

Problem 1

WTS: For any r.v. X , if $g(x)$ is a convex function, then: $Eg(X) \geq g(EX)$

Let $g(x)$ be a convex function

Suppose $l(x) = a + bx$ is a line tangent to $g(x)$ at $x = EX$

Since g is convex, it lies above the line $l(x)$

Which means $g(x) > l(x) \forall x$ except at $x = EX$

Thus $E(g(x)) \geq E(l(x)) = E(a + bX) = a + bE(X) = l(E(X)) = g(E(X))$

Then $Eg(X) > g(EX)$ unless $P(X = EX) = 1$

Problem 2

(a)

$$f_{XY}(x, y) = \left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1} \\ * \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

$$\text{Let } z = \frac{y-\mu_Y}{\sigma_Y} \quad dy = \sigma_Y dz \quad v = \frac{x-\mu_X}{\sigma_X}$$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}[v^2 - 2\rho vz + z^2]\right\} \sigma_Y dz \\ &= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}[-2\rho vz + z^2]\right\} dz \\ &= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}[-2\rho vz + z^2 + \rho^2 v^2 - \rho^2 v^2]\right\} dz \\ &= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}[(z-\rho v)^2 - \rho^2 v^2]\right\} dz \\ &= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right) \exp\left(\frac{-\rho^2 v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2\right\} dz \\ &= \frac{e^{-v^2/2}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2\right\} dz \end{aligned}$$

Since the integrand is the $N(\rho v, 1-\rho^2)$ we have:

$$f_X(x) = \frac{e^{-v^2/2}}{2\pi\sigma_X\sqrt{1-\rho^2}}\sqrt{2\pi}\sqrt{1-\rho^2}$$

$$= \frac{e^{-v^2/2}}{\sqrt{2\pi}\sigma_X}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)$$

Which is the $N(\mu_X, \sigma_X^2)$ pdf

(b)

$$\begin{aligned} \text{WTS: } f(Y|X)(y|x) &= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y} e^{\frac{-[y-\mu_Y-(\rho\sigma_Y/\sigma_X)(x-\mu_X)]^2}{2\sigma_Y^2(1-\rho^2)}} \\ f(Y|X)(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}{\frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)} \\ &= \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)} \\ &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] + \frac{(x-\mu_X)^2}{2\sigma_X^2}\right\}} \\ &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - (1-\rho^2)\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}} \\ &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}} \\ &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\}} \\ &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2\sigma_Y^2(1-\rho^2)}[y-\mu_Y-(\rho\sigma_Y/\sigma_X)(x-\mu_X)]^2\right\}} \end{aligned}$$

Which is the pdf of $N[\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_X), \sigma_Y^2(1-\rho^2)]$

(c)

$$E(a_X + b_Y) = aEX + bEY = a\mu_X + b\mu_Y$$

$$\begin{aligned} Var(a_X + b_Y) &= a^2Var(X) + b^2Var(Y) + 2abCov(X, Y) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y \end{aligned}$$

Starting with the standard bivariate normal pdf we have :

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right]$$

$$\text{Let } U = aX + bY \quad V = Y$$

$$\text{Then } X = (1/a)(U - bV) \quad Y = V$$

$$J = \begin{bmatrix} 1/a & -b/a \\ 0 & 1 \end{bmatrix} = 1/a$$

$$\begin{aligned} f_{UV}(u, v) &= \frac{1}{2a\pi\sqrt{1-\rho^2}} \exp\left[-\frac{[(1/a)(u - bv)]^2 - 2\rho(1/a)(u - bv)v + v^2}{2(1-\rho^2)}\right] \\ &= \frac{1}{2a\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{u^2 - 2bvu + b^2v^2 - 2\rho a uv + 2\rho abv^2 + a^2v^2}{a^2}\right]} \\ &= \frac{1}{2a\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{u^2 - 2uv(b + \rho a) + v^2(b^2 + 2\rho ab + a^2)}{a^2}\right]} \\ &= \frac{1}{2a\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{b^2 + 2\rho ab + a^2}{a^2}\left[\frac{u^2}{b^2 + 2\rho ab + a^2} - 2uv\frac{(b + \rho a)}{b^2 + 2\rho ab + a^2} + v^2\right]\right]} \end{aligned}$$

Which is the joint bivariate normal pdf since:

$$\mu_U = \mu_V = 0 \quad \sigma_u^2 = b^2 + 2\rho ab + a^2, \sigma_V^2 = 1$$

$$\rho_{UV} = \frac{Cov(U, V)}{\sigma_U\sigma_V} = E((U - \mu_u)(V - \mu_v)) = E(UV) = E(aXY + bY^2)$$

$$Cov(U, V) = E((U - \mu_u)(V - \mu_v)) = E(UV) = E(aXY + bY^2) = a\rho + b$$

$$\rho_{UV} = \frac{a\rho + b}{\sqrt{a^2 + b^2 + 2ab\rho}}$$

$$1 - \rho_{UV}^2 = 1 - \left(\frac{a\rho + b}{\sqrt{a^2 + b^2 + 2ab\rho}}\right)^2$$

$$= 1 - \frac{a^2\rho^2 + b^2 + 2ab\rho}{a^2 + b^2 + 2ab\rho}$$

$$= \frac{(1 - \rho^2)a^2}{a^2 + b^2 + 2ab\rho} = \frac{(1 - \rho^2)a^2}{\sigma_U^2}$$

$$\text{define } \rho_{UV} = \rho^*$$

Then $a\sqrt{1-\rho^2} = \sigma_U\sqrt{1-\rho^{*2}}$

Remembering $\sigma_V^2 = 1$ we have:

$$f_{UV}(u, v) = \frac{1}{2\sigma_U\sigma_V\pi\sqrt{1-\rho^{*2}}} e^{-\frac{1}{2(1-\rho^{*2})} \left[\frac{u^2}{\sigma_U^2} - 2\rho^* \frac{uv}{\sigma_U\sigma_V} + \frac{v^2}{\sigma_V^2} \right]}$$

Which is the bivariate normal pdf

From part a we know that the marginal distribution of U is $N(\mu_u, \sigma_u^2)$

Which means that the distribution of $aX+bY$ is $N(a\mu_x+b\mu_y, a^2\sigma_x^2+b^2\sigma_y^2+2ab\rho\sigma_x\sigma_y)$

Based on the mean and variance we calculated at the beginning of the problem

Problem 3

(a)

$$\psi_{X,Y}(t, u) = e^{2t+3u+t^2+atu+2u^2}$$

$$\text{Let } J = X + 2Y \quad K = 2X - Y$$

$$M_{J,K}(l, m) = E(e^{lJ+mK}) = E(e^{l(X+2Y)+m(2X-Y)})$$

$$= E(e^{lX+2lY+2mX-mY})$$

$$= E(e^{(l+2m)X+(2l-m)Y})$$

$$= M_{X,Y}(l+2m, 2l-m)$$

$$= e^{2l+4m+8l-3m+(l+2m)^2+a(l+2m)(2l-m)+2(2l-m)^2}$$

$$= e^{8l+m+l^2+4ml+4m^2+8l^2+2m^2-8lm+2al^2+3alm-2am^2}$$

$$= e^{8l+m+9l^2+6m^2-4ml+2al^2+3alm-2am^2}$$

$$M_{X+2Y}(l, 0) = e^{8l+9l^2+2al^2}$$

$$M_{2X-Y}(0, m) = e^{m+6m^2-2am^2}$$

$$M_{X+2Y}(l)M_{2X-Y}(m) = e^{8l+9l^2+2al^2} e^{m+6m^2-2am^2}$$

$$= e^{8l+9l^2+2al^2+m+6m^2-2am^2}$$

$$\text{If independent: } M_{X+2Y, 2X-Y}(l, m) = M_{X+2Y}(l)M_{2X-Y}(m)$$

$$e^{8l+9l^2+2al^2+m+6m^2-2am^2} = e^{8l+m+9l^2+6m^2-4ml+2al^2+3alm-2am^2}$$

$$0 = 3alm - 4ml$$

$$0 = (3a - 4)ml \quad 3a - 4 = 0$$

$$a = 4/3$$

(b)

Let $Z = (2X - Y) - (X + 2Y)$

$$P(X + 2Y < 2X - Y) = P(Z > 0) = P(X - 3Y > 0)$$

$$\begin{aligned} M_Z(\theta) &= M_{X-3Y}(\theta) = M_{X,Y}(\theta, -3\theta) \\ &= e^{2\theta - 9\theta + \theta^2 + (4/3)(-3\theta^2) + 18\theta^2} \\ &= e^{-7\theta + 15\theta^2} \end{aligned}$$

This is in the form of the mgf of the normal distribution

$$e^{\mu t + (1/2)\sigma^2 t^2}$$

Plugging in $\mu = -7$ $\sigma^2 = 30$ we have:

$$e^{-7\theta + (1/2)30\theta^2} = e^{-7\theta + 15\theta^2}$$

Thus $Z \sim N(-7, 30)$

$$P(Z > 0) = 1 - \text{pnorm}(q = 0, \text{mean} = -7, \text{sd} = \text{sqr}(30)) = .1006 \quad (\text{using R})$$

Problem 4

(a)

$X_1, X_2 \sim N(0, 1)$ and independent

$$f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{2\pi} e^{-x_1^2/2} e^{-x_2^2/2}$$

$$Y_1 = X_1 - 3X_2 + 2$$

$$Y_2 = 2X_1 - X_2 - 1$$

$$\text{Then } X_1 = (-1/5)Y_1 + (3/5)Y_2 + 1 = g_1(y_1, y_2)$$

$$X_2 = (-2/5)Y_1 + (1/5)Y_2 + 1 = g_2(y_1, y_2)$$

$$J = \begin{bmatrix} -1/5 & 3/5 \\ -2/5 & 1/5 \end{bmatrix} = 1/5$$

$$= f_{X_1 X_2}(g_1(y_1, y_2), g_2(y_1, y_2)) |J|$$

$$= \frac{1}{2\pi} e^{(-1/2)((-1/5)y_1 + (3/5)y_2 + 1)^2} e^{(-1/2)((-2/5)y_1 + (1/5)y_2 + 1)^2} (1/5)$$

$$= \frac{1}{10\pi} e^{(-1/2)[(-1/5)y_1 + (3/5)y_2 + 1]^2 + (-2/5)y_1 + (1/5)y_2 + 1]^2}$$

$$f_Y(y_1, y_2) = \frac{1}{10\pi} e^{(-1/10)[y_1^2 + 2y_1 y_2 - 6y_1 + 2y_2^2 + 8y_2 + 10]}$$

(b)

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_Y(y_1, y_2) dy_1$$

$$\begin{aligned}
&= \frac{1}{10\pi} \int_{-\infty}^{\infty} e^{(-1/10)[y_1^2 + 2y_1 y_2 - 6y_1 + 2y_2^2 + 8y_2 + 10]} dy_1 \\
&= \frac{1}{10\pi} \sqrt{10\pi} e^{-(1/10)(y_2+1)^2} \\
f_{Y_2}(y_2) &= \frac{1}{\sqrt{10\pi}} e^{-(1/10)(y_2+1)^2} \\
f_{Y_1|Y_2} &= \frac{f_{Y_1 Y_2}}{f_{Y_2}} \\
&= \frac{1}{10\pi} e^{(-1/10)[y_1^2 + 2y_1 y_2 - 6y_1 + 2y_2^2 + 8y_2 + 10]} / \left[\frac{1}{\sqrt{10\pi}} e^{-(1/10)(y_2+1)^2} \right] \\
&= \frac{1}{\sqrt{10\pi}} e^{(-1/10)[y_1^2 + 2y_1 y_2 - 6y_1 + 2y_2^2 + 8y_2 + 10]} e^{(1/10)(y_2+1)^2} \\
&= \frac{1}{\sqrt{10\pi}} e^{-(1/10)[y_1^2 + 2y_1 y_2 - 6y_1 + 2y_2^2 + 8y_2 + 10 - y_2^2 - 2y_2 - 1]} \\
f_{Y_1|Y_2} &= \frac{1}{\sqrt{10\pi}} e^{-(1/10)[y_1^2 + 2y_1 y_2 - 6y_1 + y_2^2 + 6y_2 + 9]}
\end{aligned}$$

Problem 5

Problem 6