

**Problem 1**

$$p_K(k) = \begin{cases} \frac{1}{2n+1} & \text{for } k \in \{-n, -n+1, \dots, n\} \\ 0 & \text{otherwise} \end{cases}$$

$$p_{|K|}(|k|) = \begin{cases} \frac{2}{2n+1} & \text{for } |k| \in \{1, \dots, n\} \\ \frac{1}{2n+1} & \text{for } |k| = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X = a^{|K|} \quad a > 0$$

$$Y = \log(X) = |K| \log(a)$$

PMF of Y:

$$p_Y(y) = \begin{cases} \frac{2}{2n+1} & \text{for } y \in \{\log(a), 2\log(a), \dots, n\log(a)\} \\ \frac{1}{2n+1} & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 2**

$$\begin{aligned} \text{(a)} \quad \sum_{x=-3}^3 x^2/a &= 1 \\ \sum_{x=-3}^3 x^2 &= a \\ a &= 2(1^2 + 2^2 + 3^2) = 28 \\ a &= 28 \end{aligned}$$

$$\begin{aligned} E[X] &= \sum_{x=-3}^3 x^3/28 \\ &= (1/28)[(-1)^3 + (-2)^3 + (-3)^3 + 0 + 1^3 + 2^3 + 3^3] = 0 \\ E[X] &= 0 \end{aligned}$$

(b)

$$\begin{aligned} Z &= (X - E[X])^2 \\ &= X^2 - (2X)E[X] + (E[X])^2 \\ &= X^2 - 2X(0) + 0^2 \\ Z &= X^2 \end{aligned}$$

$$p_Z(z) = \begin{cases} (2z)/28 & \text{if } z = 1, 4, 9 \\ z/28 & \text{if } z = 0 \end{cases}$$

(c)

$$\begin{aligned}
E[X^2] &= \sum_{z:P(z)>0} p_Z(z) * z \\
&= (1/14)1 + (4/14)4 + (9/14)9 + (0/28)0 \\
&= 1/14 + 16/14 + 81/14 \\
E[X^2] &= 7 \\
Var(X) &= E[X^2] - (E[X])^2 \\
&= 7 - 0^2 \\
Var(X) &= 7
\end{aligned}$$

(d)

$$\begin{aligned}
Var(x) &= \sum_{x=-3}^3 (x - E[X])^2 p_X(x) \\
&= \sum_{x=-3}^3 (x - 0)^2 p_X(x) \\
&= \sum_{x=-3}^3 x^2 p_X(x) \\
&= [9(9/28) + 4(4/28) + 1(1/28)] * 2 + 0 \\
Var(X) &= 7
\end{aligned}$$

### Problem 3

There are  $a + b - 1$  values:  $\{2^a, a^{a+1}, \dots, 2^b\}$

$$E[X] = \frac{\sum_{k=a}^b 2^k}{b - a + 1}$$

We have a geometric series where  $r=2$ ,  $n=a-b+1$ ,  $a_1 = 2^a$

$$\sum_{k=a}^b 2^k = 2^a \frac{1 - 2^{b-a+1}}{1 - 2} = 2^{b-1} - 2^a$$

$$E[X] = \frac{2^{b+1} - 2^a}{b - a + 1}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \frac{\sum_{k=a}^b (2^2)^k}{b - a + 1}$$

$$E[X^2] = \frac{4^{b+1} - 4^a}{3(b - a + 1)}$$

$$Var(X) = \frac{4^{b+1} - 4^a}{3(b - a + 1)} - \left[ \frac{2^{b-1} - 2^a}{b - a + 1} \right]^2$$

**Problem 4**

Let  $X$  be a random variable where  $X \sim \text{Geom}(p)$

This is the distribution of candy bars eaten till the first success (getting the golden ticket)

$$\begin{aligned}f_X &= pq^{x-1} \\E[X] &= \sum_{x=1}^{\infty} x * pq^{x-1} \\&= \sum_{x=1}^{\infty} p \frac{d}{dq}(q^x) \\&= p \frac{d}{dq} \sum_{x=1}^{\infty} q^x \quad \text{since the series converges uniformly} \\&= p \frac{d}{dq} \left( \frac{1}{1-q} - 1 \right) \\&= p \frac{1}{(1-q)^2} = \frac{1}{p} \\E[X] &= \frac{1}{p}\end{aligned}$$

$$\begin{aligned}
 E[X^2] &= \sum_{x=1}^{\infty} x^2 q^{x-1} p \\
 &= \sum_{x=1}^{\infty} ((x-1) + 1)^2 q^{x-1} p
 \end{aligned}$$

since  $((x-1) + 1)^2 = (x-1)^2 + 2(x-1) + 1$  we have:

$$\begin{aligned}
 &= \sum_{x=1}^{\infty} (x-1)^2 q^{x-1} p + \sum_{x=1}^{\infty} 2(x-1) q^{x-1} p + \sum_{x=1}^{\infty} q^{x-1} p \\
 &= \sum_{x=1}^{\infty} (x-1)^2 q^{x-1} p + [2 \sum_{x=1}^{\infty} (x-1) q^{x-1} p] + 1
 \end{aligned}$$

let  $i = x - 1$

$$= \sum_{i=0}^{\infty} i^2 q^i p + [2 \sum_{i=0}^{\infty} i q^i p] + 1$$

since  $i$  is just a letter we can replace it with  $x$

$$\begin{aligned}
 &= [\sum_{x=1}^{\infty} x^2 q^x p + 0] + [2 \sum_{x=1}^{\infty} x q^x p + 0] + 1 \\
 &= q \sum_{x=1}^{\infty} x^2 q^{x-1} p + [2q \sum_{x=1}^{\infty} x q^{x-1} p] + 1 \\
 &= qE[X^2] + 2qE[X] + 1
 \end{aligned}$$

Since  $E[X] = \frac{1}{p}$

$$E[X^2] = qE[X^2] + (2q)\frac{1}{p} + 1$$

$$(1 - q)E[X^2] = \frac{2q}{p} + 1$$

$$E[X^2] = \frac{2q + p}{p^2} = \frac{q + 1}{p^2}$$

$$Var(X) = \frac{q + 1}{p^2} - \frac{1}{p^2}$$

$$Var(X) = \frac{q}{p^2} = \frac{1 - p}{p^2}$$

**Problem 5**

Let  $X$  be a random variable where  $X \sim \text{Geom}(1/2)$

This is the distribution of coin tosses until the first success (getting a tails)

$$P\{X = n\} = p_X(n) = (1 - p)^{n-1}p = p^n = (1/2)^n$$

$$\text{Payout} = Y = 2^n$$

$$P\{Y = n\} = P\{X = n\}$$

Thus  $Y$  has the same PMF as  $X$

$E[Y]$  is the expected payout

$$E[Y] = \sum_{n=1}^{\infty} y p_Y(n)$$

$$E[Y] = \sum_{n=1}^{\infty} 2^n (1/2)^n = \sum_{n=1}^{\infty} 1 = \infty$$

$$E[Y] = \infty$$

## Problem 6

(a)

$$\begin{aligned}
 Y &= X^2 & X &= \sqrt{Y} \\
 f_X(x) &= 1 & \text{for } 0 < x < 1 \\
 f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right| & y \in \mathcal{Y} \\
 f_Y(y) &= (1)|(1/2)y^{-1/2}| \\
 f_Y(y) &= \frac{1}{2\sqrt{y}} & 0 < y < 1
 \end{aligned}$$

(b)

$$\begin{aligned}
 f_X(x) &= \frac{(n+m+1)!}{n!m!} x^n (1-x)^m & 0 < x < 1 \\
 Y &= g(X) = -\log(X) \\
 x &= e^{-y} & g^{-1}(y) = e^{-y} \\
 -\log(0) &= \infty & -\log(1) = 0 \\
 0 &< y < \infty \\
 f_Y(y) &= \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases} \\
 f_Y(y) &= \begin{cases} f_X(e^{-y}) \left| \frac{d}{dy}(e^{-y}) \right| & 0 < y < \infty \\ 0 & \text{otherwise} \end{cases} \\
 f_Y(y) &= \frac{(n+m+1)!}{n!m!} e^{-ny} (1-e^{-y})^m (e^{-y}) \\
 f_Y(y) &= \frac{(n+m+1)!}{n!m!} e^{-y(n+1)} (1-e^{-y})^m & 0 < y < \infty
 \end{aligned}$$

(c)

$$\begin{aligned}
Y &= g(X) = e^x \\
x &= \log(y) \quad g^{-1}(y) = \log(y) \\
f_X(x) &= \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2} \quad 0 < x < \infty \\
e^0 &= 1 \quad e^\infty = \infty \\
1 &< y < \infty \\
f_Y(y) &= \begin{cases} f_X(\log(y)) \left| \frac{d}{dy}(\log(y)) \right| & 1 < y < \infty \\ 0 & \text{otherwise} \end{cases} \\
f_Y(y) &= \frac{1}{\sigma^2} \log(y) e^{-(\log(y)/\sigma)^2/2} (1/y) \quad 1 < y < \infty
\end{aligned}$$

## Problem 7

(a) Using theorem 2.1.8 we have:

$$\begin{aligned}
&A_0, A_1, A_2 \text{ of } \chi \\
&A_0 = \{0\} \quad A_1 = (-\infty, 0) \quad A_2 = (0, \infty) \\
&P(X \in A_0) = 0 \\
&g_1(x) = |x|^3 = -x^3 \text{ on } A_1 \quad g_2(x) = |x|^3 = x^3 \text{ on } A_2 \\
&f_X(x) = (1/2)e^{-|x|} \quad -\infty < x < \infty \\
&Y = g_i(X) \\
&g_1^{-1}(X) = -Y^{1/3} \\
&g_2^{-1}(X) = Y^{1/3} \\
&f_Y(y) = \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy}(g_i^{-1}(y)) \right| \quad y \in \mathcal{Y} \\
&f_Y(y) = 2 * f_X(y^{1/3}) \left| \frac{d}{dy}(y^{1/3}) \right| \quad 0 < y < \infty \\
&f_Y(y) = 2 * (1/2)e^{-y^{1/3}} ((1/3)y^{-2/3}) \quad 0 < y < \infty \\
&f_Y(y) = (1/3)e^{-y^{1/3}} (y^{-2/3}) \quad 0 < y < \infty
\end{aligned}$$

(b)

 $A_0, A_1, A_2$  of  $\chi$ 

$$A_0 = \{0\} \quad A_1 = (-1, 0) \quad A_2 = (0, 1)$$

$$P(X \in A_0) = 0$$

$$Y = g_i(X)$$

$$g_1(x) = 1 - X^2 \text{ on } A_1$$

$$g_1^{-1}(x) = -\sqrt{1 - Y}$$

$$g_2(x) = 1 - X^2 \text{ on } A_2$$

$$g_2^{-1}(x) = \sqrt{1 - Y}$$

$$f_X(x) = (3/8)(x+1)^2 \quad -1 < x < 1$$

$$f_Y(y) = \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy}(g_i^{-1}(y)) \right| \quad y \in \mathcal{Y}$$

$$f_Y(y) = (3/8)(\sqrt{1-y}+1)^2(1/2)(1-y)^{-1/2} \\ + (3/8)(-\sqrt{1-y}+1)^2(1/2)(1-y)^{-1/2}$$

$$f_Y(y) = (3/8)[(1-y)^{1/2} + (1-y)^{-1/2}] \quad 0 < y < 1$$

(c)

 $A_0, A_1, A_2$  of  $\chi$ 

$$A_0 = \{0\} \quad A_1 = (-1, 0) \quad A_2 = (0, 1)$$

$$f_X(x) = (3/8)(x+1)^2 \quad -1 < x < 1$$

$$Y = 1 - X^2 \text{ if } X \leq 0 \quad Y = 1 - X \text{ if } X > 0$$

$$Y = g_i(X)$$

$$g_1(x) = 1 - X^2 \text{ on } A_1$$

$$g_1^{-1}(x) = -\sqrt{1 - Y}$$

$$g_2(x) = 1 - X \text{ on } A_2$$

$$g_2^{-1}(x) = 1 - Y$$

$$f_Y(y) = \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy}(g_i^{-1}(y)) \right| \quad y \in \mathcal{Y}$$

$$f_Y(y) = (3/8)(1-y+1)^2(1) \\ + (3/8)(-\sqrt{1-y}+1)^2(1/2)(1-y)^{-1/2}$$

$$f_Y(y) = (3/8)(2-y)^2 + (3/16)(1-(1-y)^{1/2})^2(1-y)^{-1/2} \quad 0 < y < 1$$



**Problem 8**

Define  $Y = F_X(X)$ , which means  $Y$  is uniformly distributed on  $(0, 1)$

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$
$$\int_1^x \frac{x-1}{2} = (1/4)(x-1)^2$$
$$u(x) = F_X(x) = \begin{cases} 0 & x \leq 1 \\ (1/4)(x-1)^2 & 1 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

**Problem 9**

(a)

**Problem 10**

(a)

$$\begin{aligned} \int_0^\infty (1 - F_X(x))dx &= \int_0^\infty P(X > x)dx \\ \text{Since } P(X > x) &= \int_x^\infty f_X(y) dy \\ \int_0^\infty P(X > x)dx &= \int_0^\infty \int_x^\infty f_X(y) dydx \\ &= \int_0^\infty \int_0^y dx f_X(y)dy \\ &= \int_0^\infty (y - 0)f_X(y)dy \\ &= \int_0^\infty yf_X(y)dy \\ &= E[X] \end{aligned}$$

(b)

$$\text{Since } 1 - F_X(k) = \sum_{k+1}^{\infty} f_X(k)$$

$$\begin{aligned}\text{Then } \sum_{k=0}^{\infty} (1 - F_X(k)) &= \sum_{k=0}^{\infty} \left[ \sum_{k+1}^{\infty} f_X(k) \right] \\ &= \left( f_x(1) + f_x(2) + \dots \right) + \left( f_x(2) + f_x(3) + \dots \right) + \left( \dots \right) \\ &= 0 + 1f_x(1) + 2f_x(2) + 3f_x(3) + \dots \\ &= \sum_{k=0}^{\infty} k f_x(k) = E[X]\end{aligned}$$

## 1 Problem 11

(a)

$$\begin{aligned}\int_0^m 3x^2 &= \int_m^1 3x^2 = 1/2 \\ \left. x^3 \right|_0^m &= \left. x^3 \right|_m^1 = 1/2 \\ m^3 &= 1 - m^3 = 1/2 \\ m &= 2^{-1/3} = 0.7937005\end{aligned}$$

(b)

$$\begin{aligned}\int_{-\infty}^m \frac{1}{\pi(1+x^2)} &= \int_m^{\infty} \frac{1}{\pi(1+x^2)} = 1/2 \\ (1/\pi) \left. \arctan(x) \right|_{-\infty}^m &= (1/\pi) \left. \arctan(x) \right|_m^{\infty} = 1/2 \\ (1/\pi)(\arctan(m) + \pi/2) &= (1/\pi)(\pi/2 - \arctan(m)) = 1/2 \\ \arctan(m) &= -\arctan(m) = 0 \\ m &= \tan(0) = -\tan(0) = 0 \\ m &= 0\end{aligned}$$

## Problem 12

WTS: If  $X$  is a continuous random variable:  $\min_a E|X - a| = E|X - m|$

$$\text{We know } E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$$

$$\begin{aligned} E|X - a| &= \int_{-\infty}^{\infty} |x - a|f(x)dx \\ &= \int_{-\infty}^a -(x - a)f(x)dx + \int_a^{\infty} (x - a)f(x)dx \end{aligned}$$

Setting the first derivative equal to 0 we have:

$$\frac{d}{da} E|X - a| = \int_{-\infty}^a f(x)dx - \int_a^{\infty} f(x)dx = 0$$

$$\text{The solution is: } \int_{-\infty}^a f(x)dx = \int_a^{\infty} f(x)dx = 1/2$$

Which means  $a=m$  is the median by definition

taking the second derivative we have:

$$\begin{aligned} \frac{d}{da} \left[ \int_{-\infty}^a f(x)dx - \int_a^{\infty} f(x)dx \right] \\ &= \frac{d}{da} \left[ \left. F(X) \right|_{-\infty}^a - \left. F(X) \right|_a^{\infty} \right] \\ &= \frac{d}{da} [F(a) - F(-\infty) - F(\infty) + F(a)] \\ &= 2f(a) > 0 \end{aligned}$$

Thus  $E|X - m|$  is a minimum by the second derivative test

$$\text{Therefore } \min_a E|X - a| = E|X - m|$$

## Problem 13

(a) The uniform distribution pdf  $U(0, 1)$  is symmetric about  $a = 1/2$

Standard Normal pdf is symmetric about  $a = 0$

The binomial distribution  $Bin(10, .5)$  pdf is symmetric about  $a = 5$

(b)

Let  $X \sim f(x)$  be symmetric at point  $a$ That is  $\forall \epsilon > 0, f(a + \epsilon) = f(a - \epsilon)$ WTS  $a$  is the median:

$$\int_{-\infty}^a f(x)dx = \int_a^{\infty} f(x)dx = 1/2$$

**Proof:** Let  $\epsilon = x - a$  Then:

$$\begin{aligned} \int_a^{\infty} f(x)dx &= \int_0^{\infty} f(a + \epsilon)d\epsilon \\ &= \int_0^{\infty} f(a - \epsilon)d\epsilon = \int_{-\infty}^a f(x)dx \quad (\text{letting } x = a - \epsilon) \end{aligned}$$

$$\text{Thus } \int_a^{\infty} f(x)dx = \int_{-\infty}^a f(x)dx$$

$$\text{We know } \int_{-\infty}^a f(x)dx + \int_a^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1$$

$$\text{Therefore } \int_{-\infty}^a f(x)dx = \int_a^{\infty} f(x)dx = 1/2$$

Thus  $a = \text{median}$ 

(c)

Given  $X \sim f(x)$  and symmetric and  $E[X]$  existsShow  $E[X] = a$ 

$$E[X] - a = E(X - a)$$

$$= \int_{-\infty}^{\infty} (x - a)f(x)dx$$

From part b we know  $a$  is the median therefore:

$$= \int_{-\infty}^a (x - a)f(x)dx + \int_a^{\infty} (x - a)f(x)dx$$