Problem 1

a

Each point in the sample space is the result of the coin toss for each of the four tosses. $S = \{X_1, X_2, X_3, X_4\}$ where X_i represents the outcome of the ith toss, either H or T. There are 16 sample points, thus it is a finite sample space.

b

$S = \{0, 1, 2, 3, \dots\}$ Since the number of damaged leaves is an integer greater than or equal to 0. This is a countably infinite sample space.

c

$S = \{t : t \geq 0\}$ Where t is the time in hours. This is an uncountably infinite sample space.

d

$S = \{w : w > 0\}$ Where w is the weight of the rate in the chosen measurement unit, possibly ounces or grams. There would reasonably be an upper bound on the weight of a 10 day old rat. This is an uncountably infinite sample space.

e

$S = \{0/n, 1/n, 2/n, \dots\}$ where n is the total number of components. This is a countably infinite sample space.

Problem 2

a

$S = \{IC_1, IC_2, \dots, IC_i\}$ where I is either $(0, 1)$ and C is either (g, f, s).

b

$A = \{0s, 1s\}$

c

$B = \{0g, 0f, 0s\}$

d

$B^c \cup A = \{1g, 1f, 1s, 0s\}$

Problem 3

a

$A^c = \{x : 0 < x \leq .5\}$

b

$A^c = \big\{(x, y) : x^2 + y^2 \geq 2,\ |x| + |y| \leq 2\big\}$

c

$$\left(\bigcap_{n=1}^{\infty} B_n\right)^c = \bigcup_{n=1}^{\infty} {B_n}^c$$

${B_n}^c = \{x : x \notin (0, 1/n)\}$

$$\bigcup_{n=1}^{\infty} \{x : x \notin (0, 1/n)\} = \mathbb{R}$$

Problem 4

$\dashrightarrow$

If $C = A\Delta B$

Then we have four cases:

Case 1: $w \in A \cap B$

Then $w \notin A\Delta B$ so $w \notin C$

Thus $w \in B\Delta C$

Case 2: $w \in A \cap B^c$

Then $w \in A\Delta B$ so $w \in C$

Thus $w \in B\Delta C$

Case 3: $w \in A^c \cap B$

Then $w \in A\Delta B$ so $w \in C$

Thus $w \notin B\Delta C$

Case 4: $w \in A^c \cap B^c$

Then $w \notin A\Delta B$ so $w \notin C$

Thus $w \in B\Delta C$

Conclude $A = B\Delta C$

$\dashleftarrow$

Suppose $A = B\Delta C$

Then we have four cases:

Case 1: $w \in B \cap C$

Then $w \notin B\Delta C$ so $w \notin A$

Thus $w \in A\Delta B$

Case 2: $w \in B \cap C^c$

Then $w \in B\Delta C$ so $w \in A$

Thus $w \notin A\Delta B$

Case 3: $w \in B^c \cap C$

Then $w \in B\Delta C$ so $w \in A$

Thus $w \in A\Delta B$

Case 4: $w \in B^c \cap C^c$

Then $w \notin B\Delta C$ so $w \notin A$

Thus $w \notin A\Delta B$

Conclude $C = A\Delta B$

Problem 5

a

$$\dashrightarrow$$

$$\begin{aligned}
\text{Suppose } & x \in (\cup_\alpha A_\alpha)^c \\
\text{Then } & x \notin \cup_\alpha A_\alpha \\
\text{That is } & x \notin A_\alpha \ \forall\ \alpha \in \Gamma \\
\text{Thus } & x \in {A_\alpha}^c \ \forall\ \alpha \in \Gamma \\
\text{Therefore } & x \in \cap_\alpha {A_\alpha}^c
\end{aligned}$$

$$\dashleftarrow$$

$$\begin{aligned}
\text{Suppose } & x \in \cap_\alpha {A_\alpha}^c \\
\text{Then } & x \in {A_\alpha}^c \ \forall\ \alpha \in \Gamma \\
\text{Thus } & x \notin A_\alpha \ \forall\ \alpha \in \Gamma \\
\text{Implying } & x \notin \cup_\alpha A_\alpha \\
\text{Therefore } & x \in (\cup_\alpha A_\alpha)^c
\end{aligned}$$

b

$$\dashrightarrow$$

$$\begin{aligned}
\text{Suppose } & x \in (\cap_\alpha A_\alpha)^c \\
\text{Then } & x \notin \cap_\alpha A_\alpha \\
\text{That is } & x \notin A_\alpha \text{ for some } \alpha \in \Gamma \\
\text{Thus } & x \in {A_\alpha}^c \text{ for some } \alpha \in \Gamma \\
\text{Therefore } & x \in \cup_\alpha {A_\alpha}^c
\end{aligned}$$

$$\dashleftarrow$$

$$\begin{aligned}
\text{Suppose } & x \in \cup_\alpha {A_\alpha}^c \\
\text{Then } & x \in {A_\alpha}^c \text{ for some } \alpha \in \Gamma \\
\text{Thus } & x \notin A_\alpha \text{ for some } \alpha \in \Gamma \\
\text{Implying } & x \notin \cap_\alpha A_\alpha \\
\text{Therefore } & x \in (\cap_\alpha A_\alpha)^c
\end{aligned}$$

Problem 6

a

$$\text{Suppose } x \in \limsup(A_n \cap B_n)$$
$$\text{That is } x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (A_k \cap B_k)$$
$$\text{Then } x \in \bigcup_{k=n}^{\infty} (A_k \cap B_k) \ \forall\ n$$
$$\text{Therefore } x \in \bigcup_{k=n}^{\infty} A_k \ \forall\ n \text{ and } x \in \bigcup_{k=n}^{\infty} B_k \ \forall\ n$$
$$\text{Thus } x \in \limsup A_n \text{ and } x \in \limsup B_n$$
$$\text{Therefore } x \in (\limsup A_n) \cap (\limsup B_n)$$

b

$$\dashrightarrow$$

$$\text{Suppose } x \in (\limsup A_n) \cup (\limsup B_n)$$
$$\text{Then } x \in \bigcup_{k=n}^{\infty} A_k \ \forall\ n \text{ or } x \in \bigcup_{k=n}^{\infty} B_k \ \forall\ n$$
$$\text{This implies } x \in \bigcup_{k=n}^{\infty} (A_k \cup B_k) \ \forall\ n$$
$$\text{Therefore } x \in \limsup(A_n \cup B_n)$$

$$\dashleftarrow$$

$$\text{Suppose } x \in \limsup(A_n \cup B_n)$$
$$\text{Then } x \in \bigcup_{k=n}^{\infty} (A_k \cup B_k) \ \forall\ n$$
$$\text{Thus } x \in \bigcup_{k=n}^{\infty} A_k \ \forall\ n \text{ or } x \in \bigcup_{k=n}^{\infty} B_k \ \forall\ n$$
$$\text{Which implies } x \in \limsup A_n \text{ or } x \in \limsup B_n$$
$$\text{Therefore } x \in (\limsup A_n) \cup (\limsup B_n)$$

Problem 7

$$\text{Suppose } x \in \liminf A_n$$
$$\text{That is } x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$
$$\text{Then } \exists\ N \text{ such that } x \in \bigcap_{k=N}^{\infty} A_k$$
$$\text{By definiton } x \in \limsup A_n \iff x \in \bigcup_{k=m}^{\infty} A_k\ \forall\ m$$
$$\textbf{Case 1: } m \geq N$$
$$\text{Then } x \in A_m \text{ since } x \in \bigcap_{k=N}^{\infty} \subset A_m$$
$$\text{Therefore } x \in \bigcup_{k=m}^{\infty} A_k$$
$$\textbf{Case 2: } m < N$$
$$\text{Then } x \in A_N \text{ since } x \in \bigcap_{k=N}^{\infty} A_k \text{ and } A_N \subset \bigcup_{k=m}^{\infty} A_m$$
$$\text{Therefore } x \in \bigcup_{k=m}^{\infty} A_k$$
$$\text{Since we have proven } x \in \bigcup_{k=m}^{\infty} A_k\ \forall\ m$$
$$\text{Conclude } x \in \limsup A_n$$