

## Problem 1

From Example 4.3.1 and Theorem 4.3.2 we know:

$$X + Y \sim \text{Poisson}(\theta + \lambda)$$

$$\text{Let } U = X + Y, \quad V = Y$$

$$\text{Then } X = U - V, \quad Y = V$$

$$f(u, v) = \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^v e^{-\lambda}}{v!} \quad v = 0, 1, \dots \quad u = v, v+1, \dots$$

$$f(u) = \frac{e^{-(\theta+\lambda)}}{u!} (\theta + \lambda)^u \quad u = 0, 1, \dots$$

Finding  $Y|X + Y$

Defining  $U$  and  $V$  the same way:

$$f(y|x+y) = f(v|u)$$

$$\begin{aligned} f(v|u) &= \frac{f(u, v)}{f(u)} \\ &= \frac{\frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^v e^{-\lambda}}{v!}}{\frac{e^{-(\theta+\lambda)}}{u!} (\theta + \lambda)^u} \\ &= \frac{u!}{(u-v)! v!} \frac{e^{-(\theta+\lambda)}}{e^{-(\theta+\lambda)}} \frac{\theta^{u-v} \lambda^v}{(\theta + \lambda)^u} \\ &= \binom{u}{v} \frac{\theta^{u-v} \lambda^v}{(\theta + \lambda)^u} \\ &= \binom{u}{v} \left( \frac{\lambda}{\theta + \lambda} \right)^v \left( \frac{\theta}{\theta + \lambda} \right)^{u-v} \end{aligned}$$

Which is binomial  $\left(u, \frac{\lambda}{\theta + \lambda}\right)$

Finding  $X|X + Y$

$$\text{Define } U = X + Y, \quad V = X$$

$$\text{Then } X = U - V, \quad Y = V$$

$$f_{U,V}(u, v) = f_{X,Y}(v, u-v) = \frac{\theta^v e^{-\theta}}{v!} \frac{\lambda^{u-v} e^{-\lambda}}{(u-v)!}$$

$$\begin{aligned} f(u) &= \sum_{v=0}^u \frac{\theta^v e^{-\theta}}{v!} \frac{\lambda^{u-v} e^{-\lambda}}{(u-v)!} \\ &= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^u \binom{u}{v} \theta^v \lambda^{u-v} \end{aligned}$$

Using the binomial theorem we have:

$$\begin{aligned}
f(u) &= \frac{e^{-(\theta+\lambda)}}{u!} (\theta + \lambda)^u \\
f(x|x+y) &= f(v|u) \\
&= \frac{f(u, v)}{f(u)} \\
&= \frac{\frac{\theta^v e^{-\theta}}{v!} \frac{\lambda^{u-v} e^{-\lambda}}{(u-v)!}}{\frac{e^{-(\theta+\lambda)}}{u!} (\theta + \lambda)^u} \\
&= \binom{u}{v} \frac{\theta^v \lambda^{u-v}}{(\theta + \lambda)^u} \\
&= \binom{u}{v} \left( \frac{\theta}{\theta + \lambda} \right)^v \left( \frac{\lambda}{\theta + \lambda} \right)^{u-v}
\end{aligned}$$

Which is binomial  $\left( u, \frac{\theta}{\theta + \lambda} \right)$

## Problem 2

$$f_X(x) = p(1-p)^{x-1} \quad f_Y(y) = p(1-p)^{y-1}$$

Since X and Y are independent we have:

$$\begin{aligned}
f_{X,Y}(x, y) &= p(1-p)^{x-1} p(1-p)^{y-1} \\
&= p^2 (1-p)^{x+y-2}
\end{aligned}$$

(a) Solving  $V = X - Y$  for X we get  $X = V + Y$

If  $V > 0$  then  $X > Y$

Since  $U = \min(X, Y)$  this means that  $U = Y$

Thus we have  $Y = U$  and  $X = U + V$

$$\begin{aligned}
f_{U,V}(u, v) &= P(Y = u, X = u + v) \\
&= p^2 (1-p)^{2u+v-2}
\end{aligned}$$

Which factors to:  $(p^2(1-p)^{2u})(1-p)^{v-2}$

If  $V < 0$ , then  $X < Y$

Thus  $X = U$ ,  $Y = U - V$

$$\begin{aligned}
f_{U,V}(u, v) &= P(X = u, Y = u - v) \\
&= p^2 (1-p)^{2u-v-2}
\end{aligned}$$

Which factors to:  $(p^2(1-p)^{2u})(1-p)^{-v-2}$

If  $V = 0$  then  $X = Y$

$$f_{U,V}(u, 0) = P(X = Y = u) = p^2 (1-p)^{2u-2}$$

Which factors to:  $(p^2(1-p)^{2u})(1-p)^{-2}$

Since we can factor all of these cases in terms of  $u$  and  $v$ ,  $U$  and  $V$  are independent

(b)

$$Z = \frac{X}{(X+Y)}$$

Define  $U = X$

$$Z = r/s \text{ where } r < s$$

$$\frac{x}{x+y} = r/s$$

Then  $X = r$   $Y = s - r$

$$\begin{aligned} P(Z = r/s) &= \sum_{i=1}^{\infty} P(X = ir, Y = i(s-r)) \\ &= \sum_{i=1}^{\infty} p(1-p)^{ir-1} p(1-p)^{i(s-r)-1} \\ &= \sum_{i=1}^{\infty} p^2(1-p)^{is-2} \\ &= \left(\frac{p}{1-p}\right)^2 \sum_{i=1}^{\infty} ((1-p)^s)^i \\ &= \left(\frac{p}{1-p}\right)^2 \frac{(1-p)^s}{1-(1-p)^s} \\ &= \frac{p^2(1-p)^{s-2}}{1-(1-p)^s} \end{aligned}$$

(c)

Define  $T = X + Y$

$$\begin{aligned} f_{X,X+Y}(x, x+y) &= P(X = x, X+Y = t) = P(X = x, Y = t-x) = P(X = x)P(Y = t-x) \\ &= p^2(1-p)^{x-1+t-x-1} = p^2(1-p)^{t-2} \end{aligned}$$

### Problem 3

(a)

$X_1, X_2$  are independent and distributed as:

$$\begin{aligned} f_{X_i}(x_i) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-x_i^2/2\sigma^2} \\ f_{X_1, X_2}(x_1, x_2) &= \frac{1}{2\pi\sigma^2} e^{-x_1^2/2\sigma^2} e^{-x_2^2/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2} e^{-(x_1^2+x_2^2)/2\sigma^2} \end{aligned}$$

Since the transformation is not one to one, we must partition the support of  $(X_1, X_2)$

$$A_0 = \{-\infty < x_1 < \infty, x_2 = 0\}$$

$$A_1 = \{-\infty < x_1 < \infty, x_2 < 0\}$$

$$A_2 = \{-\infty < x_1 < \infty, x_2 > 0\}$$

Since  $Y_2 = \frac{X_1}{\sqrt{X_1^2 + X_2^2}}$  it ranges from -1 to 1

The support of  $(Y_1, Y_2)$  is  $\mathcal{B} = \{0 < y_1 < \infty, -1 < y_2 < 1\}$

$$Y_1 = X_1^2 + X_2^2 \quad Y_2 = \frac{X_1}{\sqrt{Y_1}}$$

For  $A_1$  :

$$X_1 = Y_2 \sqrt{Y_1}$$

$$X_2 = \sqrt{Y_1 - Y_2^2 Y_1}$$

$$J_1 = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ \frac{\sqrt{1-y_2^2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1} y_2}{\sqrt{1-y_2^2}} \end{bmatrix}$$

$$= \frac{1}{2\sqrt{1-y_2^2}}$$

For  $A_2$  :

$$X_1 = Y_2 \sqrt{Y_1}$$

$$X_2 = -\sqrt{Y_1 - Y_2^2 Y_1}$$

$$J_2 = -J_1$$

$$f_{Y_1, Y_2}(y_1, y_2) = 2 \left( \frac{1}{2\pi\sigma^2} e^{-y_1/2\sigma^2} \right) \frac{1}{2\sqrt{1-y_2^2}}$$

$$= \frac{1}{2\pi\sigma^2} e^{-y_1/2\sigma^2} \frac{1}{\sqrt{1-y_2^2}} \quad 0 < y_1 < \infty, -1 < y_2 < 1$$

(b)

$$f_{Y_1, Y_2}(y_1, y_2) = \left( \frac{1}{2\pi\sigma^2} e^{-y_1/2\sigma^2} \right) \left( \frac{1}{\sqrt{1-y_2^2}} \right)$$

Since the joint pdf factors into a function of  $y_1$  and a function of  $y_2$   
 $y_1$  and  $y_2$  are independent.

## Problem 4

(a)

$$\begin{aligned}
 f_{X,Y}(x,y) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1} (1-y)^{\gamma-1} \\
 0 &< x < 1 \quad 0 < y < 1 \\
 U &= XY \quad V = Y \\
 X &= U/V \quad Y = V \\
 J &= 1/v \\
 f_{U,V}(u,v) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} (u/v)^{\alpha-1} (1-u/v)^{\beta-1} v^{\alpha+\beta-1} (1-v)^{\gamma-1} (1/v) \\
 0 &< u < v < 1 \\
 f_U(u) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 v^{\beta-1} (1-v)^{\gamma-1} ((v-u)/v)^{\beta-1} dv \\
 \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1} (1-u)^{\beta+\gamma-1} & \quad 0 < u < 1 \\
 U &\sim \text{gamma}(\alpha, \beta + \gamma)
 \end{aligned}$$

## Problem 5

(a)

$$\begin{aligned}
 Y|X &\sim N(x, x^2) \\
 E(Y|X) &= X \\
 \text{Var}(Y|X) &= X^2 \\
 X &\sim U(0, 1) \\
 EY &= E(E(Y|X)) = EX \\
 f_X(x) &= 1 \quad \text{for } 0 \leq x \leq 1 \\
 EX &= \frac{1}{2} = EY \\
 \text{Var}(Y) &= E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)) = E(X^2) + \text{Var}(X) \\
 \text{Var}(X) &= \frac{1}{12} \\
 E(X^2) &= \int_0^1 x^2 dx = 1/3 \\
 \text{Var}(Y) &= 1/12 + 1/3 = \frac{5}{12} \\
 \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
 &= E(XY) - (1/2)^2 = E(XY) - 1/4
 \end{aligned}$$

$$E(XY) = E[E(XY|X)] = E(XE(Y|X)) = E(X^2) = 1/3$$

$$\text{Cov}(X, Y) = 1/3 - 1/4 = 1/12$$

(b)

$$f(Y|X = x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{(y-x)^2}{2x^2}}$$

Since  $X = 1$  we have:

$$f(Y|X = 1) = \frac{1}{\sqrt{2\pi}1} e^{-\frac{(y-1)^2}{2 \cdot 1^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Thus  $f(Y|X = x) \sim N(1, 1)$  and  $Y$  is independent of  $X$ By theorem 4.3.5 Since  $X$  and  $Y$  are independent,  $U = g(X)$  and  $V = g(Y)$  are independentThus the transformation  $Y/X$  is independent of  $X$ 

## Problem 6

Since the  $X_i$ s are i.i.d.:

We can apply Theorem 4.6.7

$$M_H(t) = (M_X(t))^n$$

$$M_H(t) = Ee^{Ht} = EE(e^{Ht}|N) = EE(d^{(X_1+\dots+X_N)t}|N)$$

$$= E\{[E(e^{X_1 t}|N)]^N\}$$

$$Ee^{X_1 t} = \sum_{x_1=1}^{\infty} e^{x_1 t} \frac{-(1-p)^{x_1}}{x_1}$$

$$= \frac{-1}{\log(p)} \sum_{x_1=1}^{\infty} \frac{(e^t(1-p))^{x_1}}{x_1}$$

$$= \frac{-1}{\log(p)} (-\log(1 - e^{t(1-p)}))$$

$$= \frac{\log(1 - e^{t(1-p)})}{\log(p)}$$

$$N \sim \frac{e^{-\lambda} \lambda^n}{n!}$$

$$E\left(\frac{\log(1 - e^{t(1-p)})}{\log(p)}\right)^N = \sum_{n=0}^{\infty} \left(\frac{\log(1 - e^{t(1-p)})}{\log(p)}\right)^n \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= e^{-\lambda} ((p-1)e^t + 1)^{\frac{\lambda}{\log(p)}}$$

$$\begin{aligned}
&= e^{\log(p)^{-\lambda/\log(p)}} \left( \frac{1}{(p-1)e^t + 1} \right)^{\frac{-\lambda}{\log(p)}} \\
&= \left( \frac{p}{1 - e^t(1-p)} \right)^{\frac{-\lambda}{\log(p)}}
\end{aligned}$$

Which is the mgf of  $negbin(-\lambda/\log(p), p)$

## Problem 7

$$\begin{aligned}
Cov(X_1 + X_2, X_2 + X_3) &= E[(X_1 + X_2)(X_2 + X_3)] - E(X_1 + X_2)E(X_2 + X_3) \\
E(X_1 + X_2)E(X_2 + X_3) &= (\mu + \mu)(\mu + \mu) = 4\mu^2 \\
E[(X_1 + X_2)(X_2 + X_3)] &= E(X_1X_2 + X_1X_3 + X_2X_3 + X_2^2) \\
&= E(X_1X_2) + E(X_1X_3) + E(X_2X_3) + E(X_2^2) \\
&= E(X_1)E(X_2) + E(X_1)E(X_3) + E(X_2)E(X_3) + E(X_2^2) \\
&= E(X_1)E(X_2) + E(X_1)E(X_3) + E(X_2)E(X_3) + E(X_2^2) - E(X_2)^2 + E(X_2)^2 \\
&= 4\mu_2 + \sigma^2 \\
Cov(X_1 + X_2, X_2 + X_3) &= 4\mu_2 + \sigma^2 - 4\mu^2 = \sigma^2 \\
Cov(X_1 + X_2, X_1 - X_2) &= E[(X_1 + X_2)(X_1 - X_2)] - E(X_1 + X_2)E(X_1 - X_2) \\
E(X_1 + X_2)E(X_1 - X_2) &= 2\mu * (\mu - \mu) = 0 \\
E[(X_1 + X_2)(X_1 - X_2)] &= E(X_1^2 - X_2^2) \\
&= E(X_1^2) - E(X_2^2) \\
&= (E(X_1^2) - E(X_1)^2) + E(X_1)^2 - E(X_2^2) - E(X_2)^2 + E(X_2)^2 \\
&= \sigma^2 + E(X_1)^2 - (E(X_2^2) - E(X_2)^2 + E(X_2)^2) \\
&= \sigma^2 + \mu^2 - \sigma^2 - \mu^2 = 0 \\
Cov(X_1 + X_2, X_2 + X_3) &= 0 - 0 = 0
\end{aligned}$$

## Problem 8

$$f(x, y) = \frac{1}{2\pi(1-p^2)^{1/2}} \exp\left(\frac{-(x^2 - 2\rho xy + y^2)}{2(1-p^2)}\right)$$

Since this is the standard bivariate normal density:

$$\mu_x = 0 \quad \mu_y = 0 \quad \sigma_x^2 = 1 \quad \sigma_y^2 = 1$$

$$\text{WTS: } Corr(X, Y) = \rho$$

$$\begin{aligned}
Corr(X, Y) &= \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} \\
&= \frac{E(XY) - E(X)E(Y)}{\sqrt{1 * 1}} \\
&= E(XY) = E(E(XY|X)) \\
&= E(XE(Y|X))
\end{aligned}$$

Since  $Y|X \sim N(\rho X, 1 - p^2)$  we have:

$$= E(\rho X^2) = \rho E(X^2)$$

$$E(X^2) = 1 \text{ since:}$$



$$\begin{aligned} E(X^2) &= E(X^2) - E(X)^2 + E(X)^2 \\ &= \text{Var}(X) + E(X)^2 = 1 + 0 = 1 \end{aligned}$$

$$\rho E(X^2) = \rho$$

$$\text{Thus } \text{Corr}(X, Y) = \rho$$

$$\text{WTS: } \text{Corr}(X^2, Y^2) = \rho^2$$

$$\text{Corr}(X^2, Y^2) = \frac{\text{Cov}(X^2, Y^2)}{\sqrt{\text{Var}(X^2)\text{Var}(Y^2)}}$$

$$\text{Cov}(X^2, Y^2) = E(X^2 Y^2) - E(X^2)E(Y^2)$$

$$\text{Since } E(Y^2) = E(X^2) = 1 \text{ we have:}$$

$$= E(X^2 Y^2) - 1$$

$$= E(E(X^2 Y^2 | X)) - 1$$

$$= E(X^2 E(Y^2 | X)) - 1$$

$$E(Y^2 | X) = E(Y^2 | X) - E(Y | X)^2 + E(Y | X)^2$$

$$= \text{Var}(Y | X) + E(Y | X)^2 = 1 - \rho^2 + \rho^2 X^2$$

$$\text{Cov}(X^2, Y^2) = E(X^2(1 - \rho^2 + \rho^2 X^2)) - 1$$

$$= E(X^2 - \rho^2 X^2 + \rho^2 X^4) - 1$$

$$= 1 - \rho^2 + \rho^2 E(X^4) - 1$$

$$= -\rho^2 + \rho^2 E(X^4)$$

$$E(X^4) = E(X)^4 + 6E(X)^2 \text{Var}(X) + 3\text{Var}(X)^2 = 3$$

$$\text{Plugging this in we have:}$$

$$\text{Cov}(X^2, Y^2) = -\rho^2 + 3\rho^2 = 2\rho^2$$

$$\sqrt{\text{Var}(X^2)\text{Var}(Y^2)} = \sqrt{(E(X^4) - E(X^2)^2)(E(Y^4) - E(Y^2)^2)}$$

$$= \sqrt{(E(X^4) - 1)(E(Y^4) - 1)}$$

$$= \sqrt{(3 - 1)(3 - 1)} = \sqrt{4} = 2$$

$$\text{Corr}(X^2, Y^2) = 2\rho^2 / 2 = \rho^2$$

## Problem 9

(a)

$$\text{WTS: } \text{Cov}(X, Y) = \text{Cov}(X, E(Y | X))$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$\text{Cov}(X, E(Y | X)) = E(XE(Y | X)) - E(X)E(E(Y | X))$$

$$\text{Since } E(E(Y | X)) = E(Y) \text{ we have:}$$

$$= E(XE(Y | X)) - E(X)E(Y)$$

$$E(XE(Y|X)) = E(E(XY|X)) = E(XY)$$

Plugging this in we have:

$$\text{Cov}(X, E(Y|X)) = E(XY) - E(X)E(Y) = \text{Cov}(X, Y)$$

(b)

$$\text{WTS: } \text{Cov}(X, Y - E(Y|X)) = 0$$

$$\begin{aligned} \text{Cov}(X, Y - E(Y|X)) &= E(X(Y - E(Y|X))) - E(X)E(Y - E(Y|X)) \\ &= E(XY - XE(Y|X)) - E(X)(E(Y) - E(E(Y|X))) \\ &= E(XY) - E(XE(Y|X)) - E(X)(E(Y) - E(Y)) \\ &= E(XY) - E(XE(Y|X)) - 0 \end{aligned}$$

$$E(XE(Y|X)) = E(E(XY|X)) = E(XY)$$

Plugging this in we have:

$$\begin{aligned} &= E(XY) - E(XY) \\ &= 0 \end{aligned}$$

(c)

$$\text{WTS: } \text{Var}(Y - E(Y|X)) = E(\text{Var}(Y|X))$$

$$\text{Var}(Y - E(Y|X)) = \text{Var}(Y) + \text{Var}(E(Y|X)) - 2\text{Cov}(Y, E(Y|X))$$

$$\text{Cov}(Y, E(Y|X)) = E(YE(Y|X)) - E(Y)E(E(Y|X))$$

$$\begin{aligned} E(Y|E(Y|X)) &= E(E(YE(Y|X)|X)) = E(E(Y|X)E(Y|X)) \\ &= E((E(Y|X))^2) \end{aligned}$$

$$E(Y)E(E(Y|X)) = (E(E(Y|X)))^2$$

Plugging this in we get:

$$\begin{aligned} \text{Cov}(Y, E(Y|X)) &= E((E(Y|X))^2) - (E(E(Y|X)))^2 \\ &= \text{Var}(E(Y|X)) \end{aligned}$$

Putting this back we have:

$$\begin{aligned} \text{Var}(Y - E(Y|X)) &= \text{Var}(Y) + \text{Var}(E(Y|X)) - 2\text{Var}(E(Y|X)) \\ &= \text{Var}(Y) - \text{Var}(E(Y|X)) \\ &= E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)) - \text{Var}(E(Y|X)) \\ &= E(\text{Var}(Y|X)) \end{aligned}$$