Practical-1

C31

Topic: Limits and Continuity.

① $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

$$\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$$

$$\lim_{x \to a} \frac{(a+2x-3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$\frac{1}{3} \lim_{x \to a} \frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{(a-x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$\frac{1}{3} \times \frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{2\sqrt{a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{\cancel{4}2\sqrt{a}}{\cancel{2}\sqrt{3}\cancel{\sqrt{x}}}$$

$$= \frac{2}{3\sqrt{3}}.$$

$$\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$$

$$\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\times\frac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right]$$

$$\lim_{y\to 0}\ \frac{a+y-a}{y\sqrt{a+y}\left(\sqrt{a+y}+\sqrt{a}\right)}$$

$$\lim_{y\to 0}\ \frac{y}{y\sqrt{a+y}\left(\sqrt{a+y}+\sqrt{a}\right)}$$

$$=\frac{1}{\sqrt{a+0}\left(\sqrt{a+0}+\sqrt{a}\right)}$$

$$=\frac{1}{\sqrt{a}\left(\sqrt{a}+\sqrt{a}\right.}$$

$$=\frac{1}{\sqrt{a}\left(2\sqrt{a}\right)}$$

$$=\frac{1}{2a}$$

(3) $\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3} \sin x}{\pi - 6x}$

By substituting $x - \frac{\pi}{6} = h$

$x = h + \frac{\pi}{6}$

where $h \to 0$

$\lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{6}\right) - \sqrt{3} \cdot \sin\left(h + \frac{\pi}{6}\right)}{\pi - 6\left(h + \frac{\pi}{6}\right)}$ Using $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$\lim_{h \to 0} \frac{\cos h \cdot \cos \frac{\pi}{6} - \sin h \cdot \sin \frac{\pi}{6} - \sqrt{3} \cdot \sin h \cdot \cos \frac{\pi}{6} + \cos h \cdot \sin \frac{\pi}{6}}{\pi - \cancel{6}\left(\frac{6h + \pi}{\cancel{6}}\right)}$

$\lim_{h \to 0} \frac{\cos h \cdot \frac{\sqrt{3}}{2} \cdot \sin \frac{h}{\sqrt{2}} - \sqrt{3}\left(\sin h \cdot \frac{\sqrt{3}}{2} + \cos h \cdot \frac{1}{\sqrt{2}}\right)}{\cancel{\pi} - 6h + \cancel{\pi}}$

$\lim_{h \to 0} \frac{\cancel{\cos \frac{\sqrt{3}h}{2}} - \sin \frac{h}{2} - \sin \frac{3h}{2} \cdot \cancel{\frac{\cos \sqrt{3}h}{2}}}{-6h}$

$\lim_{h \to 0} \frac{+\sin \frac{4h}{2}}{+6h}$

$\lim_{h \to 0} \frac{\sin 4h}{2 \cdot 6h}$

$\frac{1}{3} \lim_{h \to 0} \frac{\sin h}{h} = \frac{1}{3} \times 1 = \frac{1}{3}$

4) $\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$

By rationalizing numerator & Denominator both

$\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+3} + \sqrt{x^2+1}} \right]$

$\lim_{x \to \infty} \left[\frac{(\cancel{x^2}+5-\cancel{x^2}+3)(\sqrt{x^2+3} + \sqrt{x^2+1})}{(\cancel{x^2}+3-\cancel{x^2}-1)(\sqrt{x^2+5} + \sqrt{x^2-3})} \right]$

$\lim_{x \to \infty} \frac{8(\sqrt{x^2+3} + \sqrt{x^2+1})}{2(\sqrt{x^2+5} + \sqrt{x^2-3})}$

$4 \lim_{x \to \infty} \frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)} + \sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)} + \sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$

After applying limit we get

$= 4$

5. $f(x) = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 < x < \frac{\pi}{2}$
$= \frac{\cos x}{\pi - 2x}$, for $\frac{\pi}{2} < x < \pi$ } at $x = \pi/2$

$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \frac{\cos x}{\pi - 2x}$

By Substituting Method $x - \frac{\pi}{2} = h$

$x = h + \frac{\pi}{2}$

where $h \to 0$

$\lim_{h \to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi - 2\left(h+\frac{\pi}{2}\right)}$

$\lim_{h \to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi - \cancel{2}\left(\frac{2h+\pi}{\cancel{2}}\right)}$

$\lim_{h \to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{-2h}$

$\lim_{h \to 0} \frac{\cos h \cdot \cos\frac{\pi}{2} \cdot \sin h \cdot \sin\frac{\pi}{2}}{-2h}$

$\lim_{h \to 0} \frac{\cos h \cdot 0 - \sin h}{-2h}$

$\lim_{h \to 0} \frac{+\sin h}{+2h}$

$\frac{1}{2} \lim_{h \to 0} \frac{\sin h}{h}$

$= \frac{1}{2}$

b. $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$

$\lim_{x \to \frac{\pi}{2}^-} \frac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$

$\lim_{x \to \frac{\pi}{2}^-} \frac{2\cancel{\sin x} \cdot \cos x}{\sqrt{2}\cancel{\sin x}}$

$\lim_{x \to \frac{\pi}{2}^-} \frac{2\cos x}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} \lim_{x \to \frac{\pi}{2}^-} \cos x$

L.H.L $\neq$ R.H.L

f is not continuous at $x = \frac{\pi}{2}$

ii. $f(x) = \frac{x^2-9}{x-3} \quad 0 < x < 3$

$= x+3 \quad 3 \leq x \leq 6$

$= \frac{x^2-9}{x+3} \quad 6 \leq x < 9$

} at $x=3$ & $x=6$

at $x=3$

~~$f(3) = x^2$~~ $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x+3$

$f(3) = x+3 = 3+3 = 6$

f is define at $x=3$.

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+3) - 6.$

(2)

iii

Solu: f is continuous at $x=0$

$\lim_{x\to 0} f(x) = f(0)$

$\lim_{x\to 0} \frac{1-\cos 4x}{x^2} = k$

$\lim_{x\to 0} \frac{2\sin^2 2x}{x^2} = k$

$2\lim_{x\to 0} \frac{\sin^2 2x}{x^2} = k$

$2\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2 = k$

$2(2)^2 = k$

$\underline{k=8}$

ii $f(x) = (\sec^2 x)^{\cot^2 x}$ $\quad x \neq 0$
$\quad = k \quad x=0$ } at $x=0$

Soln:- $f(x) = (\sec^2 x)^{\cot^2 x}$

$\lim_{x\to 0} (\sec^2 x)^{\cot^2 x}$

$\lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$

we, know that $\lim_{x\to 0} (1+px)^{\frac{1}{px}} = e$

$k = e$

iii $f(x) = \frac{\sqrt{3} - \tan x}{\pi - 3x}$ $\quad x \neq \frac{\pi}{3}$
$\quad = k \quad x = \frac{\pi}{3}$ } at $x = \frac{\pi}{3}$

$x - \frac{\pi}{3} = h$

$x = h + \frac{\pi}{3}$

where $h \to 0$

$f\left(\frac{\pi}{3}+h\right) = \frac{\sqrt{3} - \tan\left(\frac{\pi}{3}+h\right)}{\pi - 3\left(\frac{\pi}{3}+h\right)}$

$\lim_{h\to 0} \frac{\sqrt{3} - \tan\left(\frac{\pi}{3}+h\right)}{\pi - 3\left(\frac{\pi}{3}+h\right)}$

$\lim_{h\to 0} \frac{\sqrt{3} - \frac{\tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3}\cdot\tan h}}{\cancel{\pi} - \cancel{\pi} - 3h}$

$\lim_{h\to 0} \frac{\frac{\sqrt{3}\left(1 - \tan\frac{\pi}{3}\cdot\tan h\right) - \left(\tan\frac{\pi}{3} + \tan h\right)}{1 - \tan\frac{\pi}{3}\cdot\tan h}}{-3h}$

$\lim_{h\to 0} \frac{\frac{\left(\sqrt{3} - \sqrt{3}\times\sqrt{3}\cdot\tan h\right) - \left(\sqrt{3} + \tan h\right)}{1 - \tan\frac{\pi}{3}\cdot\tan h}}{-3h}$

$\lim_{h\to 0} \frac{\frac{\left(\sqrt{3} - 3\cdot\tan h\right) - \left(\sqrt{3} + \tan h\right)}{1 - \sqrt{3}\cdot\tan h}}{-3h}$

$\lim_{h\to 0} \frac{\frac{\left(\sqrt{3} - 3\tan h - \sqrt{3} - \tan h\right)}{1 - \sqrt{3}\cdot\tan h}}{-3h}$

$$\lim_{x\to 0} \frac{\cancel{+}4\tan h}{\cancel{+}3h(1-\sqrt{3}\tan h)}$$

Sol: $\lim_{h\to 0} \frac{4\tan h}{3h(1-\sqrt{3}\tan h)}$

$\frac{4}{3}\lim_{h\to 0} \frac{\tan h}{h}$

$\frac{4}{3} \times \frac{1}{1-\sqrt{3}(0)}$

$= \frac{4}{3}(1) = \frac{4}{3}$//

Q7)

i) $f(x) = \frac{1-\cos 3x}{x\tan x}$ $x \neq 0$ } at $x = 0$

$= 9$ $x = 0$

ii) $f(x) = \frac{1-\cos 3x}{x\tan x}$

Sol: $\lim_{x\to 0} \frac{2\sin^2 3/2 x}{x\tan x}$

$\lim_{x\to 0} \frac{\frac{2\sin^2 \frac{3x}{2}}{x^2} \times x^2}{\frac{x\cdot \tan x}{x^2} \times x^2}$

$= 2\lim_{x\to 0} \left(\frac{3}{2}\right)^2 = 2 \times \frac{9}{4}$

$\to \frac{9}{2}$//

$\lim_{x\to 0} f(x) = \frac{9}{2}$ $9 = f(0)$

$\therefore f$ is not continuous at $x = 0$

Redefine function

$f(x) = \begin{cases} \frac{1-\cos 3x}{x\tan x} & x \neq 0 \\ \frac{9}{2} & x = 0 \end{cases}$

Now $\lim_{x\to 0} f(x) = f(0)$

$\therefore f$ has removable discontinuity at $x = 0$.

(ii) $f(x) = \frac{(e^{3x}-1)\sin x}{x^2}$ $x \neq 0$ } at $x = 0$

$= \pi/6$ $x = 0$

$\lim_{x\to 0} \frac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$

$\lim_{x\to 0} \frac{e^{3x}-1}{x} \cancel{\lim_{x\to 0}} \sin\left(\frac{\pi x}{180}\right)$

$\lim_{x\to 0} 3\cdot\frac{e^{3x}-1}{3x} \cdot \lim_{x\to 0} \sin\left(\frac{\pi x}{180}\right)$

$3\lim_{x\to 0} \frac{e^{3x}-1}{3x} \cdot \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$

$3 \log e \frac{\pi}{180} = \frac{\pi}{6} = f(0)$

$\therefore f$ is continuous at $x = 0$.

8. $f(x) = \frac{e^{x^2} - \cos x}{x^2}$ $x = 0$

is continuous at $x = 0$

Given f is continuous at $x = 0$

$\lim_{x \to 0} f(x) = f(0)$

$= \lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2} = f(x)$

$= \lim_{x \to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$

$\lim_{x \to 0} \frac{(e^{x^2} - 1) + (1 - \cos x)}{x^2}$

$\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} + \lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$\log e + \lim_{x \to 0} \frac{2 \sin^2 x/2}{x^2}$

$\log e + 2 \lim_{x \to 0} \left(\frac{\sin x/2}{x}\right)^2$

Multiplying with 2 on Num & Denominator

$= 1 + \cancel{2} \times \frac{1}{\cancel{4} 2} = \frac{3}{2} = f(0)$

9. $f(x) = \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$ $x \neq \frac{\pi}{2}$

$f(u)$ is continuous at $x = \frac{\pi}{2}$

$\lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1 + \sin x}}{\sqrt{2} + \sqrt{1 + \sin x}}$

$\lim_{x \to \frac{\pi}{2}} \frac{2 - 1 + \sin x}{\cos^2 x (\sqrt{2} + \sqrt{1 + \sin x})}$

$= \lim_{x \to \frac{\pi}{2}} \frac{1 + \sin x}{1 - \sin^2 x (\sqrt{2} + \sqrt{1 + \sin x})}$

$= \lim_{x \to \frac{\pi}{2}} \frac{1 + \cancel{\sin x}}{(1 - \sin x)(\cancel{1 + \sin x})(\sqrt{2} + \sqrt{1 + \sin x})}$

$= \lim_{x \to \frac{\pi}{2}} \frac{1}{(1 - \sin x)(\sqrt{2} + \sqrt{1 + \sin x})}$

$= \frac{1}{2(\sqrt{2} + \sqrt{2})}$

$= \frac{1}{2(2\sqrt{2})} = \frac{1}{4\sqrt{2}}$

$\therefore f(\pi/2) = \frac{1}{4\sqrt{2}}$ //.

Practical - 2

Topic: Derivation

Q.1) Show that the following function defined from IR to IR are differentiable

(i) $\cot x$

$f(x) = \cot x$

$D f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$

$= \lim_{x \to a} \frac{\frac{1}{\tan x} - \frac{1}{\tan x}}{x - a}$

$= \lim_{x \to 0} \frac{\tan a - \tan x}{(x-a) \tan x \cdot \tan a}$

put $x - a = h$

$x = a + h$

as $x \to a, h \to 0$

$D f(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(\cancel{a} + h - \cancel{a}) \tan(a+h) \tan a}$

$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h) \tan a}$

formula: $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

$= \lim_{h \to 0} \frac{(a - a - h) - (1 + \tan a + \tan(a+h))}{h \times \tan(a+h) \tan a}$

$= \lim_{h \to 0} \frac{-\tan h}{h} \times \frac{1 + \tan a \tan(a+h)}{\tan(a+h) \tan a}$

$= -1 \times \frac{1 + \tan^2 a}{\tan^2 a}$

$= \frac{\sec^2 a}{\tan^2 a}$

$= \frac{-1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a}$

$= -\operatorname{cosec}^2 a$

$\therefore D f(a) = -\cos^2 a$

$\therefore f$ is differentiable $\forall a \in R$.

(ii) $\operatorname{cosec} x$

$f(x) = \operatorname{cosec} x$

$D f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$

$= \lim_{x \to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x - a}$

$= \lim_{x \to a} \frac{\sin a - \sin x}{(x-a) \sin a \cdot \sin x}$

put $x - a = h$

$x = a + h$

as $x \to a, h \to 0$

$D f(h) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a + h - a)(\sin a \cdot \sin(a+h))}$

formula:

$\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \cdot \sin\left(\frac{C-D}{2}\right)$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right) \sin\left(\frac{a-a-h}{2}\right)}{h \sin a \cdot \sin(a+h)}$$

$$= \lim_{h \to 0} \frac{-\sin\frac{h}{2}}{\frac{h}{2}} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \cdot \sin(a+h)}$$

$$= -\frac{1}{2} \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)}$$

$$= \frac{-\cos a}{\sin^2 a} = -\cot a \cdot \operatorname{cosec} a$$

iii. $\sec x$

$f(x) = \sec x$

$$D \cdot f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sec x - \sec a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x - a}$$

$$= \lim_{x \to a} \frac{\cos a - \cos x}{(x-a)\cos a \cdot \cos x}$$

put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$D f(h) = \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cdot \cos(a+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right) \cdot \sin\left(\frac{a-a-h}{2}\right)}{h \times \cos a \cdot \cos(a+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right) \cdot \sin -\frac{h}{2}}{\cos a \cdot \cos(a+h) \times -\frac{h}{2}} \times -\frac{1}{2}$$

$$= -\frac{1}{2} \times \frac{-2\sin\left(\frac{2a+0}{2}\right)}{\cos a \cos(a+0)}$$

$$= -\frac{1}{\cancel{2}} \times \cancel{2} \frac{\sin a}{\cos a \times \cos a}$$

$$= \tan a \cdot \sec a.$$

Q.2) If $f(x) = 4x + 1$, $x \leq 2$

$= x^2 + 5$, $x > 2$, at $x = 2$ then

Find function is differentiable or not.

Soln:-

L.H.D :-

$$D \cdot f(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{4x + 1 - (4 \times 2 + 1)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{4x + 1 - 9}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{4x - 8}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{(x-2)} = 4$$

$D \cdot f(2^-) = 4$

R.H.D

$$D \cdot f(2^+) = \lim_{x \to 2^+} \frac{x^2 + 5 - 9}{x - 2}$$

$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$

$= \lim_{x \to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

$= 2+2 = 4$

$D \cdot f(2^+) = 4$

$R \cdot H \cdot D = L \cdot H \cdot D$

$\therefore f$ is differentiable at $x=2$.

Q.3) If $f(x) = 4x+7$, $x<3$
$= x^2+3x+1$, $x \geq 3$ at $x=3$, then find f is differentiable or not ?

Soln:-

R.H.D:

$D \cdot f(3^+) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x-3}$

iii.

$= \lim_{x \to 3^+} \frac{x^2+3x+1-(3^2+3 \times 3+1)}{x-3}$

$= \lim_{x \to 3^+} \frac{x^2+3x+1-19}{x-3}$

$= \lim_{x \to 3^+} \frac{x^2+3x-18}{x-3}$

$= \lim_{x \to 3^+} \frac{x^2+6x-3x-18}{x-3}$

$= \lim_{x \to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$

$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}} = 3+6 = 9.$

$D \cdot f(3^+) = 9$

$L \cdot H \cdot D = D \cdot f(3^-)$

$= \lim_{x \to 3^-} \frac{f(x)-f(3)}{x-3}$

$= \lim_{x \to 3^-} \frac{4x+7-19}{x-3}$

$= \lim_{x \to 3^-} \frac{4x-12}{x-3}$

$= \lim_{x \to 3^-} \frac{4\cancel{(x-3)}}{\cancel{x-3}}$

$D \cdot f(3^-) = 4$

$R \cdot H \cdot D \neq L \cdot H \cdot D$

f is not differentiable at $x=3$

Q.4) If $f(x) = 8x-5$, $x \leq 2$
$= 3x^2-4x+7$, $x>2$ at $x=2$, Then find f is differentiable or not

Soln:-

$f(2) = 8 \times 2 - 5 = 16-5 = 11$

R.H.D

$D \cdot f(2^+) = \lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2}$

$= \lim_{x \to 2^+} \frac{3x^2-4x+7-11}{x-2}$

$= \lim_{x \to 2^+} \frac{3x^2-4x-4}{x-2}$

$= \lim_{x \to 2^+} \frac{3x^2-6x+2x-4}{x-2}$

$= \lim_{x \to 2^+} \frac{3x(x-2)+2(x-2)}{x-2}$

$= \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

$= 3 \times 2 + 2 = 8$

$D \cdot f(2^+) = 8$

L.H.D:

$$D \cdot f(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 16}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 8$

$D \cdot f(2^-) = 8$

L.H.D = R.H.D

$\therefore$ f is differentiable at $x = 3$.

Practical - 3.

Topic :- Application of Derivative

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Q.1.) Find the intervals in which function is increasing or decreasing.

i. $f(x) = x^3 - 5x - 11$

$f'(x) = 3x^2 - 5$

f is increasing iff $f(x) > 0$

$3x^2 - 5 > 0$

$3(x^2 - \frac{5}{3}) > 0$

$(x - \sqrt{5}/3)(x + \sqrt{5}/3) > 0$

$x \in$

$+ \quad - \quad +$

$-\sqrt{5}/3 \qquad \sqrt{5}/3$

$x \in (-\infty, \sqrt{5}/3) \cup (\sqrt{5}/3, \infty)$

and f is decreasing iff $f'(x) < 0$

$3x^2 - 5 < 0$

$3(x^2 - 5/3) < 0$

$(x - \sqrt{5}/3)(x + \sqrt{5}/3) < 0$

$+ \quad - \quad +$

$-\sqrt{5}/3 \qquad \sqrt{5}/3$

$x \in (-\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3})$

ii.) $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

$f(x)$ is increasing iff $f'(x) > 0$

$\therefore 2x - 4 > 0$

$2(x - 2) > 0$

$(x - 2) > 0$

$x \in (2, \infty)$

and f is decreasing iff $f'(x) < 0$

$2x - 4 < 0$

$2(x - 2) < 0$

$x - 2 < 0$

$x \in (-\infty, 2)$

iii.) $f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

f is increasing iff $f'(x) > 0$

$\therefore 6x^2 + 2x - 20 > 0$

$2(3x^2 + x - 10) > 0$

$3x^2 + x - 10 > 0$

$3x^2 + 6x - 5x - 10 > 0$

$(x+2)(3x-5) > 0$

$x \in (-\infty, -2) \cup (\frac{5}{3}, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

$(x+2)(3x-5) < 0$

$+ \quad - \quad +$, -2, $5/3$ $\quad x \in (-2, 5/3)$

iv.) $f(x) = 3^x + 27x + 5$

$f'(x) = 3x^2 - 27$

f is increasing iff $f'(x) > 0$

$3(x^2 - 9) > 0$

$(x-3)(x+3) > 0$

$+ \quad - \quad +$, -3, 3 $\quad x \in (-\infty, -3) \cup (3, \infty)$

v.) $f(x) = 2x^3 - 9x^2 - 24x + 69$

$f'(x) = 6x^2 - 18x - 24$

f' is increasing iff $f'(x) > 0$

$6x^2 - 18x - 24 > 0$

$6(x^2 - 3x - 4) > 0$

$x^2 - 4x - x - 4 > 0$

$(x-4)(x+1) > 0$

$+ \quad - \quad +$, -1, 4 $\quad x \in (-\infty, -1) \cup (4, \infty)$

and f is decreasing iff $f'(x) < 0$

$6x^2 - 18x - 24 < 0$

$(x-4)(x+1) < 0$

$+ \quad - \quad +$, -1, 4 $\quad x \in (-1, 4)$

Q.2.)

i. $y = 3x^2 - 2x^3$

$f(x) = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$f''(x) = 6 - 12x$

f is concave upward if $f''(x) > 0$

$(6 - 12x) > 0$

$12(6/12 - x) > 0$

$x - \frac{1}{2} > 0$

$x > 1/2$

$f''(x) > 0$

$x \in (1/2, \infty)$

2. $y = x^4 - \cancel{6x^3} + 12x^2 + 5x + 7$

$f'(x) = 4x^3 - 18x^2 + 24x + 5$

$f''(x) = 12x - 36x + 24$

f' is concave upward if $f''(x) > 0$

$12x^2 - 36x + 24 > 0$

$12(x^2 - 3x + 2) > 0$

$(x-2)(x-1) > 0$

$+ \quad - \quad +$, 1, 2 $\quad x \in (-\infty, 1) \cup (2$

3. $f = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upward iff $f''(x) > 0$

$6x > 0$

$x > 0$

$x \in (0, \infty)$

4.) $y = 69 - 24x - 9x^2 + 2x^3$

$f'(x) = -24 - 18x + 6x$

$f''(x) = 12 - 18$

f is concave upward iff $f''(x) > 0$

$12x - 18 > 0$

$x - 3/2 > 0$

$x > 3/2$

$\therefore x \in (3/2, \infty)$

5. $y = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upward iff $f''(x) > 0$

$f''(x) > 0$

$12x + 2 > 0$

$12(x + 2/12) > 0$

$x + 1/6 > 0$

$x > -1/6$

$\therefore f''(x) > 0$ There exist on interval $[-1/6, \infty)$

A. 21/12/19

Practical 4

Q.1] Find maximum and minimum value of following function

i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

Now Consider,

$f'(x) = 0$

$2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$x^4 = \frac{32}{2}$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

f has minimum value at $x = 2$.

$f(2) = 2^2 + \frac{16}{2^2}$

$= 4 + \frac{16}{4} = 4 + 4$

$= 8$

$f''(-2) = 2 + \frac{96}{-2}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

∴ f has minimum value at $x = -2$

∴ Function reaches minimum value at $x = 2$, and $x = -2$.

ii) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = -15x^2 + 15x^4$

Consider,

$f'(x) = 0$

$-15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$f''(x) = -30x + 60x^3$

$f''(1) = -30 + 60$

$= 30 > 0$

f has minimum value at $x = 1$

$f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5$

$= 1$

$f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= -30 < 0$

∴ f has maximum value at $x = -1$

$f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$

$= 5$

f has the maximum value 5 at $x = -1$ and has the minimum value 1 at $x = 1$.

iii) $f(x) = x^3 - 3x^2 + 1$
$f'(x) = 3x^2 - 6x$

Consider,
$f'(x) = 0$
$3x^2 - 6x = 0$
$3x(x-2) = 0$
$3x = 0$ or $x - 2 = 0$
$x = 0$ or $x = 2$

$f''(x) = 6x - 6$
$f''(0) = 6(0) - 6$
$= -6 < 0$
$\therefore$ f has maximum value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1$
$= 1$

$f''(2) = 6(2) - 6$
$= 12 - 6$
$= 6 > 0$
f has maximum value at $x = 2$

$f(2) = (2)^3 - 3(2)^2 + 1$
$= 8 - 3(4) + 1$
$= 9 - 12$
$= -3$

$\therefore$ f has maximum value 1 or $x = 0$ and
f has minimum value -3 or $x = 2$.

iv) $f(x) = 2x^3 - 3x^2 + 2x + 1$
$f'(x) = 6x^2 - 6x - 12$

Consider,
$f'(x) = 0$
$6x^2 - 6x - 12 = 0$
$6(x^2 - x - 2) = 0$
$x^2 - x - 2 = 0$
$x^2 + x - 2x - 2 = 0$
$x(x+1) - 2(x+1) = 0$
$(x-2)(x+1) = 0$
$x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$
$f''(2) = 12(2) - 6$
$= 24 - 6$
$= 18 > 0$

f has minimum value at $x = 2$
$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$
$= 2(8) - 3(4) - 24 + 1$
$= 16 - 12 - 24 + 1$
$= -19$

$f''(-1) = 12(-1) - 6$
$= -12 - 6$
$= -8 < 0.$

f has maximum value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$
$= -2 - 3 + 12 + 1$
$= 8$

$\therefore$ f has maximum value 8 and $x = -1$ and
f has minimum value -1 and $x = 2$.

Scanned with CamScanner

5.2) Find the root of following equation by newton's method

$x_0 = 0$ = given

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$

$f'(x) = 3x^2 - 6x - 55$

By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 + \frac{9.5}{55}$$

$$x_1 = 0.1727$$

$$f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$$
$$= 0.0051 - 0.0895 - 9.4985 + 9.5$$
$$= -0.0829$$

$$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$$
$$= 0.0895 - 1.0362 - 55$$
$$= -55.9467$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{0.0829}{56.9467}$$

$$= 0.1712$$

$$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$$
$$= 0.0050 - 0.0879 - 9.416 + 9.5$$
$$= 0.0011$$

$$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$$
$$= 0.0879 - 1.0272 - 55$$
$$= -55.9393$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.1712 + \cancel{0.011}\ \frac{0.0011}{56.9393}$$

$$= 0.1712$$

The root of the equation is 0.1712.

ii. $f(x) = x^3 - 4x - 9$ [2,3]

$f'(x) = 3x^2 - 4$

$$f(2) = 2^3 - 4(2) - 9$$
$$= 8 - 8 - 9$$
$$= -9$$

$$f(3) = 3^3 - 4(3) - 9$$
$$= 27 - 12 - 9$$
$$= 6.$$

Let $x_0 = 3$ be the initial approximation,

∴ By Newton method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{2.2}{5.2}$$

$$= 2 - 0.4230$$

$$= 1.577$$

$$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$$

$$= 3.9219 - 4.4764 - 15.77 + 17$$

$$= 0.6755$$

$$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$$

$$= 7.4608 - 5.6772 - 10$$

$$= -8.2164$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.577 + \frac{0.6755}{8.2164}$$

$$= 1.577 + 0.0822$$

$$= 1.6596.$$

$$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$$

$$= 4.5677 - 4.9553 - 16.592 + 17$$

$$= 0.0204$$

$$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$$

$$= 8.2588 - 5.97312 - 10$$

$$= -7.7143$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.6592 + \frac{0.0204}{7.7143}$$

$$= 1.6592 + 0.0026$$

$$= 1.6618$$

$$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$$

$$= 4.5892 - 4.9708 - 16.618 + 17$$

$$= 0.0004$$

$$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$$

$$= 8.2847 - 5.9824 - 10$$

$$= -7.6977$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.6618 + \frac{0.0004}{7.6977}$$

$$= 1.6618.$$

∴ The root of equation is 1.6618.

Practical- 5

Topic- Integration

Q.1] solve the following integration

i] $\frac{dx}{\sqrt{x^2+2x-3}}$

ii] $\int (4e^{3x}+1)dx$

iii] $\int (2x^2-3\sin x+5\sqrt{x})dx$

iv] $\int \frac{x^3+3x+4}{\sqrt{x}} dx$

v] $\int t^7 \times \sin(2t^4)dt$

vi] $\int \sqrt{x}(x^2-1)dx$

vii] $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right)dx$

viii] $\int \frac{\cos x}{\sqrt[3]{\sin x}} dx$

ix] $\int e^{\cos^2 x} \sin 2x\,dx$

x] $\int \left(\frac{x^2-2x}{x^3-3x^2+1}\right)dx$.

1. $\int \frac{1}{\sqrt{x^2+2x-3}} dx$

$= \int \frac{1}{\sqrt{x^2+2x+1+4}}$

$a^2+2ab+b^2=(a+b)^2$

$= \int \frac{1}{\sqrt{(x^2+1)^2 - 4}} dx$

put $(x+1)=t$

$dx = \frac{1}{-2} x dt$

where $t=1$

$t = x+1$

$\int \frac{1}{t^2-4} dt$

using

$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln(|x+\sqrt{x^2-a^2}|)$

$= \ln(|t+\sqrt{t^2-4}|)$

$t = x+1$

$= \ln(|x+1+\sqrt{(x+1)^2-4}|)$

$= \ln(|x+1+\sqrt{x^2+2x-3}|)$

$= \ln(|x+1+\sqrt{x^2+2x+3}|)+C$

2) $\int (4e^{3x}+1) dx$

$= \int 4e^{3x} dx + \int 1 dx$

$= 4\int e^{3x} dx + \int 1 dx$ # $e^{ax} dx = \frac{1}{a} x e^{4x}$

$= \frac{4e^{3x}}{3} + x$

$= \frac{4e^{3x}}{3} + x + 3.$

3.) $\int 2x^2 - 3\sin(x) + 5\sqrt{x} dx$

$= \int 2x^2 - 3\sin(x) + 5x^{1/2} dx$

$= \int 2x^2 dx - \int 3\sin(x) dx + \int 5x^{1/2} dx$ # $\sqrt[n]{a^m} = a^{m/n}$

$= \frac{2x^3}{3} + 3\cos x + 10 \frac{x\sqrt{x}}{3} + C$

$= \frac{2x^3+10x\sqrt{x}}{3} + 3\cos x + C.$

4.) $\int \frac{x^3+3x+4}{\sqrt{x}} dx$

$= \int \frac{x^3+3x+4}{x^{1/2}} dx$

split the denominator

$= \int \frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}} dx$

$= \int x^{5/2} + 3x^{1/2} + 4x^{-1/2} dx$

$= \int x^{5/2} dx + \int 3x^{1/2} dx + \int \frac{4}{x^{1/2}} dx$

$= \frac{2x^3\sqrt{x}}{7} + 2x\sqrt{x} + 8\sqrt{x} + C.$

5) $\int t^7 \times \sin(2t^4) dt$

put $u = 2t^4$

$du = 2 \times 4t^3.$

$$= \int t^7 \times \sin(2t^4) \times \frac{1}{2+4t^3}\, du$$

$$= \int t^4 \sin(2t^4) \times \frac{1}{2\times 4}\, du$$

$$= \int t^4 \sin(2t^4) \times \frac{1}{8}\, du = t^4 \times \frac{\sin}{8}(2t^4)\, du$$

substituting t^4 with $\frac{u}{2}$

$$= \int \frac{u/2 + \sin(u)}{8}\, du$$

$$= \int \frac{u \times \sin(u)}{2} / 8\, du$$

$$= \int \frac{u \times \sin(u)}{16}\, du$$

$$= \frac{1}{16} \int 4x \sin(0)\, du$$

$\int u\,dv = uv - \int v\,dv$

where $U = V$

$dv = \sin(0) \times du$ $\qquad v = -\cos(v)$.

$du = 1\, du$

$$= \frac{1}{16}(4x - \cos u) - \int -\cos 4\, dx$$

$$= \frac{1}{16} \times (4x(-\cos u)) + \int \cos u\, du.$$

$\int \cos x\, dx = \sin x$

$$= \frac{1}{16} \times (4x(-\cos(u)) + \sin(u))$$

$$\ldots \quad u = 2t^4$$

$$\ldots (2t^4 \times (-\cos(2t^4)) + \sin(2t^4))$$

$$= -\frac{t^4 \times \cos(2t^4)}{8} + \frac{\sin(2t^4)}{16} + C$$

6. $\int \sqrt{x}(x^2 - 1)\, dx$

$$= \int \sqrt{x}\, x^2 - \sqrt{x}\, dx$$

$$= \int x^{\frac{1}{2}} \times x^2 - x^{1/2}\, dx$$

$$= \int x^{3/2} - x^{1/2}\, dx$$

$$= \int x^{5/2}\, dx - \int x^{1/2}\, dx$$

$$I_1 = \frac{x^{5/2+1}}{\frac{5}{2}+1} = \frac{x^{7/2}}{7/2} = \frac{2x^{7/2}}{7} = \frac{2\sqrt{x}}{7} = \frac{2x^3\sqrt{x}}{7}$$

$$I_2 = \frac{x^{1/2+1}}{\frac{1}{2}+1} = \frac{x^{3/2}}{3/2} = \frac{2x^{3/2}}{3/2} = \frac{2\sqrt{x}^3}{3}$$

$$= \frac{2x^3\sqrt{x}}{7} + \frac{2\sqrt{x}^3}{3} + C.$$

7.) $\int \frac{\cos x}{\sqrt[3]{\sin x^2}}\, dx$

$$= \int \frac{\cos x}{\sin x^{2/3}}\, dx$$

put $t = \sin x$

$t = \cos x$

$$= \int \cancel{\frac{\cos x}{\sin x^{3/2}} \times \frac{1}{\cos x}}\, dt$$

$$= \frac{1}{\sin x^{3/2}}\, dt$$

$$= \frac{1}{t^{2/3}}\, dt$$

$$I = \int \frac{1}{t^{2/3}}\, dt = \frac{-1}{(2/3 - 1)t^{2/3 - 1}} = \frac{-1}{(2/3 - 1)t^{2/3 - 1}}$$

$$= \frac{t^{\frac{-1}{3}+1}}{-\frac{1}{3}+1} = \frac{1}{\frac{1}{3}t^{-\frac{1}{3}}} = \frac{t^{1/3}}{\frac{1}{3}} = 3t^{1/3}.$$

Return substitution $t = \sin(x)$

$$= 3\sqrt[3]{\sin x} + C$$

10) $\int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} dx$

put $x^3 - 3x^2 + 1 = dt$

$$I = \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3x^2 - 3 \times 2x} dt$$

$$= \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3x^2 - 6x} dt$$

$$= \int \frac{1}{x^3 - 3x^2 + 1} \times \frac{1}{3} dt$$

$$= \int \frac{1}{3(x^2 - 3x^2 + 1)} dt$$

$$= \int \frac{1}{3t} dt = \frac{1}{3} \int \frac{1}{t} dt$$

$$\int \frac{1}{x} dx = \ln|x|$$

$$= \frac{1}{3} \times \ln|t| + C$$

$$= \frac{1}{3} \times \log(|x^3 - 3x^2 + 1|) + C.$$

AK
04/01/2020

Practical - 6

Topic: Application of integration and Numeric integration.

Find the length of the following curve.

1) $x = t \sin t, \quad y = 1 - \cos t \qquad t \in [0, 2\pi]$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1} dt$$

$$= \int_0^{2\pi} \sqrt{2 \cdot 2\cos t} dt$$

$$= \int_0^{2\pi} 2\left|\sin \frac{t}{2}\right| dt \quad \ldots \quad \sin^2 \frac{t}{2} = \frac{1 - \cos t}{2}$$

$$= \int_0^{2\pi} 2 \sin \frac{t}{2} dt$$

$$= \left(-4\cos\left(\frac{t}{2}\right)\right)_0^{2\pi} = (-4\cos\pi - (-4\cos 0))$$

$$= 4 + 4$$

$$= 8.$$

(2) $y = \sqrt{4 - x^2}$ $\quad x \in [-2, 2]$

Soln:-

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}}$$

$$l = 2\int_0^2 \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$$

$$= 2\int_0^2 \sqrt{\frac{1+x^2}{4-x^2}}\, dx$$

$$= 4\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx$$

$$= 4\left(\sin^{-1}\left(\frac{x}{2}\right)\right)_0^2$$

$$= 2\pi.$$

(3) $y = x^{3/2}$ in $[0, 4]$

Soln:-

$$f^{-1}(x) = \frac{3}{2} x^{1/2}$$

$$[f^{-1}(x)]^2 = \frac{9}{4} x$$

$$l = \int_0^4 \sqrt{1 + [\,(x)^2\,]}\, dx.$$

$$= \int_0^2 \sqrt{1 + \frac{9}{4}}\, dx$$

put $y = 1 + \frac{9}{4}x$, $du = \frac{9}{4} dx$.

$$d = \int_1^{1+\frac{2}{4}x} \frac{4}{9}\sqrt{4}\, du = \left[\frac{4}{9}\cdot\frac{2}{3}(4x^2)\right]_1^{1+\frac{2}{4}x}$$

$$= \frac{8}{27}\left[\left(1 + \frac{9x}{4}\right) - 1\right].$$

(4) $x = 3\sin t$, $y = 3\cos t$ $\quad t \in [0, 2\pi]$

$$\frac{dx}{dt} = 3\cos t \quad \frac{dy}{dt} = -3\sin t$$

$$l = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt.$$

$$= \int_0^{2\pi} 3\sqrt{x}\, dt$$

$$-3\int_0^{2\pi} dt$$

$$= 3[x]_0^{2\pi}$$

$$= 3(2\pi - 0)$$

$$= 12.6 - \pi \text{ units}$$

⑤ $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1,2]$.

$\rightarrow \frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$

$\frac{dx}{dy} = \frac{y^4 - 1}{2y^3}$

$l = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= \int_1^2 \sqrt{1 + \frac{(y^4-1)}{4y^4}}\, dy$

$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}}\, dy$

$= \int_1^2 \frac{y^4+1}{2y^2}\, dy$

$= \frac{1}{2}\int_1^2 y^2\, dy + \frac{1}{2}\int_1^2 y^{-2}\, dy$.

$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-1}}{1}\right]_1^2$

$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$

$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{2}\right]$

$= \frac{1}{2}\left[\frac{17}{6}\right]$

$l = \frac{17}{12}$ units.

Q2] Using simpson's Rule solve the following

① $\int_0^2 e^{x^2} dx$ with $n = 4$

By simpson's rule $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$

$\int_0^2 e^{x^2} dx = \frac{1/2}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$

$= \frac{1/2}{3}\left(e^{0^2} + 4e^{(0.5)^2} + 2e^{(1)^2} + 4e^{(1.5)^2} + e^{(1.5)^2}\right)$

$= 17.35362645$

② $\int_0^4 x^2 dx$ $\quad n = 4$

$h = \frac{4-0}{4} = 1$

x	0	1	2	3	4
y	0	1	4	9	16.

$\int_0^1 x^2 dx = \frac{1}{3}\left[(y_0 + y_4) + 4(y_1 + y_2) + 2y_2\right]$

$= \frac{1}{3}\left[0 + 16 + 4(1+9) + 2\times 4\right]$

$= \frac{1}{3}\left[16 + 4(10) + 8\right)$

$= \frac{64}{3}$.

$\int_0^4 x^2 dx = 21.3333$

(3) $\int_0^{\pi/3} \sqrt{\sin x}\, dx$ with $n = 6$.

$$h = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$$

x	0	$\pi/18$	$2\pi/18$	$3\pi/10$	$4\pi/18$	$5\pi/18$	$6\pi/18$
y	0	0.4187	0.5848	0.7071	0.8071	0.8652	0.430

$$\int_0^{\pi/3} \sqrt{\sin x}\, dx = \frac{h}{3}\left(y_0 + y_6 + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right)$$

$$= \frac{\pi/18}{3}\left[0.4167 + 0.4806 + 2(0.4127 + 0.7071 + 0.8752) + 2(0.5848 + 0.8471)\right]$$

$$= \pi/54\left[1.3473 + 4(1.999) + 2(1.3865)\right]$$

$$= \pi/54\left[1.3473 + 7.196 + 2.773\right]$$

$$= \frac{\pi}{54} \times 12 \times 11.63$$

$$\int_0^{\pi/3} \sqrt{\sin}\, dx = 8.7049.$$

11/01/2020

Practical - 7

Topic :- Differential Equation.

Q.1]
1. $x\frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$P(x) = \frac{1}{x}$ $\quad Q(x) = \frac{e^x}{x}$.

$I.F = e^{\int 1/x \, dx}$

$= e^{\int 1/x \, dx}$

$= e^{\ln x}$

$I.F = x$

$y(I.F) = \int Q(x)(I.F)\,dx + C$

$= \int \frac{e^x}{x} \cdot x \cdot dx + C$

$= \cancel{\int e^x dx + 1}$

$xy = e^x + C$.

2. $e^x \frac{dy}{dx} + 2e^x y = 1$ $(\div \text{ by } e^x)$

$\frac{dy}{dx} + 2\frac{e^x}{e^x} y = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$p(x) = 2$ $\quad Q(x) = e^{-x}$

$\int p(x)dx$

$I.F = e\int 2dx = e^{2x}$

$y(I.F) = \int Q(x)(I.f)\,dx + C$

$y \cdot e^x \int e^{-x} + 2x\,dx + C$

$= \int e^x dx + C$

$ye^{2x} = e^x + C$

(3) $x\frac{dy}{dx} + \frac{\cos x}{x} = 2y$

$x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\therefore \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$

$P(x) = 2/x$ $\quad Q(x) = \cos x/x^2$

$I.f = e^{\int p\,dx}$

$= e^{\int 2/x\,dx}$

$= e^{\ln|x^2|}$

$= x^2.$

$y(I.f) = \int Q(x)\cdot(I.f)dx + C$

$= \int \frac{\cos x}{x^2} \times x^2 dx + C$

$= \int \cos x\,dx + 1$

$x^2 y = \sin x + C$

(4) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$

$P(x) = 3/x$ $\quad Q(x) = \sin x/x^3$

$I.f = e^{\int P dx}$

$= e^{\int 3/x\,dx}$

$= e^{\ln|x^3|}$

$= x^3$

$y.I.f = \int Q(x)\cdot(I.f)dx + C$

$= \int \frac{\sin x}{x^3} \cdot x^3 dx + C$

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$\therefore x^3 y = -\cos x + C.$

⑤ $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

$P(x) = 2 . \quad Q(x) = 2x/e^{2x}$

$I.f = e^{\int P dx}$

$= e^{\int 2 dx}$

$= e^{2x}$

$y(I.f) = \int Q(x)(I.f)dx + C$

$y . e^{2x} = \int \frac{2x}{e^{2x}} . e^{2x} dx + C$

$= \int 2x dx + C$

$e^{2x} y = x^2 + C$

⑥ $\sec^2 x . \tan y \, dx + \sec^2 y . \tan x \, dy = 0$

$\sec^2 x . \tan y \, dx = -\sec^2 y . \tan x \, dy$

$\frac{\sec^2 x \, dx}{\tan x} dx = \frac{-\sec^2 y}{\tan y} dx$

on integrating we get.

$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy$

$\therefore \log|\tan x| = -\log|\tan y| + C$

$\log|\tan x . \tan y| = C$

$\tan x . \tan y = e^C.$

⑦ $\frac{dy}{dx} = \sin^2(x - y + 1)$

put $x - y + 1 = v$

Differentiating on both side,

$1 - \frac{dy}{dx} = \frac{dv}{dy}$

$1 - \frac{dv}{dx} = \frac{dy}{dx}$

$\frac{dv}{dx} = 1 - \sin^2 v.$

$\frac{dv}{dx} = \cos^2 v$

$\frac{dv}{\cos^2 v} = dx$

$\int \sec^2 v \, dx = \int dx$

$\tan v = x + C$

$\tan(x + y - 1) = x + C.$

(8) $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y-6}$

put $2x+3y = v$

$2 + 3\frac{dy}{dx} = \frac{dv}{dv}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$

$\frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$

$= \frac{3v+3}{v+2}$

$\frac{dv}{dx} = \frac{3(v+1)}{v+2}$

$\frac{v+2}{v+1}\, dv = 3dx$

on integrating we get,

$\int \frac{v+2}{v+1}\, dv = \int 3dx$

$\int \frac{v+1}{v}\, dv + \int \frac{1}{v+1}\, dv = 3x$

$v + \log|v+1| = 3x + C$

$2x + 3y + \log|2x+3y+1| = 3x + C$

$3y = x - \log|2x+3y+1| + C$

11/01/2020

Practical no-8

(3)

1. $\frac{dy}{dx} = y + e^x - 2$ $y(0) = 2$, $h \in 0.5$

find $y(2)$

① $y(0) = 2$ $x_0 = 0$ $h = 0.5$

find $y(0.2)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	2.1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2927	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		

$y(2) = 9.8215$

② $\frac{dy}{dn} = 1 + y^2$ $y(0) = 0$. $h = 0.2$

find $y(1)$

$y_0 = 0$	$n_0 = 0$	$h = 0.2$	
n	x_n	y_n	y_{n+1}
0	0	1	1.0894
1	0.2	1	1.2105
2	0.4	1.0894	1.3513
3	0.6	1.2105	1.5051
4	0.8	1.3513	
5	1	1.5051	

$y(1) = 1.5051$.

(6)

4. $\frac{dy}{dn} = 3n^2 + 1$ $y(1) = 2$ find $y(2)$

$h = 0.5$ & $h = 0.25$

$y_0 = 2$ $x_0 = 1$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	7.75	7.875
2	2	7.875		

$y(2) = 7.875$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4218
2	1.5	4.4218	59.6509	19.3360
3	1.75	19.3360	1122.6426	299.9966
4	2	299.996		

$y(2) = 299.996$

⑤ $\frac{dy}{dx} = \sqrt{xy + 2}$ $y(1) = 1$ find $y(1.2)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	
1	1.2	3.6		

$y(1.2) = 3.6$

AK 18/01/2023

Practical - 9.

Topic :- Limit & particular derivative.

1. Evaluate the following limit.

i) $\lim_{(x,y)\to(-4,-1)} \frac{x^3-3y+y^2-1}{xy+5}$ ii) $\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$.

iii) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2-z^2}{x^3-x^2y}$

2. Find f_x, f_y for each of the following.

i. $f(x,y)=xye^{x^2}+y^2$. ii. $f(x,y)=e^x\cos y$.

iii. $f(x,y)=x^3y^2-3x^2y^2+y^3+1$.

3. Using definition find values of f_x, f_y at $(0,0)$ for $f(x,y)=\frac{2x}{1+y^2}$.

4. Find all second order partial derivative of following. Verify whether $f_{xy}=f_{yx}$.

i. $f(x,y)=\frac{y^2-xy}{x^2}$. ii. $f(x,y)=x^3+3x^2y^2-\log(x^2+1)$

iii. $f(x,y)=\sin(xy)+e^x+y$.

5. Find the linearization at given point.

i. $f(x,y)=\sqrt{x^2+y^2}$ at $(1,1)$ ii. $f(x,y)=1-x+y+y\sin x$ at $(\frac{\pi}{2},0)$

iii. $f(x,y)=\log x+\log y$ at $(1,1)$

Q.1. $\lim_{(x,y)\to(-4,1)} \frac{x^3-3y+y^2-1}{xy+5}$

At $(-4,-1)$, Denomination $\neq 0$.

$\therefore$ By applying limit.

$= \frac{(-4)^3-3(1)+(-1)^2-1}{-4(-1)+5}$.

$= \frac{-64-3(-1)+(-1)^2-1}{-4(-1)+5}$

$= \frac{-64+3+1-1}{4+5}$

$= -\frac{61}{9}$

2. $\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$.

at $(2,0)$, Denominator $\neq 0$

$\therefore$ By applying limit,

$= \frac{(0+1)((2)^2+0-(2))}{2+0}$

$= \frac{1(4+0-8)}{2}$

$= \frac{-4}{2}$

iii) $\lim_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2-z^2}{x^3-x^2y^2}$

at $(1,1,1)$. Denomination $= 0$.

$\lim_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2z^2}{x^3-x^2y^2}$

$\lim_{(x,y,z)\to(1,1,1)} \dfrac{\cancel{(x-yz)}(x+yz)}{x^2\cancel{(x-yz)}}$

$\lim_{(x,y,z)\to(1,1,1)} \dfrac{x+yz}{xz}$

on Applying limit

$= \dfrac{1+1(1)}{(1)^2}$

$= 2$

Q 2.

i. $f(x,y) = xy \cdot e^{x^2+y^2}$

$f_x = \dfrac{\partial}{\partial x}(f(x,y))$

$= \dfrac{\partial}{\partial x}(xye^{x^2+y^2})$

$= ye^{x^2+y^2}(2x)$

$f(x) = 2xye^{x^2+y^2}$

$f_y = \dfrac{\partial}{\partial y} f((x,y))$

$= \dfrac{\partial}{\partial y}(xye^{x^2+y^2})$

$= xe^{x^2+y^2}(2y)$

$\therefore f_y = 2yxe^{x^2+y^2}$.

ii. $f(x,y) = e^x \cos y$.

$f_x = \dfrac{\partial}{\partial x}(f(x,y))$

$= \dfrac{\partial}{\partial x}(e^x \cos y)$

$\therefore f_x = e^x \cos y$.

$f_y = \dfrac{\partial}{\partial y}(f(x,y))$

$= \dfrac{\partial}{\partial y}(e^x \cos y)$

$f_y = -e^x \sin y$.

$\cancel{f_y} =$

iii $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = \dfrac{\partial}{\partial x}(f(x,y))$

$= \dfrac{\partial}{\partial x}(x^3y^2 - 3x^2y + y^3 + 1)$

$f_x = 3x^2y^2 - 6xy$

$f_y = \dfrac{\partial}{\partial y}(f(x,y))$

$= \dfrac{\partial}{\partial y}(x^3y^2 - 3x^2y^2 + y^3 + 1)$

$= 2x^3y - 3x^2 + 3y^2$

Q3. i. $f(x,y) = \frac{2x}{1+y^2}$.

$fx = \frac{\partial}{\partial x}\left(\frac{2x}{1+y^2}\right)$

$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{d}{dx}(1+y^2)}{(1+y^2)^2}$.

$= \frac{2+2y^2-0}{(1+y^2)^2}$.

$= \frac{2(1+y^2)}{(1+y^2)(1+y)^2}$

$= \frac{2}{1+y^2}$

at $(0,0)$

$= \frac{2}{1+0}$

$= 2$

$fy = \frac{\partial}{\partial y}\left(\frac{2x}{1+y^2}\right)$

$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{d}{dx}(1+y^2)}{(1+y^2)^2}$.

$= \frac{1+y(0) - 2x(2y)}{(1+y^2)^2}$

$= \frac{-4xy}{(1+y^2)^2}$

at $(0,0)$

$= \frac{-4(0)(0)}{(1+0)^2}$.

$= 0$

$f(x,y) = \frac{y^2 - xy}{x^2}$.

$fx = \frac{x^2 \frac{d}{dx}(y^2 - xy) - (y^2 - xy)\frac{d}{dx}(x^2)}{(x^2)^2}$.

$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$

$= \frac{-x^2y - 2x(y^2 - xy)}{x^4}$

$fy = \frac{2y - x}{x^2}$

$fx = \frac{d}{dx}\left(\frac{-x^2y - 2x(y^2 - xy)}{x^4}\right)$

$= \frac{x^4\left(\frac{d}{dx}(-x^2y - 2xy^2 + 2x^2y)\right) - (-x^2y - 2xy + 2x^2)\frac{dy}{dx}}{(x^4)^2}$

$= \frac{x^4(-2xy - 2y^2 + 4xy) - 4x3(-x^2y - 2xy + 2x^2y)}{x^6}$ — (1)

$fy = \frac{d}{dy}\left(\frac{2y - x}{x^2}\right)$

$= \frac{2-0}{x^2}$

$= \frac{2}{x^2}$ — (11)

$$= \frac{x^2 - 4xy + 2x^2}{x^4} \quad \text{— (iii)}$$

$$f_{yx} = \frac{-d}{dx}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{x^2 \frac{d}{dx}(2y-x) - (2y-x)\frac{d}{dx}(x^2)}{(x^2)^2} \quad \text{(iv)}$$

from (iii) and (iv)

$f_{xy} = f_{xy}$

(ii) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$$f_x = \frac{d}{dx}(x^2 + 3x^2y^2 - \log(x^2+1))$$

$$f_y = \frac{d}{dx}(x^3 + 3x^2y^2 \cdot \log(x^2+1))$$

$$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$$

$$= 0 + 6x^2y - 0$$

$$= 6x^2y \quad \text{—}$$

$$f_y = \frac{d}{dx}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$$

$$= 0 + 12xy - 0$$

$$= 12xy.$$

$$f_{yx} = \frac{d}{dx}(6x^2y)$$

$$= 12xy$$

$\therefore f_{xy} = f_{yx}$.

iii) $f(x,y) = \sin(xy) + e^{x+y}$

$$f_x = y\cos(xy) + e^{x+y}(1)$$
$$= y\cos(xy + e^{x+y})$$

$$f_y = x(\cos xy) + e^{x+y})(1)$$
$$f_y = x\cos(xy + e^{x+y})$$

$$f_{xx} = \frac{d}{dx}(y\cos(xy)) + e^{x+y})$$

$$= -y\sin(xy)(y) + e^{xy}(1)$$
$$= -y^2\sin(xy) + e^{x+y} \rightarrow \text{(i)}.$$

$$f_{yy} = \frac{d}{dx}(x\cos(x,y) + e^{x+y})$$

$$= -x\sin(xx)(x) + e^{x+y}(1)$$
$$= -x\sin(xy + e^{x+y}) \quad \text{— (ii)}$$

$$f_{xy} = \frac{d}{dx}(y\cos(xy) + e^{x+y})$$

$$= y^2\sin(x+y) + \cos(xy) + e^{x+y}) \rightarrow \text{(iii)}$$

$$f_{yx} = \frac{d}{dx}(x\cos(xy) + e^{x+y})$$

$$= -x^2\cos(x+y) + \cos(xy) + e^{x+y} \quad \text{— (iv)}$$

from (iii) and (iv).

Q5.

i. $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$\to f(1,1) = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$f_x = \frac{1}{2\sqrt{x^2+y^2}}(2x)$

$= \frac{x}{\sqrt{x^2+y^2}}$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$

$f_y = \frac{1}{2\sqrt{x^2+y^2}}(2y)$

$= \frac{y}{\sqrt{x^2+y^2}}$

f_y at $(1,1) = \frac{1}{\sqrt{2}}$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + y - 1$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{2}{\sqrt{2}}$

(ii) $f(x,y) = 1 - x + y\sin x$ at $(\frac{\pi}{2}, 0)$

$f(\frac{\pi}{2}, 0) = 1 - \frac{\pi}{2} + 0 = 1 - \frac{\pi}{2}$.

$f_x = 0 - 1 + y\cos x$

$f_y = 0 - 0 + \sin x$

f_x at $(\frac{\pi}{2}, 0) = -1 + 0$
$= -1$

f_y at $(\frac{\pi}{2}, 0) = \sin\frac{\pi}{2}$
$= 1$.

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 1 - \frac{\pi}{2} + (-1)(x - \frac{\pi}{2}) + 1(y-0)$

$= 1 - \frac{\pi}{2} - x + \frac{\pi}{2} + y$

$= 1 - x + y$.

(iii) $f(x,y) = \log x + \log y$ at $(1,1)$

$f(1,1) = \log(1) + \log(1) = 0$.

$f_x = \frac{1}{x} + 0$ $\qquad$ $f_y = 0 + \frac{1}{y}$

f_x at $(1,1) = 1$ $\qquad$ f_y at $(1,1) = 1$

$\therefore L(x,y) = f(a,b) + f(a,b)(x-a) + f(a,b)(y-b)$

$= 0 + 1(x-1) + 1(y-1)$

$= x - 1 + y - 1$

$= x + y - 2$.

25/07/2020

Q.1) Find the directed derivative of the following function at given parts and in the direction of given vector

i. $f(x,y) = x + 2y - 3 \quad a = (1,-1) \quad u = 3i - j$

Here, $u = 3i - j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = (1) + 2(-1) - 3 = 1 - 2 - 3 = -4$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= f\left(1 + \frac{3}{\sqrt{10}}\right), \left(-1, -\frac{h}{\sqrt{10}}\right)$

$f(a+hu) = \left(1 + \frac{3}{\sqrt{10}}\right), \left(-1 - \frac{h}{\sqrt{10}}\right)$

$f(a+hu) = \left(1 + \frac{3}{\sqrt{10}}\right) + 2\left(-1 - \frac{h}{\sqrt{10}}\right)$

$= 1 + \frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}}$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h \to 0} \frac{a + h/\sqrt{10} + 4}{h}$

$D_u f(a) = \frac{1}{\sqrt{10}}$ //

$f(x) = y^2 - 4x + 1 \quad a = (3,4) \quad u = i + 5j$

Here $u = i + 5j$ is not a unit vector

$|\bar{u}| = \sqrt{(1)^2 + (5)^2} = \sqrt{26}$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}}(1,5)$

$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(x) = f(3,4) = (4)^2 - 4(3) + 1 = 5$

$f(a+hu) = f(3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right)$

$f(a+hu) = \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$

$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$

$= \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - \frac{4h}{\sqrt{26}} + 5$

$= \frac{25h^2}{26} + \frac{40h - 4h}{\sqrt{26}} + 5$

$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 8 - 8}{h}$$

$$\frac{\cancel{h}\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}{\cancel{h}}$$

$$D_u f(a) = \frac{25h}{26} + \frac{36}{\sqrt{26}}$$

iii) $2x + 3y$ $\quad a = (1,2)$, $u = (3i + 4j)$

Here $u = 3i + 4j$ is not unit vector

$|u| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$

unit vector along u is $\frac{u}{|u|} = \frac{1}{5}(3,4)$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$

$f(a) = f(1,2) = 2(1) + 3(2) = 8$

$f(a+hu) = f(1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)$

$= f\left(1 + \frac{3h}{5}, 2 + \frac{4h}{5}\right)$

$f(a+hu) = 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$

$= \frac{18h}{5} + 8$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{18h}{5} + 8 - 8}{h}$$

$= \frac{18}{5}$

Q2. Find gradient vector for the following function at given point.

i) $f(x,y) = x^y + y^x \rightarrow a = (1,1)$

$f_x = y \cdot x^{y-1} + y^x \log y$

$f_y = x^y \log x + x y^{x-1}$

$Df(x,y) = (f_x, f_y)$

$= (y x^{y-1} + y^x \log y, x^y \log x + x y^{x-1})$

$f(1,1) = (1+0, 1+0)$

$= (1, 1)$

ii) $f(x,y) = (\tan^{-1} x) \cdot y^2 \quad a = (1,-1)$

$f_x = \frac{1}{1+x^2} \cdot y^2$

$f_y = 2y \tan^{-1} x$

$Df(x,y) = (f_x, f_y)$

$= \left(\frac{y^2}{1+x^2}, 2y \tan^{-1} x\right)$

$f(1,-1) = \left(\frac{1}{2}, \tan^{-1}(1)(-2)\right)$

$= \left(\frac{1}{2}, \frac{\pi}{4}(-2)\right)$

$= \left(\frac{1}{2}, \frac{-\pi}{2}\right)$

iii-) $f(x,y,z) = xyz - e^{x+y+z}$, $a \in (1,-1,0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

D.f $(x,y,z) = f_x, f_y, f_z$

$f = yz - e^{x+y+z}, xz - e^{x+y+z}$

$f(1,-1,0) = ((-1)(0) - e^{1+(-1)+0}, (1)(0) - e^{1+(-1)+0}, (1)(-1) - e^{1+(-1)+0})$

$= (0 - e^0, 0 - e^0, -1 - e^0)$

$= (-1, -1, -2)$

Q.3 Find the equation of tangent and normal to each of the following using curve at given point.

i. $x^2 \cos y + e^{xy} = 2$ at $(1,0)$

$f_x = \cos y \, 2x + e^{xy} y$

$f_y = x^2(-\sin y) + e^{xy} \cdot x$

$(x_0, y_0) = (1,0)$ $\therefore x_0 = 1, y_0 = 0$

eqn of tangent

$f_x(x - x_0) + f_y(y - y_0) = 0$

$f_x(x_0, y_0) = \cos 0 \, 2(1) + e^0 \, 0$

$= 1(2) + 0$

$= 2$

$f_y(x_0, y_0) = (1)^2(-\sin 0) + e^0 \cdot 1$

$= 0 + 1 \cdot 1$

$= 1$

$2(x-1) + 1(y-0) = 0$

$2x - 2 + y = 0$

$2x + y - 2 = 0$ $\Rightarrow$ It is the required eqn of tang

equation of normal

$= ax + by + c = 0$

$= bx + ay + d = 0$

$1(1) + 2(y) + d = 0$

$1 + 2y + d = 0$ at $(1,0)$

$= 1 + 2(0) + d = 0$

$d + 1 = 0$

$\therefore d = -1$

ii. $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2,-2)$

$f_x = 2x + 0 - 2 + 0 + 0$

$= 2x - 2$

$f_y = 0 + 2y - 0 + 3 + 0$

$= 2y + 3$

$(x_0, y_0) = (2,-2)$ $\therefore x_0 = 2, y_0 = -2$

$f_x(x_0, y_0) = 2(2) - 2 = 2$

$f_y(x_0, y_0) = 2(-2) + 3 = -1$

eq^n of tangent

$f_x(x-x_0)+f_y(y-y_0)=0$

$2(x-2)-1(y+2)=0$

$2x-2-y-2=0$

$2x-y-4=0 \Rightarrow$ It is required of tangent.

eq^n of normal

$= ax+by+c=0$

$bx+ay+d=0$

$-1(x)+2(y)+d=0$

$-x+2y+d=0$ at $(2,-2)$

$-2+2(-2)+d=0$

$-2-4+d=0$

$-6+d=0$

$\therefore d=6$

Q.4 Find the eq^n of tangent & normal line to each of the following surface.

i. $x^2-2yz+3y+xz=7$ at $(2,1,0)$

$f_x = 2x-0+0+z$

$f_x = \cancel{2x+z}$

$f_y = 0-2z+3+0$

$= 2z+3$

$f_z = 0-2y+0+x$

$= -2y+x$

$(x_0,y_0,z_0)=(2,1,0) \therefore x_0=2, y_0=1, z_0=0.$

$f_x(x_0,y_0,z_0) = 2(2)+0=4$

$f_y(x_0,y_0,z_0)=2(0)+3=3$

$f_z(x_0,y_0,z_0)=-2(1)+2=0$

eq^n of tangent

$f_x(x_0-x_0)+f_y(y_0-y_0)+f_z(z_0-z_0)=0$

$=4(x-2)+3(y-1)+0(z-0)=0$

$=4x-8+3y-3=0$

$4x+3y-11=0 \Rightarrow$ This required eq^n of tangent.

Eq^n of normal at $(4,3,-11)$

$\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}$

$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z+11}{0}$

ii) $3xyz-x-y+z=-4$ at $(1,-1,2)$

$3xyz-x-y+z-4=0$ at $(1,-18,2)$

$f_x = 3yz-1-0+0+0$

$=3yz-1$

$f_y = 3xz-0-1+0+0$

$3xz-1$

$f_z = 3xy-0-0+1+0$

$=3xy+1$

$(x_0,y_0,z_0)=(1,-1,2)$ $x_0=1, y_0=-1, z_0=2$

$f_x(x_0,y_0,z_0)=3(-1)(2)-1=-7$

$f_y(x_0,y_0,z_0)=3(+1)(2)-1=5$

$f_z(x_0,y_0,z_0)=3(1)(-1)+1=-2$

eq^n of tangent

$-7(x-1)+5(y+1)-2(z-2)=0$

$-7x+7+5y+5-2z+4=0$

$-7x+5y-2z+16=0 \Rightarrow$ This is require tangent.

Eq^n of normal at $(-7, 5, -2)$

$$\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}$$

$$= \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$

Q5] Find the local maxima & minima for the following function

i. $f(x,y) = 3x^2+y^2-3xy+6x-4y$

$f_x = 6x+0-3y+6-0$
$= 6x-3y+6$

$f_y = 0+2y-3x+0-4$
$= 2y-3x-4$

$f_x = 0$
$6x-3y+6=0$
$3(2x-y+2)=0$
$2x-y+2=0$
$2x-y=-2 \rightarrow 1$

$f_y = 0$
$2y-3x-4=0$
$2y-3x=4 \rightarrow 2$

Multiply eq 1 with 2
$4x-\cancel{2y}=-4$
$\cancel{2y}-3x=4$
$x=0$

Substitute value of x in eq 1
$2(0)-y=-2$
$-y=-2$.

$\therefore$ Critical points are $(0,2)$.

$r = f_{xx} = 6$
$t = f_{yy} = 2$
$s = f_{xy} = -3$.

Here $r > 0$.
$= rt - s^2$
$= 6(2) - (-3)^2$
$= 12-9$
$= 3 > 0$.

f has maximum at $(0,2)$.

$3x^2+y^2-3xy+6x-4y$ at $(0,2)$
$3(0)^2+(2)^2-3(0)(2)+6(0)-4(2)$
$0+4-0+0-8$
$= \underline{\underline{-4}}$.

ii. $f(x,y) = 2x^4+3x^2y-y^2$
$f_x = 8x^3+6xy$
$f_y = 3x^2-2y$
$f_x = 0$.
$8x^3+6xy=0$.
$\cancel{2x}(4x^2+3y)=0$
$4x^2+3y=0 \rightarrow 1$.

$f_y = 0$
$3x^2-2y=0 \rightarrow 2$

Multiply eq 1 with 3.
2 with 4.

$12x^2+9y=0$
$-12x^2+-8y=0$
$y=0$
$\underline{y=0.}$

Substitute value of y in eq ①
$4x^2+3(0)=0$
$4x^2=0$
$x=0$

Critical point is $(0,0)$
$r = f_{xx} = 24x^2+6x$
$t = f_{yy} = 0-2=-2$
$s = f_{xy} = 6x-0=6x=6(0)=0$

r at $(0,0)$
$=24(0)+6(0)=0$
$r=0$

$f(x,y)$ at $(0,0)$
$2(0)^4+3(0)^2(0)-10$
$=0+0-0$
$=0.$

$rt-s^2 = 0(-2)-(s)^2$
$0-0=0$
$r=0$ & $rt-s^2=0$
(nothing to say)

$f(x,y) = x^2-y^2-2x+8y-70$
$f(x) = 2x+2$
$f_y = -2y+8$

$f_x=0$ $\therefore 2x+2=0$
$x=-\frac{2}{2}$ $\therefore x=-1.$

$f_y=0.$ $-2y+8=0$
$y=\frac{-8}{-2}$ 4
$y=4$

$\therefore$ Critical point is $(-1,4)$.

$r=f_{xx}=2$
$t=f_{yy}=-2$
$s=f_{xy}=0$
$r>0$
$rt-s^2 = 2(-2)-(0)^2$
$=-4-0$
$=-4<0.$

$f(x,y)$ at $(-1,4)$
$(-1)^2-(4)^2+2(-1)+8(4)-70$
$1+16-2+32-70$
$=17+30-70$
$=37-70$
$=33.$

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