Problem 1

$$E(X^n) = c \quad \forall\ n \geq 1$$

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} c = ce^t$$

$M_X(0) = 1$ by defintion, so we need to find a constant k that satisfys this equation

$$M_X(0) = ce^0 + k = 1$$

$$k = 1 - c$$

Thus $M_X(t) = ce^t + 1 - c$

Which is the same as the mgf of the Bernoulli distribution with parameter c

Problem 2

$$E[X^n] = \frac{2^n}{n+1} \text{ for } n \geq 1$$

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{2^n}{n+1} = \sum_{n=0}^{\infty} \frac{(2t)^n}{(n+1)!} = \frac{e^{2t} - 1}{2t}$$

Since $M_X(t)$ is undefined for $t = 0$ and $M_X(0)$ must equal 1, we define the mgf as:

$$M_X(t) = \begin{cases} \dfrac{e^{2t} - 1}{2t} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

This is the same as the mgf for the Uniform (0,2) distribution.

Problem 3

(a)

Given $f_X(x) = f_X(-x)\ \forall\ x$

WTS: X and $-X$ are identically distributed

$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = \int_{-\infty}^{\infty} f_X(-x)\mathrm{d}x$$

Thus $F_X(x) = F_X(-x)\ \forall x$

Since the CDFs are equal, X and -X are identically distributed

(b)

Given $f_X(x) = f_X(-x)\ \forall\ x$

WTS: $M_X(t) \forall\ \epsilon > 0\ M_X(0+\epsilon) = M_X(0-\epsilon)$

$$\begin{aligned}
M_X(0+\epsilon) &= \int_{-\infty}^{\infty} e^{(0+\epsilon)x} f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} e^{\epsilon x} f_X(x)\mathrm{d}x + \int_{0}^{\infty} e^{\epsilon x} f_X(x)\mathrm{d}x \\
&= \int_{0}^{\infty} e^{\epsilon(-x)} f_X(-x)\mathrm{d}x + \int_{-\infty}^{0} e^{\epsilon(-x)} f_X(-x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{\epsilon(-x)} f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{(0-\epsilon)x} f_X(x)\mathrm{d}x = M_X(0-\epsilon)
\end{aligned}$$

Thus $M_X(t)$is symmetric about 0

Problem 4

A distribution does not exist because:
$M_X(0) = \frac{0}{1-0} = 0$
But $M_X(0) = 1$ always by definition.

Problem 5

$$EX = \frac{d}{dt}S(t)\bigg|_{t=0} = \frac{d}{dt}log(M_X(t))\bigg|_{t=0} = \frac{\frac{d}{dt}M_X(t)}{M_X(t)}\bigg|_{t=0}$$

Since $M_X(0) = 1$ always and $\frac{d}{dt}M_X(t) = EX$ we have:

$$\frac{\frac{d}{dt}M_X(t)}{M_X(t)}\bigg|_{t=0} = \frac{EX}{1} = EX$$

$$Var(X) = \frac{d^2}{dt^2}S(t)\bigg|_{t=0} = \frac{d}{dt}\frac{M_X^{'}(t)}{M_X(t)}\bigg|_{t=0} = \frac{M_X^{''}(t)M_X(t) - M_X^{'}(t)^2}{M_X(t)^2}\bigg|_{t=0}$$
$$= \frac{E[X^2]*1 - E[X]^2}{1^2} = E[X^2] - E[X]^2 = Var(X)$$

Problem 6

(a)

$$M_X(t) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}e^{tx} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

Since $\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{\lambda}$ we have:

$$e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

$$E(X) = \frac{d}{dt}e^{\lambda(e^t-1)}\bigg|_{t=0} = e^{-\lambda}\lambda e^{\lambda e^t + t}\bigg|_{t=0} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$
$$E(X^2) = \frac{d}{dt}e^{-\lambda}\lambda e^{\lambda e^t+t}\bigg|_{t=0} = e^{-\lambda}\lambda(\lambda e^t + 1)e^{\lambda e^t + t}\bigg|_{t=0}$$
$$= e^{-\lambda}\lambda(\lambda+1)e^{\lambda} = \lambda(\lambda+1)$$
$$Var(X) = E(X^2) - E(X)^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

(b)

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{\infty} e^{tx} p(1-p)^x = p\sum_{x=0}^{\infty}(e^t(1-p))^x \\
&= \frac{p}{pe^t - e^t + 1} = \frac{p}{1-(1-p)e^t} \\
E(X) &= \frac{d}{dt}\frac{p}{1-(1-p)e^t}\bigg|_{t=0} = \frac{p(1-p)e^t}{(1-(1-p)e^t)^2}\bigg|_{t=0} \\
E(X) &= \frac{p(1-p)}{p^2} = \frac{1-p}{p} \\
E(X^2) &= \frac{d}{dt}\frac{p(1-p)e^t}{(1-(1-p)e^t)^2}\bigg|_{t=0} \\
\\
&= \frac{2p(1-p)^2e^{2x}}{(1-(1-p)e^x)^3} + \frac{p(1-p)e^x}{(1-(1-p)e^x)^2}\bigg|_{t=0} \\
&= \frac{2(1-p)^2 + p(1-p)}{p^2} \\
Var(X) &= \frac{2(1-p)^2 + p(1-p)}{p^2} - (\frac{1-p}{p})^2 = \frac{(1-p)^2 + p(1-p)}{p^2} = \frac{1-p}{p^2}
\end{aligned}
$$

(c)

$$
\begin{aligned}
M_X(t) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-(x^2-2x\mu-2\sigma^2 tx+\mu^2)/2\sigma^2}\mathrm{d}x \\
M_X(t) &= e^{\mu t+\sigma^2t^2/2} \\
E(X) &= \frac{d}{dt}e^{\mu t+\sigma^2t^2/2}\bigg|_{t=0} = e^{\mu t+\sigma^2t^2/2}(\mu+\sigma^2t/2)\bigg|_{t=0} \\
E(X) &= e^0(\mu+0) = \mu \\
E(X^2) &= \frac{d}{dt}e^{\mu t+\sigma^2t^2/2}(\mu+\sigma^2t/2)\bigg|_{t=0} = e^{\mu t+\sigma^2t^2/2}(\mu+\sigma^2t)^2 + \sigma^2e^{\mu t+\sigma^2t/2}\bigg|_{t=0} \\
E(X^2) &= e^0(\mu)^2 + \sigma^2e^0 = \mu^2+\sigma^2 \\
Var(X) &= \mu^2+\sigma^2-\mu^2 = \sigma^2
\end{aligned}
$$

Problem 7

$$\text{Let } f(x) = \frac{1}{x}$$
$$g(x) = M_X(t)$$
$$\text{on the interval } (0, \infty),\ g(x) \geq f(x)$$

Using the comparision test we can show $f(x)$ diverges, thus $g(x)$ must also diverge

$$\int_0^\infty \frac{1}{x} = \log(x)\Big|_0^\infty \text{ diverges}$$
$$\text{Thus } M_X(t) \text{ also diverges}$$
$$\text{Therefore } M_X(t) \text{ does not exist}$$

Problem 8

(a)

$$\begin{aligned}
M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x \\
&= \frac{p^r}{(r-1)!} \sum_{x=0}^{\infty} \frac{(r+x-1)! e^{tx} (1-p)^x}{x!} \\
M_X(t) &= \frac{p^r}{(r-1)!}(r-1)!(1-(1-p)e^t)^{-r} = \left(\frac{p}{1-(1-p)e^t}\right)^r \\
M_X(t) &= \left(\frac{p}{1-(1-p)e^t}\right)^r
\end{aligned}$$

(b)

$$\begin{aligned}
M_Y(t) &= E(e^{tY}) = E(e^{2ptX}) = M_X(2pt) \\
&= \left(\frac{p}{1-(1-p)e^{2pt}}\right)^r
\end{aligned}$$

Using L'Hospital's Rule to find the limit as p goes to 0

$$\lim_{p\to 0} \left(\frac{\frac{d}{dp}p}{\frac{d}{dp}1-(1-p)e^{2pt}}\right)^r$$
$$= \lim_{p\to 0}\left(\frac{1}{e^{2pt}(2(p-1)t+1)}\right)^r$$
$$= \left(\frac{1}{1-2t}\right)^r$$

Problem 9

(a)

$$P(X=x)=\begin{cases} p^x(q)^{1-x} & x=0,1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \phi_X(t)=E(e^{itx}) &= \sum_{x=0}^{1} e^{itx}p_X(x) \\ &= e^{it*1}p_X(1)+e^{it*0}*p_X(0) \\ &= pe^{it}+q \end{aligned}$$

(b)

$$P(X=x)=\begin{cases} \binom{n}{x}p^xq^{n-x} & x=0,1,\dots,n \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \phi_X(t)=E(e^{itx}) &= \sum_{x=0}^{n} e^{itx}\binom{n}{x}p^xq^{n-x} \\ &= \sum_{x=0}^{n}\binom{n}{x}(e^{it}p)^xq^{n-x} \end{aligned}$$

Using the binomial theorem we have :

$(pe^{it}+q)^n$

(c)

$$P(X=x)=\begin{cases} p(q)^x & x=0,1,2,\dots \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \phi_X(t)=E(e^{itx}) &= \sum_{x=0}^{\infty} e^{itx}p(q)^x \\ &= p\sum_{x=0}^{\infty}(qe^{it})^x=\frac{p}{1-qe^{it}} \end{aligned}$$

(d)

$$\begin{aligned} P(X=x) &= \frac{e^{-\lambda}\lambda^x}{x!} \\ \phi_X(t)=E(e^{itx}) &= \sum_{x=0}^{\infty} e^{itx}\frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{it})^x}{x!}=e^{-\lambda}e^{\lambda e^{it}}=e^{\lambda e^{it}-\lambda} \\ &= e^{\lambda(e^{it}-1)} \end{aligned}$$