Problem 1

(a) $P(\text{Doubles}) = \frac{1}{6} * \frac{1}{6} * 6 = \frac{1}{6}$

(b) $P(D) = \text{Doubles} = \frac{1}{6} \quad P(A) = \text{Sum of 4 or less} = \frac{6}{36} = \frac{1}{6}$

$$P(D|A) = \frac{P(A|D)P(D)}{P(A)}$$

$$P(D|A) = \frac{(1/3) * (1/6)}{(1/6)} = \frac{1}{3}$$

(c) $P(\text{at least one 6}) = 1 - P(\text{no 6's}) = 1 - \frac{5}{6} * \frac{5}{6} = 1 - \frac{25}{36} = \frac{11}{36}$

(d) $P(B) = \text{Different numbers} = 1 - P(\text{Doubles}) = \frac{5}{6}$

$P(A) = \text{At least one six} = \frac{11}{36}$

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)} = \frac{(10/11) * (11/36)}{(5/6)} = \frac{1}{3}$$

Problem 2

$$\begin{aligned}
P(W_1) &= \frac{m}{m+n} \\
P(W_k) &= P(W_{k-1})P(W_k|W_{k-1}) + P({W_{k-1}}^c)P(W_k|{W_{k-1}}^c) \\
&= \frac{m}{m+n}\frac{m+1}{m+n+1} + \frac{n}{m+n}\frac{m}{m+n+1} \\
&= \frac{m(m+1)+nm}{(m+n)(m+n+1)} \\
&= \frac{m(m+n+1)}{(m+n)(m+n+1)} \\
&= \frac{m}{m+n} \\
&\text{Thus } P(W_1) = P(W_k)
\end{aligned}$$

Problem 3

We want to show:

$$P(A|B) = P(C|B)P(A|B \cap C) + P(C^c|B)P(A|B \cap C^c)$$

Using the definition of conditional probability:

$$P(C|B) = \frac{P(C \cap B)}{P(B)}$$

$$P(A|B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$

$$P(C^c|B) = \frac{P(C^c \cap B)}{P(B)}$$

$$P(A|B \cap C^c) = \frac{P(A \cap B \cap C^c)}{P(B \cap C^c)}$$

Putting this together we can write

$P(C|B)P(A|B \cap C) + P(C^c|B)P(A|B \cap C^c)$ as:

$$\frac{P(C \cap B)}{P(B)} \frac{P(A \cap B \cap C)}{P(B \cap C)} + \frac{P(C^c \cap B)}{P(B)} \frac{P(A \cap B \cap C^c)}{P(B \cap C^c)}$$

$$= \frac{P(A \cap B \cap C)}{P(B)} + \frac{P(A \cap B \cap C^c)}{P(B)}$$

$$= \frac{P(A \cap B)}{P(B)}$$

$$= P(A|B)$$

Problem 4

(a)

We want to prove: $P(A \cap B^c) = P(A)P(B^c)$

Given $P(A \cap B) = P(A) + P(B)$

We can write A as:

$$A = (A \cap B) \cup (A \cap B^c)$$

Since the union is disjoint we can write:

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

$$P(A) = P(A)P(B) + P(A \cap B^c)$$

$$P(A \cap B^c) = P(A) - P(A)P(B)$$

$$P(A \cap B^c) = P(A)(1 - P(B))$$

$$P(A \cap B^c) = P(A)P(B^c)$$

Therefore A and B^c are independent

(b)

We want to prove: $P(A^c \cap B^c) = P(A^c)P(B^c)$

We can write B^c as:

$$B^c = (B^c \cap A) \cup (B^c \cap A^c)$$

Since the union is disjoint we can write:

$$P(B^c) = P(B^c \cap A) + P(B^c \cap A^c)$$

Using part a we can write:

$$P(B^c) = P(A)P(B^c) + P(B^c \cap A^c)$$
$$P(B^c \cap A^c) = P(B^c) - P(A)P(B^c)$$
$$P(B^c \cap A^c) = P(B^c)(1 - P(A))$$
$$P(B^c \cap A^c) = P(B^c)P(A^c)$$

Therefore A^c and B^c are independent

Problem 5

Game 1: P(win)=P(tie)=(.4)(.5)
P(loss)=(.6)

Game 2: P(win)=P(tie)=(.7)(.5)
P(loss)=(.3)

PMF: $f(y) = P\{Y(\omega) = y\}$
$P(0) = (.6)(.3) = .18$
$P(1) = (.4)(.5)(.3) + (.6)(.7)(.5) = .27$
$P(2) = (.4)(.5)(.7)(.5) + (.4)(.5)(.3) + (.6)(.7)(.5) = .34$
$P(3) = (.4)(.5)(.7)(.5)(2) = .14$
$P(4) = (.4)(.5)(.7)(.5) = .07$

Problem 6

(a) P(Harry Wins)= $\sum_1^{10}(.3)^n$

$= (.3) + (.3)^2 + .(3)^3 + (.3)^4 + (.3)^5 + .(3)^6 + (.3)^7 + (.3)^8 + .(3)^9 + (.3)^{10}$
≈ 0.4285689

There is a .3 probability Harry wins the first game plus the probability the first game is drawn times .3 probability he wins the second and so on all the way to 10 games.

(b) $P(1) = .4 + .3$
$P(2) = (.3)(.4) + (.3)(.3)$
$P(3) = [\prod_{i=1}^{2} .3^i](.4 + .3)$
$P(4) = [\prod_{i=1}^{3} .3^i](.4 + .3)$
$P(5) = [\prod_{i=1}^{4} .3^i](.4 + .3)$
$P(6) = [\prod_{i=1}^{5} .3^i](.4 + .3)$
$P(7) = [\prod_{i=1}^{6} .3^i](.4 + .3)$
$P(8) = [\prod_{i=1}^{7} .3^i](.4 + .3)$
$P(9) = [\prod_{i=1}^{8} .3^i](.4 + .3)$
$P(10) = [\prod_{i=1}^{9} .3^i](.4 + .3) + .3^{10}$

Problem 7

$P(M) = P(W) = .5$
$P(M \cap C) = .05$
$P(W \cap C) = .0025$
$P(C|M) = \dfrac{P(C \cap M)}{P(M)} = .05/.5 = .1$
$P(C|W) = \dfrac{P(C \cap W)}{P(W)} = .0025/.5 = .005$
$P(C) = P(M)P(C|M) + P(W)P(C|W) = (.5)(.1) + (.5)(.005) = .0525$

$$P(M|C) = \frac{P(C|M)P(M)}{P(C)} = (.1)(.5)/(.0525) = .952381$$

Problem 8

(a) $P(H) = 1/5$
$P(H \geq 2) = 1 - [P(H = 0) + P(H = 1)]$
$P(H = 0) = (4/5)^{10}$
$P(H = 1) = (4/5)^9(1/5)(10)$
$P(H \geq 2) = 1 - [(4/5)^{10} + (4/5)^9(1/5)(10)] \approx .6241904$

(b) $P(H \geq 1) = 1 - P(H = 0) \approx .8926258$

$$P(H \geq 2|H \geq 1) = \frac{P(H \geq 2 \cap H \geq 1)}{P(H \geq 1)}$$

$$= \frac{P(H \geq 2)}{P(H \geq 1)} \approx .6992744$$

Problem 9