TCSS 343 A Homework #1

Daniel Ty

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1 2.1 Understand

1. Prove the theorem below. Use a direct proof to find constants that satisfy the definition of Big-Theta or else use the limit test. Make sure your proof is complete, concise, clear and precise.

Theorem 1.

$$81n^5 - 54n^3 - 26n^2 \in \Theta(n^5)$$

Limit test proof:

$$\lim_{n\to\infty} \frac{81n^5 - 54n^3 - 26n^2}{n^5}$$

$$= \lim_{n\to\infty} \frac{81n^5}{n^5} - \lim_{x\to\infty} \frac{54n^3}{n^5} - \lim_{x\to\infty} \frac{26n^2}{n^5}$$

$$= \lim_{x\to\infty} 81 - \lim_{x\to\infty} \frac{54}{n^2} - \lim_{x\to\infty} \frac{26}{n^3}$$

$$= 81 - 0 - 0 = 81$$

The Theorem 1 holds true because the $\lim_{x\to\infty} \frac{81n^5 - 54n^3 - 26n^2}{n^5} = 81$. The limit theorem states that if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$, where c is some positive constant, then $f(x) \in \Theta(g(x))$. So, because 81 is a positive constant, Theorem 1. $81n^5 - 54n^3 - 26n^2 \in \Theta(n^5)$ holds true.

2. Prove Gauss's sum using induction on n. Make sure to include a base case for $n = 1$ and an inductive hypothesis and an inductive step for $n > 1$.

Theorem 2.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof by Induction)

Base Case: $n = 1$

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1$$

Inductive Hypothesis: $n = k$, $n > 1$

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Inductive Step: $n = k+1$, $n > 1$

$$\sum_{i=1}^{k+1} i =? \frac{(k+1)[(k+1)+1]}{2}$$

$$\sum_{i=1}^{k+1} i = 1 + 2 + ... + k + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)}{2} + \frac{2k+2}{2}$$

$$= \frac{k(k+1) + 2k + 2}{2} = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)[(k+1)+1]}{2}$$

This is of the Theorem 2's form, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, where $n = k+1$ and $n > 1$. This proves Gauss's theorem by induction.

3. Prove the following extension to Gauss's sum using induction on n. Make sure to include a base case for $n = 1$ and an inductive hypothesis and an inductive step for $n > 1$.

Theorem 3.

$$\sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

Proof by Induction)

Basic Step: $n = 1$

$$\sum_{i=1}^{1} i^4 = \frac{1^5}{5} + \frac{1^4}{2} + \frac{1^3}{3} - \frac{1}{30} = \frac{6}{30} + \frac{15}{30} + \frac{10}{30} - \frac{1}{30} = \frac{30}{30} = 1$$

Inductive Hypothesis: $n = k$, $n > 1$

$$\sum_{i=1}^{k} i^4 = \frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30}$$

Inductive Step: $n = k + 1$, $n > 1$

$$\sum_{i=1}^{k+1} i^4 =? \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{(k+1)}{30}$$

Left-hand side:

$$\sum_{i=1}^{k+1} i^4 = 1^4 + 2^4 + ... + k^4 + (k+1)^4 = \frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} + (k+1)^4$$

$$= \frac{6k^5 + 15k^4 + 10k^3 - k + 30(k+1)^4}{30} = \frac{6k^5 + 15k^4 + 10k^3 - k + 30k^4 + 120k^3 + 180k^2 + 120k + 30}{30}$$

$$= \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30}$$

Right-hand side:

$$\frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{(k+1)}{30} = \frac{6(k+1)^5 + 15(k+1)^4 + 10(k+1)^3 - (k+1)}{30}$$

$$= \frac{6(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) + 15(k^4 + 4k^3 + 6k^2 + 4k + 1) + 10(k^3 + 3k^2 + 3k + 1) - (k+1)}{30}$$

$$= \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30}$$

The Left-hand side and Right-hand side are equal, meaning that $\sum_{i=1}^{k+1} i^4 = ?\frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{(k+1)}{30}$ is true, and Theorem 3, $\sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$, has been proven by induction.

4. Prove the following theorem using the definition of $\Omega(f(n))$ and $O(f(n))$, or else the limit test.

Theorem 4.
Let $d \geq 1$ be an integer. Then $\sqrt[4d]{n} \in O(\sqrt[d]{n})$ and $\sqrt{n} \in \Omega((\ln n)^d)$

Theorem: $\sqrt[4d]{n} \in O(\sqrt[d]{n})$
Definition of $O(f(n))$: A function $f(n) \in O(g(n))$ if there exist positive constants b and n_0 such that $f(n) \leq b \times g(n)$ for all $n > n_0$.
Proof: For all $n > 1$ and $d \geq 1$,

$$n^{\frac{1}{4d}} \leq (1)n^{\frac{1}{d}}$$

So, if we select $n_0 = 1$ and $b = 1$, the definition holds and proves $\sqrt[4d]{n} \in O(\sqrt[d]{n})$.

Theorem:$\sqrt{n} \in \Omega((\ln n)^d)$
Definition of $\Omega(f(n))$: A function $f(n) \in \Omega(g(n))$ if there exist positive constants a and n_0 such that $a \times g(n) \leq f(n)$ for all $n > n_0$.
Proof: For all $d \geq 1$, there exists an n_0 such that for all $n > n_0$,

$$(\ln n)^d \leq \sqrt{n}$$

I am unsure how to calculate n_0 for all $d \geq 1$, but I think it is exponential because the n_0, when $\sqrt{n}$ passes $(\ln n)^d$ for all positive n values greater than n_0, becomes very large as d increases even slightly.

2 2.2 Explore

1. Place these functions in order from slowest asymptotic growth to fastest asymptotic growth. You will want to simplify them algebraically before comparing them. Give a short justification (a proof is not necessary) of how you came to this ordering. The notation $\lg n$ stands for $\log_2 n$.

Slowest

$f_5(n) = \ln(\ln n + 1)$
$f_8(n) = \ln n + 1$
$f_1(n) = 8^{\lg n} = 2^{3\lg n} = 2^{\lg n^3} = n^3$
$f_4(n) = (\frac{n}{\lg n})^6 = \frac{n^6}{(\lg n)^6}$
$f_6(n) = n^{\lg n}$
$f_2(n) = n^6 - n^4 - n^3 - n$
$f_0(n) = 2020^{\sqrt[3]{n}}$
$f_7(n) = 8^{n/3} = 2^{3\frac{n}{3}} = 2^n$
$f_3(n) = 3^{2n}$
Fastest

I ordered the functions by their largest terms, from slowest asymptotic growth to fastest, in order of logarithmic ($\log_a n$), then polynomial ($\sqrt{n}, n, n\log n, n^2, ...$), then exponential ($2^n, 3^n, n!, n^n$). If terms were of equal order, I ordered by examining the largest term within the 'parent' term. If I was unsure, I inputted very large integers for n to check. Whichever function had the greater answer was determined to be faster.

3 2.3 Expand

1. Prove the theorem below using the techniques of binding the term and splitting the sum to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.

Theorem 5.

$$\sum_{i=1}^{n} i^d \in \Theta(n^{d+1})$$

Proof by Binding and Splitting (Squeeze Theorem):

Upper Bound

$$\sum_{i=1}^{n} i^d \leq \sum_{i=1}^{n} n^d = n(n^d) = n^{d+1} \in O(n^{d+1})$$

Lower Bound

$$\sum_{i=1}^{n} i^d = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i^d + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} i^d \geq \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} i^d$$

$$\geq \lceil \frac{n}{2} \rceil (\lfloor \frac{n}{2} \rfloor + 1)^d \geq \frac{n}{2} (\frac{n}{2})^d = (\frac{n}{2})^{d+1} = (2^{-1})^{d+1} n^{d+1} = 2^{-1}(2^{-d}) n^{d+1}$$

$$= \frac{1}{2(2^d)} n^{d+1} \geq \frac{1}{3(2^d)} n^{d+1}$$

For all $n > 0$, therefore, $\sum_{i=1}^{n} i^d \in \Omega(n^{d+1})$

Combining the upper and lower bounds, we get, $\frac{1}{3(2^d)} n^{d+1} \leq \sum_{i=1}^{n} i^d \leq n^{d+1}$ for all $n > 0$, so, we can conclude $\sum_{i=1}^{n} i^d \in \Theta(n^{d+1})$.

2. Prove the theorem below using the techniques of binding the term and splitting the sum to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise. The notation $\lg n$ stands for $\log_2 n$.

Theorem 6.

$$\sum_{i=1}^{\lg n} \lg i \in \Theta(\lg n \times (\lg n))$$

Proof by Binding and Splitting (Squeeze Theorem):

Upper Bound

$$\sum_{i=1}^{\lg n} \lg i \leq \sum_{i=1}^{\lg n} \lg n = \lg n \times \lg n \in O(\lg n \times (\lg n))$$

Lower Bound

$$\sum_{i=1}^{\lg n} \lg i = \sum_{i=1}^{\lfloor \frac{\lg n}{2} \rfloor} \lg i + \sum_{i=\lfloor \frac{\lg n}{2} \rfloor + 1}^{\lg n} \lg i \geq \sum_{i=\lfloor \frac{\lg n}{2} \rfloor + 1}^{\lg n} \lg i$$

$$\geq \lceil \frac{\lg n}{2} \rceil \lg(\lfloor \frac{\lg n}{2} \rfloor + 1) \geq \frac{\lg n}{2} \lg(\frac{\lg n}{2}) = \frac{\lg n}{2} (\lg(\lg n) - 1)$$

$$\geq \frac{1}{10}(\lg n \times (\lg n))$$

For $n > 9$, therefore, $\sum_{i=1}^{\lg n} \lg i \in \Omega(\lg n \times (\lg n))$.

Combining the upper and lower bounds, we get, $\frac{1}{10}(\lg n \times (\lg n)) \leq \sum_{i=1}^{\lg n} \lg i \leq \lg n \times (\lg n)$ for all $n > 9$, so we can conclude, $\sum_{i=1}^{\lg n} \lg i \in \Theta(\lg n \times (\lg n))$.