Problem 1

Let $U = X/Y$ and $V = Y$

Then $X = UV, \quad Y = V$

where X and Y are independent χ^2 r.v.s with m and n degrees of freedom respectively

WTS: $U \sim F_{m,n}$ that is:

$$f_U(u) = \frac{m}{n} \frac{\Gamma(\dfrac{m+n}{2})(\dfrac{m}{n}u)^{m/2-1}}{\Gamma(m/2)\Gamma(n/2)(1+u(m/n))^{(m+n)/2}} \; u > 0$$

Let $c = m/2$ and $d = n/2$

$$f_X(x) = \frac{1}{\Gamma(c)2^c} x^{c-1} e^{-x/2} \; x > 0$$

$$f_Y(y) = \frac{1}{\Gamma(d)2^d} y^{d-1} e^{-y/2} \; y > 0$$

$$f_{XY}(x,y) = \left(\frac{1}{\Gamma(c)2^c} x^{c-1} e^{-x/2}\right)\left(\frac{1}{\Gamma(d)2^d} y^{d-1} e^{-y/2}\right)$$

$$= \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} x^{c-1} y^{d-1} e^{-x/2} e^{-y/2} \; x, y > 0$$

$$J(u,v) = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} = v$$

$$f_{UV}(u,v) = \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} (uv)^{c-1} v^{d-1} e^{-(uv+v)/2} |v| \; u, v > 0$$

$$f_U(u) = \int_0^\infty f_{UV}(u,v)\; dv = \int_0^\infty \frac{v}{\Gamma(c)\Gamma(d)2^{c+d}} (uv)^{c-1} v^{d-1} e^{-(uv+v)/2} \; dv$$

$$= \int_0^\infty \frac{u^{c-1}}{\Gamma(c)\Gamma(d)2^{c+d}} v^{c+d-1} e^{-v(u+1)/2} \; dv$$

$$f_U(u) = \frac{u^{c-1}}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty v^{c+d-1} e^{-v(u+1)/2} \; dv$$

$$\text{Let } z = \frac{v(u+1)}{2}$$

$$dz = \frac{u+1}{2} \; dv$$

$$dv = \frac{2}{u+1} \; dz$$

$$f_U(u) = \frac{u^{c-1}}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty \frac{2z}{u+1}^{c+d-1} e^{-z} \frac{2}{u+1} \; dz$$

$$= \left(\frac{2}{u+1}\right)^{c+d} \frac{u^{c-1}}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty z^{c+d-1} e^{-z} \; dz$$

$$
\begin{aligned}
&= \left(\frac{2}{u+1}\right)^{c+d} \frac{u^{c-1}}{\Gamma(c)\Gamma(d)2^{c+d}} \Gamma(c+d) \\
&= \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} \frac{u^{c-1}}{(u+1)^{c+d}} \\
f_{(n/m)(UV)(u)} &= \frac{m}{n} f_U(\frac{m}{n}u) \\
&= \frac{m}{n} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(m/2)\Gamma(n/2)} \frac{((m/n)u)^{m/2-1}}{((m/n)u+1)^{(m+n)/2}} \\
\text{Thus } & \frac{X/m}{Y/n} \sim F(m,n)
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
f_x(X) &= \begin{cases} \frac{2}{2e-5} x_1^2 x_2 e^{x_1 x_2 x_3} & 0 < x_1, x_2, x_3 < 1 \\ 0 & \text{otherwise} \end{cases} \\
\text{Let } Y_1 &= X_1 * X_2 * X_3 \\
Y_2 &= X_1 \\
Y_3 &= X_2 \\
\text{Then } X_1 &= Y_2 \\
X_2 &= Y_3 \\
X_3 &= \frac{Y_1}{Y_2 Y_3} \\
J(y_1, y_2, y_3) &= \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_3}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_3}{\partial y_2} \\ \frac{\partial x_1}{\partial y_3} & \frac{\partial x_2}{\partial y_3} & \frac{\partial x_3}{\partial y_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/(y_2 y_3) \\ 1 & 0 & -(y_1/y_3) y_1^{-2} \\ 0 & 1 & -(y_1/y_2) y_3^{-2} \end{bmatrix} \\
J(y_1, y_2, y_3) &= \frac{1}{y_2 y_3} \\
f_y(Y) &= \frac{2}{2e-5} y_2^2 y_3 e^{y_1} \left| \frac{1}{y_2 y_3} \right| \\
f_y(Y) &= \begin{cases} \frac{2}{2e-5} y_2 e^{y_1} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \\
0 &< y_1 < y_2 < 1 \\
\frac{y_1}{y_2} &< y_3 < 1
\end{aligned}
$$

$$\begin{aligned}
f_y(y_1) &= \int_{y_1}^{1}\int_{y_1/y_2}^{1} \frac{2}{2e-5} y_2 e^{y_1}\ dy_3 dy_2 \\
&= \frac{2}{2e-5}\int_{y_1}^{1} y_2 e^{y_1} - y_1 e^{y_1}\ dy_2 \\
&= \frac{2}{2e-5}(1/2)e^{y_1}(1-2y_1+y_1^2) \\
&= \frac{1}{2e-5}e^{y_1}(1-y_1)^2 \\
f_y(y_1) &= \begin{cases} \frac{1}{2e-5}e^{y_1}(1-y_1)^2 & 0<y_1<1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Thus the pdf of $X_1 * X_2 * X_3$ is $f_y(y_1)$

Problem 3

$$\begin{aligned}
f(x) &= e^{-x} \\
f_y &= e^{-y} \\
f_{X,Y}(x,y) &= e^{-x}e^{-y} = e^{-(x+y)}\ x,y>0 \\
&\text{Let } U = X+Y \text{ and } V = X \\
X &= V \quad Y = U-V \\
J(u,v) &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = -1 \\
f_{U,V}(u,v) &= f_{X,Y}(v,u-v) = e^{-(v+u-v)} = e^{-u},\ 0<v<u<\infty \\
f_{V|U=c}(v) &= \frac{f_{U,V}(c,v)}{f_U(c)} = \frac{e^{-c}}{ce^{-c}} = \frac{1}{c},\ 0<v<c
\end{aligned}$$

Which is distributed as $U(0,c)$

Problem 4

(a)

$$\begin{aligned}
-2 \leq x \leq 4 &\quad 7 \text{ values} \\
x-1 \leq y \leq x+1 &\quad 3 \text{ values} \\
3*7 = 21 &\text{ possible } (x,y) \text{ pairs} \\
x &\in [-2,4] \\
y &\in [-3,5] \\
p_{X,Y}(x,y) &= \begin{cases} 1/21 & -2 \leq x \leq 4 \quad x-1 \leq y \leq x+1 \\ 0 & \text{otherwise} \end{cases} \\
p_X(x) &= \sum_{y=x-1}^{x+1} p(x,y) = p_X(x) = \begin{cases} 1/7 & -2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \\
p_Y(y) &= \sum_{x=-2}^{4} p(x,y) = p_Y(y) = \begin{cases} 1/21 & y = -3, 5 \\ 2/21 & y = -2, 4 \\ 1/7 & -1 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases} \\
E[X] &= \sum_{-2}^{4} xp(x) = \frac{1}{7}(-2 + -1 + 0 + 1 + 2 + 3 + 4) = 1 \\
E[Y] &= \sum_{-3}^{5} yp(y) = \frac{1}{21}(-3+5) + \frac{2}{21}(-2+4) + \frac{1}{7}(-1+0+1+2+3) = 1
\end{aligned}$$

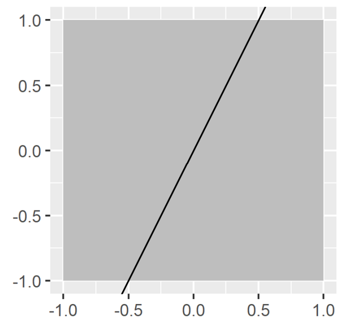
(b)

$$\begin{aligned}
100 * E[X] + 200 * E[Y] &= 100 + 200 = 300 \\
\text{Mean of traders profit } &= 300
\end{aligned}$$

Problem 5

We have a 2 by 2 square with an area of 4

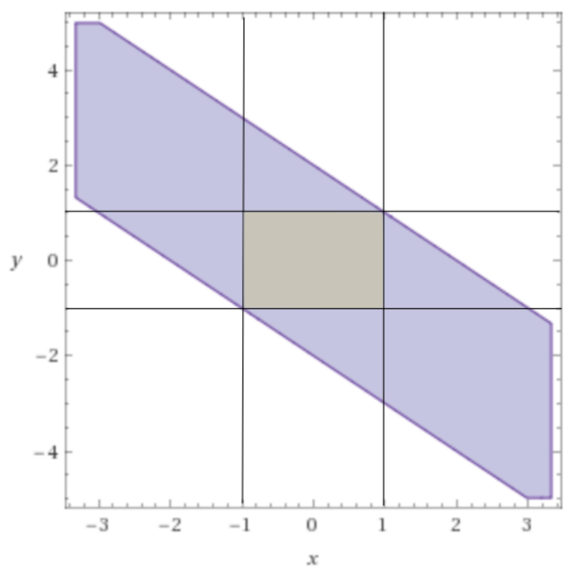
(a) $X^2 + Y^2 < 1$
Since the distribution is continuous this is the same as:
$X^2 + Y^2 \leq 1$
Which is a circle with area π
Since the total area is 4, $P(X^2 + Y^2 < 1) = \dfrac{\pi}{4}$



(b)

$2X + Y > 0$
$Y < 2X$
Which is the area below the line $y = 2x$
Since this line cuts the square in half
$P(2X - Y > 0) = \frac{2}{4} = \frac{1}{2}$

(c) Looking at the plot of $|X + Y| < 2$:



We can see that this includes the entire rectangle, thus
$P(|X + Y| < 2) = 1$

Problem 6

(a)

$$
\begin{aligned}
C\int_0^2\int_0^1 x+2y\ dy\mathrm{d}x &= 1\\
C\int_0^2 x+1\ \mathrm{d}x &= 1\\
4C &= 1\\
C &= 1/4
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_X(x) &= (1/4)\int_0^1 x+2y\ dy\\
&= (1/4)(x+1)\\
f_X(x) &= \begin{cases} 1/4(x+1) & 0<x<2\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

(c)

$$
\begin{aligned}
F_{XY}(x,y) &= \int_{-\infty}^x\int_{-\infty}^y f(s,t)\ dt\ ds\\
&= (1/4)\int_0^x\int_0^y s+2t\ dt\ ds\\
&= (1/8)x^2y+(1/4)xy^2
\end{aligned}
$$

In order to get the complete CDF we must consider the boundaries

setting $x=2$ we have:

$(1/4)y+(1/2)y^2$

setting $y=1$ we have:

$(1/8)x^2+(1/4)x$

$$
F_{XY}(x,y)=\begin{cases}
0 & x\le 0 \text{ or } y\le 0\\
(1/8)x^2y+(1/4)xy^2 & 0<x<2 \text{ and } 0<y<1\\
(1/2)y+(1/2)y^2 & x\ge 2 \text{ and } 0<y<1\\
(1/8)x^2+(1/4)x & 0\le x\le 2 \text{ and } y\ge 1\\
1 & x\ge 2 \text{ and } y\ge 1
\end{cases}
$$

(d)

$$f_X(x)=1/4(x+1)\quad 0<x<2$$

$$\begin{aligned}
Z = g(x) &= \frac{9}{(x+1)^2} \\
\text{When } x &= 0 \; z = 9 \\
\text{When } x &= 2 \; z = 1 \\
1 &< z < 9 \\
X &= 3z^{-1/2} \\
\frac{dx}{dz} &= (-3/2)z^{-3/2} \\
f_z(z) &= f_x(g^{-1}(z))\left|\frac{dx}{dz}\right| \\
&= (9/8)z^{-2} \quad 1 < z < 9
\end{aligned}$$

Problem 7

(a)

$$\begin{aligned}
P(X > \sqrt{Y}) &= \int_0^1 \int_{\sqrt{y}}^1 x + y \ \mathrm{d}x dy \\
&= \int_0^1 1/2 + (1/2)y - y^{3/2} \ dy \\
&= 1/2 + 1/4 - 2/5 = 7/20
\end{aligned}$$

(b)

$$\begin{aligned}
X^2 &< Y < X \\
X &< \sqrt{Y} \ X > Y \\
Y &< X < \sqrt{Y} \\
P(X^2 < Y < X) &= \int_0^1 \int_y^{\sqrt{y}} 2x \ \mathrm{d}x dy \\
&= \int_0^1 y - y^2 \ dy \\
&= 1/2 - 1/3 = 1/6
\end{aligned}$$

Problem 8

$$\begin{aligned}
X &\sim U(0, 30) \\
Y &\sim U(40, 50) \\
30 - 0 &= 30 \quad 50 - 40 = 10 \\
30 * 10 &= 300 \\
X + Y &< 60 \\
X &< 60 - Y \\
P(X + Y < 60) &= \int_{40}^{50} \int_0^{60-y} \frac{1}{300} \ \mathrm{d}x dy \\
&= \frac{1}{300} \int_{40}^{50} 60 - y \ dy \\
&= (1/300)(60 * 50 - 50^2/2 - 60 * 40 + 40^2/2) \\
&= 150/300 = 1/2
\end{aligned}$$

Problem 9

Given $F_{XY}(x,y) = F_X(x)F_Y(y)$

WTS: For any (a,b) and (c,d) :$P(a \leq X \leq b, c \leq Y \leq d) = P(a \leq X \leq b)P(c \leq Y \leq d)$

We know X and Y are independent since the joint cdf equals the marginal cdfs multiplied together

$$
\begin{aligned}
& P(a \leq X \leq b)P(c \leq Y \leq d) \\
&= [P(X \leq b) - P(X \leq a)] * [P(Y \leq d) - P(Y \leq c)] \\
&= P(X \leq b)P(Y \leq d) - P(X \leq b)P(Y \leq c) - P(X \leq a)P(Y \leq d) + P(X \leq a)P(Y \leq c) \\
&= F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c) \\
&= F(b,d) - F(b,c) - F(a,d) + F(a,c) \\
&= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c) - [P(X \leq a, Y \leq d) - P(X \leq a, Y \leq c)] \\
&= P(X \leq b, c \leq Y \leq d) - P(X \leq a, c \leq Y \leq d) \\
&= P(a \leq X \leq b, c \leq Y \leq d)
\end{aligned}
$$