Problem 1

WTS: For any r.v. X, if $g(x)$ is a convex function, then: $Eg(X) \geq g(EX)$
Let $g(x)$ be a convex function
Suppose $a + bx$ is a line tangent to $g(x)$ at $x = EX$
and $g(x) > a + bx$ except at $x = EX$
Then $Eg(X) > g(EX)$ unless $P(X = EX) = 1$

Problem 2

(a)

$$\begin{aligned}
f_{XY}(x,y) &= \left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1} \\
&\quad * \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\} \\
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy \\
\text{Let } z &= \frac{y-\mu_Y}{\sigma_Y} \quad dy = \sigma_Y dz \quad v = \frac{x-\mu_X}{\sigma_X} \\
f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[v^2 - 2\rho vz + z^2\right]\right\}\sigma_Y\ dz \\
&= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-2\rho vz + z^2\right]\right\} dz \\
&= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-2\rho vz + z^2 + \rho^2 v^2 - \rho^2 v^2\right]\right\} dz \\
&= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[(z-\rho v)^2 - \rho^2 v^2\right]\right\} dz \\
&= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)\exp\left(\frac{-\rho^2 v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2\right\} dz \\
&= \frac{e^{-v^2/2}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2\right\} dz
\end{aligned}$$

Since the integrand is the $N(\rho v, 1-\rho^2)$ we have:

$$f_X(x) = \frac{e^{-v^2/2}}{2\pi\sigma_X\sqrt{1-\rho^2}}\sqrt{2\pi}\sqrt{1-\rho^2}$$

$$= \frac{e^{-v^2/2}}{\sqrt{2\pi}\sigma_X}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)$$

Which is the $N(\mu_X, \sigma_X^2)$ pdf

(b)

WTS: $f(Y|X)(y|x) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y} e^{\dfrac{-[y-\mu_Y-(\rho\sigma_Y/\sigma_X)(x-\mu_X)]^2}{2\sigma_Y^2(1-\rho^2)}}$

$$f(Y|X)(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$= \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}{\frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)}$$

$$= \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] + \frac{(x-\mu_X)^2}{2\sigma_X^2}\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - (1-p^2)\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2\sigma_Y^2(1-\rho^2)}[y-\mu_Y-(\rho\sigma_Y/\sigma_X)(x-\mu_X)]^2\right\}}$$

Which is the pdf of $N[\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_X), \sigma_Y^2(1-\rho^2)]$

(c)