Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 8.20

2. C&B 8.22

3. C&B 8.28

4. C&B 8.31

5. Let $X_1, \ldots, X_n$ be random sample of size n having a pdf of the form $f(x|\theta) = 1/\theta$, $0 < x < \theta$, and zero elsewhere. Let $X_{(n)}$ be the maximum order statistics. One would reject $H_0 : \theta = 1$ and accept $H_1 : \theta \neq 1$ if either $X_{(n)} \leq 1/2$ and $X_{(n)} > 1$. Find the power function $\beta(\theta)$ of the test for $\theta > 0$,.

   Solution: The cdf of $X_{(n)}$ given a value of θ is $P(X_{(n)} \leq y) = \{P(X_1 \leq y)\}^n = y^n/\theta^n$. The power $\beta(\theta)$ at the null hypothesis $\theta = 1$ is

   $$\beta(1) = P(X_{(n)} \leq 1/2|\theta = 1) + P(X_{(n)} > 1|\theta = 1) = P(X_{(n)} \leq 1/2|\theta = 1) = 2^{-n}.$$

   When $\theta > 1$, the power function $\beta(\theta)$ becomes

   $$\begin{aligned}\beta(\theta) &= P(X_{(n)} \leq 1/2|\theta > 1) + P(X_{(n)} > 1|\theta > 1) = \int_0^{1/2} f_{X_n}(y)dy + \int_1^{\theta} f_{X_n}(y)dy \\ &= 2^{-n}\theta^{-n} + 1 - \theta^{-n}.\end{aligned}$$

   One can see that when $\theta \to \infty$, $\beta(\infty) = 1$, which makes sense. When $0 < \theta \leq 1/2$, the power function $\beta(\theta)$ becomes

   $$\begin{aligned}\beta(\theta) &= P(X_{(n)} \leq 1/2|0 < \theta \leq 1/2) + P(X_{(n)} > 1|0 < \theta \leq 1/2) \\ &= P(X_{(n)} \leq 1/2|0 < \theta \leq 1/2) = \int_0^{\theta} f_{X_n}(y)dy = 1.\end{aligned}$$

   When $1/2 < \theta < 1$, the power function $\beta(\theta)$ becomes

   $$\begin{aligned}\beta(\theta) &= P(X_{(n)} \leq 1/2|1/2 < \theta < 1) + P(X_{(n)} > 1|1/2 < \theta < 1) \\ &= P(X_{(n)} \leq 1/2|1/2 < \theta < 1) = \int_0^{1/2} f_{X_n}(y)dy = 2^{-n}\theta^{-n}.\end{aligned}$$

   Accordingly,

   $$\beta(\theta) = \begin{cases} 1 & 0 < \theta \leq 1/2 \\ 2^{-n}\theta^{-n} & 1/2 < \theta \leq 1 \\ 2^{-n}\theta^{-n} + 1 - \theta^{-n} & \theta > 1. \end{cases}$$

The power equaling 1 for $0 < \theta \leq 1/2$ is interesting. That means, when the true θ lies in between 0 and 1/2, the proposed rejection rule $X_{(n)} \leq 1/2$ or $X_{(n)} > 1$ makes no errors. Why? That is because when the actual θ lies in between 0 and 1/2, the $X_{(n)}$ is always smaller than 1/2, so we always reject the null hypothesis, which is always the right decision since the null hypothesis is wrong ($\theta \neq 1$).

6. Let $X_1, \ldots, X_n$ be a random sample from a population with probability density function

$$f(x|\theta) = \frac{1}{\sqrt{2\pi}\theta x} \exp\left\{-\frac{1}{2}\left(\frac{\log x}{\theta}\right)^2\right\}, \quad x > 0, \quad \theta > 0.$$

This is a pdf of what we call "log-normal distribution" and could be a possible distribution other than exponential (or Gamma family) for a variable having only positive values (with only positive domain). You may check C&B to see its relationship with the normal distribution.

(a) Let $T = \sum_{i=1}^n (\log X_i)^2$. Show that $P(T > t|\theta = \theta_2) > P(T > t|\theta = \theta_1)$, for all $\theta_2 > \theta_1$ and any constant value $t > 0$.

Solution: Let $Y_i = \log X_i$. One can show that Y_i follows $\mathrm{N}(0, \theta^2)$ for $i = 1, \ldots, n$. Therefore,

$$\frac{T}{\theta^2} = \frac{\sum_{i=1}^n Y_i^2}{\theta^2} \sim \chi_n^2.$$

This result gives

$$P\left(\frac{T}{\theta_2^2} > \frac{t}{\theta_2^2}|\theta = \theta_2\right) > P\left(\frac{T}{\theta_1^2} > \frac{t}{\theta_1^2}|\theta = \theta_1\right),$$

since $t/\theta_2^2 < t/\theta_1^2$ when $\theta_2 > \theta_1$.

(b) Show that there is an uniformly most powerful test of null hypothesis $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, and find the rejection region of such test.

Solution: It is not difficult to show that T is a sufficient statistic for θ^2 and has an MLR property using the pdf of $\mathrm{N}(0, \theta^2)$. We hence can use Karlin-Rubin theorem to derive the UMP test with a test function

$$\delta(\boldsymbol{X}) = \begin{cases} 1 & \text{if } T > c \\ 0 & \text{otherwise,} \end{cases}$$

where c satisfies $\sup_{\theta \leq \theta_0} P(T > c|\theta) = \alpha$. Hence,

$$\alpha = \sup_{\theta \leq \theta_0} P(T > c|\theta) = P(T > c|\theta = \theta_0) = P\left(\frac{T}{\theta_0^2} > \frac{c}{\theta_0^2}\right).$$

One may have $c/\theta_0 = \chi^2_{n,1-\alpha}$ and choose $c = \theta_0\chi^2_{n,1-\alpha}$.

7. [Bios 763 class discussion] Let $X_1, \dots, X_n$ be a random variable having probability density $f(x|\theta) = \exp\{w(\theta)t(x) - \xi(\theta)\}h(x)$, where $w(\theta)$ is an increasing and differentiable function of $\theta \in \Theta \subset \mathcal{R}$.

   (a) Show that $\ell(\hat{\theta}) - \ell(\theta_0)$ is increasing (or decreasing) in t when $\hat{\theta} > \theta_0$ (or $\hat{\theta} < \theta_0$), where $\ell(\theta)$ is the log-likelihood function, $\hat{\theta}$ is the MLE of θ, and $\theta_0 \in \Theta$.

   Solution: The log-likelihood function $\ell(\theta)$ can be written as

$$\ell(\theta) = w(\theta)T - n\xi(\theta) + \sum_{i=1}^{n} \log\{h(x_i)\},$$

where $T = \sum_{i=1}^n t(x_i)$. The MLE $\hat{\theta}$ has to satisfy

$$w'(\hat{\theta})T - n\xi'(\hat{\theta}) = 0, \tag{1}$$

and

$$w''(\hat{\theta})T - n\xi''(\hat{\theta}) < 0.$$

We can conclude that $\hat{\theta}$ is an increasing function of T since, by differentiating on both sides of (1), we have

$$w''(\hat{\theta})\frac{d\hat{\theta}}{dT}T + w'(\hat{\theta}) - n\xi''(\hat{\theta})\frac{d\hat{\theta}}{dT} = 0,$$

and

$$\frac{d\hat{\theta}}{dT} = -\frac{w'(\hat{\theta})}{w''(\hat{\theta})T - n\xi''(\hat{\theta})} > 0.$$

Therefore, for $\theta_0 \in \Theta$,

$$\begin{aligned}\frac{d}{dT}\{\log\ell(\hat{\theta}) - \log\ell(\theta_0)\} &= \frac{d}{dT}\{w(\hat{\theta})T - n\xi(\hat{\theta}) - w(\theta_0)T - n\xi(\theta_0)\}\\ &= w'(\hat{\theta})\frac{d\hat{\theta}}{dT}T + w(\hat{\theta}) - n\xi'(\hat{\theta})\frac{d\hat{\theta}}{dT} - w(\theta_0)\\ &= \frac{d\hat{\theta}}{dT}\{w'(\hat{\theta})T - n\xi'(\hat{\theta})\} + w(\hat{\theta}) - w(\theta_0)\\ &= w(\hat{\theta}) - w(\theta_0),\end{aligned}$$

which is positive (increasing) if $\hat{\theta} > \theta_0$ or negative (decreasing) when $\hat{\theta} < \theta_0$.

(b) For testing $H_0 : \theta_1 \leq \theta \leq \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$, show that there is a likelihood ratio test whose rejection region is equivalent to $T(X) < c_1$ or $T(X) > c_2$ for some constant c_1 and c_2.

Solution: Since $\ell(\theta)$ is increasing when $\theta \leq \hat{\theta}$ and decreasing when $\hat{\theta} < \theta$, we can write that

$$\lambda(\boldsymbol{x}) = \frac{\sup_{\theta_1 \leq \theta \leq < \theta_2} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)} = \begin{cases} \frac{\exp\{\ell(\theta_1)\}}{\exp\{\ell(\hat{\theta})\}} & \hat{\theta} < \theta_1 \\ 1 & \theta_1 \leq \hat{\theta} \leq \theta_2 \\ \frac{\exp\{\ell(\theta_2)\}}{\exp\{\ell(\hat{\theta})\}} & \hat{\theta} > \theta_2 \end{cases}$$

for $\theta_1 \leq \theta_2$. Hence, the rejection region of LRT as $R = \{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\}$ is equivalent to $R = \{\boldsymbol{x} : \hat{\theta} > c_1^* \text{ or } \hat{\theta} < c_2^*\}$. Since $\hat{\theta}$ is an increasing function of T, the critical region is further equivalent to $R = \{\boldsymbol{x} : T > c_1 \text{ or } T < c_2\}$ for some constant c_1 and c_2.