Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 7.9

2. C&B 7.12

3. C&B 7.19

4. [master exam 2010] Let $X_1, \dots, X_n$ be a random sample from a Poisson distribution with mean $\mu \in \{1, 2\}$. Note that the parameter space contains only two points; it is not the usual parameter space for the Poisson.

   (a) Let $W = \sum_{i=1}^n X_i$ and $V = (1 + 3n)W - W^2 - 2n^2$. Find $E(V|\mu = 1)$ and $E(V|\mu = 2)$.

   Solution: Given $\mu = 1$, W follows a Poisson distribution with mean n and

   $$E(V|\mu = 1) = E\{(1 + 3n)W - W^2 - 2n^2\} = n(1 + 3n) - (n + n^2) - 2n^2 = 0.$$

   Given $\mu = 1$, W follows a Poisson distribution with mean $2n$ and

   $$E(V|\mu = 2) = E\{(1 + 3n)W - W^2 - 2n^2\} = 2n(1 + 3n) - (2n + 4n^2) - 2n^2 = 0.$$

   (b) Find a minimal sufficient statistic. Is it complete? Hint: Part (a).

   Solution: Since X_i follows a Poisson distribution, which belongs to an exponential family, one can claim $\sum_{i=1}^n X_i$ is a sufficient statistic (it does not matter whether if μ is in an open space). By the definition of minimal sufficiency, one can prove $\sum_{i=1}^n X_i$ is a minimal sufficient statistic. However, since $E(V|\mu = 1) = 0$ and $E(V|\mu = 2) = 0$ in (a), $W = \sum_{i=1}^n X_i$ is not a complete statistic since one can find a function that is not a zero function but is unbiased for 0.

   (c) Derive the maximum likelihood estimator (MLE) of μ. Is it unique?

   Solution: The MLE of μ is either 1 or 2, depending on which likelihood is larger. That is,

   $$\hat{\mu} = \begin{cases} 1, & \text{if } L(\mu = 1|x) > L(\mu = 2|x) \\ 2, & \text{if } L(\mu = 2|x) \geq L(\mu = 1|x), \end{cases}$$

   where

   $$L(\mu = 1|x) = \prod_{i=1}^n \frac{e^{-1}}{x_i!},$$

and

$$L(\mu = 2|x) = \prod_{i=1}^{n} \frac{2^{x_i} e^{-2}}{x_i!}.$$

Since it is unlikely that $L(\mu = 1|x) = L(\mu = 2|x)$, one can claim $\hat{\mu}$ is unique. One can see, in (d), if $L(\mu = 1|x) = L(\mu = 2|x)$, then $\sum_{i=1}^{n} x_i = n/\log(2)$, which is not possible since $\sum_{i=1}^{n} x_i$ is an integer.

(d) For $n = 3$, compute the exact numerical values of the mean and variance of the MLE $\hat{\mu}$ when $\mu = 1$ and $\mu = 2$.

Solution: The mean value of $\hat{\mu}$ depends on the probability of $\hat{\mu}$ being 1 or 2. That is,

$$\begin{aligned} E(\hat{\mu}) &= 1 \times P(\hat{\mu} = 1) + 2 \times P(\hat{\mu} = 2) \\ &= P(L(\mu = 1|x) > L(\mu = 2|x)) + 2\{1 - P(L(\mu = 1|x) > L(\mu = 2|x))\}, \end{aligned}$$

where $P(L(\mu = 1|x) > L(\mu = 2|x))$ equals

$$\begin{aligned} &P(\ell(\mu = 1|x) > \ell(\mu = 2|x)) \\ &\quad = P\left(\sum_{i=1}^{n}\{(-1) - \log(X_i!)\} > \sum_{i=1}^{n}\{X_i \log(2) - 2 - \log(X_i!)\}\right) \\ &\quad = P\left(\sum_{i=1}^{n} X_i < n/\log(2)\right). \end{aligned}$$

Since $W = \sum_{i=1}^{n} X_i$ follows a Poisson distribution with mean $n\mu$, the probability above depends on the value of μ. Given $n = 3$ and $\mu = 1$, W follows a Poisson distribution with mean 3, the probability above becomes

$$P\left(W < 3/\log(2)\right) = 0.815.$$

The mean value of $\hat{\mu}$, given $\mu = 1$, is

$$E(\hat{\mu}|\mu = 1) = 0.815 + 2 \times (1 - 0.815) = 1.185,$$

and the variance is

$$\text{Var}(\hat{\mu}|\mu = 1) = (1 - 1.185)^2 \times 0.815 + (2 - 1.185)^2 \times (1 - 0.815) = 0.151.$$

Similarly, $E(\hat{\mu}|\mu = 2) = 1.715$ and $\text{Var}(\hat{\mu}|\mu = 2) = 0.204$

5. Let $X_1, \ldots, X_n$ be iid random variables from the $\mathrm{N}(\theta, \theta)$ distribution, $\theta > 0$.

   (a) Find the Cramér-Rao Lower Bound (CRLB) for any unbiased estimator of θ.

   Solution: For one observation,
$$L(\theta|x_i) = (2\pi\theta)^{-1/2} \exp\left\{-\frac{(x_i-\theta)^2}{2\theta}\right\}.$$
   One has
$$\ell(\theta|x_i) = -\frac{x_i^2}{2\theta} - \frac{\theta}{2} - \frac{1}{2}\log\theta + x_i - \frac{1}{2}\log(2\pi).$$
   Taking the first derivative, one has
$$\frac{\partial}{\partial\theta}\ell(\theta|x_i) = \frac{1}{2\theta^2}\{x_i^2 - (\theta^2+\theta)\}.$$
   Taking the second derivative, one has
$$\frac{\partial^2}{\partial\theta^2}\ell(\theta|x_i) = -\frac{1}{\theta^2}\left(\frac{x_i^2}{\theta} - \frac{1}{2}\right).$$
   The denominator of CRLB from a single observation is
$$E\left\{-\frac{\partial^2}{\partial\theta^2}\ell(\theta|x_i)\right\} = \frac{1}{\theta^2}\left(\frac{\theta^2+\theta}{\theta} - \frac{1}{2}\right) = \frac{2\theta+1}{2\theta^2}.$$
   Since $\tau(\theta) = \theta$ and $d\tau(\theta)/d\theta = 1$, the CRLB for θ is $2\theta^2/\{n(2\theta+1)\}$.

   (b) Find an explicit expression for MLE $\hat{\theta}$. In this case, it is not easy to verify the $\hat{\theta}$ that solves the first derivative of the log-likelihood function is indeed the maximizer. You may skip the verification.

   Solution: The likelihood function is
$$L(\theta|\boldsymbol{x}) = (2\pi\theta)^{-n/2} \exp\left\{\frac{-\sum(x_i-\theta)^2}{2\theta}\right\}.$$
   One has the log-likelihood function written as
$$\ell(\theta|\boldsymbol{x}) = \left\{-\frac{t(\boldsymbol{x})}{2\theta} - \frac{\theta}{2} - \frac{1}{2}\log\theta + \bar{X} - \frac{1}{2}\log(2\pi)\right\} n,$$
   where $t(\boldsymbol{x}) = n^{-1}\sum_{i=1}^n x_i^2$. This is an exponential family. Hence, $T(\boldsymbol{x}) = n^{-1}\sum_{i=1}^n X_i^2$ is a complete sufficient statistic. Taking the first derivative, one has
$$\frac{\partial}{\partial\theta}\ell(\theta|\boldsymbol{x}) = \frac{n}{2\theta^2}\{t(\boldsymbol{x}) - (\theta^2+\theta)\}.$$

The MLE, solving $(\partial/\partial\theta)\ell(\theta|\boldsymbol{x}) = 0$, is

$$\hat{\theta} = \frac{1}{2}\left\{(4T+1)^{1/2} - 1\right\}.$$

Taking the second derivative, one has

$$\frac{\partial^2}{\partial\theta^2}\ell(\theta|\boldsymbol{x}) = -\frac{n}{\theta^2}\left\{\frac{t(\boldsymbol{x})}{\theta} - \frac{1}{2}\right\}$$

One can see that $(\partial/\partial\theta)\ell(\theta|\boldsymbol{x}) > 0$ for $\theta \in [0, \hat{\theta})$. That means $\ell(\theta|\boldsymbol{x})$ is monotone increasing in $\theta \in [0, \hat{\theta})$. Similarly, one can see that $\ell(\theta|\boldsymbol{x})$ is monotone decreasing when $\theta > \hat{\theta}$. That concludes $\hat{\theta}$ reaches the maximum over the domain.

(c) One can estimate θ using $\bar{X} = \sum_{i=1}^n X_i/n$ and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$. Derive $\text{Var}(\bar{X})$ and $\text{Var}(S^2)$. Is one uniformly better than the other?

Solution: One can have

$$\text{var}(\bar{X}) = \frac{1}{n}\text{Var}(X_1) = \frac{\theta}{n},$$

while

$$\text{var}(S^2) = \frac{\theta^2}{(n-1)^2}\text{var}((n-1)S^2/\theta) = \frac{\theta^2}{(n-1)^2}2(n-1) = \frac{2\theta^2}{(n-1)},$$

since $(n-1)S^2/\theta$ follows a chi-squared distribution with degree of freedom $(n-1)$. Given a finite n, say $n = 2$, $\bar{X}$ is better than S^2 if $\theta > 1/4$. Asymptotically, $\bar{X}$ is better than S^2 if $\theta < \theta^2$, i.e., if $\theta > 1/2$. In conclusion, one estimator is not uniformly better than the other.

(d) Let $T(X) = n^{-1}\sum_{i=1}^n X_i^2$. Find $\tau(\theta)$, for which $T(X)$ is an unbiased estimator.

Solution: One may try $T(X) = n^{-1}\sum_{i=1}^n X_i^2$. One can show

$$E\{T(X)\} = n^{-1}\sum_{i=1}^n E(X_i^2) = n^{-1}\sum_{i=1}^n(\theta^2 + \theta) = \theta^2 + \theta \equiv \tau(\theta).$$

(e) Does $T(X)$ have the smallest variance among unbiased estimators of $\tau(\theta)$?

Solution: Taking $Z_i = (X_i - \theta)/\sqrt{\theta}$, we know that $Z_i \sim \text{N}(0,1)$. That gives

$$X_i^2 = (\sqrt{\theta}Z_i + \theta)^2 = \theta Z_i^2 + \theta^2 + 2\theta\sqrt{\theta}Z_i.$$

Then, one has

$$\begin{aligned}\text{var}(X_i^2) &= \theta^2\text{var}(Z_i^2) + 4\theta^3\text{var}(Z_i) + 4\theta^2\sqrt{\theta}\text{cov}(Z_i, Z_i^2)\\ &= 2\theta^2 + 4\theta^3 + 0 = 2\theta^2(2\theta+1).\end{aligned}$$

Since $\text{cov}(Z_i, Z_i^2) = E(Z_i^3) - E(Z_i)E(Z_i^2) = 0 - 0 \times 1 = 0$, we have

$$\text{var}\{T(X)\} = n^{-2}\sum_{i=1}^n \text{var}(X_i^2) = 2\theta^2(2\theta+1)/n.$$

The CRLB for $\tau(\theta) = \theta^2 + \theta$ is

$$\frac{\{d\tau(\theta)/d\theta\}^2}{nE\left\{-\frac{\partial^2}{\partial\theta^2}\ell(\theta|x_1)\right\}} = \frac{(2\theta+1)^2}{n(2\theta+1)/(2\theta^2)} = 2\theta^2(2\theta+1)/n.$$

We hence can conclude the variance of $T(X) = n^{-1}\sum_{i=1}^n X_i^2$ reaches the CRLB and $T(X)$ has the smallest variance among unbiased estimators of $\tau(\theta)$.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a $N(0, \sigma^2)$.

   (a) Develop an explicit expression for an unbiased estimator $\hat{\theta}$ of the unknown parameter $\theta = \sigma^r$, where r is some known positive integer.

   Solution: Let $T = \sum_{i=1}^n X_i^2$. One can see that

$$\frac{T}{\sigma^2} = \sum_{i=1}^n \left(\frac{X_i}{\sigma}\right)^2 = \sum_{i=1}^n Z_i^2 \sim \chi_n^2.$$

One can show

$$E\left(\frac{T}{\sigma^2}\right)^{r/2} = \frac{\Gamma(n/2+r/2)}{\Gamma(n/2)}2^{r/2}.$$

Hence,

$$E(T^{r/2}) = \frac{\Gamma(n/2+r/2)}{\Gamma(n/2)}2^{r/2}\sigma^r,$$

and

$$E\left(T^{r/2}2^{-r/2}\frac{\Gamma(n/2)}{\Gamma(n/2+r/2)}\right) = \sigma^r.$$

(b) Derive an explicit expression for the CRLB for the variance of any unbiased estimator of the parameter θ.

Solution: To see how the formula of the CRLB can be applied, one can express $\theta = \tau(\sigma^2)$, where $\tau(x) = x^{r/2}$. Hence, the numerator of the CRLB is $\{d\tau(x)/dx\}^2 = r^2x^{r-2}/4$. To derive the denominator of the CRLB, one have

$$f(x_1|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x_1^2}{2\sigma^2}\right),$$

and

$$\log f(x_1|\sigma^2) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{x_1^2}{2\sigma^2}.$$

Then, letting $\eta = \sigma^2$, one have

$$\frac{\partial}{\partial\eta}\log f(x_1|\eta) = -\frac{1}{2\eta} + \frac{x_1^2}{2\eta^2},$$

and

$$\frac{\partial^2}{\partial\eta^2}\log f(x_1|\eta) = \frac{1}{2\eta^2} - \frac{x_1^2}{\eta^3},$$

The denominator of the CRLB is

$$nE\left\{-\frac{\partial^2}{\partial\eta^2}\log f(x_1|\eta)\right\} = -\frac{n}{2\eta^2} + \frac{nE(x_1^2)}{\eta^3} = \frac{n}{2\eta^2}$$

and

$$\text{CRLB} = \frac{r^2\eta^r}{2n} = \frac{r^2\sigma^{2r}}{2n}.$$

(c) Find a particular value of r for which the variance of $\hat{\theta}$ actually achieve the CRLB.

Solution: Choose $r = 2$. One have

$$E(T/n) = \sigma^2$$

and

$$\text{var}(T/n) = \text{var}(X_1^2)/n = 2\sigma^4/n,$$

which achieves CRLB.

7. [Bios 673/740 in class, C&B 7.37] Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by
$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} | p\right),$$
which is the probability that the first n observations exceed the $(n+1)$st.

   (a) Show that
   $$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & \text{otherwise} \end{cases}$$
   is an unbiased estimator of $h(p)$.

   (b) Find the best unbiased estimator of $h(p)$.