

Problem 1

Let $X \sim N(\mu, \sigma^2)$

Then $X = \sigma(z) + \mu$

From theorem 4.8 we know:

$$\phi_{aX+b}(t) = e^{ibt} \phi_X(at)$$

The characteristic function standard normal distribution $Z \sim N(0, 1)$ is:

$$\phi_Z(t) = e^{-t^2/2}$$

Plugging in $a = \sigma$ and $b = \mu$ into theorem 4.8 we have:

$$\phi_X(t) = \phi_{\sigma Z + \mu}(t) = e^{i\mu t} \phi_Z(\sigma t)$$

$$\phi_X(t) = e^{i\mu t} e^{-(\sigma t)^2/2} = e^{i\mu t - \sigma^2 t^2/2}$$

Problem 2

$$p_X(k) = \begin{cases} e^{-\lambda} \lambda^k / k! & k = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

We can write:

$$\frac{P(X = k)}{P(X = k - 1)} = \frac{\lambda}{k}$$

p_X is increasing monotonically for $\frac{\lambda}{k} > 1$

p_X is decreasing monotonically for $\frac{\lambda}{k} < 1$

Let i = the largest integer value not exceeding λ

Thus p_x is increasing for $k = 0, 1, \dots, i$

and p_x is decreasing for $k = i + 1, i + 2, \dots$

Since $p_X(i) \geq p_X(k) \forall k$ in the support of X

p_X has a maximum at $k = i$

Since the Poisson distribution is unimodal, i is also the mode

Problem 3

A car must pass at $t=4$ seconds so that the pedestrian does not start crossing before 4 seconds and no cars can pass at $t=5, 6, 7$ seconds in order for the pedestrian to only have to wait 4 seconds before crossing.

Thus we have: $p(1 - p)^3$

Looking at the first 3 seconds, at least one car must pass during the period so the pedestrian does not start crossing immediately.

The probability of this is: $1 - (1 - p)^3$

Multiplying the two probabilities together we have:

$$(1 - (1 - p)^3)p(1 - p)^3$$

Problem 4

The probability of a success is .8 if the new and old drugs are equally effective. Assuming all trials are independent we can use the binomial distribution

$$P(X \geq 85) = \sum_{85}^{100} \binom{100}{k} .8^k .2^{100-k} = .1285055$$

Since the probability of observing 85 or more success is almost 13%, this is not that unlikely so we cannot conclude the new drug is more effective.

Problem 7

- (a) There are $\binom{N}{4}$ ways to select 4 packets of cocaine over $\binom{N+M}{4}$ total ways to select 4 packets. The probability of selecting 4 packets of cocaine is

$$\frac{\binom{N}{4}}{\binom{N+M}{4}}$$

There are $\binom{M}{2}$ ways to select 2 non cocaine packets after selecting 4 cocaine packets over $\binom{N+M-4}{2}$ total ways to select 2 packets after having already selected 4 packets.

The probability of selecting 2 non cocaine packets is $\frac{\binom{M}{2}}{\binom{N+M-4}{2}}$

Thus the probability the defendant is innocent equals the above probabilities multiplied together.

$$P(\text{Defendent Innocent}) = \frac{\binom{N}{4}\binom{M}{2}}{\binom{N+M}{4}\binom{N+M-4}{2}}$$

- (b) Since $M = 496 - N$ we have:

$$\begin{aligned} \frac{\binom{N}{4}\binom{496-N}{2}}{\binom{496}{4}\binom{492}{2}} \quad N = 4, \dots, 496 \\ \frac{P_{N+1}}{P_N} &= \frac{\binom{N+1}{4}\binom{496-N-1}{2}}{\binom{N}{4}\binom{496-N}{2}} \\ &= \frac{(N+1)(494-N)}{(N-3)(496-N)} \end{aligned}$$

Plugging in $N = 331$ we have :

$$\frac{(331+1)(494-331)}{(331-3)(496-331)} \approx 1.00547 > 1$$

Plugging in $N = 332$ we have :

$$\frac{(332+1)(494-332)}{(332-3)(496-332)} \approx .9998 < 1$$

Thus $N = 331$ is a maximum

Plugging in $N = 331$ into the the original equation we get:

$$\frac{\binom{331}{4}\binom{496-331}{2}}{\binom{496}{4}\binom{492}{2}} = .02208168 \approx .022$$

The maximum innocence probability is .022 and this is attained at $N=331$

Problem 9

WTS: Limiting $r \rightarrow \infty$, $p \rightarrow 1$, $r(1-p) \rightarrow \lambda$

In the Negative Binomial mgf: $M(t) = \left(\frac{p}{1 - (1-p)e^t} \right)^r$

We have the Poisson mgf: $M(t) = e^{\lambda(e^t-1)}$

Let $a = r(1-p)$ then we have:

$$\begin{aligned} M(t) &= \left(\frac{1 - a/r}{1 - (a/r)e^t} \right)^r \\ &= \left(\frac{1 - (a/r)e^t + (a/r)e^t - (a/r)}{1 - (a/r)e^t} \right)^r \\ &= \left(\frac{(1 - (a/r)e^t) + (a/r)(e^t - 1)}{1 - (a/r)e^t} \right)^r \\ &= \left(\frac{r(1 - (a/r)e^t) + a(e^t - 1)}{r(1 - (a/r)e^t)} \right)^r \\ &= \left(1 + \frac{a(e^t - 1)}{r(1 - (a/r)e^t)} \right)^r \\ &= \left(1 + \frac{a(e^t - 1)}{1 - (a/r)e^t} \right)^r \end{aligned}$$

$$\text{Let } a_n = \frac{a(e^t - 1)}{1 - (a/r)e^t}$$

applying the limits:

$$\begin{aligned} a_n \text{ goes to } & \frac{\lambda(e^t - 1)}{1 - (1-1)e^t} \\ &= \lambda(e^t - 1) \end{aligned}$$

And since we know from Lemma(2.3.14) that:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{a_n}{n} \right)^n = e^a$$

where a_n is a sequence where $\lim_{n \rightarrow \infty} a_n = a$

$$\left(1 + \frac{a_n}{r} \right)^r \text{ goes to } e^{\lambda(e^t-1)}$$

Which is the Poisson mgf

Thus the Negative Binomial mgf converges to the Poisson mgf

Problem 10

(a)

$$\text{WTS: } \Gamma(a+1) = a\Gamma(a)$$

$$\begin{aligned}\Gamma(a+1) &= \int_0^\infty t^a e^{-t} dt \\ &= \left[-t^a e^{-t} + a \int_0^\infty t^{a-1} e^{-t} dt \right]_0^\infty \\ \text{since } \Gamma(a) &= \int_0^\infty t^{a-1} e^{-t} dt \text{ we have:} \\ &= 0 + a\Gamma(a) = a\Gamma(a)\end{aligned}$$

(b)

$$\Gamma(1/2) = \int_0^\infty t^{1/2-1} e^{-t} dt = \int_0^\infty t^{-1/2} e^{-t} dt$$

doing a change of variables where $u = \sqrt{2t}$, $t = u^2/2$, $du = t^{-1/2}/\sqrt{2}$ we have:

$$\int_0^\infty \sqrt{2} e^{-u^2/2} du$$

from (3.3.14) we know:

$$\int_0^\infty e^{-u^2/2} du = \sqrt{\pi/2} \text{ so we have:}$$

$$\sqrt{\pi/2} \sqrt{2} = \sqrt{\pi}$$

Problem 11

Let $X \sim \text{binom}(1000, 1/6)$

Using the normal approximation to the binomial

$$p = 1/6 \quad n = 1000 \quad \mu = np = 1000 * 1/6 = 500/3$$

$$\sigma = \sqrt{(500/3)(5/6)} = \sqrt{1250/3}$$

then we have $Y \sim \text{normal}(500/3, \sqrt{1250/3})$

$$P(150 \leq Y \leq 200) = P(Y \leq 200) - P(Y \leq 150)$$

$$= P(Z \leq (200 - 500/3)/\sqrt{1250/3}) - P(Z \leq (150 - 500/3)/\sqrt{1250/3})$$

$$= \Phi(2.87) - \Phi(-1.46) = \Phi(2.87) - 1 + \Phi(-1.46) = .9258$$

Problem 12

Let k be the number of years old the radio is

$$P(X > 8 + k | X > 8) = P(X > 8)$$

by the memoryless property of the exponential distribution

$$P(X > 8) = \int_8^\infty (1/8)e^{-(1/8)x} dx$$

$$= \left| -e^{-(1/8)x} \right|_8^\infty = e^{-1}$$

Problem 13

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda-t)x} dx$$

$$= \frac{-\lambda}{\lambda-t} \Big|_0^\infty \text{ for } t < \lambda$$

$$M_X(t) = \frac{\lambda}{\lambda-t} \text{ for } t < \lambda$$

$$M_X(t) = \sum_{k=0}^\infty \frac{t^k}{k!} E(X^k)$$

Letting $E(X^k) = \frac{k!}{\lambda^k}$ we have:

$$M_X(t) = \sum_{k=0}^\infty \frac{t^k}{k!} \frac{k!}{\lambda^k}$$

$$= \sum_{k=0}^\infty \frac{t^k}{\lambda^k} = \frac{\lambda}{\lambda-t} \text{ by sum of infinite geometric series}$$

$$\text{Thus } E(X^k) = \frac{k!}{\lambda^k}$$

Problem 14

WTS: If $X \sim \exp(\lambda)$, $cX \sim \exp(\lambda/c)$ $c > 0$

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

$$P(X \leq x) = F(x) = 1 - e^{-\lambda x} \quad x \geq 0$$

Let $Y = cX$

$$\text{Then } P(Y \leq y) = P(cX \leq y) = P(X \leq y/c) = F_X(y/c) = 1 - e^{-\lambda y/c}$$

$$f_Y(y) = \frac{d}{dy} 1 - e^{-\lambda y/c} = \frac{\lambda}{c} e^{-\lambda y/c} \quad y \geq 0$$

Thus $cX \sim \exp(\lambda/c)$

Problem 15

$$F(s) = 1 - \exp\left(-\left(\frac{s-v}{\alpha}\right)^\beta\right) \quad s \geq v = 0$$

$$1 - F(s) = \exp\left(-\left(\frac{s-0}{\alpha}\right)^\beta\right)$$

$$\log(1 - F(s)) = \log\left(\exp\left(-\left(\frac{s}{\alpha}\right)^\beta\right)\right) = \left(-\frac{s}{\alpha}\right)^\beta$$

$$\log(1 - F(s))^{-1} = -\left(\frac{s}{\alpha}\right)^{-\beta}$$

$$\log(\log(1 - F(s))^{-1}) = \log\left(-\left(\frac{s}{\alpha}\right)^{-\beta}\right)$$

$$= -\beta \log\left(-\left(\frac{s}{\alpha}\right)\right)$$

$$= \beta \log(s) - \beta \log(\alpha)$$

Plotting this against $\log(s)$ we have a linear function

$$P(s \leq \alpha) = 1 - \exp\left(-\left(\frac{\alpha}{\alpha}\right)^\beta\right)$$

$$= 1 - \exp(-(1)^\beta) = 1 - \exp(-1)$$

$$= .6321206$$