Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 7.9
2. C&B 7.12
3. C&B 7.19
4. [master exam 2010] Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\mu \in \{1, 2\}$. Note that the parameter space contains only two points; it is not the usual parameter space for the Poisson.
   (a) Let $W = \sum_{i=1}^n X_i$ and $V = (1 + 3n)W - W^2 - 2n^2$. Find $E(V|\mu = 1)$ and $E(V|\mu = 2)$.
   (b) Find a minimal sufficient statistic. Is it complete? Hint: Part (a).
   (c) Derive the maximum likelihood estimator (MLE) of μ. Is it unique?
   (d) For $n = 3$, compute the exact numerical values of the mean and variance of the MLE $\hat{\mu}$ when $\mu = 1$ and $\mu = 2$.
5. Let $X_1, \ldots, X_n$ be iid random variables from the $\mathrm{N}(\theta, \theta)$ distribution, $\theta > 0$.
   (a) Find the Cramér-Rao Lower Bound (CRLB) for any unbiased estimator of θ.
   (b) Find an explicit expression for MLE $\hat{\theta}$. In this case, it is not easy to verify the $\hat{\theta}$ that solves the first derivative of the log-likelihood function is indeed the maximizer. You may skip the verification.
   (c) One can estimate θ using $\bar{X} = \sum_{i=1}^n X_i/n$ and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$. Derive $\mathrm{Var}(\bar{X})$ and $\mathrm{Var}(S^2)$. Is one uniformly better than the other?
   (d) Let $T(X) = n^{-1} \sum_{i=1}^n X_i^2$. Find $\tau(\theta)$, for which $T(X)$ is an unbiased estimator.
   (e) Does $T(X)$ have the smallest variance among unbiased estimators of $\tau(\theta)$?
6. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a $N(0, \sigma^2)$.
   (a) Develop an explicit expression for an unbiased estimator $\hat{\theta}$ of the unknown parameter $\theta = \sigma^r$, where r is some known positive integer.
   (b) Derive an explicit expression for the CRLB for the variance of any unbiased estimator of the parameter θ.
   (c) Find a particular value of r for which the variance of $\hat{\theta}$ actually achieve the CRLB.

7. [Bios 673/740 in class, C&B 7.37] Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by

$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} | p\right),$$

which is the probability that the first n observations exceed the $(n+1)$st.

   (a) Show that

   $$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

   is an unbiased estimator of $h(p)$.

   (b) Find the best unbiased estimator of $h(p)$.