Problem 1

WTS: For any r.v. X, if $g(x)$ is a convex function, then: $Eg(X) \geq g(EX)$
Let $g(x)$ be a convex function
Suppose $l(x) = a + bx$ is a line tangent to $g(x)$ at $x = EX$
Since g is convex, it lies above the line $l(x)$
Which means $g(x) > l(x)$ $\forall x$ except at $x = EX$
Thus $E(g(x)) \geq E(l(x)) = E(a + bX) = a + bE(X) = l(E(X)) = g(E(X))$
Then $Eg(X) > g(EX)$ unless $P(X = EX) = 1$

Problem 2

(a)

$$\begin{aligned}
f_{XY}(x,y) &= \left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1} \\
&\quad * \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\} \\
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy \\
\text{Let } z &= \frac{y-\mu_Y}{\sigma_Y} \quad dy = \sigma_Y dz \quad v = \frac{x-\mu_X}{\sigma_X} \\
f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[v^2 - 2\rho vz + z^2\right]\right\}\sigma_Y\ dz \\
&= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-2\rho vz + z^2\right]\right\} dz \\
&= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-2\rho vz + z^2 + \rho^2v^2 - \rho^2v^2\right]\right\} dz \\
&= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[(z-\rho v)^2 - \rho^2v^2\right]\right\} dz \\
&= \frac{\exp\left(-\frac{v^2}{2(1-\rho^2)}\right)\exp\left(\frac{-\rho^2v^2}{2(1-\rho^2)}\right)}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2\right\} dz \\
&= \frac{e^{-v^2/2}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}(z-\rho v)^2\right\} dz
\end{aligned}$$

Since the integrand is the $N(\rho v, 1-\rho^2)$ we have:

$$f_X(x) = \frac{e^{-v^2/2}}{2\pi\sigma_X\sqrt{1-\rho^2}}\sqrt{2\pi}\sqrt{1-\rho^2}$$
$$= \frac{e^{-v^2/2}}{\sqrt{2\pi}\sigma_X}$$
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)$$

Which is the $N(\mu_X, \sigma_X^2)$ pdf

(b)

$$\text{WTS: } f(Y|X)(y|x) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y} e^{\dfrac{-[y-\mu_Y-(\rho\sigma_Y/\sigma_X)(x-\mu_X)]^2}{2\sigma_Y^2(1-\rho^2)}}$$

$$f(Y|X)(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$= \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}{\frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)}$$

$$= \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]+\frac{(x-\mu_X)^2}{2\sigma_X^2}\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 -(1-p^2)\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2\sigma_Y^2(1-\rho^2)}[y-\mu_Y-(\rho\sigma_Y/\sigma_X)(x-\mu_X)]^2\right\}}$$

Which is the pdf of $N[\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_X), \sigma_Y^2(1-\rho^2)]$

(c)

$$E(a_X + b_Y) = aEX + bEY = a\mu_X + b\mu_Y$$

$$Var(a_X + b_Y) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$$

$$= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$$

Starting with the standard bivariate normal pdf we have :

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right]$$

Let $U = aX + bY \quad V = Y$

Then $X = (1/a)(U - bV) \quad Y = V$

$$J = \begin{bmatrix} 1/a & -b/a \\ 0 & 1 \end{bmatrix} = 1/a$$

$$f_{UV}(u, v) = \frac{1}{2a\pi\sqrt{1-\rho^2}} \exp\left[-\frac{[(1/a)(u - bv)]^2 - 2\rho(1/a)(u - bv)v + v^2}{2(1-\rho^2)}\right]$$

$$= \frac{1}{2a\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{u^2 - 2bvu + b^2v^2 - 2\rho auv + 2\rho abv^2 + a^2v^2}{a^2}\right]}$$

$$= \frac{1}{2a\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{u^2 - 2uv(b+\rho a) + v^2(b^2 + 2\rho ab + a^2)}{a^2}\right]}$$

$$= \frac{1}{2a\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{b^2 + 2\rho ab + a^2}{a^2}\left[\frac{u^2}{b^2 + 2\rho ab + a^2} - 2uv\frac{(b+\rho a)}{b^2 + 2\rho ab + a^2} + v^2\right]\right]}$$

Which is the joint bivariate normal pdf since:

$$\mu_U = \mu_V = 0 \quad \sigma_u^2 = b^2 + 2\rho ab + a^2, \sigma_V^2 = 1$$

$$\rho_{UV} = \frac{Cov(U, V)}{\sigma_U\sigma_V} = E((U - \mu_u)(V - \mu_v)) = E(UV) = E(aXY + bY^2)$$

$$Cov(U, V) = E((U - \mu_u)(V - \mu_v)) = E(UV) = E(aXY + bY^2) = a\rho + b$$

$$\rho_{UV} = \frac{a\rho + b}{\sqrt{a^2 + b^2 + 2ab\rho}}$$

$$1 - \rho_{UV}^2 = 1 - \left(\frac{a\rho + b}{\sqrt{a^2 + b^2 + 2ab\rho}}\right)^2$$

$$= 1 - \frac{a^2\rho^2 + b^2 + 2ab\rho}{a^2 + b^2 + 2ab\rho}$$

$$= \frac{(1 - p^2)a^2}{a^2 + b^2 + 2ab\rho} = \frac{(1 - p^2)a^2}{\sigma_U^2}$$

define $\rho_{UV} = \rho^*$

$$\text{Then } a\sqrt{1-\rho^2} = \sigma_U\sqrt{1-p^{*2}}$$
$$\text{Remembering } \sigma_V^2 = 1 \text{ we have:}$$
$$f_{UV}(u,v) = \frac{1}{2\sigma_U\sigma_V\pi\sqrt{1-\rho^{*2}}} e^{-\frac{1}{2(1-p^{*2})}\left[\frac{u^2}{\sigma_U^2} - 2\rho^* \frac{uv}{\sigma_U\sigma_V} + \frac{v^2}{\sigma_V^2}\right]}$$

Which is the bivarate normal pdf

From part a we know that the marginal distribution of U is $N(\mu_u, \sigma_u^2)$

Which means that the distribution of aX+bY is $N(a\mu_x + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y)$

Based on the mean and variance we calculated at the beginning of the problem

Problem 3

(a)

$$\psi_{X,Y}(t,u) = e^{2t+3u+t^2+atu+2u^2}$$
$$\text{Let } J = X+2Y \quad K = 2X-Y$$
$$M_{J,K}(l,m) = E(e^{lJ+mK}) = E(e^{l(X+2Y)+m(2X-Y)})$$
$$= E(e^{lX+2lY+2mX-mY})$$
$$= E(e^{(l+2m)X+(2l-m)Y})$$
$$= M_{X,Y}(l+2m, 2l-m)$$
$$= e^{2l+4m+8l-3m+(l+2m)^2+a(l+2m)(2l-m)+2(2l-m)^2}$$
$$= e^{8l+m+l^2+4ml+4m^2+8l^2+2m^2-8lm+2al^2+3alm-2am^2}$$
$$= e^{8l+m+9l^2+6m^2-4ml+2al^2+3alm-2am^2}$$
$$M_{X+2Y}(l,0) = e^{8l+9l^2+2al^2}$$
$$M_{2X-Y}(0,m) = e^{m+6m^2-2am^2}$$
$$M_{X+2Y}(l)M_{2X-Y}(m) = e^{8l+9l^2+2al^2}e^{m+6m^2-2am^2}$$
$$= e^{8l+9l^2+2al^2+m+6m^2-2am^2}$$
$$\text{If independent: } M_{X+2Y,2X-Y}(l,m) = M_{X+2Y}(l)M_{2X-Y}(m)$$
$$e^{8l+9l^2+2al^2+m+6m^2-2am^2} = e^{8l+m+9l^2+6m^2-4ml+2al^2+3alm-2am^2}$$
$$0 = 3alm - 4ml$$
$$0 = (3a-4)ml \quad 3a-4=0$$
$$a = 4/3$$

(b)

Let $Z = (2X - Y) - (X + 2Y)$

$$P(X + 2Y < 2X - Y) = P(Z > 0) = P(X - 3Y > 0)$$
$$M_Z(\theta) = M_{X-3Y}(\theta) = M_{X,Y}(\theta, -3\theta)$$
$$= e^{2\theta - 9\theta + \theta^2 + (4/3)(-3\theta^2) + 18\theta^2}$$
$$= e^{-7\theta + 15\theta^2}$$

This is in the form of the mgf of the normal distribution

$$e^{\mu t + (1/2)\sigma^2 t^2}$$

Plugging in $\mu = -7 \quad \sigma^2 = 30$ we have:

$$e^{-7\theta + (1/2)30\theta^2} = e^{-7\theta + 15\theta^2}$$

Thus $Z \sim N(-7, 30)$

$$P(Z > 0) = 1 - pnorm(q = 0, mean = -7, sd = sqrt(30)) = .1006 \quad \text{(using R)}$$

Problem 4

(a)

$X_1, X_2 \sim N(0,1)$ and independent

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{2\pi} e^{-x_1^2/2} e^{-x_2^2/2}$$
$$Y_1 = X_1 - 3X_2 + 2$$
$$Y_2 = 2X_1 - X_2 - 1$$
$$\text{Then } X_1 = (-1/5)Y_1 + (3/5)Y_2 + 1 = g_1(y_1, y_2)$$
$$X_2 = (-2/5)Y_1 + (1/5)Y_2 + 1 = g_2(y_1, y_2)$$
$$J = \begin{bmatrix} -1/5 & 3/5 \\ -2/5 & 1/5 \end{bmatrix} = 1/5$$
$$= f_{X_1X_2}(g_1(y_1, y_2), g_2(y_1, y_2))|J|$$
$$= \frac{1}{2\pi} e^{(-1/2)((-1/5)y_1 + (3/5)y_2 + 1)^2} e^{(-1/2)((-2/5)y_1 + (1/5)y_2 + 1)^2} (1/5)$$
$$= \frac{1}{10\pi} e^{(-1/2)\left[((-1/5)y_1 + (3/5)y_2 + 1)^2 + (-2/5)y_1 + (1/5)y_2 + 1)^2\right]}$$
$$f_{\mathbf{Y}}(y_1, y_2) = \frac{1}{10\pi} e^{(-1/10)[y_1^2 + 2y_1y_2 - 6y_1 + 2y_2^2 + 8y_2 + 10]}$$

(b)

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{\mathbf{Y}}(y_1, y_2)\, dy_1$$

$$= \frac{1}{10\pi} \int_{-\infty}^{\infty} e^{(-1/10)[y_1^2+2y_1y_2-6y_1+2y_2^2+8y_2+10]}\ dy_1$$

$$= \frac{1}{10\pi} \sqrt{10\pi} e^{-(1/10)(y_2+1)^2}$$

$$f_{Y_2}(y_2) = \frac{1}{\sqrt{10\pi}} e^{-(1/10)(y_2+1)^2}$$

$$f_{Y_1|Y_2} = \frac{f_{Y_1Y_2}}{f_{Y_2}}$$

$$= \frac{1}{10\pi} e^{(-1/10)[y_1^2+2y_1y_2-6y_1+2y_2^2+8y_2+10]} / \left[\frac{1}{\sqrt{10\pi}} e^{-(1/10)(y_2+1)^2}\right]$$

$$= \frac{1}{\sqrt{10\pi}} e^{(-1/10)[y_1^2+2y_1y_2-6y_1+2y_2^2+8y_2+10]} e^{(1/10)(y_2+1)^2}$$

$$= \frac{1}{\sqrt{10\pi}} e^{-(1/10)[y_1^2+2y_1y_2-6y_1+2y_2^2+8y_2+10-y_2^2-2y_2-1]}$$

$$f_{Y_1|Y_2} = \frac{1}{\sqrt{10\pi}} e^{-(1/10)[y_1^2+2y_1y_2-6y_1+y_2^2+6y_2+9]}$$

Problem 5

Let $f_x(x) = \frac{1}{\theta}$

$$\text{Then } F_X(x) = \int_0^x \frac{1}{\theta}\ \mathrm{d}x = \frac{x}{\theta} \quad 0 < x < \theta$$

$$\text{Let } Y = X_{(n)},\ Z = X_{(1)}$$

Then using theorem 5.4.6:

$$f_{Z,Y}(z,y) = \frac{n!}{0!(n-2)!0!} \frac{1}{\theta} \frac{1}{\theta} \left(\frac{z}{\theta}\right)^0 \left(\frac{y-z}{\theta}\right)^{n-2} \left(1 - \frac{y}{\theta}\right)^0$$

$$= n(n-1) \frac{1}{\theta^2} \left(\frac{1}{\theta}(y-z)\right)^{n-2}$$

$$= n(n-1) \frac{1}{\theta^2} \frac{1}{\theta^{n-2}} (y-z)^{n-2}$$

$$f_{Z,Y}(z,y) = \frac{n(n-1)}{\theta^n} (y-z)^{n-2} \quad 0 < z < y < \theta$$

$$\text{Let } W = Z/Y \quad Q = Y$$

$$\text{Then } Y = Q \quad Z = WQ$$

$$|J| = \begin{bmatrix} 1 & 0 \\ w & q \end{bmatrix} = q$$

Thus we have:

$$f_{W,Q}(w,q) = \frac{n(n-1)}{\theta^n}(q-wq)^{n-2}(q)$$
$$= \frac{n(n-1)}{\theta^n}(q(1-w))^{n-2}(q)$$
$$= \frac{n(n-1)}{\theta^n}(1-w)^{n-2}q^{n-2}(q)$$
$$f_{W,Q}(w,q) = \frac{n(n-1)}{\theta^n}(1-w)^{n-2}q^{n-1} \quad 0<w<1,\ 0<q<\theta$$
$$f_{W,Q}(w,q) = g(w)h(q) = \left[\frac{n(n-1)}{\theta^n}(1-w)^{n-2}\right]\left[q^{n-1}\right]$$

Since $f_{W,Q}(w,q)$ can be factored into functions of w and q, W and Q are independent

and since $W = \frac{X_{(1)}}{X_{(n)}}, \quad Q = X_{(n)}$

Thus $\frac{X_{(1)}}{X_{(n)}}$ and $X_{(n)}$ are independent random variables

Problem 6

Assume X_1 and X_2 are iid $geom(p)$ random variables

WTS: $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent

Let the PMFs of X_1 and X_2 be:

$$f_X(x) = (1-p)^{x-1}p$$

Then $F_X(x) = 1-(1-p)^x$

Let $Y = X_{(1)} \quad Z = X_{(2)}$

$$f_{Y,Z}(y,z) = \frac{n!}{(1-1)!(2-1-1)!(n-2)!}p(1-p)^{y-1}p(1-p)^{z-1}$$
$$*\,[1-(1-p)^y]^{1-1}[(1-(1-p)^z)-(1-(1-p)^y)]^{2-1-1}[1-(1-(1-p)^z)]^{n-2}$$
$$= \frac{n!}{(n-2)!}p(1-p)^{y-1}p(1-p)^{z-1}[(1-p)^z]^{n-2}$$
$$f_{Y,Z}(y,z) = n(n-1)p^2(1-p)^{z(n-1)+y-2}$$

Let $V = Z-U \quad U = Y$

Then $Z = V+U \quad Y = U$

$$|J| = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

$$f_{U,V}(u,v) = n(n-1)p^2(1-p)^{(v+u)(n-1)+u-2}$$
$$= n(n-1)p^2(1-p)^{vn-v+un-2}$$
$$f_{U,V}(u,v) = g(u)h(v) = [n(n-1)p^2(1-p)^{vn-v}][(1-p)^{un-2}]$$

Since $f_{U,V}(u,v)$ can be factored into functions of u and v, U and V are independent

and since $U = X_{(1)}, \quad V = X_{(2)} - X_{(1)}$

Thus $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent