Problem 1

$$X_1, X_2, X_3 \sim exp(\lambda)$$
$$Y \sim max(X_1, X_2, X_3)$$
$$P(Y \leq y) = P(max(x_1, x_2, x_3) \leq y)$$
$$= P(X_1 \leq y \text{ and } X_2 \leq y \text{ and } X_3 \leq y)$$
Since the components are independent:
$$= P(X_1 \leq y)P(X_2 \leq y)P(X_2 \leq y)$$
Each components probability is:
$$P(X \leq y) = \int_0^y \lambda e^{-\lambda x} \mathrm{d}x = 1 - e^{-\lambda y} \quad 0 < y < \infty \quad \lambda > 0$$
$$P(Y \leq y) = (1 - e^{-\lambda y})^3 \quad 0 < y < \infty$$
$$f_y(y) = \begin{cases} 3(1 - e^{-\lambda y})^2 e^{-\lambda y} & y > 0 \\ 0 \quad \text{otherwise} \end{cases}$$

Problem 2

Want to find $g(S^2)$ such that $E[g(S^2)] = \sigma$

Let $g(S^2) = c\sqrt{S^2}$ then:
$$E[(c\sqrt{S^2})] = E\left(c\sqrt{S^2}\sqrt{\frac{\sigma^2}{n-1}}\sqrt{\frac{n-1}{\sigma^2}}\right)$$
$$= c\sqrt{\frac{\sigma^2}{n-1}}E\left(\sqrt{\frac{S^2(n-1)}{\sigma^2}}\right)$$
Let $Z = \frac{S^2(n-1)}{\sigma^2}$
$$Z \sim \chi^2_{n-1}$$
Use transformation $Y = \sqrt{Z}$
$$Y = \sqrt{\frac{S^2(n-1)}{\sigma^2}}$$
$$Z = Y^2 \quad \frac{dy}{dz} = 2y$$

$$f_Y(y) = f_Z(g^{-1}(z))\left|\frac{dx}{dy}\right|$$
$$= \frac{1}{2^{(n-1)/2}\Gamma((n-1)/2)}(y^2)^{((n-1)/2-1)}e^{-y^2/2}|2y|$$
$$f_Y(y) = \frac{2}{2^{(n-1)/2}\Gamma((n-1)/2)}y^{n-2}e^{-y^2/2}$$
$$E[(c\sqrt{S^2})] = c\sqrt{\frac{\sigma^2}{n-1}}\int_0^\infty 2y\frac{1}{2^{(n-1)/2}\Gamma((n-1)/2)}y^{n-2}e^{-y^2/2}\ dy$$
$$= c\sqrt{\frac{\sigma^2}{n-1}}\int_0^\infty \frac{1}{2^{(n-1)/2}\Gamma((n-1)/2)}y^{n-2}e^{-y^2/2}(2y)\ dy$$
$$\text{Let } w = y^2 \quad dw = 2y\ dy \quad y = w^{1/2}$$
$$= c\sqrt{\frac{\sigma^2}{n-1}}\int_0^\infty \frac{1}{2^{(n-1)/2}\Gamma((n-1)/2)}w^{n/2-1}e^{-w/2}\ dw$$
$$= c\frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)}\sqrt{\frac{\sigma^2}{n-1}}\int_0^\infty \frac{1}{2^{n/2}\Gamma(n/2)}w^{n/2-1}e^{-w/2}\ dw$$
$$E[(c\sqrt{S^2})] = \sigma = c\frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)}\sqrt{\frac{\sigma^2}{n-1}}$$
$$c = \sigma * 1/\left(\frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)}\sqrt{\frac{\sigma^2}{n-1}}\right)$$
$$c = \frac{\sqrt{n-1}\Gamma((n-1)/2)}{\sqrt{2}\Gamma(n/2)}$$

Problem 3

$$P(Z > z) = \sum_{x=1}^{\infty} P(Z > z|x)P(X = x)$$
$$= \sum_{x=1}^{\infty} P(U_1 > z, \ldots, U_x > z|x)P(X = x)$$
$$\text{Since } U_i \text{ are independent:}$$
$$= \sum_{x=1}^{\infty}\prod_{i=1}^{x} P(U_i > z)P(X = x)$$

$$= \sum_{x=1}^{\infty} P(U_i > z)^x P(X = x)$$

$$= \sum_{x=1}^{\infty} (1-z)^x \frac{1}{x!(e-1)}$$

$$= \frac{1}{e-1} \sum_{x=1}^{\infty} \frac{(1-z)^x}{x!}$$

$$\text{Since } \sum_{x=1}^{\infty} z^x/x! = e^z - 1$$

$$\sum_{x=1}^{\infty} (1-z)^x/x! = e^{1-z} - 1$$

Thus we have:

$$= \frac{e^{1-z}-1}{e-1} \quad 0 < z < 1$$

Problem 4

(a)

$$T|V = v \sim \frac{U}{\sqrt{v/p}} \sim N(0, p/v)$$

$$f_{T|V=v}(t) = \frac{\sqrt{v}}{\sqrt{2p\pi}} e^{\left(\frac{-t^2 v}{2p}\right)}$$

$$f_{T,V}(t,v) = f_{T|V}(t) f_V(v)$$

$$f_{T,V}(t,v) = \frac{\sqrt{v}}{\sqrt{2p\pi}} e^{\left(\frac{-t^2 v}{2p}\right)} \frac{1}{\Gamma(p/2)2^{p/2}} v^{p/2-1} e^{-v/2}$$

$$f_{T,V}(t,v) = \frac{1}{\Gamma(p/2)\sqrt{p\pi}2^{(p-1)/2}} e^{\left(-v/2(1+t^2/p)\right)} v^{(p-1)/2}$$

$$f_T(t) = \int_0^{\infty} f_{T,V}(t,v)\ dv$$

$$\frac{1}{\Gamma(p/2)\sqrt{p\pi}2^{(p-1)/2}} \int_0^{\infty} e^{\left(-v/2(1+t^2/p)\right)} v^{(p-1)/2}\ dv$$

$$\text{Let } z = v/2(1+t^2/p) \quad dz = 1/2(1+t^2/p)\ dv \text{ then } v = \frac{2z}{1+t^2/p}$$

$$f_T(t) = \frac{2^{(p-1)/2+1}}{\sqrt{p\pi}} \frac{(1+t^2/p)^{-[(p-1)/2+1]}}{\Gamma(p/2)2^{(p+1)/2}} \int_0^\infty e^{-z} z^{(p-1)/2}\ dz$$
$$= \frac{(1+t^2/p)^{-(p+1)/2}}{\Gamma(p/2)\sqrt{p\pi}} \int_0^\infty e^{-z} z^{(p+1)/2-1}\ dz$$
$$f_T(t) = \frac{\Gamma((p+1)/2)}{\Gamma(p/2)\sqrt{p\pi}}(1+t^2/p)^{-(p+1)/2}$$

Which is the t-distribution with p degrees of freedom

(b)

$$E(T) = E\left(\frac{U}{\sqrt{V/p}}\right)$$
$$= E\left(E\left(\frac{U}{\sqrt{V/p}}|V=v\right)\right)$$
$$= E\left(\frac{1}{\sqrt{v/p}}E(U)\right)$$
$$= E\left(\sqrt{\frac{p}{v}} * 0\right) = 0$$
$$E(T) = 0$$
$$Var(T) = E\left(Var\left(\frac{U}{\sqrt{V/p}}|V=v\right)\right) + Var\left(E\left(\frac{U}{\sqrt{V/p}}|V=v\right)\right)$$
$$= E\left(\frac{p}{V}Var(U)\right) + 0$$
$$= E\left(\frac{p}{V} * 1\right)$$
$$= pE(1/V)$$
$$Var(T) = pE(v^{-1}) = p\int_0^\infty v^{-1}\frac{1}{\Gamma(p/2)2^{p/2}} v^{p/2-1} e^{-v/2}\ dv$$
$$= p\int_0^\infty \frac{1}{\Gamma(p/2)2^{p/2}} v^{(p/2-1)-1} e^{-v/2}\ dv$$
$$= p\frac{\Gamma(p/2-1)2^{p/2-1}}{\Gamma(p/2)2^{p/2}} \int_0^\infty \frac{1}{\Gamma(p/2-1)2^{p/2-1}} v^{(p/2-1)-1} e^{-v/2}\ dv$$
$$Var(T) = p\frac{\Gamma(p/2-1)2^{p/2-1}}{\Gamma(p/2)2^{p/2}}$$
$$= \frac{p}{2}\frac{\Gamma(p/2-1)}{\Gamma(p/2)}$$

Since $p/2-1$ and $p/2$ are whole numbers we have:

$$= \frac{p}{2}\frac{(p/2-1)!}{(p/2-1)!}$$

$$= \frac{p}{2(p/2-1)} = \frac{p}{p-2}$$

$$Var(T) = \frac{p}{p-2}$$

(c)

$$U \sim N(0,1) \quad V \sim \chi^2_p$$

$$f_{UV}(u,v) = f_U(u)f_V(v) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\frac{1}{\Gamma(p/2)2^{p/2}}v^{p/2-1}e^{-v/2}$$

$$\text{Let } T = \frac{U}{\sqrt{V/p}} \quad W = V$$

$$V = W \quad U = T\sqrt{W/p}$$

$$J = \begin{bmatrix} \sqrt{w/p} & \dfrac{-t}{2\sqrt{pw}} \\ 0 & 1 \end{bmatrix} = |\sqrt{w/p}|$$

$$f_{TW}(t,w) = f_{UV}(t\sqrt{w/p}, w)(\sqrt{w/p})$$

$$= \frac{1}{\sqrt{2\pi}}e^{-(t\sqrt{w/p})^2/2}\frac{1}{\Gamma(p/2)2^{p/2}}w^{p/2-1}e^{-w/2}(\sqrt{w/p})$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\left(\frac{t^2w}{2p}\right)}\frac{1}{\Gamma(p/2)2^{p/2}}w^{(p-1)/2}e^{-w/2}(1/\sqrt{p})$$

$$f_T(t) = \int_0^\infty f_{TW}\ dw$$

$$= \frac{1}{\sqrt{2p\pi}}\frac{1}{\Gamma(p/2)2^{p/2}}\int_0^\infty e^{(-w/2)\left(t^2/p+1\right)}w^{(p-1)/2}\ dw$$

$$\text{Let } z = (1+t^2/p)(w/2) \quad dz = 1/2(1+t^2/p)\ dw \text{ Then } w = \frac{2z}{1+t^2/p}$$

$$f_T(t) = \frac{2^{(p-1)/2+1}}{\sqrt{p\pi}}\frac{(1+t^2/p)^{-[(p-1)/2+1]}}{\Gamma(p/2)2^{(p+1)/2}}\int_0^\infty e^{-z}z^{(p-1)/2}\ dz$$

$$= \frac{(1+t^2/p)^{-(p+1)/2}}{\Gamma(p/2)\sqrt{p\pi}}\int_0^\infty e^{-z}z^{(p+1)/2-1}\ dz$$

$$f_T(t) = \frac{\Gamma((p+1)/2)}{\Gamma(p/2)\sqrt{p\pi}}(1+t^2/p)^{-(p+1)/2}$$

Which is the t-distribution with p degrees of freedom

Problem 5

(a)

$$U \sim min(X_1, X_2)$$
$$P(U \geq t) = P(min(x_1, x_2) \geq t)$$
$$\pi_0(t) = P(X_1 \geq t, X_2 \geq t)$$
Since X_i are independent:
$$= P(X_1 \geq t)P(X_2 \geq t)$$
Each probability is:
$$P(X \geq t) = 1 - \int_0^t \alpha e^{-\alpha x} \mathrm{d}x$$
$$= 1 + \Big|_0^t e^{-\alpha x} = e^{-\alpha t}$$
$$\pi_0(t) = 1 - F_U(t) = e^{(-\alpha t)^2} = e^{-2\alpha t} \quad t \geq 0$$
$\pi_0(t)$ is the probability that both organs are still functioning at time t

(b)

$$1 - f_U(u) = \frac{d}{du} 1 - e^{-2\alpha u} = 2\alpha e^{-2\alpha u}$$
$$f_{UV}(u, v) = f_V(V|U = u)f_U(u)$$
$$f_{UV}(u, v) = \beta e^{-\beta(v-u)}(2\alpha)e^{-2\alpha u}$$
$$= 2\alpha\beta e^{-\beta v} e^{-(2a-\beta)u}$$
$$\pi_1(t) = P(U \leq t, V \geq t) = \int_0^t \int_t^\infty f_{UV}(u, v)\ dv\ du$$
$$= \int_0^t \int_t^\infty 2\alpha\beta e^{-\beta v} e^{-(2a-\beta)u}\ dv\ du$$
$$= 2\alpha\beta \int_0^t \Big|_t^\infty \frac{-1}{\beta} e^{-\beta v} e^{-(2a-\beta)u}\ du$$
$$= 2\alpha \int_0^t e^{-\beta t} e^{-(2a-\beta)u}\ du$$
$$= 2\alpha \frac{-1}{2\alpha - \beta} \Big|_0^t e^{-\beta t} e^{-(2a-\beta)u}$$

$$= \frac{2\alpha}{\beta - 2\alpha} e^{-\beta t}(e^{-(2a-\beta)t} - 1)$$
$$\pi_1(t) = \frac{2\alpha}{\beta - 2\alpha}(e^{-2at} - e^{-\beta t}) \quad t \geq 0$$

Which is the probability that exactly one kidney is functioning at time t

(c)

$$f_V(t) = f_T(t)$$
$$f_V(t) = \int_0^t f_{UV}(u, v)\ du$$
$$= \int_0^t 2\alpha\beta e^{-\beta t} e^{-(2a-\beta)u}\ du$$
$$= \Big|_0^t \frac{2\alpha\beta}{\beta - 2\alpha} e^{-\beta t} e^{-(2a-\beta)u}$$
$$= \frac{2\alpha\beta}{\beta - 2\alpha} e^{-\beta t}(e^{-(2a-\beta)t} - 1)$$
$$f_T(t) = f_V(t) = \frac{2\alpha\beta}{\beta - 2\alpha}(e^{-2at} - e^{-\beta t}) \quad t \geq 0$$