Problem 1

Let $X \sim N(\mu, \sigma^2)$
Then $X = \sigma(z) + \mu$
From theorem 4.8 we know:
$\phi_{aX+b}(t) = e^{ibt}\phi_X(at)$
The characteristic function standard normal distribution $Z \sim N(0, 1)$ is:
$\phi_Z(t) = e^{-t^2/2}$
Plugging in $a = \sigma$ and $b = \mu$ into theorem 4.8 we have:

$\phi_X(t) = \phi_{\sigma Z+\mu}(t) = e^{i\mu t}\phi_Z(\sigma t)$
$\phi_X(t) = e^{i\mu t}e^{-(\sigma t)^2/2} = e^{i\mu t-\sigma^2 t^2/2}$

Problem 2

$$p_X(k) = \begin{cases} e^{-\lambda}\lambda^k/k! & k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

We can write:

$$\frac{P(X = k)}{P(X = k-1)} = \frac{\lambda}{k}$$

p_X is increasing monotonically for $\frac{\lambda}{k} > 1$

p_X is decreasing monotonically for $\frac{\lambda}{k} < 1$

Let $i =$ the largest integer value not exceding λ

Thus p_x is increasing for $k = 0, 1, \ldots, i$

and p_x is decreasing for $k = i + 1, i + 2, \ldots$

Since $p_X(i) \geq p_X(k)\ \forall\ k$ in the support of X

p_X has a maximum at $k = i$

Since the Poisson distribution is unimodal, i is also the mode

Problem 3

A car must pass at t=4 seconds so that the pedestrian does not start crossing before 4 seconds and no cars can pass at t=5,6,7 seconds in order for the pedestrian to only have to wait 4 seconds before crossing.
Thus we have: $p(1 - p)^3$
Looking at the first 3 seconds, at least one car must pass during the period so the pedestrian does not start crossing immediately.
The probability of this is: $1 - (1 - p)^3$
Multiplying the two probabilities together we have:
$(1 - (1 - p)^3)p(1 - p)^3$

Problem 4

The probability of a success is .8 if the new and old drugs are equally effective. Assuming all trials are independent we can use the binomial distribution

$$P(X \geq 85) = \sum_{85}^{100} \binom{100}{k} .8^k .2^{100-k} = .1285055$$

Since the probability of observing 85 or more success is almost 13%, this is not that unlikely so we cannot conclude the new drug is more effective.

Problem 7

(a) There are $\binom{N}{4}$ ways to select 4 packets of cocaine over $\binom{N+M}{4}$ total ways to select 4 packets. The probability of selecting 4 packets of cocaine is $\frac{\binom{N}{4}}{\binom{N+M}{4}}$

There are $\binom{M}{2}$ ways to select 2 non cocaine packets after selecting 4 cocaine packets over $\binom{N+M-4}{2}$ total ways to select 2 packets after having already selected 4 packets.

The probability of selecting 2 non cocaine packets is $\frac{\binom{M}{2}}{\binom{N+M-4}{2}}$

Thus the probability the defendant is innocent equals the above probabilities multiplied together.

$$P(\text{Defendent Innocent}) = \frac{\binom{N}{4}\binom{M}{2}}{\binom{N+M}{4}\binom{N+M-4}{2}}$$

(b) Since $M = 496 - N$ we have:

$$\frac{\binom{N}{4}\binom{496-N}{2}}{\binom{496}{4}\binom{492}{2}} \quad N = 4, \ldots, 496$$

$$\frac{P_{N+1}}{P_N} = \frac{\binom{N+1}{4}\binom{496-N-1}{2}}{\binom{N}{4}\binom{496-N}{2}}$$

$$= \frac{(N+1)}{(N-3)}\frac{(494-N)}{(496-N)}$$

Plugging in $N = 331$ we have :

$$\frac{(331+1)(494-331)}{(331-3)(496-331)} \approx 1.00547 > 1$$

Plugging in $N = 332$ we have :

$$\frac{(332+1)(494-332)}{(332-3)(496-332)} \approx .9998 < 1$$

Thus $N = 331$ is a maximum

Plugging in $N = 331$ into the the original equation we get:

$$\frac{\binom{331}{4}\binom{496-331}{2}}{\binom{496}{4}\binom{492}{2}} = .02208168 \approx .022$$

The maximum innocence probability is .022 and this is attained at N=331

Problem 9

WTS: Limiting $r \to \infty,\ p \to 1,\ r(1-p) \to \lambda$

In the Negative Binomial mgf: $M(t) = \left(\dfrac{p}{1-(1-p)e^t}\right)^r$

We have the Poisson mgf: $M(t) = e^{\lambda(e^t-1)}$

Let $a = r(1-p)$ then we have:

$$M(t) = \left(\frac{1-a/r}{1-(a/r)e^t}\right)^r$$
$$= \left(\frac{1-(a/r)e^t+(a/r)e^t-(a/r)}{1-(a/r)e^t}\right)^r$$
$$= \left(\frac{(1-(a/r)e^t)+(a/r)(e^t-1)}{1-(a/r)e^t}\right)^r$$
$$= \left(\frac{r(1-(a/r)e^t)+a(e^t-1)}{r(1-(a/r)e^t)}\right)^r$$
$$= \left(1+\frac{a(e^t-1)}{r(1-(a/r)e^t)}\right)^r$$
$$= \left(1+\frac{\frac{a(e^t-1)}{1-(a/r)e^t}}{r}\right)^r$$

Let $a_n = \dfrac{a(e^t-1)}{1-(a/r)e^t}$

applying the limits:

a_n goes to $\dfrac{\lambda(e^t-1)}{1-(1-1)e^t}$

$= \lambda(e^t-1)$

And since we know from Lemma(2.3.14) that:

$$\lim_{n\to\infty}(1+\frac{a_n}{n})^n = e^a$$

where a_n is a sequence where $\lim\limits_{n\to\infty} a_n = a$

$\left(1+\dfrac{a_n}{r}\right)^r$ goes to $e^{\lambda(e^t-1)}$

Which is the Poisson mgf

Thus the Negative Binomial mgf converges to the Poisson mgf

Problem 10

(a)

$$\begin{aligned}
\text{WTS: } \Gamma(a+1) &= a\Gamma(a) \\
\Gamma(a+1) &= \int_0^\infty t^a e^{-t}\ dt \\
&= \Big|_0^\infty - t^a e^{-t} + a\int_0^\infty t^{a-1}e^{-t}\ dt \\
\text{since } \Gamma(a) &= \int_0^\infty t^{a-1}e^{-t}\ dt \text{ we have:} \\
&= 0 + a\Gamma(a) = a\Gamma(a)
\end{aligned}$$

(b)

$$\Gamma(1/2) = \int_0^\infty t^{1/2-1}e^{-t}\ dt = \int_0^\infty t^{-1/2}e^{-t}$$

doing a change of variables where $u=\sqrt{2t},\ t=u^2/2,\ du=t^{-1/2}/\sqrt{2}$ we have:

$$\int_0^\infty \sqrt{2}e^{-u^2/2}\ du$$

from (3.3.14) we know:

$$\int_0^\infty e^{-u^2/2}\ du = \sqrt{\pi/2} \text{ so we have:}$$

$$\sqrt{\pi/2}\sqrt{2} = \sqrt{\pi}$$

Problem 11

Let $X \sim binom(1000, 1/6)$
Using the normal approximation to the binomial
$p = 1/6 \quad n = 1000 \quad \mu = np = 1000 * 1/6 = 500/3$
$\sigma = \sqrt{(500/3)(5/6)} = \sqrt{1250}/3$
then we have $Y \sim normal(500/3, \sqrt{1250}/3)$
$P(150 \leq Y \leq 200) = P(Y \leq 200) - P(Y \leq 150)$
$= P(Z \leq (200 + .5 - 500/3)/\sqrt{1250}/3) - P(Z \leq (150 - .5 - 500/3)/\sqrt{1250}/3)$
$= \Phi(2.87) - \Phi(-1.46) = \Phi(2.87) - 1 + \Phi(-1.46) = .9258$

Problem 14

$$\begin{aligned}
&\text{WTS: If } X \sim exp(\lambda),\ cX \sim exp(\lambda/c)\ c > 0\\
f_X(x) &= \lambda e^{-\lambda x}\ x \geq 0\\
P(X \leq x) &= F(x) = 1 - e^{-\lambda x}\ x \geq 0\\
&\text{Let } Y = cX\\
\text{Then } P(Y \leq y) &= P(cX \leq y) = P(X \leq y/c) = F_X(y/c) = 1 - e^{-\lambda y/c}\\
f_Y(y) &= \frac{d}{dy} 1 - e^{-\lambda y/c} = \frac{\lambda}{c} e^{-\lambda y/c}\ y \geq 0\\
&\text{Thus } cX \sim exp(\lambda/c)
\end{aligned}$$

Problem 15

$$\begin{aligned}
F(s) &= 1 - \exp(-(\frac{s-v}{\alpha})^{\beta}) \quad s \geq v = 0\\
1 - F(s) &= \exp(-(\frac{s-0}{\alpha})^{\beta})\\
\log(1 - F(s)) &= \log(\exp(-(\frac{s}{\alpha})^{\beta})) = (-\frac{s}{\alpha})^{\beta}\\
\log(1 - F(s))^{-1} &= -(\frac{s}{\alpha})^{-\beta}\\
\log(\log(1 - F(s))^{-1}) &= \log(-(\frac{s}{\alpha})^{-\beta})\\
&= -\beta \log(-(\frac{s}{\alpha}))\\
&= \beta \log(s) - \beta \log(\alpha)\\
\text{Plotting this against } &\log(s) \text{ we have a linear function}\\
P(s \leq \alpha) &= 1 - \exp(-(\frac{\alpha}{\alpha})^{\beta})\\
&= 1 - \exp(-(1)^{\beta}) = 1 - \exp(-1)\\
&= .6321206
\end{aligned}$$