Problem 1

$$\text{Since the } Xs \sim U(0,\theta)$$
$$E(X)=\frac{\theta}{2} \text{ and } Var(X)=\frac{\theta^2}{12}$$
$$E(X)=\frac{\theta}{2} \quad M=\bar{X}$$
$$\theta/2=\bar{X}$$
$$\hat{\theta}_{MM}=2\bar{X}$$
$$E(\hat{\theta}_{MM})=E(2\bar{X})=2E(X)=\theta \text{ (unbiased estimator)}$$
$$Var(\hat{\theta}_{MM})=Var(2\bar{X})=4Var(\bar{X})=\frac{4\theta^2/12}{n}=\frac{\theta^2}{3n}$$
$$L(\theta|x)=\prod_{i=1}^{n}\frac{1}{\theta}I(0\le x_i\le\theta)$$
$$L(\theta|x)=\theta^{-n}I(0\le x_{(n)}\le\theta)$$

For $\theta \ge x_{(n)},$ the likelihood funciton is decreasing and thus is maximized at $\hat{\theta}=X_{(n)}$

$$\hat{\theta}_{MLE}=X_{(n)}$$
$$f_{X_{(n)}}=\frac{nx^{n-1}}{\theta^n} \quad 0\le x\le\theta$$
$$E(\hat{\theta}_{MLE})=\frac{n}{\theta^n}\int_0^{\theta}x^n=\frac{\theta^{n+1}}{\theta^n}\frac{n}{n+1}=\frac{\theta n}{n+1} \text{ (biased estimator)}$$
$$E(\hat{\theta}^2_{MLE})=\frac{n}{\theta^n}\int_0^{\infty}x^{n+1}=\frac{\theta^2 n}{n+2}$$
$$Var(\hat{\theta}_{MLE})=E(\hat{\theta}^2_{MLE})-[E(\hat{\theta}_{MLE})]^2=\frac{\theta^2 n}{n+2}-\left(\frac{\theta n}{n+1}\right)^2=\frac{n\theta^2}{(n+1)^2(n+2)}$$
$$Var(\hat{\theta}_{MM})>Var(\hat{\theta}_{MLE})$$
$$\hat{\theta}_{MM} \text{ is an unbiased estimator but its variance is larger than } \hat{\theta}_{MLE}$$
$$\hat{\theta}_{MLE} \text{ is a biased estimator}$$
$$MSE(\hat{\theta}_{MM})=Var(\hat{\theta}_{MM})+bias(\hat{\theta}_{MM})^2=\frac{\theta^2}{3n}+0^2=\frac{\theta^2}{3n}$$
$$bias(\hat{\theta}_{MLE})=\frac{\theta n}{n+1}-\theta=\theta\left(\frac{n}{n+1}-1\right)=\frac{-\theta}{n+1}$$
$$bias^2(\hat{\theta}_{MLE})=\frac{\theta^2}{(n+1)^2}$$
$$MSE(\hat{\theta}_{MLE})=\frac{n\theta^2}{(n+1)^2(n+2)}+\frac{\theta^2}{(n+1)^2}=\frac{n\theta^2+(n+2)\theta^2}{(n+1)^2(n+2)}$$

$$= \frac{\theta^2(2n+2)}{(n+1)^2(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$$

$$MSE(\hat{\theta}_{MLE}) = \frac{2\theta^2}{n^2+3n+2} \quad MSE(\hat{\theta}_{MM}) = \frac{\theta^2}{3n}$$

Comparing the two MSE, the MSE of $\hat{\theta}_{MLE}$ gets smaller as n gets larger

For large n, $\hat{\theta}_{MLE}$ is preferrable

For smaller n, $\hat{\theta}_{MM}$ is preferable since the bias is quite large for $\hat{\theta}_{MLE}$ and $\hat{\theta}_{MM}$ is unbiased.

Problem 2

(a)

$$X_1, \ldots, X_n \sim Bern(\theta) \quad 0 \le \theta \le 1/2$$

$$E(X) = \theta M = \bar{X}$$

$$\hat{\theta}_{MM} = \bar{X}$$

$$L(\theta|x) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}$$

$$\ell(\theta|x) = \sum_{i=1}^{n} x_i \log(\theta) + (n - \sum_{i=1}^{n} x_i)\log(1-\theta)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{\sum_{i=1}^n x_i}{\theta} - \frac{n - \sum_{i=1}^n x_i}{1-\theta} = 0$$

$$\frac{\sum_{i=1}^n x_i}{\theta} = \frac{n - \sum_{i=1}^n x_i}{1-\theta}$$

$$\frac{1-\theta}{\theta} = \frac{n - \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i}$$

$$\frac{1}{\theta} - 1 = \frac{n}{\sum_{i=1}^n x_i} - 1$$

$$\hat{\theta} = \frac{\sum_{i=1}^n x_i}{n} = \bar{X}$$

$$\frac{\partial \ell}{\partial \theta^2} = -\left[\frac{\sum_{i=1}^n x_i}{\theta^2} + \frac{n - \sum_{i=1}^n x_i}{(1-\theta)^2}\right] < 0$$

Since $0 \le \theta \le 1/2$:

When $\bar{X} \le 1/2, \hat{\theta}_{MLE} = \bar{X}$ since $\bar{X}$ is the overall maximum

When $\bar{X} > 1/2, L(\theta|x)$ is an increasing function of θ on the interval $[0, 1/2]$

$$\text{and is therefore maximized by the upper bound of } \theta,\ 1/2$$
$$\text{Thus when } \bar{X} \leq 1/2,\ \hat{\theta}_{MLE} = \bar{X} \text{ when } \bar{X} > 1/2,\ \hat{\theta}_{MLE} = 1/2$$

(b)

$$E(\bar{X}) = (1/n)E(\sum_{i=1}^{n} x_i) = n/nE(X_1) = \theta \text{ (unbiased)}$$
$$bias(\bar{X}) = 0$$
$$Var(\sum_{i=1}^{n} x_i/n) = n/n^2 Var(X_1) = \frac{\theta(1-\theta)}{n}$$
$$MSE(\hat{\theta}_{MM}) = Var(\hat{\theta}_{MM}) + bias(\hat{\theta}_{MM})^2$$
$$= \frac{\theta(1-\theta)}{n} + 0^2$$
$$MSE(\hat{\theta}_{MM}) = \frac{\theta(1-\theta)}{n}$$
$$MSE(\hat{\theta}_{MLE}) = E[(\hat{\theta}_{MLE} - \theta)^2]$$
$$\text{Let } y = \sum_{i=1}^{n} x_i$$
$$E[g(x)] = \sum g(x) f_X(x)$$
$$E[(\hat{\theta}_{MLE} - \theta)^2] = \sum_{y=0}^{n} (\hat{\theta}_{MLE} - \theta)^2 \binom{n}{y} \theta^y (1-\theta)^{n-y}$$
$$MSE(\hat{\theta}_{MLE}) = \sum_{y=0}^{k} \left(\frac{y}{n} - \theta\right)^2 \binom{n}{y} \theta^y (1-\theta)^{n-y} + \sum_{y=k+1}^{n} (1/2-\theta)^2 \binom{n}{y} \theta^y (1-\theta)^{n-y}$$
$$\text{where } k = \begin{cases} n/2 & \text{if n is even} \\ (n-1)/2 & \text{if n is odd} \end{cases}$$

(c)

Defining y and k the same way as in the previous part:

$$MSE(\hat{\theta}_{MM}) = E(\bar{X}-\theta)^2 = \sum_{y=0}^{k}\left(\frac{y}{n}-\theta\right)^2\binom{n}{y}\theta^y(1-\theta)^{n-y}$$

$$MSE(\hat{\theta}_{MM}) - MSE(\hat{\theta}_{MLE}) =$$

$$\sum_{y=0}^{n}\left(\frac{y}{n}-\theta\right)^2\binom{n}{y}\theta^y(1-\theta)^{n-y}-\sum_{y=0}^{k}\left(\frac{y}{n}-\theta\right)^2\binom{n}{y}\theta^y(1-\theta)^{n-y}-\sum_{y=k+1}^{n}(1/2-\theta)^2\binom{n}{y}\theta^y(1-\theta)^{n-y}$$

$$=\sum_{y=k+1}^{n}[(\frac{y}{n}-\theta)^2-(1/2-\theta)^2]\binom{n}{y}\theta^y(1-\theta)^{n-y}$$

$$=\sum_{k+1}^{n}\left(\frac{y}{n}+1/2-2\theta\right)\left(\frac{y}{n}-1/2\right)\binom{n}{y}\theta^y(1-\theta)^{n-y}$$

Thus $MSE(\hat{\theta}_{MLE}) < MSE(\hat{\theta}_{MM})$ for all $\theta \in (0, 1/2]$

Problem 3

(a)

$$Y_i = \beta x_i + \epsilon_i \text{ xs are fixed constants, } \epsilon_i \sim N(0,\sigma^2)$$

$$\epsilon_i = Y_i - \beta x_i$$

$$L(\beta,\sigma^2|\boldsymbol{y}) = \prod_{i=1}^{n}(2\pi\sigma^2)^{-1/2}\exp\left(-\frac{1}{2\sigma^2}(y_i-\beta x_i)^2\right)$$

$$= (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta x_i)^2\right)$$

$$= (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i^2-2\beta y_i x_i+\beta^2 x_i^2)\right)$$

Writing $L(\beta,\sigma^2|\boldsymbol{y})$ in the form of $f(\boldsymbol{y}|\theta) = g(T(\boldsymbol{y})|\theta)h(\boldsymbol{y})$:

$$= (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{\beta^2\sum_{i=1}^{n}x_i^2}{2\sigma^2}\right)\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}y_i^2+\frac{\beta}{\sigma^2}\sum_{i=1}^{n}y_i x_i\right)$$

where $h(\boldsymbol{y}) = 1$ and

$$g(T(\boldsymbol{y})|\theta) = (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{\beta^2\sum_{i=1}^{n}x_i^2}{2\sigma^2}\right)\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}y_i^2+\frac{\beta}{\sigma^2}\sum_{i=1}^{n}y_i x_i\right)$$

$$T_1(y) = \sum_{i=1}^{n} Y_i^2 \quad T_2(y) = \sum_{i=1}^{n} x_i Y_i$$

$$\text{Thus } (\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} x_i Y_i) \text{ is an SS for } (\beta, \sigma^2)$$

(b)

$$L(\beta, \sigma^2 | \boldsymbol{y}) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{\beta^2 \sum_{i=1}^n x_i^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2} \sum_{i=1}^{n} y_i x_i\right)$$

$$\ell(\beta, \sigma^2 | \boldsymbol{y}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} y_i x_i$$

$$\text{Fixing } \sigma^2 :$$

$$\frac{\partial \ell}{\partial \beta} = -\frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i x_i = 0$$

$$\frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i x_i$$

$$\hat{\beta} = \frac{\sum_{i=1}^n y_i x_i}{\sum_{i=1}^n x_i^2} \quad (\hat{\beta} \text{ does not depend on } \sigma^2)$$

$$\frac{\partial \ell}{\partial \beta^2} = -\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 < 0 \quad \text{thus } \hat{\beta} \text{ is a maximum}$$

$$\text{Thus } \hat{\beta} \text{ is the MLE}$$

$$E(\hat{\beta}) = E\left(\frac{\sum_{i=1}^n y_i x_i}{\sum_{i=1}^n x_i^2}\right) = \frac{\sum_{i=1}^n x_i E(Y_i)}{\sum_{i=1}^n x_i^2} = \frac{\sum_{i=1}^n x_i \beta x_i}{\sum_{i=1}^n x_i^2} = \beta$$

$$\text{Therefore } \hat{\beta} \text{ is unbiased}$$

(c)

$$E(\hat{\beta}) = \beta$$

$$\text{Let } c_i = \frac{x_i}{\sum x_j^2} \text{ (constants)}$$

$$\hat{\beta} = \sum_{i=1}^{n} c_i Y_i$$

$$Var(\hat{\beta}) = Var(\sum_{i=1}^{n} c_i Y_i) = \sum_{i=1}^{n} c_i^2 Var(Y_i) = \sum_{i=1}^{n} \left(c_i^2\right) \sigma^2$$
$$= \sum_{i=1}^{n} \left(\frac{x_i}{\sum x_j^2}\right)^2 \sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{\left(\sum x_j^2\right)^2} \sigma^2 = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$
$$Var(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

Since a linear combination of independent normal r.v.s is normally distributed:

$$\hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right)$$

Problem 4

(a)

$$X_1, \ldots, X_n \sim Pois(\mu) \quad u \in \{1, 2\}$$
$$W = \sum_{i=1}^{n} X_i \quad V = (1+3n)W - W^2 - 2n^2$$
$$E(V|\mu = 1) = (1+3n)E(W|\mu = 1) - E(W^2|\mu = 1) - 2n^2$$
$$E(W) = n\mu$$
$$E(W^2) = Var(W) + [E(W)]^2 = n\mu + (n\mu)^2$$
$$E(V|\mu = 1) = (1+3n)(1n) - (1n + (1n))^2) - 2n^2$$
$$= n + 3n^2 - n - n^2 - 2n^2 = 0$$
$$E(V|\mu = 1) = 0$$
$$E(V|\mu = 2) = (1+3n)(2n) - (2n + (2n)^2) - 2n^2$$
$$= 2n + 6n^2 - 2n - 4n^2 - 2n^2 = 0$$
$$E(V|\mu = 2) = 0$$

(b)

$$f(x_i|\mu) = \prod_{i=1}^{n} f(x_i|\mu) = \prod_{i=1}^{n} \frac{u^{x_i} e^{-\mu}}{x_i!}$$
$$= \left(\prod_{i=1}^{n} \frac{1}{x_i!}\right) \mu^{\sum_{i=1}^{n} x_i} e^{-n\mu}$$
Writing $f(x_i|\mu)$ in the form of:
$$h(x)c(\mu)\exp\left(\sum_{j=1}^{k} w_j(\theta)t_j(x)\right)$$
$$= \left(\prod_{i=1}^{n} \frac{1}{x_i!}\right) e^{-n\mu} \exp\left(\log(\mu)\sum_{i=1}^{n} x_i\right)$$
Where $h(x) = \prod_{i=1}^{n} \frac{1}{x_i!} \quad c(\mu) = e^{-n\mu} \quad t(x) = \sum_{i=1}^{n} x_i \quad w(\mu) = \log(\mu)$

Thus $W = \sum_{i=1}^{n} X_i$ is a minimal sufficient statistic

Let $g(W) = V$

Then from part a we have: $E(g(W)) = 0$

Since $g(W) = V$ is not always equal to zero, W is not complete

(c)

MLE of $\mu = \{1, 2\}$

$\hat{\mu}_{MLE} = 1$ or $\hat{\mu}_{MLE} = 2$

$$\hat{\mu}_{MLE} = \begin{cases} 1 \text{ if } L(\mu = 1|x) > L(\mu = 2|x) \\ 2 \text{ if } L(\mu = 1|x) \leq L(\mu = 2|x) \end{cases}$$
$$L(\mu|x) = \prod_{i=1}^{n} f(x_i|\mu)$$
$$= \prod_{i=1}^{n} \frac{u^{x_i} e^{-\mu}}{x_i!}$$
$$= \left(\prod_{i=1}^{n} \frac{1}{x_i!}\right) \mu^{\sum_{i=1}^{n} x_i} e^{-n\mu}$$

$$L(\mu=1|x)=\left(\prod_{i=1}^{n}\frac{1}{x_i!}\right)e^{-n}$$
$$L(\mu=2|x)=\left(\prod_{i=1}^{n}\frac{1}{x_i!}\right)2^{\sum_{i=1}^{n}x_i}e^{-2n}$$
$$e^{-n}\stackrel{?}{=}2^{\sum_{i=1}^{n}x_i}e^{-2n}$$
$$e^{n}\stackrel{?}{=}2^{\sum_{i=1}^{n}x_i}$$
$$n\neq\sum_{i=1}^{n}x_i log(2)$$

Thus the MLE is unique

(d)

$$\ell(\mu=1|x)=-n$$
$$\ell(\mu=2|x)=\sum_{i=1}^{n}x_i\log(2)-2n$$
$$E(\hat{\mu}|\mu=1)=P(\hat{\mu}=1)+2P(\hat{\mu}=2)$$
$$P(\hat{\mu}=1)=P(L(\mu=1|x)>L(\mu=2|x))$$
$$=P(\ell(\mu=1|x)>\ell(\mu=2|x))$$
$$=P(-n>\sum_{i=1}^{n}x_i\log(2)-2n)$$
$$=P(n>\sum_{i=1}^{n}x_i\log(2))$$
$$=P\left(\sum_{i=1}^{n}x_i<\frac{n}{\log(2)}\right)$$
$$\text{Since }\sum_{i=1}^{n}x_i\sim Pois(n\mu)=Pois(3)$$
$$=ppois(lambda=3,q=3/log(2))\approx .815\text{ (using R)}$$
$$E(\hat{\mu}|\mu=1)=P(\hat{\mu}=1)+2P(\hat{\mu}=2)=.815+2*.185=1.185$$
$$Var(\hat{\mu}|\mu=1)=E(\hat{\mu}-1.185)^2=(1-1.185)^2P(\hat{\mu}=1)+(2-1.185)^2P(\hat{\mu}=2)$$
$$=(1-1.185)^2*.815+(2-1.185)^2*(1-.815)\approx .151$$
$$E(\hat{\mu}|\mu=2)=P(\hat{\mu}=1)+2P(\hat{\mu}=2)$$
$$P(\hat{\mu}=2)=P(\ell(\mu=2|x)>\ell(\mu=1|x))$$

$$= P\left(\sum_{i=1}^{n} x_i > \frac{3}{\log(2)}\right)$$

$$\text{Since } \sum_{i=1}^{n} x_i \sim Pois(n\mu) = Pois(6)$$

$$= ppois(lambda = 6, q = 3/log(2)) \approx .285$$

$$E(\hat{\mu}|\mu = 2) = .285 + 2 * (1 - .285) = 1.715$$

$$Var(\hat{\mu}|\mu = 2) = E(\hat{\mu} - 1.715)^2 = (1 - 1.715)^2 P(\hat{\mu} = 1) + (2 - 1.715)^2 P(\hat{\mu} = 2)$$

$$= (1 - 1.715)^2 * (1 - .715) + (2 - 1.715)^2 * (.715) \approx .204$$

Problem 5

(a)

$$f(x_i|\theta) = (2\pi\theta)^{-1/2} e^{-\frac{(x_i-\theta)^2}{2\theta}}$$

$$\log(x_i|\theta) = (-1/2)\log(2\pi) - (1/2)\log(\theta) - \frac{1}{2\theta}(x_i - \theta)^2$$

$$= (-1/2)\log(2\pi) - (1/2)\log(\theta) - \frac{1}{2\theta}x_i^2 + x_i - \frac{\theta}{2}$$

$$\frac{\partial}{\partial\theta}\log(x_i|\theta) = -\frac{1}{2\theta} + \frac{1}{2\theta^2}x_i^2 - \frac{1}{2}$$

$$\frac{\partial}{\partial\theta^2}\log(x_i|\theta) = \frac{1}{2\theta^2} - \frac{1}{\theta^3}x_i^2 = -\frac{1}{\theta^2}\left(\frac{x_i^2}{\theta} - \frac{1}{2}\right)$$

$$nE\left[\frac{1}{\theta^2}\left(\frac{x_i^2}{\theta} - \frac{1}{2}\right)\right]$$

$$= \frac{n}{\theta^2}\left(\frac{E(x_i^2)}{\theta} - \frac{1}{2}\right)$$

$$= \frac{n}{\theta^2}\left(\frac{\theta + \theta^2}{\theta} - \frac{1}{2}\right)$$

$$= \frac{n(2\theta + 1)}{2\theta^2}$$

$$CRLB = 1/\frac{n(2\theta + 1)}{2\theta^2} = \frac{2\theta^2}{n(2\theta + 1)}$$

(b)

$$L(\theta|x) = \prod_{i=1}^{n} (2\pi\theta)^{-1/2} e^{-\dfrac{(x_i-\theta)^2}{2\theta}}$$

$$\ell(\theta|x) = (-n/2)\log(2\pi) - (n/2)\log(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\theta)^2$$

$$= (-n/2)\log(2\pi) - (n/2)\log(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i - \frac{n\theta}{2}$$

$$\frac{\partial \ell}{\partial \theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2 - \frac{n}{2}$$

$$\text{let } t(x) = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{2\theta^2}\big(t(x) - (\theta^2+\theta)\big) = 0$$

$$\frac{n}{2\theta^2} t(x) = (\theta^2+\theta)\frac{n}{2\theta^2}$$

$$\theta^2 + \theta = t(x)$$

$$\hat{\theta}_{MLE} = (1/2)[(4t(x)+1)^{1/2} - 1]$$

(c)

$$\text{Estimate } \theta \text{ with: } \begin{cases} \bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} x_i \\ S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \end{cases}$$

$$Var(\bar{X}) = \sigma^2/n = \theta/n$$

$$Var(\chi^2_{n-1}) = 2(n-1)$$

$$Var\left(\frac{n-1}{\sigma^2}S^2\right) = Var(\chi^2_{n-1})$$

$$= \frac{(n-1)^2}{\sigma^4} Var(S^2) = 2(n-1)$$

$$Var(S^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1} = \frac{2\theta^2}{n-1}$$

$$Var(\bar{X}) < Var(S^2) \text{ when } \frac{\theta}{n} < \frac{2\theta^2}{n-1}$$

$$Var(\bar{X}) > Var(S^2) \text{ when } \frac{\theta}{n} > \frac{2\theta^2}{n-1}$$
$$\frac{1}{n} < \frac{2\theta}{n-1}$$
$$\theta > \frac{n-1}{2n}$$
$$\text{Thus when } \theta > \frac{n-1}{2n}\ Var(S^2) \text{ is smaller}$$
$$\text{When } \theta < \frac{n-1}{2n}\ Var(\bar{X}) \text{ is smaller}$$
Therefore one estimate is not uniformly better than the other

(d)

$$T(X) = (1/n)\sum_{i=1}^{n} X_i^2$$
$$E\left((1/n)\sum_{i=1}^{n} X_i^2\right) = \theta + \theta^2$$
$$E(T(X)) = \theta^2 + \theta = \tau(\theta)$$
$$\text{Thus for } \tau(\theta) = \theta^2 + \theta,\ T(X) \text{ is an unbiased estimator}$$

(e)

$$X \sim N(\theta, \theta)$$
$$\text{Let } Y = \frac{X_i\theta}{\sqrt{\theta}} \sim N(0,1)$$
$$X_i = \sqrt{\theta}Y + \theta$$
$$X_i^2 = (\sqrt{\theta}Y + \theta)^2 = \theta Y^2 + 2\theta\sqrt{\theta}Y + \theta^2$$
$$Y^2 \sim \chi_1^2$$
$$Var(X_i^2) = Var(\theta Y^2 + 2\theta\sqrt{\theta}Y + \theta^2)$$
$$= \theta^2(2) + 4\theta^3(1) + 0$$
$$Var(X_i^2) = 2\theta^2 + 4\theta^3$$
$$\frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2) = \frac{n}{n^2}(2\theta^2 + 4\theta^3) = \frac{2\theta^2 + 4\theta^3}{n}$$

$$\frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{2\theta^2+4\theta^3}{n}$$

$$\text{Thus } Var(T(X)) = \frac{2\theta^2(2\theta+1)}{n}$$

$$CRLB = \frac{\left(\dfrac{\partial(\theta^2+\theta)}{\partial\theta}\right)^2}{nE[(\frac{\partial^2}{\partial\theta^2}\log f(x|\theta))]}$$

$$= \frac{(1+2\theta)^2}{nE[(\frac{\partial^2}{\partial\theta^2}\log f(x|\theta))]}$$

$$\text{From part a we have: } nE[(\frac{\partial^2}{\partial\theta^2}\log f(x|\theta))^2] = \frac{n(2\theta+1)}{2\theta^2}$$

$$CRLB = \frac{(1+2\theta)^2}{\left(\dfrac{n(2\theta+1)}{2\theta^2}\right)}$$

$$CRLB = \frac{2\theta^2(2\theta+1)}{n}$$

Thus $T(X)$ has the smallest variance among unbiased estimators of $\tau(\theta)$