Problem 1

x	1	2	3	4	5	6	7
$f(x\|H_0)$	.01	.01	.01	.01	.01	.01	.94
$f(x\|H_1)$	.06	.05	.04	.03	.02	.01	.79

By (N-P) lemma, the UMP test has a rejection region:

$$R = \{x : \frac{f(x|H_1)}{f(x|H_0)} > c\}$$

x	1	2	3	4	5	6	7
$\frac{f(x\|H_1)}{f(x\|H_0)}$	6	5	4	3	2	1	.84

$$\alpha = .04 = P(X \leq c|H_0)$$

$$P(X \leq 4|H_0) = .04 \text{ thus } c = 4$$

$$P(\text{Type II Error}) = P(X \geq 5|H_1) = .02 + .01 + .79 = .82$$

Problem 2

(a)

$$X_1, \ldots, X_{10} \sim Bern(p)$$

$$H_0 : p = 1/2 \text{ vs } H_1 : p = 1/4 \quad \alpha = .0547$$

$$L(p|x) = p^{\sum_{i=1}^n x_i}(1-p)^{1-\sum_{i=1}^n x_i} \quad x = 0, 1 \; p \in [0, 1]$$

$$\sum_{i=1}^n x_i \text{ is an SS for p}$$

$$Y = \sum_{i=1}^n x_i \sim Bin(10, p)$$

$$f(y|p) = \frac{10!}{(10-y)!y!} p^y (1-p)^{10-y} \quad y = 0, 1, \ldots, 10$$

By (N-P) lemma, the UMP test has a rejection region:

$$R = \{y : \frac{f(y|1/4)}{f(y|1/2)} > c\}$$

$$\frac{f(y|1/4)}{f(y|1/2)} = \frac{(1/4)^y(1-1/4)^{10-y}}{(1/2)^y(1-1/2)^{10-y}} = (1/2)^y(3/2)^{10-y} = (1/2)^{10}3^{10-y} = (3/2)^{10}3^{-y}$$

$$R = \{y : (3/2)^{10}3^{-y} > c\}$$

$$= \{y : 10\log(3/2) - y\log(3) > \log(c)\}$$

$$R = \{y : y < \frac{-\log(c) + 10\log(3/2)}{\log(3)}\}$$

$$\alpha = .0547 = P(Y \leq c^*|p = 1/2)$$

$$f(y|1/2) = \frac{10!}{(10-y)!y!}(1/2)^{10}$$

$$P(y = 0|1/2) = (1/2)^{10} \quad P(y = 1|1/2) = (1/2)^{10}*10 \quad P(y = 2|1/2) = (1/2)^{10}*45$$

$$P(Y \leq 2|1/2) = (1/2)^{10}(1 + 10 + 45) = .0546875 \approx .0547$$

$$\text{Thus } c^* = 2$$

$$Power = 1-P(Y > 2|p = 1/4) = P(Y \leq 2|p = 1/4) = pbinom(2, 10, 1/4) \approx .526$$

(b)

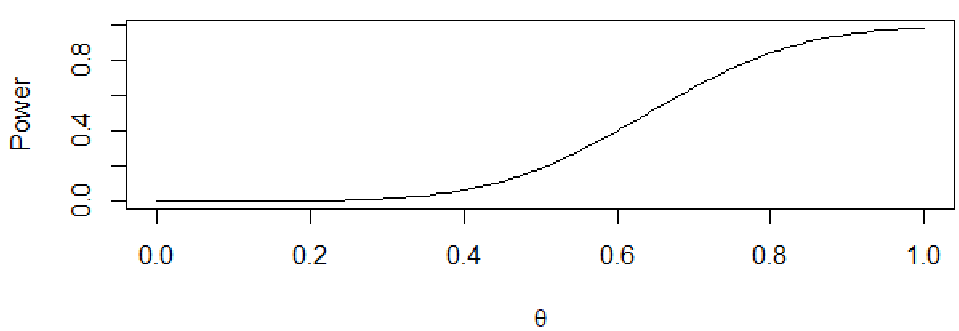
$$H_0 : p \leq 1/2 \text{ vs } H_1 : p > 1/2$$

$$Y = \sum_{i=1}^{n} x_i$$

$$R = \{y : y \geq 6\}$$

$$P(Y \geq 6|p = 1/2) = \sum_{k=6}^{10} \binom{10}{k} (1/2)^k (1/2)^{10-k} = sum(dbinom(6 : 10, 10, 1/2)) \approx .377$$

$$B(\theta) = \sum_{k=6}^{10} \binom{10}{k} \theta^k (1-\theta)^{10-k}$$



(c)

$$f(y|1/2) = \frac{10!}{(10-y)!y!}(1/2)^{10}$$
$$(1/2)^{10} = \frac{1}{1024}$$
$$\alpha_i = P(Y \le i|p = 1/2) \quad 0 \le i \le 10$$
$$\alpha_i = \frac{1}{1024}\sum_{y=0}^{i}\frac{10!}{(10-y)!y!}$$
$$\frac{\alpha_y}{1024} = 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$$
$$\frac{\alpha_i}{1024} = 1, 11, 56, 176, 386, 638, 848, 968, 1013, 1023, 1024$$
$$\alpha_i = \left(\frac{1}{1024}, \frac{11}{1024}, \frac{56}{1024}, \frac{176}{1024}, \frac{386}{1024}, \frac{638}{1024}, \frac{848}{1024}, \frac{968}{1024}, \frac{1013}{1024}, \frac{1023}{1024}, 1\right) \text{ also } \alpha \text{ could equal } 0$$

Problem 3

(a)

$$f(x|\theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2} \quad -\infty < x < \infty,\ -\infty < \theta < \infty$$
$$\text{For } \theta_2 > \theta_1$$
$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{e^{x-\theta_2}}{(1+e^{x-\theta_2})^2}\frac{(1+e^{x-\theta_1})^2}{e^{x-\theta_1}}$$
$$= e^{\theta_1-\theta_2}\left(\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right)^2$$
$$T(X) = \frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}$$
$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = (e^{\theta_1-\theta_2})[T(X)]^2$$
$$\frac{d}{dx}T(X) = \frac{e^{x-\theta_1}-e^{x-\theta_2}}{(1+e^{x-\theta_2})^2}$$

because $\theta_2 > \theta_1,\ e^{x-\theta_1} > e^{x-\theta_2}$ thus the numerator is positive, making the derivative positive. Thus $T(X)$ is increasing. Since the likelihood ratio depends only on x through $T(X)$ and is a monotone increasing function of $T(X),$ the family has MLR

(b)

$$H_0 : \theta = 0 \text{ vs } H_1 : \theta = 1 \quad \alpha = .2$$

Using (N-P) lemmma, the rejection region for the UMP level α test is:

$$R = \{x : f(x|1)/f(x|0) > c\}$$

based on results from part a, this ratio is increasing in x and thus equivalent to:

$$R^* = \{x : x > c^*\}$$

$$\alpha = .2 = P(x > c^*|\theta = 0)$$

$(-1)f(x|\theta) \sim Logistic(\theta, -1)$ thus:

$$F(x|\theta) = \frac{1}{1 + e^{-x+\theta}}$$

$$.2 = 1 - F(c^*|\theta = 0) \Rightarrow .2 = 1 - \frac{1}{1 + e^{-c^*}}$$

$$.2 = \frac{e^{-c^*}}{1 + e^{-c^*}} \Rightarrow .2 + .2e^{-c^*} = e^{-c^*} \Rightarrow .2 = .8e^{-c^*} \Rightarrow .25 = e^{-c^*}$$

$$c^* = -\log(.25) \approx 1.386$$

$$\beta = P(x \leq c^*|\theta = 1)$$

$$\beta = F(c^*|\theta = 1)$$

$$\beta = \frac{1}{1 + e^{-c^*+1}} \Rightarrow \frac{1}{1 + e^{\log(.25)+1}} \approx 0.5954$$

$$\beta = .5954$$

(c)

$$H_0 : \theta \leq \theta_0 \text{ vs } H_1 : \theta > \theta_0$$

Since from part a, we have the MLR property holds and

$T(x)$ is a sufficient statistic then by K-R thm, it is a UMP level α test

This is true in general for the logistic location family

Problem 4

(a)

$$X_1, \dots, X_n \sim Pois(\lambda)$$
$$H_0 : \lambda \leq \lambda_0 \text{ vs } H_1 : \lambda > \lambda_0$$
$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, \dots, \; 0 \leq \lambda < \infty$$
$$T(X) = \sum_{i=1}^{n} x_i \text{ is a sufficient statistic}$$
$$T(X) = \sum_{i=1}^{n} x_i \sim Pois(n\lambda) \text{ the MLR property holds for } T(X)$$
By K-R thm, the UMP level alpha test is:
$$R = \{x : \sum_{i=1}^{n} x_i > c\}$$
$$\alpha = P(\sum_{i=1}^{n} X_i > c|\lambda = \lambda_0)$$

(b)

$$H_0 : \lambda \leq 1 \text{ vs } H_1 : \lambda > 1$$
$$R = \{x : \sum_{i=1}^{n} x_i > c\}$$
$$Z = \frac{\sum_{i=1}^{n} X_i - n\lambda}{\sqrt{n\lambda}} \sim N(0, 1)$$
$$\alpha = .05 = P(\sum_{i=1}^{n} X_i > c|\lambda = 1)$$
$$.05 = P(Z > (c - n)/\sqrt{n})$$
$$qnorm(1 - .05) \approx 1.645$$
$$\frac{c - n}{\sqrt{n}} = 1.645$$
$$c = 1.645\sqrt{n} + n$$
$$\alpha = .9 = P(\sum_{i=1}^{n} X_i > c|\lambda = 2)$$

$$.9 = P(Z > (c-2n)/\sqrt{2n})$$
$$qnorm(1-.9) \approx -1.28$$
$$\frac{c-2n}{\sqrt{2n}} = -1.28$$
Plugging in c from the first equation:
$$\frac{1.645\sqrt{n}+n-2n}{\sqrt{2n}} = -1.28 \Rightarrow 1.645\sqrt{n}-n = -1.28\sqrt{2n}$$
$$n = 1.645\sqrt{n}+1.28\sqrt{2}\sqrt{n} \Rightarrow \sqrt{n} = 1.645+1.28\sqrt{2} \Rightarrow n = (1.645+1.28\sqrt{2})^2$$
$$n = 11.93836 \approx 12$$
Plugging $n = 12$ into the first equation:
$$(c-12)/\sqrt{12} = 1.645 \Rightarrow c = 1.645 * \sqrt{12}+12$$
$$c = 17.69845 \approx 17.7$$
$$\text{Thus } n = 12,\ c = 17.7$$

Problem 5

$$X_1, \dots, X_n \sim \frac{1}{\theta} \quad 0 < x < \theta$$
$$\beta(\theta) = P(x_{(n)} \leq 1/2 \text{ or } x_{(n)} > 1|\theta) \quad \theta \in \Theta$$
$$H_0 : \theta = 1 \text{ vs } H_1 : \theta \neq 1$$
Four Cases: $\theta = 1 \quad \theta > 1 \quad 0 < \theta < 1/2 \quad 1/2 < \theta < 1$

Case 1: $\theta = 1$
$$\beta(1) = P(x_{(n)} \leq 1/2 \text{ or } x_{(n)} > 1|\theta = 1)$$
$$= P(x_{(n)} \leq 1/2|\theta = 1) + P(x_{(n)} > 1|\theta = 1)$$
$$F_{x_{(n)}}(x) = P(x_{(n)} \leq x) = \{P(X_1 \leq x)\}^n = \left(\frac{x}{\theta}\right)^n = \frac{x^n}{\theta^n}$$
$$P(x_{(n)} \leq 1/2|\theta = 1) = \frac{(1/2)^n}{1^n} = 2^{-n}$$
$$P(x_{(n)} > 1|\theta = 1) = 0$$
$$\beta(1) = 2^{-n} + 0 = 2^{-n}$$
$$\beta(\theta) = P(x_{(n)} \leq 1/2|\theta) + P(x_{(n)} > 1|\theta)$$
Case 2: $\theta > 1$
$$\beta(\theta) = \frac{(1/2)^n}{\theta^n} + 1 - P(x_{(n)} \leq 1|\theta)$$
$$= \frac{2^{-n}}{\theta^n} + 1 - \frac{1}{\theta^n} = \frac{2^{-n}-1}{\theta^n} + 1 \text{ (increasing function of } \theta)$$

$$\theta \to \infty \quad \beta(\theta) \to 1 \text{ since:}$$

$$\frac{(1/2)^n - 1}{\theta^n} \to 0$$

Case 3: $0 < \theta < 1/2$

$$\beta(\theta) = P(x_{(n)} \leq 1/2 | 0 < \theta < 1/2) + P(x_{(n)} > 1 | 0 < \theta < 1/2)$$

$$P(x_{(n)} \leq 1/2 | 0 < \theta < 1/2) = \int_0^\theta f_{x_{(n)}}(x)\mathrm{d}x = \frac{x^n}{\theta^n} = \frac{1^n}{1^n} = 1$$

$$P(x_{(n)} > 1 | 0 < \theta < 1/2) = 0$$

$$\beta(\theta) = 1 + 0 = 1$$

Case 4: $1/2 < \theta < 1$

$$\beta(\theta) = P(x_{(n)} \leq 1/2 | 1/2 < \theta < 1) + P(x_{(n)} > 1 | 1/2 < \theta < 1)$$

$$\int_0^{1/2} f_{x_{(n)}}(x)\mathrm{d}x = \frac{(1/2)^n}{\theta^n} = \frac{2^{-n}}{\theta^n} \text{ (decreasing function of } \theta\text{) since:}$$

$$\theta \to \infty \quad \frac{2^{-n}}{\theta^n} \to 0$$