Notes 14

Convolution

If X and Y are independent continuous r.v.s with pdfs $f_X(x)$ and $f_Y(y)$, then the pdf of $Z = X + Y$ is:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\, dw$$

Sum of Two Independent Poissons

$X \sim Pois(\lambda_1),\ Y \sim Pois(\lambda_2)$
$U = X + Y\ V = Y$
$X = U - V\ Y = V$
Joint PMF of U and V is:
$f_{U,V}(u,v) = f_{X,Y}(u-v,v) = \dfrac{e^{-\lambda_1}\lambda_1^{u-v}}{(u-v)!}\dfrac{e^{-\lambda_2}\lambda_2^{v}}{v!}$
The distribution of $U = X + Y$ is the marginal:
$f_U(u) = \sum_{v=0}^{u} \dfrac{e^{-\lambda_1}\lambda_1^{u-v}}{(u-v)!}\dfrac{e^{-\lambda_2}\lambda_2^{v}}{v!}$
$= \dfrac{e^{-(\lambda_1+\lambda_2)}}{u!} \sum_{v=0}^{u} \binom{u}{v} \lambda_1^{u-v}\lambda_2^{v}$
Because of the binomial theorem
$= \dfrac{e^{-(\lambda_1+\lambda_2)}}{u!}(\lambda_1+\lambda_2)^u$
$U \sim Pois(\lambda_1+\lambda_2)$

Jacobian

$J(u,v)$ is the Jacobian of the transformation $(x,y) \to (u,v)$ given by:

$J(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}$

Functions of Independent Random Variables

Let X and Y be independent r.v.s
Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be functions
Then the r.v.s $U = g(X)$ and $V = h(Y)$ are independent

Ratio of Two Independent Normals

Let $X \sim N(0,1)$ and $Y \sim N(0,1)$
The ratio X/Y has the Cauchy distribution

Let $U = X/Y$ and $V = Y$ Then $X = UV$ and $Y = V$ $J(u,v) = v$
$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$
$f_{U,V}(uv,v) = \frac{1}{2\pi} e^{-[(uv)^2+v^2]/2} * |v| = \frac{|v|}{2\pi} e^{-(u^2+1)v^2/2}$
$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\ dv = 2\int_0^{\infty} \frac{v}{2\pi} e^{-(u^2+1)v^2/2}\ dv$
$= \frac{1}{\pi} \int_0^{\infty} e^{-(u^2+1)z}\ dz = \frac{1}{\pi(u^2+1)}$

Sum of Two Independent Random Variables

Suppose X and Y are independent, find distribution of $Z = X + Y$
In general: $F_Z(z) = P(X + Y \leq z) = P(\{(x,y) \text{ such that } x + y \leq z\})$
Approaches:

- Bivariate transformation method (continuous and discrete)
- Discrete convolution:
  $f_Z(z) = \sum_{x+y=z} f_X(x) f_Y(y) = \sum_x f_X(x) f_Y(z-x)$
- Continuous convolution
- Mgf/cf method (continuous and discrete)
  $\phi_Z\theta = \phi_X(\theta)\phi_Y(\theta)$
  $Z = X - Y \quad \phi_Z\theta = \phi_X(\theta)\phi_Y(-\theta)$

Notes 15

Conditional Expectation and Variance

Iterative Expectation Formula
$EX = E(E(X|Y))$
Variance
$Var[g(Y)] = E[g(y) - E(g(Y))]^2$

$VarX = E(Var(X|Y)) + Var(E(X|Y))$
$Var(g(Y)|X) = E\{[g(Y) - E(g(Y)|X)]^2|X\}$
where both expectations are taken with respect to $f_{Y||X}(y)$

- $E(Var(X|Y)) = E\{[X - E(X|Y)]^2\}$
- $Var(E(X|Y)) = E\{[E(X|Y) - EX]^2\}$

Covariance and Correlation

$Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)] = \sigma_{XY}$
Correlation $= \rho_{XY} = \frac{Cov(X,Y)}{\sqrt{VarX\ VarY}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$
$= E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)\left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right]$

X and Y are uncorrelated iff:
$Cov(X,Y) = 0$ or equivalently $\rho_{XY} = 0$
$Cov(X,Y) = E(XY) - E(X)E(Y)$
If X and Y are independent and $Cov(X,Y)$ exists, then $Cov(X,Y) = 0$
If X and Y are uncorrelated this does not imply independence.

Linear Combinations

$Cov(aX + B_Y, Z) = aCov(X,Z) + bCov(Y,Z)$

$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X,Y)$

$Corr(aX + b, cY + d) = \dfrac{ac}{|ac|} Corr(X,Y)$

Standard Bivariate Normal

$f_{XY}(x,y) = \dfrac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\dfrac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right]$
Both X and Y have marginal distributions are $N(0,1)$
Correlation of X and Y is ρ
Conditional Distribution are normal:
$Y|X \sim N(\rho X, 1-\rho^2) \quad X|Y \sim N(\rho Y, 1-\rho^2)$
The means are the regression lines of Y on X and X on Y respectively.

Bivariate Normal

Let $\tilde{X}$ and $\tilde{Y}$ have a standard bivariate normal distribution with correlation ρ
Let $X = \mu_X + \sigma_X \tilde{X} \quad Y = \mu_Y + \sigma_Y \tilde{Y}$
Then (X,Y) has the bivariate normal density:
$f_{XY}(x,y) =$
$$\left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$
Marginal distributions: $N(\mu_X, \sigma_X^2) \quad N(\mu_Y, \sigma_Y^2)$
$Corr(X,Y) = \rho$

Conditional distributions are normal:
$Y|X \sim N[\mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_X), \sigma_Y^2(1-\rho^2)]$

Distribution of $aX + bY$ is:
$N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y)$

Multivariate Distributions

$\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$
If $\boldsymbol{X}$ is discrete then:
$P(\boldsymbol{X} \in A) = \sum_{\boldsymbol{X} \in A} f(\boldsymbol{X})$
where $f(\boldsymbol{X})$ is the joint pmf

If $\boldsymbol{X}$ is continuous then:
$P(\boldsymbol{X} \in A) = \int \cdots \int_A f(x_1, \ldots, x_n) dx_1, \ldots dx_n$

Marginals and Conditionals

The **marginal** pdf or pmf of any subset of coordinates is found by integrating or summing the joint pdf or pmf over all possible values of the other coordinates.

The **conditional** pdf or pmf of a subset of coordinates given the values of the remaining coordinates is found by dividing the full joint pdf or pmf by the joint pdf or pmf of the remaining variables.

Multivariate Independence

Independent Random Vectors:
Let $\boldsymbol{X_1}, \ldots, \boldsymbol{X_n}$ be random vectors with joint pdf or pmf $f(\boldsymbol{X_1}, \ldots, \boldsymbol{X_n})$
Let $f\boldsymbol{X_j}(\boldsymbol{x_j})$ be the marginal pdf or pmf of $\boldsymbol{X_j}$.
Then $\boldsymbol{X_1}, \ldots, \boldsymbol{X_n}$ are **mutually independent** random vectors if:
$\forall$ $(\boldsymbol{X_1}, \ldots, \boldsymbol{X_n})$: $\quad f(\boldsymbol{X_1}, \ldots, \boldsymbol{X_n}) = \prod_{j=1}^{n} f\boldsymbol{X_j}(\boldsymbol{x_j})$

Notes 16

Order Statistics

Theorem 5.4.6
Let $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics of a random sample, $X_1, \ldots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$. Then the joint pdf of $X_{(i)}$ and $X_{(j)}$, $1 \leq i < j \leq n$, is:

$$f_{X_{(i)},X_{(j)}}(u,v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f_X(u) f_X(v) [F_X(u)]^{i-1} [F_X(v) - F_X(u)]^{j-1-i} [1 - F_X(v)]^{n-j}$$

for $-\infty < u, v < \infty$
The joint pdf of all the order statistics is:

$$f_{X_{(1)} \ldots, X_{(n)}}(x_1, \ldots, x_n) = \begin{cases} n! f_X(x_1) \cdots f_X(x_n) & -\infty < x_1 < \cdots < x_n < \infty \\ 0 \quad \text{otherwise} \end{cases}$$