Problem 1

$$X_1, \ldots, X_n \sim Bern(p)$$

$$L(p|x) = \prod_{i=1}^{n} p^{x_1}(1-p)^{1-x_i}$$

$$= p^{\sum_{i=1}^{n} x_i}(1-p)^{\sum_{i=1}^{n}(1-x_i)}$$

$$\ell(p|x) = \sum_{i=1}^{n} x_i \log(p) + (n - \sum_{i=1}^{n} x_i) \log(1-p)$$

$$\frac{\partial \ell}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0$$

$$\frac{1-p}{p} = \frac{n - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}$$

$$\frac{1}{p} - 1 = \frac{n}{\sum_{i=1}^{n} x_i} - 1$$

$$\hat{p}_{MLE} = \sum_{i=1}^{n} x_i/n = \bar{x}$$

$$\frac{\partial \ell}{\partial p^2} = -\left(\frac{\sum_{i=1}^{n} x_i}{p^2} + \frac{n - \sum_{i=1}^{n} x_i}{(1-p)^2}\right) < 0$$

$$\text{Thus } \hat{p} \text{ is the MLE}$$

$$\log(f(x_1)) = x \log(p) + (1-x) \log(1-p)$$

$$\frac{\partial}{\partial p} \log(f(x_1)) = \frac{x}{p} - \frac{1-x}{1-p}$$

$$\frac{\partial^2}{\partial p^2} \log(f(x_1)) = -\left(\frac{x}{p^2} + \frac{1-x}{(1-p)^2}\right)$$

$$nE[-\frac{\partial^2}{\partial p^2} \log(f(x_1))] = nE\left(\frac{x}{p^2} + \frac{1-x}{(1-p)^2}\right)$$

$$= n\left(\frac{p}{p^2} + \frac{1-p}{(1-p)^2}\right)$$

$$= n(\frac{1}{p} + \frac{1}{1-p}) = \frac{n}{p(1-p)}$$

$$CRLB = 1/\frac{n}{p(1-p)} = \frac{p(1-p)}{n}$$

$$Var(\hat{p}_{MLE}) = Var(\bar{X}) = \frac{p(1-p)}{n}$$

$$Var(\hat{p}_{MLE}) = CRLB$$

$$\text{Thus } \bar{X} \text{ attains the CRLB and is therefore the UMVUE of p}$$

Problem 2

$$X_1, \ldots, X_n \sim N(\theta, 1)$$

$$L(\theta|x) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(x_i - \theta)^2}{2}\right)$$

$$= (2\pi)^{-n/2} \exp\left(-\frac{(\sum_{i=1}^{n} x_i - n\theta)^2}{2}\right)$$

$$\ell(\theta|x) = (-n/2)\log(2\pi) - \frac{(\sum_{i=1}^{n} x_i - n\theta)^2}{2}$$

$$\propto -\frac{(\sum_{i=1}^{n} x_i - n\theta)^2}{2} = (-1/2)\left[(\sum_{i=1}^{n} x_i)^2 + n^2\theta^2 - 2n\theta \sum_{i=1}^{n} x_i\right]$$

$$\frac{\partial \ell}{\partial \theta} = -n^2\theta + n\sum_{i=1}^{n} x_i = 0$$

$$\theta = \frac{n\sum_{i=1}^{n} x_i}{n^2}$$

$$\hat{\theta}_{MLE} = \bar{x}$$

$$\frac{\partial \ell}{\partial \theta^2} = -n^2 < 0$$

Thus $\hat{\theta}$ is the MLE

$$\tau(\theta) = \theta^2$$

By invariance property: $\tau(\hat{\theta})_{MLE} = \tau(\hat{\theta}_{MLE}) = (\hat{\theta}_{MLE})^2 = \bar{x}^2$

$$\hat{\theta}^2_{MLE} = \bar{x}^2$$

$$E(\bar{X}^2) = Var(\bar{X}) + E(\bar{X})^2 = 1/n + \theta^2 \text{ (biased)}$$

$$\theta^{2*} = \bar{X}^2 - \frac{1}{n} \text{ (unbiased)}$$

$$\sum_{i=1}^{n} x_i \text{ is a CSS for } \theta$$

θ^{2*} is an unbiased estimator and a function of $\displaystyle\sum_{i=1}^{n} x_i$

Thus θ^{2*} is the UMVUE by Lehmann-Sheffe Theorem

$$\log(f(x_1|\theta)) = (-1/2)[\log(2\pi) + (x - \theta)^2]$$

$$\frac{\partial}{\partial \theta}\log(f(x_1|\theta)) = x - \theta$$

$$\frac{\partial^2}{\partial\theta^2}\log(f(x_1|\theta)) = -1$$

$$nE(-\frac{\partial^2}{\partial\theta^2}\log(f(x_1|\theta))) = n$$

$$\left(\frac{d\tau(\theta)}{d\theta}\right)^2 = 4\theta^2$$

$$CRLB = \frac{4\theta^2}{n}$$

$$Var(\theta^{2*}) = Var(\bar{X}^2 - 1/n) = Var(\bar{X}^2) = E(\bar{X}^4) - E(\bar{X}^2)^2 = E(\bar{X}^4) - (1/n + \theta^2)^2$$

Using Steins Lemma:

$$E(\bar{X}^4) = E[\bar{X}^3(\bar{X} - \theta + \theta)] = E[\bar{X}^3(\bar{X} - \theta)] + \theta E(\bar{X}^3)$$

$$E[\bar{X}^3(\bar{X} - \theta)] = \sigma^2 E(3\bar{X}^2) = (3/n)(1/n + \theta^2)$$

$$\theta E(\bar{X}^3) = \theta E(\bar{X}^2 - \theta + \theta) = \theta E(\bar{X}^2(\bar{X} - \theta) + \theta\bar{X}^2) = 2\theta\sigma^2 E(\bar{X}) + \theta^2 E(\bar{X}^2)$$

$$= (2/n)\theta^2 + \theta^2(1/n + \theta^2) = \theta^2(3/n + \theta^2)$$

$$Var(\theta^{2*}) = (3/n)(1/n + \theta^2) + \theta^2(3/n + \theta^2) - (1/n + \theta^2)^2$$

$$= (1/n + \theta^2)(3/n - 1/n - \theta^2) + \theta^2(3/n + \theta^2)$$

$$= (1/n + \theta^2)(2/n - \theta^2) + \theta^2(3/n + \theta^2)$$

$$= 2/n^2 - \theta^4 + (1/n)\theta^2 + (3/n)\theta^2 + \theta^4$$

$$Var(\theta^{2*}) = \frac{2}{n^2} + \frac{4\theta^2}{n}$$

$$\frac{2}{n^2} + \frac{4\theta^2}{n} > \frac{4\theta^2}{n}$$

Thus the variance of the UMVUE is greater than the CRLB

Problem 3

(a)

$$X_1, \ldots, X_n \sim Pareto(v, \theta)$$

$$E(X_i) = \frac{\theta v}{\theta - 1} \quad Var(X_i) = \frac{\theta v^2}{(\theta + 1)^2(\theta - 2)}$$

$$f(x|\theta, v) = \frac{\theta v^\theta}{x^{\theta+1}} I[v, \infty)(x)$$

$$L(\theta, v|x) = \prod_{i=1}^{n} \frac{\theta v^\theta}{x_i^{\theta+1}} I[v, \infty)(x_i)$$

$$\ell(\theta, v|x) = \sum_{i=1}^{n} [\log(\theta) + \theta \log(v) - (\theta + 1) \log(x_i)], v \le x_{(1)}$$

$$\ell(\theta, v|x) = n \log(\theta) + n\theta \log(v) - (\theta + 1) \sum_{i=1}^{n} \log(x_i), v \le x_{(1)}$$

Where $x_{(1)} = \min_i(x_i)$

For all θ this is an increasing function of v, $v \le x_{(1)}$

Thus $\hat{v}_{MLE} = x_{(1)}$

Fixing v:

$$\ell(\theta, x_{(1)}|x) = n \log(\theta) + n\theta \log(x_{(1)}) - (\theta + 1) \sum_{i=1}^{n} \log(x_i)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + n \log(x_{(1)}) - \sum_{i=1}^{n} \log(x_i) = 0$$

$$1/\theta + \log(X_{(1)}) = \frac{\log\left(\prod x_i\right)}{n}$$

$$1/\theta = \frac{\log\left(\prod x_i\right) - n \log(x_{(1)})}{n}$$

$$\theta = \frac{n}{\log\left(\prod x_i\right) - n \log(x_{(1)})}$$

$$\hat{\theta}_{MLE} = \frac{n}{\log\left(\frac{\prod_{i=1}^{n}}{x_{(1)}^{n}}\right)} = \frac{n}{T}$$

Where $T = \log\left(\frac{\prod_{i=1}^{n}}{[\min_i(x_i)]^n}\right)$

$$\frac{\partial \ell}{\partial \theta^2}{}^{2} = -\frac{n}{\theta^2} < 0$$

Thus $\hat{\theta}$ is the MLE of θ

(b)

$$L(\theta, x_{(1)}|x) = \prod_{i=1}^{n} \frac{\theta x_{(1)}^{\theta}}{x_i^{\theta+1}} = \frac{\theta^n x_{(1)}^{n\theta}}{\left(\prod_{i=1}^{n} x_i\right)^{\theta+1}}$$

$$H_0 : \theta = 1, v \text{ unknown} \quad H_1 : \theta \ne 1, v \text{ unknown}$$

$$\lambda(x) = \frac{L(\theta = 1, v = x_{(1)})}{L(\theta = n/T, v = x_{(1)})}$$

$$= \frac{1^n x_{(1)}^{n}}{\left(\prod_{i=1}^{n} x_i\right)^2} \Big/ \frac{(n/T)^n x_{(1)}^{n^2/T}}{\left(\prod_{i=1}^{n} x_i\right)^{n/T+1}}$$

$$= \frac{x_{(1)}{}^n}{(\prod_{i=1}^n x_i)^2} * \frac{(\prod_{i=1}^n x_i)^{n/T+1}}{(n/T)^n x_{(1)}{}^{n^2/T}}$$

$$= x_{(1)}^{n-n^2/T}(\prod_{i=1}^n x_i)^{n/T-1}(T/n)^n$$

$$= \frac{(\prod_{i=1}^n x_i)^{n/T-1}}{x_{(1)^n}^{n/T-1}}(T/n)^n$$

Since $e^T = \dfrac{\prod_{i=1}^n x_i}{x_{(1)}^n}$:

$$\lambda(x) = (e^T)^{n/T-1}(T/n)^n = e^{-T+n}(T/n)^n$$

$$R = \{x : \lambda(x) \le c\} = \{x : e^{-T+n}(T/n)^n \le c\}$$

Since $\lambda(x)$ is in the form of $e^{-T}T^n$, $\lambda(x)$ is concave

Which means the rejection region is equivalent to:

$$\{x : T \le c_1^* \text{ or } T \ge c_2^*\}$$

Problem 4

(a)

$$Y_1, \ldots, Y_n \sim Pois(\theta x_i) \quad \theta > 0$$

$$L(\theta|y) = \prod_{i=1}^n \frac{e^{-\theta x_i}(\theta x_i)^{y_i}}{y_i!}$$

$$\ell(\theta|y) = \sum_{i=1}^n (-\theta x_i + y_i \log(\theta) + y_i \log(x_i) - \log(y_i!))$$

$$= -\theta \sum_{i=1}^n x_i + \sum_{i=1}^n y_i \log(\theta) + \sum_{i=1}^n y_i \log(x_i) - \sum_{i=1}^n \log(y_i!)$$

$$\frac{\partial \ell}{\partial \theta} = -\sum_{i=1}^n x_i + \frac{\sum_{i=1}^n y_i}{\theta} = 0$$

$$\sum_{i=1}^n x_i = \frac{\sum_{i=1}^n y_i}{\theta}$$

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i} \quad \frac{\partial \ell}{\partial \theta^2}^2 = -\frac{\sum_{i=1}^n y_i}{\theta^2} < 0$$

Thus $\hat{\theta}$ is the MLE

$$E(\hat{\theta}) = \frac{1}{\sum_{i=1}^{n} x_i} E(\sum_{i=1}^{n} y_i) = \frac{1}{\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} E(y_i) = \frac{\theta \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i} = \theta \text{ (unbiased)}$$

$$Var(\hat{\theta}) = \frac{1}{(\sum_{i=1}^{n} x_i)^2} \sum_{i=1}^{n} Var(y_i) = \frac{\theta \sum_{i=1}^{n} x_i}{(\sum_{i=1}^{n} x_i)^2} = \frac{\theta}{\sum_{i=1}^{n} x_i}$$

(b)

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}$$

$$\text{Since } E(\hat{\theta}_{MLE}) = \theta :$$

$$\hat{\theta}_{MLE} \text{ is an unbiased estimator and a function of } \sum_{i=1}^{n} y_i \text{ and a postive constant}$$

$$\sum_{i=1}^{n} y_i \text{ is a CSS by exponential family}$$

$$\text{Thus } \hat{\theta}_{MLE} \text{ is the UMVUE by Lehmann-Sheffe Theorem}$$

(c)

$$\frac{\partial}{\partial \theta} \log(f(\boldsymbol{x}|\theta)) = -\sum_{i=1}^{n} x_i + \frac{\sum_{i=1}^{n} y_i}{\theta}$$

$$E(\frac{\partial}{\partial \theta} \log(f(\boldsymbol{x}|\theta))^2) = E([-\sum_{i=1}^{n} x_i + \frac{\sum_{i=1}^{n} y_i}{\theta}]^2)$$

$$= E\left((\sum_{i=1}^{n} x_i)^2 + \frac{(\sum_{i=1}^{n} y_i)^2}{\theta^2} - \frac{2 \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\theta} \right)$$

$$= (\sum_{i=1}^{n} x_i)^2 + \frac{E[(\sum_{i=1}^{n} y_i)^2]}{\theta^2} - \frac{2 \sum_{i=1}^{n} x_i E[\sum_{i=1}^{n} y_i]}{\theta}$$

$$E[(\sum_{i=1}^{n} y_i)^2] = Var(\sum_{i=1}^{n} y_i) + E(\sum_{i=1}^{n} y_i)^2 = \theta \sum_{i=1}^{n} x_i + \theta^2 (\sum_{i=1}^{n} x_i)^2$$

$$E(\frac{\partial}{\partial \theta} \log(f(\boldsymbol{x}|\theta))^2) = (\sum_{i=1}^{n} x_i)^2 + \frac{\theta \sum_{i=1}^{n} x_i + \theta^2 (\sum_{i=1}^{n} x_i)^2}{\theta^2} - \frac{2 \sum_{i=1}^{n} x_i \theta \sum_{i=1}^{n} x_i}{\theta}$$

$$= (\sum_{i=1}^{n} x_i)^2 + \frac{\sum_{i=1}^{n} x_i}{\theta} + (\sum_{i=1}^{n} x_i)^2 - 2(\sum_{i=1}^{n} x_i)^2$$

$$E(\frac{\partial}{\partial \theta} \log(f(\boldsymbol{x}|\theta))^2) = \frac{\sum_{i=1}^{n} x_i}{\theta}$$

$$CRLB = 1/\frac{\sum_{i=1}^{n} x_i}{\theta} = \frac{\theta}{\sum_{i=1}^{n} x_i}$$

$$\frac{\theta}{\sum_{i=1}^{n} x_i} = \frac{\theta}{\sum_{i=1}^{n} x_i}$$

Thus the variance of $\hat{\theta}$ achieves the CRLB

Problem 5

(a)

$$f(x|\theta) = e^{-(x-\theta)}, I(x \geq \theta)$$

$$L(\theta|x) = \exp\left(-\sum_{i=1}^{n} x_i + n\theta\right), \theta \leq x_{(1)}$$

$$\ell(\theta|x) = -\sum_{i=1}^{n} x_i + n\theta, \theta \leq x_{(1)}$$

This is an increasing function of θ for $\theta \leq x_{(1)}$

Thus $\hat{\theta} = x_{(1)}$ maximizes the function

$$\hat{\theta}_{MLE} = x_{(1)}$$

$$H_0 : \theta = \theta_0 \quad H_1 : \theta \neq \theta_0$$

If $x_{(1)} < \theta_0$, $L(\theta) = 0$, thus reject H_0

$$\lambda(x) = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\exp\left(-\sum_{i=1}^{n} x_i + n\theta_0\right), I(x_{(1)} \geq \theta_0)}{\exp\left(-\sum_{i=1}^{n} x_i + nx_{(1)}\right)}$$

$$\lambda(x) = \exp\left(-n(x_{(1)} - \theta_0)\right), I(x_{(1)} \geq \theta_0)$$

(b)

WTS: $R = \{x : \lambda(x) \leq c\}$ and $R^* = \{x : x_{(1)} \geq c^* \text{ or } x_{(1)} < \theta_0\}$ are equivalent

$$H_0 : \theta = \theta_0 \quad H_1 : \theta \neq \theta_0$$

$$R = \{x : \lambda(x) \leq c\} = \{x : \exp(-n(x_{(1)} - \theta_0)) \leq c \text{ or } x_{(1)} < \theta_0\}$$

$$= \{x : x_{(1)} - \theta_0 \geq -\log(c)/n \text{ or } x_{(1)} < \theta_0\} = \{x : x_{(1)} \geq \theta_0 - \log(c)/n \text{ or } x_1 < \theta_0\}$$

$$c^* = \theta_0 - \log(c)/n$$

$$R^* = \{x : x_{(1)} \geq c^* \text{ or } x_{(1)} < \theta_0\}$$

Thus R and R^* are equivalent

(c)

$$\alpha = \sup_{\theta \in \Theta} P(\boldsymbol{X} \in R^* | H_0)$$
$$\alpha = P(\lambda(x) \leq c | \theta = \theta_0)$$
$$= P(x_{(1)} \geq c^* \text{ or } x_1 < \theta_0 | \theta = \theta_0)$$
$$= P(x_{(1)} \geq c^* | \theta = \theta_0) + P(x_1 < \theta_0 | \theta = \theta_0)$$
$$\alpha/2 = P(x_{(1)} \geq c^* | \theta = \theta_0)$$
$$f_{x_{(1)}}(x) = n(f_x)[1 - F_X]^{n-1}$$
$$F_X = e^\theta - e^{-x+\theta}$$
$$n(e^{-x+\theta})[1 - e^\theta + e^{-x+\theta}]^{n-1}$$

(d)