1 Supremum

Bounded Above: Let S be a set of real numbers. If $\exists\ b$ such that $x \leq b\ \forall\ x \in S$, then b is an **upper bound** for S. Then S is **bounded above** by b.

Supremum(or least upper bound): Let S be a set of real numbers bounded above. b is a least upper bound for S ($b = sup\ S$) if:

- b is an upper bound for S
- No number less than b is an upper bound for S

If S has a **maximum element** then $max\ S = sup\ S$

2 The Completeness Axiom

The Completeness Axiom: Every nonempty set S of real numbers which is bounded above has a sup; that is, there is a real number b such that $b = sup\ S$.

It follows that every nonempty set of real numbers which is **bounded below** has an **infimum**.

3 Some Properties of the Supremum

Approximation property: Let S be a nonempty set of real numbers with a supremum, $b = sup\ S$. Then $\forall\ a < b\ \exists\ x \in S$ such that $a < x \leq b$.

Additive Property: Given nonempty subsets A and B of R, let C denote the set:

$$C = \{x + y : x \in A, y \in B\}$$

If each of A and B has a supremum, then C has a supremum and:

$$sup\ C = sup\ A + sup\ B$$

Comparison Property: Given nonempty subsets S and T of R such that $s \leq t\ \forall\ s \in S, t \in T$. If T has a supremum then S has a supremum and:

$$sup\ S \leq sup\ T$$