Problem 1

In order for the sum to be greater than or equal to k, we need at least k ones. The number of ways to get k ones is:

$$\left(\sum_{i=1}^{n} x_i = k\right) \to \binom{n}{k}$$

Then we have to take in account all of the ways to get $k+1, k+2, ..., n$ ones. Summing all of these possibilities gives us:

$$\sum_{i=k}^{n} \binom{n}{i}$$

Problem 2

We want to show $\displaystyle\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$ for $n > 0$

The binomial theorem states:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \forall\ n \in \mathbb{N} \tag{1}$$

Let $x = -1, y = 1, k = i$

Then (1) becomes $\displaystyle\sum_{i=0}^{n} \binom{n}{i}(-1)^i(1)^{n-i} = (-1+1)^n$

Then we have $\displaystyle\sum_{i=0}^{n} \binom{n}{i}(-1)^i(1) = 0^n$

$$= \sum_{i=0}^{n} \binom{n}{i}(-1)^i = 0$$

Therefore $\displaystyle\sum_{i=0}^{n} \binom{n}{i}(-1)^i = 0$ for $n > 0$

Problem 3

(a) $\dfrac{4\binom{13}{5}}{\binom{52}{5}} \approx .001980$

(b) $\dfrac{13\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}} \approx .422569$

(c) $\dfrac{\binom{13}{2}\binom{4}{2}\binom{4}{2}11*4}{\binom{52}{5}} \approx .047539$

(d) $\dfrac{13\binom{4}{3}\binom{12}{2}*4^2}{\binom{52}{5}} \approx .021128$

(e) $\dfrac{13*12*4}{\binom{52}{5}} \approx .000240$

Problem 4

$$\frac{16*4}{52*51}+\frac{4*16}{52*51}=\frac{16*4*2}{52*51}\approx .048265$$

Problem 5

Let E_i be the event that the ith couple sit next to each other
$P\left(\bigcup_{i=1}^{4} E_i\right)$ =P(at least one couple sits together)
P(no couples sit together) = 1-P(at least 1 couple sits together)
P(no couples sit together) =$1-P\left(\bigcup_{i=1}^{4} E_i\right)$
Using the inclusion-exclusion principle we have:

$$P\left(\bigcup_{i=1}^{4} E_i\right)=\sum E_i-\sum E_i\cap E_j+\sum E_i\cap E_j\cap E_k-\sum E_i\cap E_j\cap E_k\cap E_l$$

$$P(E_i)=\frac{2*7!}{8!}$$

$$P(E_i\cap E_j)=\frac{2^2*6!}{8!}$$

$$P(E_i\cap E_j\cap E_k)=\frac{2^3*5!}{8!}$$

$$P(E_i\cap E_j\cap E_k\cap E_l)=\frac{2^4*4!}{8!}$$

$$P\left(\bigcup_{i=1}^{4} E_i\right)=\binom{4}{1}\frac{2*7!}{8!}-\binom{4}{2}\frac{2^2*6!}{8!}+\binom{4}{3}\frac{2^3*5!}{8!}-\binom{4}{4}\frac{2^4*4!}{8!}$$

$$P\left(\bigcup_{i=1}^{4} E_i\right)=1-\frac{3}{7}+\frac{4}{42}-\frac{1}{105}$$

$$1-P\left(\bigcup_{i=1}^{4} E_i\right)=1-\left(1-\frac{3}{7}+\frac{4}{42}-\frac{1}{105}\right)$$

$$=\frac{3}{7}-\frac{4}{42}+\frac{1}{105}$$

$$\text{P(no couples sit together)}=\frac{12}{35}=.3428571$$

Problem 6

There are N^n equally likely arrangements. There are $\binom{n}{m}$ ways to select m balls for the first compartment. This leaves $N-1$ compartments and $n-m$ balls giving us $(N-1)^{n-m}$ possible arrangements for the remaining balls. Putting this all together we have:

$$\frac{\binom{n}{m}(N-1)^{n-m}}{N^n}$$

Problem 7

$$\frac{\binom{6}{3}\binom{6}{3}}{\binom{12}{6}} = .4329004$$

Problem 8

There are $(N - r)$ empty parking spaces that could be on adjacent to one side of his car over $(N - 1)$ total spaces (since we don't count his car). This then leaves $(N - r - 1)$ empty parking spaces that could be adjacent to the other side of his car over $(N - 2)$ total spaces. Putting this together we have:

$$\frac{(N-r)(N-r-1)}{(N-1)(N-2)}$$

Problem 9

In order to find the conditional probability of the coin having a heads and tails side given that one side turned up heads, we will use Bayes Rule combined with the Law of Total Probability.

$$\text{3 coins: } \{HH, TT, HT\}$$

$$P(HT|H) = \frac{P(H|HT)P(HT)}{P(H|HT)P(HT) + P(H|HT^c)P(HT^c)}$$

$$P(HT|H) = \frac{1/2 * 1/3}{1/2 * 1/3 + 1/2 * 2/3}$$

$$P(HT|H) = 1/3$$