Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 5.21
2. C&B 5.24
3. C&B 5.25
4. Let $X_1, \ldots, X_n$ be independent random variables having exponential distribution with respective parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ and probability density functions

$$f_{X_i}(x_i) = \alpha_i e^{-\alpha_i x_i}, \quad x_i > 0, \quad \alpha_i > 0, \quad i = 1, \ldots, n.$$

   (a) Show that the minimum order statistic $X_{(1)} = \min\{X_1, \ldots, X_n\}$ has an exponential distribution with parameter $\sum_{i=1}^n \alpha_i$ and pdf

$$f_{X_{(1)}}(x) = \left(\sum_{i=1}^n \alpha_i\right) e^{-(\sum_{i=1}^n \alpha_i)x}.$$

   [Note: The pdf formula for the order statistics does not work since the random variables are not iid. Use CDF method.]

   (b) Show that

$$P(X_{(1)} = X_k) = \frac{\alpha_k}{\sum_{i=1}^n \alpha_i}, \quad k \geq 1.$$

5. Suppose that iid random variables $X_1, \ldots, X_n$ follow a uniform distribution on the interval $(0, 1)$ with pdf

$$f_X(x) = 1, \quad 0 < x < 1.$$

   Let random variables $U = X_{(1)}$ and $V = 1 - X_{(n)}$, where $X_{(1)} = \min_i X_i$ and $X_{(n)} = \max_i X_i$ are minimum and maximum order statistics, respectively.

   (a) Find an explicit expression for the joint distribution of the random variables U and V.

   (b) Let $R = nU$ and $S = nV$. Show that

$$P(R > r, S > s) = \left(1 - \frac{r}{n} - \frac{s}{n}\right)^n.$$

   (c) Following the result in (b), show that R and S are asymptotically independent. You may need the fact that

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

(d) What is the asymptotic distribution of R and S?

6. Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution with pdf

$$\beta^{-1} e^{(\alpha - x)/\beta}, \quad \alpha < x < \infty,$$

where $\alpha \in \mathcal{R}$ and $\beta > 0$ are parameters. Let $X_{(1)} \leq \cdots \leq X_{(n)}$ be order statistics, and let $Z_1 = X_{(1)}$ and $Z_i = X_{(i)} - X_{(i-1)}$ for $i = 2, \ldots, n$. Show that

(a) $Z_1, \ldots, Z_n$ are independent and $2(n - i + 1)Z_i/\beta$ has the χ^2_2 distribution.

(b) $X_{(1)}$ and Y are independent, where $Y = (n-1)^{-1} \sum_{i=1}^n (X_i - X_{(1)})$.

(c) (Bios 673 class material, no need to return) $T = (X_{(1)} - \alpha)/Y$ has a pdf

$$f_Y(t) = n \left(1 + \frac{nt}{n-1}\right)^{-n},$$

for $0 < t < \infty$ and 0 otherwise.

7. (Bios 673 class material) Let $X_1, \ldots, X_n$ be a random sample from a distribution with unknown mean $\mu \in \mathcal{R}$ and unknown variance $\sigma^2 > 0$. Let $\bar{X}$ and S^2 be the sample mean and sample variance, respectively. One is interested in comparing three estimators, $T_1 = \bar{X}^2$, $T_2 = \bar{X}^2 - S^2/n$, and $T_3 = \max\{0, T_2\}$ for μ^2.

(a) When $\mu \neq 0$, show that three estimators have the same limiting distribution, i.e.,

$$\sqrt{n}(T_i - \mu^2) \to_d N(0, \tau^2),$$

for $i = 1, 2, 3$. Express τ^2 as a function of μ and σ^2.

(b) When $\mu = 0$, show that

$$nT_1 \to_d \sigma^2 W,$$

and

$$nT_2 \to_d \sigma^2 (W - 1),$$

where W follows a χ^2_1 distribution.

(c) When $\mu = 0$, show that T_2 has a smaller asymptotic mean square error (AMSE) than T_1, where AMSE is defined by EX^2/a_n^2 when $a_n X_n \to_d X$ with $EX^2 < \infty$.

(d) When $\mu = 0$, T_3 is in fact a better estimator of μ^2 with the smallest AMSE. Without any theoretical proof, comment on why this is true.