Problem 1

(a)

Using X, Y for X_1, X_2

$$\text{Given } X,\ Y \sim n(0,1)$$
$$\text{We will first find } X - Y$$
$$\text{Let } U = X + Y \quad V = X - Y$$
$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2}$$
$$X = \frac{U+V}{2} \quad Y = \frac{U-V}{2}$$
$$\text{Since this solution is unique we have a one-to-one transformation}$$
$$J = |-1/2|$$
$$f_{UV}(u,v) = \frac{1}{2\pi} e^{-\left(\frac{u+v}{2}\right)^2/2} e^{-\left(\frac{u-v}{2}\right)^2/2} (1/2)$$
$$f_{UV}(u,v) = \left(\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-u^2/4}\right)\left(\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-v^2/4}\right)$$
$$f_{UV} \text{ can be factored into functions of U and V, they are independent}$$
$$f_V(v) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-v^2/4}$$
$$X - Y \sim n(0,2)$$
$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-(x-y)/4}$$
$$\frac{X-Y}{\sqrt{2}} \sim n(0,1)$$
$$\text{Let } Z = \frac{X-Y}{\sqrt{2}}$$
$$\text{Using the transformation } W = Z^2$$
$$g(z) = z^2 \text{ is monotone on } (-\infty, 0) \text{ and } (0, \infty)$$
$$g_1(z) = z^2 \quad g_1^{-1}(w) = -\sqrt{w}$$
$$g_2(z) = z^2 \quad g_2^{-1}(w) = \sqrt{w}$$
$$f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{w})^2/2} \left| -\frac{1}{2\sqrt{w}} \right| + \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{w})^2/2} \left| \frac{1}{2\sqrt{w}} \right|$$
$$f_W(w) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{w}} e^{-w/2} \quad 0 < w < \infty$$
$$\text{Thus } W \sim \chi_1^2$$

(b)

$$\text{Let } U = \frac{X_1}{X_1 + X_2} \quad V = X_1 + X_2$$
$$X_1 = UV \quad X_2 = V - UV$$
$$J = \begin{bmatrix} v & u \\ -v & 1-u \end{bmatrix} = |v|$$
$$f_{X_1X_2}(x_1, x_2) = \frac{1}{\Gamma(\alpha_1)} x_1^{\alpha_1 - 1} e^{-x_1} \frac{1}{\Gamma(\alpha_2)} x_2^{\alpha_2 - 1} e^{-x_2}$$
$$u \in (0,1) \quad v \in (0, \infty)$$
$$f_{UV}(uv) = \frac{1}{\Gamma(\alpha_1)} (uv)^{\alpha_1 - 1} e^{-uv} \frac{1}{\Gamma(\alpha_2)} [v(1-u)]^{\alpha_2 - 1} e^{-v(1-u)}(v)$$
$$f_U(u) = u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty v^{\alpha_1 + \alpha_2 - 1} e^{-v} \, dv$$
$$f_U(u) = u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1} \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$
$$\frac{X_1}{X_1 + X_2} = U \sim beta(\alpha_1, \alpha_2) \quad U \in (0,1)$$
$$\frac{X_2}{X_1 + X_2} = 1 - U \sim beta(\alpha_2, \alpha_1) \quad 1 - U \in (0,1)$$

Since the beta distribution has reflection symmetry

Problem 2

(a)

$$f_{X,Y}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1}$$
$$0 < x < 1 \quad 0 < y < 1$$
$$U = XY \quad V = Y$$
$$X = U/V \quad Y = V$$
$$J = 1/v$$
$$f_{U,V}(u,v) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} (u/v)^{\alpha-1}(1-u/v)^{\beta-1} v^{\alpha+\beta-1}(1-v)^{\gamma-1}(1/v)$$
$$0 < u < v < 1$$
$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 v^{\beta-1}(1-v)^{\gamma-1}((v-u)/v)^{\beta-1} \, dv$$
$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 (1-v)^{\gamma-1}(v-u)^{\beta-1} \, dv$$

$$\text{Let } w=\frac{v-u}{1-u} \quad \text{Then } dw=\frac{1}{1-u}\ dv \text{ and } 1-w=\frac{1-v}{1-u}$$

$$f_U(u)=\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}(1-u)^{\beta}(1-u)^{\gamma-1}\int_0^1(1-w)^{\gamma-1}(w)^{\beta-1}\ dw$$

$$\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1}\ 0<u<1$$

$$\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1}\ 0<u<1$$

$$U\sim beta(\alpha,\beta+\gamma)$$

(b)

$$f_{X,Y}(x,y)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)}y^{\alpha+\beta-1}(1-y)^{\gamma-1}$$

$$0<x<1 \quad 0<y<1$$

$$U=XY \quad V=X/Y$$

$$X=\sqrt{UV} \quad Y=\sqrt{\frac{U}{V}}$$

$$J=|-(1/2)v^{-1}|$$

$$f_{U,V}(u,v)=\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}(\sqrt{uv})^{\alpha-1}(1-\sqrt{uv})^{\beta-1}\sqrt{\frac{u}{v}}^{\alpha+\beta-1}\left(1-\sqrt{\frac{u}{v}}\right)^{\gamma-1}\frac{1}{2v}$$

$$0<u<v<1/u \quad 0<u<1$$

$$f_U(u)=\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1}\int_u^{1/u}\left(\frac{\sqrt{u/v}-u}{1-u}\right)^{\beta-1}\left(\frac{1-\sqrt{u/v}}{1-u}\right)^{\gamma-1}\frac{\sqrt{u/v}}{2v(1-u)}\ dv$$

$$\text{Let } z=\frac{\sqrt{u/v}-u}{1-u}\ dz=\frac{-\sqrt{u/v}}{2v(1-u)}\ dv \text{ and } 1-z=\frac{1-\sqrt{u/v}}{1-u}$$

$$f_U(u)=\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1}\int_0^1 z^{\beta-1}(1-z)^{\gamma-1}\ dz$$

$$f_U(u)=\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$f_U(u)=\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1} \quad 0<u<1$$

$$U\sim beta(\alpha,\beta+\gamma)$$

Problem 3

(a)

$$Z = X - Y \quad W = Y$$
$$X = Z + W \quad Y = W$$
$$f_{Z,W}(z,w) = f_{XY}(z+w,w) = f_X(z+w)f_Y(w)$$
$$J = 1$$
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z+w)f_Y(w)\ dw$$

(b)

$$Z = XY \quad W = X$$
$$X = W \quad Y = Z/W$$
$$f_{Z,W}(z,w) = f_{XY}(w,z/w)(1/w) = f_X(w)f_Y(z/w)(1/w)$$
$$J = 1/w$$
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w)f_Y(z/w)(1/w)\ dw$$

(c)

$$Z = X/Y \quad W = X$$
$$X = W \quad Y = W/Z$$
$$J = |-w/z^2|$$
$$f_{Z,W}(z,w) = f_X(w)f_Y(w/z)(w/z^2)$$
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w)f_Y(w/z)(w/z^2)\ dw$$

Problem 4

(a) $\mathcal{S} = \{(x_1,x_2) : 0 < x_1, x_2 < 1\}$

(b) $\mathcal{T} = \{(y_1,y_2) : y_2 < y_1, y_2 < 2 - y_1, y_2 > y_1 - 2, y_2 > -y_1\}$

(c)

$$Y_1 = X_1 + X_2 \quad Y_2 = X_1 - X_2$$
$$X_1 = \frac{Y_1 + Y_2}{2} \quad X_2 = \frac{Y_1 - Y_2}{2}$$

$$J = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = |-1/2| = 1/2$$
$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2})(1/2) = 1/2$$
$$f_{Y_1Y_2}(y_1, y_2) = \begin{cases} 1/2 & (y_1, y_2) \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$f_{y_1}(y_1) = \int f_{y_1,y_2}(y_1, y_2)\ dy_2$$
$$\text{for } 0 < y_1 < 1 : \int_{-y_1}^{y_1} 1/2\ dy_2 = \Big|_{-y_1}^{y_1} (1/2)y_2 = y_1$$
$$\text{for } 1 \le y_1 < 2 : \int_{y_1-2}^{2-y_1} 1/2\ dy_1 = \Big|_{y_1-2}^{2-y_1} (1/2)y_2 = 2 - y_1$$
$$f_{y_1}(y_1) = \begin{cases} y_1 & 0 < y_1 < 1 \\ 2 - y_1 & 1 \le y_1 < 2 \\ 0 & \text{otherwise} \end{cases}$$
$$f_{y_2}(y_2) = \int f_{y_1,y_2}(y_1, y_2)\ dy_1$$
$$\text{for } 0 \le y_2 < 1 : \int_{y_2}^{2-y_2} 1/2\ dy_1 = \Big|_{y_2}^{2-y_2} (1/2)y_1 = 1 - y_2$$
$$\text{for } -1 < y_2 < 0 : \int_{-y_2}^{2+y_2} 1/2\ dy_1 = \Big|_{-y_2}^{2+y_2} (1/2)y_1 = 1 + y_2$$
$$f_{y_2}(y_2) = \begin{cases} 1 - y_2 & 0 \le y_2 < 1 \\ 1 + y_2 & -1 < y_2 < 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5

(a)

$Y_1 = X_1/X_2 \quad Y_2 = X_2$

$$-\infty < x_1, x_2 < \infty$$
$$X_1 = Y_1Y_2 \quad X_2 = Y_2$$
$$-\infty < y_2 < \infty$$

$$J = \begin{bmatrix} Y_2 & 0 \\ Y_1 & 1 \end{bmatrix} = |Y_2|$$

(b)

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi} e^{-\left(\frac{x_1^2 + x_2^2}{2}\right)}$$

$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(y_1y_2, y_2) = \frac{1}{2\pi} e^{-\left(\frac{(y_1y_2)^2 + (y_2)^2}{2}\right)} y_2$$

$$f_{Y_1Y_2}(y_1, y_2) = \frac{1}{2\pi} e^{-\left(\frac{y_1^2y_2^2 + y_2^2}{2}\right)} y_2$$

$$-\infty < y_1, y_2 < \infty$$

(c)

$$f_{y_1} = \frac{2}{2\pi} \int_0^\infty e^{-\left(\frac{y_2^2(1+y_1^2)}{2}\right)} (y_2)\ dy_2$$

$$\text{Let } u = \frac{y_2^2(1+y_1^2)}{2} \text{ then } du = y_2(1+y_1^2)$$

$$f_{y_1} = \frac{1}{\pi} \frac{1}{1+y_1^2} \int_0^\infty e^{-u}\ du$$

$$f_{y_1} = \frac{1}{\pi} \frac{1}{1+y_1^2} \qquad -\infty < y_1 < \infty$$

$$Y_1 \sim cauchy(0, 1)$$