Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 5.30
2. C&B 5.32
3. Two *sufficient* conditions for consistency, $T_n \to_p \theta$, are

   i. $\lim_{n\to\infty} E(T_n) = \theta$;

   ii. $\lim_{n\to\infty} Var(T_n) = 0$.

   Assume the distribution of income (in thousands of dollars) in a large U.S. city follows a Pareto density function

   $$f_Y(y) = \theta\gamma^\theta y^{-(\theta+1)}, \quad 0 < \gamma < y < \infty, 2 < \theta < \infty,$$

   where γ and θ are known parameters. Let $Y_1, \cdots, Y_n$ be a random sample from $f_Y(y)$. Show that the sample minimum $Y_{(1)}$ is a consistent estimator of γ, i.e., $Y_{(1)} \to_p \gamma$.

4. (midterm 1 in 2014) Suppose that $X_1, X_2, \ldots, X_n$ are iid random variables distributed as Poisson with mean $\mu > 0$. Denote $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$. We are interested in constructing a confidence interval for μ.

   (a) State the central limit theorem for $\bar{X}_n$.

   (b) What is the asymptotic variance of $T_n = \sqrt{n}(\bar{X}_n - \mu)$?

   (c) What is the appropriate function $h(\bar{X}_n)$ so that $h(\bar{X}_n)T_n \to_d N(0,1)$? What theorem(s) are needed to justify such claim?

   (d) Use the last part to construct an approximate 95% confidence interval for μ. Give the upper and lower limits in explicit form.

   (e) Another approach to eliminate μ from the asymptotic variance is to find a function g such that $\sqrt{n}(g(\bar{X}_n) - g(\mu)) \to_d N(0,1)$. Find an explicit expression for $g(\mu)$.

   (f) (For discussion, no need to return) Use the last part to construct an approximate 95% confidence interval for μ. Give the upper and lower limits in explicit form.

5. (midterm 1 in 2015) Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, 1)$.

   (a) Find the limiting distribution of $U_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i - \mu)$ by the central limit theorem.

   (b) Show that $V_n = \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 \to 1$ in probability by the weak law of large numbers.

(c) Find the limiting distribution of $W_n = U_n/V_n$.

(d) Find the limiting distribution of $\sqrt{n}(\bar{X}^2 - \mu^2)$, where $\bar{X} = n^{-1}\sum_{i=1}^n X_i$.

(e) Construct a 95% confidence interval for μ^2, either under a finite n (exact) or $n \to \infty$ (limiting).

6. Suppose that X_n is a random variable following a binomial distribution $B(n, \theta)$, where $\theta \in (0, 1)$. Let

$$Y_n = \begin{cases} \log(X_n/n), & X_n \geq 1, \\ 1, & X_n = 0. \end{cases}$$

Show that $\lim_{n\to\infty} Y_n = \log\theta$ a.s. and $\sqrt{n}(Y_n - \log\theta) \to_d N(0, (1-\theta)/\theta)$.

7. (Bios 673 class material) Let $X_1, \ldots, X_n$ be independent random variables. Suppose that $\sum_{i=1}^n (X_i - EX_i)/\sigma_n \to_d N(0, 1)$, where $\sigma_n^2 = \mathrm{Var}(\sum_{i=1}^n X_i)$. Show that $n^{-1}\sum_{i=1}^n (X_i - EX_i) \to_p 0$ if and only if $\lim_{n\to\infty} \sigma_n/n = 0$.