Problem 1

$$X_1, \dots, X_n \sim Bern(p)$$
$$L(p|x) = \prod_{i=1}^{n} p^{x_1}(1-p)^{1-x_i}$$
$$= p^{\sum_{i=1}^{n} x_i}(1-p)^{\sum_{i=1}^{n}(1-x_i)}$$
$$\ell(p|x) = \sum_{i=1}^{n} x_i \log(p) + (n - \sum_{i=1}^{n} x_i)\log(1-p)$$
$$\frac{\partial \ell}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0$$
$$\frac{1-p}{p} = \frac{n - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}$$
$$\frac{1}{p} - 1 = \frac{n}{\sum_{i=1}^{n} x_i} - 1$$
$$\hat{p}_{MLE} = \sum_{i=1}^{n} x_i/n = \bar{x}$$
$$\frac{\partial \ell}{\partial p^2} = -\left(\frac{\sum_{i=1}^{n} x_i}{p^2} + \frac{n - \sum_{i=1}^{n} x_i}{(1-p)^2}\right) < 0$$

Thus $\hat{p}$ is the MLE

$$\log(f(x_1)) = x\log(p) + (1-x)\log(1-p)$$
$$\frac{\partial}{\partial p}\log(f(x_1)) = \frac{x}{p} - \frac{1-x}{1-p}$$
$$\frac{\partial^2}{\partial p^2}\log(f(x_1)) = -\left(\frac{x}{p^2} + \frac{1-x}{(1-p)^2}\right)$$
$$nE[-\frac{\partial^2}{\partial p^2}\log(f(x_1))] = nE\left(\frac{x}{p^2} + \frac{1-x}{(1-p)^2}\right)$$
$$= n\left(\frac{p}{p^2} + \frac{1-p}{(1-p)^2}\right)$$
$$= n(\frac{1}{p} + \frac{1}{1-p}) = \frac{n}{p(1-p)}$$
$$CRLB = 1/\frac{n}{p(1-p)} = \frac{p(1-p)}{n}$$
$$Var(\hat{p}_{MLE}) = Var(\bar{X}) = \frac{p(1-p)}{n}$$
$$Var(\hat{p}_{MLE}) = CRLB$$

Thus $\bar{X}$ attains the CRLB and is therefore the UMVUE of p

Problem 2

$$X_1, \dots, X_n \sim N(\theta, 1)$$

$$L(\theta|x) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(x_i-\theta)^2}{2}\right)$$

$$= (2\pi)^{-n/2} \exp\left(-\frac{(\sum_{i=1}^n x_i - n\theta)^2}{2}\right)$$

$$\ell(\theta|x) = (-n/2)\log(2\pi) - \frac{(\sum_{i=1}^n x_i - n\theta)^2}{2}$$

$$\propto -\frac{(\sum_{i=1}^n x_i - n\theta)^2}{2} = (-1/2)\left[(\sum_{i=1}^n x_i)^2 + n^2\theta^2 - 2n\theta\sum_{i=1}^n x_i\right]$$

$$\frac{\partial \ell}{\partial \theta} = -n^2\theta + n\sum_{i=1}^n x_i = 0$$

$$\theta = \frac{n\sum_{i=1}^n x_i}{n^2}$$

$$\hat{\theta}_{MLE} = \bar{x}$$

$$\frac{\partial \ell}{\partial \theta^2} = -n^2 < 0$$

$$\text{Thus } \hat{\theta} \text{ is the MLE}$$

$$\tau(\theta) = \theta^2$$

$$\text{By invariance property: } \tau(\hat{\theta})_{MLE} = \tau(\hat{\theta}_{MLE}) = (\hat{\theta}_{MLE})^2 = \bar{x}^2$$

$$\hat{\theta}^2_{MLE} = \bar{x}^2$$

$$E(\bar{X}^2) = Var(\bar{X}) + E(\bar{X})^2 = 1/n + \theta^2 \text{ (biased)}$$

$$\theta^{2*} = \bar{X}^2 - \frac{1}{n} \text{ (unbiased)}$$

$$\sum_{i=1}^n x_i \text{ is a CSS for } \theta$$

$$\theta^{2*} \text{ is an unbiased estimator and a function of } \sum_{i=1}^n x_i$$

$$\text{Thus } \theta^{2*} \text{ is the UMVUE by Lehmann-Sheffe Theorem}$$

$$\log(f(x_1|\theta)) = (-1/2)[\log(2\pi) + (x-\theta)^2]$$

$$\frac{\partial}{\partial \theta}\log(f(x_1|\theta)) = x - \theta$$

$$\frac{\partial^2}{\partial\theta^2}\log(f(x_1|\theta)) = -1$$
$$nE(-\frac{\partial^2}{\partial\theta^2}\log(f(x_1|\theta))) = n$$
$$\left(\frac{d\tau(\theta)}{d\theta}\right)^2 = 4\theta^2$$
$$CRLB = \frac{4\theta^2}{n}$$
$$Var(\theta^{2*}) = Var(\bar{X}^2 - 1/n) = Var(\bar{X}^2) = E(\bar{X}^4) - E(\bar{X}^2)^2 = E(\bar{X}^4) - (1/n + \theta^2)^2$$

Using Steins Lemma:

$$E(\bar{X}^4) = E[\bar{X}^3(\bar{X} - \theta + \theta)] = E[\bar{X}^3(\bar{X} - \theta)] + \theta E(\bar{X}^3)$$
$$E[\bar{X}^3(\bar{X} - \theta)] = \sigma^2 E(3\bar{X}^2) = (3/n)(1/n + \theta^2)$$
$$\theta E(\bar{X}^3) = \theta E(\bar{X}^2 - \theta + \theta) = \theta E(\bar{X}^2(\bar{X} - \theta) + \theta\bar{X}^2) = 2\theta\sigma^2 E(\bar{X}) + \theta^2 E(\bar{X}^2)$$
$$= (2/n)\theta^2 + \theta^2(1/n + \theta^2) = \theta^2(3/n + \theta^2)$$
$$Var(\theta^{2*}) = (3/n)(1/n + \theta^2) + \theta^2(3/n + \theta^2) - (1/n + \theta^2)^2$$
$$= (1/n + \theta^2)(3/n - 1/n - \theta^2) + \theta^2(3/n + \theta^2)$$
$$= (1/n + \theta^2)(2/n - \theta^2) + \theta^2(3/n + \theta^2)$$
$$= 2/n^2 - \theta^4 + (1/n)\theta^2 + (3/n)\theta^2 + \theta^4$$
$$Var(\theta^{2*}) = \frac{2}{n^2} + \frac{4\theta^2}{n}$$
$$\frac{2}{n^2} + \frac{4\theta^2}{n} > \frac{4\theta^2}{n}$$

Thus the variance of the UMVUE is greater than the CRLB

Problem 3

(a)

$$X_1, \ldots, X_n \sim Pareto(v, \theta)$$
$$E(X_i) = \frac{\theta v}{\theta - 1} \quad Var(X_i) = \frac{\theta v^2}{(\theta+1)^2(\theta - 2)}$$
$$f(x|\theta, v) = \frac{\theta v^\theta}{x^{\theta+1}} I[v, \infty)(x)$$
$$L(\theta, v|x) = \prod_{i=1}^n \frac{\theta v^\theta}{x_i^{\theta+1}} I[v, \infty)(x_i)$$

$$\ell(\theta, v|x) = \sum_{i=1}^{n}[\log(\theta) + \theta\log(v) - (\theta+1)\log(x_i)], v \leq x_{(1)}$$

$$\ell(\theta, v|x) = n\log(\theta) + n\theta\log(v) - (\theta+1)\sum_{i=1}^{n}\log(x_i), v \leq x_{(1)}$$

Where $x_{(1)} = \min_i(x_i)$

For all θ this is an increasing function of v, $v \leq x_{(1)}$

Thus $\hat{v}_{MLE} = x_{(1)}$

Fixing v:

$$\ell(\theta, x_{(1)}|x) = n\log(\theta) + n\theta\log(x_{(1)}) - (\theta+1)\sum_{i=1}^{n}\log(x_i)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + n\log(x_{(1)}) - \sum_{i=1}^{n}\log(x_i) = 0$$

$$1/\theta + \log(X_{(1)}) = \frac{\log\left(\prod x_i\right)}{n}$$

$$1/\theta = \frac{\log\left(\prod x_i\right) - n\log(x_{(1)})}{n}$$

$$\theta = \frac{n}{\log\left(\prod x_i\right) - n\log(x_{(1)})}$$

$$\hat{\theta}_{MLE} = \frac{n}{\log\left(\frac{\prod_{i=1}^{n}}{{x_{(1)}}^{n}}\right)} = \frac{n}{T}$$

$$\text{Where } T = \log\left(\frac{\prod_{i=1}^{n}}{[\min_i(x_i)]^n}\right)$$

$$\frac{\partial \ell}{\partial \theta^2}^{2} = -\frac{n}{\theta^2} < 0$$

Thus $\hat{\theta}$ is the MLE of θ

(b)

$$L(\theta, x_{(1)}|x) = \prod_{i=1}^{n}\frac{\theta {x_{(1)}}^{\theta}}{x_i^{\theta+1}} = \frac{\theta^n {x_{(1)}}^{n\theta}}{(\prod_{i=1}^{n} x_i)^{\theta+1}}$$

$$H_0 : \theta = 1, v \text{ unknown} \quad H_1 : \theta \neq 1, v \text{ unknown}$$

$$\lambda(x) = \frac{L(\theta = 1, v = x_{(1)})}{L(\theta = n/T, v = x_{(1)})}$$

$$= \frac{1^n {x_{(1)}}^{n}}{(\prod_{i=1}^{n} x_i)^2} \Big/ \frac{(n/T)^n {x_{(1)}}^{n^2/T}}{(\prod_{i=1}^{n} x_i)^{n/T+1}}$$

$$= \frac{{x_{(1)}}^n}{(\prod_{i=1}^n x_i)^2} * \frac{(\prod_{i=1}^n x_i)^{n/T+1}}{(n/T)^n {x_{(1)}}^{n^2/T}}$$

$$= x_{(1)}^{n-n^2/T} (\prod_{i=1}^n x_i)^{n/T-1} (T/n)^n$$

$$= \frac{(\prod_{i=1}^n x_i)^{n/T-1}}{x_{(1)^n}^{n/T-1}} (T/n)^n$$

$$\text{Since } e^T = \frac{\prod_{i=1}^n x_i}{x_{(1)}^n} :$$

$$\lambda(x) = (e^T)^{n/T-1}(T/n)^n = e^{-T+n}(T/n)^n$$

$$R = \{x : \lambda(x) \leq c\} = \{x : e^{-T+n}(T/n)^n \leq c\}$$

Since $\lambda(x)$ is in the form of $e^{-T}T^n, \lambda(x)$ is concave

Which means the rejection region is equivalent to:

$$\{x : T \leq c_1^* \text{ or } T \geq c_2^*\}$$

Problem 4

(a)

$$Y_1, \ldots, Y_n \sim Pois(\theta x_i) \quad \theta > 0$$

$$L(\theta|y) = \prod_{i=1}^n \frac{e^{-\theta x_i}(\theta x_i)^{y_i}}{y_i!}$$

$$\ell(\theta|y) = \sum_{i=1}^n (-\theta x_i + y_i \log(\theta) + y_i \log(x_i) - \log(y_i!))$$

$$= -\theta \sum_{i=1}^n x_i + \sum_{i=1}^n y_i \log(\theta) + \sum_{i=1}^n y_i \log(x_i) - \sum_{i=1}^n \log(y_i!)$$

$$\frac{\partial \ell}{\partial \theta} = -\sum_{i=1}^n x_i + \frac{\sum_{i=1}^n y_i}{\theta} = 0$$

$$\sum_{i=1}^n x_i = \frac{\sum_{i=1}^n y_i}{\theta}$$

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i}$$

$$\frac{\partial \ell}{\partial \theta^2}^2 = -\frac{\sum_{i=1}^n y_i}{\theta^2} < 0$$

Thus $\hat{\theta}$ is the MLE

$$E(\hat{\theta}) = \frac{1}{\sum_{i=1}^n x_i} E(\sum_{i=1}^n y_i) = \frac{1}{\sum_{i=1}^n x_i} \sum_{i=1}^n E(y_i) = \frac{\theta \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i} = \theta \text{ (unbiased)}$$

$$Var(\hat{\theta}) = \frac{1}{(\sum_{i=1}^n x_i)^2} \sum_{i=1}^n Var(y_i) = \frac{\theta \sum_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^2} = \frac{\theta}{\sum_{i=1}^n x_i}$$

(b)

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i}$$

$$\text{Since } E(\hat{\theta}_{MLE}) = \theta :$$

$$\hat{\theta}_{MLE} \text{ is an unbiased estimator and a function of } \sum_{i=1}^n y_i \text{ and a postive constant}$$

$$\sum_{i=1}^n y_i \text{ is a CSS by exponential family}$$

$$\text{Thus } \hat{\theta}_{MLE} \text{ is the UMVUE by Lehmann-Sheffe Theorem}$$

(c)

$$\frac{\partial}{\partial\theta} \log(f(\boldsymbol{x}|\theta)) = -\sum_{i=1}^n x_i + \frac{\sum_{i=1}^n y_i}{\theta}$$

$$E(\frac{\partial}{\partial\theta} \log(f(\boldsymbol{x}|\theta))^2) = E([-\sum_{i=1}^n x_i + \frac{\sum_{i=1}^n y_i}{\theta}]^2)$$

$$= E\left((\sum_{i=1}^n x_i)^2 + \frac{(\sum_{i=1}^n y_i)^2}{\theta^2} - \frac{2\sum_{i=1}^n x_i \sum_{i=1}^n y_i}{\theta} \right)$$

$$= (\sum_{i=1}^n x_i)^2 + \frac{E[(\sum_{i=1}^n y_i)^2]}{\theta^2} - \frac{2\sum_{i=1}^n x_i E[\sum_{i=1}^n y_i]}{\theta}$$

$$E[(\sum_{i=1}^n y_i)^2] = Var(\sum_{i=1}^n y_i) + E(\sum_{i=1}^n y_i)^2 = \theta\sum_{i=1}^n x_i + \theta^2(\sum_{i=1}^n x_i)^2$$

$$E(\frac{\partial}{\partial\theta} \log(f(\boldsymbol{x}|\theta))^2) = (\sum_{i=1}^n x_i)^2 + \frac{\theta\sum_{i=1}^n x_i + \theta^2(\sum_{i=1}^n x_i)^2}{\theta^2} - \frac{2\sum_{i=1}^n x_i \theta \sum_{i=1}^n x_i}{\theta}$$

$$= (\sum_{i=1}^n x_i)^2 + \frac{\sum_{i=1}^n x_i}{\theta} + (\sum_{i=1}^n x_i)^2 - 2(\sum_{i=1}^n x_i)^2$$

$$E(\frac{\partial}{\partial\theta}\log(f(\boldsymbol{x}|\theta))^2) = \frac{\sum_{i=1}^n x_i}{\theta}$$
$$CRLB = 1/\frac{\sum_{i=1}^n x_i}{\theta} = \frac{\theta}{\sum_{i=1}^n x_i}$$
$$\frac{\theta}{\sum_{i=1}^n x_i} = \frac{\theta}{\sum_{i=1}^n x_i}$$

Thus the variance of $\hat{\theta}$ achieves the CRLB

Problem 5

(a)

$$f(x|\theta) = e^{-(x-\theta)}, I(x \geq \theta)$$
$$L(\theta|x) = \exp\left(-\sum_{i=1}^n x_i + n\theta\right), \theta \leq x_{(1)}$$
$$\ell(\theta|x) = -\sum_{i=1}^n x_i + n\theta, \theta \leq x_{(1)}$$

This is an increasing function of θ for $\theta \leq x_{(1)}$

Thus $\hat{\theta} = x_{(1)}$ maximizes the function

$$\hat{\theta}_{MLE} = x_{(1)}$$
$$H_0 : \theta = \theta_0 \quad H_1 : \theta \neq \theta_0$$

If $x_{(1)} < \theta_0,\ L(\theta) = 0,$ thus reject H_0

$$\lambda(x) = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\exp\left(-\sum_{i=1}^n x_i + n\theta_0\right), I(x_{(1)} \geq \theta_0)}{\exp\left(-\sum_{i=1}^n x_i + nx_{(1)}\right)}$$
$$\lambda(x) = \exp\left(-n(x_{(1)} - \theta_0)\right), I(x_{(1)} \geq \theta_0)$$

(b)

WTS: $R = \{x : \lambda(x) \leq c\}$ and $R^* = \{x : x_{(1)} \geq c^* \text{ or } x_{(1)} < \theta_0\}$ are equivalent

$$H_0 : \theta = \theta_0 \quad H_1 : \theta \neq \theta_0$$
$$R = \{x : \lambda(x) \leq c\} = \{x : \exp(-n(x_{(1)} - \theta_0)) \leq c \text{ or } x_{(1)} < \theta_0\}$$
$$= \{x : x_{(1)} - \theta_0 \geq -\log(c)/n \text{ or } x_{(1)} < \theta_0\} = \{x : x_{(1)} \geq \theta_0 - \log(c)/n \text{ or } x_1 < \theta_0\}$$
$$c^* = \theta_0 - \log(c)/n$$

$R^* = \{x : x_{(1)} \geq c^* \text{ or } x_{(1)} < \theta_0\}$ Thus R and R^* are equivalent

(c)

$$\alpha = \sup_{\theta \in \Theta} P(\boldsymbol{X} \in R^*|H_0)$$
$$\alpha = P(\lambda(x) \leq c|\theta = \theta_0)$$
$$= P(x_{(1)} \geq c^* \text{ or } x_{(1)} < \theta_0|\theta = \theta_0)$$
$$= P(x_{(1)} \geq c^*|\theta = \theta_0) + P(x_{(1)} < \theta_0|\theta = \theta_0)$$
$$P(x_1 < \theta_0|\theta = \theta_0) = 0$$
$$\alpha = P(x_{(1)} \geq c^*|\theta = \theta_0)$$
$$F(x|\theta) = \int_\theta^x e^{-(t-\theta)}dt = 1 - e^{-(x-\theta)}$$
$$f_{x_{(1)}}(x) = \frac{n!}{(n-1)!}e^{-(x-\theta)}[e^{-(x-\theta)}]^{n-1}$$
$$= ne^{-n(x-\theta)}, x > \theta$$
$$\alpha = P(x_{(1)} \geq c^*|\theta = \theta_0)$$
$$= \int_{c^*}^{\infty} ne^{-n(x-\theta_0)}\mathrm{d}x = \Big|_{c^*}^{\infty} - e^{-n(x-\theta_0)}$$
$$\alpha = e^{-n(c^*-\theta_0)}$$
$$\frac{\log(\alpha)}{-n} = c^* - \theta_0$$
$$c^* = -\frac{\log(\alpha)}{n} + \theta_0$$

(d)

$$R^* = \{x : x_{(1)} \geq \theta_0 - \frac{\log(\alpha)}{n} \text{ or } x_{(1)} < \theta_0\}$$
$$\beta(\theta) = P\left(x_{(1)} \geq \theta_0 - \frac{\log(\alpha)}{n} \text{ or } x_{(1)} < \theta_0|\theta \in \Theta\right)$$
$$P\left(x_{(1)} \geq \theta_0 - \frac{\log(\alpha)}{n}|\theta \geq \theta_0\right) + P(x_{(1)} \leq \theta_0|\theta \geq \theta_0)$$
$$= P\left(x_{(1)} \geq \theta_0 - \frac{\log(\alpha)}{n}|\theta \geq \theta_0\right) + 0$$
$$= \exp\left(-n(\theta_0 - \log(\alpha)/n - \theta)\right) \text{ (increasing function of } \theta)$$
$$\exp\left(-n(\theta_0 - \log(\alpha)/n - \theta)\right) + P(x_{(1)} \leq \theta_0|\theta < \theta_0)$$

$$
\begin{aligned}
&= \exp\left(-n(\theta_0 - \log(\alpha)/n - \theta)\right) + 1 - \exp\left(-n(\theta_0 - \theta)\right) \\
&= -\exp\left(-n(\theta_0 - \theta)\right)(1-\alpha) + 1 \text{ (decreasing function of } \theta\text{)}
\end{aligned}
$$