Random Samples

$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$
Computation formula of S^2
$(n-1)S^2 = \sum_{i=1}^n X_i^2 - \left(\sum_{i=1}^n X_i\right)^2 / n$
Unbiased Estimator
If $ET(X_1, \dots, X_n) = \theta$ T is an unbiased estimator of θ
ex: if $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$
Then $\bar{X}$ is unbiased estimator of μ and S^2 of σ^2
Samples From Normal Distribution
$X_1, \dots, X_n$ is a random sample from $N(\mu, \sigma^2)$
Then $\bar{X} \sim N(\mu, \sigma^2/n)$

Order Statistics

Sample Max:
$F_{X_{(n)}}(x) = P(X_{(n)} \leq x) = P(X_1 \leq x, \dots, X_n \leq x) = \{F(x)\}^n$
If continuous:
$f_{X_{(n)}}(x) = nf(x)\{F(x)\}^{n-1}$
Sample Min:
$F_{X_{(1)}}(x) = 1 - P(X_{(1)} \leq x) = P(X_1 > x, \dots, X_n > x) = 1 - \{1 - F(x)\}^n$
If continuous:
$f_{X_{(1)}}(x) = nf(x)\{1 - F(x)\}^{n-1}$
Joint pdf of order statistics:
$f_{X_{(1)},\dots,X_{(n)}}(y_1, \dots, y_n) = n! \prod_{i=1}^n f_X(y_i)$ for $y_1 < \cdots < y_n$
Distribution of $X_{(j)}$
$f_{X_{(j)}}(x) = \frac{n!}{(i-1)!(n-1)!} f(x)\{F(x)\}^{j-1}\{1 - F(x)\}^{n-j}$
Joint Distribution of $(X_{(i)}, X_{(j)})$
$f_{X_{(i)},X_{(j)}}(u, v) =$
$\frac{n!}{(j-1)!(j-i-1)!(n-j)!} f(u)f(v)\{F(u)\}^{i-1}\{F(v) - F(u)\}^{j-i-1}\{1 - F(v)\}^{n-j}$

Convergence

$\bar{X_n} = \frac{1}{n} \sum_{i=1}^n X_i$
Probability
WLLN
Let $X_1, \dots, X_n$ be iid rv with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$ Then for every $\epsilon > 0$
$\lim_{n \to \infty} P(|\bar{X_n} - \mu| < \epsilon) = 1$ which is the same as:
$\lim_{n \to \infty} P(|\bar{X_n} - \mu| \geq \epsilon) = 0$
$\bar{X_n} \xrightarrow{p} \mu$ (consistency of $\bar{X_n}$)

Convergence in probability to a constant is the same as convergence in distribution to a constant

Distribution

$X_n \xrightarrow{d} X$ as $n \to \infty$

$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

CLT

Let $X_1, \ldots, X_n$ be iid rv with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$

Let $Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$

Then $Z_n \xrightarrow{d} N(0,1)$ as $n \to \infty$

Or $Z_n = \sqrt{n}(\bar{X} - \mu)$

Then $Z_n \xrightarrow{d} N(0,\sigma^2)$ as $n \to \infty$

$X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X$

Slutsky's Theorem

If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} a$ Then:

$Y_n X_n \xrightarrow{d} aX \quad Y_n + X_n \xrightarrow{d} a + X \quad X_n/Y_n \xrightarrow{d} X/a$

Slutsky's Thm allows substituting consistent estimators when proving $\xrightarrow{d}$

X_n doesnt have to be independent of Y_n

Convergence of Transformed Sequences

If $X_n \xrightarrow{p} X$ then $h(X_n) \xrightarrow{p} h(X)$

If $X_n \xrightarrow{d} X$ then $h(X_n) \xrightarrow{d} h(X)$

$S_n^2 \xrightarrow{a.s.} \sigma^2$ as $n \to \infty$

Delta Method

$\{T_n\}$ is a random sequence with $\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0,\sigma^2)$

g is a function with $g^{'}(\theta)$ exists and not 0. Then:

$\sqrt{n}\{g(T_n) - g(\theta)\} \xrightarrow{d} N(0, \{g^{'}(\theta)\}^2\sigma^2)$

θ is the asymptotic mean of T_n

SSLN

Same as WLLN but with $\xrightarrow{a.s.}$

Data Reduction

Sufficient Statistics

$T(X)$ is an SS for θ if $P(X = x|T(X) = t)$ does not depend on θ

$f(x|\theta) = h(x)c(\theta)\exp\left(\sum_{j=1}^{k} w_j(\theta)t_j(x)\right)$

T(X)=$(\sum_{i=1}^{n} t_1(X_i), \ldots, \sum_{i=1}^{n} t_k(X_i))$

Minimal SS

an SS is a minSS if it is a function of every other SS

Any one to one tranformation of a min SS is also a min SS.

Let $f(x|\theta)$ be joint pdf or pmf of X. Suppose $T(X)$ exists such that for every two sample points x and y:

$f(x|\theta)/f(y|\theta)$ does not depend on $\theta \iff T(x) = T(y)$

Then $T(X)$ is a min SS for θ

Ancillary Statistic for θ
A statistic whose distribution does not depend on the parameter θ
Complete Statistic
Completeness means that $E(g(T)) \neq 0$ (except for 0 function)
Complete if can be written as exponential family
For our purposes an SS is complete only if it is minimal.
Basu's Theorem
If $T(X)$ is a complete and minimally sufficient statistic, then $T(X)$ is independent of every ancillary statistic

Other Stuff

Exponential CDF $1 - e^{-\lambda x}$
$E(X) = E(E(X|Y))$
$\Gamma(n) = (n-1)\Gamma(n-1)$
$\Gamma(n) = (n-1)!$ (n is an whole number)
$Var(X) = E(X^2) - E(X)]^2$
$Var(X) = E[(X-\mu)^2]$
Law of total variance ($Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$)
$\lim_{n\to\infty}(1-x/n)^n = e^{-x}$
$P(a < X < b) = F_x(b) - F_x(a)$
$\beta(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
Conditional Probability
$P\{Y=y|X=x\} = \dfrac{P(X=x,Y=y)}{P(X=x)}$
$= \dfrac{f_{X,Y}(x,y)}{f_X(x)}$
Convolution
$X \perp Y$
$Z = X + Y$
$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\ dy$
$\Gamma(1/2) = \sqrt{\pi}$
$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\ dt$