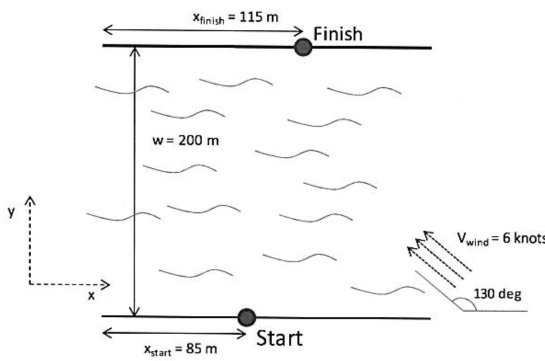
MECH622 Problem Statement.pdf

Optimal Control Qual Exam for Tim Coon
Take-Home (48 hours)

Numerical Optimization of a River Crossing:

Your task is to cross a river in minimum time using a sailboat (you may assume a point mass model). A diagram of the river crossing is shown in the below figure.



The river is flowing *from left to right* with $V_{river} = 4*V_{max}*y*(w-y)/w^2$, where V_{max} is 32 knots. Your solution to this problem must include a diagram of the crossing profile (e.g. path as a function of x and y) and a plot of the sail angle vs. time for the optimal path. You may not use GPOPS for this problem, but you may use any method within Matlab (e.g. *fmincon*, *fsolve*, etc.) that you deem appropriate. In addition to the above mentioned plots, please include:

a) A formal statement of the discretized static optimization problem that you used to solve for your optimal path. Include your objective function and ALL constraints.
b) The augmented Lagrangian function. Write out each of the terms expanded as much as possible, but leave in terms of *i*. That is, expand the state terms but not time other than the first and last time steps as is traditional.
c) The similarly expanded Hamiltonian, if you did not expand as part of b).
d) Perform an analysis (experimenting is sufficient) to determine the value of V_{max} that results in an optimal path with the minimum crossing time (an optimum of optimums, so to speak). Why is the river velocity that results in this not zero?
e) What is the minimum crossing time if you are not required to finish at a specific location on the far bank (use V_{max} = 32 knots)? Which far bank location results in this minimum crossing time?
f) *Optional:* If you solve this problem by supplying the gradients to the optimizer in Matlab, you will automatically get full credit. You must also state why your code does not require the explicit calculation of the lagrange multiplier, v.

b/c $(x-x_f)+(y-y_f)=0 \equiv (x-x_f)=0$ & $(y-y_f)=0$

INCLUDE ANY CODE YOU WRITE (comments are helpful)

b/c $(x-x_f)\le 0$ & $(y-y_f)\le 0$

Only Dr. Cobb or Maj. Dillsaver may answer questions. Open book, (your) open notes. You may use any code from Mech 622 that you consider helpful. Do not use the internet.

Figure 1: caption

Problem #1

For the differential equation

$$\ddot{x} + x = \epsilon x^2, \qquad x(0) = a, \qquad \dot{x}(0), \qquad 0 < \epsilon \ll 1 \tag{1}$$

assume this issue extends to higher-order derivatives for special analysis purposes of irregular systems. For this system, the stationary points are $(0, 0)$ and $(0, frac1\epsilon)$. To perform linear stability analysis, first calculate the Jacobian Matrix.

$$J = \begin{bmatrix} \dfrac{\partial f_1}{\partial z_1} & \dfrac{\partial f_1}{\partial z_2} \\ \dfrac{\partial f_2}{\partial z_1} & \dfrac{\partial f_2}{\partial z_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ (2\epsilon z_1 - 1) & 0 \end{bmatrix}$$

Now, evaluate the Jacobian at each stationary point and find the eigenvalues. The eigenvalues indicate the action of the exponentials of the solutions. What is the Jacobian system? The solution to the Jacobian system describes the behavior of the states (Reference Baker's stability notes (9.7)).

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \lambda_1 = \pm i$$
$$J(\frac{1}{\epsilon}, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \lambda_1 = \pm 1$$

The stationary point $(0, 0)$ has purely imaginary eigenvalues of multiplicity 1, so the point is considered stable (marginally stable). The stationary point $(\frac{1}{\epsilon})$ has one positive and one negative eigenvalue, thus, the exponential growth takes over and the point is unstable.

Poincarè Approximate Solution

Assume the nonlinear differential equation has a solution of the general form:

$$x(t, \epsilon) = \delta_0(\epsilon)x_0(t) + \delta_1(\epsilon)x_1(t) + \delta_2(\epsilon)x_2(t) + \cdots$$

To simplify, we assume a more specific form with $\delta_i(\epsilon) = \epsilon^i$ and the solution becomes:

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{3}$$

Take derivatives and substitute (3) into (**??**) and order with coefficients of powers of epsilon: (note: higher order terms are not evaluated)

$$(\ddot{x}_0 + x_0) + \epsilon(\ddot{x}_1 + x_1) + \epsilon^2(\ddot{x}_2 + x_2) = \epsilon \left[x_0^2 + 2\epsilon x_0 x_1 + h.o.t\right] \tag{4}$$

By equating coefficients of like-power terms of ϵ, develop differential equations to be solved in sequential order.

$$\mathcal{O}(\epsilon^0): \qquad \ddot{x}_0 + x_0 = 0, \qquad x_0(0) = a, \qquad \dot{x}_0(0) = 0 \tag{5}$$

$$x_0(t) = a\cos(t) \tag{6}$$

$$\begin{aligned} \mathcal{O}(\epsilon^1): \qquad \ddot{x}_1 + x_1 &= x_0^2, \qquad x_1(0) = 0, \qquad \dot{x}_1(0) = 0 \\ &= a^2\cos^2(t) \\ &= \frac{a^2}{2}(1+\cos(2t)) \end{aligned} \tag{7}$$

Solve using the method of undetermined coefficients ("judicial guessing")

$$\textit{homogeneous}: \qquad x_{1h}(t) = a_1\cos(t) + b_1\sin(t) \tag{8}$$

$$\textit{particular}: \qquad x_{1p}(t) = A_1 + B_1\cos(2t) \tag{9}$$

$$= \frac{a^2}{2}(1 - \frac{1}{3}\cos(2t)) \tag{10}$$

$$\textit{total}: \qquad x_1(t) = a_1\cos(t) + b_1\sin(t) + \frac{a^2}{2}(1 - \frac{1}{3}\cos(2t)) \tag{11}$$

Apply initial conditions to find:

$$x_1(t) = a^2\left[\frac{q}{2} - \frac{1}{3}\cos(t) - \frac{1}{6}\cos(2t)\right] \tag{12}$$

Substitute into (3) to find:

$$x(t,\epsilon) = \frac{a^2}{2}\epsilon + (a - \frac{a^2}{3}\epsilon)\cos(t) - \frac{a^2}{6}\epsilon\cos(2t) + \mathcal{O}(\epsilon^2) \tag{13}$$

This second-order approximation is a well-behaved periodic solution.

a) Apply Linstedt's Method to obtain an approximate solution. State the effect of the nonlinear term on the strained frequency and the development of secular terms.

b) Produce an energy function for this system and use it to sketch the phase plane.

c) Show that the center of oscillation can be approximated by $(x_c, 0)$