Problem 1

From Example 4.3.1 and Theorem 4.3.2 we know:

$$
\begin{aligned}
X+Y &\sim \text{Poisson}(\theta+\lambda)\\
\text{Let } U &= X+Y,\ V=Y\\
\text{Then } X &= U-V,\ Y=V\\
f(u,v) &= \frac{\theta^{u-v}e^{-\theta}}{(u-v)!}\frac{\lambda^v e^{-\lambda}}{v!} \ \ v=0,1,\dots \ \ u=v,v+1,\dots\\
f(u) &= \frac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u \ \ u=0,1,\dots
\end{aligned}
$$

Finding $Y|X+Y$

Defining U and V the same way:

$$
\begin{aligned}
f(y|x+y) &= f(v|u)\\
f(v|u) &= \frac{f(u,v)}{f(u)}\\
&= \frac{\dfrac{\theta^{u-v}e^{-\theta}}{(u-v)!}\dfrac{\lambda^v e^{-\lambda}}{v!}}{\dfrac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u}\\
&= \frac{u!}{(u-v)!v!}\frac{e^{-(\theta+\lambda)}}{e^{-(\theta+\lambda)}}\frac{\theta^{u-v}\lambda^v}{(\theta+\lambda)^u}\\
&= \binom{u}{v}\frac{\theta^{u-v}\lambda^v}{(\theta+\lambda)^u}\\
&= \binom{u}{v}\left(\frac{\lambda}{\theta+\lambda}\right)^v\left(\frac{\theta}{\theta+\lambda}\right)^{u-v}
\end{aligned}
$$

Which is binomial $\left(u, \dfrac{\lambda}{\theta+\lambda}\right)$

Finding $X|X+Y$

Define $U=X+Y,\ V=X$

Then $X=U-V,\ X=V$

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(v,u-v) = \frac{\theta^v e^{-\theta}}{v!}\frac{\lambda^{u-v}e^{-\lambda}}{(u-v)!}\\
f(u) &= \sum_{v=0}^{u}\frac{\theta^v e^{-\theta}}{v!}\frac{\lambda^{u-v}e^{-\lambda}}{(u-v)!}\\
&= \frac{e^{-(\theta+\lambda)}}{u!}\sum_{v=0}^{u}\binom{u}{v}\theta^v\lambda^{u-v}
\end{aligned}
$$

Using the binomial theorem we have:

$$
\begin{aligned}
f(u) &= \frac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u \\
f(x|x+y) &= f(v|u) \\
&= \frac{f(u,v)}{f(u)} \\
&= \frac{\dfrac{\theta^v e^{-\theta}}{v!}\dfrac{\lambda^{u-v}e^{-\lambda}}{(u-v)!}}{\dfrac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u} \\
&= \binom{u}{v}\frac{\theta^v\lambda^{u-v}}{(\theta+\lambda)^u} \\
&= \binom{u}{v}\left(\frac{\theta}{\theta+\lambda}\right)^v\left(\frac{\lambda}{\theta+\lambda}\right)^{u-v} \\
\text{Which is binomial}&\left(u, \frac{\theta}{\theta+\lambda}\right)
\end{aligned}
$$

Problem 2

$f_X(x) = p(1-p)^{x-1} \quad f_Y(y) = p(1-p)^{y-1}$
Since X and Y are independent we have:
$f_{X,Y}(x,y) = p(1-p)^{x-1}p(1-p)^{y-1}$
$= p^2(1-p)^{x+y-2}$

(a) Solving $V = X - Y$ for X we get $X = V + Y$
If $V > 0$ then $X > Y$
Since $U = min(X,Y)$ this means that $U = Y$
Thus we have $Y = U$ and $X = U + V$

$$
\begin{aligned}
f_{U,V}(u,v) &= P(Y=u, X=u+v) \\
&= p^2(1-p)^{2u+v-2} \\
\text{Which factors to: } &(p^2(1-p)^{2u})((1-p)^{v-2}) \\
&\text{If } V<0, \text{ then } X<Y
\end{aligned}
$$

Thus $X = U, \quad Y = U - V$

$$
\begin{aligned}
f_{U,V}(u,v) &= P(X=u, Y=u-v) \\
&= p^2(1-p)^{2u-v-2} \\
\text{Which factors to: } &(p^2(1-p)^{2u})((1-p)^{-v-2}) \\
&\text{If } V=0 \text{ then } X=Y \\
f_{U,V}(u,0) &= P(X=Y=u) = p^2(1-p)^{2u-2} \\
\text{Which factors to: } &(p^2(1-p)^{2u})((1-p)^{-2})
\end{aligned}
$$

Since we can factor all of these cases in terms of u and v, U and V are independent

(b)

$$Z = \frac{X}{(X+Y)}$$
$$\text{Define } U = X$$
$$\text{Then } X = U \quad Y = U/Z - U$$

(c)

$$\text{Define } T = X + Y$$
$$f_{X,X+Y}(x, x+y) = P(X=x, X+Y=t) = P(X=x, Y=t-x) = P(X=x)P(Y=t-x)$$
$$= p^2(1-p)^{x-1+t-x-1} = p^2(1-p)^{t-2}$$

Problem 3

(a)

X_1,X_2 are independent and distributed as:

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x_i^2/2\sigma^2}$$
$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma^2} e^{-x_1^2/2\sigma^2} e^{-x_2^2/2\sigma^2}$$
$$= \frac{1}{2\pi\sigma^2} e^{-(x_1^2+x_2^2)/2\sigma^2}$$

Since the transformation is not one to one, we must partition the support of (X_1, X_2)

$$A_0 = \{-\infty < x_1 < \infty, x_2 = 0\}$$
$$A_1 = \{-\infty < x_1 < \infty, x_2 < 0\}$$
$$A_2 = \{-\infty < x_1 < \infty, x_2 > 0\}$$

Since $Y_2 = \dfrac{X_1}{\sqrt{X_1^2 + X_2^2}}$ it ranges from -1 to 1

The support of (Y_1, Y_2) is $\mathcal{B} = \{0 < y_1 < \infty, -1 < y_2 < 1\}$

$$Y_1 = X_1^2 + X_2^2 \quad Y_2 = \frac{X_1}{\sqrt{Y_1}}$$

For A_1 :

$$X_1 = Y_2\sqrt{Y_1}$$
$$X_2 = \sqrt{Y_1 - {Y_2}^2 Y_1}$$

$$
\begin{aligned}
J_1 &= \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \dfrac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ \dfrac{\sqrt{1-y_2^2}}{2\sqrt{y_1}} & \dfrac{\sqrt{y_1}y_2}{\sqrt{1-y_2^2}} \end{bmatrix} \\
&= \frac{1}{2\sqrt{1-y_2^2}} \\
&\text{For } A_2: \\
X_1 &= Y_2\sqrt{Y_1} \\
X_2 &= -\sqrt{Y_1 - {Y_2}^2 Y_1} \\
J_2 &= -J_1 \\
f_{Y_1,Y_2}(y_1,y_2) &= 2\left(\frac{1}{2\pi\sigma^2}e^{-y_1/2\sigma^2}\right)\frac{1}{2\sqrt{1-y_2^2}} \\
&= \frac{1}{2\pi\sigma^2}e^{-y_1/2\sigma^2}\frac{1}{\sqrt{1-y_2^2}} \quad 0 < y_1 < \infty, -1 < y_2 < 1
\end{aligned}
$$

(b)

$$
f_{Y_1,Y_2}(y_1,y_2) = \left(\frac{1}{2\pi\sigma^2}e^{-y_1/2\sigma^2}\right)\left(\frac{1}{\sqrt{1-y_2^2}}\right)
$$

Since the joint pdf factors into a function of y_1 and a function of y_2 y_1 and y_2 are independent.

Problem 4

(a)

$$f_{X,Y}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)}y^{\alpha+\beta-1}(1-y)^{\gamma-1}$$
$$0 < x < 1 \quad 0 < y < 1$$
$$U = XY \quad V = Y$$
$$X = U/V \quad Y = V$$
$$J = 1/v$$
$$f_{U,V}(u,v) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)}(u/v)^{\alpha-1}(1-u/v)^{\beta-1}v^{\alpha+\beta-1}(1-v)^{\gamma-1}(1/v)$$
$$0 < u < v < 1$$
$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}\int_u^1 v^{\beta-1}(1-v)^{\gamma-1}((v-u)/v)^{\beta-1}\, dv$$
$$\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1} \; 0 < u < 1$$
$$U \sim gamma(\alpha, \beta+\gamma)$$

Problem 5

(a)

$$Y|X \sim N(x, x^2)$$
$$E(Y|X) = X$$
$$Var(Y|X) = X^2$$
$$X \sim U(0,1)$$
$$EY = E(E(Y|X)) = EX$$
$$f_X(x) = 1 \quad \text{for } 0 \le x \le 1$$
$$EX = \frac{1}{2} = EY$$
$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)) = E(X^2) + Var(X)$$
$$Var(X) = \frac{1}{12}$$
$$E(X^2) = \int_0^1 X^2 \,\mathrm{d}x = 1/3$$
$$Var(Y) = 1/12 + 1/3 = \frac{5}{12}$$
$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= E(XY) - (1/2)^2 = E(XY) - 1/4$$

$$E(XY) = E[E(XY|X)] = E(XE(Y|X)) = E(X^2) = 1/3$$
$$Cov(X,Y) = 1/3 - 1/4 = 1/12$$

(b)

$$f(Y|X=x) = \frac{1}{\sqrt{2\pi}x} e^{-\dfrac{(y-x)^2}{2x^2}}$$

Since $X = 1$ we have:

$$f(Y|X=1) = \frac{1}{\sqrt{2\pi}1} e^{-\dfrac{(y-1)^2}{2*1^2}}$$
$$= \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Thus $f(Y|X=x) \sim N(1,1)$ and Y is independent of X

By theorem 4.3.5 Since X and Y are independent, $U = g(X)$ and $V = g(y)$ are independent

Thus the transformation Y/X is indepedent of X

Problem 6

Since the X_is are i.i.d.:

We can apply Theorem 4.6.7

$$M_H(t) = (M_{\mathbb{X}}(t))^n$$
$$M_H(t) = Ee^{Ht} = EE(e^{Ht}|N) = EE(d^{(X_1+\cdots+X_N)t}|N)$$
$$= E\{[E(e^{X_1 t}|N)]^N\}$$
$$Ee^{X_1 t} = \sum_{x_1=1}^{\infty} e^{x_1 t} \frac{-(1-p)^{x_1}}{x_1}$$
$$= \frac{-1}{\log(p)} \sum_{x_1=1}^{\infty} \frac{(e^t(1-p))^{x_1}}{x_1}$$
$$= \frac{-1}{\log(p)}(-\log(1-e^{t(1-p)}))$$
$$= \frac{\log(1-e^t(1-p))}{\log(p)}$$
$$N \sim \frac{e^{-\lambda}\lambda^n}{n!}$$
$$E(\frac{\log(1-e^t(1-p))}{\log(p)})^N = \sum_{n=0}^{\infty}(\frac{\log(1-e^t(1-p))}{\log(p)})^n \frac{e^{-\lambda}\lambda^n}{n!}$$
$$= e^{-\lambda}((p-1)e^t+1)^{\dfrac{\lambda}{\log(p)}}$$

$$= e^{\log(p)^{-\lambda/\log(p)}} \left(\frac{1}{(p-1)e^t + 1}\right)^{\frac{-\lambda}{\log(p)}}$$

$$= \left(\frac{p}{1 - e^t(1-p)}\right)^{\frac{-\lambda}{\log(p)}}$$

Which is the mgf of $negbin(-\lambda/\log(p), p)$

Problem 7

$$\begin{aligned}
Cov(X_1+X_2,X_2+X_3) &= E[(X_1+X_2)(X_2+X_3)] - E(X_1+X_2)E(X_2+X_3)\\
E(X_1+X_2)E(X_2+X_3) &= (\mu+\mu)(\mu+\mu) = 4\mu^2\\
E[(X_1+X_2)(X_2+X_3)] &= E(X_1X_2+X_1X_3+X_2X_3+X_2^2)\\
&= E(X_1X_2)+E(X_1X_3)E(X_2X_3)+E(X_2^2)\\
&= E(X_1)E(X_2)+E(X_1)E(X_3)+E(X_2)E(X_3)+E(X_2^2)\\
&= E(X_1)E(X_2)+E(X_1)E(X_3)+E(X_2)E(X_3)+E(X_2^2)-E(X_2)^2+E(X_2)^2\\
&= 4\mu_2+\sigma^2\\
Cov(X_1+X_2,X_2+X_3) &= 4\mu_2+\sigma^2-4\mu^2=\sigma^2\\
Cov(X_1+X_2,X_1-X_2) &= E[(X_1+X_2)(X_1-X_2)] - E(X_1+X_2)E(X_1-X_2)\\
E(X_1+X_2)E(X_1-X_2) &= 2\mu*(\mu-\mu)=0\\
E[(X_1+X_2)(X_1-X_2)] &= E(X_1^2-X_2^2)\\
&= E(X_1^2)-E(X_2^2)\\
&= (E(X_1^2)-E(X_1)^2)+E(X_1)^2-E(X_2^2)-E(X_2)^2+E(X_2)^2\\
&= \sigma^2+E(X_1)^2-(E(X_2^2)-E(X_2)^2+E(X_2)^2)\\
&= \sigma^2+\mu^2-\sigma^2-\mu^2=0\\
Cov(X_1+X_2,X_2+X_3) &= 0-0=0
\end{aligned}$$

Problem 8

$$f(x,y)=\frac{1}{2\pi(1-p^2)^{1/2}}\exp\left(\frac{-(x^2-2\rho xy+y^2)}{2(1-\rho^2)}\right)$$

Since this is the standard bivariate normal density:

$$\mu_x=0\ \mu_y=0\ \sigma_x^2=1\ \sigma_y^2=1$$

$$\begin{aligned}
\text{WTS: } Corr(X,Y) &= \rho\\
Corr(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}\\
&= \frac{E(XY)-E(X)E(Y)}{\sqrt{1*1}}\\
&= E(XY)=E(E(XY|X))\\
&= E(XE(Y|X))
\end{aligned}$$

Since $Y|X \sim N(\rho X, 1-p^2)$ we have:

$$= E(\rho X^2)=\rho E(X^2)$$

$E(X^2)=1$ since:

$$
\begin{aligned}
E(X^2) &= E(X^2) - E(X)^2 + E(X)^2 \\
&= Var(X) + E(X)^2 = 1 + 0 = 1 \\
\rho E(X^2) &= \rho \\
&\text{Thus } Corr(X,Y) = \rho
\end{aligned}
$$

$$
\begin{aligned}
\text{WTS: } & Corr(X^2,Y^2) = \rho^2 \\
Corr(X^2,Y^2) &= \frac{Cov(X^2,Y^2)}{\sqrt{Var(X^2)Var(Y^2)}} \\
Cov(X^2,Y^2) &= E(X^2Y^2) - E(X^2)E(Y^2) \\
&\text{Since } E(Y^2) = E(X^2) = 1 \text{ we have:} \\
&= E(X^2Y^2) - 1 \\
&= E(E(X^2Y^2|X)) - 1 \\
&= E(X^2E(Y^2|X)) - 1 \\
E(Y^2|X) &= E(Y^2|X) - E(Y|X)^2 + E(Y|X)^2 \\
&= Var(Y|X) + E(Y|X)^2 = 1 - \rho^2 + \rho^2X^2 \\
Cov(X^2,Y^2) &= E(X^2(1 - \rho^2 + \rho^2X^2)) - 1 \\
&= E(X^2 - \rho^2X^2 + \rho^2X^4) - 1 \\
&= 1 - \rho^2 + \rho^2E(X^4) - 1 \\
&= -\rho^2 + \rho^2E(X^4) \\
E(X^4) &= E(X)^4 + 6E(X)^2Var(X) + 3Var(X)^2 = 3 \\
&\text{Plugging this in we have:} \\
&Cov(X^2,Y^2) = -\rho^2 + 3\rho^2 = 2\rho^2 \\
\sqrt{Var(X^2)Var(Y^2)} &= \sqrt{(E(X^4) - E(X^2)^2)(E(Y^4) - E(Y^2)^2)} \\
&= \sqrt{(E(X^4) - 1)(E(Y^4) - 1)} \\
&= \sqrt{(3-1)(3-1)} = \sqrt{4} = 2 \\
Corr(X^2,Y^2) &= 2\rho^2/2 = \rho^2
\end{aligned}
$$

Problem 9

(a)

$$
\begin{aligned}
\text{WTS: } & Cov(X,Y) = Cov(X, E(Y|X)) \\
Cov(X,Y) &= E(XY) - E(X)E(Y) \\
Cov(X,E(Y|X)) &= E(XE(Y|X)) - E(X)E(E(Y|X)) \\
&\text{Since } E(E(Y|X)) = E(Y) \text{ we have:} \\
&= E(XE(Y|X)) - E(X)E(Y)
\end{aligned}
$$

$$E(XE(Y|X)) = E(E(XY|X)) = E(XY)$$
Plugging this in we have:
$$Cov(X, E(Y|X)) = E(XY) - E(X)E(Y) = Cov(X, Y)$$

(b)

$$\begin{aligned}
&\text{WTS: } Cov(X, Y - E(Y|X)) = 0 \\
Cov(X, Y - E(Y|X)) &= E(X(Y - E(Y|X))) - E(X)E(Y - E(Y|X)) \\
&= E(XY - XE(Y|X)) - E(X)(E(Y) - E(E(Y|X)) \\
&= E(XY) - E(XE(Y|X)) - E(X)(E(Y) - E(Y)) \\
&= E(XY) - E(XE(Y|X)) - 0 \\
E(XE(Y|X)) &= E(E(XY|X)) = E(XY) \\
&\text{Plugging this in we have:} \\
&= E(XY) - E(XY) \\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
&\text{WTS: } Var(Y - E(Y|X)) = E(Var(Y|X)) \\
Var(Y - E(Y|X)) &= Var(Y) + Var(E(Y|X)) - 2Cov(Y, E(Y|X)) \\
Cov(Y, E(Y|X)) &= E(YE(Y|X)) - E(Y)E(E(Y|X)) \\
E(Y|E(Y|X)) &= E(E(YE(Y|X)|X)) = E(E(Y|X)E(Y|X) \\
&= E((E(Y|X))^2) \\
E(Y)E(E(Y|X)) &= (E(E(Y|X)))^2 \\
&\text{Plugging this in we get:} \\
Cov(Y, E(Y|X)) &= E((E(Y|X))^2) - (E(E(Y|X)))^2 \\
&= Var(E(Y|X)) \\
&\text{Putting this back we have:} \\
Var(Y - E(Y|X)) &= Var(Y) + Var(E(Y|X)) - 2Var(E(Y|X)) \\
&= Var(Y) - Var(E(Y|X)) \\
&= E(Var(Y|X)) + Var(E(X|Y)) - Var(E(Y|X)) \\
&= E(Var(Y|X))
\end{aligned}$$