Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 4.55
2. C&B 5.13
3. C&B 5.23
4. Suppose that a random variable U follows $N(0,1)$ and a random variable V follows a χ^2 distribution with p degrees of freedom. Assuming that U and V are independent, one can show that a random variable

$$T = \frac{U}{\sqrt{V/p}}$$

follows a t-distribution with p degrees of freedom.

(a) Find the conditional density function of T, given $V = v$, and use the result to derive the marginal density function of T.

Solution: The conditional density of T given $V = v$ is $N(0, p/v)$, so the conditional pdf is

$$f_T(t|V = v) = \frac{1}{\sqrt{2\pi p/v}} \exp\left(-\frac{t^2}{2p/v}\right).$$

The marginal pdf of T can be derived as

$$\begin{aligned}
f_T(t) &= \int_0^\infty f_T(t|V = v) f_V(v) dv \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi p/v}} \exp\left(-\frac{t^2}{2p/v}\right) \frac{1}{\Gamma(p/2)2^{p/2}} v^{p/2-1} \exp(-v/2) du \\
&= \frac{1}{\sqrt{2\pi p}} \frac{1}{\Gamma(p/2)2^{p/2}} \int_0^\infty v^{\frac{p+1}{2}-1} \exp\left\{-\left(\frac{t^2}{2p} + \frac{1}{2}\right) v\right\} dv \\
&= \frac{\Gamma(\frac{p+1}{2})}{\sqrt{2\pi p}} \frac{1}{\Gamma(p/2)2^{p/2}} \left(\frac{t^2}{2p} + \frac{1}{2}\right)^{-(p+1)/2} \\
&= \frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi p}\Gamma(p/2)} \left(1 + \frac{t^2}{p}\right)^{-(p+1)/2}, \quad -\infty < t < \infty.
\end{aligned}$$

(b) Find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$ without using the marginal density function of T.

Solution: Since U and V are independent,

$$\mathrm{E}(T) = E(U)E(p^{1/2}V^{-1/2}) = 0$$

and

$$\mathrm{Var}(T) = E(T^2) = \mathrm{E}(U^2)\mathrm{E}(pV^{-1}) = p\mathrm{E}(V^{-1}) = \frac{p}{p-2}, \quad p > 2.$$

where

$$\begin{aligned}
\mathrm{E}(V^{-1}) &= \int_0^\infty v^{-1} \frac{1}{\Gamma(p/2)2^{p/2}} v^{p/2-1} \exp(-v/2) du \\
&= \frac{1}{\Gamma(p/2)2^{p/2}} \int_0^\infty \frac{1}{\Gamma(p/2-1)2^{p/2-1}} v^{(p/2-1)-1} \exp(-v/2) du \Gamma(p/2-1)2^{p/2-1} \\
&= \frac{1}{p-2}.
\end{aligned}$$

(c) Use the transformation method to find the density function of T, as suggested in the course slides.

Solution: Since Z and U are independent, the joint density of (Z, U) is

$$f_{Z,U}(z, u) = f_Z(z) f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{1}{\Gamma(v/2)2^{v/2}} u^{v/2-1} e^{-u/2}.$$

Using transformation from (Z, U) to $T = \frac{Z}{\sqrt{U/v}}$ and $W = U$, one can derive the inverse function as $Z = T\sqrt{W/v}$ and $U = W$ with Jacobian $\sqrt{w/v}$. Then the join pdf of (T, W) becomes

$$\begin{aligned}
f_{T,W}(t, w) &= f_{Z,U}(t\sqrt{w/v}, w)\sqrt{w/v} \\
&= \frac{1}{\sqrt{2\pi}} e^{-t^2 w/(2v)} \frac{1}{\Gamma(v/2)2^{v/2}} w^{v/2-1} e^{-w/2} \sqrt{w/v} \\
&= \frac{1}{\sqrt{2\pi v/w}} \exp\left(-\frac{t^2}{2v/w}\right) \frac{1}{\Gamma(v/2)2^{v/2}} w^{v/2-1} \exp(-w/2),
\end{aligned}$$

which is the same as $f_{T,U}(t, u)$ in (a). The same as (a), we can obtain $f_T(t)$ by integrating out w from $f_{T,W}(t, w)$.

5. For patients receiving a double kidney transplant, let X_i be the lifetime (in months) of the ith kidney, $i = 1, 2$. Also, assume that X_i follows exponential distribution with density

$$f_{X_i}(x_i) = \alpha e^{-\alpha x_i}, \quad x_i > 0, \quad \alpha > 0, \quad i = 1, 2.$$

Assume X_1 and X_2 are independent, and define a new variable V, which is the lifetime of the remaining functional kidney as soon as one of the two kidneys fails, having a conditional density function

$$f_V(v|U = u) = \beta e^{-\beta(v-u)}, \quad 0 < u < v < \infty, \quad \beta > 2\alpha,$$

where $U = \min(X_1, X_2)$.

(a) Show that the probability that both organs are still functioning at time t is equal to

$$\pi_0(t) = e^{-2\alpha t}, \quad t \geq 0.$$

Solution:

$$P(X_1 \geq t, X_2 \geq t) = P(X_1 \geq t)P(X_2 \geq t) = \left(\int_t^\infty \alpha e^{-\alpha x} dx \right)^2 = e^{-2\alpha t}.$$

(b) Show that the probability that exactly one organ is still functioning at time t is equal to

$$\pi_1(t) = \frac{2\alpha}{(\beta - 2\alpha)} \left(e^{-2\alpha t} - e^{-\beta t} \right), \quad t \geq 0.$$

[Hint: P(exactly one kidney is functioning at time t)=$P(U \leq t, V \geq t)$].

Solution: From (a), $F_U(u) = P(U \leq u) = 1 - e^{-2\alpha u}$, and $f_U(u) = 2\alpha e^{-2\alpha u}$, $u > 0$. Hence,

$$f_{U,V}(u, v) = f_U(u) f_V(v|U = u) = \left(2\alpha e^{-2\alpha u}\right) \left\{ \beta e^{-\beta(v-u)} \right\}, 0 < u < v < \infty.$$

Now,

$$P(U \leq t, V \geq t) = \int_0^t \int_t^\infty f_{U,V}(u, v) dv du = \frac{2\alpha}{(\beta - 2\alpha)} \left(e^{-2\alpha t} - e^{-\beta t} \right), t \geq 0.$$

(c) Find $f_T(t)$, where T is the length of time (in months) until both kidneys have failed.

Solution: One has $F_T(t) = P(T \leq t) = 1 - \pi_0(t) - \pi_1(t)$, and

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{2\alpha\beta}{(\beta - 2\alpha)} \left(e^{-2\alpha t} - e^{-\beta t} \right), \quad t \geq 0. \tag{1}$$

Notice that, V also indicates the time of both kidneys failure. Hence, the marginal pdf of V shows the same result.

$$f_V(v) = \int_0^v f_{U,V}(u, v) du = \int_0^v \left(2\alpha e^{-2\alpha u}\right) \beta e^{-\beta(v-u)} du = \frac{2\alpha\beta}{(\beta - 2\alpha)} \left(e^{-2\alpha v} - e^{-\beta v} \right).$$

Also, $V = \max(X_1, X_2)$. We know that

$$\begin{aligned} F_V(t) &= P(V \leq t) = P(\max(X_1, X_2 \leq t) \\ &= P(X_1 \leq t, X_2 \leq t) \\ &= P(X_1 \leq t) P(X_2 \leq t) \\ &= (1 - e^{-\alpha t})^2. \end{aligned}$$

The pdf of T (or V) is

$$f_T(t) = \frac{d}{dt}F_V(t) = 2\alpha e^{-\alpha t}(1 - e^{-\alpha t}),$$

which is same as the result in (1) if $\beta = \alpha$. That means, the conditional pdf of V given U is not necessarily indexed by β. It makes sense since exponential distribution has a memoryless property.

6. Let $X = (X_1, \ldots, X_n)$ be a random vector having the joint distribution as a multivariate normal distribution $N(\mu J, D)$, where J is a vector of 1 and

$$D = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix},$$

and $|\rho| < 1$. Solve the following items.

(a) Let

$$A = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2\times 1}} & \frac{-1}{\sqrt{2\times 1}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3\times 2}} & \frac{1}{\sqrt{3\times 2}} & \frac{-2}{\sqrt{3\times 2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n\times(n-1)}} & \frac{1}{\sqrt{n\times(n-1)}} & \frac{1}{\sqrt{n\times(n-1)}} & \frac{1}{\sqrt{n\times(n-1)}} & \cdots & \frac{-(n-1)}{\sqrt{n\times(n-1)}} \end{pmatrix}.$$

Show that $AA^T = A^T A = I$, where I is an $n \times n$ identity matrix, and that

$$ADA^T = \sigma^2 \begin{pmatrix} 1 + (n-1)\rho & 0 & \cdots & 0 \\ 0 & 1 - \rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \rho \end{pmatrix}.$$

(b) Derive the distribution of $Y = AX$ and show that $Y = (Y_1, \ldots, Y_n)$ are mutually independent.

Solution: Since Y is a linear combination of X, which follows a normal distribution, Y follows a normal distribution with mean $E(Y) = E(AX) = AE(X) =$

$\mu AJ = (\sqrt{n}\mu, 0, \ldots, 0)'$ and variance $\text{var}(Y) = \text{var}(AX) = ADA'$. Since one can see that $\text{cov}(Y_i, Y_j) = 0$ for $i \neq j$ in the covariance matrix ADA' above, we can conclude $Y_1, \ldots, Y_n$ are mutually independent because they are normally distributed random variables.

(c) Show that $Y_1 = \sqrt{n}\bar{X}$ and that $\bar{X}$ has the normal distribution $N(\mu, \frac{1+(n-1)\rho}{n}\sigma^2)$.

Solution: By the result of (b), one can see Y_1 follows $N(\sqrt{n}\mu, \{1+(n-1)\rho\}\sigma^2)$. Hence, it is straightforward to conclude $\bar{X}$ follows $N(\mu, \{1+(n-1)\rho\}\sigma^2/n)$.

(d) Show that $W = \sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=2}^n Y_i^2$ and that $W/\{(1-\rho)\sigma^2\}$ follows χ^2_{n-1}.

Solution: First, one can show that $\sum_{i=1}^n Y_i^2 = Y'Y = (AX)'AX = X'A'AX = X'X = \sum_{i=1}^n X_i^2$. One can then show that $W = \sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2 = \sum_{i=1}^n Y_i^2 - Y_1^2 = \sum_{i=2}^n Y_i^2$. Since $Y_i^2/\{(1-\rho)\sigma^2\}$ follows χ^2_1, one can know $W/\{(1-\rho)\sigma^2\} = \sum_{i=2}^n Y_i^2/\{(1-\rho)\sigma^2\}$ follows χ^2_{n-1} since $Y_2, \ldots, Y_n$ are mutually independent.

(e) Show that $\bar{X}$ and W are independent.

Solution: Since $\bar{X}$ is a function of Y_1 and W is a function of $Y_2, \ldots, Y_n$, we can conclude $\bar{X}$ and W are independent since $Y_1, Y_2, \ldots, Y_n$ are mutually independent.