Interval Estimation

Feng-Chang Lin

Department of Biostatistics
University of North Carolina at Chapel Hill

flin@bios.unc.edu

(C&B §9)

Introduction

- **Example 1** Suppose $X_1, \ldots, X_n$ are iid from N$(\theta, 1)$.
- We know that $P_\theta(\bar{X} = \theta) = 0$ since $\bar{X}$ is a continuous random variable.
- Therefore, even though $\bar{X}$ is a good estimator of θ, it is never equal to θ.

Introduction (cont'd)

- **Example 2** $X \sim \text{Binomial}(n, \theta)$, $\theta \in (0, 1)$.
- X/n is the MLE of θ.
- $P_\theta(X/n = \theta)$ will be 0 unless θ is one of $\{1/n, 2/n, \ldots, (n-1)/n\}$.
- If $\theta = i/n$ for some $i \in \{1, 2, \ldots, n-1\}$, then

$$P_\theta(X/n = \theta) = P(X = i) = \binom{n}{i} \left(\frac{i}{n}\right)^i \left(1 - \frac{i}{n}\right)^{n-i}.$$

- This probability can be very small, especially for large n. For example, if $n = 20$, $\theta = 1/2$, then $P_\theta(X/n = \theta)$ is about 0.18, and if $n = 100$, it is about 0.08.

Introduction (cont'd)

- In many situations point estimators have low (or zero) probability of being equal to the parameter they estimate.
- If one considers estimators that are intervals instead of single points, that shortcoming can be overcome.
- In the normal mean problem, the interval $(\bar{X} - 1.96/\sqrt{n}, \bar{X} + 1.96/\sqrt{n})$ has probability 0.95 of containing the true parameter value θ.

Interval Estimator

- **Interval Estimator** $(L(X), U(X))$, where $L(X)$ and $U(X)$ are statistics, $L(X) < U(X)$.
- Denoted by either $(L(X), U(X))$ or $[L(X), U(X)]$.
- One-sided intervals: e.g. $L(X) = -\infty$ or $U(X) = \infty$ (depending on Θ).
- **Coverage probability** for $(L(X), U(X))$:

$$CP(\theta) = P_\theta(\theta \in (L(X), U(X))),$$

  as a function of θ.
- **Confidence Coefficient (Confidence Level)**: $\inf_{\theta \in \Theta} CP(\theta)$.

Interval Estimator (cont'd)

- **Example** $X \sim \text{Bernoulli}(\theta)$, $\theta \in [0,1]$. If one has a confidence interval $[0.4, 0.5 + 0.2X]$

$$CP(\theta) = \begin{cases} 0, & 0 \leq \theta < 0.4, \\ 1, & 0.4 \leq \theta \leq 0.5, \\ \theta, & 0.5 < \theta \leq 0.7, \\ 0, & 0.7 < \theta \leq 1. \end{cases}$$

- Confidence coefficient = 0.

How to Find a Confidence Interval

- **Inverting a test**: Consider $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. By inverting the acceptance region of a level α test

  $$A(\theta_0) = \{x : \delta(x, \theta_0, \alpha) = 0\}$$

  with a test function $\delta(x)$, written as $\delta(x, \theta, \alpha)$, one can define

  $$C(x) = \{\theta \in \Theta : \delta(x, \theta, \alpha) = 0\},$$

  as a subset of Θ.
- Then,

  $$\begin{aligned} P_\theta(\theta \in C(x)) &= P_{\theta_0}(\delta(X, \theta_0, \alpha) = 0) \\ &= 1 - P_{\theta_0}(\delta(X, \theta_0, \alpha) = 1) \geq 1 - \alpha. \end{aligned}$$

- Thus $C(x)$ is a $1 - \alpha$ confidence interval of θ.

How to Find a Confidence Interval (cont'd)

- **Example** X is a random sample of size n from N$(\theta, 1)$. $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ with $\alpha = 0.05$.

$$\delta(X, \theta_0, \alpha) = I(|\bar{X} - \theta_0| > 1.96/\sqrt{n}).$$

- That means,

$$P_{\theta_0}(\bar{X} - 1.96\frac{1}{\sqrt{n}} \leq \theta_0 \leq \bar{X} + 1.96\frac{1}{\sqrt{n}}) = 0.95.$$

- The statement is true for every θ_0. Hence, we can write

$$P_{\theta}(\bar{X} - 1.96\frac{1}{\sqrt{n}} \leq \theta \leq \bar{X} + 1.96\frac{1}{\sqrt{n}}) = 0.95.$$

- Hence, the 0.95 confidence interval is

$$C(x) = (\bar{x} - 1.96/\sqrt{n}, \bar{x} + 1.96/\sqrt{n}).$$

How to Find a Confidence Interval (cont'd)

- **Example** X is a random sample of size n from Exponential(θ).
- To test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, the acceptance region of the likelihood ratio test (LRT) statistic is

$$A(\theta_0) = \left\{ \boldsymbol{x} : \left(\frac{\sum_{i=1}^n x_i}{\theta_0} \right)^n e^{-\sum_{i=1}^n x_i/\theta_0} \geq c \right\}$$

- Inverting this acceptance region gives the $1 - \alpha$ confidence interval

$$C(x) = \left\{ \theta : \left(\frac{\sum_{i=1}^n x_i}{\theta} \right)^n e^{-\sum_{i=1}^n x_i/\theta} \geq c \right\}.$$

- Check C&B on how to find the upper and lower bound for θ.

How to Find a Confidence Interval (cont'd)

- **Lower confidence bounds**: $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$. Inverting a test gives the interval $[L(X), \infty)$.
- **Upper confidence bounds**: $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$. Inverting a test gives the interval $(-\infty, U(X)]$.
- **Example** Let $X_1, \cdots, X_n$ be a random sample from $\mathrm{N}(\theta, \sigma^2)$.
- Consider constructing a $1 - \alpha$ upper confidence bound for μ.
- The size α test acceptance region is

$$A(\theta_0) = \left\{ \boldsymbol{x} : \frac{\bar{x} - \theta_0}{s/\sqrt{n}} \geq t_{n-1,\alpha} \right\}.$$

- The $1 - \alpha$ confidence region (or set) is

$$C(x) = \left\{ \theta : \bar{x} - t_{n-1,\alpha} \frac{s}{\sqrt{n}} \geq \theta \right\}$$

How to Find a Confidence Interval (cont'd)

- **Pivot (Pivotal Quantity)** $Q(X, \theta)$ is a pivot if the distribution of $Q(X, \theta)$ does not depend on θ.
- **Examples**

Family	Density	Pivot
Location	$f(x-\mu)$	$\bar{X}-\mu$
Scale	$\frac{1}{\sigma}f(\frac{1}{\sigma})$	$\frac{\bar{X}}{\sigma}$
Location-Scale	$\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$	$\frac{\bar{X}-\mu}{\sigma}$

Pivotal Quantity

- **Example** If $X_1, \cdots, X_n$ are iid N(μ, σ^2), then $(\bar{X} - \mu)/(\sigma/\sqrt{n})$ is a pivot.
- If σ^2 is known, we can use this pivot to calculate a confidence interval for μ.
- Let $z_{1-\alpha/2}$ be the $(1 - \alpha/2)$th percentile of a standard norm distribution. One has

$$
\begin{aligned}
1 - \alpha &= P\left(-z_{1-\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{1-\alpha/2}\right) \\
&= P\left(\bar{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)
\end{aligned}
$$

- The $1 - \alpha$ confidence interval is $(\bar{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})$.

Pivotal Quantity (cont'd)

- What if σ^2 is unknown, what pivot can we use to calculate a confidence interval for μ?

Pivotal Quantity (cont'd)

- **Example** X is a random sample of size n from exponential(θ).
- Construct a 95% ($1-\alpha = 0.95$) confidence interval for θ.
- This is a scale family. Why?
- Let $Q(X,\theta) = 2n\bar{X}/\theta \sim \chi^2_{2n}$. Then,

$$\begin{aligned} 1-\alpha &= P\left(a < Q(X,\theta) < b\right) = P\left(a < 2n\bar{X}/\theta < b\right) \\ &= P\left(2n\bar{X}/b < \theta < 2n\bar{X}/a\right). \end{aligned}$$

- Hence, the $1-\alpha$ confidence interval for θ is $(2n\bar{X}/b, 2n\bar{X}/a)$.
- How to choose a and b? One may let $a = F^{-1}(\alpha_1)$ and $b = F^{-1}(1-\alpha_2)$, where $\alpha_1 + \alpha_2 = \alpha$.

Minimization of Expected Length

- How to choose α_1 and α_2? A convenient choice is $\alpha_1 = \alpha_2 = \alpha/2$.
- One possible criterion is “the shortest interval".
- Since the length can be considered as a function of $\bar{X}$, we may calculate the “expected length"

$$E\left(2n\bar{X}/a - 2n\bar{X}/b\right) = 2n\theta\left(\frac{1}{a} - \frac{1}{b}\right).$$

- We choose a and b (or equivalently, α_1 and α_2) such that the expected length is minimized.
- For a fixed θ, the solution depends on n.
- Examples: for $n = 1$, $\alpha_1 = 0.05$, $\alpha_2 = 0$; for $n = 10$, $\alpha_1 = 0.044, \alpha_2 = 0.006$; for $n = 20$, $\alpha_1 = 0.04$, $\alpha_2 = 0.01$.

Another Example from Scale Family

- **Example** X is a random sample of size n from N(μ, σ^2).
- How do we construct a $1-\alpha$ confidence interval for σ^2?
- If μ is unknown, the pivot is

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

- What if μ is known? What is the pivot?

Pivoting the CDF

- Suppose T is a statistic with cdf F_T. Using $F_T(t|\theta)$ as a pivot is feasible if $F_T(t|\theta)$ is a decreasing or increasing function in θ for each fixed t.
- If $F_T(t|\theta)$ is a decreasing function of θ, to construct a $1-\alpha$ confidence interval, we find $U(t)$ and $L(t)$ such that

$$P(T \leq t|\theta = U(t)) = \alpha_1, \text{ and } P(T \geq t|\theta = L(t)) = \alpha_2.$$

  with “tail probability" α_1 and α_2 satisfying $\alpha_1 + \alpha_2 = \alpha$.
- One can prove $\{\theta : \alpha_1 \leq F_T(t|\theta) \leq 1-\alpha_2\}$=$\{\theta : L(t) \leq \theta \leq U(t)\}$ (Theorem 9.2.12 in C&B).

Pivoting the CDF (cont'd)

- **Example** If $X_1, \ldots, X_n$ are iid with pdf $f(x|\mu) = e^{-(x-\mu)} I_{[\mu,\infty)}(x)$.
- Then, $Y = X_{(1)}$ is sufficient for μ with pdf

$$f_Y(y|\mu) = ne^{-n(y-\mu)} I_{[\mu,\infty)}(y).$$

- Since the CDF $F_Y(y|\mu) = 1 - e^{-n(y-\mu)}$, $\mu \le y < \infty$, is decreasing in μ, we can have

$$\int_{U(y)}^{y} ne^{-n(u-U(y))} du = \frac{\alpha}{2}, \quad \text{and} \quad \int_{y}^{\infty} ne^{-n(u-L(y))} du = \frac{\alpha}{2}.$$

- The solutions for $L(y)$ and $U(y)$ are

$$L(y) = y + \frac{1}{n} \log(\alpha/2), \quad \text{and} \quad U(y) = y + \frac{1}{n} \log(1 - \alpha/2).$$

Pivoting the CDF (cont'd)

- The $1 - \alpha$ confidence interval for μ is

$$C(y) = \left\{ \mu : y + \frac{1}{n}\log(\alpha/2) \leq \mu \leq y + \frac{1}{n}\log(1-\alpha/2) \right\}.$$

- Can we invert the acceptance region of the LRT test to obtain the confidence interval?
- Can we use the pivotal quantity to obtain the confidence interval? What is the pivot?
- If these intervals are different, which one has a shorter length?
- Check Exercise 9.25 in C&B.

Evaluating Interval Estimators

- **Optimizing the length**: Minimization of $|a - b|$ is generally not easy.
- **(Theorem 9.3.2 in C&B)** For any unimodal density g with mode in $[a, b]$, subject to total tail area $\alpha_1 + \alpha_2 = \alpha$. Then $|a - b|$ is minimized by a and b with $g(a) = g(b)$.
- **Optimizing the expected length**: we have seen the example.
- Check Example 9.3.4 for which the application of Theorem 9.3.2 will not give the shortest confidence interval.

Exact versus Approximate Confidence Intervals

- **Exact confidence interval**:

$$P(L(X) < \theta < U(X)) = 1 - \alpha$$

- **Approximate confidence interval**:

$$P(L(X) < \theta < U(X)) \approx 1 - \alpha$$

- Let $X_1, \ldots, X_n$ be iid $\mathrm{N}(\mu, \sigma^2)$. The $1 - \alpha$ confidence interval for μ could be $\bar{X} \pm t_{n-1,1-\alpha/2} S/\sqrt{n}$. Exact or approximate?
- The $1 - \alpha$ confidence interval for σ^2 could be $((n-1)S^2/b, (n-1)S^2/a)$ for some a and b. Exact or approximate?

Exact versus Approximate CI (cont'd)

- Let $X_1, \ldots, X_n$ be iid Beroulli(θ). The MLE of θ is $\hat{\theta} = \bar{X}$. According to the CLT,
$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow_d N(0, \sigma^2),$$
where
$$\sigma^2 = Var(X_1) = \theta(1 - \theta).$$
- With $\hat{\sigma}^2 = \bar{X}(1 - \bar{X})$, one can construct a $1 - \alpha$ **approximate** confidence interval
$$\bar{X} \pm z_{1-\alpha/2}\sqrt{\frac{\bar{X}(1 - \bar{X})}{n}}.$$

Exact versus Approximate CI (cont'd)

- In fact, an **exact** confidence interval can be constructed but may not have “exactly" $1-\alpha$ confidence level.
- Check Example 9.2.11 in C&B for another binomial case.
- Check Example 9.2.15 in C&B for a Poisson case.