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Modal coupling in linear control systems using robust eigenstructure assignment [1]

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Abstract

Eigenvalue assignment methods are used widely in the design of control and state-estimation systems. The corresponding eigenvectors can be selected to ensure robustness. For specific applications, eigenstructure assignment can also be applied to achieve more general performance criteria. In this paper a new output feedback design approach using robust eigenstructure assignment to achieve prescribed mode input and output coupling is described. A minimisation technique is developed to improve both the mode coupling and the robustness of the system, whilst allowing the precision of the eigenvalue placement to be relaxed. An application to the design of an automatic flight control system is demonstrated. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

The inverse problem of eigenvalue (or pole) assignment by output feedback arises frequently in control system design. Robustness of the design can be ensured by assigning the eigenvectors (or modal vectors), as well as the eigenvalues, of the system [1,2]. For specific applications, eigenstructure assignment can be used to achieve more general performance criteria [3]. In the design of aircraft guidance and control systems, prescribed mode input and mode output coupling is a major objective. Current techniques for achieving the desired

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coupling do not, however, ensure the robustness of the system and, although exact mode output coupling can be partially attained, the mode input coupling may be unsatisfactory [4,5].

A new design approach is developed in this paper, which exploits fully the degrees of freedom in the system in order to improve both mode input/output coupling and robustness, whilst allowing the precision of the eigenvalue placement to be relaxed. The corresponding nonlinear eigenstructure assignment problem is shown to be equivalent to a linear least-squares problem that can be solved directly by standard techniques. The application of this procedure to an aircraft design problem is described. In Section 2 notation is introduced and the mathematical problem is stated. Current methods for treating the problem are reviewed in Section 3. The new algorithm is established in Section 4 and its application is illustrated in Section 5. Section 6 concludes with a summary of the results.

2. Statement of the problem

We consider the linear time-invariant system governed by the equations

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$ are the state, input and output variables, respectively. Matrices A, B, C are assumed constant with B and C of full rank.

In practice we are concerned with *nonlinear* systems. The governing equations (1) for the nonlinear problem are obtained by linearization about a steady-state or equilibrium condition, and the system variables $\mathbf{x}$, $\mathbf{u}$, $\mathbf{y}$ then denote displacements from the equilibrium.

The output response of the system (1) can be written as

$$\mathbf{y}(t) = C\mathbf{x}(t) = \sum_{i=1}^{n}(C\mathbf{v}_i)\mathrm{e}^{\lambda_i t}\mathbf{w}_i^{\mathrm{T}}\mathbf{x}_0 + \sum_{i=1}^{n}(C\mathbf{v}_i)(\mathbf{w}_i^{\mathrm{T}}B)\int_0^t \mathrm{e}^{\lambda_i(t-s)}\mathbf{u}(s)\,\mathrm{d}s, \tag{2}$$

where λ_i are the eigenvalues and $\mathbf{v}_i, \mathbf{w}_i^{\mathrm{T}}$ are the corresponding right and left eigenvectors of A, respectively. We write $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$, $W^{\mathrm{T}} = [\mathbf{w}_1, \ldots, \mathbf{w}_n]^{\mathrm{T}}$ and assume that $V^{-1} = W^{\mathrm{T}}$, where the columns of V are normalised to unit length.

From Eq. (2) it is seen that the response of the system depends on:
- the eigenvalues, which determine the *decay/growth rate* of the response;
- the eigenvectors, which determine the *shape* of the response;
- the initial condition of the system, which determines the *degree* to which each mode participates in the free response.

We note that the vector $(C\mathbf{v}_i)$ in Eq. (2) determines the outputs participating in the response of each mode and the vector $(\mathbf{w}_i^{\mathrm{T}}B)$ determines those state variables in each mode that are affected by each input. These vectors are defined (as in [5]), to be the *mode output coupling vectors* and the *mode input coupling vectors*, respectively.

The interactions between the outputs and inputs of the system are dependent upon the mode coupling vectors and can be determined directly from the matrices CV and $W^{\mathrm{T}}B$. Specifically, since the ith mode of the system is excited by the jth input in proportion to the element $(W^{\mathrm{T}}B)_{i,j}$ and the kth output depends on the ith mode in proportion to the element $(CV)_{k,i}$, then the jth input u_j and the kth output y_k are completely decoupled if and only if the mode coupling vectors of the system are such that

$$\sum_{i=1}^{n}(CV)_{k,i}(W^{\mathrm{T}}B)_{i,j} = 0.$$

It is immediately apparent that it is not possible to specify all of the mode output and input coupling vectors independently because of the relationship $V^{-1} = W^{\mathrm{T}}$. In practice, only the mode coupling vectors associated with a subset of the eigenvalues may be of significance in the design process. We denote

$$G_0 = CV_1, \qquad G_1 = W_1^{\mathrm{T}}B,$$

where the columns of $V_1 = [\mathbf{v}_1, \ldots, \mathbf{v}_q]$ and $W_1^{\mathrm{T}} = [\mathbf{w}_1, \ldots, \mathbf{w}_q]^{\mathrm{T}}$ correspond, respectively, to the right and left eigenvectors associated with a specified subset $L_q = \{\lambda_1, \ldots, \lambda_q\}$ of the eigenvalues of the system. The aim of the design process is to achieve the desired mode output and mode input coupling of the system, as defined by the matrices G_0 and G_1, respectively, by assigning the eigenstructure of the system appropriately.

The following example illustrates the interactions between the modes and the inputs/outputs for a specific choice of G_0 and G_1.

Example 2.1. We consider a system of dimensions $n = 7$, $m = 2$, $p = 4$; the desired $q = p = 4$ mode input and output coupling vectors are:

$$G_1 = W_1^{\mathrm{T}}B = \begin{array}{c} \text{Inputs}(j) \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \text{Modes}(i), \tag{3}$$

$$G_0 = CV_1 = \begin{array}{c} \text{Modes}(i) \\ \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \text{Outputs}(k), \tag{4}$$

where $*$ denotes an arbitrary value. Here the first input excites the first mode (since $(G_1)_{1,1} = 1$), which is directly coupled to the first output $((G_0)_{1,1} = *)$ but does not affect the second output $((G_0)_{2,1} = 0)$. The second input, however, excites the third mode $((G_1)_{3,2} = 1)$, which does affect the second output $((G_0)_{2,3} = *)$. Examining all of the elements $(G_0 G_1)_{k,j}$ shows that the first and third outputs are coupled only to the first input, while the second and fourth outputs are coupled only to the second input.

In order to achieve the desired mode output and input coupling, output feedback is used to alter the response of the system and to assign the required eigenstructure. The feedback takes the form

$$\mathbf{u} = K\mathbf{y} - \mathbf{r} = KC\mathbf{x} - \mathbf{r},$$

where $\mathbf{r}$ is the reference (or demand) vector. The system (1) is transformed by the feedback into the closed loop system

$$\dot{\mathbf{x}} = (A + BKC)\mathbf{x} - B\mathbf{r}. \tag{5}$$

The design objective is to select the feedback gain matrix K to assign a specified set of eigenvalues and corresponding sets of mode output and input coupling vectors to the closed loop system matrix $(A + BKC)$. As previously indicated, the entire eigenstructure cannot be assigned arbitrarily and only a subset of the eigenvalues and eigenvectors may be specified. Robustness of the closed loop system is also important, in the sense that the assigned eigenstructure needs to be insensitive to perturbations in the system matrices.

The design problem can be stated as follows:

Problem 2.2. Given the real triple (A, B, C) and a self-conjugate set of scalars $L_q = \{\lambda_1, \dots, \lambda_q\}$, together with corresponding self-conjugate sets of n-dimensional mode coupling vectors, $G_0 = [\mathbf{g}_{01}, \dots, \mathbf{g}_{0q}], G_1 = [\mathbf{g}_{11}, \dots \mathbf{g}_{1q}]$, find a real $(m \times p)$ matrix K such that L_q is a subset of the eigenvalues of the closed loop system matrix $(A + BKC)$ with corresponding mode output and input coupling vectors, G_0 and G_1, respectively, and such that the closed loop system is stable and some measure of the robustness of the system is minimized.

It can be shown [2] that the right and left eigenvectors $\mathbf{v}_i, \mathbf{w}_i^{\mathrm{T}}$ corresponding to an assigned eigenvalue λ_i of $A + BKC$ must be such that

$$\mathbf{v}_i \in \mathscr{S}_i \equiv \mathscr{N}[U_1^{\mathrm{T}}(A - \lambda_i I)], \quad \mathbf{w}_i \in \mathscr{T}_i \equiv \mathscr{N}[P_1^{\mathrm{T}}(A^{\mathrm{T}} - \lambda_i I)], \tag{6}$$

respectively, where $\mathscr{N}(\cdot)$ denotes right null space, and U_1^{T}, P_1 are determined by the QR decompositions of B and C, given respectively by

$$B = [U_0, U_1]\begin{bmatrix} Z_B \\ 0 \end{bmatrix}, \qquad C = [Z_C, 0]\begin{bmatrix} P_0^{\mathrm{T}} \\ P_1^{\mathrm{T}} \end{bmatrix}. \tag{7}$$

These conditions are both necessary and sufficient for the existence of a real feedback matrix K that assigns a set $L_n = \{\lambda_1, \ldots, \lambda_n\}$ of n prescribed eigenvalues and a corresponding nonsingular matrix of prescribed right eigenvectors $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$, where the left eigenvectors are determined by $W = [\mathbf{w}_1, \ldots, \mathbf{w}_n] = V^{-T}$.

It is evident from this result that for a given set of eigenvalues, it is not possible to assign arbitrary eigenvectors to the closed loop system. If the system is controllable and observable, it is possible to construct a feedback to assign exactly p eigenvalues and p corresponding right eigenvectors, (where at most $\min\{m, p\}$ components are specified in each eigenvector) [4,5]. Alternatively, it is possible to assign a full set of n arbitrarily prescribed eigenvalues and corresponding right eigenvectors approximately and to control the sensitivity of the whole eigenstructure so as to ensure the desired accuracy of the eigenvalues [2]. In Section 3 we review numerical algorithms for achieving these results, and in Section 4 we develop a new approach for assigning both right and left eigenvectors to match desired mode output and input coupling vectors as accurately as possible in a least squares sense.

3. Eigenstructure assignment methods

We now review two techniques that are frequently used in practice to treat the eigenstructure assignment problem. The first of these exactly assigns a prescribed set of p eigenvalues and assigns as accurately as possible a corresponding prescribed set of p right eigenvectors. This approach is commonly used to achieve desired output coupling vectors [4,5]. The alternative approach is to assign a full set of n prescribed eigenvalues approximately. The corresponding eigenvectors are selected to ensure the robustness of the closed loop system [2].

3.1. Exact partial assignment for output coupling

The aim is to assign p prescribed eigenvalues λ_i and p desired right eigenvectors $\mathbf{v}_{di}, i = 1, \ldots, p$. Each eigenvector must lie in the corresponding subspace $\mathscr{S}_i$, given by Eq. (6) and, therefore, an arbitrary vector $\mathbf{v}_{di}$ may not be achievable. The best achievable vector $\mathbf{v}_{ai}$, in a least squares sense, is given by the projection of the desired vector into the required subspace. If the columns of the matrix S_i give an orthonormal basis for $\mathscr{S}_i$, then the best achievable (unnormalised) vector can be written as

$$\mathbf{v}_{ai} = S_i S_i^{+} \mathbf{v}_{di}, \tag{8}$$

where $(\cdot)^{+}$ denotes the Moore–Penrose pseudo-inverse.

We let $V_1 = [\mathbf{v}_{a1}, \ldots, \mathbf{v}_{ap}]$ be the matrix of achievable eigenvectors and let $\Lambda_1 = \mathrm{diag}\{\lambda_1, \ldots, \lambda_p\}$. It can be shown [4,6] that if V_1 is of full rank and CV_1

is invertible, the prescribed eigenvalues and the best achievable eigenvectors are assigned exactly by the feedback matrix

$$K = Z_B^{-1} U_0^{\mathrm{T}} (V_1 \Lambda_1 - A V_1)(C V_1)^{-1}, \tag{9}$$

where Z_B and U_0 are determined by the decomposition (7) of B.

In practice, the mode output coupling vectors, are prescribed, instead of the eigenvectors, in order to ensure the desired transient output response of the closed loop system. A complete specification of the desired coupling vectors is, in general, neither required nor known and the designer is interested only in certain elements of each vector. Following the theory in [4] a desired mode output coupling vector can be written in the form

$$\mathbf{g}_{0\mathrm{d}} = [g_{0\mathrm{d}1}, \ldots, *, g_{0\mathrm{d}j}, \ldots, *, g_{0\mathrm{d}k}]^{\mathrm{T}},$$

where $g_{0\mathrm{d}j}$ are designer specified components (usually 0 or 1 which represent decoupling and coupling respectively) and $*$ is an unspecified component.

From Eq. (6), the vector $\mathbf{g}_{0\mathrm{d}}$ must reside in $C\mathscr{S}_i$, where $\mathscr{S}_i$ is associated with the corresponding eigenvalue λ_i. The desired coupling vectors may not lie in the required subspace, however, and hence may not be achievable. Instead a 'best possible' choice is made by projecting the specified part of $\mathbf{g}_{0\mathrm{d}}$ into the corresponding part of $C\mathscr{S}_i$. We define a permutation matrix, $\mathscr{P}$, such that

$$\mathscr{P}\mathbf{g}_{0\mathrm{d}} = \begin{bmatrix} \mathbf{d} \\ \mathbf{n} \end{bmatrix}, \qquad \mathscr{P}CS_i = \begin{bmatrix} D_i \\ N_i \end{bmatrix},$$

where $\mathbf{d}$ and $\mathbf{n}$ are the vectors of specified and unspecified components, respectively. The best achievable vector corresponding to a desired vector is then

$$\mathbf{g}_{0\mathrm{a}} = CS_i D_i^{+} \mathbf{d}. \tag{10}$$

This vector is exactly equal to the desired vector $\mathbf{g}_{0\mathrm{d}}$ if the number of prescribed elements is $k \leqslant m$ and the matrix D_i has full rank equal to k.

From Eq. (10) we can construct the corresponding best achievable eigenvector as $\mathbf{v}_{\mathrm{a}i} = S_i D_i^{+} \mathbf{d}$ (since $\mathbf{g}_{0\mathrm{a}} = C\mathbf{v}_{\mathrm{a}i}$). If the matrix $V_1 = [\mathbf{v}_{\mathrm{a}1}, \ldots, \mathbf{v}_{\mathrm{a}p}]$, constructed from these vectors, is of full rank and CV_1 is invertible, then the feedback matrix K given by Eq. (9) exactly assigns the prescribed eigenvalues and the best achievable mode output coupling vectors to the closed loop system (5) [6]. General conditions on G_0 ensuring that CV_1 is invertible for a given choice of L_p are difficult to formulate. An approximate result can, however, always be achieved by using the Moore–Penrose pseudo-inverse of CV_1. (See Section 4.2.) In practical applications it seems that realistic choices of G_0 and L_p lead generically to invertible matrices CV_1.

Although this approach allows us to assign some components in the output coupling modes exactly, it suffers from a number of disadvantages:

- the unspecified eigenvalues and eigenvectors cannot be controlled and the resulting closed loop system may display poor behaviour, even becoming *unstable*;
- the robustness of the closed loop system is not guaranteed and the system may be highly sensitive to small disturbances, parameter estimations and/or nonlinear effects;
- the desired mode input coupling vectors are not generally achieved through this procedure and the forced response of the closed loop system to the inputs may be unacceptable.

An alternative approach is to assign all of the eigenvalues approximately, and to select the eigenvectors to ensure robustness. A technique for achieving this result is described in Section 3.2.

3.2. *Approximate full assignment for robustness*

The objective is now to assign a full set of n prescribed eigenvalues λ_i and to assign a corresponding set of right eigenvectors $\mathbf{v}_i, i = 1, \ldots, n$, such that the eigenstructure of the closed loop system is robust, or as insensitive as possible to perturbations. In general, an arbitrary set of n eigenvalues cannot be assigned exactly by output feedback, since the corresponding right and left eigenvectors must lie *simultaneously* in the spaces $\mathscr{S}_i$ and $\mathscr{T}_i$, defined by Eq. (6). We aim, therefore, to select n right eigenvectors from the subspaces $\mathscr{S}_i$ such that the distances of the corresponding left eigenvectors from the subspaces $\mathscr{T}_i$ are minimised and such that robustness is ensured. It can be shown [2] that a feedback matrix can then be constructed so that the eigenvalues of the closed loop system are approximately equal to the prescribed values.

A measure of the robustness of the eigenstructure of the closed loop system is given by the Frobenius condition number of the matrix V of its right eigenvectors [7]. If the assigned eigenvectors $\mathbf{v}_i$ are independent and normalized to unit length, then the robustness measure is given by

$$J_1 = \|V^{-1}\|_F^2. \tag{11}$$

If the prescribed eigenvalues are distinct and the system (1) is completely controllable, then the right eigenvectors can be selected to be independent. If multiple eigenvalues are to be assigned, then the maximum feasible multiplicity is equal to m, the number of inputs. If defective eigenvalues are assigned then the system is *not* robust. If the system (1) is not completely controllable, then, as long as the uncontrollable poles are included in the set to be assigned, the corresponding eigenvectors can be reassigned to improve the robustness [1].

If $\mathbf{w}_i^{\mathrm{T}} = \mathbf{e}_i^{\mathrm{T}} V^{-1}$ are the left eigenvectors corresponding to the assigned right eigenvectors $\mathbf{v}_i$, for $i = 1, \ldots, n$, where $\mathbf{e}_i$ denotes the ith unit vector, and if the columns of the matrices $T_i, \hat{T}_i$ give orthonormal bases for the space $\mathscr{T}_i$ and its complement, respectively, then $\|\mathbf{w}_i^{\mathrm{T}} \hat{T}_i\|_2$ measures the minimum distance between $\mathbf{w}_i$ and the subspace $\mathcal{T}_i$. The sum of the squares of the distances is written

$$J_2 = \sum_{i=1}^{n} \|\mathbf{e}_i^{\mathrm{T}} V^{-1} \hat{T}_i\|_2^2. \tag{12}$$

To determine the required feedback, the right eigenvectors $\mathbf{v}_i$ comprising V are selected from the subspaces $\mathcal{S}_i$ to minimize a weighted sum of the robustness and distance measures. The objective functional is given by

$$J = \omega_1^2 J_1 + \omega_2^2 J_2, \tag{13}$$

where J_1 and J_2 are given by Eqs. (11) and (12), respectively, and $\omega_i, i = 1, 2$, are weights to be chosen. The aim is to minimize J over $\mathbf{v}_i \in \mathcal{S}_i$, subject to $\|\mathbf{v}_i\|_2 = 1$, for $i = 1, \dots, n$. This problem can be reduced to a least squares problem that is solvable by standard techniques [2].

The feedback matrix K is then constructed from the solution V to the optimization problem using the decompositions (7). The feedback is given explicitly by

$$K = Z_B^{-1} U_0^{\mathrm{T}} (V \Lambda V^{-1} - A) P_0 Z_C^{-1}, \tag{14}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$, and U_0, Z_B, P_0, and Z_C are determined from the decompositions of B and C, respectively. If the left eigenvectors corresponding to V lie in the required subspaces, then Eq. (14) exactly assigns the prescribed poles and $J_2 = 0$. If this is not the case, then the feedback K satisfies the equation

$$(A + BKC)V - V\Lambda = -EV, \tag{15}$$

where

$$E = V(\Lambda W^{\mathrm{T}} - W^{\mathrm{T}} A) P_1 P_1^{\mathrm{T}} \tag{16}$$

with P_1 given by Eq. (7). This holds since the right eigenvectors $\mathbf{v}_i$ are selected to lie in the required subspaces $\mathcal{S}_i$ and therefore, for each $i = 1, \dots, n$,

$$U_0 U_0^{\mathrm{T}} (\lambda_i I - A)\mathbf{v}_i = (I - U_1 U_1^{\mathrm{T}})(\lambda_i I - A)\mathbf{v}_i = (\lambda_i I - A)\mathbf{v}_i.$$

From the definition of the spaces $\mathcal{T}_i$ we also find that

$$\mathbf{w}_i^{\mathrm{T}} (\lambda_i I - A) P_1 P_1^{\mathrm{T}} = \mathbf{w}_i^{\mathrm{T}} \hat{T}_i R_i$$

for some nonsingular matrix R_i, $i = 1, \dots, n$, and it follows that

$$\|E\|_F^2 \leqslant \sum_{i=1}^{n} r_i^2 \|\mathbf{e}_i^{\mathrm{T}} V^{-1} \hat{T}_i\|_2^2, \tag{17}$$

where the constants r_i, $i = 1, \dots, n$, are independent of V. From this result it can be deduced [2], using the Bauer–Fike Theorem [7], that the minimum difference between an eigenvalue of the closed loop system and one of the prescribed eigenvalues is bounded in terms of the robustness measure and the distances between the left eigenvectors of the closed loop system and the subspaces $\mathcal{T}_i$. By adjusting the ratio ω_1/ω_2 of the weights in the objective functional (13) the robustness of the system can be traded off against the accuracy of the eigenvalue assignment. (For example, see [2].)

Although this approach allows all of the eigenvalues of the closed loop system to be controlled and also ensures that the assigned eigenstructure is robust to perturbations (within limits defined by the data), the freedom to shape the mode output and input coupling is lost. In aircraft control system design this is a significant restriction [8]. Ideally we should like to be able both to shape the mode coupling vectors and to ensure that the closed loop syst n is robust and displays satisfactory, *stable* behaviour overall. In Section 4, an algorithm is established that aims to combine the advantages of the two different approaches currently used in practice.

4. Algorithm for robust modal coupling

In general the modal coupling assignment problem, Problem 2.2, cannot be solved exactly. We may assign precisely p output mode coupling vectors corresponding to p prescribed eigenvalues, as described in Section 3.1, by assigning p right eigenvectors to the closed loop system using an appropriate feedback matrix K. The required input mode coupling vectors will not generally be attained. If we relax the constraints on the exact placement of the eigenvalues and output mode coupling vectors, however, then there exist additional degrees of freedom in the problem that can be exploited in order to assign the remaining $n-p$ eigenvectors of the system. We aim to use these extra degrees of freedom to improve the mode input coupling vectors and to ensure that the closed loop system is robust.

We partition

$$V \;\; = [V_1, V_2] = [\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_n],$$

$$W^{\mathrm{T}} = [W_1, W_2]^{\mathrm{T}} = [\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{w}_{p+1}, \ldots, \mathbf{w}_n]^{\mathrm{T}},$$

where we assume $W^{\mathrm{T}} = V^{-1}$. We select V_1 as in Section 3.1 to obtain exactly p prescribed output coupling vectors $G_{0\mathrm{d}} = CV_1$ with p corresponding (self-conjugate) eigenvalues. We then choose V_2 with $n-p$ corresponding (self-conjugate) eigenvalues in order to match p prescribed input mode coupling vectors $G_{1\mathrm{d}} = W_1^{\mathrm{T}}B$ and to ensure robustness of the eigenstructure. Since V_1 is not altered by this choice, the original output coupling vectors $G_{0\mathrm{d}} = CV_1$ are unaffected.

Unfortunately, we cannot now find a feedback matrix that exactly assigns the selected full set of eigenvalues and eigenvectors. If we select the vectors in V_2 such that the corresponding vectors $\mathbf{w}_i, i = 1, \dots, n$ lie close to the subspaces $\mathscr{T}_i$ corresponding to the prescribed eigenvalues, then a feedback matrix K can be constructed that *approximately* assigns the selected eigenstructure to the closed loop system and retains the robustness property. The construction follows as in Section 3.2.

The columns $\mathbf{v}_i, i = p+1, \dots, n$, of V_2 are selected from the subspaces $\mathscr{S}_i$, defined in Eq. (6), to minimize a weighted sum of squares of three measures:

- the error in the mode input coupling vectors, measured by

$$J_0 = \|G_{\mathrm{Id}} - W_1^{\mathrm{T}} B\|_F^2 \equiv \|G_{\mathrm{Id}} - [I_p, 0] V^{-1} B\|_F^2; \tag{18}$$

- the conditioning of the eigenvectors, measured as in Section 3.2 by

$$J_1 = \|V^{-1}\|_F^2;$$

- the distance of the left eigenvectors from the required subspaces $\mathscr{T}_i$ measured by

$$J_2 = \sum_{i=1}^{n} \|\mathbf{e}_i^{\mathrm{T}} V^{-1} \hat{T}_i\|_2^2,$$

  which controls the accuracy of the eigenvalue assignment as discussed in Section 3.2.

The functional to be minimised is given by

$$J = [\omega_0^2 J_0 + \omega_1^2 J_1 + \omega_2^2 J_2], \tag{19}$$

where $V = [V_1, V_2]$ and the weights $\omega_i^2, i = 0, 1, 2$ are chosen according to the design specifications. For given V_1, the optimal V_2 to minimize J is found by an iterative procedure. At each step of the iteration, the aim is to minimize J over a column $\mathbf{v}_i \in \mathscr{S}_i$, of V_2, subject to $\|\mathbf{v}_i\|_2 = 1$, where $i \in \{p+1, \dots, n\}$. This problem is a *nonlinear* least squares problem. Using a special structure for V^{-1} (see [9]), we can reduce the nonlinear problem to a *linear* least squares problem, which can be solved by highly efficient standard methods. The reduction is established in the following subsection. The construction of the feedback matrix K is described subsequently.

4.1. Main results

The objective is to minimize the functional J with respect to one of the columns of V_2 at each step of an iteration process. We consider the case where $\mathbf{v}_n$ is the vector to be up-dated and assume throughout that $p < n$. We show then that the problem can be reduced to a linear least squares problem. The nonlinear problem is specified by Problem 4.1.

Problem 4.1. Minimize J given by Eq. (19) over $\mathbf{v}_n \in \mathscr{S}_n$, subject to $\|\mathbf{v}_n\|_2 = 1$.

The reduction to the linear problem is shown in three steps. First, the problem is expressed in terms of the vector $\mathbf{v}_n$, using a QR decomposition. Next the functional is rewritten using a lemma from [9] and, finally, a scaling technique developed in [1] is used to obtain the linear formulation.

We denote $V_- = [\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}]$ and let

$$V_- = Q\begin{bmatrix} R \\ \mathbf{0}^{\mathrm{T}} \end{bmatrix},$$

where $Q = [Q_1, \mathbf{q}]$ is orthogonal and R is square and invertible. The inverse of matrix V can then be written as

$$V^{-1} = [V_-, \mathbf{v}_n]^{-1} = \begin{bmatrix} R & Q_1^{\mathrm{T}}\mathbf{v}_n \\ \mathbf{0}^{\mathrm{T}} & \mathbf{q}^{\mathrm{T}}\mathbf{v}_n \end{bmatrix}^{-1} Q^{\mathrm{T}} = \begin{bmatrix} R^{-1} & -\rho M\mathbf{v}_n \\ \mathbf{0}^{\mathrm{T}} & \rho \end{bmatrix} \begin{bmatrix} Q_1^{\mathrm{T}} \\ \mathbf{q}^{\mathrm{T}} \end{bmatrix}, \tag{20}$$

where $\rho = (\mathbf{q}^{\mathrm{T}}\mathbf{v}_n)^{-1}$ and $M = R^{-1}Q_1^{\mathrm{T}}$. We have the following lemma.

Lemma 4.2. *The functionals J_0, J_1 and J_2, given respectively by Eqs. (18), (11) and (12), can be written as:*

$$J_0 = \|L_0 + M_0\rho\mathbf{v}_n\mathbf{z}_0^{\mathrm{T}}\|_F^2, \tag{21a}$$

$$J_1 = \|M\rho\mathbf{v}_n\|_2^2 + \|\rho\mathbf{v}_n\|_2^2 + \alpha_1, \tag{21b}$$

$$J_2 = \sum_{i=1}^{n-1} \|\mathbf{e}_i^{\mathrm{T}}(L_i - M\rho\mathbf{v}_n\mathbf{z}_i^{\mathrm{T}})\|_2^2 + \|\rho\mathbf{z}_n^{\mathrm{T}}\|_2^2, \tag{21c}$$

where

$$M_0 = [I_p, 0]M, \qquad L_0 = G_{1\mathrm{d}} - M_0 B, \qquad \mathbf{z}_0^{\mathrm{T}} = \mathbf{q}^{\mathrm{T}}B,$$

$$L_i = M\hat{T}_i, \quad \mathbf{z}_i^{\mathrm{T}} = \mathbf{q}^{\mathrm{T}}\hat{T}_i, \qquad i = 1, 2, \ldots,$$

and $\alpha_1 = \|R^{-1}\|_F^2$ is constant, independent of $\mathbf{v}_n$.

Proof. The proof follows directly by substituting Eq. (20) for V^{-1} into Eqs. (18), (11) and (12) and simplifying. The definition of the Frobenius norm is used to expand J_1 and the assumption that $\|\mathbf{v}_n\|_2 = 1$ is applied in order to write $|\rho|^2 = \|\rho\mathbf{v}_n\|_2^2$. □

The next step in the reduction is established using Lemma 4.3.

Lemma 4.3. (Kautsky and Nichols [9]). *For matrices A, B of appropriate orders and vectors $\mathbf{z}, \mathbf{w} \neq 0$,*

$$\|A + B\mathbf{w}\mathbf{z}^{\mathrm{T}}\|_F^2 = \|\beta^{-1}A\mathbf{z} + \beta B\mathbf{w}\|_2^2 + \alpha, \tag{22}$$

where

$$\beta = \|\mathbf{z}\|_2, \qquad \alpha = \sum_{i=1}^{n} \mathbf{e}_i^{\mathrm{T}} A(I - \beta^{-2}\mathbf{z}\mathbf{z}^{\mathrm{T}})A^{\mathrm{T}}\mathbf{e}_i.$$

Proof. See [9]. □

Applying Lemma 4.3 then gives Lemma 4.4.

Lemma 4.4. *The functionals J_0 and J_2 can be written as:*

$$J_0 = \|\beta_0 M_0 \rho \mathbf{v}_n + \mathbf{f}_0\|_2^2 + \alpha_0, \tag{23a}$$

$$J_2 = \|DM\rho\mathbf{v}_n - \mathbf{f}\|_2^2 + \|\beta_n \rho \mathbf{v}_n\|_2^2 + \alpha_2, \tag{23b}$$

where

$$D = \mathrm{diag}\{\beta_1, \dots, \beta_{n-1}\}, \qquad \beta_i = \|\mathbf{z}_i\|_2, \quad i = 0, 1, \dots, n,$$

$$\mathbf{f}_0 = \beta_0^{-1} L_0 \mathbf{z}_0, \quad \mathbf{f} = \{f_i\} = \{\beta_i^{-1}\mathbf{e}_i^{\mathrm{T}} L_i \mathbf{z}_i, \quad i = 1, \dots, n-1\},$$

and α_0 and α_2 are constants independent of $\mathbf{v}_n$.

Proof. Eq. (23a) follows directly by applying Lemma 4.3 to Eq. (21a) and using the definition given for β_0 and $\mathbf{f}_0$. Applying Lemma 4.3 to Eq. (21c) gives

$$J_2 = \sum_{i=1}^{n-1} |\mathbf{e}_i^{\mathrm{T}}(\beta_i^{-1} L_i \mathbf{z}_i - \beta_i M \rho \mathbf{v}_n)|^2 + |\beta_n \rho|^2 + \alpha_2,$$

where $\beta_i = \|\mathbf{z}_i\|_2$, $i = 1, \dots, n$. The sum of squares is then just equal to the square of the L_2-norm of the vector $\mathbf{f} - DM\rho\mathbf{v}_n$, where $\mathbf{f}$ and D are defined in the lemma. Finally, the assumption that $\|\mathbf{v}_n\|_2 = 1$ is used to write $|\beta_n \rho|^2 = \|\beta_n \rho \mathbf{v}_n\|_2^2$. □

We summarize these results in the following Theorem.

Theorem 4.5. *The cost function J, defined by Eq. (19), can be written as*

$$\begin{aligned} J = {} & \omega_0^2(\|\beta_0 M_0 \rho \mathbf{v}_n + \mathbf{f}_0\|_2^2) + \omega_1^2(\|M\rho\mathbf{v}_n\|_2^2 + \|\rho\mathbf{v}_n\|_2^2) \\ & + \omega_2^2(\|DM\rho\mathbf{v}_n - \mathbf{f}\|_2^2 + \|\beta_n\rho\mathbf{v}_n\|_2^2) + \alpha_4, \end{aligned} \tag{24}$$

where $M = R^{-1}Q_1^{\mathrm{T}}$ is determined by the QR decomposition of V_-, M_0, D, $\mathbf{f}_0$, $\mathbf{f}$, β_0, β_n are as defined in Lemmata 4.2 and 4.4 and α_4 is a constant, independent of $\mathbf{v}_n$.

Proof. The proof follows directly from Lemmata 4.2 and 4.4. □

The functional defined in Eq. (24) remains nonlinear due to the term ρ. To reduce J to a linear form, finally, we use the fact that an eigenvector can be arbitrarily scaled. At the same time we rewrite the constrained minimization problem, Problem 4.1, as an unconstrained problem. Theorem 4.6 gives the solution to Problem 4.1 in terms of the equivalent formulation.

Theorem 4.6. *The solution to Problem 4.1 is given by*

$$\mathbf{v}_n = S_n H \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix} \Big/ \left\| S_n H \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix} \right\|_2, \tag{25}$$

where $\mathbf{u}$ *minimizes*

$$\tilde{J} = \left\| \begin{pmatrix} \omega_0 \beta_0 M_0 S_n H_2 \\ \omega_1 M S_n H_2 \\ \omega_1 S_n H_2 \\ \omega_2 D M S_n H_2 \\ \omega_2 \beta_n S_n H_2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{pmatrix} \right\|_2^2 \tag{26}$$

over $\mathbf{u} \in \mathbb{R}^{n-1}$. *Here*

$$\begin{aligned} \mathbf{r}_0 &= \omega_0 \beta_0 M_0 S_n \mathbf{h}_1 + \sigma \mathbf{f}_0, \\ \mathbf{r}_1 &= \omega_1 M S_n \mathbf{h}_1, \\ \mathbf{r}_2 &= \omega_1 S_n \mathbf{h}_1, \\ \mathbf{r}_3 &= \omega_2 D M S_n \mathbf{h}_1 - \sigma \mathbf{f}, \\ \mathbf{r}_4 &= \omega_2 \beta_n S_n \mathbf{h}_1 \end{aligned} \tag{27}$$

and $H = [\mathbf{h}_1, H_2]$ *is a Householder matrix such that*

$$\mathbf{q}^{\mathrm{T}} S_n H = \sigma \mathbf{e}_1^{\mathrm{T}}. \tag{28}$$

Proof. Since we require $\mathbf{v}_n \in \mathscr{S}_n$, subject to $\|\mathbf{v}_n\|_2 = 1$, we may write

$$\mathbf{v}_n = S_n \mathbf{s}_n, \qquad \rho^{-1} = \mathbf{q}^{\mathrm{T}} \mathbf{v}_n = \mathbf{q}^{\mathrm{T}} S_n \mathbf{s}_n,$$

where the columns of S_n give an orthonormal basis for $\mathscr{S}_n$, $\mathbf{s}_n \in \mathbb{R}^n$ and $\|\mathbf{s}_n\|_2 = 1$ must hold. From the definition (28) of the Householder matrix H, we then have $\rho^{-1} = \sigma \mathbf{e}_1^{\mathrm{T}} H^{\mathrm{T}} \mathbf{s}_n$. Hence the first component of the vector $\sigma H^{\mathrm{T}} \rho \mathbf{s}_n$ is unity and we may write

$$(\rho \mathbf{s}_n) = \sigma^{-1} H \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix} = \sigma^{-1} [\mathbf{h}_1, H_2] \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix} = \sigma^{-1} (\mathbf{h}_1 + H_2 \mathbf{u}), \tag{29}$$

where $\mathbf{u} \in \mathbb{R}^{n-1}$. Substituting $\rho \mathbf{v}_n = S_n (\rho \mathbf{s}_n)$ into Eq. (24) and using Eq. (29) then gives

$$\begin{aligned}J =&\omega_0^2(\|\beta_0\sigma^{-1}M_0S_n(\mathbf{h}_1 + H_2\mathbf{u}) + \mathbf{f}_0\|_2^2) + \omega_1^2(\|\sigma^{-1}MS_n(\mathbf{h}_1 + H_2\mathbf{u})\|_2^2 \\ &+ \|\sigma^{-1}S_n(\mathbf{h}_1 + H_2\mathbf{u})\|_2^2) + \omega_2^2(\|\sigma^{-1}DMS_n(\mathbf{h}_1 + H_2\mathbf{u}) - \mathbf{f}\|_2^2 \\ &+ \|\beta_n\sigma^{-1}S_n(\mathbf{h}_1 + H_2\mathbf{u})\|_2^2) + \alpha_4. \end{aligned} \tag{30}$$

Rearranging and combining terms and neglecting the constant term α_4 then gives the equivalent functional (26).

The vector $\mathbf{v}_n = S_n\mathbf{s}_n$ such that $\|\mathbf{v}_n\|_2^2 = 1$ is reconstructed from the solution $\mathbf{u}$ that minimizes $\tilde{J}$. From Eq. (29) we find that

$$\mathbf{v}_n = (\sigma\rho)^{-1}S_nH\begin{bmatrix}1\\ \mathbf{u}\end{bmatrix}, \quad \text{where } \sigma\rho = \left\|S_nH\begin{bmatrix}1\\ \mathbf{u}\end{bmatrix}\right\|_2,$$

which gives the solution (25). □

We have thus reduced the nonlinear problem of finding the vector $\mathbf{v}_n$ that minimizes the functional J to the linear problem of finding the vector $\mathbf{u}$ that minimizes the functional $\tilde{J}$. The vectors $\mathbf{v}_i$, $i = p+1, \dots, n-1$, can each be updated in turn by solving a corresponding linear least squares problem. Since the value of the functional is reduced at each step of the iteration, the process is convergent. We remark, however, that without any further constraints on the solution, the computed columns of V_2 may not form a self-conjugate set. To ensure the self-conjugacy property, if λ_i and λ_{i+1} are prescribed complex conjugate eigenvalues, then we update $\mathbf{v}_i$ by solving the minimization problem and let $\mathbf{v}_{i+1} = \bar{\mathbf{v}}_i$. The iteration process is not guaranteed to converge in this case, but in practice convergence is observed and good results are obtained after a small number of iterations.

If the eigenvalues λ_i, for $i = p+1, \dots, n$, are not explicitly prescribed, then the same process can be applied, with $\mathscr{S}_i = I$ and $\mathscr{T}_i = I$ for each i. The rate of convergence of the iteration for determining V_2 is found to be faster in this case, since the constraints on the solution are weaker. Both the matrix K and the unspecified eigenvalues then have to be reconstructed, however, subject to the constraint that the closed loop system is *stable*.

The iteration process is easy to implement using a package such as Matlab and, since only orthogonal transformations (QR and SVD decompositions) are used, the steps of the procedure are all numerically stable. The algorithm is found to be more efficient in general than the standard nonlinear packages for solving the nonlinear minimization problem directly. In [6] results from the process described here are compared to results from the Matlab Optimization Toolbox for a number of test problems, with different starting vectors and different weightings in the objective functional and different convergence tolerances. In the case where the eigenvalues are not prescribed, the experiments show that the new process is two to four times faster than the standard package.

4.2. Construction of feedback

After determining the optimal set, V_2, we must construct a feedback K to assign the eigenvectors $V = [V_1, V_2]$. We consider two constructions given by

$$K = K_1 \equiv B^+(V\Lambda V^{-1} - A)C^+, \qquad K = K_2 \equiv B^+(V\Lambda - AV)(CV)^+, \tag{31}$$

where K_1 is used when the left eigenspace error is small, and K_2 is used otherwise. Here the Moore–Penrose pseudo-inverses $B^+ = Z_B^{-1}U_0^{\mathrm{T}}$ and $C^+ = P_0 Z_C^{-1}$, are determined from the decompositions (7), and a further decomposition is needed to determine $(CV)^+$.

We remark that the first construction, K_1, is invariant under scalings of the matrix V of right eigenvectors. The error introduced into the eigenstructure of the closed loop system satisfies Eq. (15), as shown in [2], where

$$E = E_1 \equiv V(\Lambda W^{\mathrm{T}} - W^{\mathrm{T}}A)(I - C^+C). \tag{32}$$

A bound on the error E is given by Eq. (17), where the constants r_i, $i = 1, \dots, n$, are independent of V. The error in the assigned eigenstructure can thus be controlled by the selecting the weights ω_i, $i = 0, 1, 2$, in the objective functional $\tilde{J}$ appropriately and, in particular, by forcing the distance of the left eigenvectors from the required subspaces to be small.

The second construction, K_2, is similar to the construction given by Eq. (9), discussed in Section 3.1. The assumption that CV is invertible is not required. It can be shown by arguments similar to those in [2] (see [6]) that the error introduced by this construction also satisfies Eq. (15), where we now have

$$E = E_2 \equiv V(\Lambda W^{\mathrm{T}} - W^{\mathrm{T}}A)(I - V(CV)^+(CV)W^{\mathrm{T}}). \tag{33}$$

The error E can again be bounded by an expression of the form of Eq. (17), where the constants r_i, $i = 1, \dots, n$, are different from those of the first construction but remain independent of V.

The errors introduced by the constructions K_1 and K_2 are related by

$$E_2 = E_1(I - V(CV)^+(CV)W^{\mathrm{T}}). \tag{34}$$

This result is obtained using $(CV)(CV)^+(CV) = CV$, which holds by the definition of the Moore–Penrose pseudo-inverse (see [6]). It follows that if the left eigenvectors are in the required subspaces, i.e. $\mathbf{w}_i \in \mathscr{T}_i$ for all i, then $E_1 = 0$ and hence $E_2 = 0$.

We remark also that the two feedback matrices K_1 and K_2 are identical in the case where the eigenvector matrix V is unitary. In this case $(CV)^+ = V^{-1}C^+$ and the result follows immediately. In general the two constructions lead to different closed loop systems. The second construction is not invariant under scalings of the matrix V and an improved match to the input coupling vectors can be achieved by an a posteriori scaling of the left eigenvectors comprising W_1. If the scaled vectors are denoted by $W_1 D_1^{-1}$, where D_1 is a diagonal matrix, then K_2

is constructed using $V = [D_1 V_1, V_2]$. Evidence suggests that whilst the scaling improves the fit to the desired input coupling vectors, the accuracy of the eigenvalue assignment is decreased, due to an increase in the distance of the left eigenvectors from the required subspaces (see [6] for details).

The numerical computation of the feedback matrices K_1 and K_2 requires the inversion of the matrices V and CV, respectively. The accuracy of the computation thus depends on the conditioning of these matrices, reinforcing the need for a robust solution to the problem. The effects of the numerical error introduced by the computation can be represented by additional terms in the errors E_1 and E_2 dependent on the machine precision and on V^{-1} and $(CV)^+$, respectively.

If the eigenvalues associated with the unspecified coupling modes are not prescribed, then another method for constructing the feedback matrix K is needed. We now aim to select both $K = K_3$ and $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ to solve the equation

$$V^{-1}(A + BK_3C)V - \Lambda = 0$$

subject to $\mathrm{Re}\{\lambda_i\} \leqslant \gamma < 0$ for all i, where γ is a specified tolerance. These equations are written as an over-determined linear system for the $mp + n$ unknown variables consisting of the components of K_3 together with the diagonal elements of Λ. The equations are then solved in a least squares sense, subject to the constraints, using a standard procedure. If complex conjugate pairs of eigenvalues are allowed, then Λ is written as a block diagonal matrix, with 2×2 diagonal blocks representing the complex conjugate pairs. The matrix V of eigenvectors is written in *real* form, where the columns of V represent the real and imaginary parts of the corresponding pairs of complex conjugate eigenvectors. Details of the procedure are described in [6]. Experimental evidence shows that in practice this approach is more successful in some cases than in others. The results that can be achieved depend ultimately on the prescribed eigenvalues associated with the desired output and input mode coupling vectors.

5. Examples

The test system considered here is a lateral axis model of an L-1011 aircraft at cruise condition taken from [4]. For this system $n = 7$, $m = 2$, $p = 4$, and the model matrices are given by:

$$A = \begin{bmatrix} -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.744 & -0.032 & 0 & -0.154 & -0.0042 & 1.54 & 0 \\ 0.337 & -1.12 & 0 & 0.249 & -1 & -5.2 & 0 \\ 0.02 & 0 & 0.0386 & -0.996 & -0.0003 & -0.1170 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & -0.5 \end{bmatrix}, \quad (35)$$

$$B = \begin{bmatrix} 20 & 0 \\ 0 & 25 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{36}$$

5.1. Example 1

In the first test case the prescribed eigenvalues are $L_p = \{-6 \pm \mathrm{i}, -1 \pm 2\mathrm{i}\}$, and the corresponding desired mode input and output coupling vectors are those given by Eqs. (3) and (4), respectively.

The method of Section 3.1 is applied initially to obtain V_1. The corresponding matrix K that assigns the required output coupling vectors is then determined. With this feedback the closed loop eigenvalues, to four decimal places, are given by $L_n = \{-6 \pm \mathrm{i}, -1 \pm 2\mathrm{i}, -23.9954, -8.1679, -0.6077\}$. The prescribed eigenvalues are thus attained and, although the desired output coupling cannot be achieved exactly, only a small amount of coupling between the second and third outputs is introduced. The sensitivity of the closed loop eigenstructure is proportional to $\kappa_F(V) = 6.66 \times 10^4$, however, and the robustness of the system is poor. The input coupling is also unsatisfactory, with undesirable coupling between the first input and the third and fourth modes. The errors in the matching of the output and input coupling vectors are given by

$$\begin{aligned} &\|G_{0\mathrm{d}} - G_{0\mathrm{a}}\|_F^2 = 4.5860 \times 10^{-4}, \\ &\|G_{1\mathrm{d}} - G_{1\mathrm{a}}\|_F^2 = 23.0735, \end{aligned} \tag{37}$$

respectively.

The objective now is find a feedback to reduce the error, $\|G_{1\mathrm{d}} - G_{1\mathrm{a}}\|_F^2$, whilst retaining the accuracy of the mode output coupling vectors and achieving robustness. The full set of eigenvalues

$$L_n\{-6 \pm \mathrm{i}, -1 \pm 2\mathrm{i}, -23.9954, -8.1679, -0.6077\}$$

obtained in the first step of the process is reassigned. Since the aim is primarily to match the desired input coupling vectors, the weights in the objective functional J are selected to be

$$(\omega_0^2, \omega_1^2, \omega_2^2) = (10^4, 1, 1).$$

The minimization algorithm described in Section 4.1 is applied and a new matrix V_2 of corresponding eigenvectors is found. The feedback gain matrix K is determined from $V = [V_1, V_2]$ by the first construction described in Section 4.2.

The feedback obtained by this procedure is

$$K_1 = \begin{bmatrix} 12.7270 & -0.4798 & -56.7815 & -1.2742 \\ -1.5221 & 0.4292 & -1.4384 & 1.4316 \end{bmatrix}. \tag{38}$$

The eigenvalues of the corresponding closed loop system are given by

$$L_n = \{-7.29 \pm 9.28\mathrm{i}, -0.7070 \pm 1.0144\mathrm{i}, -24.6274, -5.5598, -0.5859\}$$

and the sensitivity of the closed loop eigenstructure is proportional to

$$\kappa_F(V) = 3.80 \times 10^2,$$

giving a considerable improvement. The corresponding mode input coupling vectors are

$$G_{1a} = \begin{bmatrix} 1 & -0.0130 - 0.0133\mathrm{i} \\ 1 & -0.0130 + 0.0133\mathrm{i} \\ 0.0271 + 0.0443\mathrm{i} & 1 \\ 0.0271 - 0.0443\mathrm{i} & 1 \end{bmatrix} \tag{39}$$

and the output coupling vectors are

$$G_{0a} = \begin{bmatrix} 1 & 0 \mp 0.0296\mathrm{i} \\ -0.687 \pm 0.123\mathrm{i} & 1 \\ 0.027 \mp 0.064\mathrm{i} & 0.002 \pm 0.007\mathrm{i} \\ 0.028 \mp 0.052\mathrm{i} & -0.463 \mp 0.663\mathrm{i} \end{bmatrix}. \tag{40}$$

We have thus calculated a feedback that gives the desired level of input coupling at the expense of increased coupling between the second and third outputs. The assigned eigenvalues are not close to those prescribed, as expected with the relatively low weighting of ω_2^2, but the robustness of the closed loop system is increased. The improvement in the input coupling and the system robustness is therefore balanced by a loss of accuracy in the output coupling and in the prescribed eigenvalues. The new result gives a more satisfactory over-all design, however, with improved input–output decoupling.

5.2. Example 2

In the second test case the set of prescribed poles is $L_p = \{-7 \pm 5\mathrm{i}, -15 \pm 4\mathrm{i}\}$, and the corresponding desired mode input and output coupling vectors are again given by Eqs. (3) and (4), respectively. The partial eigenstructure assignment method of Section 3.1 in this case produces a highly *unstable* closed loop system with closed loop eigenvalues given, to four decimals, by $L_n = \{-7 \pm 5\mathrm{i}, -15 \pm 4\mathrm{i}, -6.2805, -0.5785, 4.0879\}$. The sensitivity of the assigned eigenstructure is proportional to $\kappa_F(V) = 6.43 \times 10^4$ and the robustness of the closed loop system is thus also very poor. The errors in the matching of the mode output and input coupling vectors are

$$\begin{aligned} \|G_{0d} - G_{0a}\|_F^2 &= 3.7495 \times 10^{-4}, \\ \|G_{1d} - G_{1a}\|_F^2 &= 5.0074, \end{aligned} \tag{41}$$

respectively.

The objective now is to find a *stable* closed loop system which is also robust and has the desired mode coupling behaviour. The minimization algorithm described in Section 4.1 is applied, without constraints on the prescribed poles, to find the matrix V_2. The iteration is initiated using an orthogonal basis for the complement of the space spanned by the matrix V_1, selected in the first stage of the process to assign the prescribed set L_p. The weights in the objective functional J are chosen to be

$$(\omega_0^2, \omega_1^2, \omega_2^2) = (100, 1, 0).$$

The feedback gain matrix $K = K_3$ and a matrix $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$ are then determined from $V = [V_1, V_2]$ by the third construction method described in Section 4.2. (We remark that the eigenvalues of the closed loop system $A + BK_3C$ are not, in general, equal to the diagonal components of Λ, since a least squares fit is used in the construction.)

The feedback produced by this procedure is

$$K_3 = \begin{bmatrix} 9.0815 & -0.1286 & -28.9725 & 0.1228 \\ 3.1673 & 5.5682 & -14.3302 & 1.0733 \end{bmatrix}. \tag{42}$$

The eigenvalues of the corresponding closed loop system are

$$L_n = \{-7.41 \pm 4.39\mathrm{i}, -12.91 \pm 3.36\mathrm{i}, -5.3813, -0.5908, -0.1607\},$$

and the sensitivity of the eigenstructure is proportional to

$$\kappa_F(V) = 6.85 \times 10^2.$$

The new mode input and output coupling vectors are

$$G_{1a} = \begin{bmatrix} 1 & 0.0297 + 0.0278\mathrm{i} \\ 1 & 0.0297 - 0.0278\mathrm{i} \\ -0.0769 + 0.0843\mathrm{i} & 1 \\ -0.0769 - 0.0843\mathrm{i} & 1 \end{bmatrix}, \tag{43}$$

$$G_{0a} = \begin{bmatrix} 1 & 0.0048 \mp 0.0143\mathrm{i} \\ -0.0580 \mp 0.0602\mathrm{i} & 1 \\ 0.0717 \pm 0.0547\mathrm{i} & 0.0014 \mp 0.0005\mathrm{i} \\ 0.0022 \pm 0.0094\mathrm{i} & -0.0725 \mp 0.0189\mathrm{i} \end{bmatrix}, \tag{44}$$

respectively. The system has thus been made stable and more robust. In addition we have retained the desired level of output coupling and also reduced the level of input coupling to a satisfactory level.

These two examples demonstrate that the additional freedom in the design problem obtained by mildly relaxing the requirements on the output coupling and on the prescribed eigenvalues can be used effectively to improve stability, robustness and input coupling. Additional examples illustrating the behaviour of the algorithm are presented in [6].

6. Conclusions

We have developed a new feedback design technique for achieving mode input and output coupling by eigenstructure assignment. A linear least squares minimization procedure has been derived for simultaneously improving the mode coupling and the robustness of the system, whilst controlling the precision of the eigenvalue assignment. The method allows the accuracy of the prescribed output mode coupling and the corresponding eigenvalues of the system to be balanced against the accuracy of the prescribed mode input coupling and the robustness of the system. The balance is achieved by selecting the weights in the objective functional to be minimized. An application of the method to the design of an automatic flight control system has been presented.

Whilst allowing for flexibility in the design process, the weights in the objective functional are not easy to select in order to obtain the desired balance, although rules of thumb can be provided. The initial selection of the right eigenvectors to ensure the desired output coupling, on the other hand, imposes a strong restriction on the freedom to achieve a good balance. A better strategy might be to include the error in the output coupling vectors in the objective functional, as a weak constraint, and then to minimize over the entire set of right eigenvectors.

The optimal solution that can be obtained ultimately depends on the prescribed eigenvalues associated with the system. The alternative technique described here for constructing the feedback (the third method of Section 4.2) allows the assigned eigenvalues to be determined as part of the optimization process, although this process is expensive and may only be suitable for small problems. The scaling of the eigenvectors used in matching the desired input and output coupling vectors also affects the accuracy of the solution and deserves further investigation. The method derived here offers advantages over the processes currently used in practice, but the design process could be improved by further development. The test results demonstrate clearly that the freedom in the design problem can be used effectively to improve stability, robustness and input coupling of the closed loop system with only a small loss in the desired output coupling.

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