Problem 1

(a)

$$E(X^2) = Var(X) + [E(X)]^2 = \sigma^2 + \mu^2 = \sigma^2 + 0$$
$$E(X^2) = \sigma^2$$
$$\text{Thus } E(X^2) \text{ is an unbiased estimator of } \sigma^2$$

(b)

$$L(\sigma|x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$
$$\ell(\sigma|x) = (-1/2)\log(2\pi) - \log(\sigma) - \frac{x^2}{2\sigma^2}$$
$$\propto -\log(\sigma) - \frac{x^2}{2\sigma^2}$$
$$\frac{\partial \ell(\sigma)}{\partial \sigma} = \frac{-1}{\sigma} + \frac{x^2}{\sigma^3} = 0$$
$$\sigma^2 = x^2$$
$$\hat{\sigma} = \sqrt{x^2} = |x|$$
$$\frac{\partial \ell(\sigma^2)}{\partial \sigma^2} = \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4}$$
$$\frac{\partial \ell(\sigma^2)}{\partial \sigma^2}(\hat{\sigma}) = \frac{1}{|x|^2} - \frac{3x^2}{|x|^4} = \frac{1}{x^2} - \frac{3}{x^2} = \frac{-2}{x^2} < 0$$
$$\text{Therefore } \hat{\sigma} \text{ is the MLE}$$

(c)

$$M_1 = 0 \quad M_2 = \sigma^2$$
$$E(X^2) = \sigma^2 = \frac{1}{n}\sum_{i=1}^{1} X_i^2 = X^2$$
$$\sigma^2 = X^2$$
$$\sqrt{\sigma^2} = \sqrt{X^2}$$
$$\hat{\sigma} = |X|$$

Problem 2

(a)

$$L(\theta = 0|x) = \prod_{i=1}^{n} 1 = 1 \quad 0 < x_i < 1$$
$$L(\theta = 1|x) = \prod_{i=1}^{n} (1/2)x_i^{-1/2} = (1/2)^n \prod_{i=1}^{n} x_i^{-1/2} \quad 0 < x_i < 1$$
$$MLE = 0 \text{ if } L(\theta = 0|x) \geq L(\theta = 1|x)$$
$$\text{Which is the same as } 1 \geq (1/2)^n \prod_{i=1}^{n} x_i^{-1/2}$$
$$MLE = 1 \text{ if } L(\theta = 0|x) < L(\theta = 1|x)$$
$$\text{Which is the same as } 1 < (1/2)^n \prod_{i=1}^{n} x_i^{-1/2}$$

(b)

$$\text{Given } n = 10 \quad \sum_{i=1}^{n} \log(x_i) = -10.7$$
$$L(1|x) = (1/2)^{10} \prod_{i=1}^{10} x_i^{-1/2}$$
$$= (1/2)^{10} \exp\left(\sum_{i=1}^{10} \log(x_i^{-1/2})\right) \text{ since } \prod_{i=1}^{n} a_n = \exp\left(\sum_{i=1}^{n} \log(a_n)\right)$$
$$= (1/2)^{10} \exp\left((-1/2)\sum_{i=1}^{10} \log(x_i)\right)$$
$$= (1/2)^{10} \exp[(-1/2)(-10.7)]$$
$$= \frac{1}{1024}e^{5.35} \approx .20567$$
$$L(1|x) \approx .20567 \quad 0 < x_i < 1$$
$$L(0|x) = 1 \quad 0 < x_i < 1$$
$$\text{Since } 1 > .20567 \text{ the MLE of } \theta = 0$$

Problem 3

(a)

$$X_1, \ldots, X_n \sim$$
$$P(X_i \leq x|\alpha, \beta) = \begin{cases} 0 & x < 0 \\ (x/\beta)^\alpha & 0 \leq x \leq \beta \\ 1 & x > \beta \end{cases}$$
$$\alpha, \beta > 0$$
$$L(\alpha, \beta|x) = f(x|\theta) = \prod_{i=1}^{n} \frac{\alpha}{\beta^\alpha}(x_i)^{\alpha-1} \ I[0, \beta](x_i)$$
$$= \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \ I(x_{(n)} < \beta) I(x_{(1)} > 0)$$
$$f(\boldsymbol{x}|\theta) = g(T(\boldsymbol{x})|\theta)h(\boldsymbol{x}) \text{ where}$$
$$h(\boldsymbol{x}) = I(x_{(1)} > 0)$$
$$g(T(\boldsymbol{x})|\theta) = \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \ I(x_{(n)} < \beta)$$
$$T(\boldsymbol{x}) = \prod_{i=1}^{n} x_i, x_{(n)}$$
$$\text{Thus by the factorization theorem, } \prod_{i=1}^{n} x_i, x_{(n)} \text{ are sufficient}$$

(b)

$$L(\alpha, \beta|x) = \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}$$
$$\text{Fixing } \alpha L(\alpha, \beta|x) = 0 \text{ if } \beta < x_{(n)}$$
$$L(\alpha, \beta|x) \text{ is a decreasing function of } \beta \text{ if } \beta \geq x_{(n)}$$
$$\text{Therefore } \hat{\beta} = x_{(n)} \text{ is the MLE of } \beta$$
$$\ell(\alpha, \beta|x) = \log\left(\left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}\right)$$
$$= n\log\alpha - \alpha n\log(\beta) + (\alpha - 1)\sum_{i=1}^{n} \log(x_i)$$

$$\frac{\partial \ell(\alpha)}{\partial \alpha} = \frac{n}{\alpha} - n\log(\beta) + \sum_{i=1}^{n} \log(x_i) = 0$$

$$\text{Plug in } \hat{\beta} = x_{(n)}$$

$$\frac{n}{\alpha} = n\log(x_{(n)}) - \sum_{i=1}^{n} \log(x_i)$$

$$\hat{\alpha} = \frac{n}{n\log(x_{(n)}) - \sum_{i=1}^{n} \log(x_i)}$$

$$\hat{\alpha} = \left[\frac{n\log(x_{(n)}) - \sum_{i=1}^{n} \log(x_i)}{n}\right]^{-1}$$

$$\hat{\alpha} = \left[\frac{1}{n}\sum_{i=1}^{n}(\log(x_{(n)}) - \log(x_i))\right]^{-1}$$

$$\frac{\partial \ell(\alpha^2)}{\partial \alpha^2} = \frac{n}{\alpha} - n\log(x_{(n)}) + \sum_{i=1}^{n} \log(x_i)$$

$$= \frac{-n}{\alpha^2} < 0$$

$$\text{Thus } \hat{\alpha} \text{ is the MLE}$$

(c)

$$\hat{\beta}_{MLE} = X_{(n)} = 25$$

$$\sum_{i=1}^{n} \log(x_i) = 40.81287$$

$$\hat{\alpha}_{MLE} = \frac{13}{13\log(25) - 40.81287} = 12.59055$$

Problem 4

(a)

$$f_y(y) = \frac{1}{\lambda}e^{-y/\lambda} \quad f_z(z) = \frac{1}{\mu}e^{-z/\mu}$$

Since $Y \perp Z$ we can solve for the MLE individually

$$L(\lambda|x) = \prod_{i=1}^{n} \frac{1}{\lambda}e^{-y_i/\lambda} = (1/\lambda)^n e^{-(1/\lambda)\sum_{i=1}^n y_i}$$

$$\ell(\lambda|x) = -n\log(\lambda) - (1/\lambda)\sum_{i=1}^{n} y_i$$

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{-n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n} y_i = 0$$

$$\frac{n}{\lambda} = \frac{1}{\lambda^2}\sum_{i=1}^{n} y_i = 0$$

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{Y}$$

$$\frac{\partial \ell(\lambda^2)}{\partial \lambda^2}(\hat{\lambda}) = \frac{n}{(\frac{1}{n}\sum_{i=1}^n y_i)^2} + \frac{-2}{(\frac{1}{n}\sum_{i=1}^n y_i)^3}\sum_{i=1}^{n} y_i = \frac{n}{(\frac{1}{n}\sum_{i=1}^n y_i)^2} + \frac{-2n}{(\frac{1}{n}\sum_{i=1}^n y_i)^2}$$

$$= \frac{-n}{(\frac{1}{n}\sum_{i=1}^n y_i)^2} < 0$$

Thus $\hat{\lambda}$ is the MLE

Same for $\hat{\mu}_{MLE}$ thus:

$$\hat{\lambda}_{MLE} = \bar{Y} \quad \hat{\mu}_{MLE} = \bar{Z}$$

(b)

$$X_i = \min(Y_i, Z_i)$$

$$\delta_i = \begin{cases} 1, \text{ if } X_i = Y_i \\ 0, \text{ if } X_i = Z_i \end{cases}$$

$$f_{X,\delta}(x,\delta) = \left(\frac{1}{\lambda}\right)^{\delta} \left(\frac{1}{\mu}\right)^{1-\delta} e^{-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)x}$$

$$L(\lambda,\mu|x,\delta)=\prod_{i=1}^{n}\left(\frac{1}{\lambda}\right)^{\delta_i}\left(\frac{1}{\mu}\right)^{1-\delta_i}e^{-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)x_i}$$

$$=\left(\frac{1}{\lambda}\right)^{\sum_{i=1}^{n}\delta_i}\left(\frac{1}{\mu}\right)^{n-\sum_{i=1}^{n}\delta_i}e^{-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\sum_{i=1}^{n}x_i}$$

$$\ell(\lambda,\mu|x,\delta)=-\sum_{i=1}^{n}\delta_i\log(\lambda)-(n-\sum_{i=1}^{n}\delta_i)\log(\mu)-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\sum_{i=1}^{n}x_i$$

$$\frac{\partial\ell(\lambda,\mu)}{\partial\lambda}=-\frac{1}{\lambda}\sum_{i=1}^{n}\delta_i+\frac{1}{\lambda^2}\sum_{i=1}^{n}x_i=0$$

$$\frac{1}{\lambda}\sum_{i=1}^{n}\delta_i=\frac{1}{\lambda^2}\sum_{i=1}^{n}x_i$$

$$\hat{\lambda}=\frac{\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}\delta_i}$$

$$\frac{\partial\ell^2(\lambda,\mu)}{\partial\lambda^2}(\hat{\lambda})=\frac{1}{\left[\frac{\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}\delta_i}\right]^2}\sum_{i=1}^{n}\delta_i+\frac{-2}{\left[\frac{\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}\delta_i}\right]^3}\sum_{i=1}^{n}x_i$$

$$=\frac{\left(\sum_{i=1}^{n}\delta_i\right)^3}{\left(\sum_{i=1}^{n}x_i\right)^2}-2\frac{\left(\sum_{i=1}^{n}\delta_i\right)^3}{\left(\sum_{i=1}^{n}x_i\right)^2}=-\frac{\left(\sum_{i=1}^{n}\delta_i\right)^3}{\left(\sum_{i=1}^{n}x_i\right)^2}<0$$

Thus $\hat{\lambda}$ is the MLE

$$\frac{\partial\ell(\lambda,\mu)}{\partial\mu}=-(n-\sum_{i=1}^{n}\delta_i)\frac{1}{\mu}+\frac{1}{\mu^2}\sum_{i=1}^{n}x_i=0$$

$$(n-\sum_{i=1}^{n}\delta_i)\frac{1}{\mu}=\frac{1}{\mu^2}\sum_{i=1}^{n}x_i$$

$$\hat{\mu}=\frac{\sum_{i=1}^{n}x_i}{n-\sum_{i=1}^{n}\delta_i}$$

$$\frac{\partial\ell^2(\lambda,\mu)}{\partial\mu^2}(\hat{\mu})=(n-\sum_{i=1}^{n}\delta_i)\left(\frac{n-\sum_{i=1}^{n}\delta_i}{\sum_{i=1}^{n}x_i}\right)^2-2\left(\frac{n-\sum_{i=1}^{n}\delta_i}{\sum_{i=1}^{n}x_i}\right)^3\sum_{i=1}^{n}x_i$$

$$=\frac{\left(n-\sum_{i=1}^{n}\delta_i\right)^3}{\left(\sum_{i=1}^{n}x_i\right)^2}-2\frac{\left(n-\sum_{i=1}^{n}\delta_i\right)^3}{\left(\sum_{i=1}^{n}x_i\right)^2}=-\frac{\left(n-\sum_{i=1}^{n}\delta_i\right)^3}{\left(\sum_{i=1}^{n}x_i\right)^2}<0$$

Thus $\hat{\mu}$ is the MLE

Problem 5

(a)

$$
\begin{gathered}
X_1,\ldots,X_n \sim N(\mu_1) \quad \mu \text{ unknown} \\
Y_1,\ldots,Y_n \text{ where } Y_i = I(X_i > 0) \\
Y_i = \begin{cases} 1 \text{ if } X_i > 0 \\ 0 \text{ if } X_i < 0 \end{cases} \\
Y_i \sim Ber(P(Y_i = 1)) \\
\text{Let } \tau = P(Y_i = 1) = P(X_i > 0) \\
Y_i \sim Ber(\tau) \\
L(\tau|y) = \prod_{i=1}^{n} \tau^{y_i}(1-\tau)^{1-y_i}
\end{gathered}
$$

(b)

$$
\begin{aligned}
L(\tau|y) &= \prod_{i=1}^{n} \tau^{y_i}(1-\tau)^{1-y_i} \\
&= \tau^{\sum_{i=1}^n y_i}(1-\tau)^{\sum_{i=1}^n 1-y_i} \\
\ell(\tau|y) &= \sum_{i=1}^{n} y_i \log(\tau) + (\sum_{i=1}^{n}(1-y_i))\log(1-\tau) \\
\frac{\partial \ell(\tau)}{\partial \tau} &= \frac{\sum_{i=1}^n y_i}{\tau} - \frac{\sum_{i=1}^n (1-y_i)}{1-\tau} = 0 \\
\frac{\sum_{i=1}^n y_i}{\tau} &= \frac{\sum_{i=1}^n (1-y_i)}{1-\tau} \\
\frac{1-\tau}{\tau} &= \frac{\sum_{i=1}^n (1-y_i)}{\sum_{i=1}^n y_i} \\
\frac{1}{\tau} &= \frac{\sum_{i=1}^n (1-y_i)}{\sum_{i=1}^n y_i} + 1 \\
\hat{\tau} &= \frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n (1-y_i) + \sum_{i=1}^n y_i} \\
&= \frac{\sum_{i=1}^n y_i}{n - \sum_{i=1}^n (y_i) + \sum_{i=1}^n y_i} \\
\hat{\tau} &= \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{Y}
\end{aligned}
$$

$$\frac{\partial \ell(\tau^2)}{\partial \tau^2} = \frac{-\sum_{i=1}^n y_i}{\tau^2} - \frac{n - \sum_{i=1}^n y_i}{(1-\tau)^2}$$

Plugging in $\hat{\tau} = \bar{y}$

$$= -\left(\frac{n\bar{y}}{\bar{y}^2} + \frac{n(1-\bar{y})}{(1-\bar{y})^2} \right)$$

$$= -n \left(\frac{1}{\bar{y}} + \frac{1}{1-\bar{y}} \right) < 0$$

Since the term inside the parenthesis is positive because $\bar{y} \leq 1$

Thus $\hat{\tau}$ is the MLE

(c)

$$\tau = P(Y_i = 1) = P(X_i > 0)$$

$$\tau = P\left(\frac{X_i - \mu}{\sqrt{1}} > \frac{0-\mu}{\sqrt{1}} \right) \sim N(0,1)$$

$$\tau = 1 - \Phi(-\mu)$$

$$\mu = -\Phi^{-1}(1-\tau)$$

$$\hat{\mu}_{MLE} = -\Phi^{-1}(1 - \hat{\tau}_{MLE}) = -\Phi^{-1}(1 - \bar{Y})$$