

## Notes 14

### Sum of Two Independent Poissons

$$X \sim \text{Pois}(\lambda_1), Y \sim \text{Pois}(\lambda_2)$$

$$U = X + Y \quad V = Y$$

$$X = U - V \quad Y = V$$

Joint PMF of U and V is:

$$f_{U,V}(u, v) = f_{X,Y}(u-v, v) = \frac{e^{-\lambda_1} \lambda_1^{u-v}}{(u-v)!} \frac{e^{-\lambda_2} \lambda_2^v}{v!}$$

The distribution of  $U = X + Y$  is the marginal:

$$\begin{aligned} f_U(u) &= \sum_{v=0}^u \frac{e^{-\lambda_1} \lambda_1^{u-v}}{(u-v)!} \frac{e^{-\lambda_2} \lambda_2^v}{v!} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)}}{u!} \sum_{v=0}^u \binom{u}{v} \lambda_1^{u-v} \lambda_2^v \end{aligned}$$

Because of the binomial theorem

$$= \frac{e^{-(\lambda_1 + \lambda_2)}}{u!} (\lambda_1 + \lambda_2)^u$$

$$U \sim \text{Pois}(\lambda_1 + \lambda_2)$$

### Jacobian

$J(u, v)$  is the Jacobian of the transformation  $(x, y) \rightarrow (u, v)$  given by:

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

### Functions of Independent Random Variables

Let X and Y be independent r.v.s

Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  be functions

Then the r.v.s  $U = g(X)$  and  $V = h(Y)$  are independent

### Ratio of Two Independent Normals

Let  $X \sim N(0, 1)$  and  $Y \sim N(0, 1)$

The ratio  $X/Y$  has the Cauchy distribution

Let  $U = X/Y$  and  $V = Y$  Then  $X = UV$  and  $Y = V$   $J(u, v) = v$

$$f_{X,Y}(x, y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$f_{U,V}(u, v) = \frac{1}{2\pi} e^{-[(uv)^2+v^2]/2} * |v| = \frac{|v|}{2\pi} e^{-(u^2+1)v^2/2}$$

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u, v) dv = 2 \int_0^{\infty} \frac{v}{2\pi} e^{-(u^2+1)v^2/2} dv \\ &= \frac{1}{\pi} \int_0^{\infty} e^{-(u^2+1)z} dz = \frac{1}{\pi(u^2+1)} \end{aligned}$$

## Sum of Two Independent Random Variables

Suppose  $X$  and  $Y$  are independent, find distribution of  $Z = X + Y$

In general:  $F_Z(z) = P(X + Y \leq z) = P(\{(x, y) \text{ such that } x + y \leq z\})$

Approaches:

- Bivariate transformation method (continuous and discrete)
- Discrete convolution:  

$$f_Z(z) = \sum_{x+y=z} f_X(x)f_Y(y) = \sum_x f_X(x)f_Y(z-x)$$
- Continuous convolution
- Mgf/cf method (continuous and discrete)  

$$\phi_Z\theta = \phi_X(\theta)\phi_Y(\theta)$$

$$Z = X - Y \quad \phi_Z\theta = \phi_X(\theta)\phi_Y(-\theta)$$

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### Conditional Expectation and Variance

#### Iterative Expectation Formula

$$EX = E(E(X|Y))$$

#### Variance

$$Var[g(Y)] = E[g(y) - E(g(Y))]^2$$

$$VarX = E(Var(X|Y)) + Var(E(X|Y))$$

$$Var(g(Y)|X) = E\{[g(Y) - E(g(Y)|X)]^2|X\}$$

where both expectations are taken with respect to  $f_{Y||X}(y)$

- $E(Var(X|Y)) = E\{[X - E(X|Y)]^2\}$
- $Var(E(X|Y)) = E\{[E(X|Y) - EX]^2\}$

### Covariance and Correlation

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \sigma_{XY}$$

$$\text{Correlation} = \rho_{XY} = \frac{Cov(X, Y)}{\sqrt{VarX} \sqrt{VarY}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$= E \left[ \left( \frac{X - \mu_X}{\sigma_X} \right) \left( \frac{Y - \mu_Y}{\sigma_Y} \right) \right]$$

$X$  and  $Y$  are uncorrelated iff:

$$Cov(X, Y) = 0 \text{ or equivalently } \rho_{XY} = 0$$

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

If  $X$  and  $Y$  are independent and  $Cov(X, Y)$  exists, then  $Cov(X, Y) = 0$

If  $X$  and  $Y$  are uncorrelated this does not imply independence.

## Linear Combinations

$$\text{Cov}(aX + bY, Z) = a\text{Cov}(X, Z) + b\text{Cov}(Y, Z)$$

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y)$$

$$\text{Corr}(aX + b, cY + d) = \frac{ac}{|ac|} \text{Corr}(X, Y)$$

## Standard Bivariate Normal

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right]$$

Both X and Y have marginal distributions are  $N(0, 1)$

Correlation of X and Y is  $\rho$

Conditional Distribution are normal:

$$Y|X \sim N(\rho X, 1 - \rho^2) \quad X|Y \sim N(\rho Y, 1 - \rho^2)$$

The means are the regression lines of Y on X and X on Y respectively.

## Bivariate Normal

Let  $\tilde{X}$  and  $\tilde{Y}$  have a standard bivariate normal distribution with correlation  $\rho$

$$\text{Let } X = \mu_X + \sigma_X \tilde{X} \quad Y = \mu_Y + \sigma_Y \tilde{Y}$$

Then  $(X, Y)$  has the bivariate normal density:

$$f_{XY}(x, y) = \left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

Marginal distributions:  $N(\mu_X, \sigma_X^2) \quad N(\mu_Y, \sigma_Y^2)$

$$\text{Corr}(X, Y) = \rho$$

Conditional distributions are normal:

$$Y|X \sim N[\mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_X), \sigma_Y^2(1 - \rho^2)]$$

Distribution of  $aX + bY$  is:

$$N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y)$$

## Multivariate Distributions

$$\mathbf{X} = (X_1, X_2, \dots, X_n)$$

If  $\mathbf{X}$  is discrete then:

$$P(\mathbf{X} \in A) = \sum_{\mathbf{X} \in A} f(\mathbf{X})$$

where  $f(\mathbf{X})$  is the joint pmf

If  $\mathbf{X}$  is continuous then:

$$P(\mathbf{X} \in A) = \int \cdots \int_A f(x_1, \dots, x_n) dx_1, \dots, dx_n$$

## Marginals and Conditionals

The **marginal** pdf or pmf of any subset of coordinates is found by integrating or summing the joint pdf or pmf over all possible values of the other coordinates.

The **conditional** pdf or pmf of a subset of coordinates given the values of the remaining coordinates is found by dividing the full joint pdf or pmf by the joint pdf or pmf of the remaining variables.

## Multivariate Independence

Independent Random Vectors:

Let  $\mathbf{X}_1, \dots, \mathbf{X}_n$  be random vectors with joint pdf or pmf  $f(\mathbf{X}_1, \dots, \mathbf{X}_n)$

Let  $f_{\mathbf{X}_j}(\mathbf{x}_j)$  be the marginal pdf or pmf of  $\mathbf{X}_j$ .

Then  $\mathbf{X}_1, \dots, \mathbf{X}_n$  are **mutually independent** random vectors if:

$$\forall (\mathbf{X}_1, \dots, \mathbf{X}_n): \quad f(\mathbf{X}_1, \dots, \mathbf{X}_n) = \prod_{j=1}^n f_{\mathbf{X}_j}(\mathbf{x}_j)$$