

Problem 1

In order for the sum to be greater than or equal to k , we need at least k ones. The number of ways to get k ones is:

$$\left(\sum_{i=1}^n x_i = k \right) \rightarrow \binom{n}{k}$$

Then we have to take in account all of the ways to get $k+1, k+2, \dots, n$ ones. Summing all of these possibilities gives us:

$$\sum_{i=k}^n \binom{n}{i}$$

Problem 2

We want to show $\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$ for $n > 0$

The binomial theorem states:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \quad \forall n \in \mathbb{N} \quad (1)$$

Let $x = -1, y = 1, k = i$

Then (1) becomes $\sum_{i=0}^n \binom{n}{i} (-1)^i (1)^{n-i} = (-1+1)^n$

Then we have $\sum_{i=0}^n \binom{n}{i} (-1)^i (1) = 0^n$

$$= \sum_{i=0}^n \binom{n}{i} (-1)^i = 0$$

Therefore $\sum_{i=0}^n \binom{n}{i} (-1)^i = 0$ for $n > 0$

Problem 3

$$(a) \frac{4 \binom{13}{5}}{\binom{52}{5}} \approx .001980$$

$$(b) \frac{13 \binom{4}{2} \binom{12}{3} 4^3}{\binom{52}{5}} \approx .422569$$

$$(c) \frac{\binom{13}{2} \binom{4}{2} \binom{4}{2} 11 * 4}{\binom{52}{5}} \approx .047539$$

$$(d) \frac{13 \binom{4}{3} \binom{12}{2} * 4^2}{\binom{52}{5}} \approx .021128$$

$$(e) \frac{13 * 12 * 4}{\binom{52}{5}} \approx .000240$$

Problem 4

$$\frac{16 * 4}{52 * 51} + \frac{4 * 16}{52 * 51} = \frac{16 * 4 * 2}{52 * 51} \approx .048265$$

Problem 5

Let E_i be the event that the i th couple sit next to each other

$P\left(\bigcup_{i=1}^4 E_i\right) = P(\text{at least one couple sits together})$

$P(\text{no couples sit together}) = 1 - P(\text{at least 1 couple sits together})$

$P(\text{no couples sit together}) = 1 - P\left(\bigcup_{i=1}^4 E_i\right)$

Using the inclusion-exclusion principle we have:

$$P\left(\bigcup_{i=1}^4 E_i\right) = \sum E_i - \sum E_i \cap E_j + \sum E_i \cap E_j \cap E_k - \sum E_i \cap E_j \cap E_k \cap E_l$$

$$P(E_i) = \frac{2 * 7!}{8!}$$

$$P(E_i \cap E_j) = \frac{2^2 * 6!}{8!}$$

$$P(E_i \cap E_j \cap E_k) = \frac{2^3 * 5!}{8!}$$

$$P(E_i \cap E_j \cap E_k \cap E_l) = \frac{2^4 * 4!}{8!}$$

$$P\left(\bigcup_{i=1}^4 E_i\right) = \binom{4}{1} \frac{2 * 7!}{8!} - \binom{4}{2} \frac{2^2 * 6!}{8!} + \binom{4}{3} \frac{2^3 * 5!}{8!} - \binom{4}{4} \frac{2^4 * 4!}{8!}$$

$$P\left(\bigcup_{i=1}^4 E_i\right) = 1 - \frac{3}{7} + \frac{4}{42} - \frac{1}{105}$$

$$1 - P\left(\bigcup_{i=1}^4 E_i\right) = 1 - \left(1 - \frac{3}{7} + \frac{4}{42} - \frac{1}{105}\right)$$

$$= \frac{3}{7} - \frac{4}{42} + \frac{1}{105}$$

$$P(\text{no couples sit together}) = \frac{12}{35} = .3428571$$

Problem 6

$$\frac{\frac{n!}{(n-m)!} (N-1)^{n-m}}{N^n}$$

Problem 7

$$\frac{\binom{6}{3}\binom{6}{3}}{\binom{12}{6}} = .4329004$$

Problem 8

$$\frac{(N-r)(N-r-1)}{(N-1)(N-2)}$$

Problem 9

3 coins: $\{HH, TT, HT\}$

$$P(HT|H) = \frac{P(H|HT)P(HT)}{P(H|HT)P(HT) + P(H|HT^c)P(HT^c)}$$

$$P(HT|H) = \frac{1/2 * 1/3}{1/2 * 1/3 + 1/2 * 2/3}$$

$$P(HT|H) = 1/3$$