Problem 1

(a)

$$X_1, \ldots, X_n \sim P(X_i \leq x|\alpha, \beta) = \begin{cases} 0 & x < 0 \\ (x/\beta)^\alpha & 0 \leq x \leq \beta \\ 1 & x > \beta \end{cases}$$
$$\alpha,\ \beta \text{ are positive}$$
$$\hat{\beta}_{MLE} = X_{(n)} \text{ (from problem 7.10)}$$
$$\alpha = \alpha_0$$
$$\text{Pivot} = X_{(n)}/\beta$$
$$.05 = P(X_{(n)}/\beta \leq c) = P(X_1, \ldots, X_n \leq c\beta) = \left(\frac{c\beta}{\beta}\right)^{\alpha_0 n} = c^{\alpha_0 n}$$
$$.05^{1/(\alpha_0 n)} = c$$
$$.95 = P(X_{(n)}/\beta > c) = P(X_{(n)}/c > \beta)$$
$$= P(X_{(n)}/.05^{1/(\alpha_0 n)} > \beta)$$
$$\{\beta : \beta < X_{(n)}/(.05^{1/(\alpha_0 n)})\}$$

(b)

$$\hat{\beta}_{MLE} = X_{(n)} = 25$$
$$\hat{\alpha}_{MLE} = 12.59 \quad n = 14$$
$$\{\beta : \beta < 25/(.05^{1/(12.59*14)})\}$$
$$25/(.05^{1/(12.59*14)}) = 25.42853 = 25.43$$
$$\beta < 25.43$$
$$\text{Since } \beta \text{ is positive, the lower bound cannot be below 0}$$
$$\text{Thus the interval for } \beta \text{ is } (0, 25.43)$$

Problem 2

(a)

$$X_1,\dots,X_n \sim N(0,\sigma_X^2) \quad Y_1,\dots,Y_m \sim N(0,\sigma_Y^2) \quad X \perp Y$$
$$\lambda = \sigma_Y^2/\sigma_X^2$$
$$H_0 : \lambda = \lambda_0 \text{ vs } H_1 : \lambda \neq \lambda_0$$
$$\lambda(x,y) = \frac{\sup_{\lambda=\lambda_0} L(\sigma_X^2,\sigma_Y^2|x,y)}{\sup_{\lambda\in\Theta} L(\sigma_X^2,\sigma_Y^2|x,y)}$$
$$\text{Unrestricted MLEs:}$$
$$\hat{\sigma^2}_{X\,MLE} = \sum_{i=1}^n X_i^2/n \quad \hat{\sigma^2}_{Y\,MLE} = \sum_{i=1}^n Y_i^2/m$$
$$L(\sigma_X^2,\sigma_Y^2|x,y) = (2\pi)^{-(n+m)/2}(\sigma_X^2)^{-n/2}(\sigma_Y^2)^{-m/2}\exp\left(-\sum x_i^2/(2\sigma_X^2)\right)\exp\left(-\sum y_i^2/(2\sigma_Y^2)\right)$$
$$\text{Under } H_0: \quad \lambda = \lambda_0 = \sigma_Y^2/\sigma_X^2$$
$$\sigma_Y^2 = \lambda_0\sigma_X^2$$
$$L(\sigma_X^2,\lambda_0\sigma_Y^2|x,y) = (2\pi\sigma_X^2)^{-n/2}(2\pi\lambda_0\sigma_X^2)^{-m/2}\exp\left(-\sum x_i^2/(2\sigma_X^2)\right)\exp\left(-\sum y_i^2/(2\lambda_0\sigma_X^2)\right)$$
$$= (2\pi)^{-(n+m)/2}(\sigma_X^2)^{-(n+m)/2}\lambda_0^{-m/2}\exp\left(-\left[\lambda_0\sum x_i^2+\sum y_i^2\right]/(2\lambda_0\sigma_X^2)\right)$$
$$\ell = -((n+m)/2)\log(2\pi)-((n+m)/2)\log(\sigma_X^2)-(m/2)\log(\lambda_0)-\left[\lambda_0\sum x_i^2+\sum y_i^2\right]/(2\lambda_0\sigma_X^2)$$
$$\frac{\partial\ell}{\partial\sigma_X^2} = -\frac{(n+m)/2}{\sigma_X^2}+\frac{\lambda_0\sum x_i^2+\sum y_i^2}{2\lambda_0(\sigma_X^2)^2} = 0$$
$$\frac{(n+m)/2}{\sigma_X^2} = \frac{\lambda_0\sum x_i^2+\sum y_i^2}{2\lambda_0(\sigma_X^2)^2}$$
$$\hat{\sigma_0^2} = \frac{\lambda_0\sum x_i^2+\sum y_i^2}{\lambda_0(n+m)}$$
$$\frac{\partial\ell^2}{\partial(\sigma_X^2)^2} = \frac{(n+m)/2}{(\sigma_X^2)^2}-\frac{\lambda_0\sum x_i^2+\sum y_i^2}{\lambda_0(\sigma_X^2)^3}$$
$$\text{Plugging in } \hat{\sigma_0^2}:$$
$$= \left(\frac{\lambda_0(n+m)}{\lambda_0\sum x_i^2+\sum y_i^2}\right)^2\left[(n+m)/2-\frac{\lambda_0\sum x_i^2+\sum y_i^2}{\lambda_0}\frac{\lambda_0(n+m)}{\lambda_0\sum x_i^2+\sum y_i^2}\right]$$
$$= \left(\frac{\lambda_0(n+m)}{\lambda_0\sum x_i^2+\sum y_i^2}\right)^2(-(n+m)/2)$$
$$= -\frac{(\sigma_0^2)^2(n+m)}{2} < 0$$
$$\text{Thus } \hat{\sigma_0^2} \text{ is the MLE}$$

$$\lambda(x,y)=\frac{(\hat{\sigma_0^2})^{-(n+m)/2}\lambda_0^{-m/2}\exp\left(-\left[\lambda_0\sum x_i^2+\sum y_i^2\right]/(2\lambda_0\hat{\sigma_0^2})\right)}{(\hat{\sigma_X^2})^{-n/2}(\hat{\sigma_Y^2})^{-m/2}\exp\left(-\sum x_i^2/(2\hat{\sigma_X^2})-\sum y_i^2/(2\hat{\sigma_Y^2})\right)}$$

$$=\frac{(\hat{\sigma_X^2})^{n/2}(\hat{\sigma_Y^2})^{m/2}\exp\left(-\left[\lambda_0\sum x_i^2+\sum y_i^2\right]/(2\lambda_0\hat{\sigma_0^2})+\sum x_i^2/(2\hat{\sigma_X^2})+\sum y_i^2/(2\hat{\sigma_Y^2})\right)}{(\hat{\sigma_0^2})^{(n+m)/2}\lambda_0^{m/2}}$$

$$=\frac{(\hat{\sigma_X^2})^{n/2}(\hat{\sigma_Y^2})^{m/2}\exp\left(-(n+m)/2+(n/2)+(m/2)\right)}{(\hat{\sigma_0^2})^{(n+m)/2}\lambda_0^{m/2}}$$

$$\lambda(x,y)=\frac{(\hat{\sigma_X^2})^{n/2}(\hat{\sigma_Y^2})^{m/2}}{(\hat{\sigma_0^2})^{(n+m)/2}\lambda_0^{m/2}}$$

$$R=\{(x,y):\lambda(x,y)<c\}\text{ Where c is chosen to give the test size }\alpha$$

(b)

$$\text{Under }H_0:\ \lambda_0=\lambda=\sigma_Y^2/\sigma_X^2$$

$$\text{Let }A=\sum Y_i^2/(\lambda_0\sigma_X^2)=\sum Y_i^2/\sigma_Y^2\sim\chi^2_m$$

$$\text{Let }B=\sum X_i^2/\sigma_X^2\sim\chi^2_n$$

$$A\perp B$$

$$\text{Let }F=\frac{A}{B}\frac{n}{m}=\frac{(\sum Y_i^2/\sigma_Y^2)/m}{(\sum X_i^2/\sigma_X^2)/n}=\frac{\sum Y_i^2/(\lambda_0 m)}{\sum X_i^2/n}\sim F_{m,n}$$

$$\lambda(x,y)=\left(\frac{\hat{\sigma_X^2}}{\hat{\sigma_0^2}}\right)^{n/2}\left(\frac{\hat{\sigma_Y^2}}{\hat{\sigma_0^2}\lambda_0}\right)^{m/2}$$

$$\frac{\hat{\sigma_X^2}}{\hat{\sigma_0^2}}=\frac{n+m}{n}\frac{\sum X_i^2\lambda_0}{\lambda_0\sum X_i^2+\sum Y_i^2}$$

$$=\frac{n+m}{n}\frac{\sum X_i^2\lambda_0}{\lambda_0\sum X_i^2+\sum Y_i^2}$$

$$=\frac{n+m}{n}\frac{1}{1+(\sum Y_i^2/\sigma_Y^2)/(\sum X_i^2/\sigma_X^2)}$$

$$=\frac{n+m}{n+nA/B}$$

$$=1/\left(\frac{n+nA/B}{n+m}\right)$$

$$=1/\left(\frac{n}{n+m}+\frac{1}{m+n}\frac{mnA}{mB}\right)$$

$$=1/\left(\frac{n}{n+m}+\frac{m}{m+n}F\right)$$

$$\frac{\hat{\sigma_Y^2}}{\hat{\sigma_0^2}\lambda_0} = \frac{n+m}{m}\frac{\sum Y_i^2}{\lambda_0 \sum X_i^2 + \sum Y_i^2}$$

$$\frac{n+m}{m}\frac{1}{1+(\sigma_Y^2 \sum X_i^2)/(\sum Y_i^2 \sigma_X^2)}$$

$$= 1/\left(\frac{m}{n+m}\left[1+\frac{\sigma_Y^2 \sum X_i^2}{\sigma_X^2 \sum Y_i^2}\right]\right)$$

$$= 1/\left(\frac{m}{n+m}(1+B/A)\right)$$

$$= 1/\left(\frac{m}{n+m}+\frac{1}{n+m}\frac{mB}{A}\frac{n}{n}\right)$$

$$= 1/\left(\frac{m}{n+m}+\frac{n}{n+m}\frac{mB}{nA}\right)$$

$$= 1/\left(\frac{m}{n+m}+\frac{n}{n+m}F^{-1}\right)$$

$$\lambda(x,y) = \left(\frac{1}{\frac{n}{n+m}+\frac{m}{m+n}F}\right)^{n/2}\left(\frac{1}{\frac{m}{n+m}+\frac{n}{n+m}F^{-1}}\right)^{m/2}$$

$$R = \left\{(x,y) : \left(\frac{1}{\frac{n}{n+m}+\frac{m}{m+n}F}\right)^{n/2}\left(\frac{1}{\frac{m}{n+m}+\frac{n}{n+m}F^{-1}}\right)^{m/2} < c_\alpha\right\}$$

Where c_α satisfies:

$$P\left(\left(\frac{1}{\frac{n}{n+m}+\frac{m}{m+n}F}\right)^{n/2}\left(\frac{1}{\frac{m}{n+m}+\frac{n}{n+m}F^{-1}}\right)^{m/2} < c_\alpha\right) = \alpha$$

(c)

$$\lambda(x,y) = \left(\frac{n}{n+m}+\frac{m}{n+m}\frac{\sum Y_i^2}{\sum X_i^2 \lambda}\frac{n}{m}\right)^{-n/2}\left(\frac{m}{n+m}+\frac{n}{n+m}\frac{\sum X_i^2 \lambda}{\sum Y_i^2}\frac{m}{n}\right)^{-m/2}$$

$$= \left(\frac{n}{n+m}+\frac{n}{n+m}\frac{\sum Y_i^2}{\sum X_i^2 \lambda}\right)^{-n/2}\left(\frac{m}{n+m}+\frac{m}{n+m}\frac{\sum X_i^2 \lambda}{\sum Y_i^2}\right)^{-m/2}$$

Let $D = \frac{n}{n+m}$ and $E = D\frac{\sum Y_i^2}{\sum X_i^2} = \frac{n}{n+m}\frac{\sum Y_i^2}{\sum X_i^2}$

$$\lambda(x,y) = \left(D+\frac{E}{\lambda}\right)^{-n/2}\left((1-D)+(1-D)\frac{D\sum Y_i^2}{D\sum X_i^2}\lambda\right)^{-m/2}$$

$$\lambda(x,y) = \left(D + \frac{E}{\lambda}\right)^{-n/2} \left((1-D) + (1-D)D\frac{\lambda}{E}\right)^{-m/2}$$

The acceptance region is:

$$A = \left\{\lambda(x,y) : \left(D + \frac{E}{\lambda}\right)^{-n/2} \left((1-D) + (1-D)D\frac{\lambda}{E}\right)^{-m/2} \geq c_a\right\}$$

Inverting the acceptance region we have the $1-\alpha$ CI for $\lambda:$

$$C(\lambda) = \left\{\lambda : \left(D + \frac{E}{\lambda}\right)^{-n/2} \left((1-D) + (1-D)D\frac{\lambda}{E}\right)^{-m/2} \geq c_a\right\}$$

Multiplying both sides by $\left(\frac{E}{1-D}\right)^{-m/2}$ we have:

$$\left(D + \frac{E}{\lambda}\right)^{-n/2} \left([(1-D) + (1-D)D\frac{\lambda}{E}]\left(\frac{E}{1-D}\right)\right)^{-m/2} \geq c_a \left(\frac{E}{1-D}\right)^{-m/2}$$

$$\left(D + \frac{E}{\lambda}\right)^{-n/2} (E + D\lambda)^{-m/2} \geq c_a \left(\frac{E}{1-D}\right)^{-m/2}$$

taking the derivative of the log with respect to λ of the left side:

$$\frac{\partial}{\partial\lambda}(-n/2)\log(D + E/\lambda) - (m/2)\log(E + D\lambda)$$

$$= \frac{nE - mD\lambda}{2\lambda(E + D\lambda)} \Rightarrow (1/2)\frac{n\frac{n}{n+m}\frac{\sum Y_i}{\sum X_i} - m\frac{n}{n+m}\lambda}{\lambda(\frac{n}{n+m}\frac{\sum Y_i}{\sum X_i} + \frac{n}{n+m}\lambda)}$$

$$\text{Let } S = \frac{\sum Y_i}{\sum X_i}$$

$$= (1/2)\frac{nS - m\lambda}{\lambda(S+\lambda)} \Rightarrow (1/2)\frac{nS/\lambda - m}{S+\lambda}$$

For $\lambda \geq 0$ the derivative changes sign from positive to negative,
thus $C(\lambda)$ increases and decreases and is therefore an interval
The graph of $C(\lambda)$ is a parabola

Problem 3

(a)

$$X_1, \ldots, X_n \sim f(x|\theta) = 1, \quad \theta - 1/2 < x < \theta + 1/2$$

$$Y = X - \theta$$

$$f_Y(y) = 1 \quad -1/2 < y < 1/2 \quad Y \perp \theta$$

$$Y \sim U(-1/2, 1/2)$$
$$1-\alpha = P(\alpha_1 - 1/2 \leq X - \theta \leq 1/2 - \alpha_2)$$
$$1-\alpha = P(X_1 - (1/2 - \alpha_2) \leq \theta \leq X_1 - (\alpha_1 - 1/2))$$
$$1-\alpha \text{ CI} = (\alpha_2 - 1/2, 1/2 - \alpha_1)$$

(b)

$$X_1, \ldots, X_n \sim f(x|\theta) = 2x/\theta^2, \quad 0 < x < \theta,\ \theta > 0$$
$$Y = X/\theta \quad \frac{dy}{dx} = 1/\theta$$
$$f_y(y) = f_x(\theta y) = \frac{1}{|1/\theta|} = \frac{2y\theta}{\theta^2}\theta$$
$$f_Y(y) = 2y \quad 0 \leq y \leq 1 \quad Y \perp \theta$$
$$P(a \leq X/\theta \leq b) = \int_a^b 2y\ dy = b^2 - a^2$$
$$b^2 - a^2 = 1 - \alpha \Rightarrow b^2 - a^2 = \sqrt{1-\alpha/2}^2 - \sqrt{\alpha/2}^2$$
$$b = \sqrt{1-\alpha/2} \quad a = \sqrt{\alpha/2}$$

Problem 4

(a)

$$X_1, \ldots, X_n \sim f(x|\theta) = (a/\theta)(x/\theta)^{a-1} \quad 0 < x < \theta$$
$$CDF \sim U(0,1)$$
$$1-\alpha = P(\alpha_1 < F_{X_{(n)}}(x) < 1 - \alpha_2)$$
$$F_{X_{(n)}}(x) = P(X_{(n)} \leq x) = [P(X_1 \leq x)]^n$$
$$P(X_1 \leq x) = \int_0^x (a/\theta)(y/\theta)^{a-1}\ dy = (x/\theta)^a$$
$$[P(X_1 \leq x)]^n = (x/\theta)^{an}$$
$$1-\alpha = P(\alpha_1 < (X_{(n)}/\theta)^{an} < 1 - \alpha_2)$$
$$= P(\alpha_1^{1/(an)} < X_{(n)}/\theta < (1-\alpha_2)^{1/(an)})$$
$$= P(X_{(n)}/(1-\alpha_2)^{1/(an)} < \theta < X_{(n)}/\alpha_1^{1/(an)})$$
$$1-\alpha \text{ CI } = \left(\frac{X_{(n)}}{(1-\alpha_2)^{1/(an)}}, \frac{X_{(n)}}{\alpha_1^{1/(an)}}\right)$$

(b)

$$Y = \left(\frac{X_{(n)}}{\theta}\right)^{na}$$
$$F_Y(y) = P((X_{(n)}/\theta)^{na} \le y) = [P(X_{(n)} \le \theta y^{1/na})]^n$$
$$= \left(\frac{\theta y^{1/(na)}}{\theta}\right)^{na} = y \Rightarrow Y \sim U(0,1)$$
$$F_Y(y) \sim U(0,1) \perp \theta$$
$$1-\alpha = P(\alpha_1 \le (X_{(n)}/\theta)^n a < 1-\alpha_1)$$

(c)

The intervals are the same

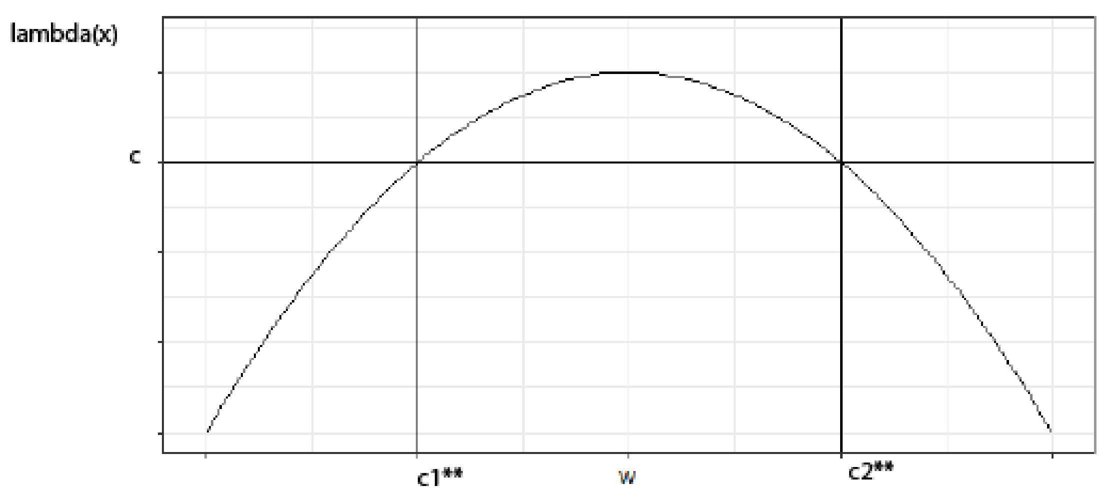
Problem 5

(a)

$$X_1, \dots, X_m \sim f(x|\mu_1) = \frac{1}{\mu_1}\exp(-x/\mu_1)$$
$$Y_1, \dots, Y_n \sim f(y|\mu_2) = \frac{1}{\mu_2}\exp(-y/\mu_2)$$
$$H_0 : \mu_1 = \mu_2 = \mu_0 \text{ vs } H_1 : \mu_1 \ne \mu_2$$
$$\lambda(x) = \frac{\sup_{\theta\in\Theta_0} L(\theta)}{\sup_{\theta\in\Theta} L(\theta)}$$
$$= \frac{L(\hat{\mu_0})}{L(\hat{\mu_1}, \hat{\mu_2})} = \frac{L(\hat{\mu_0})}{L(\bar{X}, \bar{Y})}$$
$$L(\mu_1, \mu_2) = \prod \frac{1}{\mu_1} e^{-x_i/\mu_1} \prod \frac{1}{\mu_2} e^{-y_i/\mu_2}$$
$$L(\mu_0) = (1/\mu_0)^{m+n} e^{-\sum x_i/\mu_0} e^{-\sum y_i/\mu_0}$$
$$= (1/\mu_0)^{m+n} e^{-(\sum x_i + \sum y_i)/\mu_0}$$
$$\hat{\mu_0} = \frac{\sum x_i + \sum y_i}{m+n}$$

$$\frac{L(\hat{\mu_0})}{L(\bar{X},\bar{Y})} = \frac{\left(\frac{m+n}{\sum x_i + \sum y_i}\right)^{m+n} e^{-(m+n)}}{(1/\bar{x})^m e^{-m}(1/\bar{y})^n e^{-n}}$$
$$= \left(\frac{m+n}{\sum x_i + \sum y_i}\right)^{m+n} \bar{x}^m \bar{y}^n$$
$$= \frac{(m+n)^{m+n}}{n^n m^m} \frac{(\sum x_i)^m (\sum y_i)^n}{(\sum x_i + \sum y_i)^{m+n}}$$
$$= \frac{(m+n)^{m+n}}{n^n m^m} \left(\frac{\sum x_i}{\sum x_i + \sum y_i}\right)^m \left(\frac{\sum y_i}{\sum x_i + \sum y_i}\right)^n$$
$$= \frac{(m+n)^{m+n}}{n^n m^m} \left(\frac{\sum x_i + \sum y_i}{\sum x_i}\right)^{-m} \left(\frac{\sum x_i + \sum y_i}{\sum y_i}\right)^{-n}$$
$$= \frac{(m+n)^{m+n}}{n^n m^m} \left(1 + \frac{\sum y_i}{\sum x_i}\right)^{-m} \left(1 + \frac{\sum x_i}{\sum y_i}\right)^{-n}$$
$$= \frac{(m+n)^{m+n}}{n^n m^m} \left(1 + \frac{n\bar{y}}{m\bar{x}}\right)^{-m} \left(1 + \frac{m\bar{x}}{n\bar{y}}\right)^{-n}$$
$$\text{Let } r = \bar{y}/\bar{x} \quad w = m/(m+nr)$$
$$= \frac{(m+n)^{m+n}}{n^n m^m} \left(1 + \frac{nr}{m}\right)^{-m} \left(1 + \frac{m}{nr}\right)^{-n}$$
$$= \frac{(m+n)^{m+n}}{n^n m^m} \left(\frac{m}{m+nr}\right)^m \left(\frac{nr}{m+nr}\right)^n$$
$$= \frac{(m+n)^{m+n}}{n^n m^m} w^m (1-w)^n$$

(b)



$$w = \frac{m}{m + n\left(\frac{m\sum y_i}{n\sum y_i}\right)} = \frac{\sum x_i}{\sum x_i + \sum y_i} \Rightarrow 0 \le w \le 1$$

$$\lambda(x,y) \text{ is a quadratic function of } w$$

$$\text{w is a monotone decreasing function of r}$$

$$R = \{\lambda(x,y) \le c\} \Leftrightarrow \{w < c_1^*\} \text{ or } \{w > c_2^*\}$$

$$\Leftrightarrow \{r < c_1^{**}\} \text{ or } \{r > c_2^{**}\}$$

$$X_i \sim Exp(\mu_1) = Gamma(1, \mu_1)$$

$$\sum X_i \sim Gamma(m, \mu_1) \Rightarrow 2\sum X_i/\mu_1 \sim Gamma(m, 2) = \chi^2_{2m}$$

$$2\sum Y_i/\mu_2 \sim \chi^2_{2n}$$

$$\frac{2\sum X_i/\mu_1/2m}{2\sum Y_i/\mu_2/2n} = \frac{n\sum X_i}{m\sum Y_i}\frac{\mu_2}{\mu_1} = \frac{\bar{X}\mu_2}{\bar{Y}\mu_1} \sim F_{2m,2n}$$

$$\alpha_1 = P(\frac{\bar{Y}}{\bar{X}} < c_1^* | H_0) = P(\frac{\mu_1\bar{Y}}{\mu_2\bar{X}} < \frac{c_1^*\mu_1}{\mu_2} | H_0)$$

$$\frac{\mu_1\bar{Y}}{\mu_2\bar{X}} \sim F_{2n,2m} \quad \frac{c_1^*\mu_1}{\mu_2} = c_1^* \text{ (since } \mu_1 = \mu_2 \text{ under } H_0)$$

$$c_1^* = F_{2n,2m,\alpha_1}$$

$$\alpha_2 = P(\frac{\bar{Y}}{\bar{X}} > c_2^* | H_0) = P(\frac{\mu_1\bar{Y}}{\mu_2\bar{X}} > \frac{c_2^*\mu_1}{\mu_2} | H_0)$$

$$c_2^* = F_{2n,2m,1-\alpha_2}$$

(c)

$$\psi = \mu_2/\mu_1$$

$$\frac{\psi\bar{X}}{\bar{Y}} = \frac{\mu_2\bar{X}}{\mu_1\bar{Y}} \sim F_{2m,2n} \perp \mu_1, \mu_2 \ (\perp \psi)$$

$$\text{Thus } \frac{\psi\bar{X}}{\bar{Y}} \text{ is a pivotal quantity}$$

$$\text{Exact 95\% CI for } \psi:$$

$$1 - \alpha = P(F_{2m,2n,\alpha_1} < \frac{\psi\bar{X}}{\bar{Y}} < F_{2m,2n,1-\alpha_2})$$

$$95\% \text{ CI} = P(\frac{\bar{Y}}{\bar{X}}F_{2m,2n,\alpha_1} < \psi < \frac{\bar{Y}}{\bar{X}}F_{2m,2n,1-\alpha_2})$$