Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 4.19
2. C&B 4.23
3. C&B 5.6
4. Suppose (X_1, X_2) have the joint pdf

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 1 & 0 < x_1 < 1, \quad 0 < x_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Let the support of (X_1, X_2) be denoted by the set $\mathcal{S} = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\}$. Draw $\mathcal{S}$ on the xy-plane.

(b) Suppose $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ with a support $\mathcal{T}$. Draw $\mathcal{T}$ on the xy-plane.

Solution: The inverse of the transformation is $X_1 = \frac{1}{2}(Y_1 + Y_2)$ and $X_2 = \frac{1}{2}(Y_1 - Y_2)$. Therefore,

$$\begin{aligned} \mathcal{T} &= \{(y_1, y_2) : 0 < \frac{1}{2}(y_1 + y_2) < 1, \quad 0 < \frac{1}{2}(Y_1 - Y_2) < 1\} \\ &= \{(y_1, y_2) : -y_1 < y_2, \quad y_2 < 2 - y_1, \quad y_2 < y_1, \quad y_1 - 2 < y_2\}. \end{aligned}$$

(c) Derive the joint distribution of (Y_1, Y_2) using Jacobin method.

Solution:

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} f_{X_1,X_2}(\frac{1}{2}(y_1 + y_2), \frac{1}{2}(y_1 - y_2))|J| = \frac{1}{2} & (y_1, y_2) \in \mathcal{T}, \\ 0 & \text{otherwise.} \end{cases}$$

(d) Derive the marginal distribution (pdf) of Y_1 and Y_2.

Solution:

$$f_{Y_1}(y_1) = \begin{cases} \int_{-y_1}^{y_1} \frac{1}{2} dy_2 = y_1 & 0 < y_1 \le 1 \\ \int_{y_1-2}^{2-y_1} \frac{1}{2} dy_2 = 2 - y_1 & 1 < y_1 < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$f_{Y_2}(y_2) = \begin{cases} \int_{-y_2}^{y_2+2} \frac{1}{2} dy_1 = y_2 + 1 & -1 < y_2 \le 0 \\ \int_{y_2}^{2-y_2} \frac{1}{2} dy_1 = 1 - y_2 & 0 < y_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

5. Suppose that random variables X_1 and X_2 are mutually independent and follow $N(0,1)$. Show that the random variable $Y_1 = X_1/X_2$ follows Cauchy distribution with pdf

$$f_{Y_1}(y_1) = \frac{1}{\pi}\frac{1}{1+y_1^2}, \quad -\infty < y_1 < \infty.$$

Answer the following questions step by step toward the final solution.

(a) Let $Y_2 = X_2$. Show that the Jacobian of the inverse function of y_1 and y_2 is y_2, where $-\infty < y_2 < \infty$.

Solution: The inverse function of y_1 and y_2 is $X_1 = Y_1Y_2$ and $X_2 = Y_2$ with Jacobian matrix

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix}.$$

The determinant is y_2. Pay attend to the domain of y_2, which is from $-\infty$ to ∞.

(b) Derive the joint pdf of Y_1 and Y_2 using the Jacobian method.

Solution: The joint pdf of Y_1 and Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(y_1y_2, y_2)|y_2| \\ &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_1^2y_2^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_2^2}{2}\right)|y_2|. \end{aligned}$$

(c) Find the marginal distribution of Y_1 from the joint distribution of Y_1 and Y_2 in (b).

Solution: The marginal distribution of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_1^2y_2^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_2^2}{2}\right)|y_2|dy_2 \\ &= \frac{1}{\pi}\int_0^{\infty}\exp\left(-\frac{(1+y_1^2)y_2^2}{2}\right)y_2dy_2 \\ &\text{(set } \ t = y_2^2) \\ &= \frac{1}{2\pi}\int_0^{\infty}\exp\left(-\frac{(1+y_1^2)t}{2}\right)dt \\ &= \frac{1}{\pi}\frac{1}{1+y_1^2}, \end{aligned}$$

where $-\infty < y_1 < \infty$.

6. Let $X_1, \ldots, X_n$ constitute a random sample of size $n(n \geq 3)$ from the parent population

$$f_X(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty, \quad 0 < \lambda < \infty$$

(a) Find the conditional density function of $X_1, \ldots, X_n$ given that $S = \sum_{i=1}^n X_i = s$.

Solution: Since $X_1, \ldots, X_n$ follows exponential distribution with mean λ^{-1}, $S = \sum_{i=1}^n X_i$ follows Gamma distribution with pdf

$$f_S(s) = \frac{1}{\Gamma(n)\lambda^{-n}} s^{n-1} e^{-s\lambda}.$$

The conditional distribution of $X_1, \ldots, X_n$ given that $S = s$ is

$$\begin{aligned} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n | S = s) &= \frac{f_{X_1,\ldots,X_n,S}(x_1, \ldots, x_n, s)}{f_S(s)} \\ &= \frac{f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)}{f_S(s)} \\ &= \frac{(n-1)!}{s^{n-1}}. \end{aligned}$$

(b) Consider the $(n-1)$ random variables

$$Y_1 = \frac{X_1}{S}, Y_2 = \frac{X_1 + X_2}{S}, \ldots, Y_{n-1} = \frac{X_1 + X_2 + \cdots + X_{n-1}}{S}.$$

Find the joint distribution of $Y_1, Y_2, \ldots, Y_{n-1}$ given that $S = s$.

Solution: The inverse function of the transformation is

$$X_1 = Y_1 S, \quad , X_2 = (Y_2 - Y_1)S, \ldots, X_{n-1} = (Y_{n-1} - Y_{n-2})S.$$

The Jacobian is S^{n-1}. Hence, the joint pdf of $Y_1, Y_2, \ldots, Y_{n-1}$ given that $S = s$ is

$$\begin{aligned} & f_{Y_1,Y_2,\ldots,Y_{n-1}}(y_1, y_2, \ldots, y_{n-1} | S = s) \\ &\quad = f_{X_1,X_2,\ldots,X_{n-1}}(y_1 s, (y_2 - y_1)s, \ldots, (y_{n-1} - y_{n-2})s | S = s) s^{n-1} \\ &\quad = f_{X_1,X_2,\ldots,X_n}(y_1 s, (y_2 - y_1)s, \ldots, (y_{n-1} - y_{n-2})s, (1 - y_{n-1})s | S = s) s^{n-1} \\ &\quad = \frac{(n-1)!}{s^{n-1}} s^{n-1} = (n-1)!, \quad y_1 < y_2 < \ldots < y_{n-1}. \end{aligned}$$

The second equation stands because $f_{X_1,X_2,\ldots,X_{n-1}}(x_1, x_2, \ldots, x_{n-1} | S = s) = f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n | S = s)$ since s is given and when one knows $x_1, \ldots, x_{n-1}$, $x_n = s - \sum_{i=1}^{n-1} x_i$ is fixed, i.e., no randomness in X_n.

(c) When $n = 3$ and when $n = 4$, find the marginal distribution of Y_1 given that $S = s$, and then use these results to infer the structure of the marginal distribution of Y_1 given that $S = s$ for any $n \geq 3$.

Solution: When $n = 3$,

$$f_{Y_1}(y_1|S = s) = \int_{y_2}^{1} 2!dy_2 = 2(1 - y_1), \quad 0 < y_1 < 1.$$

When $n = 4$,

$$f_{Y_1}(y_1|S = s) = \int_{y_1}^{1}\int_{y_2}^{1} 3!dy_3dy_2 = 3(1 - y_1)^2, \quad 0 < y_1 < 1.$$

For a general n,

$$f_{Y_1}(y_1|S = s) = (n - 1)(1 - y_1)^{n-2}, \quad 0 < y_1 < 1.$$

7. (Bios 673 class material) A certain simple biological system involves exactly two independently functioning components. If one of these two components fails, then entire systems fails. For $i = 1, 2$, let Y_i be the random variable representing the time to failure of the ith component, with the pdf of Y_i being

$$f_{Y_i}(y_i) = \theta_i e^{-\theta_i y_i}, \quad 0 < y_i < \infty, \quad \theta_i > 0.$$

Clearly, if this biological system fails, then only two random variables are observable, namely U and W, where $U = \min(Y_1, Y_2)$ and

$$W = \begin{cases} 1, & \text{if } \ Y_1 < Y_2, \\ 0, & \text{if } \ Y_2 < Y_1. \end{cases}$$

(a) Show that the joint distribution $f_{U,W}(u, w)$ of random variables U and W is

$$f_{U,W}(u, w) = \theta_1^{(1-w)}\theta_2^{w}e^{-(\theta_1+\theta_2)u}, \quad 0 < u < \infty, \quad w = 0, 1.$$

Solution: Working on the derivation of $P(U \leq u, W = 0)$ and $P(U \leq u, W = 1)$, one have

$$\begin{aligned} P(U \leq u, W = 0) &= P(Y_2 \leq u, Y_2 < Y_1) \\ &= \int_0^u \int_{y_2}^{\infty} \left(\theta_1 e^{-\theta_1 y_1}\right)\left(\theta_2 e^{-\theta_2 y_2}\right) dy_1 dy_2 \\ &= \left(\frac{\theta_1}{\theta_1 + \theta_2}\right)\left\{1 - e^{-(\theta_1+\theta_2)u}\right\} \end{aligned}$$

$$f_{U,W}(u, w = 0) = \theta_1 e^{-(\theta_1+\theta_2)u}, \quad 0 < u < \infty. \tag{1}$$

$$\begin{aligned} P(U \le u, W = 1) &= P(Y_1 \le u, Y_1 < Y_2) \\ &= \int_0^u \int_{y_1}^{\infty} \left(\theta_1 e^{-\theta_1 y_1}\right) \left(\theta_2 e^{-\theta_2 y_2}\right) dy_2 dy_1 \\ &= \left(\frac{\theta_2}{\theta_1+\theta_2}\right) \left\{1 - e^{-(\theta_1+\theta_2)u}\right\} \end{aligned}$$

$$f_{U,W}(u, w = 1) = \theta_2 e^{-(\theta_1+\theta_2)u}, \quad 0 < u < \infty. \tag{2}$$

Combining (1) and (2), one has

$$f_{U,W}(u, w) = \theta_1^{(1-w)} \theta_2^{w} e^{-(\theta_1+\theta_2)u}, \quad 0 < u < \infty, \quad w = 0, 1.$$

(b) Find the marginal distribution $f_W(w)$ of the random variable W.

Solution: By definition of marginal distribution, one can have

$$\begin{aligned} f_W(w) &= \int_0^{\infty} f_{U,W}(u, w) du = \int_0^{\infty} \theta_1^{(1-w)} \theta_2^{w} e^{-(\theta_1+\theta_2)u} du \\ &= \theta_1^{(1-w)} \theta_2^{w} (\theta_1+\theta_2)^{-1} = \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{(1-w)} \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{w} \end{aligned}$$

(c) Find the marginal distribution $f_U(u)$ of the random variable U.

Solution: Similar to (b), one can have

$$f_U(u) = \sum_{w=0}^{1} f_{U,W}(u, w) = (\theta_1+\theta_2) e^{-(\theta_1+\theta_2)u}.$$

(d) Show that U and W are independent.

Solution: Since $f_{U,W}(u, w) = f_U(u) f_W(w)$, one can claim U are W are independent.