Problem 1

$$P(max(X_1, X_2) > m) = 1 - P(X_1 \le m, X_2 \le m)$$
Since the Xs are iid we have:
$$= 1 - P(X_1 \le m)P(X_2 \le m)$$
$$= 1 - P(X_1 \le m)^2$$
$$= 1 - (1/2)^2 = 3/4$$
Generalizing this result:
$$P(max(X_1, \dots, X_n) \le m) = 1 - P(X_i \le m,\ i = 1, \dots, n)$$
$$= 1 - P(X_1)P(X_2 \le m) \cdots P(X_n \le m)$$
$$= 1 - [P(X_1 \le m)]^n = 1 - (1/2)^n$$

Problem 2

$$f_X(x) = \frac{1}{\theta} \quad 0 < x < \theta \quad F_X(x) = \frac{1}{\theta}x$$
Let $U = X_{(1)} \quad V = X_{(n)}$
$$f_{U,V}(u, v) = \frac{n!}{(1-1)!(n-1-1)!(n-n)!}\frac{1}{\theta^2}[\frac{1}{\theta}u]^{1-1}[\frac{1}{\theta}(u-v)]^{n-1-1}[1-\frac{1}{\theta}v]^{n-n}$$
$$f_{U,V}(u, v) = \frac{n(n-1)}{\theta^n}(v-u)^{n-2} \quad 0 < u < v < \theta$$
Let $Z = U/V \quad W = V$

Then $U = ZW \quad V = W$

$0 < z < 1$(since $u < v$) $\quad 0 < w < \theta$
$$J = \begin{bmatrix} w & 0 \\ z & 1 \end{bmatrix} = |w|$$
$$f_{Z,W}(z, w) = \frac{n(n-1)}{\theta^n}(w - zw)^{n-2}|w|$$
$$= \frac{n(n-1)}{\theta^n}w^{n-2}(1-z)^{n-2}w$$
$$f_{Z,W}(z, w) = \frac{n(n-1)}{\theta^n}w^{n-1}(1-z)^{n-2} \quad 0 < z < 1,\ 0 < w < \theta$$
Since f_{ZW} can be factored into $(f_Z)(f_W)$ they are independent

Thus $\frac{X_{(1)}}{X_{(n)}}$ and $X_{(n)}$ are independent random variables

Problem 3

$$f_X(x) = \frac{a}{\theta^a} x^{a-1} \quad 0 < x < \theta$$
$$F_X(x) = \frac{1}{\theta^a} x^{a} \quad 0 < x < \theta$$
$$f_{X_{(1)},\ldots,X_{(n)}}(u_1,\ldots,u_n) = n! \prod_{i=1}^{n} \frac{a}{\theta^a} u_i^{a-1} \text{ for } u_1 < \cdots < u_n < \theta$$
$$= \frac{n! a^n}{\theta^{an}} u_1^{a-1} \cdots u_n^{a-1} \text{ for } u_1 < \cdots < u_n < \theta$$
$$\text{Let } Y_1 = \frac{X_{(1)}}{X_{(2)}} \quad Y_2 = \frac{X_{(2)}}{X_{(3)}}$$
$$\cdots\cdots$$
$$Y_{n-1} = \frac{X_{(n-1)}}{X_{(n)}} \quad Y_n = X_{(n)}$$
$$\text{Then } X_{(1)} = Y_1 Y_2 \cdots Y_n \quad X_{(2)} = Y_2 \cdots Y_n$$
$$\cdots\cdots$$
$$X_{(n-1)} = Y_{n-1} Y_n \quad X_{(n)} = Y_n$$
$$J = \begin{bmatrix} \frac{dx_1}{dy_1} & \cdots & \cdots & \frac{dx_n}{dy_1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{dx_1}{dy_n} & \cdots & \cdots & \frac{dx_n}{dy_n} \end{bmatrix}$$

J is a lower triangular matrix, thus its determinant is the product of its diagonal values

$$\begin{bmatrix} y_2 \cdots y_n & 0 & 0 & 0 \\ y_1 y_3 \cdots y_n 6 & y_3 \cdots y_n & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ & & & \end{bmatrix}$$
$$J = |y_2 y_3^2 \cdots y_n^{n-1}|$$
$$f(y_1,\ldots,y_n) = \frac{n! a^n}{\theta^{an}} (y_1 \cdots y_n)^{a-1} (y_2 \cdots y_n)^{a-1} \cdots (y_n)^{a-1} (y_2 y_3^2 \cdots y_n^{n-1})$$
$$= \frac{n! a^n}{\theta^{an}} y_1^{a-1} y_2^{2a-1} \cdots y_n^{na-1} \quad 0 < y_i < 1 \quad i = 1,\ldots,n-1 \quad 0 < y_n < \theta$$

Since the joint pdf can be factored, all of the y's are mutually independent

The marginal distributions of Y_1 is obtained by

integrating out all the other ys and solving for the constant of integration

$$f_{Y_1}(y_1) = c_1 y_1^{a-1} \quad 0 < y_1 < 1$$
$$\int_0^1 c_1 y_1^{a-1}\, dy_1 = 1$$
$$= \frac{c_1}{a}\Big|_0^1 y_1^a = 1$$
$$= \frac{c_1}{a} = 1$$
$$c_1 = a$$
$$f_{Y_1}(y_1) = a y_1^{a-1} \quad 0 < y_1 < 1$$
The result holds for $f_{Y_i}(y_1)$ for $i = 1, 2, \ldots, n-1$
$$f_{Y_i}(y_i) = i a y_i^{ia-1} \quad 0 < y_i < 1$$
$$f_{Y_n}(y_n) = \frac{na}{\theta^{an}} y_n^{na-1} \quad 0 < y_n < \theta$$

Problem 4

(a)

$$f_X(x) = \alpha_i e^{-\alpha_i x} \quad x > 0 \ \alpha_i > 0 \ i = 1, \ldots, n$$
$$F_{X_{(1)}}(x) = P(X_{(1)} \leq x)$$
$$= 1 - P(X_{(1)} \geq x)$$
$$1 - P(X_1 \geq x)P(X_2 \geq x) \cdots P(X_n \geq x)$$
$$= 1 - e^{-\alpha_1 x} e^{-\alpha_2 x} \cdots e^{\alpha_n x}$$
$$F_{X_{(1)}}(x) = 1 - e^{-(\sum_{i=1}^n \alpha_i)x}$$
$$f_{X_{(1)}}(x) = \left(\sum_{i=1}^n \alpha_i\right) e^{-(\sum_{i=1}^n \alpha_i)x}$$
$$X_{(1)} \sim Exp\left(\sum_{i=1}^n \alpha_i\right)$$

(b)

$$f_{X_k}(x_k) = \alpha_k e^{-\alpha_k x}$$
$$P(X_{(1)} = X_k) = \int_0^\infty P(X_{(1)} = X_k, X_k = x_k) \mathrm{d}x_k$$

$$\int_0^\infty P(X_1 > X_k, X_2 > X_k, \ldots, X_n > X_k, X_k = x_k)\mathrm{d}x_k$$
$$\int_0^\infty P(X_1 > X_k, X_2 > X_k, \ldots, X_n > X_k | X_k = x_k) f_{X_k}(x_k)\mathrm{d}x_k$$
$$= \int_0^\infty \prod_{i=1}^n P(X_i > k) f_{X_k}(x)\mathrm{d}x \quad i \neq k$$
$$= \int_0^\infty \prod_{i=1}^n e^{-\alpha_i x}\alpha_k e^{-\alpha_k x}\mathrm{d}x \quad i \neq k$$
$$= \alpha_k \int_0^\infty e^{-(\sum_{i=1}^n \alpha_i)x} e^{-\alpha_k x}\mathrm{d}x \quad i \neq k$$
$$= \alpha_k \int_0^\infty e^{-(\sum_{i=1}^n \alpha_i)x}\mathrm{d}x$$
$$= \frac{a_k}{-\sum_{i=1}^n \alpha_i}\Big|_0^\infty e^{-(\sum_{i=1}^n \alpha_i)x}$$
$$\frac{a_k}{-\sum_{i=1}^n \alpha_i}(e^{-\infty} - 1)$$
$$P(X_{(1)} = X_k) = \frac{a_k}{\sum_{i=1}^n \alpha_i} \quad k \geq 1$$

Problem 5

(a)

$$X_{(1)} = min_i X_i \quad X_{(n)} = max_i X_i$$
$$f_X(x) = 1 \quad 0 < x < 1$$
$$F_X(x) = x$$
$$\text{Let } W = X_{(1)} \quad Z = X_{(n)}$$
$$f_{X_{(1)},X_{(n)}}(w,z) = \frac{n!}{(n-2)!} w^0 \{z - w\}^{n-2} 1 - z^0$$
$$f_{X_{(1)},X_{(n)}}(w,z)4 = n(n-1)(z-w)^{n-2} \quad 0 < w < z < 1$$
$$U = W \quad V = 1 - Z$$
$$W = U \quad Z = 1 - V$$
$$v < 1 - u$$
$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |-1| = 1$$
$$f_{U,V}(u,v) = n(n-1)([1-v] - u)^{n-2} * 1$$
$$f_{U,V}(u,v) = n(n-1)(1 - v - u)^{n-2} \quad 0 < v < 1 - u < 1$$

(b)

$$P(R > r, S > s) = P(nu > r, nv > s)$$
$$= P(u > r/n, v > s/n)$$
$$\text{Since } v < 1 - u \quad u < 1 - v$$
$$\text{Thus } r/n < u < 1 - v \quad s/n < v < 1 - r/n$$
$$= \int_{s/n}^{1-r/n} \int_{r/n}^{1-v} f_{UV}(uv)\ du\ dv$$
$$= \int_{s/n}^{1-r/n} \int_{r/n}^{1-v} n(n-1)(1-v-u)^{n-2}\ du\ dv$$
$$= (1 - r/n - s/n)^n$$

(c)

$$\lim_{n\to\infty} P(R > r, S > s) = \lim_{n\to\infty} (1 - r/n - s/n)^n$$
$$= \lim_{n\to\infty} (1 + -(r+s)/n)^n$$
$$\text{Let } x = -(r+s)$$
$$= (1 + x/n)^n$$
$$\text{Since } \lim_{n\to\infty} (1 + x/n)^n = e^x \text{ we have:}$$
$$= e^{-(r+s)}$$
$$= e^{-r}e^{-s}$$
$$\text{Thus } R \perp S$$

R and S are asymptotically independent

(d)

Since the asymptotic distributions of R and S are:

$$e^{-r} \text{ and } e^{-s}$$
$$R \sim Exp(1) \quad S \sim Exp(1)$$