# Notes 14

## Convolution

If X and Y are independent continuous r.v.s with pdfs $f_X(x)$ and $f_Y(y)$, then the pdf of $Z = X + Y$ is:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\, dw$$

## Sum of Two Independent Poissons

$X \sim Pois(\lambda_1),\ Y \sim Pois(\lambda_2)$
$U = X + Y\ V = Y$
$X = U - V\ Y = V$
Joint PMF of U and V is:
$f_{U,V}(u,v) = f_{X,Y}(u-v, v) = \frac{e^{-\lambda_1}\lambda_1^{u-v}}{(u-v)!}\frac{e^{-\lambda_2}\lambda_2^{v}}{v!}$
The distribution of $U = X + Y$ is the marginal:
$f_U(u) = \sum_{v=0}^{u} \frac{e^{-\lambda_1}\lambda_1^{u-v}}{(u-v)!}\frac{e^{-\lambda_2}\lambda_2^{v}}{v!}$
$= \frac{e^{-(\lambda_1+\lambda_2)}}{u!} \sum_{v=0}^{u} \binom{u}{v} \lambda_1^{u-v}\lambda_2^{v}$
Because of the binomial theorem
$= \frac{e^{-(\lambda_1+\lambda_2)}}{u!}(\lambda_1+\lambda_2)^u$
$U \sim Pois(\lambda_1 + \lambda_2)$

## Jacobian

$J(u,v)$ is the Jacobian of the transformation $(x,y) \to (u,v)$ given by:

$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

## Functions of Independent Random Variables

Let X and Y be independent r.v.s
Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be functions
Then the r.v.s $U = g(X)$ and $V = h(Y)$ are independent

## Ratio of Two Independent Normals

Let $X \sim N(0,1)$ and $Y \sim N(0,1)$
The ratio $X/Y$ has the Cauchy distribution

Let $U = X/Y$ and $V = Y$ Then $X = UV$ and $Y = V$ $J(u,v) = v$
$f_{X,Y}(x,y) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}\dfrac{1}{\sqrt{2\pi}}e^{-y^2/2} = \dfrac{1}{2\pi}e^{-(x^2+y^2)/2}$
$f_{U,V}(uv,v) = \dfrac{1}{2\pi}e^{-[(uv)^2+v^2]/2} * |v| = \dfrac{|v|}{2\pi}e^{-(u^2+1)v^2/2}$
$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\ dv = 2\int_0^{\infty} \dfrac{v}{2\pi}e^{-(u^2+1)v^2/2}\ dv$
$= \dfrac{1}{\pi}\int_0^{\infty} e^{-(u^2+1)z}\ dz = \dfrac{1}{\pi(u^2+1)}$

### Sum of Two Independent Random Variables

Suppose X and Y are independent, find distribution of $Z = X + Y$
In general: $F_Z(z) = P(X + Y \leq z) = P(\{(x,y) \text{ such that } x + y \leq z\})$
Approaches:

- Bivariate transformation method (continuous and discrete)
- Discrete convolution:
  $f_Z(z) = \sum_{x+y=z} f_X(x)f_Y(y) = \sum_x f_X(x)f_Y(z-x)$
- Continuous convolution
- Mgf/cf method (continuous and discrete)
  $\phi_Z\theta = \phi_X(\theta)\phi_Y(\theta)$
  $Z = X - Y$ $\phi_Z\theta = \phi_X(\theta)\phi_Y(-\theta)$

## Notes 15

### Conditional Expectation and Variance

**Iterative Expectation Formula**
$EX = E(E(X|Y))$
**Variance**
$Var[g(Y)] = E[g(y) - E(g(Y))]^2$
$VarX = E(Var(X|Y)) + Var(E(X|Y))$
$Var(g(Y)|X) = E\{[g(Y) - E(g(Y)|X)]^2|X\}$
where both expectations are taken with respect to $f_{Y||X}(y)$

- $E(Var(X|Y)) = E\{[X - E(X|Y)]^2\}$
- $Var(E(X|Y)) = E\{[E(X|Y) - EX]^2\}$

### Covariance and Correlation

$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \sigma_{XY}$
Correlation $=\rho_{XY} = \dfrac{Cov(X,Y)}{\sqrt{VarX\ VarY}} = \dfrac{\sigma_{XY}}{\sigma_X\sigma_Y}$
$= E\left[\left(\dfrac{X-\mu_X}{\sigma_X}\right)\left(\dfrac{Y-\mu_Y}{\sigma_Y}\right)\right]$

X and Y are uncorrelated iff:
$Cov(X,Y) = 0$ or equivalently $\rho_{XY} = 0$
$Cov(X,Y) = E(XY) - E(X)E(Y)$
If X and Y are independent and $Cov(X,Y)$ exists, then $Cov(X,Y) = 0$
If X and Y are uncorrelated this does not imply independence.

## Linear Combinations

$Cov(aX + B_Y, Z) = aCov(X,Z) + bCov(Y,Z)$

$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$

$Corr(aX + b, cY + d) = \frac{ac}{|ac|}Corr(X,Y)$

## Standard Bivariate Normal

$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right]$
Both X and Y have marginal distributions are $N(0,1)$
Correlation of X and Y is $\rho$
Conditional Distribution are normal:
$Y|X \sim N(\rho X, 1-\rho^2) \quad X|Y \sim N(\rho Y, 1-\rho^2)$
The means are the regression lines of Y on X and X on Y respectively.

## Bivariate Normal

Let $\tilde{X}$ and $\tilde{Y}$ have a standard bivariate normal distribution with correlation $\rho$
Let $X = \mu_X + \sigma_X\tilde{X} \quad Y = \mu_Y + \sigma_Y\tilde{Y}$
Then $(X,Y)$ has the bivariate normal density:
$f_{XY}(x,y) =$

$$\left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

Marginal distributions: $N(\mu_X, \sigma_X^2) \quad N(\mu_Y, \sigma_Y^2)$
$Corr(X,Y) = \rho$

Conditional distributions are normal:
$Y|X \sim N[\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_X), \sigma_Y^2(1-\rho^2)]$

Distribution of $aX + bY$ is:
$N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y)$

## Multivariate Distributions

$\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$
If $\boldsymbol{X}$ is discrete then:
$P(\boldsymbol{X} \in A) = \sum_{\boldsymbol{X} \in A} f(\boldsymbol{X})$
where $f(\boldsymbol{X})$ is the joint pmf

If $\boldsymbol{X}$ is continuous then:
$P(\boldsymbol{X} \in A) = \int \cdots \int_A f(x_1, \ldots, x_n) dx_1, \ldots dx_n$

### Marginals and Conditionals

The **marginal** pdf or pmf of any subset of coordinates is found by integrating or summing the joint pdf or pmf over all possible values of the other coordinates.

The **conditional** pdf or pmf of a subset of coordinates given the values of the remaining coordinates is found by dividing the full joint pdf or pmf by the joint pdf or pmf of the remaining variables.

### Multivariate Independence

Independent Random Vectors:
Let $\boldsymbol{X_1}, \ldots, \boldsymbol{X_n}$ be random vectors with joint pdf or pmf $f(\boldsymbol{X_1}, \ldots, \boldsymbol{X_n})$
Let $f\boldsymbol{X_j}(\boldsymbol{x_j})$ be the marginal pdf or pmf of $\boldsymbol{X_j}$.
Then $\boldsymbol{X_1}, \ldots, \boldsymbol{X_n}$ are **mutually independent** random vectors if:
$\forall\ (\boldsymbol{X_1}, \ldots, \boldsymbol{X_n})$: $\quad f(\boldsymbol{X_1}, \ldots, \boldsymbol{X_n}) = \prod_{j=1}^{n} f\boldsymbol{X_j}(\boldsymbol{x_j})$

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### Inequalities

**Chebychev Inequality**
$p[g(X) \geq r] leq \dfrac{E[g(X)]}{r}$
If X is nonnegative and g is a positive non-decreasing function then:
$P\{X \geq a\} \leq \dfrac{E[g(X)]}{g(a)}$
**Special Cases:**
$X \geq 0 \quad P\{X \geq a\} \leq \dfrac{E(e^{tX})}{e^{ta}}$
$L^p$ Space- consists of all r.v.s whose $p^{th}$ absolute power is integrable,
$E(|X|^p) < \infty$
**Triangle Inequality**
$|a+b| \leq |a| + |b|$
**Convex Functions**
A function $g : I \to R$ is convex for any $\lambda \in [0,1]$ and any points x and y in I
$g[\lambda x + (1-\lambda)y] \leq \lambda g(x) + (1-\lambda) g(y)$
A differentiable function g is convex iff it lays above all tangents.
A twice differentiable function g is convex iff its second derivative is non-negative
concave if -g is convex on I
**Jensen's Inequality**

Let $X \in L^1$ and $g(x)$ be a convex function where $E[g(X)]$ exists. Then:
$E[g(X)] \geq g[EX]$
with equality iff for every line $a + bx$ tangent to $g(x)$ at
$x = EX, P[g(X) = a + bX] = 1$
direction of inequality is reversed if g is concave
**Young's Inequality**
Let $a, b > 0$ and $p, q > 1$ with $1/p + 1/q = 1$ Then:
$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$
with equality iff $a^p = b^q$
**Holder's Inequality**
Suppose $X \in L^p, Y \in L^q$ where $p, q > 1$ and $1/p + 1/q = 1$ Then:
$E[|XY|] \leq [E|X|^p]^{1/p} E[|Y|^q]^{1/q}$
with equality if $X^p = cY^q$ for some $c \in \mathbb{R}$
**Cauchy-Schwartz Inequality**
corollary of Holders where $p = q = 2$
$E[|XY|] \leq [E|X|^2]^{1/2} E[|Y|^2]^{1/2} = \sqrt{E[X^2]E[Y^2]}$
with equality if $X = cY$
**Lyapunov's Inequality**
corrallary of Holders
for $1 \leq r \leq s$ and $X \in L^s$
$E[|X|^r]^{1/r} \leq E[|X|^s]^{1/s}$
**Application of Cauchy-Schwartz**
$p = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$
Then $|p| \leq 1$ with equality iff $Y - \mu_Y = c(X - \mu_X)$
**Minkowski's Inequality**
Suppose $X, Y \in L^p, p \geq 1$ Then $(X + Y) \in L^p$ and
$[E|X + Y|^p]^{1/p} \leq [E|X|^p]^{1/p} + [E|Y|^p]^{1/p}$

## Order Statistics

**Distribution of the Maximum**
The cdf of $Z = max(Y_1, \ldots, Y_n)$ is

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} \\ &= P\{Y_1 \leq z, Y_2 \leq z, \ldots, Y_n \leq z\} \\ &= \prod_{j=1}^{n} P\{Y_j \leq z\} \text{ indep} \\ &= F_Y(z)^n \text{ ident. distrib} \end{aligned}$$

Thus the pmf is:
$f_Z(z) = nF_Y(z)^{n-1} f_Y(z)$
**Distribution of the Minimum**

$W = min(Y_1, \dots, Y_n)$
$F_W(w) = 1 - (1 - F_Y(w))^n$
$f_W(w) = n(1 - F_Y(w))^{n-1} f_Y(w)$
**Order Statistics**
Let $Y_1, Y_2, \dots, Y_n$ be iid with pdf $f_Y(x)$
Order the observations:
$Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$
The $Y_{(i)}$ are called order statistics. Minimum is $Y_{(1)}$ max is $Y_{(n)}$
We are interested in finding the distribution of an arbitrary $Y_{(i)}$ as well as the joint distributions of sets of $Y_{(i)}$s and $Y_{(j)}$s
ex: Range=$Y_{(n)} - Y_{(1)}$
$r^{th}$ **order statistic**
We need to find the density of $Y_{(r)}$ at a value y
Consider 3 intervals $(-\infty, y), [y, y + dy), [y + dy, \infty)$
The number of observations in each of the intervals follows the tri-nomial distribution:
$f(s_1, s_2, s_3) = \dfrac{n!}{s_1!s_2!s_3!} p_1^{s_1} p_2^{s_2} p_3^{s_3}$
The event that $y \leq Y_{(r)} < y + dy$ is the event we have:
$(r - 1)$ observations are less than y,
$(n - r)$ observations are greater than y
1 observation is in interval $y, y + dy$
In the trinomial distribution this corresponds to:
$s_1 = r - 1, s_2 = 1, s_3 = n - r$
$p_1 = F_Y(y), p_2 = f_Y(y)dy, p_3 = 1 - F_Y(y + dy)$
**Theorem 5.4.6**
Let $X_{(1)}, \dots, X_{(n)}$ denote the order statistics of a random sample, $X_1, \dots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$. Then the joint pdf of $X_{(i)}$ and $X_{(j)}$, $1 \leq i < j \leq n$, is:

$$f_{X_{(i)},X_{(j)}}(u, v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f_X(u) f_X(v) [F_X(u)]^{i-1} [F_X(v) - F_X(u)]^{j-1-i} [1 - F_X(v)]^{n-j}$$

for $-\infty < u, v < \infty$
The joint pdf of all the order statistics is:

$$f_{X_{(1)} \dots, X_{(n)}}(x_1, \dots, x_n) = \begin{cases} n! f_X(x_1) \cdots f_X(x_n) & -\infty < x_1 < \cdots < x_n < \infty \\ 0 \quad \text{otherwise} \end{cases}$$