Problem 1

(a)

$$X_1, \ldots, X_n \text{ is a random sample from population with pdf:}$$
$$f(x|\theta) = \theta x^{\theta-1} \quad 0 < x < 1,\ \theta > 0$$
$$f(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta {x_i}^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$
$$T(X) = \sum_{i=1}^{n} X_i$$
$$g(T(X)|\theta) = \sum_{i=1}^{n} \theta {x_i}^{\theta-1} = n\theta \left(\sum_{i=1}^{n} x_i\right)^{\theta-1}$$
$$\text{Since we cannot factor the joint pdf as: } g(T(X)|\theta)h(x)$$
$$T(X) = \sum_{i=1}^{n} X_i \text{ is not an SS for } \theta$$

(b)

$$f(x|\theta) = \theta x^{\theta-1} \quad 0 < x < 1,\ \theta > 0$$
$$f(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta {x_i}^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$
$$\text{Thus the joint pdf can be factored as an exponential family:}$$
$$h(x) = \prod_{i=1}^{n} I(0 < x_i < 1) \quad c(\theta) = \theta^n \quad w_j(\theta) = \theta - 1 \quad t_j(x) = \log\left(\prod_{i=1}^{n} x_i\right)$$
$$\text{Thus } \log\left(\prod_{i=1}^{n} x_i\right) \text{ is a CSS for } \theta$$
$$\text{Since any 1 to 1 function of a CSS is also a CSS:}$$
$$\text{We have that } \prod_{i=1}^{n} X_i \text{ is a complete and sufficient statistic for } \theta$$

Problem 2

(a)

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \quad 0 \leq x < \infty \quad \alpha, \beta > 0$$

$$L(\beta|x) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} {x_i}^{\alpha-1} e^{-x_i/\beta}$$

$$= \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} e^{-(1/\beta)\sum_{i=1}^{n} x_i}$$

$$\log(L(\beta|x)) = -\log(\Gamma(\alpha))^n - n\alpha\log(\beta) + (\alpha-1)\log\left(\prod_{i=1}^{n} x_i \right) - \frac{\sum_{i=1}^{n} x_i}{\beta}$$

$$\frac{\partial \ell(\beta)}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2}$$

Setting the partial derviative to 0 and solving for β :

$$0 = -\frac{n\alpha}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2}$$

$$\frac{n\alpha}{\beta} = \frac{\sum_{i=1}^{n} x_i}{\beta^2}$$

$$\beta(n\alpha) = \sum_{i=1}^{n} x_i$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{na} \text{ (unique extrema)}$$

$$\frac{\partial^2 \log(L)}{\partial^2 \beta} = \frac{na}{\beta^2} - \frac{2\sum_{i=1}^{n} x_i}{\beta^3}$$

Plugging in $\hat{\beta}$ for β :

$$\frac{na}{\left[\frac{\sum_{i=1}^{n} x_i}{na}\right]^2} - \frac{2\sum_{i=1}^{n} x_i}{\left[\frac{\sum_{i=1}^{n} x_i}{na}\right]^3}$$

$$= \frac{(na)^3}{\left[\sum_{i=1}^{n} x_i\right]^2} - \frac{2(na)^3}{\left[\sum_{i=1}^{n} x_i\right]^2}$$

Since $\frac{\partial^2 \log(L)}{\partial^2 \beta}(\hat{\beta}) = -\frac{(na)^3}{\left[\sum_{i=1}^{n} x_i\right]^2} < 0 \quad \hat{\beta}$ is a maximum

Since $\hat{\beta}$ is a maximum and unique extrema, it is the global maximum

Thus $\hat{\beta}$ is the MLE

Problem 3

(a)

$$f(x|\theta) = \prod_{i=1}^{n} (\theta {x_i}^{-2}) I(\theta \leq x_i < \infty)$$
$$= \theta^n \left(\prod_{i=1}^{n} {x_i}^{-2} \right) I(\theta \leq x_i < \infty)$$

Using the factorization theorem and rewriting as: $f(x|\theta) = g(T(x)|\theta)h(x)$

$$\theta^n \left(\prod_{i=1}^{n} {x_i}^{-2} \right) I(\theta \leq x_{(1)} < \infty)$$
$$T(X) = x_{(1)}$$
$$g(T(x)|\theta) = \theta^n I(\theta \leq T(x) < \infty)$$
$$h(x) = \left(\prod_{i=1}^{n} {x_i}^{-2} \right)$$

Thus by the factorization theorem $x_{(1)}$ is a sufficient statistic for θ

(b)

$$L(\theta|x) = \prod_{i=1}^{n} (\theta {x_i}^{-2}) I(\theta \leq x_i < \infty)$$

$\theta \leq x_i$ whichs means $\theta \leq x_{(1)}$ (the min)

$$= \theta^n \left(\prod_{i=1}^{n} {x_i}^{-2} \right) I(\theta \leq x_{(1)} < \infty)$$

$\propto \theta^n$ which is increasing for all θ

$$L(\theta|x) = 0 \text{ if } \theta > x_{(1)}$$

Thus $\theta = x_{(1)}$ maximizes $L(\theta|x)$

$$\hat{\theta} = x_{(1)}$$

(c)

$$
\begin{gathered}
E(X)=\int_{\theta}^{\infty} x\theta x^{-2}\ \mathrm{d}x\\
=\int_{\theta}^{\infty} \theta x^{-1}\ \mathrm{d}x\\
=\Big|_{\theta}^{\infty}\theta\log(x)\\
=\theta\log(\infty)-\theta\log(\theta)=\infty\\
E(X)=\infty\\
\text{Therefore } \hat{\theta}_{MM} \text{ DNE}
\end{gathered}
$$

Problem 4

(a)

$$
\begin{gathered}
Y_x\sim N(x\mu, x^3\sigma^2)\quad x=1,2,\dots,n\\
\sigma^2 \text{ known}\\
\frac{1}{n}\sum_{x=1}^{n}Y_x=E\left(\frac{1}{n}\sum_{x=1}^{n}Y_x\right)\\
=\frac{1}{n}\sum_{x=1}^{n}X_x\mu\\
=\frac{\mu}{n}\frac{n(n+1)}{2}\\
\frac{1}{n}\sum_{x=1}^{n}Y_x=\frac{\mu(n+1)}{2}\\
\hat{\mu}_1=\frac{2}{n(n+1)}\sum_{x=1}^{n}Y_x
\end{gathered}
$$

(b)

$$
\begin{aligned}
L(\mu|y) &= \prod_{x=1}^{n} f(y_x|\mu) \\
&= \prod_{x=1}^{n} \frac{1}{\sqrt{2\pi x^3\sigma^2}} \exp\left(-\frac{(y_x - x\mu)^2}{2x^3\sigma^2}\right) \\
&= \left(\frac{1}{\sqrt{2\pi x^3\sigma^2}}\right)^n \exp\left(-\sum_{x=1}^{n} \frac{(y_x - x\mu)^2}{2x^3\sigma^2}\right) \\
&\propto \exp\left(-\sum_{x=1}^{n} \frac{(y_x - x\mu)^2}{2x^3\sigma^2}\right) \\
\log(L) &= -\sum_{x=1}^{n} \frac{(y_x - x\mu)^2}{2x^3\sigma^2} \\
&= \sum_{x=1}^{n} [(-y_x^2 + 2x\mu y_x - x^2\mu^2)/(2x^3\sigma^2)] \\
&= \frac{1}{2\sigma^2} \sum_{x=1}^{n} (-y_x^2/x^{-3} + 2x^{-2}\mu y_x - \mu^2 x^{-1}) \\
&\propto \sum_{x=1}^{n} -y_x^2/x^{-3} + 2\mu \sum_{x=1}^{n} x^{-2} y_x - \mu^2 \sum_{x=1}^{n} x^{-1} \\
\frac{\partial \ell(\mu)}{\partial \mu} &= 2\sum_{x=1}^{n} x^{-2} y_x - 2\mu \sum_{x=1}^{n} x^{-1} = 0 \\
\hat{\mu}_2 &= \frac{\sum_{x=1}^{n} x^{-2} y_x}{\sum_{x=1}^{n} x^{-1}}
\end{aligned}
$$

(c)

Since the sum of normal r.v.s times constants is normal:

$\hat{\mu_1}, \hat{\mu_2}$ follow normal distributions

$$
\begin{aligned}
E(\hat{\mu_1}) &= \frac{2}{n(n+1)} E(\sum_{x=1}^{n} Y_x) \\
&= \frac{2}{n(n+1)} \sum_{x=1}^{n} \mu x \\
&= \frac{2}{n(n+1)} \frac{n(n+1)}{2} \mu = \mu
\end{aligned}
$$

$$E(\hat{\mu_1}) = \mu \text{ (unbiased)}$$
$$E(\hat{\mu_2}) = E\left(\frac{\sum_{x=1}^n x^{-2} y_x}{\sum_{x=1}^n x^{-1}}\right)$$
$$= \frac{\sum_{x=1}^n x^{-2}}{\sum_{x=1}^n x^{-1}} E(y_x)$$
$$= \frac{\sum_{x=1}^n x^{-2}}{\sum_{x=1}^n x^{-1}} \mu \sum_{x=1}^n x$$
$$= \frac{\sum_{x=1}^n x^{-1}}{\sum_{x=1}^n x^{-1}} \mu = \mu$$
$$E(\hat{\mu_2}) = \mu \text{ (unbiased)}$$
$$Var(\hat{\mu_1}) = Var\left(\frac{2}{n(n+1)} \sum_{x=1}^n Y_x\right)$$
$$= \left(\frac{2}{n(n+1)}\right)^2 Var\left(\sum_{x=1}^n Y_x\right)$$
$$= \left(\frac{2}{n(n+1)}\right)^2 \sigma^2 \sum_{x=1}^n x^3$$
$$= \sigma^2 \left(\frac{2}{n(n+1)}\right)^2 \left(\frac{n(n+1)}{2}\right)^2 = \sigma^2$$
$$Var(\hat{\mu_1}) = \sigma^2$$
$$Var(\hat{\mu_2}) = Var\left(\frac{\sum_{x=1}^n x^{-2} y_x}{\sum_{x=1}^n x^{-1}}\right)$$
$$= \left(\frac{1}{\sum_{x=1}^n x^{-1}}\right)^2 Var\left(\sum_{x=1}^n x^{-2} y_x\right)$$
$$= \left(\frac{1}{\sum_{x=1}^n x^{-1}}\right)^2 \left(\sum_{x=1}^n x^{-2} Var(y_x)\right)$$
$$= \left(\frac{1}{\sum_{x=1}^n x^{-1}}\right)^2 \left(\sum_{x=1}^n x^{-2} x^3 \sigma^2\right)$$
$$= \sigma^2 \left(\frac{1}{\sum_{x=1}^n x^{-1}}\right)^2 \left(\sum_{x=1}^n x\right)$$
$$Var(\hat{\mu_2}) = \frac{\sigma^2}{\sum_{x=1}^n x^{-1}}$$
$$\hat{\mu_1} \sim N(\mu, \sigma^2)$$
$$\hat{\mu_2} \sim N\left(\mu, \frac{\sigma^2}{\sum_{x=1}^n x^{-1}}\right)$$

(d)

$$Var(\hat{\mu_1}) = \sigma^2$$
$$Var(\hat{\mu_2}) = \frac{\sigma^2}{\sum_{x=1}^{n} x^{-1}}$$
$$\text{Since } \sum_{x=1}^{n} x^{-1} > 1$$
$$Var(\hat{\mu_1}) > Var(\hat{\mu_2})$$

Problem 5

(a)

$$E(Y_1) = \theta x_1 \quad Var(Y_i) = \sigma^2$$
$$xs \text{ are known nonzero constants}$$
$$f(y_1, \ldots, y_n|\theta, \sigma^2) = \prod_{i=1}^{n} f(x_i|\theta, \sigma^2)$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - \theta x_i)^2}{2\sigma^2}\right)$$
$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{\sum_{i=1}^{n}(y_i - \theta x_i)^2}{2\sigma^2}\right)$$
$$= g(T(y)|\theta)h(y)$$
$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(\frac{-\sum_{i=1}^{n} y_i^2 + 2\theta \sum_{i=1}^{n} x_i y_i - \theta^2 \sum_{i=1}^{n} x_i^2}{2\sigma^2}\right)$$
$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{\theta^2 \sum_{i=1}^{n} x_i^2}{2\sigma^2}\right) \exp\left(-\frac{\sum_{i=1}^{n} y_i^2 - 2\theta \sum_{i=1}^{n} x_i y_i}{2\sigma^2}\right)$$
$$T(y|\theta) = \exp\left(-\frac{\sum_{i=1}^{n} y_i^2 - 2\theta \sum_{i=1}^{n} x_i y_i}{2\sigma^2}\right)$$
$$= \exp\left(-\frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2}\right) + \exp\left(-\frac{\theta \sum_{i=1}^{n} x_i y_i}{\sigma^2}\right)$$
$$T(y) = \left(\sum_{i=1}^{n} y_i^2, \sum_{i=1}^{n} x_i y_i\right)$$
$$\text{Thus } T(y) \text{ is a sufficient statistic for } (\theta, \sigma^2)$$

(b)

$$\sigma^2 \text{ is fixed}$$
$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta)$$
$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{\sum_{i=1}^n (y_i - \theta x_i)^2}{2\sigma^2}\right)$$
$$\propto \exp\left(-\frac{\sum_{i=1}^n (y_i - \theta x_i)^2}{2\sigma^2}\right)$$
$$\ell(\theta) = -\frac{\sum_{i=1}^n (y_i - \theta x_i)^2}{2\sigma^2}$$
$$= \frac{-\sum_{i=1}^n y_i^2 + 2\theta \sum_{i=1}^n x_i y_i - \theta^2 \sum_{i=1}^n x_i^2}{2\sigma^2}$$
$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{\sum_{i=1}^n x_i y_i - \theta \sum_{i=1}^n x_i^2}{\sigma^2} = 0$$
$$\theta \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i y_i$$
$$\hat{\theta} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$
$$E(\hat{\theta}) = E\left(\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}\right)$$
$$c_i = \frac{x_i}{\sum_{i=1}^n x_i^2}$$
$$= \sum_{i=1}^n c_i E(y_i)$$
$$= \sum_{i=1}^n \frac{x_i}{\sum_{i=1}^n x_i^2} \theta x_i$$
$$= \theta \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2} = \theta$$
$$E(\hat{\theta}) = \theta$$

Thus $\hat{\theta}$ is an unbiased estimator of θ

(c)

$$
\begin{aligned}
E(\hat{\theta}) &= \theta \\
Var(\hat{\theta}) &= Var\left(\sum_{i=1}^{n} c_i y_i\right) \\
&= \sigma^2 \sum_{i=1}^{n} c_i^2 \\
&= \sigma^2 \sum_{i=1}^{n} \left(\frac{x_i}{\sum_{i=1}^{n} x_i^2}\right)^2 \\
&= \sigma^2 \frac{\sum_{i=1}^{n} x_i^2}{\left(\sum_{i=1}^{n} x_i^2\right)^2} \\
Var(\hat{\beta}) &= \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2} \\
\hat{\theta} &\sim N\left(\theta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right)
\end{aligned}
$$

(d)

$$
\begin{aligned}
&\theta \text{ is fixed, let } \eta = \sigma^2 \\
L(\eta) &= \left(\frac{1}{2\pi\eta}\right)^{n/2} \exp\left(-\frac{Q(\theta)}{2\eta}\right) \\
\ell(\eta) &= (-n/2)\log(2\pi\eta) - \frac{Q(\theta)}{2\eta} \\
\frac{\partial \ell(\eta)}{\partial \eta} &= \frac{-n\pi}{2\pi\eta} + \frac{Q(\theta)}{2\eta^2} = 0 \\
\frac{n}{2\eta} &= \frac{Q(\theta)}{2\eta^2} \\
2Q(\theta)(\eta) &= 2n(\eta^2) \\
\hat{\eta} &= \frac{Q(\theta)}{n} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \theta x_i)^2 \\
\frac{\partial^2 \ell(\hat{\eta})}{\eta^2} &= \frac{n}{2\hat{\eta^2}} - \frac{Q(\theta)}{4\hat{\eta}^3} < 0 \\
\text{Thus } \hat{\eta} &= \frac{1}{n}\sum_{i=1}^{n}(y_i - \theta x_i)^2
\end{aligned}
$$

(e)

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

$$\hat{\sigma_e^2} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\theta}x_i)^2$$

$$\hat{\sigma_e^2} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}x_i)^2$$