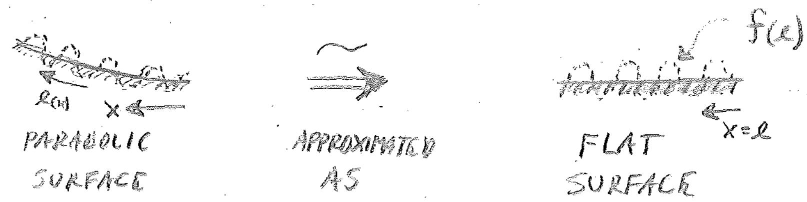
f_i's << MIRROR THICKNESS

$\ell_0 \equiv$ ARC LENGTH OF MIRROR SURFACE

ASSUME WE HAVE A PARABOLIC MIRROR SEGMENT. THE EXTENT OF THE MIRROR IS SMALL wrt THE RADIUS OF CURVATURE. INTRODUCE A SURFACE ROUGHNESS TERM.

$$f(\ell) = \sum_{i=1}^{\infty} f_i \sin\left(\pi i \frac{\ell}{\ell_0}\right)$$



FURTHERMORE, FOLLOWING THE DEVELOPMENT OF THE GENERALIZED PUPIL FUNCTION IN GOODMAN, WE NEED AN ABERRATION FUNCTION, $W(x)$, A PATHLENGTH DEVIATION NORMAL TO THE GAUSSIAN REFERENCE SPHERE IN THE EXIT PUPIL. ASSUME $W(x) = f(\ell)$. FROM GOODMAN, (5-14) GIVES THE DIFF PATT AS THE FRAUNHOFER INTEGRAL OF THE PUPIL FUNCTION AND THIS IS THE SAME FOR THE GENERALIZED PUPIL FUNCTION. THE INTENSITY PATTERN IN THE FOCAL PLANE IS, THEN

$$I_f(u) = \frac{A^2}{\lambda^2 f^2} \mathcal{F}\{\mathcal{P}(x)\}^2$$

SUB (6-33) W/ $P(x) = 1$

$$I_f(u) = \frac{A^2}{\lambda f^2} \mathcal{F}\{e^{jkW(x)}\}^2$$

$$= \frac{A^2}{\lambda f^2} \mathcal{F}\{e^{jkf(\ell)}\}^2$$

BUT, TAKING A FOURIER TRANSFORM OF A SINUSOIDAL EXPONENTIAL IS NOT POSSIBLE, ANALYTICALLY. THUS, EXPAND THE EXPONENTIAL IN A TAYLOR SERIES BY FIRST APPLYING EULER'S FORMULA

$$e^{jkf(\ell)} = e^{j\theta}$$

$$= \cos(\theta) + j\sin(\theta)$$

$$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right] + j\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right]$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

SUBSTITUTE

$$I_f(u) = \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{\left[1 - \frac{\theta^2}{2!}\right] + j\left[\theta - \frac{\theta^3}{3!}\right]\right\}^2$$

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{\left[1 - \frac{(kf(\ell))^2}{2!}\right] + j\left[kf(\ell) - \frac{(k^3 f(\ell))^3}{3!}\right]\right\}^2$$

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{\left[1 - \frac{k^2 f_i \sin(\pi \frac{\ell}{\ell_0})}{2}\right] + j\left[k f_i \sin(\pi \tfrac{\ell}{\ell_0}) - \frac{k^3 \sin^3(\pi \frac{\ell}{\ell_0})}{3!}\right]\right\}^2$$

$f = R/2 \equiv$ FOCAL LENGTH

$$I_{f}(u) = \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{\left[1 - \frac{\theta^2}{2!}\right] + j\left[\theta - \frac{\theta^3}{3!}\right]\right\}^2$$

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{\left[1 - \frac{K^2 f^2(\ell)}{2!}\right] + j\left[K f(\ell) - \frac{K^3 f^3(\ell)}{3!}\right]\right\}^2$$

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{\left[1 - \frac{K^2 f_1 \sin\left(\pi \frac{\ell}{\ell_0}\right)}{2}\right] + j\left[K f_1 \sin\left(\pi \frac{\ell}{\ell_0}\right) - \frac{K^3 \sin^3\left(\pi \frac{\ell}{\ell_0}\right)}{6}\right]\right\}^2$$

TSE SINE

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots$$

SUBSTITUTE TSE OF SINE

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{\left[1 - \frac{1}{2} K^2 f_1 \left(\pi \frac{\ell}{\ell_0} - \frac{1}{6}\left(\pi \frac{\ell}{\ell_0}\right)^3\right)\right]\right.$$

$$+ j\left[K\right.$$

GOODMAN Pg 146

$\mathcal{P}(x,y) \equiv$ COMPLEX AMPLITUDE TRANSMITTANCE FUNCTION
(GENERALIZED PUPIL FUNCTION)

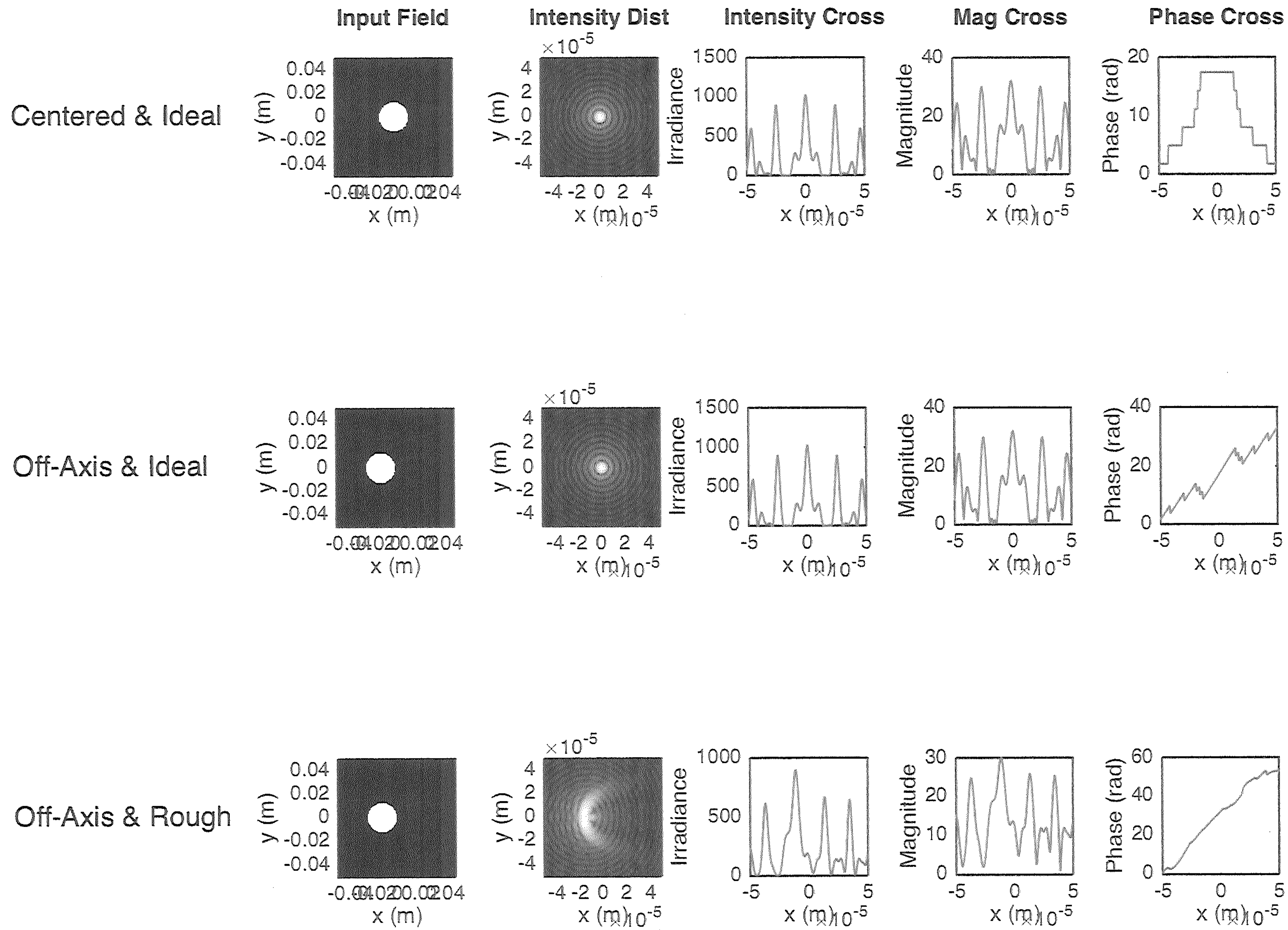
$$\mathcal{P}(x,y) = P(x,y)\, e^{jkW(x,y)}$$

$P(x,y) \equiv$ PUPIL FUNCTION
$W(x,y) \equiv$ EFFECTIVE PATH-LENGTH ERROR
($\perp$ TO GAUSS. REF SPHERE)

QUESTION:

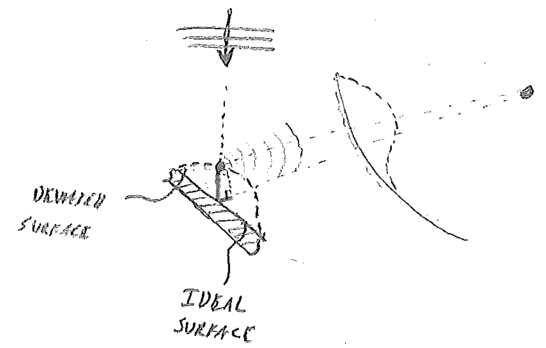
HOW DOES THE DEVIATION OF A PARABOLIC MIRROR TRANSLATE TO AN EFFECTIVE PATH-LENGTH ERROR, $W(x,y)$?

Focal Plane Diffraction Pattern



WHAT DOES PHASE MEAN ?

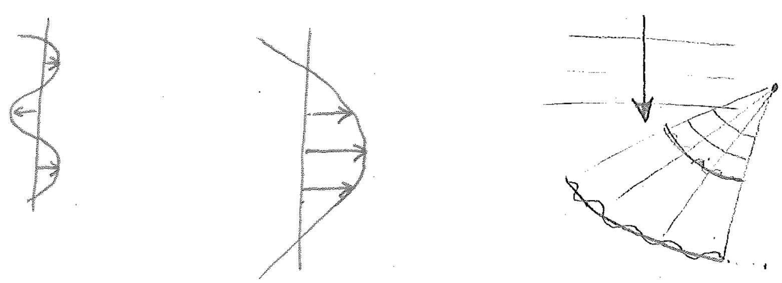
THE IDEAL PARABOLIC MIRROR REFLECTS A FLAT WAVEFRONT TO A CIRCULAR WAVEFRONT CENTERED ON THE FOCAL POINT. HYUGEN'S PRINCIPLE CAN REVEAL MUCH ABOUT THE WAVEFRONT REFLECTED BY THE DEVIATED SURFACE

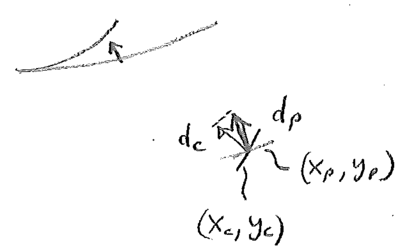


LET'S JUST SAY THE WAVEFRONT DEVIATION FROM THE GAUSSIAN REF SPHERE IN THE EXIT PUPIL IS EQUAL TO THE SURFACE ROUGHNESS TERM

CHANGE DEVIATION NORMAL TO PARABOLIC SURF $(d_p(R(t)))$ INTO DEVIATION IN RADIAL (DIR OF REFL. RAY)

$$\vec{d}_p = |\vec{d}_p|\, \hat{n}_p(x,y)$$

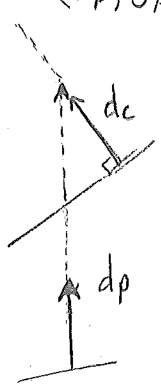




LET US ASSUME THE MIRROR IMPARTS AN EFFECTIVE PATH LENGTH ERROR EQUAL TO $W(\ell)$ PROJECTED ONTO THE CIRCLE (GAUSS REF SPH) $\Rightarrow$ PROJECT DEVIATION DEFINED NORMAL TO PARABOLA ONTO NORMAL OF CIRCLE

PARAMETRIC PARABOLA	PARAMETRIC CIRCLE
$x_i(t) = 2ft$	$x_i(s) = \frac{1-s^2}{1+s^2}$
$y_i(t) = ft^2$	$y_i(s) = \frac{2s}{1+s^2}$

PROJECT: $\vec{d}_p(x_{p_i}, y_{p_i}) = \langle x_p, y_p\rangle - \langle x_{p_i}, y_{p_i}\rangle$ ONTO CIRCLE OF RADIUS (f)



* JUST ASSUME DEVIATION OF SURFACE NORMAL TO IDEAL PARABOLIC ARC IS EQUAL TO THE DEVIATION FROM GAUSS REF SPHERE IN EXIT PUPIL

$\Rightarrow$ IMAGE INTENSITY DISTRIBUTION IS

$$(5\text{-}16)\ \&\ (5\text{-}14) \Rightarrow I_f(u,v) = \frac{A^2}{\lambda^2 f^2}\left[\iint_{-\infty}^{\infty} t_A(x,y)\, \underbrace{\mathcal{P}(x,y)}_{} \exp\left[-j\frac{2\pi}{\lambda f}(xu+yv)\right] dx\,dy\right]^2 \quad \mathcal{P}(x,y) \,/\!/\, W(x,y)$$

$x_p = -2ft$

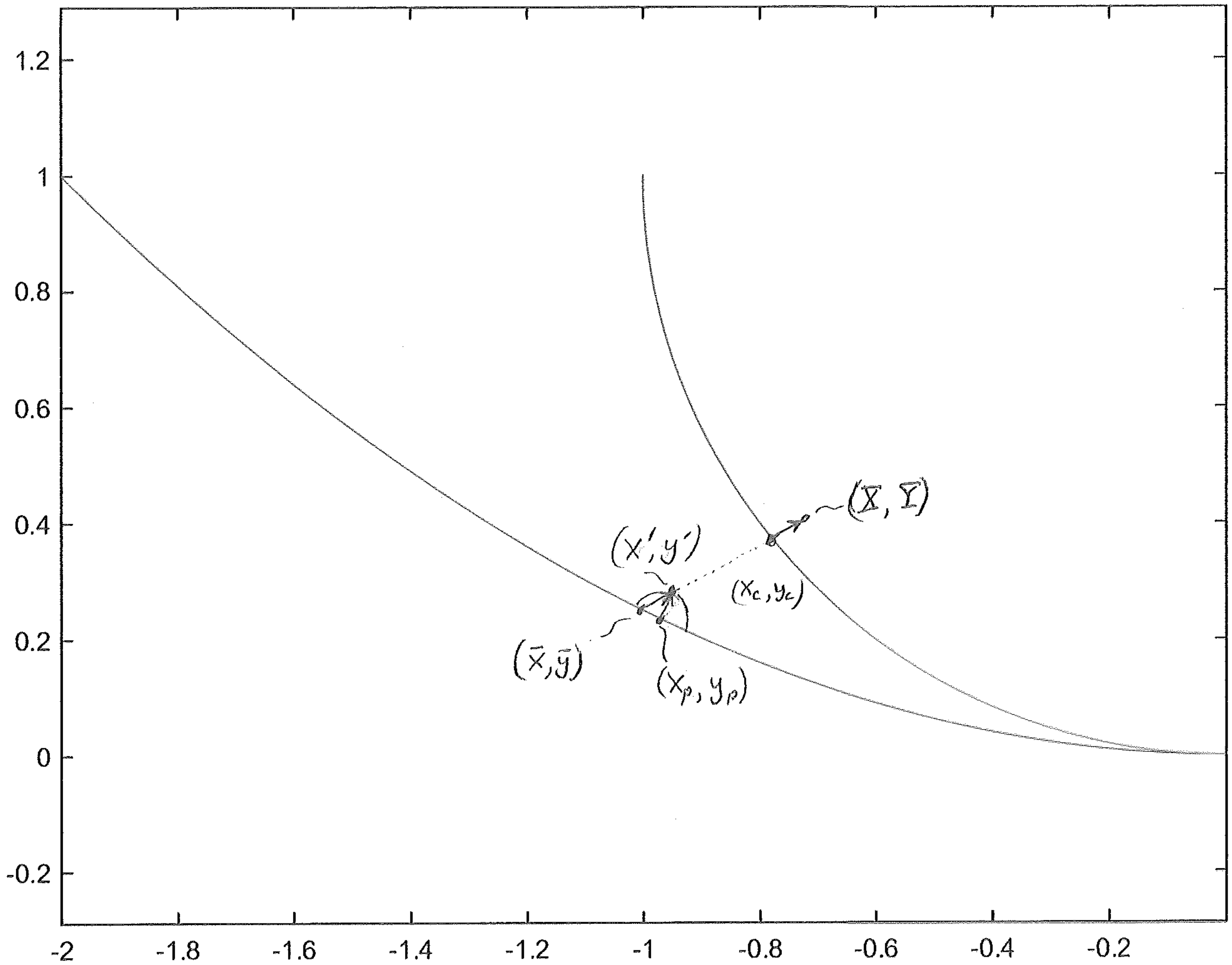
$y_p = ft^2$

$x_c = -\frac{1-t^2}{1+t^2}$

$y_c = 1 - \frac{2t}{1+t^2}$

$$y_c(x_c) = \sqrt{r^2 + x_c^2}$$
$$= \sqrt{r^2 + \left(\frac{1-t^2}{1+t^2}\right)^2}$$
$$= \sqrt{r^2 + \frac{1-2t^2+t^4}{1+2t^2+t^4}}$$

$$\begin{array}{r|l} & 1 - \frac{4}{t^2} \\ t^4+2t^2+1 & t^4-2t^2+1 \\ & -(t^4+2t^2+1) \\ & -4t^2 \\ & -(-4t^2-8-\frac{4}{t^2}) \\ & 8+\frac{4}{t^2} \end{array}$$



$$\vec{d}_p(x,y) = \langle x', y' \rangle - \langle x_p, y_p \rangle$$

$$\vec{d}_c(x,y) = \langle x', y' \rangle - \langle \bar{x}, \bar{y} \rangle$$

$$I_f = \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{ t_A(x)\, \mathcal{P}(x) \right\}^2$$

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{ e^{-j \frac{k}{2f}(x^2)} e^{j k f_1 \sin(x)} \right\}^2$$

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{ e^{jk\left[f_1 \sin(x) - \frac{x^2}{2f}\right]} \right\}^2$$

$$I_f(u) = \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{\mathcal{P}(x)\right\}^2$$

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{e^{jkW(x)}\right\}$$

$$\Rightarrow W(\ell) = \sum_{i=1}^{\infty} c_i \sin\left(\pi i \frac{\ell}{\ell_0}\right)$$

FOR PARABOLIC ARC, IF $\ell \equiv$ ARCLENGTH

$$\ell(x) = \frac{1}{2}\left| x\sqrt{\frac{x^2}{f^2}+1} + \frac{1}{f}\,\operatorname{asinh}\left(\frac{x}{f}\right)\right|$$

FOR SPHERE ARC; IF $\ell \equiv$ ARCLENGTH

$$\ell(x) = \int_0^x \left[1 + \frac{\xi^2}{1-2\xi^2+\xi^4}\right]^{1/2} d\xi$$

FOR ℓ = RADIUS = X, $\ell_0 = R$, i.e. $W_{(x)} \equiv$ DEVIATION IN SURFACE IN DIR. OF FOCAL POINT w/ RADIAL POSITION

$$W(\ell) = W(x) = \sum_{i=1}^{\infty} c_i \sin\left(\pi i \frac{x}{R}\right)$$

$\Rightarrow$ TAYLOR EXPANSION OF $e^{jkW(x)} = g_i($

CONVERGENCE TEST

FOR INCREASING TSK EXPANSION

DIAMOND TURNING

— PSF GIVES DIST. IN CONJUGATE PLANE

— CONJUGATE PLANE TO FOCAL PLANE IS INFINITELY FAR

— A POINT SOURCE INFINITELY FAR MAKES PLANE WAVE AT APERATURE

FRAUNHOFER INTEGRAL OF $\mathcal{P}$ IS PSF OF ABERRATED SYS

$$e^{jkW} = \cos(kW) + i\sin(kW)$$

$$e^{j\theta} = \left[1 + \frac{-\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^8}{8!} + \cdots\right] + i\left[\theta + \frac{-\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots\right]$$

ANALYTIC SOLUTION TO FRAUNHOFER INTEGRAL W/ ABERRATION

$$I_f(u) = \frac{A^2}{\lambda^2 f^2}\left[\int_{-\infty}^{\infty} t_A(x)\, \mathcal{P}(x) \exp\left[-j\frac{2\pi}{\lambda f} xu\right] dx\right]^2$$

$$t_A = 1 \qquad \mathcal{P}(x) = P(x)\, e^{jkW(x)} = e^{jkW(x)}$$

$$I_f(u) = \frac{A^2}{\lambda^2 f^2}\left[\int_{-R}^{0} \exp\left[jk\left(W(x) - \frac{xu}{f}\right)\right] dx\right]^2$$

$$W(x) = \sum_{i=1}^{\infty} c_i \sin\left(\pi i \frac{\ell(x)}{\ell_0}\right)$$

$$\Rightarrow \ell(x) = \left|\frac{x\sqrt{\frac{x^2}{f^2}+1}}{2} + \frac{\operatorname{asinh}\left(\frac{x}{f}\right)}{2/f}\right|$$

$$= \frac{1}{2}\left|x\sqrt{\frac{x^2}{f^2}+1} + \frac{1}{f}\operatorname{asinh}\left(\frac{x}{f}\right)\right|$$

$$= \sum_{i=1}^{\infty} c_i \sin\left(\pi i \frac{1}{\ell_0}\frac{1}{2}\left(|x|\sqrt{\frac{x^2}{f^2}+1} + \frac{1}{f}\operatorname{asinh}\left(\frac{x}{f}\right)\right)\right)$$

$$I_f(u) = \frac{A^2}{\lambda^2 f^2}\mathcal{F}\{\mathcal{P}(x)\}^2 = \int_{-\infty}^{\infty} \mathcal{P}(x) \exp\left[-j2\pi f_x x\right] dx = \int_{-\infty}^{\infty} \mathcal{P}(x) \exp\left[-j\frac{2\pi}{\lambda f} u x\right] dx$$

$$= \mathcal{F}\{e^{jkW(x)}\} = \mathcal{F}\{e^{j\pi\frac{2}{\lambda}W(x)}\}$$

$$= \mathcal{F}\{e^{j\pi}$$

$$I_f(u) = \frac{A^2}{\lambda^2 f^2} \mathcal{F}\{t_A(x)\,\mathcal{P}(x)\}^2$$

$$t_A = \begin{cases} 1 & \text{INSIDE} \\ 0 & \text{OUTSIDE} \end{cases} \qquad \mathcal{P}(x) = P(x)\, e^{jkW(x)} = e^{jkW(x)} \quad \text{(INSIDE)}$$

$$I_f(u) = \frac{A^2}{\lambda^2 f^2} \mathcal{F}\{e^{jkW(x)}\}$$

$\ell = X \qquad \ell_0 = R$

$$\Rightarrow \quad W(x) = \sum_{i=1}^{\infty} c_i \sin\left(\pi i \frac{\ell}{\ell_0}\right)$$

$$\cancel{\Rightarrow \quad \ell(x) = \int_0^X \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}\, d\xi}$$

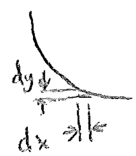
$y_c = \sqrt{r^2 - x_c^2}$

$y_c' = \frac{1}{2}(r^2 - x_c^2)^{-1/2}(-2x_c)$

$= -\frac{x_c}{\sqrt{r^2 - x_c^2}}$

$$\cancel{= \int_0^X \left[1^2 + \left(-\frac{\xi}{\sqrt{1-\xi^2}}\right)^2\right]^{1/2} d\xi}$$

$$\cancel{= \int_0^X \left[1 + \frac{\xi^2}{1 - 2\xi^2 + \xi^4}\right]^{1/2} d\xi}$$



LET $W(x) = 10^{-5} \sin\left(\pi \frac{x}{R}\right)$

$$I_f(u) = \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{e^{jk(10^{-5}\sin(\frac{\pi}{R}x))}\right\}$$

$k = \frac{2\pi}{\lambda}$

$$= \frac{A^2}{\lambda^2 f^2} \mathcal{F}\left\{e^{j\pi\left(\frac{2}{\lambda}\times 10^{-5}\sin\left(\frac{\pi}{R}x\right)\right)}\right\}$$

$f_x = \frac{u}{\lambda f}$

$$= \frac{A^2}{\lambda^2 f^2} \left| \int_{-\infty}^{\infty} e^{jk\, c_1 \sin\left(\frac{\pi}{R}x\right)} e^{-j\frac{2\pi}{\lambda f}xu}\, dx \right|^2$$

$\int e^u du = e^u + C$

$$= \frac{A}{\lambda^2 f^2} \left| \int_{-\infty}^{\ell} e^{jk\left(c_1 \sin\left(\frac{\pi}{R}x\right) - \frac{1}{f}xu\right)}\, dx \right|^2$$

$$H(f_x) = \mathcal{P}(\lambda z_i f_x) = P(\lambda z_i f_x)\, \exp[jkW(\lambda z_i f_x)]$$

$$H(u) = P(u)$$

A (10×1) B (10×2)

A(i) B(i, :)

$\mathcal{P}(\ell) \cong 1 + jKf_1 \sin(\pi \frac{\ell}{\ell_0})$

$= 1 + jKf_1 \pi \frac{\ell}{\ell_0}$

$\mathcal{F}\{(-j\ell)^n f(\ell)\} = \frac{d^n F(u)}{du^n}$

$\mathcal{F}\{j\ell\} = \delta(u)' (2\pi)$?

$\hookrightarrow \mathcal{F}\{1 + jKf_1 \pi \frac{\ell}{\ell_0}\} = F(u)$

$F(u) = \int_{-\infty}^{\infty} (1) e^{-j\ell u} du + \int_{-\infty}^{\infty} (jKf_1 \pi \frac{\ell}{\ell_0}) e^{-j\ell u} du$

$= 2\pi \delta(u) - \frac{jKf_1\pi}{\ell_0} (2\pi) \delta'(u)$

$\int_{-\infty}^{\infty} (j\ell) e^{-j\ell u} du = \left[\frac{j\ell}{-j\ell} e^{-j\ell u}\right]_{-\infty}^{\infty}$

$= \left[-e^{-j\ell u}\right]_{-\infty}^{\infty}$

$= -e^{-j\ell\infty} + e^{j\ell\infty}$

$= 2j\sin(\ell\infty)$

$\int_D v' f \, dx = -\int_D w f' dx$

QUALITATIVE ANALYSIS OF AN ANALYTIC SOLUTION PRESENTED QUANTITATIVELY

FUCK THAT.

THE ANALYTIC SOLUTION OF THE TSE DOES NOT EXIST

↳ THIS IS BECAUSE WE MUST $\mathcal{F}\{j\ell^n\}$ ANY POLYNOMIAL APPROX (e.g. gPC) IS GOING TO HAVE THE SAME PROBLEM

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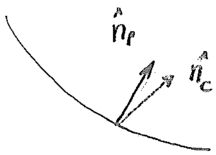
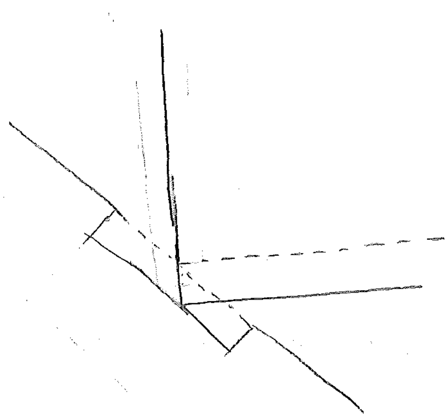


EXIT PUPIL PATHLENGTH DEV = ROUGHNESS TERM

GENERALIZED PUPIL FUNCTION (ASSUME $W(x) = d_c(x)$)



$$\cancel{d(\ell(t)) = d(t)}$$

$$y = \frac{1}{2f}x^2 = \frac{1}{2f}(2ft^2) = ft^2 \quad \checkmark$$

$$y' = \frac{x}{f}$$

ARC LENGTH AS A FUNCTION OF RADIUS

$$\ell(x) = \int_0^x \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2}\, d\tau$$

$$= \int_0^x \left[1^2 + \left(\frac{\tau}{f}\right)^2\right]^{1/2} d\tau$$

$$\ell_0 = \ell(|x|)\Big|_0^1$$

$$\ell_0 = \frac{\operatorname{asinh}\left(\frac{1}{f}\right)}{2/f} + \frac{\sqrt{\frac{1}{f}+1}}{2}$$

$$\rightarrow \ell(x) = \left| \frac{x\sqrt{\frac{x^2}{f^2}+1}}{2} + \frac{\operatorname{asinh}\left(\frac{x}{f}\right)}{2/f} \right|$$

$$K = \frac{2\pi}{\lambda}$$

$$\underbrace{d(\ell(x))}_{= W(x)} = \sum_{i=1}^{\infty} f_i \sin\left(\pi i \frac{\ell(x)}{\ell_0}\right)$$

$$I_f(u) = \frac{A^2}{\lambda^2 f^2}\left| \int_{-\infty}^{\infty} t_A(x)\, \mathcal{P}(x) \exp\left[-j\frac{2\pi}{\lambda f}xu\right] dx \right|^2$$

$$I_f(u) = \frac{A^2}{\lambda^2 f^2}\left| \int_0^1 \exp\left[jK\left(W(x) - \frac{xu}{f}\right)\right] dx \right|^2$$

$$t_A = 1 \qquad \mathcal{P}(x) = P(x)e^{jKW(x)} = e^{jKW(x)}$$

$\mathcal{F}\{e^{j\sin(t)}\}$ $\qquad\qquad$ $\mathcal{F}\{e^{-j2\pi t\omega_0}\} = \delta(\omega-\omega_0)$

$$e^{j\sin(t)} = \left[1-\frac{\sin(t)}{2}\right] + j\left[\sin(t) - \frac{\sin^3(t)}{6}\right] = \cos(\sin(t)) + j\sin(\sin(t))$$

$$\sin(t) = t - \frac{t^3}{6}$$

$$\mathcal{F}\left\{\left[1-\tfrac{1}{2}\left(t-\tfrac{1}{6}t^3\right)\right] + j\left[\left(t-\tfrac{1}{6}t^3\right) - \tfrac{1}{6}\left(t-\tfrac{1}{6}t^3\right)^3\right]\right\}$$

$$\mathcal{F}\left\{\left[1-\tfrac{1}{2}t+\tfrac{1}{12}t^3\right] + j\left[t-\tfrac{1}{6}t^3-\tfrac{1}{6}\underbrace{\left(t^2-\tfrac{1}{3}t^4+\tfrac{1}{36}t^6\right)\left(t-\tfrac{1}{6}t^3\right)}_{\left(t^3-\frac{1}{3}t^5+\frac{1}{36}t^6-\frac{1}{6}t^5+\frac{1}{18}t^7-\frac{1}{216}t^9\right)}\right]\right\}$$

$$\mathcal{F}\left\{\left[1-\tfrac{1}{2}t+\tfrac{1}{12}t^3\right] + j\left[t-\tfrac{1}{6}t^3-\tfrac{1}{6}t^3+\tfrac{1}{18}t^5-\tfrac{1}{216}t^6+\tfrac{1}{36}t^5-\tfrac{1}{108}t^7+\tfrac{1}{1296}t^9\right]\right\}$$

$$\mathcal{F}\left\{\left[1-\tfrac{1}{2}t+\tfrac{1}{12}t^3\right] + j\left[t-\tfrac{1}{3}t^3+\tfrac{1}{12}t^5-\tfrac{1}{216}t^6-\tfrac{1}{108}t^7+\tfrac{1}{1296}t^9\right]\right\}$$

$$\Rightarrow \mathcal{F}\{(-jt)^n f(t)\} = \frac{d^n F(\omega)}{d\omega^n}$$

$$\hookrightarrow \mathcal{F}\{-jt(-1)\} = -2\pi\,\delta'(\omega)$$

$\mathcal{F}\{e^{j\sin(t)}\}$ $\qquad$ $\mathcal{F}\{e^{jt^2}\}$ $\qquad$ $\mathcal{F}\{e^{j\sin(\theta)}\}$

$\mathcal{F}\{e^{j(t-\frac{1}{6}t^3)}\}$ $\qquad$ t_A $\qquad$ $\approx \mathcal{F}\{e^{j(\theta-\frac{\theta^3}{3!})}\}$

$\mathcal{F}\{e^{jt} \div e^{j\frac{1}{6}t^3}\}$ $\qquad$ $\mathcal{F}\{e^{jK(f_1\sin(\pi\frac{\ell}{\ell_0}))}\}$

$\mathcal{F}\{$ $\qquad$ PADÉ APPROX $\qquad$ $\approx \mathcal{F}\left\{e^{jK\left(f_1\left(\frac{\pi}{\ell_0}\ell-\frac{1}{6}\left(\frac{\pi}{\ell_0}\ell\right)^3\right)\right)}\right\}$