Problem 1

$$p_K(k) = \begin{cases} \dfrac{1}{2n+1} & \text{for k } \in \{-n, -n+1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

$$p_{|K|}(|k|) = \begin{cases} \dfrac{2}{2n+1} & \text{for } |k| \in \{1, \ldots, n\} \\ \dfrac{1}{2n+1} & \text{for } |k| = 0 \\ 0 & \text{otherwise} \end{cases}$$

$X = a^{|K|} \quad a > 0$
$Y = \log(X) = |K| \log(a)$
PMF of Y:

$$p_Y(y) = \begin{cases} \dfrac{2}{2n+1} & \text{for y } \in \{\log(a), 2\log(a), \ldots, n\log(a)\} \\ \dfrac{1}{2n+1} & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2

(a) $\sum_{x=-3}^{3} = x^2/a = 1$
$\sum_{x=-3}^{3} = x^2 = a$
$a = 2(1^2 + 2^2 + 3^2) = 28$
$a = 28$

$E[X] = \sum_{x=-3}^{3} x^3/28$
$= (1/28)[(-1)^3 + (-2)^3 + (-3)^3 + 0 + 1^3 + 2^3 + 3^3] = 0$
$E[X] = 0$

(b)

$$\begin{aligned} Z &= (X - E[X])^2 \\ &= X^2 - (2X)E[X] + (E[X])^2 \\ &= X^2 - 2X(0) + 0^2 \\ Z &= X^2 \end{aligned}$$

$$p_Z(z) = \begin{cases} (2z)/28 & \text{if } z = 1, 4, 9 \\ z/28 & \text{if } z = 0 \end{cases}$$

(c)

$$\begin{aligned}
E[X^2] &= \sum_{z:P(z)>0} p_Z(z) * z \\
&= (1/14)1 + (4/14)4 + (9/14)9 + (0/28)0 \\
&= 1/14 + 16/14 + 81/14 \\
E[X^2] &= 7 \\
Var(X) &= E[X^2] - (E[X])^2 \\
&= 7 - 0^2 \\
Var(X) &= 7
\end{aligned}$$

(d)

$$\begin{aligned}
Var(x) &= \sum_{x=-3}^{3} (x - E[X])^2 p_X(x) \\
&= \sum_{x=-3}^{3} (x - 0)^2 p_X(x) \\
&= \sum_{x=-3}^{3} x^2 p_X(x) \\
&= [9(9/28) + 4(4/28) + 1(1/28)] * 2 + 0 \\
Var(X) &= 7
\end{aligned}$$

Problem 3

There are $a + b - 1$ values:$\{2^a, a^{a+1}, \ldots, 2^b\}$

$E[X] = \dfrac{\sum_{k=a}^{b} 2^k}{b - a + 1}$

We have a geometric series where r=2, n=a-b+1, $a_1 = 2^a$

$\sum_{k=a}^{b} 2^k = 2^a \dfrac{1 - 2^{b-a+1}}{1 - 2} = 2^{b-1} - 2^a$

$E[X] = \dfrac{2^{b+1} - 2^a}{b - a + 1}$

$Var(X) = E[X^2] - (E[X])^2$

$E[X^2] = \dfrac{\sum_{k=a}^{b} (2^2)^k}{b - a + 1}$

$E[X^2] = \dfrac{4^{b+1} - 4^a}{3(b - a + 1)}$

$Var(X) = \dfrac{4^{b+1} - 4^a}{3(b - a + 1)} - \left[\dfrac{2^{b-1} - 2^a}{b - a + 1}\right]^2$

Problem 4

Let X be a random variable where $X \sim Geom(p)$
This is the distribution of candy bars eaten till the first success (getting the golden ticket)

$$
\begin{aligned}
f_X &= pq^{x-1} \\
E[X] &= \sum_{x=1}^{\infty} x * pq^{x-1} \\
&= \sum_{x=1}^{\infty} p \frac{d}{dq}(q^x) \\
&= p \frac{d}{dq} \sum_{x=1}^{\infty} q^x \quad \text{since the series converges uniformly} \\
&= p \frac{d}{dq}(\frac{1}{1-q} - 1) \\
&= p \frac{1}{(1-q)^2} = \frac{1}{p} \\
E[X] &= \frac{1}{p}
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \sum_{x=1}^{\infty} x^2 q^{x-1} p \\
&= \sum_{x=1}^{\infty} ((x-1)+1)^2 q^{x-1} p \\
\text{since } ((x-1)+1)^2 &= (x-1)^2 + 2(x-1) + 1 \text{ we have:} \\
&= \sum_{x=1}^{\infty} (x-1)^2 q^{x-1} p + \sum_{x=1}^{\infty} 2(x-1) q^{x-1} p + \sum_{x=1}^{\infty} q^{x-1} p \\
&= \sum_{x=1}^{\infty} (x-1)^2 q^{x-1} p + [2 \sum_{x=1}^{\infty} (x-1) q^{x-1} p] + 1 \\
\text{let } i &= x-1 \\
&= \sum_{i=0}^{\infty} i^2 q^i p + [2 \sum_{i=0}^{\infty} i q^i p] + 1 \\
\text{since i is just a letter we can replace it with x} \\
&= [\sum_{x=1}^{\infty} x^2 q^x p + 0] + [2 \sum_{x=1}^{\infty} x q^x p + 0] + 1 \\
&= q \sum_{x=1}^{\infty} x^2 q^{x-1} p + [2q \sum_{x=1}^{\infty} x q^{x-1} p] + 1 \\
&= qE[X^2] + 2qE[X] + 1 \\
\text{Since } E[X] &= \frac{1}{p} \\
E[X^2] &= qE[X^2] + (2q)\frac{1}{p} + 1 \\
(1-q)E[X^2] &= \frac{2q}{p} + 1 \\
E[X^2] &= \frac{2q+p}{p^2} = \frac{q+1}{p^2} \\
Var(X) &= \frac{q+1}{p^2} - \frac{1}{p^2} \\
Var(X) &= \frac{q}{p^2} = \frac{1-p}{p^2}
\end{aligned}
$$

Problem 5

Let X be a random variable where $X \sim Geom(1/2)$
This is the distribution of coin tosses until the first success (getting a tails)
$P\{X = n\} = p_X(n) = (1 - p)^{n-1}p = p^n = (1/2)^n$
Payout=$Y = 2^n$
$P\{Y = n\} = P\{X = n\}$
Thus Y has the same PMF as X
$E[Y]$ is the expected payout
$E[Y] = \sum_{n=1}^{\infty} y \; p_Y(n)$
$E[Y] = \sum_{n=1}^{\infty} 2^n(1/2)^n = \sum_{n=1}^{\infty} 1 = \infty$
$E[Y] = \infty$
Even though the expected payout is infinite there is a very good chance of losing all of your money. Because of this I would not be willing to risk very much. I would be willing to pay \$4 to play since the expected number of tosses is 2.

Problem 6

(a)

$$
\begin{aligned}
Y &= X^2 \quad X = \sqrt{Y} \\
f_X(x) &= 1 \quad \text{for } 0 < x < 1 \\
f_Y(y) &= f_x(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}(g^{-1}(y)) \right| \quad y \in \mathcal{Y} \\
f_Y(y) &= (1)|(1/2)y^{-1/2}| \\
f_Y(y) &= \frac{1}{2\sqrt{y}} \quad 0 < y < 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_X(x) &= \frac{(n+m+1)!}{n!m!} x^n (1-x)^m \quad 0 < x < 1 \\
Y &= g(X) = -\log(X) \\
x &= e^{-y} \quad g^{-1}(y) = e^{-y} \\
-\log(0) &= \infty \quad -log(1) = 0 \\
0 <& y < \infty \\
f_Y(y) &= \begin{cases} f_X(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}(g^{-1}(y)) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases} \\
f_Y(y) &= \begin{cases} f_X(e^{-y}) \left| \frac{\mathrm{d}}{\mathrm{d}y}(e^{-y}) \right| & 0 < y < \infty \\ 0 & \text{otherwise} \end{cases} \\
f_Y(y) &= \frac{(n+m+1)!}{n!m!} e^{-ny}(1-e^{-y})^m (e^{-y}) \\
f_Y(y) &= \frac{(n+m+1)!}{n!m!} e^{-y(n+1)}(1-e^{-y})^m \quad 0 < y < \infty
\end{aligned}
$$

(c)

$$\begin{aligned}
Y &= g(X) = e^x \\
x &= log(y) \quad g^{-1}(y) = \log(y) \\
f_X(x) &= \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2} \quad 0 < x < \infty \\
e^0 &= 1 \quad e^{\infty} = \infty \\
1 &< y < \infty \\
f_Y(y) &= \begin{cases} f_X(\log(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}(\log(y)) \right| & 1 < y < \infty \\ 0 & \text{otherwise} \end{cases} \\
f_Y(y) &= \frac{1}{\sigma^2} \log(y) e^{-(\log(y)/\sigma)^2/2} (1/y) \quad 1 < y < \infty
\end{aligned}$$

Problem 7

(a) Using theorem 2.1.8 we have:

$$\begin{aligned}
&A_0, A_1, A_2 \text{ of } \chi \\
A_0 &= \{0\} \quad A_1 = (-\infty, 0) \quad A_2 = (0, \infty) \\
P(X \in A_0) &= 0 \\
g_1(x) &= |x|^3 = -x^3 \text{ on } A_1 \quad g_2(x) = |x|^3 = x^3 \text{ on } A_2 \\
f_X(x) &= (1/2) e^{-|x|} \quad -\infty < x < \infty \\
Y &= g_i(X) \\
g_1^{-1}(X) &= -Y^{1/3} \\
g_2^{-1}(X) &= Y^{1/3} \\
f_Y(y) &= \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}(g_i^{-1}(y)) \right| \quad y \in \mathcal{Y} \\
f_Y(y) &= 2 * f_X(y^{1/3}) \left| \frac{\mathrm{d}}{\mathrm{d}y}(y^{1/3}) \right| \quad 0 < y < \infty \\
f_Y(y) &= 2 * (1/2) e^{-y^{1/3}} ((1/3) y^{-2/3}) \quad 0 < y < \infty \\
f_Y(y) &= (1/3) e^{-y^{1/3}} (y^{-2/3}) \quad 0 < y < \infty \\
\int_0^{\infty} (1/3) e^{-y^{1/3}} (y^{-2/3}) dy &= \Big|_0^{\infty} e^{-y^{1/3}} = 0 + 1 = 1
\end{aligned}$$

(b)

$$\begin{aligned}
A_0, A_1, A_2 &\text{ of } \chi \\
A_0 &= \{0\} \quad A_1 = (-1,0) \quad A_2 = (0,1) \\
P(X \in A_0) &= 0 \\
Y &= g_i(X) \\
g_1(x) &= 1 - X^2 \text{ on } A_1 \\
g_1^{-1}(x) &= -\sqrt{1-Y} \\
g_2(x) &= 1 - X^2 \text{ on } A_2 \\
g_2^{-1}(x) &= \sqrt{1-Y} \\
f_X(x) &= (3/8)(x+1)^2 \quad -1 < x < 1 \\
f_Y(y) &= \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}(g_i^{-1}(y)) \right| \quad y \in \mathcal{Y} \\
f_Y(y) &= (3/8)(\sqrt{1-y}+1)^2(1/2)(1-y)^{-1/2} \\
&\quad + (3/8)(-\sqrt{1-y}+1)^2(1/2)(1-y)^{-1/2} \\
f_Y(y) &= (3/8)[(1-y)^{1/2} + (1-y)^{-1/2}] \quad 0 < y < 1 \\
\int_0^1 (3/8)[(1-y)^{1/2} + (1-y)^{-1/2}]dy &= \Big|_0^1 (1/4)(1-y)^{1/2}(y-4) = 0 - (1/4 * -4) = 1
\end{aligned}$$

(c)

A_0, A_1, A_2 of χ

$$\begin{aligned}
A_0 &= \{0\} \quad A_1 = (-1, 0) \quad A_2 = (0, 1) \\
f_X(x) &= (3/8)(x+1)^2 \quad -1 < x < 1 \\
Y &= 1 - X^2 \text{ if } X \leq 0 \quad Y = 1 - X \text{ if } X > 0 \\
Y &= g_i(X) \\
g_1(x) &= 1 - X^2 \text{ on } A_1 \\
g_1^{-1}(x) &= -\sqrt{1-Y} \\
g_2(x) &= 1 - X \text{ on } A_2 \\
g_2^{-1}(x) &= 1 - Y \\
f_Y(y) &= \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}(g_i^{-1}(y)) \right| \quad y \in \mathcal{Y} \\
f_Y(y) &= (3/8)(1 - y + 1)^2(1) \\
&\quad + (3/8)(-\sqrt{1-y} + 1)^2(1/2)(1-y)^{-1/2} \\
f_Y(y) &= (3/8)(2-y)^2 + (3/16)(1 - (1-y)^{1/2})^2(1-y)^{-1/2} \quad 0 < y < 1 \\
&\int_0^1 (3/8)(2-y)^2 + (3/16)(1-(1-y)^{1/2})^2(1-y)^{-1/2} dy \\
&= (3/16)\int_0^1 6 - 8y + 2y^2 + (1-y)^{1/2} + (1-y)^{-1/2} dy \\
&= (3/16)\Big|_0^1 6y - 4y^2 + (2/3)y^3 - (2/3)(1-y)^{3/2} - (1-y)^{1/2} = 1
\end{aligned}$$

Problem 8

Define $Y = F_X(X)$, which means Y is uniformly distributed on $(0, 1)$

$$f(x) = \begin{cases} \dfrac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_1^x \frac{x-1}{2} = (1/4)(x-1)^2$$

$$u(x) = F_X(x) = \begin{cases} 0 & x \leq 1 \\ (1/4)(x-1)^2 & 1 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

Problem 9

(a)

Problem 10

(a)

$$\begin{aligned}
\int_0^\infty (1 - F_X(x))\mathrm{d}x &= \int_0^\infty P(X > x)\mathrm{d}x \\
\text{Since } P(X > x) &= \int_x^\infty f_X(y)\ dy \\
\int_0^\infty P(X > x)\mathrm{d}x &= \int_0^\infty \int_x^\infty f_X(y)\ dy\mathrm{d}x \\
&= \int_0^\infty \int_0^y \mathrm{d}x f_X(y) dy \\
&= \int_0^\infty (y - 0) f_X(y) dy \\
&= \int_0^\infty y f_X(y) dy \\
&= E[X]
\end{aligned}$$

(b)

$$\begin{aligned}
\text{Since } 1 - F_X(k) &= \sum_{k+1}^\infty f_X(k) \\
\text{Then } \sum_{k=0}^\infty (1 - F_X(k)) &= \sum_{k=0}^\infty \left[\sum_{k+1}^\infty f_X(k)\right] \\
&= \left(f_x(1) + f_x(2) + \dots\right) + \left(f_x(2) + f_x(3) + \dots\right) + \left(\dots\right) \\
&= 0 + 1f_x(1) + 2f_x(2) + 3f_x(3) + \dots \\
&= \sum_{k=0}^\infty k f_x(k) = E[X]
\end{aligned}$$

1 Problem 11

(a)

$$\begin{aligned}\int_0^m 3x^2 &= \int_m^1 3x^2 = 1/2 \\ \Big|_0^m x^3 &= \Big|_m^1 x^3 = 1/2 \\ m^3 &= 1 - m^3 = 1/2 \\ m &= 2^{-1/3} = 0.7937005\end{aligned}$$

(b)

$$\begin{aligned}\int_{-\infty}^m \frac{1}{\pi(1+x^2)} &= \int_m^\infty \frac{1}{\pi(1+x^2)} = 1/2 \\ (1/\pi)\Big|_{-\infty}^m \arctan(x) &= (1/\pi)\Big|_m^\infty \arctan(x) = 1/2 \\ (1/\pi)(\arctan(m) + \pi/2) &= (1/\pi)(\pi/2 - \arctan(m)) = 1/2 \\ \arctan(m) &= -\arctan(m) = 0 \\ m &= \tan(0) = -\tan(0) = 0 \\ m &= 0\end{aligned}$$

Problem 12

WTS: If X is a continuous random variable: $\min_a E|X-a| = E|X-m|$

$$\text{We know } E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$$

$$E|X-a| = \int_{-\infty}^{\infty} |x-a| f(x)dx$$

$$= \int_{-\infty}^{a} -(x-a)f(x)\mathrm{d}x + \int_{a}^{\infty} (x-a)f(x)\mathrm{d}x$$

Setting the first derivative equal to 0 we have:

$$\frac{d}{da}E|X-a| = \int_{-\infty}^{a} f(x)\mathrm{d}x - \int_{a}^{\infty} f(x)\mathrm{d}x = 0$$

$$\text{The solution is: } \int_{-\infty}^{a} f(x)\mathrm{d}x = \int_{a}^{\infty} f(x)\mathrm{d}x = 1/2$$

Which means a=m is the median by definition

taking the second derivative we have:

$$\frac{d}{da}\left[\int_{-\infty}^{a} f(x)\mathrm{d}x - \int_{a}^{\infty} f(x)\mathrm{d}x\right]$$

$$= \frac{d}{da}\left[\Big|_{-\infty}^{a} F(X) - \Big|_{a}^{\infty} F(X)\right]$$

$$= \frac{d}{da}\left[F(a) - F(-\infty) - F(\infty) + F(a)\right]$$

$$= 2f(a) > 0$$

Thus $E|X-m|$ is a minimum by the second derivative test

Therefore $\min_a E|X-a| = E|X-m|$

Problem 13

(a) The uniform distribution pdf $U(0,1)$ is symmetric about $a = 1/2$
Standard Normal pdf is symmetric about $a = 0$
The Cauchy distribution pdf is symmetric about $a = 0$

(b)

Let $X \sim f(x)$ be symmetric at point a

That is $\forall\ \epsilon > 0, f(a+\epsilon) = f(a-\epsilon)$

WTS a is the median:

$$\int_{-\infty}^{a} f(x)\mathrm{d}x = \int_{a}^{\infty} f(x)\mathrm{d}x = 1/2$$

Proof: Let $\epsilon = x - a$ Then:

$$\int_{a}^{\infty} f(x)\mathrm{d}x = \int_{0}^{\infty} f(a+\epsilon)d\epsilon$$

$$= \int_{0}^{\infty} f(a-\epsilon)d\epsilon = \int_{-\infty}^{a} f(x)\mathrm{d}x \text{ (letting } x = a-\epsilon)$$

$$\text{Thus } \int_{a}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{a} f(x)\mathrm{d}x$$

$$\text{We know } \int_{-\infty}^{a} f(x)\mathrm{d}x + \int_{a}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$$

$$\text{Therefore } \int_{-\infty}^{a} f(x)\mathrm{d}x = \int_{a}^{\infty} f(x)\mathrm{d}x = 1/2$$

Thus $a = median$

(c)

Given $X \sim f(x)$ and symmetric and $E[X]$ exists

Show $E[X] = a$

$$E[X] - a = E[X-a]$$

$$= \int_{-\infty}^{\infty} (x-a)f(x)\mathrm{d}x$$

From part b we know a is the median therefore:

$$= \int_{-\infty}^{a} (x-a)f(x)\mathrm{d}x + \int_{a}^{\infty} (x-a)f(x)\mathrm{d}x$$

making the change of variables $\epsilon = a - x$ for the first integral and $\epsilon = x - a$ for the second.

$$= \int_{0}^{\infty} -\epsilon f(a-\epsilon)d\epsilon + \int_{0}^{\infty} \epsilon f(a+\epsilon)d\epsilon$$

Since $f(x)$ is symmetric $f(a+e) = f(a-e)\forall\ \epsilon > 0$

$$\text{Thus we have: } -\int_{0}^{\infty} \epsilon f(a-\epsilon)d\epsilon + \int_{0}^{\infty} \epsilon f(a-\epsilon)d\epsilon = 0$$

Therefore $E[X] - a = E[X-a]$

Which means $E[X] = a$

(d)

$$\text{Let } f(x) = e^{-x},\ x \geq 0$$
$$\text{WTS: For } a > \epsilon > 0\ f(a+\epsilon) \neq f(a-\epsilon)$$
$$\text{Since } x \geq 0 \text{ we only need to consider positive a}$$
$$f(a-\epsilon) = e^{-(a-\epsilon)} = e^{-a+\epsilon}$$
$$f(a+\epsilon) = e^{-(a+\epsilon)} = e^{-a-\epsilon}$$
$$f(a-\epsilon) > f(a+\epsilon)$$

Thus $f(x)$ is not symmetric about $a > 0$

(e)

$$f(x) = e^{-x},\ x \geq 0$$
$$\text{WTS: } median < E[X]$$
$$\int_0^m e^{-x} = \int_m^\infty = 1/2$$
$$\Big|_0^m - e^{-x} = \Big|_m^\infty - e^{-x} = 1/2$$
$$-e^{-m} + e^0 = -e^{-\infty} + e^{-m} = 1/2$$
$$-e^{-m} + 1 = e^{-m} = 1/2 log(e^{-m}) \qquad = log(1/2)$$
$$m = -log(1/2) = log(2)$$
$$f(x) = (1/\lambda)e^{-x/\lambda},\ \lambda = 1$$
$$E[X] = \int_0^\infty e{-}x/\lambda \mathrm{d}x = \lambda = 1$$
$$log(2) < 1$$
$$\text{Therefore } median < E[X]$$

Problem 14

(a) The standard normal distribution is unimodal and has a unique mode, 0.

(b) The standard uniform distribution $U(0,1)$ is unimodal and does not have a unique mode since every value in the interval is part of the mode.

(c) Given $f(x)$ is symmetric at point a and unimodal
WTS: a is the mode
Two Cases:
Case 1: Mode=b is unique
Let a be the point of symmetry, $b = mode$
Assume $a \neq b,\ a = b + \epsilon, \epsilon > 0$ that is a is not the mode

By definition of a mode, $f(b) > f(b+\epsilon) \geq f(b+2\epsilon)$
Which means $f(a-\epsilon) > f(a) \geq f(a+\epsilon)$
This contradicts $f(x)$ being symmetric
Therefore a is the mode.
Case 2: Mode is not unique, b is a mode
That is, there exists an interval (x_1, x_2) where $f(x)$ has the same value in the whole interval.
Assume $a \notin (x_1, x_2),\ a = (b+\epsilon), \epsilon > 0$, that is a is not a mode
Since b is a mode, $f(b) > f(b+\epsilon) \geq f(b+2\epsilon)$
Which means $f(a) - \epsilon) > f(a) \geq f(a+\epsilon)$ Which contradicts $f(x)$ being symmetric.
Thus a is a mode.

(d) Given $f(x) = e^{-x},\ x \geq 0$
WTS: $f(x)$ is unimodal
$f(x)$ is decreasing for $x \geq 0$
Thus for $0 \leq x_1 \leq x_2,\ f(0) \geq f(x_1) \geq f(x_2)$
Therefore $f(x)$ is unimodal with mode $a = 0$