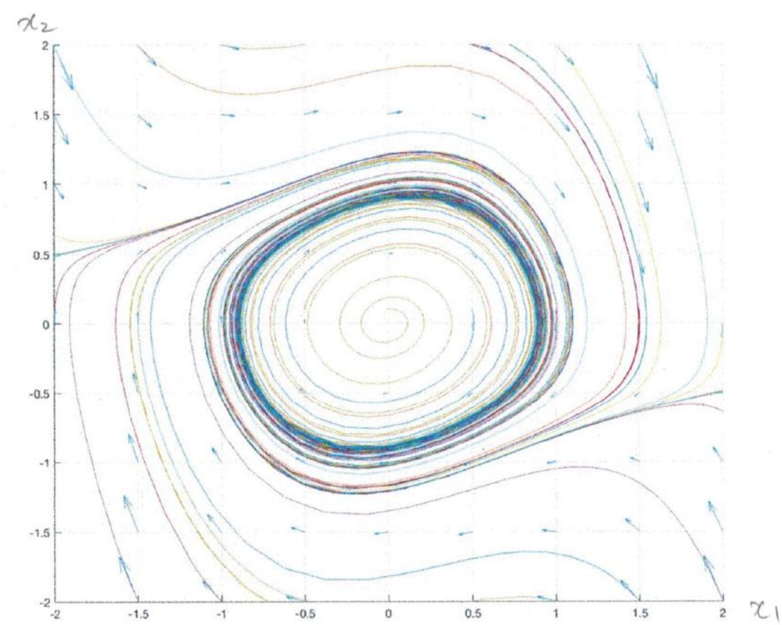
mu=-0.2



mu=0



mu=0.2



c. Phenomenon: Hopf bifurcation.

4. *Courtesy of Nasir Hariri*

The state derivative equations for the second order nonlinear system that represents the Van der Pol equation are:

$$\dot{x}_1 = -x_2 \tag{7}$$
$$\dot{x}_2 = x_1 - (1 - x_1^2)x_2 \tag{8}$$

(a) The phase portrait of the nonlinear system:

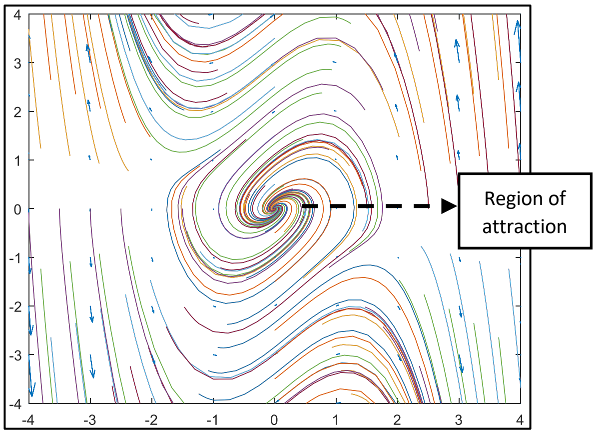


Figure 4: Zoomed-in area of the limit cycle.

(b) Is the limit cycle stable?

The limit cycle is unstable since the motion trajectories around the limit cycle are diverging (moving away) from the path of the limit cycle.

(c) Determine the stability of the equilibrium point at the origin, and its region of attraction if it is asymptotically stable:

For further investigation of the stability of the linearized system about the equilibrium point. the linearization of the nonlinear system about the singular “equilibrium” point (0,0) can be computed; thus, the system matrix **A**:

$$A = \frac{\partial f}{\partial x}\bigg|_{0,0} = \begin{bmatrix} 0 & -1 \\ 1 + 2 * x1.* x2 & -1 + x1.\hat{}2 \end{bmatrix}_{0,0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \tag{9}$$

Therefore, the eigenvalues of the linearized system about the equilibrium point can be found as shown below, where both eigenvalues have a negative real part that indicates that the linearized system is asymptotically stable (stable focus) around the equilibrium point (0,0):

```
eigenvalue_A_1_equili =

  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
```

Region of attraction: the whole area inside the limit cycle.