Bios 661: 1 – 5; Bios 673: 2 – 6.

1. C&B 4.19
2. C&B 4.23
3. C&B 5.6
4. Suppose (X_1, X_2) have the joint pdf

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 1 & 0 < x_1 < 1, \quad 0 < x_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Let the support of (X_1, X_2) be denoted by the set $\mathcal{S} = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\}$. Draw $\mathcal{S}$ on the xy-plane.
   (b) Suppose $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ with a support $\mathcal{T}$. Draw $\mathcal{T}$ on the xy-plane.
   (c) Derive the joint distribution of (Y_1, Y_2) using Jacobin method.
   (d) Derive the marginal distribution (pdf) of Y_1 and Y_2.

5. Suppose that random variables X_1 and X_2 are mutually independent and follow $N(0, 1)$. Show that the random variable $Y_1 = X_1/X_2$ follows Cauchy distribution with pdf

$$f_{Y_1}(y_1) = \frac{1}{\pi} \frac{1}{1 + y_1^2}, \quad -\infty < y_1 < \infty.$$

   Answer the following questions step by step toward the final solution.

   (a) Let $Y_2 = X_2$. Show that the Jacobian of the inverse function of y_1 and y_2 is y_2, where $-\infty < y_2 < \infty$.
   (b) Derive the joint pdf of Y_1 and Y_2 using the Jacobian method.
   (c) Find the marginal distribution of Y_1 from the joint distribution of Y_1 and Y_2 in (b).

6. Let $X_1, \ldots, X_n$ constitute a random sample of size $n (n \geq 3)$ from the parent population

$$f_X(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty, \quad 0 < \lambda < \infty$$

   (a) Find the conditional density function of $X_1, \ldots, X_n$ given that $S = \sum_{i=1}^n X_i = s$.
   (b) Consider the $(n - 1)$ random variables

$$Y_1 = \frac{X_1}{S}, Y_2 = \frac{X_1 + X_2}{S}, \ldots, Y_{n-1} = \frac{X_1 + X_2 + \cdots + X_{n-1}}{S}.$$

   Find the joint distribution of $Y_1, Y_2, \ldots, Y_{n-1}$ given that $S = s$.

(c) When $n = 3$ and when $n = 4$, find the marginal distribution of Y_1 given that $S = s$, and then use these results to infer the structure of the marginal distribution of Y_1 given that $S = s$ for any $n \geq 3$.

7. (Bios 673 class material) A certain simple biological system involves exactly two independently functioning components. If one of these two components fails, then entire systems fails. For $i = 1, 2$, let Y_i be the random variable representing the time to failure of the ith component, with the pdf of Y_i being

$$f_{Y_i}(y_i) = \theta_i e^{-\theta_i y_i}, \quad 0 < y_i < \infty, \quad \theta_i > 0.$$

Clearly, if this biological system fails, then only two random variables are observable, namely U and W, where $U = \min(Y_1, Y_2)$ and

$$W = \begin{cases} 1, & \text{if } \ Y_1 < Y_2, \\ 0, & \text{if } \ Y_2 < Y_1. \end{cases}$$

(a) Show that the joint distribution $f_{U,W}(u, w)$ of random variables U and W is

$$f_{U,W}(u, w) = \theta_1^{(1-w)} \theta_2^{w} e^{-(\theta_1+\theta_2)u}, \quad 0 < u < \infty, \quad w = 0, 1.$$

(b) Find the marginal distribution $f_W(w)$ of the random variable W.

(c) Find the marginal distribution $f_U(u)$ of the random variable U.

(d) Show that U and W are independent.