Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 8.12
2. C&B 8.7(a)
3. C&B 8.15
4. [From 2011 master exam] Let $X_1, \ldots, X_n$ be i.i.d. random variables from the pdf

$$f(x|\theta) = (1-\theta) + \frac{\theta}{2\sqrt{x}}, \quad 0 < x < 1, \quad 0 \le \theta \le 1.$$

That is, we have a random sample of size n from the population f. The parameter θ is unknown.

   (a) Derive the uniformly most powerful level α test ($0 < \alpha < 1$) for $H_0 : \theta = 0$ against $H_1 : \theta = 1$. Specify the critical region as concisely and as explicitly as possible. Justify your answers. Do not use any approximations.

   (b) For the special case of $n = 5$ and $\alpha = 0.01$, find the critical region (exactly). Also, find the (exact) power of the test.

   (c) An investigator wants to design a study in which the test derived above will be applied. The investigator desires a Type I Error probability of 0.01 and a Type II Error probability of 0.01. Find the minimum required sample size n (exact, or approximate, whichever is easier).

   (d) This part pertains to the special case of $n = 1$ (sample size = 1). Find $\hat{\theta}$, the maximum likelihood estimator (MLE) of θ. Show that the MLE is biased. Then find constants a and b such that $T(X_1) = a + b\hat{\theta}$ is unbiased for θ. Do you see any potential problems with $T(X_1)$ as an estimator of θ?

5. An epidemiologist gathers data (x_i, Y_i) on each of n randomly chosen noncontiguous and demographically similar cities in the United States, where x_i, $i = 1, \ldots, n$, is the known population size (in millions of people) in city i, and where Y_i is the random variable denoting the number of people in city i with colon cancer. It is reasonable to assume that Y_i, $i = 1, \ldots, n$, has a Poisson distribution with mean $E(Y_i) = \theta x_i$, where $\theta > 0$ is an unknown parameter, and that $Y_1, Y_2, \ldots, Y_n$ are mutually independent random variables.

   (a) Use the available data (x_i, Y_i), $i = 1, \ldots, n$, construct a uniformly most powerful (UMP) level α test for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

   (b) Use the available data (x_i, Y_i), $i = 1, \ldots, n$, construct a uniformly most powerful (UMP) level α test for $H_0 : \theta \le 1$ versus $H_1 : \theta > 1$. Is this critical region the same as the one used in (a)?

(c) One can show that $S = \sum_{i=1}^n Y_i$ follows Poisson$(\theta \sum_{i=1}^n x_i)$. If one observes $\sum_{i=1}^n x_i = 0.8$, find c^* in the critical region $\mathcal{R} = \{S : S \geq c^*\}$ which is a level $\alpha = 0.05$ test.

(If $X \sim$ Poisson(0.8), then $P(X = 0) = 0.449$, $P(X \leq 1) = 0.808$, $P(X \leq 2) = 0.952$, $P(X \leq 3) = 0.990$, $P(X \leq 4) = 0.999$).

(d) What is the power when $\theta = 5$, using the critical region in (c) and $\sum_{i=1}^n x_i = 0.8$?

(If $X \sim$ Poisson(4), then $P(X = 0) = 0.018$, $P(X \leq 1) = 0.092$, $P(X \leq 2) = 0.238$, $P(X \leq 3) = 0.433$, $P(X \leq 4) = 0.628$).

6. Suppose that $Y_1, \ldots, Y_n$, $n > 1$, is a random sample from the pdf

$$f_Y(y|\theta) = \frac{4}{\sqrt{\pi}} \theta^{-3} y^2 \exp\left(-\frac{y^2}{\theta^2}\right), \quad 0 < y < \infty, \quad 0 < \theta < \infty.$$

(a) Show that Y_i^2, $i = 1, \ldots, n$, follows a Gamma distribution $\Gamma(3/2, \theta^2)$.

(b) Derive the uniformly most powerful size α test, $0 < \alpha < 1$, of $H_0 : \theta = 1$ against $H_1 : \theta > 1$. Specify the rejection region as $R = \{\boldsymbol{y} : \sum_{i=1}^n y_i^2 \geq c^*\}$ with some constant c^*.

(c) Derive the likelihood ratio test statistic $\lambda(\boldsymbol{y})$ for $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$, and show that the rejection region $R = \{\boldsymbol{y} : \lambda(\boldsymbol{y}) \leq c\}$ is equivalent to $R = \{\boldsymbol{y} : \sum_{i=1}^n y_i^2 \leq c_1^* \text{ or } \sum_{i=1}^n y_i^2 \geq c_2^*\}$. Find the cutoff c_1^* and c_2^* explicitly given size $\alpha = 0.05$.

(d) Determine whether one rejects the null hypothesis if $n = 25$ and $\hat{\theta} = 1.2$ (observed value of θ) at the 0.05 level.

7. [Bios 673/740 class discussion] Let $X_1, \ldots, X_n$ be a random sample from the discrete uniform distribution on points $1, \ldots, \theta$, where $\theta = 1, 2, \ldots$.

(a) Consider $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where $\theta_0 > 0$ is known. Show that

$$\delta^*(X) = \begin{cases} 1 & X_{(n)} > \theta_0 \\ \alpha & X_{(n)} \leq \theta_0 \end{cases}$$

is a UMP size α test.

(b) Consider $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Show that

$$\delta^*(X) = \begin{cases} 1 & X_{(n)} > \theta_0 \text{ or } X_{(n)} \leq \theta_0 \alpha^{1/n} \\ \alpha & \text{otherwise} \end{cases}$$

is a UMP size α test.