Definition A.4.1. *For a sequence of sets A_n, $n \in \mathbb{N}$, we define*

$$\begin{aligned}\inf_{k \geq n} A_k &= \bigcap_{k=n}^{\infty} A_k \\ \sup_{k \geq n} A_k &= \bigcup_{k=n}^{\infty} A_k \\ \liminf_{n \to \infty} A_n &= \bigcup_{n \in \mathbb{N}} \inf_{k \geq n} A_k = \bigcup_{n \in \mathbb{N}} \bigcap_{k=n}^{\infty} A_k \\ \limsup_{n \to \infty} A_n &= \bigcap_{n \in \mathbb{N}} \sup_{k \geq n} A_k = \bigcap_{n \in \mathbb{N}} \bigcup_{k=n}^{\infty} A_k.\end{aligned}$$

Applying De-Morgan's law (Proposition A.1.1) we have

$$\left(\liminf_{n \to \infty} A_n\right)^c = \limsup_{n \to \infty} A_n^c.$$

Definition A.4.2. *If for a sequence of sets A_n, $n \in \mathbb{N}$, we have $\liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n$, we define the limit of A_n, $n \in \mathbb{N}$ to be*

$$\lim_{n \to \infty} A_n = \liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n,$$

The notation $A_n \to A$ is equivalent to the notation $\lim_{n \to \infty} A_n = A$.

Example A.4.1. For the sequence of sets $A_k = [0, k/(k+1))$ from Example A.1.6 we have

$$\begin{aligned}\inf_{k \geq n} A_k &= [0, n/(n+1)) \\ \sup_{k \geq n} A_k &= [0, 1) \\ \limsup_{n \to \infty} A_n &= [0, 1) \\ \liminf_{n \to \infty} A_n &= [0, 1) \\ \lim A_n &= [0, 1).\end{aligned}$$

We have the following interpretation for the $\liminf$ and $\limsup$ limits.

Proposition A.4.1. *Let $A_n, n \in \mathbb{N}$ be a sequence of subsets of Ω. Then*

$$\begin{aligned}\limsup_{n \to \infty} A_n &= \left\{\omega \in \Omega : \sum_{n \in \mathbb{N}} I_{A_n}(\omega) = \infty\right\} \\ \liminf_{n \to \infty} A_n &= \left\{\omega \in \Omega : \sum_{n \in \mathbb{N}} I_{A_n^c}(\omega) < \infty\right\}.\end{aligned}$$

In other words, $\limsup_{n \to \infty} A_n$ is the set of $\omega \in \Omega$ that appear infinitely often (abbreviated i.o.) in the sequence A_n, and $\liminf_{n \to \infty} A_n$ is the set of $\omega \in \Omega$ that always appear in the sequence A_n except for a finite number of times.

Proof. We prove the first part. The proof of the second part is similar. If $\omega \in \limsup_{n\to\infty} A_n$ then by definition for all n there exists a k_n such that $\omega \in A_{k_n}$. For that ω we have $\sum_{n\in\mathbb{N}} I_{A_n}(\omega) = \infty$. Conversely, if $\sum_{n\in\mathbb{N}} I_{A_n}(\omega) = \infty$, there exists a sequence $k_1, k_2, \ldots$ such that $\omega \in A_{k_n}$, implying that for all $n \in \mathbb{N}$, $\omega \in \cup_{i\geq n} A_i$.

Corollary A.4.1.

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.$$

Definition A.4.3. *A sequence of sets $A_n, n \in \mathbb{N}$ is monotonic non-decreasing if $A_1 \subset A_2 \subset A_3 \subset \cdots$ and monotonic non-increasing if $\cdots \subset A_3 \subset A_2 \subset A_1$. We denote this as $A_n \nearrow$ and $A_n \searrow$, respectively. If $\lim A_n = A$, we denote this as $A_n \nearrow A$ and $A_n \searrow A$, respectively.*

Proposition A.4.2. *If $A_n \nearrow$ then $\lim_{n\to\infty} A_n = \cup_{n\in\mathbb{N}} A_n$ and if $A_n \searrow$ then $\lim_{n\to\infty} A_n = \cap_{n\in\mathbb{N}} A_n$.*

Proof. We prove the first statement. The proof of the second statement is similar. We need to show that if A_n is monotonic non-decreasing, then $\limsup A_n = \liminf A_n = \cup_n A_n$. Since $A_i \subset A_{i+1}$, we have $\cap_{k\geq n} A_k = A_n$, and

$$\begin{aligned}
\liminf_{n\to\infty} A_n &= \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k = \bigcup_{n\in\mathbb{N}} A_n \\
\limsup_{n\to\infty} A_n &= \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \subset \bigcup_{k\in\mathbb{N}} A_k = \liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.
\end{aligned}$$

The following corollary of the above proposition motivates the notations $\liminf$ and $\limsup$.

Corollary A.4.2. *Since $B_n = \cup_{k\geq n} A_k$ and $C_n = \cap_{k\geq n} A_k$ are monotonic sequences*

$$\begin{aligned}
\liminf_{n\to\infty} A_n &= \lim_{n\to\infty} \inf_{k\geq n} A_n \\
\limsup_{n\to\infty} A_n &= \lim_{n\to\infty} \sup_{k\geq n} A_n.
\end{aligned}$$