Problem 1

$$A_n = \left\{(x,y)|(x-(-1)^n/n)^2+(y)^2<1\right\}$$
$$\liminf A_n = \left\{(x,y)|x^2+y^2<1\right\}$$

lim inf is contained by the unit circle

$$\limsup A_n = \left\{(x,y)|x^2+y^2\leq 1\right\} - \{(0,1),(0,-1)\}$$

lim sup is equal to the unit circle except for the points $(0,1)$ and $(0,-1)$

Figure 1: The center of the circle that contains A_n

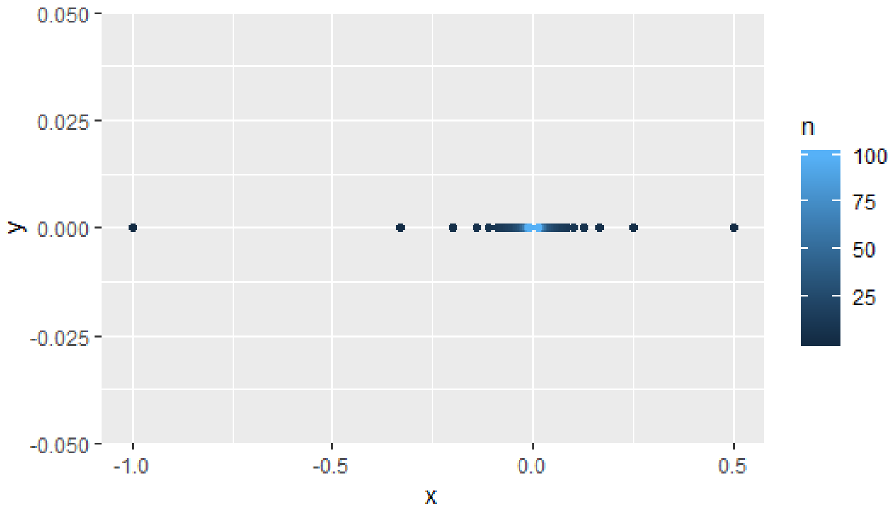
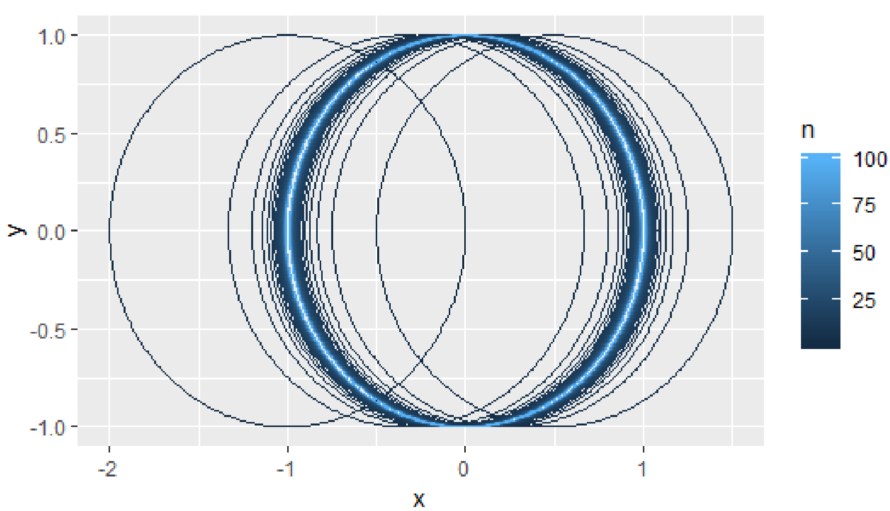


Figure 2: The circle that contains A_n



Problem 2

Let $\mathcal{F} = \{A : A \text{ is countable or } A^c \text{ is countable}\}$

We know that $\emptyset$ is countable

Therefore $\emptyset \in \mathcal{F}$

Thus $\mathcal{F}$ is nonempty

Let $B_n =$ All A_n that is countable

Then $B_n \in \mathcal{F}$ and all ${B_n}^c$ have countable complements

Which means all ${B_n}^c \in \mathcal{F}$

Therefore $\mathcal{F}$ is closed under complementation

We have two cases:

Case 1: All A_n is countable

Then $\bigcup\limits_{A_n}$ is countable

Therefore $\bigcup\limits_{A_n} \in \mathcal{F}$

Thus $\mathcal{F}$ is closed under countable union

Case 2: At least one A_n is not countable

$$\text{Then } \bigcup_{A_n} \in \mathcal{F} \iff \left(\bigcup_{A_n}\right)^c \in \mathcal{F} \tag{1}$$

Then by DeMorgan's Rule $\left(\bigcup\limits_{A_n}\right)^c = \bigcap\limits_{{A_n}^c}$

$\bigcap\limits_{{A_n}^c}$ is countable because at least one ${A_n}^c$ is countable

Thus $\bigcap\limits_{{A_n}^c} \in \mathcal{F}$

By (1) and DeMorgan's Rule, $\bigcup\limits_{A_n} \in \mathcal{F}$

Therefore $\mathcal{F}$ is closed under countable union

Conclude $\mathcal{F}$ is a σ field

Problem 3

Suppose χ_1, χ_2 are σ fields

Let $A \in \chi_1 \cap \chi_2$

Then $\chi_1 \cap \chi_2$ is nonempty and

$A \in \chi_1$ and χ_2

Which means $A^c \in \chi_1$ and χ_2

Thus $A^c \in \chi_1 \cap \chi_2$

Which means $\chi_1 \cap \chi_2$ is closed under complementation

Also if $A_1, A_2, \ldots \in \chi_1 \cap \chi_2$

Then $A_1, A_2, \ldots \in \chi_1, \chi_2$

Which means $\displaystyle\bigcup_{i=1}^{\infty} A_i \in \chi_1, \chi_2$

Thus $\displaystyle\bigcup_{i=1}^{\infty} A_i \in \chi_1 \cap \chi_2$

Therefore $\chi_1 \cap \chi_2$ is closed under countable unions

Conclude $\chi_1 \cap \chi_2$ is a σ field

Problem 4

Suppose G is a collection of σ fields

Let $G = \chi_1, \chi_2, \ldots, \chi_n$ where each χ is a σ field

We know from problem 3 if:

χ_1, χ_2 are σ fields then $\chi_1 \cap \chi_2$ is a σ field

Therefore $\forall\ \chi_i \cap \chi_j$ where $\chi_i, \chi_j \in G$

Then each of these $\chi_i \cap \chi_j$ is also a σ field

and thus any of these $\chi_i \cap \chi_j$ interesected

with another sigma field in G is also a σ field

Therefore $\displaystyle\bigcap_{\chi \in G} \chi$ is also a σ field

Problem 5

$$\begin{aligned}
\text{Suppose } \chi_1 = \{\emptyset, A, A^c, \Omega\} &\text{ and } \chi_2 = \{\emptyset, B, B^c, \Omega\} \\
\text{and } \chi_1, \chi_2 &\text{ are } \sigma \text{ fields} \\
\text{Then } \chi_1 \cup \chi_2 &= \{\emptyset, A, A^c, B, B^c, \Omega\} \\
\text{Since } A \cup B^c &\notin \chi_1 \cup \chi_2 \\
\chi_1 \cup \chi_2 &\text{ is not closed under countable unions} \\
\text{Therefore } \chi_1 \cup \chi_2 &\text{ is not a } \sigma \text{ field}
\end{aligned}$$

Problem 6

Dr. Cai's increasing sequence of sets proof from class

We want to prove if $\{E_n\}$ is an increasing sequence of sets and $E_n \in \mathcal{A}$ then

$$P(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} P(E_n)$$

Proof: Let $F_1 = E_1$ and let $F_j = E_j - E_{j-1}$ for $j > 1$

$$\begin{aligned}
\text{Then } \lim_{n\to\infty} E_n &= \bigcup_{n=1}^{\infty} F_n \text{ and} \\
P(\lim_{n\to\infty} E_n) &= P\left(\bigcup_{n=1}^{\infty} F_n\right) \\
&= \sum_{n=1}^{\infty} P(F_n) \text{ by countable additivity} \\
&= \sum_{n=1}^{\infty} [P(E_n) - P(E_n - 1)] \\
&= \lim_{n\to\infty} P(E_n)
\end{aligned}$$

Therefore we have proven $P(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} P(E_n)$

Problem 6 Proof

$$\text{We want to prove } P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

$$\text{Let } \{A_n\} \text{ be a decreasing sequence of sets}$$

$$\text{Which means } A_1 \supset A_2 \supset \cdots \supset A_n$$

$$\text{Thus } {A_1}^c \subset {A_2}^c \subset \cdots \subset {A_n}^c$$

$$\text{Therefore } \{{A_n}^c\} \text{ is an increasing sequence of sets}$$

$$\text{This means } \lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

$$\text{and } P\left(\lim_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} A_n\right) \tag{1}$$

$$\text{Also } \lim_{n\to\infty} {A_n}^c = \bigcup_{n=1}^{\infty} {A_n}^c$$

$$\text{and } P\left(\lim_{n\to\infty} {A_n}^c\right) = P\left(\bigcup_{n=1}^{\infty} {A_n}^c\right) \tag{2}$$

$$\text{From the proof of the increasing sequence of sets from class,}$$

$$\text{we know that } (2) = \lim_{n\to\infty} P({A_n}^c)$$

$$\text{Using DeMorgan's Law } \bigcup_{n=1}^{\infty} {A_n}^c = \left(\bigcap_{n=1}^{\infty} A_n\right)^c$$

Substituting this into the right side of (2) we get

$$P\left(\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right) = \lim_{n\to\infty} P({A_n}^c)$$

$$1 - P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1 - \lim_{n\to\infty} P(A_n)$$

$$\text{Using algebra we obtain } P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n) \tag{3}$$

$$\text{Using (1) and (3) we can conclude}$$

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

Problem 7

$$\begin{aligned}
P(E \cup F \cup G) &= P[(E \cup F) \cup G] \\
&= P(E \cup F) + P(G) - P[(E \cup F) \cap G] \\
&= P(E \cup F) + P(G) - P([E \cap G] \cup [F \cap G]) \\
&= P(E) + P(F) - P(E \cap F) + P(G) - P([E \cap G] \cup [F \cap G]) \\
&= P(E) + P(F) - P(E \cap F) + P(G) - P(E \cap G) - P(F \cap G) + P([E \cap G] \cap [F \cap G]) \\
&= P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)
\end{aligned}$$

Problem 8

a

$$P(A \cup B) = P(A) + P(B) \quad \text{where } A \in \mathcal{B} \text{ and } B \in \mathcal{B} \text{ are disjoint (Finite additivity)} \tag{1}$$

$$\text{If } A_1, A_2, \cdots \in B \text{ are pairwise disjoint, then } P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \text{(Countable additivity)} \tag{2}$$

$$\text{We want to prove (2)} \implies \text{(1)}$$

$$\text{Let } A \text{ and } B \text{ be disjoint sets}$$

$$\text{Let } A_1 = A, A_2 = B, \text{and } A_i = \emptyset \text{ for } i > 2$$

$$\text{Since } A_i \cap A_j = \emptyset \quad \forall\ i \neq j$$

$$\text{Then by (2)} \quad P(A \cup B) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\text{Since } P(\emptyset) = 0, \quad \sum_{i=1}^{\infty} P(A_i) = P(A) + P(B)$$

$$\text{Therefore we have proven (2)} \implies \text{(1)}$$

b

$$\lim_{n\to\infty} A_n = \emptyset \implies \lim_{n\to\infty} P(A_n) = 0 \quad \text{(Continuity)} \tag{3}$$

We want to prove (1) and (3) $\implies$ (2)

Let $A_1, A_2, \ldots$ be pairwise disjoint

$$\begin{aligned}
\text{Then } P\left(\bigcup_{i=1}^{\infty} A_i\right) &= P\left(\bigcup_{i=1}^{n} A_i \cup \bigcup_{i=n+1}^{\infty} A_i\right) \\
&= P\left(\bigcup_{i=1}^{n} A_i\right) + P\left(\bigcup_{i=n+1}^{\infty} A_i\right) \quad \text{since all } A_i \text{ is disjoint} \\
&= \sum_{i=1}^{n} P(A_i) + P\left(\bigcup_{i=n+1}^{\infty} A_i\right) \quad \text{using finite additivity}
\end{aligned} \tag{4}$$

$$\text{Let } B_k = \bigcup_{i=k}^{\infty} A_i$$

Then $B_k \supset B_{k+1} \quad \forall\ k$

That is $\lim_{k\to\infty} B_k = \emptyset$

because $\bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_n = \emptyset$ since all A_i is disjoint

Thus $\limsup A_n = \emptyset$

Since $\liminf A_n \subset \limsup A_n$

because $\limsup A_n = \emptyset$ and the only thing $\emptyset$ can contain is $\emptyset$

Which means $\liminf A_n = \emptyset$

Thus $\limsup A_n = \liminf A_n = \emptyset$

Therefore $\lim_{n\to\infty} A_n = \emptyset$

That is $\lim_{k\to\infty} B_k = \emptyset$

Then from (3) we have $\lim_{k\to\infty} P(B_k) = 0$

$$\begin{aligned}
\text{Therefore } P\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{\infty} A_i\right) \\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{n} P(A_i) + P\left(\bigcup_{i=n+1}^{\infty} A_i\right)\right] \quad \text{from (4)}
\end{aligned}$$

Since $\bigcup_{i=n+1}^{\infty} A_i = B_{n+1}$ we can write the above equation as

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty}\left[\sum_{i=1}^{n} P(A_i) + P(B_{n+1})\right]$$

Since $\lim_{n\to\infty} P(B_{n+1}) = 0$

$$\text{We have } P\left(\bigcup_{i=1}^{\infty} A_1\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\text{We have shown } P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \text{where } A_i \cap A_j = \emptyset \quad \forall\ i \neq j$$

Therefore we have proven (1) and (3) $\implies$ (2)

Problem 9

a

$$\begin{aligned}
&\text{If } P(E) = .9 \text{ and } P(F) = .8 \\
&\text{Then by the inclusion exclusion identity:} \\
&P(E \cup F) = P(E) + P(F) - P(E \cap F) \\
&\text{Then } P(E \cup F) = .9 + .8 - P(E \cap F) \\
&\text{Then } P(E \cup F) = 1.7 - P(E \cap F) \\
&\text{Rearranging we get } P(E \cap F) = 1.7 - P(E \cup F) \\
&\text{Since } P(E \cup F) \leq 1 \\
&\text{and } 1.7 - 1 = .7 \\
&P(E \cap F) \geq .7
\end{aligned}$$

b

$$\begin{aligned}
&\text{We want to show } P(E \cap F) \geq P(E) + P(F) - 1 \\
&\text{By the inclusion exclusion identity:} \\
&P(E \cup F) = P(E) + P(F) - P(E \cap F) \\
&\text{Rearranging we get } P(E \cap F) = P(E) + P(F) - P(E \cup F) \\
&\text{Since } P(E \cup F) \leq 1 \\
&P(E) + P(F) - P(E \cup F) \geq P(E) + P(F) - 1 \\
&\text{Therefore } P(E \cap F) \geq P(E) + P(F) - 1
\end{aligned}$$

Problem 10

Proof by induction:

$\forall\ n \in \mathbb{N}$ let $P(n)$ be:

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) \geq P(E_1) + P(E_2) + \cdots + P(E_n) - (n-1)$$

Basis Step: $P(1) = P(E_1) \geq P(E_1) - (1-1)$

Thus $P(1)$ is true

Inductive Step: Let $k \in \mathbb{N}$ and assume $P(k)$ is true:

$$P(E_1 \cap E_2 \cap \cdots \cap E_k) \geq P(E_1) + P(E_2) + \cdots + P(E_k) - (k-1) \tag{1}$$

We will prove $P(k+1)$ is true:

$$P(E_1 \cap E_2 \cap \cdots \cap E_{k+1}) \geq P(E_1) + P(E_2) + \cdots + P(E_{k+1}) - ((k+1)-1)$$

$$P(E_1 \cap E_2 \cap \cdots \cap E_{k+1}) \geq P(E_1) + P(E_2) + \cdots + P(E_{k+1}) - k \tag{2}$$

We can rewrite the left side using associativity of intersections

$$\begin{aligned} P(E_1 \cap E_2 \cap \cdots \cap E_{k+1}) &= P(E_1 \cap E_2 \cap \cdots \cap E_{k-1} \cap (E_k \cap E_{k+1})) \\ &\geq P(E_1) + \cdots + P(E_{k-1}) + P(E_k \cap E_{k+1}) - (k-1) \end{aligned} \tag{3}$$

We know from problem 9 that $P(E_k \cap E_{k+1}) \geq P(E_k) + P(E_{k+1}) - 1$

So we can write (3) as

$$\begin{aligned} P(E_1 \cap E_2 \cap \cdots \cap E_{k+1}) &\geq P(E_1) + \cdots + P(E_{k-1}) + P(E_k) + P(E_{k+1}) - 1 - (k-1) \\ &\geq P(E_1) + \cdots + P(E_k) + P(E_{k+1}) - k \end{aligned}$$

Which is the same as (2)

Hence the inductive step has been established

and by PMI we have proven that:

$\forall\ n \in \mathbb{N}$

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) \geq P(E_1) + P(E_2) + \cdots + P(E_n) - (n-1)$$

Problem 11

a $\dfrac{3}{5}$

b $\dfrac{3}{5} * \dfrac{2}{5} + \dfrac{2}{5} * \dfrac{3}{4} = \dfrac{3}{5}$

c $\dfrac{3}{5} * \dfrac{2}{4} = \dfrac{3}{10}$