Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 6.22
2. C&B 7.2(a)
3. C&B 7.6
4. Suppose that $Y_x \sim N(x\mu, x^3\sigma^2)$, $x = 1, 2, \ldots, n$, and assume that $\{Y_1, Y_2, \ldots, Y_n\}$ constitute a set of n mutually independent random variables, and that σ^2 is a known positive constant.
   (a) Derive the method of moments estimator $\hat{\mu}_1$;
   (b) Derive the maximum likelihood estimator $\hat{\mu}_2$.
   (c) Determine the exact distribution of $\hat{\mu}_1$ and $\hat{\mu}_2$.
   (d) Which one has a smaller variance?
5. Suppose that the random variables $Y_1, \ldots, Y_n$, $n > 2$ are independent and normally distributed with $E(Y_i) = \theta x_i$, where $x_1, \ldots, x_n$ are known non-zero constants, and $\text{var}(Y_i) = \sigma^2$. Both $\theta \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$ are unknown.
   (a) Find a two-dimensional sufficient statistic for (θ, σ^2).
   (b) With σ^2 fixed (in some sense as known), find the MLE $\hat{\theta}$ of θ and show that it is an unbiased estimator of θ.
   (c) Find the distribution of the MLE $\hat{\theta}$.
   (d) With θ fixed at the MLE $\hat{\theta}$ in (b), find the MLE $\hat{\sigma}^2$ of σ^2.
   (e) When both θ and σ^2 are unknown, the MLE of (θ, σ^2) are $\hat{\theta}$ in (b) and $\hat{\sigma}_e^2 = n^{-1}\sum_{i=1}^n (Y_i - \hat{\theta}x_i)^2$, which is $\hat{\sigma}^2$ in (d) but with θ replaced by $\hat{\theta}$. Show that $\hat{\theta}$ and $\hat{\sigma}_e^2$ are independent.
6. Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x|\theta) = h(x)c(\theta)\exp\{\theta t(x)\}$ (an exponential family with $w(\theta) = \theta$).
   (a) Show that $E\{T(X_1)\} = -c'(\theta)/c(\theta)$.
   (b) Show that the MLE for θ has to satisfy $E\{\sum_{i=1}^n T(X_i)\} = \sum_{i=1}^n T(X_i)$. That is, the equation holds when one plugs in $\theta = \hat{\theta}$ in the left hand side of the equation.
   (c) Suppose $f(x|\beta) = \beta^{-1}\exp(-x/\beta)$ (exponential distribution with mean β). Use the result in (b) to find the MLE for β.

7. (Bios 673 class material) The same result holds for the general exponential family

$$f(x|\theta) = h(x)c(\theta)\exp\left\{\sum_{j=1}^{k} w_j(\theta)t_j(x)\right\},$$

where $\theta = (\theta_1, \ldots, \theta_k)'$. Suppose that the two-dimensional vectors $(X_1, Y_1), \ldots, (X_n, Y_n)$ follows a bivariate normal distribution

$$N\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}\right).$$

Find the MLE for $\theta = (\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)'$ using the result in 6(b) (You may find it is much easier than using partial derivatives).