Hypothesis Testing

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(C&B §8)

Introduction

- **Example** Current standard treatment for a given disease has success probability 0.7. A new drug has success probability θ (unknown). Is the new drug better than the current treatment?
- Hypothesis: $H_0 : \theta \leq 0.7$ and $H_1 : \theta > 0.7$.
- A *hypothesis* is a statement about a population parameter.
- The parameter space is divided into two disjoint sets:

$$\Theta = \Theta_0 \cup \Theta_0^c$$

- The *null hypothesis* is $H_0 : \theta \in \Theta_0$.
- The *alternative hypothesis* is $H_1 : \theta \in \Theta_0^c$.

Introduction (cont'd)

- Assume $X_1, \ldots, X_n \sim \text{Bernoulli}(\theta)$:
  - **Simple versus simple** $\Theta = \{1/4, 3/4\}$, $\Theta_0 = \{1/4\}$, $\Theta_0^c = \{3/4\}$.
  - **Simple versus composite, one-sided** $\Theta = [1/4, 1]$, $\Theta_0 = \{1/4\}$, $\Theta_0^c = (1/4, 1]$.
  - **Simple versus composite, two-sided** $\Theta = [0, 1]$, $\Theta_0 = \{1/4\}$, $\Theta_0^c = [0, 1/4) \cup (1/4, 1]$.
  - **Composite versus composite** $\Theta = [0, 1]$, $\Theta_0 = [0, 1/4]$, $\Theta_0^c = (1/4, 1]$.
  - **Composite versus composite** $\Theta = [0, 1/4] \cup [3/4, 1]$, $\Theta_0 = [0, 1/4]$, $\Theta_0^c = [3/4, 1]$.

Hypothesis Testing

- *Hypothesis testing*: Use data to decide whether to reject H_0 as false or accept H_0 as true (do not reject H_0).
- A *hypothesis testing procedure* is a rule that specifies for which values of $\boldsymbol{X}$ are to reject H_0 or not.
- *Test function*: $\delta(\boldsymbol{X})$ is either 0 or 1.
- Decision rule: If $\delta(\boldsymbol{X}) = 1$, H_0 is rejected; If $\delta(\boldsymbol{X}) = 0$, H_0 is not rejected.

Rejection Region

- $\delta(\boldsymbol{X})$ divides the sample space into two regions.
- The *rejection region* or *critical region* R is the region over which $\delta(\boldsymbol{x}) = 1$, and H_0 is rejected.
- The acceptance region is the region R^c (the complement of R) over which $\delta(\boldsymbol{x}) = 0$, and H_0 is accepted.

$$R = \{\boldsymbol{x} : \delta(\boldsymbol{x}) = 1\}, \text{ and, } R^c = \{\boldsymbol{x} : \delta(\boldsymbol{x}) = 0\}.$$

- *Type I error*: Reject H_0 when it is true.
- Size of the test (the largest type-I error one can make):

$$\alpha = \sup_{\theta \in \Theta_0} P(\delta(\boldsymbol{x}) = 1).$$

- *Type II error*: Do not reject H_0 when it is false.

Likelihood Ratio Test

- The *likelihood ratio statistic* for testing $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_0^c$ is

$$\lambda(\boldsymbol{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta|\boldsymbol{x})}{\sup_{\theta \in \Theta} L(\theta|\boldsymbol{x})}$$

- let $\hat{\theta}_0$ denote the restricted MLE over Θ_0, and Let $\hat{\theta}$ denote the unrestricted MLE over $\Theta = \Theta_0 \cup \Theta_0^c$.
- The likelihood ratio statistic:

$$\lambda(\boldsymbol{x}) = \frac{L(\hat{\theta}_0|\boldsymbol{x})}{L(\hat{\theta}|\boldsymbol{x})}.$$

- A *likelihood ratio test* (LRT) is any test with

$$R = \{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\} \text{ for some } c \in [0, 1].$$

Likelihood Ratio Test (cont'd)

- A test with test function $\delta(\boldsymbol{x}) = I(\lambda(\boldsymbol{x}) \leq c)$ for some $c \in [0, 1]$.
- Use the test size, say α, to find c, where

$$\alpha = \sup_{\theta \in \Theta_0} P(\lambda(\boldsymbol{X}) \leq c).$$

- However, we usually do not know about the distribution of $\lambda(\boldsymbol{X})$.
- We intend to find an equivalent region using unrestricted MLE $\hat{\theta}$ with

$$R = \{\boldsymbol{x} : \lambda(\boldsymbol{X}) \leq c\} \Longleftrightarrow R^* = \{\boldsymbol{x} : \hat{\theta} \geq c^* \ \text{ or } \ \hat{\theta} \leq c^*\}.$$

- $\hat{\theta} \geq c^*$ or $\hat{\theta} \leq c^*$ follows the direction of H_1.

LRT under Normal Distribution

- Let $X_1, \cdots, X_n$ be iid N$(\theta, 1)$. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.
- Under H_0, θ_0 is a fixed number determined by the researcher, so the numerator of $\lambda(\boldsymbol{x})$ is

$$\sup_{\theta \in \Theta_0} L(\theta|\boldsymbol{x}) = L(\theta_0|\boldsymbol{x}).$$

- Under the unrestricted parameter space $\Theta = \Theta \cup \Theta^c$, the MLE is $\bar{X}$, so the denominator of $\lambda(\boldsymbol{x})$ is

$$\sup_{\theta \in \Theta} L(\theta|\boldsymbol{x}) = L(\bar{x}|\boldsymbol{x}).$$

LRT under Normal Distribution (cont'd)

- The LRT statistic is

$$\lambda(\boldsymbol{x}) = \frac{(2\pi)^{-n/2}\exp\{-\sum_{i=1}^{n}(x_i-\theta_0)^2/2\}}{(2\pi)^{-n/2}\exp\{-\sum_{i=1}^{n}(x_i-\bar{x})^2/2\}}$$
$$= \exp\left\{\left[-\sum_{i=1}^{n}(x_i-\theta_0)^2 + \sum_{i=1}^{n}(x_i-\bar{x})^2\right]/2\right\}.$$

- Since $\sum_{i=1}^{n}(x_i-\theta_0)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\theta_0)^2$, the LRT statistic is simplified to

$$\lambda(\boldsymbol{x}) = \exp\{-n(\bar{x}-\theta_0)^2/2\}.$$

- The rejection region is $\{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\}$, which can be written by

$$\{\boldsymbol{x} : |\bar{x}-\theta_0| \geq \sqrt{-2(\log c)/n}\}.$$

LRT under Exponential Distribution

- Let $X_1, \cdots, X_n$ be a random sample from an exponential distribution with pdf

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & x \geq \theta \\ 0 & x < \theta, \end{cases}$$

where $-\infty < \theta < \infty$. The likelihood function is

$$L(\theta|\boldsymbol{x}) = \begin{cases} e^{-\sum_{i=1}^{n} x_i + n\theta} & \theta \leq x_{(1)} \\ 0 & \theta > x_{(1)}. \end{cases}$$

- Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where θ_0 is a value specified by the researcher.
- Unrestricted MLE (denominator) is more straightforward. The unrestricted maximum of $L(\theta|x)$ is $L(x_{(1)}|\boldsymbol{x}) = e^{-\sum x_i + n x_{(1)}}$.

LRT under Exponential Distribution (cont'd)

- Under H_0, finding maximum of $L(\theta|\boldsymbol{x})$ is more complicated. Drawing $L(\theta|\boldsymbol{x})$ helps.
- If $x_{(1)} \leq \theta_0$, the numerator of $\lambda(\boldsymbol{x})$ is also $L(x_{(1)}|\boldsymbol{x})$.
- If $x_{(1)} > \theta_0$, the numerator of $\lambda(\boldsymbol{x})$ is $L(\theta_0|\boldsymbol{x})$.
- Therefore, the likelihood ratio test statistic is

$$\lambda(\boldsymbol{x}) = \begin{cases} 1 & x_{(1)} \leq \theta_0 \\ e^{-n(x_{(1)}-\theta_0)} & x_{(1)} > \theta_0. \end{cases}$$

- One rejects H_0 if $\lambda(\boldsymbol{x}) \leq c$.
- The rejection region $\{\boldsymbol{x} : x_{(1)} \geq \theta_0 - \log(c)/n\}$.

Evaluating Tests

- The *power function* of a hypothesis test is

$$\beta(\theta) = P_\theta(\boldsymbol{X} \in R) = E_\theta \delta(\boldsymbol{X})$$

- *Type I error*: $\beta(\theta)$, $\theta \in \Theta_0$.
- *Type II error*: $1 - \beta(\theta)$, $\theta \in \Theta_0^c$.
- A **size** α test if

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha.$$

- A **level** α test if

$$\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha.$$

Power Function under Normal Distribution

- Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\theta, \sigma^2)$ with known σ^2.
- To test $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$, one would reject H_0 if $(\bar{X} - \theta_0)/(\sigma/\sqrt{n}) > c$ by LRT.
- The power function of this test is

$$
\begin{aligned}
\beta(\theta) &= P_\theta\left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > c\right) = P_\theta\left(\frac{\bar{X} - \theta}{\sigma/\sqrt{n}} > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) \\
&= P_\theta\left(Z > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) = 1 - \Phi\left(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right).
\end{aligned}
$$

- $\lim_{\theta \to -\infty} \beta(\theta) = 0$ and $\lim_{\theta \to \infty} \beta(\theta) = 1$
- $\beta(\theta_0) = \alpha$ if $\Phi(c) = 1 - \alpha$.

Power Function under Binomial Distribution

- $X \sim \text{Binomial}(3, \theta)$, $\Theta = (0, 1)$,
- $H_0 : \theta \leq 1/4$ versus $H_1 : \theta > 1/4$.
- The test defined by $\delta(x) = I(x = 3)$ has a power function

$$\beta(\theta) = P_\theta(X = 3) = \theta^3.$$

- The size of $\delta(x)$ is $\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(1/4) = 1/64$.
- Another test defined by $\delta^*(x) = I(x \geq 2)$ has a power function

$$\beta^*(\theta) = P_\theta(X \in \{2, 3\}) = 3\theta^2(1 - \theta) + \theta^3.$$

- The size of $\delta^*(x)$ is $\beta^*(1/4) = 10/64$.
- Clearly, $\beta^*(\theta) > \beta(\theta)$ for all $\theta \in (0, 1)$.

Size of a Binomial Test

- $X \sim \text{Binomial}(3, \theta)$, $\Theta = \{1/4, 3/4\}$.
- $H_0 : \theta = \theta_0 = 1/4$ versus $H_1 : \theta = \theta_1 = 3/4$.
- Under H_0, $P_{\theta_0}(X = 0) = 27/64$, $P_{\theta_0}(X = 1) = 27/64$, $P_{\theta_0}(X = 2) = 9/64$, and $P_{\theta_0}(X = 3) = 1/64$.
- Any test function $\delta(x)$ will simply partition the set $\{0, 1, 2, 3\}$ into two subsets.
- Hence, no matter what $\delta(X)$ is, the test size

$$\sup_{\theta \in \Theta_0} \beta(\theta) = P_{\theta_0}(\delta(X) = 1)$$

  will be the sum of one or more of the numbers in $\{0, 27/64, 27/64, 9/64, 1/64\}$.
- Can we have the test size exactly equals $\alpha = 0.05$?

Nonexistence of a Size α Test

- A size α test may not always exist (for example, discreteness).
- Solutions:
  (1) Practical: Settle for a size α^* test with α^* being the largest possible size that is less than or equal to α.
  (2) Mathematical: Randomized tests. Find c such that $\alpha^* + c(1 - \alpha^*) = \alpha$. If the test with size α^* does not reject H_0, draw $U \sim \text{Uniform}(0, 1)$ and reject H_0 if $U < c$.

Desirable Properties of Tests

- Error probabilities as small as possible.
- Error probabilities that are uniformly 0 are impossible except in trivial cases.
- **Example of a trivial case**: $X \sim \text{Bernoulli}(\theta)$, $\Theta = \{0, 1\}$. If $H_0 : \theta = 0$ against $H_1 : \theta = 1$.
- The test $\delta(X) = X$ has error probabilities uniformly 0. Why?

Uniformly Most Powerful (UMP) Level α Test

- Fix type I error at α, then minimize type II error uniformly in θ.
- Restrict to the class of level α tests, then find the uniformly most powerful test.
- **Neyman-Pearson Lemma**: X (scalar or vector) has pdf or pmf $f(x|\theta)$, $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$. Suppose a test has a rejection region

$$R = \left\{ x \,\middle|\, \frac{f(x|\theta_1)}{f(x|\theta_0)} > c \right\},$$

and acceptance region

$$R^c = \left\{ x \,\middle|\, \frac{f(x|\theta_1)}{f(x|\theta_0)} < c \right\},$$

for some $c \geq 0$ and has size $\alpha = P_{\theta_0}(X \in R)$.

UMP Level α Test (cont'd)

- Then,
  (a) Any such test is a UMP level α test.
  (b) If such a test exists with $c > 0$ then every UMP size α test has the same test function (except on a set that has probability 0).
- Proof of (a): Given a level α test $\delta^*(x)$, we want to show that $\beta(\theta_1) - \beta^*(\theta_1) \geq 0$. The inequality

$$\{\delta(x) - \delta^*(x)\}\{f(x|\theta_1) - cf(x|\theta_0)\} \geq 0.$$

  holds for each of the four cases: $\delta(x)$, $\delta^*(x) = 0, 1$.
- Integrating out x gives

$$\beta(\theta_1) - c\beta(\theta_0) - \beta^*(\theta_1) + c\beta^*(\theta_0) \geq 0,$$

  which can be written as

$$\beta(\theta_1) - \beta^*(\theta_1) \geq c\{\beta(\theta_0) - \beta^*(\theta_0)\}.$$

UMP Level α Test (cont'd)

- Since $\beta(\theta_0) = \alpha$, $\beta^*(\theta_0) \leq \alpha$ and $c \geq 0$, it follows that $c\{\beta(\theta_0) - \beta^*(\theta_0)\} \geq 0$ and

$$\beta(\theta_1) \geq \beta^*(\theta_1).$$

- **Example** $X_1, \ldots, X_n \sim \mathrm{N}(\theta, 1)$. Find the UMP level α test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1 > \theta_0$ (simple versus simple).
- This test is also UMP test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ (simple versus composite) since the rejection region does not depend on $\theta_1 > \theta_0$.

UMP Level α Test (cont'd)

- The statement of the Neyman-Pearson Lemma does not say what to do (reject or accept H_0) if

$$X \in \left\{ x \,\middle|\, \frac{f(x|\theta_1)}{f(x|\theta_0)} = c \right\}.$$

- If X is continuous, the probability of this event is zero, and we do not need to worry about it.
- If X is discrete, the event may or may not have positive probability.
- If it does have positive probability, the lemma does not say anything about such tests.
- The implication is that, when deriving UMP tests based on discrete X, we simply avoid using such values of c.

Monotone Likelihood Ratio (MLR)

- The MLR property is said to hold if the likelihood ratio

$$\frac{L(\theta_2|x)}{L(\theta_1|x)} = \frac{f_X(x|\theta_2)}{f_X(x|\theta_1)},$$

  depends on x only through a statistic $T(x)$, and is monotone increasing function of $T(x)$ for every $\theta_2 > \theta_1$.
- We will say that the likelihood has a MLR property in $T(X)$.
- **Example**: $X_1, \cdots, X_n$ are iid Poisson(θ), $\theta > 0$. The likelihood ratio

$$\frac{f_X(x|\theta_2)}{f_X(x|\theta_1)} = \exp\left\{\left(\log\frac{\theta_2}{\theta_1}\right)\left(\sum_{i=1}^{n} x_i\right) - n(\theta_2 - \theta_1)\right\},$$

  is clearly a monotone increasing function in $T(X) = \sum_{i=1}^{n} X_i$ since $\log(\theta_2/\theta_1) > 0$ for all $\theta_2 > \theta_1 > 0$.

Karlin-Rubin Theorem

- Consider testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.
- Suppose that T is *sufficient*, and the *MLR property holds*, then $\delta(X) = I(T > c)$ defines a UMP level α test.
- The theorem can be restated for the reversed testing problem.
- For testing $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$, a UMP level α test has test function $\delta(X) = I(T < t_0)$.
- The value of t_0 needs to be chosen so that the test has the desired size α in the continuous case.
- Or, the largest possible size $\alpha^* \leq \alpha$ in the discrete case.

Unbiased Tests

- Uniformly most powerful (UMP) level α tests do not always exist.
- **Example 8.3.19** Let $X_1, \cdots, X_n$ be iid N(θ, σ^2), σ^2 known. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.
- A size α test that rejects for large values of $\bar{X}$ is most powerful for $\theta > \theta_0$ but not for $\theta < \theta_0$.
- One way out of the nonexistence of UMP is to restrict to smaller classes of tests.

Unbiased Tests (cont'd)

- We define *unbiased tests* as:

$$\sup_{\theta \in \Theta_0} \beta(\theta) \leq \inf_{\theta \in \Theta_0^c} \beta(\theta).$$

- **Example** X is a random sample of size n from the N$(\theta, 1)$ distribution. $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.
- A uniformly most powerful (UMP) unbiased level α test is defined by $\delta(X) = I(\sqrt{n}|\bar{X} - \theta_0| > c)$ for some $c \geq 0$.
- Since $\Theta_0 = \{\theta_0\}$, the size of the test is $E_{\theta_0}\delta(X) = 2\Phi(-c)$.
- If we want the size to be 0.05, we choose $c = 1.96$.

P-value

- **Definition**: Given a sample $\boldsymbol{X}$, a *p-value* is a test statistic $p(\boldsymbol{X}) \in [0, 1]$ such that small values support H_1 over H_0.
- A p-value is *valid* if, for every $\theta \in \Theta_0$, and every $0 \leq \alpha \leq 1$,

$$P_\theta(p(\boldsymbol{X}) \leq \alpha) \leq \alpha.$$

- That means, if $p(\boldsymbol{X})$ is a valid p-value, a test that rejects H_0 if $p(\boldsymbol{X}) \leq \alpha$ is a level α test.

P-value (cont'd)

- **Example** $\boldsymbol{X}$ is a random sample of size n from the $\mathrm{N}(\theta, 1)$ distribution. $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.
- Let $\boldsymbol{x}$ be an observed sample and $\bar{x}$ be the observed sample mean.
- Consider a function:

$$p(\boldsymbol{x}) = 1 - \Phi\left(\sqrt{n}(\bar{x} - \theta_0)\right).$$

- $p(\boldsymbol{X})$ is a statistic since θ_0 is a specified known number (not an unknown parameter), and n is also known.
- $p(\boldsymbol{x})$ can be interpreted as the probability that the random variable $\bar{X}$ exceeds the observed value $\bar{x}$ if $\theta = \theta_0$.

P-value (cont'd)

- $p(\boldsymbol{x})$ is decreasing in $\bar{x}$, so large values of $\bar{x}$, which would support H_1 over H_0, go with small values of $p(\boldsymbol{x})$.
- Also,

$$
\begin{aligned}
P_\theta(p(\boldsymbol{X}) \le \alpha) &= P_\theta(\bar{X} \ge \theta_0 + \Phi^{-1}(1-\alpha)/\sqrt{n}) \\
&= P_\theta(\sqrt{n}(\bar{X} - \theta) \ge \sqrt{n}(\theta_0 - \theta) + \Phi^{-1}(1-\alpha)) \\
&= 1 - \Phi(\sqrt{n}(\theta_0 - \theta) + \Phi^{-1}(1-\alpha)).
\end{aligned}
$$

- That means $P_{\theta_0}(p(\boldsymbol{X}) \le \alpha) = \alpha$, and $P_\theta(p(\boldsymbol{X}) \le \alpha) < \alpha$ for $\theta < \theta_0$.
- Hence, this is a valid p-value, and the test with test function $\delta(X) = I(p(\boldsymbol{X}) \le \alpha)$ has size α.

P-value (cont'd)

- In general, if a hypothesis test rejects $H_0 : \theta = \theta_0$ for large values of a statistic $T(X)$, the p-value can be defined to be

$$p(x) = P_{\theta_0}(T(X) \geq T(x)),$$

where $T(x)$ is observed value of $T(X)$.

Union-Intersection Test

- Union-intersection and intersection-union tests are ways of combining many simpler hypothesis tests into a single more complicated test.
- In some problems, the null hypothesis is the intersection of two or more simpler null hypotheses,

$$H_0 : \theta \in \bigcap_{j \in J} \Theta_j \text{ against } H_1 : \theta \in \bigcup_{j \in J} \Theta_j^c.$$

- J may be finite or infinite.

Union-Intersection Test (cont'd)

- Suppose that for each individual problem of testing

$$H_{0j} : \theta \in \Theta_j \ \text{ against } \ H_{1j} : \theta \in \Theta_j^c,$$

  $j \in J$, with rejection region R_j. Then, the union-intersection test has rejection region

$$R = \bigcup_{j \in J} R_j$$

- That is, the union-intersection test rejects H_0 if any of the individual hypotheses H_{0j} is rejected.
- The null hypothesis is an intersection while the rejection region is a union.

Example for Union-Intersection Test

- $X \sim \mathrm{N}(\theta, 1)$. Test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$.
- Suppose that the simpler hypothesis tests are

$$H_{01} : \theta \geq 1 \text{ against } H_{11} : \theta < 1,$$

with rejection region $R_1 = \{x : x < a\}$, and

$$H_{02} : \theta \leq 1 \text{ against } H_{12} : \theta > 1,$$

with rejection region $R_2 = \{x : x > b\}$, where a and b are specified constants with $a < b$.

- Then the union-intersection test has critical region

$$R = R1 \bigcup R2 = \{x : x \notin [a, b]\}.$$

Intersection-Union Test

- In intersection-union tests, the null hypothesis is a union while the rejection region is an intersection,

$$H_0 : \theta \in \bigcup_{j \in J} \Theta_j \text{ against } H_1 : \theta \in \bigcap_{j \in J} \Theta_j^c,$$

and

$$R = \bigcap_{j \in J} R_j.$$

Example for Intersection-Union Test

- Suppose we observe a pair of random variables for each patient.
- X is an indicator of response to treatment, while Y is an indicator of severe side effects.
- Let $\theta_1 = P(X = 1)$ and $\theta_2 = P(Y = 1)$.
- One may test

$$H_0 : \theta_1 < 0.8 \;\text{ or }\; \theta_2 > 0.15 \;\text{ against }\; H_1 : \theta_1 \geq 0.8 \;\text{ and }\; \theta_2 \leq 0.15.$$

- Suppose that the simpler hypothesis tests are

$$H_{01} : \theta_1 < 0.8 \;\text{ against }\; H_{11} : \theta_1 \geq 0.8,$$

with rejection region $R_1 = \{x : \sum_{i=1}^{n} x_i > a\}$,

Example for Intersection-Union Test (cont'd)

- and
$$H_{02} : \theta_2 > 0.15 \text{ against } H_{12} : \theta_2 \leq 0.15,$$
with rejection region $R_2 = \{y : \sum_{i=1}^{n} y_i < b\}$.
- Then the intersection-union test has critical region
$$R = R_1 \bigcap R_2 = \{(x, y) : \sum_{i=1}^{n} x_i > a \text{ and } \sum_{i=1}^{n} y_i < b\},$$
and it rejects H_0 if the observed (x, y) falls within R, i.e. if both simpler null hypotheses are rejected.