Problem 4.63

$EZ = 0$

$$Z = g(X) = \log(X)$$
$$\text{Since } g(X) \text{ is concave:}$$
$$E(g(X)) \leq g(EX)$$
$$0 \leq g(EX)$$
$$0 \leq \log(EX)$$
$$e^0 \leq e^{\log EX}$$
$$1 \leq EX$$
$$EX > 1 \text{ unless for every line } a{+}bx \text{ that is tangent to } g(X) \text{ at } x = EX,\ P(g(X) = a{+}bX) = 1$$
$$\log(x) \text{ is linear on an interval only if the interval is a single point}$$
$$\text{Thus } EX > 1 \text{ unless } P(X = EX) = 1$$

$Z = \log(X)$

$$EZ = 0$$
$$X = e^Z$$
$$g(z) = e^z$$
$$\text{Since } e^z \text{ is convex :}$$
$$EX = E(g(Z)) \geq g(EZ)$$
$$EX \geq g(0) = e^0 = 1$$
$$E(X) \geq 1$$
$$E(X) = 1 \text{ iff there is an interval I with } P(Z \in I) = 1 \text{ and } g(z) \text{ is linear on I}$$
$$e^z \text{ is only linear on an interval if the interval is a single point}$$
$$\text{Thus } E(X) > 1 \text{ unless } P(Z = EZ = 0) = 1$$

Problem 4.64

(a)

$$|a+b|^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$
$$(|a| + |b|)^2 = |a|^2 + 2|ab| + |b|^2$$
$$a^2 + 2ab + b^2 \leq |a|^2 + 2|ab| + |b|^2$$
$$|a+b|^2 \leq (|a| + |b|)^2$$

$$\sqrt{|a+b|^2} \leq \sqrt{(|a|+|b|)^2}$$
$$|a+b| \leq |a|+|b|$$

(b)

$$\text{WTS: } E|X+Y| \leq E|X|+E|Y|$$
$$|X+Y| \leq |X|+|Y| \quad \text{plugging X and Y into the triangle inequality}$$
$$E|X+Y| \leq E[|X|+|Y|] \quad \text{taking the expectation of both sides}$$
$$E|X+Y| \leq E|X|+E|Y| \quad \text{by linearity of expectation}$$

Problem 4.65

$$\text{WTS: } E(XY) \leq EXEY \text{ if X is nondecreasing, Y is nonincreasing}$$
$$Cov(X,Y) = E(XY) - EXEY \leq 0 \text{ since there is negative correlation between X and Y (or no correlation)}$$
$$\text{Thus } E(XY) \leq EXEY$$
$$\text{If X and Y are both nonincreasing or nondecreasing :}$$
$$\text{There is positive correlation (or no correlation)}$$
$$Cov(X,Y) = E(XY) - EXEY \geq 0$$
$$\text{Thus } E(XY) \geq EXEY$$