Problem 1

a

Proof by contradiction

Let A be a countably infinite sample space

$$A = \{A_1, A_2, \dots\} = \bigcup_{i=1}^{\infty} P(A_i)$$

Suppose all outcomes are equally likely $P(A_i) = c > 0$

$$(\text{If } c = 0 \text{ then } P(A) = \sum_{i=1}^{\infty} 0 = 0 \neq 1)$$

$$\text{Then } P(A) = \bigcup_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{\infty} c = \infty$$

This contradicts $P(A) = 1$

Therefore all outcomes cannot be equally likely

b

Let A be a countably infinite sample space

Where the probability of each point is given by p_n

Where p_n is an infinite geomtric series with common ratio $r = 1/2$ and first term $p_1 = 1/2$

$$P(A) = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \frac{1}{2}^n$$

We can see as $n \to \infty$ the terms in p_n approach 0 but never reach it

Since $|r| < 1$ the series converges to:

$$\frac{p_1}{(1-r)} = \frac{1/2}{1-1/2} = 1$$

Thus $p_n > 0 \quad \forall\ n$ and $P(A) = 1$

Therefore we have proven it is possible to have a countably infinite sample space where all points have positive probability

Problem 2

(a) $26^3 = 17576$

(b) $26 + 26^2 + 26^3 = 18278$

(c) $26 + 26^2 + 26^3 + 26^4 = 475254$

Problem 3

(a) $\dfrac{2!(n-1)(n-2)!}{n!} = \dfrac{2(n-1)!}{n!} = \dfrac{2}{n}$

(b) $\dfrac{3!(n-2)(n-3)!}{n!} = \dfrac{6(n-2)!}{n!} = \dfrac{6}{(n-1)(n)} = \dfrac{6}{n^2-n}$

Problem 4

$\sum_{i=1}^{20}(i-1) = \frac{19(20)}{2} = 190$

Problem 5

$$\binom{10}{3}7! = 604800$$

Problem 6

(a) $x_1 + x_2 + x_3 + x_4 = 8$
where x_i is the number of blackboards at the ith school
The number of nonnegative integer solutions is:

$$\binom{8+4-1}{4-1} = \binom{11}{3} = 165$$

(b) The number of positive integer solutions is:

$$\binom{8+-1}{4-1} = \binom{7}{3} = 35$$

Problem 7

(a) considering the minimal investments we have \$9 thousand left to invest
$(r_1+2)+(r_2+2)+(r_3+3)+(r_4+4)=20$
$r_1+r_2+r_3+r_4=9$
The number of nonnegative integer solutions is:

$$\binom{9+4-1}{4-1}=\binom{12}{3}=220$$

(b) If we do not invest in the first investment:
$(r_2+2)+(r_3+3)+(r_4+4)=20$
$r_2+r_3+r_4=11$
The number of nonnegative solutions is:

$$\binom{11+3-1}{3-1}=\binom{13}{2}=78$$

Which is the same as not investing in the second investment
If we do not invest in the third investment:
$(r_1+2)+(r_2+2)+(r_4+4)=20$
$r_1+r_2+r_4=12$
The number of nonnegative solutions is:

$$\binom{12+3-1}{3-1}=\binom{14}{2}=91$$

If we do not invest in the fourth investment:
$(r_1+2)+(r_2+2)+(r_3+3)=20$
$r_1+r_2+r_3=13$
The number of nonnegative solutions is:

$$\binom{13+3-1}{3-1}=\binom{15}{2}=105$$

We know from part a that there are 220 ways to invest in all investments.
2 Adding them all up we get $78+78+91+105+220=572$ total investment strategies

Problem 8

We want to prove $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$ for $n \in \mathbb{N}$

Proof by induction:

$\forall\ n \in \mathbb{N}$ let $P(n)$ be:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

Basis Step: $P(1) = (a+b)^1 = = \sum_{r=0}^{1} \binom{1}{r} a^r b^{1-r}$

$$a+b = \binom{1}{0}(a^0 b^{1-0}) + \binom{1}{1}(a^1 b^{1-1})$$

$$a+b = a+b$$

Thus $P(1)$ is true

Inductive Step: Let $k \in \mathbb{N}$ and assume $P(k)$ is true:

$$(a+b)^k = \sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r} \tag{1}$$

We will prove $P(k+1)$ is true:

$$(a+b)^{k+1} = \sum_{r=0}^{k+1} \binom{k+1}{r} a^r b^{k+1-r} \tag{2}$$

$$= \sum_{r=1}^{k} \binom{k+1}{r} a^r b^{k+1-r} + \binom{k+1}{k+1} a^{k+1} b^{k+1-(k+1)} + \binom{k+1}{0} a^0 b^{k+1-0}$$

$$= \left[\sum_{r=1}^{k} \binom{k+1}{r} a^r b^{k+1-r} \right] + a^{k+1} + b^{k+1} \tag{3}$$

Multiply both sides of (1) by $(a+b)$

$$(a+b)^k (a+b) = \left(\sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r} \right) (a+b)$$

$$(a+b)^{k+1} = \sum_{r=0}^{k} \binom{k}{r} a^{r+1} b^{k-r} + \sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r+1} \tag{4}$$

We will need to perform an index shift

Define $i = r+1$ then $r = i-1$

We can rewrite the first term of the right side of (4) as:

$$\sum_{r=0}^{k}\binom{k}{r}a^{r+1}b^{k-r}=\sum_{i=1}^{k+1}\binom{k}{i-1}a^{i}b^{k-i+1}$$

Since i is just a name we can replace i by r to get:

$$\sum_{r=0}^{k}\binom{k}{r}a^{r+1}b^{k-r}=\sum_{r=1}^{k+1}\binom{k}{r-1}a^{r}b^{k-r+1}$$

Plugging this result into (2) we have :

$$(a+b)^{k+1}=\sum_{r=1}^{k+1}\binom{k}{r-1}a^{r}b^{k-r+1}+\sum_{r=0}^{k}\binom{k}{r}a^{r}b^{k-r+1}$$

$$=\binom{k}{(k+1)-1}a^{k+1}b^{k+1-(k+1)}+\left[\sum_{r=1}^{k}\binom{k}{r-1}a^{r}b^{k-r+1}\right]+\left[\sum_{r=1}^{k}\binom{k}{r}a^{r}b^{k-r+1}\right]+\binom{k}{0}a^{0}b^{k-0+1}$$

$$=a^{k+1}+\sum_{r=1}^{k}\left[\binom{k}{r-1}+\binom{k}{r}\right]a^{r}b^{k-r+1}+b^{k+1}$$

We proved $\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$ in Problem 9

Substituting this in we have:

$$=\left[\sum_{r=1}^{k}\binom{k+1}{r}a^{r}b^{k-r+1}\right]+a^{k+1}+b^{k+1}$$

Which is the same as (3)

Therefore, since we know (3) = (2), the inductive step has been established and by PMI we have proven that:

$\forall\ n\in\mathbb{N}$

$$(a+b)^{n}=\sum_{r=0}^{n}\binom{n}{r}a^{r}b^{n-r}$$

Problem 9

We want to prove $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

$$\begin{aligned}
\binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(n-r+1!)(r-1)!} + \frac{n!}{(n-r)!(r)!} \\
&= n!\left(\frac{r}{(n+1-r)!(r)!} + \frac{n+1-r}{(n+1-r)!r!}\right) \\
&= n!\left(\frac{r+(n+1-r)}{(n+1-r)!(r)!}\right) \\
&= n!\left(\frac{(n+1)}{(n+1-r)!(r)!}\right) \\
&= \frac{(n+1)!}{(n+1-r)!(r)!} \\
&= \binom{n+1}{r}
\end{aligned}$$

Therefore $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$