SCHOOL OF ENGINEERING AND TECHNOLOGY

TCSS 343 – Assignment 1

September 28, 2020

1 GUIDELINES

Homework should be electronically submitted to the instructor by midnight on the due date. A submission link is provided on the course Canvas Page. The submitted document should be typeset using any common software and submitted as a PDF. We strongly recommend using LaTeX to prepare your solution. You can use any LaTeX tools such as Overleaf, MiKTeX/TeXWorks, TeXShop etc. Scans of handwritten/hand drawn solutions are not acceptable.

Each problem is worth the indicated number of points. Solutions receiving full points must be correct (no errors or omissions), clear (stated in a precise and concise way), and have a well organized presentation. Show your work as partial points will be awarded to rough solutions or solutions that make partial progress toward a correct solution.

Remember: You must write up the solutions completely on your own. You must also list the names of everyone whom you discussed the problem set with. You may not consult written materials other than the course materials in coming up with your solutions.

2 Problems

2.1 Understand

(2 points) 1. Prove the theorem below. Use a **direct proof** to find constants that satisfy the definition of Big-Theta *or else* use the **limit test**. Make sure your proof is complete, concise, clear and precise.

Theorem 1. $81n^5 - 54n^3 - 26n^2 \in \Theta(n^5)$

(2 points) 2. Prove Gauss's sum using induction on n. Make sure to include a base case for $n = 1$ and an inductive hypothesis and an inductive step for $n > 1$.

Theorem 2.
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(3 points) 3. Prove the following extension to Gauss's sum using induction on n. Make sure to include a base case for $n = 1$ and an inductive hypothesis and an inductive step for $n > 1$.

Theorem 3.
$$\sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

(1+2 points) 4. Prove the following theorem using the definition of $\Omega(f(n))$ and $O(f(n))$, or else the limit test.

Theorem 4. *Let $d \geq 1$ be an integer. Then $\sqrt[4d]{n} \in O(\sqrt[d]{n})$ and $\sqrt{n} \in \Omega\left((\ln n)^d\right)$.*

Grading You will be docked points for errors in your math, disorganization, lack of clarity, or incomplete proofs.

2.2 Explore

(10 points) 1. Place these functions in order from slowest asymptotic growth to fastest asymptotic growth. You will want to simplify them algebraically before comparing them. Give a short justification (a proof is not necessary) of how you came to this ordering. The

notation $\lg n$ stand for $\log_2 n$.

$$\begin{aligned}
f_0(n) &= 2020^{\sqrt[3]{n}} \\
f_1(n) &= 8^{\lg n} \\
f_2(n) &= n^6 - n^4 - n^3 - n \\
f_3(n) &= 3^{2n} \\
f_4(n) &= \left(\frac{n}{\lg n}\right)^6 \\
f_5(n) &= \ln(\ln n + 1) \\
f_6(n) &= n^{\lg n} \\
f_7(n) &= 8^{n/3} \\
f_8(n) &= \ln n + 1
\end{aligned}$$

Grading You will be docked points for functions in the wrong order and for lack of justification for your ordering.

2.3 Expand

(5 points) 1. Prove the theorem below using the techniques of **binding the term** and **splitting the sum** to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.

Theorem 5.

$$\sum_{i=1}^{n} i^d \in \Theta(n^{d+1})$$

(5 points) 2. Prove the theorem below using the techniques of **binding the term** and **splitting the sum** to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise. The notation $\lg n$ stand for $\log_2 n$.

Theorem 6.

$$\sum_{i=1}^{\lg n} \lg i \in \Theta\big(\lg n \cdot \lg(\lg n)\big)$$

Grading You will be docked points for errors in your math, disorganization, lack of clarity, or incomplete proofs.

2.4 Challenge

(5 points) 1. Find two strictly increasing functions $f(n)$ and $g(n)$ such that $f(n) \notin O(g(n))$ and $g(n) \notin O(f(n))$, and both involving only “elementary” operations in their description (that is, only basic arithmetic operations, powers, exponentials, and factorials). Prove that the functions you found do indeed satisfy the above relationships (pay particular attention to the requirement that the functions are strictly increasing, e.g. $f(n) = \sin n$ and $g(n) = \cos n$ trivially fail to satisfy this requirement).

Grading The restriction to elementary operations in the description is intended to facilitate the proof: the simpler, the better. You will be docked points for errors in your math, disorganization, lack of clarity, or incomplete proofs.