Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 4.55
2. C&B 5.13
3. C&B 5.23
4. Suppose that a random variable U follows $N(0,1)$ and a random variable V follows a χ^2 distribution with p degrees of freedom. Assuming that U and V are independent, one can show that a random variable

$$T = \frac{U}{\sqrt{V/p}}$$

follows a t-distribution with p degrees of freedom.

(a) Find the conditional density function of T, given $V = v$, and use the result to derive the marginal density function of T.

(b) Find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$ without using the marginal density function of T.

(c) Use the transformation method to find the density function of T, as suggested in the course slides.

5. For patients receiving a double kidney transplant, let X_i be the lifetime (in months) of the ith kidney, $i = 1, 2$. Also, assume that X_i follows exponential distribution with density

$$f_{X_i}(x_i) = \alpha e^{-\alpha x_i}, \quad x_i > 0, \quad \alpha > 0, \quad i = 1, 2.$$

Assume X_1 and X_2 are independent, and define a new variable V, which is the lifetime of the remaining functional kidney as soon as one of the two kidneys fails, having a conditional density function

$$f_V(v|U = u) = \beta e^{-\beta(v-u)}, \quad 0 < u < v < \infty, \quad \beta > 2\alpha,$$

where $U = \min(X_1, X_2)$.

(a) Show that the probability that both organs are still functioning at time t is equal to

$$\pi_0(t) = e^{-2\alpha t}, \quad t \geq 0.$$

(b) Show that the probability that exactly one organ is still functioning at time t is equal to

$$\pi_1(t) = \frac{2\alpha}{(\beta - 2\alpha)} \left(e^{-2\alpha t} - e^{-\beta t}\right), \quad t \geq 0.$$

[Hint: P(exactly one kidney is functioning at time t)=$P(U \leq t, V \geq t)$].

(c) Find $f_T(t)$, where T is the length of time (in months) until both kidneys have failed.

6. Let $X = (X_1, \ldots, X_n)$ be a random vector having the joint distribution as a multivariate normal distribution $N(\mu J, D)$, where J is a vector of 1 and

$$D = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix},$$

and $|\rho| < 1$. Solve the following items.

(a) Let

$$A = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2\times 1}} & \frac{-1}{\sqrt{2\times 1}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3\times 2}} & \frac{1}{\sqrt{3\times 2}} & \frac{-2}{\sqrt{3\times 2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n\times(n-1)}} & \frac{1}{\sqrt{n\times(n-1)}} & \frac{1}{\sqrt{n\times(n-1)}} & \frac{1}{\sqrt{n\times(n-1)}} & \cdots & \frac{-(n-1)}{\sqrt{n\times(n-1)}} \end{pmatrix}.$$

Show that $AA^T = A^T A = I$, where I is an $n \times n$ identity matrix, and that

$$ADA^T = \sigma^2 \begin{pmatrix} 1+(n-1)\rho & 0 & \cdots & 0 \\ 0 & 1-\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\rho \end{pmatrix}.$$

(b) Derive the distribution of $Y = AX$ and show that $Y = (Y_1, \ldots, Y_n)$ are mutually independent.

(c) Show that $Y_1 = \sqrt{n}\bar{X}$ and that $\bar{X}$ has the normal distribution $N(\mu, \frac{1+(n-1)\rho}{n}\sigma^2)$.

(d) Show that $W = \sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=2}^n Y_i^2$ and that $W/\{(1-\rho)\sigma^2\}$ follows χ^2_{n-1}.

(e) Show that $\bar{X}$ and W are independent.

7. (Bios 673 class material) Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution with pdf

$$\beta^{-1} e^{(\alpha - x)/\beta}, \quad \alpha < x < \infty,$$

where $\alpha \in \mathcal{R}$ and $\beta > 0$ are parameters. Let $X_{(1)} \le \cdots \le X_{(n)}$ be order statistics, and let $Z_1 = X_{(1)}$ and $Z_i = X_{(i)} - X_{(i-1)}$ for $i = 2, \ldots, n$. Show that

(a) $Z_1, \ldots, Z_n$ are independent and $2(n-i+1)Z_i/\beta$ has the χ^2_2 distribution.

(b) $X_{(1)}$ and Y are independent, where $Y = (n-1)^{-1}\sum_{i=1}^n (X_i - X_{(1)})$.

(c) $T = (X_{(1)} - \alpha)/Y$ has a pdf

$$f_Y(t) = n\left(1 + \frac{nt}{n-1}\right)^{-n},$$

for $0 < t < \infty$ and 0 otherwise.