The Complex Exponential

There is an explicit relationship between the complex exponential $e^{i\theta}$ and the trigonometric functions $\sin(\theta)$ and $\cos(\theta)$. At first this relationship may not be obvious; the exponential, complex numbers, and trigonometric functions are not intuitively related. This proof is the most well-known and straight-forward method of proving Euler's Formula, with the other main method being a geometric argument in the complex plane. This proof uses Taylor Expansion from calculus to show that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

1 Proof by Taylor Expansion

First we must define a few functions in terms of their Taylor expansions.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \end{aligned} \tag{1}$$

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots \pm \frac{x^{2k+1}}{(2k+1)!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{aligned} \tag{2}$$

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots \mp \frac{x^{2k}}{(2k)!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \end{aligned} \tag{3}$$

Using these three formulas, we are able to to show that $e^{i\theta} = \cos\theta + i\sin\theta$. The proof starts out by a substitution in the taylor series expansion of e^x to e^{ix}.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \Longrightarrow \qquad e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}$$

Then, by manipulating the power series for e^{ix}, using that ...

$$
\begin{aligned}
e^{ix} &= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \\
&= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots + \frac{(ix)^n}{n!} + \cdots \\
&= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \cdots \\
&= 1 + ix + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \cdots \qquad (4)
\end{aligned}
$$

We know (4) expands infinitely, and that we are able to regroup the real and imaginary terms.

$$
\begin{aligned}
e^{ix} &= 1 + ix + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \cdots \\
&= \left(1 + \frac{-x^2}{2!} + \frac{x^4}{4!} + \frac{-x^6}{6!} + \frac{x^8}{8!} + \cdots\right) + \left(ix + \frac{-ix^3}{3!} + \frac{ix^5}{5!} + \frac{-ix^7}{7!} + \frac{ix^9}{9!} \cdots\right) \\
&= \left(1 + \frac{-x^2}{2!} + \frac{x^4}{4!} + \frac{-x^6}{6!} + \frac{x^8}{8!} + \cdots\right) + i\left(x + \frac{-x^3}{3!} + \frac{x^5}{5!} + \frac{-x^7}{7!} + \frac{x^9}{9!} \cdots\right) \qquad (5)
\end{aligned}
$$

By grouping the series for e^{ix} in terms of real and imaginary parts, we can clearly see that it is composed of two distinct taylor series that we are familiar with: the sine and cosine functions.

$$
\begin{aligned}
e^{ix} &= \left(1 + \frac{-x^2}{2!} + \frac{x^4}{4!} + \frac{-x^6}{6!} + \frac{x^8}{8!} + \cdots\right) + i\left(x + \frac{-x^3}{3!} + \frac{x^5}{5!} + \frac{-x^7}{7!} + \frac{x^9}{9!} \cdots\right) \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad + \qquad i\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\
&= \cos(x) + i\sin(x) \qquad (6)
\end{aligned}
$$

Thus in the end we get Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.