Problem 1

From Example 4.3.1 and Theorem 4.3.2 we know:

$$X+Y \sim \text{Poisson}(\theta+\lambda)$$

Let $U = X+Y,\ V = Y$

Then $X = U-V,\ Y = V$

$$f(u,v) = \frac{\theta^{u-v}e^{-\theta}}{(u-v)!}\frac{\lambda^v e^{-\lambda}}{v!} \quad v = 0,1,\dots \quad u = v, v+1, \dots$$

$$f(u) = \frac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u \quad u = 0,1,\dots$$

Finding $Y|X+Y$

Defining U and V the same way:

$$f(y|x+y) = f(v|u)$$

$$\begin{aligned} f(v|u) &= \frac{f(u,v)}{f(u)} \\ &= \frac{\dfrac{\theta^{u-v}e^{-\theta}}{(u-v)!}\dfrac{\lambda^v e^{-\lambda}}{v!}}{\dfrac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u} \\ &= \frac{u!}{(u-v)!v!}\frac{e^{-(\theta+\lambda)}}{e^{-(\theta+\lambda)}}\frac{\theta^{u-v}\lambda^v}{(\theta+\lambda)^u} \\ &= \binom{u}{v}\frac{\theta^{u-v}\lambda^v}{(\theta+\lambda)^u} \\ &= \binom{u}{v}\left(\frac{\lambda}{\theta+\lambda}\right)^v\left(\frac{\theta}{\theta+\lambda}\right)^{u-v} \end{aligned}$$

Which is binomial $\left(u, \dfrac{\lambda}{\theta+\lambda}\right)$

Finding $X|X+Y$

Define $U = X+Y,\ V = X$

Then $X = U-V,\ X = V$

$$f_{U,V}(u,v) = f_{X,Y}(v, u-v) = \frac{\theta^v e^{-\theta}}{v!}\frac{\lambda^{u-v}e^{-\lambda}}{(u-v)!}$$

$$\begin{aligned} f(u) &= \sum_{v=0}^{u}\frac{\theta^v e^{-\theta}}{v!}\frac{\lambda^{u-v}e^{-\lambda}}{(u-v)!} \\ &= \frac{e^{-(\theta+\lambda)}}{u!}\sum_{v=0}^{u}\binom{u}{v}\theta^v\lambda^{u-v} \end{aligned}$$

Using the binomial theorem we have:

$$\begin{aligned}
f(u) &= \frac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u \\
f(x|x+y) &= f(v|u) \\
&= \frac{f(u,v)}{f(u)} \\
&= \frac{\dfrac{\theta^v e^{-\theta}}{v!}\dfrac{\lambda^{u-v}e^{-\lambda}}{(u-v)!}}{\dfrac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u} \\
&= \binom{u}{v}\frac{\theta^v\lambda^{u-v}}{(\theta+\lambda)^u} \\
&= \binom{u}{v}\left(\frac{\theta}{\theta+\lambda}\right)^v\left(\frac{\lambda}{\theta+\lambda}\right)^{u-v} \\
\text{Which is binomial} & \left(u, \frac{\theta}{\theta+\lambda}\right)
\end{aligned}$$

Problem 2

$f_X(x) = p(1-p)^{x-1} \quad f_Y(y) = p(1-p)^{y-1}$
Since X and Y are independent we have:
$f_{X,Y}(x,y) = p(1-p)^{x-1}p(1-p)^{y-1}$
$= p^2(1-p)^{x+y-2}$

(a) Solving $V = X - Y$ for X we get $X = V + Y$
If $V > 0$ then $X > Y$
Since $U = min(X, Y)$ this means that $U = Y$
Thus we have $Y = U$ and $X = U + V$

$$\begin{aligned}
f_{U,V}(u,v) &= P(Y=u, X=u+v) \\
&= p^2(1-p)^{2u+v-2} \\
\text{Which factors to: } & (p^2(1-p)^{2u})((1-p)^{v-2}) \\
\text{If } V < 0,\ & \text{then } X < Y
\end{aligned}$$

Thus $X = U, \quad Y = U - V$

$$\begin{aligned}
f_{U,V}(u,v) &= P(X=u, Y=u-v) \\
&= p^2(1-p)^{2u-v-2} \\
\text{Which factors to: } & (p^2(1-p)^{2u})((1-p)^{-v-2}) \\
\text{If } V = 0 & \text{ then } X = Y \\
f_{U,V}(u,0) &= P(X=Y=u) = p^2(1-p)^{2u-2} \\
\text{Which factors to: } & (p^2(1-p)^{2u})((1-p)^{-2})
\end{aligned}$$

Since we can factor all of these cases in terms of u and v, U and V are independent

(b)

(c)

$$\begin{aligned}
&\text{Define } T = X + Y \\
f_{X,X+Y}(x, x+y) &= P(X = x, X + Y = t) = P(X = x, Y = t - x) = P(X = x)P(Y = t - x) \\
&= p^2(1-p)^{x-1+t-x-1} = p^2(1-p)^{t-2}
\end{aligned}$$

Problem 3

(a)

$$\begin{aligned}
&X_1, X_2 \text{ are independent and distributed as:} \\
f_{X_i}(x_i) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-x_i^2/2\sigma^2} \\
Y_1 &= X_1^2 + X_2^2 \quad Y_2 = \frac{X_1}{\sqrt{Y_1}}
\end{aligned}$$