Problem 1

Starting with the pdf of $gamma(\alpha, \beta)$ we have:

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

$$\text{Then: } EX^\nu = \int_0^\infty x^\nu \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \mathrm{d}x$$

$$= \int_0^\infty \frac{x^{\nu+\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} \mathrm{d}x$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\nu+\alpha-1} e^{-x/\beta} \mathrm{d}x$$

$$\text{Let } y = -x/\beta$$

$$= \frac{(-\beta)(-\beta^{\alpha+\nu-1})}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^{\nu+\alpha-1} e^y \, dy$$

$$\text{Since } \Gamma(\alpha+\nu) = \int_0^\infty y^{\nu+\alpha-1} e^y \, dy \text{ we have:}$$

$$= \frac{\beta^{\nu+\alpha}\Gamma(\nu+\alpha)}{\Gamma(\alpha)\beta^\alpha}$$

$$= \frac{\beta^\nu \Gamma(\nu+\alpha)}{\Gamma(\alpha)}$$

Problem 2

$$Y \sim negbin(r,p)$$

$$\text{WTS: } \lim_{p\to 0} M_{pY}(t) = \left(\frac{1}{1-t}\right)^r \text{ (the gamma(r,1) mgf)}$$

$$M_Y(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$$

$$\text{Since } M_{aX+b}(t) = e^{bt} M_X(at)$$

$$M_{pY}(t) = \left(\frac{p}{1-(1-p)e^{pt}}\right)^r$$

Since the limit is in a 0/0 indeterminate form we will use L'Hospitals Rule

$$\lim_{p\to 0} \left(\frac{\frac{d}{dp}p}{\frac{d}{dp}1-(1-p)e^{pt}}\right)^r = \lim_{p\to 0} \left(\frac{1}{e^{pt}+pte^{pt}-te^{pt}}\right)^r$$

$$= \left(\frac{1}{1-t}\right)^r$$

Thus limiting p to 0, the mgf of p_Y converges to the mgf of the gamma distribution

Problem 3

(a) $E(X)$

$$
\begin{aligned}
f(x) &= \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \\
E(X) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty x e^{-x^2/2} \mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}} \Big|_0^\infty - e^{-x^2/2} \\
&= \frac{2}{\sqrt{2\pi}} (0 - -1) = \frac{2}{\sqrt{2\pi}} \\
E(X) &= \frac{2}{\sqrt{2\pi}}
\end{aligned}
$$

$Var(X)$

$$
\begin{aligned}
E(X^2) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-x^2/2} \mathrm{d}x = \frac{2}{\sqrt{2\pi}} \sqrt{\pi/2} \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} = 1 \\
Var(X) &= E(X^2) - E(X)^2 = 1^2 - \left(\frac{2}{\sqrt{2\pi}} \right)^2 \\
&= 1 - \frac{4}{2\pi} = 1 - \frac{2}{\pi}
\end{aligned}
$$

(b)

$$\text{Given } f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\text{Find } g(X) = Y \text{ so that:}$$

$$f(y|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}$$

looking at the core parts of the two pdfs, we have:

$$f_x = e^{-x^2/2} \quad f_y = (y)^{\alpha-1} e^{-y/\beta}$$

We will use the transformation $Y = X^2$

$$g^{-1}(y) = \sqrt{y} \quad dy = (1/2)y^{-1/2}$$

$$f_y(y) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-(y^{1/2})^2/2} (1/2) y^{-1/2}$$

$$= \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-1/2y}$$

Since $\Gamma(1/2) = \sqrt{\pi}$ let $\alpha = 1/2$, giving us:

$$gamma(1/2, \beta) = \frac{1}{\sqrt{\pi}\beta^{1/2}} y^{-1/2} e^{-y/\beta}$$

in order to match our transformation let $\beta = 2$

$$= \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-1/2y}$$

Thus we need the transformation $Y = X^2$ so that $Y \sim gamma(1/2, 2)$

Problem 4

$$\text{Given } h_T(t) = \lim_{\delta \to 0} P(t \leq T \leq t+\delta | T \geq t)/\delta, \text{ T is a continuos r.v.}$$

$$\text{WTS: } h_T(t) = \frac{f_T(t)}{1-F_T(t)} = -\frac{d}{dt}\log(1-F_T(t))$$

$$P(t \leq T \leq t+\delta | T \geq t) = \frac{P(t \leq T \leq t+\delta)}{P(T \geq t)}$$

$$= \frac{F_T(t+\delta) - F_T(t)}{1-F_T(t)}$$

$$\text{Thus } h_T(t) = \lim_{\delta \to 0} \frac{F_T(t+\delta) - F_T(t)}{(1-F_T(t))\delta}$$

From the definition of a derivative we know:

$$F_T'(t) = f_T(t) = \lim_{\delta \to 0} \frac{F_T(t+\delta) - F_T(t)}{\delta}$$

$$\text{Which means: } h_T(t) = \frac{f_T(t)}{1-F_T(t)} \text{ and}$$

$$-\frac{d}{dt}log(1-F_T(t)) = -\left(\frac{-f_T(t)}{1-F_T(t)}\right) = \frac{f_T(t)}{1-F_T(t)}$$

$$\text{Thus } h_T(t) = \frac{f_T(t)}{1-F_T(t)} = -\frac{d}{dt}\log(1-F_T(t))$$

Problem 5

(a)

$$f_T(t) = 1/\beta e^{-t/\beta}$$

$$F_T(t) = \int_0^t 1/\beta e^{-x/\beta} \mathrm{d}x = 1 - e^{-t/\beta}$$

$$h_T(t) = \frac{f_T(t)}{1-F_T(t)} = \frac{1/\beta e^{-t/\beta}}{1-(1-e^{-t/\beta})} = \frac{(1/\beta)e^{-t/\beta}}{e^{-t/\beta}} = 1/\beta$$

(b)

$$f_T(t) = \gamma/\beta t^{\gamma-1} e^{-t^\gamma/\beta}$$

$$F_T(t) = 1 - e^{-t^\gamma/\beta}$$

$$h_T(t) = \frac{\gamma/\beta t^{\gamma-1} e^{-t^\gamma/\beta}}{1-(1-e^{-t^\gamma/\beta})} = \frac{\gamma/\beta t^{\gamma-1} e^{-t^\gamma/\beta}}{e^{-t^\gamma/\beta}} = (\gamma/\beta)t^{\gamma-1}$$

(c)

$$f_T(t) = \frac{e^{-(t-\mu)/\beta}}{\beta(1+e^{-(t-\mu)/\beta})^2}$$
$$F_T(t) = \frac{1}{1+e^{-(t-\mu)/\beta}}$$
$$h_T(t) = \frac{\dfrac{e^{-(t-\mu)/\beta}}{\beta(1+e^{-(t-\mu)/\beta})^2}}{1-(\dfrac{1}{1+e^{-(t-\mu)/\beta}})}$$
$$= \frac{\dfrac{e^{-(t-\mu)/\beta}}{\beta(1+e^{-(t-\mu)/\beta})^2}}{\dfrac{e^{-(t-\mu)/\beta}}{1+e^{-(t-\mu)/\beta}}} = \frac{1}{\beta(1+e^{-(t-\mu)/\beta})} = (1/\beta)F_T(t)$$

Problem 6

(a)

$$f_X(x) = \frac{1}{b-a} \quad x \in [a,b]$$
$$\text{Let } m, x, y \in [a,b]$$
$$\text{Then } f(m) = f(x) = f(y)$$
$$\text{since } \frac{1}{b-a} = \frac{1}{b-a} = \frac{1}{b-a}$$
$$\text{Which means if } m \geq x \geq y \text{ or } m \leq x \leq y$$
$$\text{Then } f(m) \geq f(x) \geq f(y)$$
$$\text{Thus the pdf of } U(a,b) \text{ is unimodal}$$

(b)

$$f_X(x) = x^{\alpha-1}e^{-x/\beta} \text{ ignoring constants}$$
$$\frac{d}{dx}f_X(x) = \frac{x^{\alpha-2}e^{-x/\beta}(\beta(\alpha-1)-x)}{\beta} = 0$$
$$x = \beta(\alpha-1)$$

Since there is only one sign change, $f_X(x)$ is unimodal with mode $\beta(\alpha-1)$

(c)

$$
\begin{aligned}
f_X(x) &= e^{\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]} \text{ ignoring constants} \\
\frac{d}{dx} f_X(x) &= \frac{x-\mu}{\sigma^2} e^{-(x-\mu)^2/(2\sigma^2)} = 0 \\
x &= \mu
\end{aligned}
$$

Since there is only one sign change, $f_X(x)$ is unimodal with mode μ

(d)

$$
\begin{aligned}
f_X(x) &= x^{\alpha-1}(1-x)^{\beta-1} \text{ ignoring constants} \\
\frac{d}{dx} &= (\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} - (\beta-1)x^{\alpha-1}(1-x)^{\beta-2} \\
&= (1-x)^{\beta-2}x^{\alpha-2}((\alpha-1) - x(\alpha+\beta-2) = 0 \\
x &= \frac{\alpha-1}{\alpha+\beta-2}
\end{aligned}
$$

Since there is only one sign change, $f_X(x)$ is unimodal with mode $\dfrac{\alpha-1}{\alpha+\beta-2}$

Problem 7

(a)

$$
\begin{aligned}
f(x|\eta) &= h(x)c^*(\eta)\exp\left(\sum_{j=1}^{k}\eta_j t_j(x)\right) \\
\frac{\partial}{\partial\eta}(1) &= \frac{\partial}{\partial_\eta}\int h(x)c^*(\eta)\exp\left(\sum_{j=1}^{k}\eta_j t_j(x)\right)\mathrm{d}x \\
0 &= \int \frac{\partial}{\partial_\eta} h(x)c^*(\eta)\exp\left(\sum_{j=1}^{k}\eta_j t_j(x)\right)\mathrm{d}x \\
&= \int h(x)c^{'*}(\eta)\exp\left(\sum_{j=1}^{k}\eta_j t_j(x)\right)\mathrm{d}x + \int h(x)c^*(\eta)\exp\left(\sum_{j=1}^{k}\eta_j t_j(x)\right)\left(\sum_{j=1}^{k}\frac{\partial\eta_j(\eta)}{\partial\eta_j}t_j(x)\right)\mathrm{d}x \\
&= \int h(x)\left[\frac{\partial}{\partial_\eta}\log(c^*(\eta))\right]c^*(\eta)\exp\left(\sum_{j=1}^{k}\eta_j t_j(x)\right)\mathrm{d}x + E\left(\sum_{j=1}^{k}\frac{\partial\eta_j(\eta)}{\partial\eta_j}t_j(x)\right) \\
0 &= \left[\frac{\partial}{\partial_\eta}\log(c^*(\eta))\right] + E\left(\sum_{j=1}^{k}\frac{\partial\eta_j(\eta)}{\partial\eta_j}t_j(x)\right) \\
E\left(\sum_{j=1}^{k}\frac{\partial\eta_j(\eta)}{\partial\eta_j}t_j(x)\right) &= -\left[\frac{\partial}{\partial_\eta}\log(c^*(\eta))\right] \\
E(t_j(X)) &= E\left(\sum_{j=1}^{k}\frac{\partial\eta_j(\eta)}{\partial\eta_j}t_j(x)\right) \\
E(t_j(X)) &= -\left[\frac{\partial}{\partial_\eta}\log(c^*(\eta))\right]
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_X(x|a,b) &= \frac{1}{\Gamma(a)b^a} x^{a-1} \left(\sum_{j=1}^{k} \eta_j t_j(x) \right) \\
&= x^{-1} I(x>0) \frac{1}{\Gamma(a)b^a} e^{a \log x + (-1/b)x} \\
&= h(x) \frac{(-\eta_2)^{\eta_1}}{\Gamma(\eta_1)} e^{\eta_1 \log x + \eta_2 x} \\
EX &= -\frac{\partial}{\partial \eta_2} (\log \left(\frac{(-\eta_2)^{\eta_1}}{\Gamma(\eta_1)} \right)) \\
&= -\frac{\partial}{\partial \eta_2} (\eta_1 \log(-\eta_2) - \log \Gamma(\eta_1)) \\
&= \frac{\eta_1}{(-\eta_2)} = ab \\
EX &= ab \\
Var(X) &= -\frac{\partial^2}{\partial_{\eta_2^2}} \log \left(\frac{(-\eta_2)^{\eta_1}}{\Gamma(\eta_1)} \right) \\
&= \frac{\partial}{\partial \eta_2} (\frac{\eta_1}{-\eta_2}) = \frac{\eta_1}{\eta_2^2} = ab^2 \\
Var(x) &= ab^2
\end{aligned}
$$

Problem 8

(a)

$$
\begin{aligned}
f(x|\theta,\theta) &= \frac{1}{\sqrt{2\pi\theta}} e^{\left[\frac{-(x-\theta)^2}{2\theta^2} \right]} \\
&= \frac{1}{\sqrt{2\pi\theta}} e^{-\theta/2 + x - x^2/(2\theta)}
\end{aligned}
$$

$$
h(x) = e^x \ I_{-\infty<x<\infty}(x) \quad c(\theta) = \frac{1}{\sqrt{2\pi\theta}} \quad w_1(\theta) = 1/2\theta \quad t_1(x) = -x^2
$$

Since $\mu = \sigma^2$ the θ parameter vector lies on the nonnegative real line

(b)

$$\begin{aligned}
f(x|\theta, a\theta^2) &= \frac{1}{\sqrt{2\pi a\theta^2}} e^{\left[\frac{-(x-\theta)^2}{2a\theta^2}\right]} \\
&= \frac{1}{\sqrt{2\pi a\theta^2}} e^{-x^2/(2a\theta^2)+x/(a\theta)-1/2a} \\
h(x) &= I_{-\infty<x<\infty}(x) \quad c(\theta) = \frac{1}{\sqrt{2\pi a\theta^2}} e^{-1/2a} \\
w_1 &= 1/(2a\theta^2) \quad w_2 = 1/(a\theta) \quad t_1 = -x^2 \quad t_2 = x
\end{aligned}$$

Since $\mu^2 = a\sigma^2$ the θ parameter vector lies on a parabola

(c)

$$\begin{aligned}
f(x|a, 1/a) &= \frac{1}{\Gamma(a)(1/a)^a} x^{a-1} e^{-ax} \\
&= \frac{a^a}{\Gamma(a)} \frac{1}{x} x^a e^{-ax} \\
h(x) &= \frac{1}{x}\ I_{0<x<\infty}(x) \quad c(a) = a^a/\Gamma(a) \\
w_1(a) &= a \quad w_2(a) = a \quad t_1(x) = \log(x) \quad t_2(x) = -x
\end{aligned}$$

The θ parameter vector lies on a line

(d)

$$\begin{aligned}
f(x|\theta) &= Ce^{-(x-\theta)^4} \\
&= Ce^{-\theta^4+4\theta^3x-6\theta^2x^2+4\theta x^3-x^4} \\
h(x) &= Ce^{-x^4} I_{-\infty<x<\infty}(x) \quad c(\theta) = e^{-\theta^4} \\
w_1(\theta) &= \theta \quad w_2(\theta) = \theta^2 \quad w_3(\theta) = \theta^3 \\
t_1(x) &= 4x^3 \quad t_2(x) = -6x^2 \quad t_3(x) = 4x
\end{aligned}$$

The θ parameter vector lies on a 3 dimensional spiral

Problem 9

$$
\begin{aligned}
P(Z \geq t) &= \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} \mathrm{d}x \\
P(|Z| \geq t) &= 2P(Z \geq t) \\
&= \frac{2}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} \mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \int_t^\infty \frac{1+x^2}{1+x^2} e^{-x^2/2} \mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \left[\int_t^\infty \frac{1}{1+x^2} e^{-x^2/2} \mathrm{d}x + \int_t^\infty \frac{x^2}{1+x^2} e^{-x^2/2} \mathrm{d}x \right]
\end{aligned}
$$

integrating the second term by parts where:

$$
u = \frac{x}{1+x^2} \quad dv = xe^{-x^2/2} \quad v = -e^{-x^2/2} \quad du = \frac{1-x^2}{(1+x^2)^2}
$$

$$
\begin{aligned}
\int_t^\infty \frac{x^2}{1+x^2} e^{-x^2/2} \mathrm{d}x &= \frac{x}{1+x^2} (-e^{-x^2/2}) \Big|_t^\infty + \int_t^\infty \frac{1-x^2}{(1+x^2)^2} e^{-x^2/2} dx \\
&= \frac{t}{t+t^2} e^{-t^2/2} + \int_t^\infty \frac{1-x^2}{(1+x^2)^2} e^{-x^2/2} dx \\
P(|Z| \geq t) &= \sqrt{\frac{2}{\pi}} \left(\frac{t}{t+t^2} e^{-t^2/2} + \int_t^\infty \frac{1}{1+x^2} e^{-x^2/2} \frac{1-x^2}{(1+x^2)^2} e^{-x^2/2} dx \right) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{t}{t+t^2} e^{-t^2/2} + \int_t^\infty \frac{(1+x^2)+1-x^2}{(1+x^2)^2} e^{-x^2/2} dx \right) \\
&= \sqrt{\frac{2}{\pi}} \frac{t}{t+t^2} e^{-t^2/2} + \sqrt{\frac{2}{\pi}} \int_t^\infty \frac{2}{(1+x^2)^2} e^{-x^2/2} \mathrm{d}x \\
&\geq \sqrt{\frac{2}{\pi}} \frac{t}{t+t^2} e^{-t^2/2}
\end{aligned}
$$

Therefore $P(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \frac{t}{t+t^2} e^{-t^2/2}$