Problem 1

Using the central limit theorem $\bar{X}_1$ and $\bar{X}_2$ are approximatley normally distributed

$$\bar{X}_1 \sim N(\mu, \sigma^2/n)$$
$$\bar{X}_2 \sim N(\mu, \sigma^2/n)$$
$$\text{Since } \bar{X}_1 \text{ and } \bar{X}_2 \text{ are independent:}$$
$$\bar{X}_1 - \bar{X}_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$
$$\bar{X}_1 - \bar{X}_2 \sim N(0, 2\sigma^2/n)$$
$$\text{Let } Z = \frac{[\bar{X}_1 - \bar{X}_2]\sqrt{n}}{\sqrt{2}\sigma}$$
$$Z \sim N(0,1)$$
$$\text{We want to find n such that:}$$
$$P(|\bar{X}_1 - \bar{X}_2| < \sigma/5) \approx .99$$
$$P(|\bar{X}_1 - \bar{X}_2| < \sigma/5) = P(-\sigma/5 < \bar{X}_1 - \bar{X}_2 < \sigma/5)$$
$$= P(\frac{-\sqrt{n}\sigma}{5\sqrt{2}\sigma} < Z < \frac{\sqrt{n}\sigma}{5\sqrt{2}\sigma})$$
$$= P(\frac{-\sqrt{n}}{5\sqrt{2}} < Z < \frac{\sqrt{n}}{5\sqrt{2}})$$
$$.99 \approx P(\frac{-\sqrt{n}}{5\sqrt{2}} < Z < \frac{\sqrt{n}}{5\sqrt{2}})$$
$$.99 \approx P(Z < \frac{\sqrt{n}}{5\sqrt{2}}) - P(Z > -\frac{\sqrt{n}}{5\sqrt{2}})$$
$$.99 \approx P(Z < \frac{\sqrt{n}}{5\sqrt{2}}) - (1 - P(Z < \frac{\sqrt{n}}{5\sqrt{2}}))$$
$$.99 \approx 2P(Z < \frac{\sqrt{n}}{5\sqrt{2}}) - 1$$
$$P(Z < \frac{\sqrt{n}}{5\sqrt{2}}) \approx .995$$
$$qnorm(.995) = 2.575829 \quad \text{(using R)}$$
$$2.575829 = \frac{\sqrt{n}}{5\sqrt{2}}$$
$$n = [2.575829(5\sqrt{2})]^2 = 50 * 2.575829^2 = 331.7448$$
$$n = 332$$

Problem 2

(a)

$$\text{Given } X_n \xrightarrow{p} a$$
$$\forall\ \epsilon > 0 \quad \lim_{n\to\infty} P[|X_n - a| > \epsilon] = 0$$
$$\text{WTS: } Y_i = \sqrt{X_i} \text{ converges in probability}$$
$$\text{That is: } \lim_{n\to\infty} P(|\sqrt{X_n} - \sqrt{a} > \epsilon) = 0$$
$$P(|\sqrt{X_n} - \sqrt{a} > \epsilon) = P(|\sqrt{X_n} - \sqrt{a}||\sqrt{X_n} + \sqrt{a}| > \epsilon|\sqrt{X_n} + \sqrt{a}|)$$
$$= P(|X_n - a| > \epsilon|\sqrt{X_n} + \sqrt{a}|)$$
$$\leq P(|X_n - a| > \epsilon\sqrt{a})$$
$$P(|\sqrt{X_n} - \sqrt{a} > \epsilon) \leq P(|X_n - a| > \epsilon\sqrt{a})$$
$$\text{Since } \forall\ \epsilon > 0 \quad \lim_{n\to\infty} P[|X_n - a| > \epsilon] = 0$$
$$\text{and } \epsilon \leq \sqrt{a}\epsilon$$
$$\text{We have } \lim_{n\to\infty} P(|X_n - a| > \epsilon\sqrt{a}) = 0$$
$$\text{Which means } \lim_{n\to\infty} P(|\sqrt{X_n} - \sqrt{a} > \epsilon) \leq \lim_{n\to\infty} P(|X_n - a| > \epsilon\sqrt{a}) = 0$$
$$\text{Thus } \lim_{n\to\infty} P(|\sqrt{X_n} - \sqrt{a} > \epsilon) = 0$$
$$\text{Therefore } \sqrt{X_n} \xrightarrow{p} \sqrt{a}$$

WTS: $Y_i^{'} = a/X_i$ converges in probability

$$\text{That is: } \lim_{n\to\infty} P(|a/X_n - 1| \leq \epsilon) = 1$$
$$P(|a/X_n - 1| \leq \epsilon) = P(-\epsilon \leq a/X_n - 1 \leq \epsilon)$$
$$= P\left(\frac{1-\epsilon}{a} \leq 1/X_n \leq \frac{1+\epsilon}{a}\right)$$
$$= P\left(\frac{a}{1+\epsilon} \leq X_n \leq \frac{a}{1-\epsilon}\right)$$
$$= P\left(\frac{a + a\epsilon - a\epsilon}{1+\epsilon} \leq X_n \leq \frac{a - a\epsilon + a\epsilon}{1-\epsilon}\right)$$
$$= P\left(\frac{a(1+\epsilon) - a\epsilon}{1+\epsilon} \leq X_n \leq \frac{a(1-\epsilon) + a\epsilon}{1-\epsilon}\right)$$
$$= P\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1-\epsilon}\right)$$

$$\geq P\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1+\epsilon}\right) \text{ Since } a + \frac{a\epsilon}{1+\epsilon} < a + \frac{a\epsilon}{1-\epsilon}$$
$$= P\left(-\frac{a\epsilon}{1+\epsilon} \leq X_n - a \leq \frac{a\epsilon}{1+\epsilon}\right)$$
$$= P\left(|X_n - a| \leq \frac{a\epsilon}{1+\epsilon}\right)$$
$$P(|a/X_n - 1| \leq \epsilon) \geq P\left(|X_n - a| \leq \frac{a\epsilon}{1+\epsilon}\right)$$
$$\text{Since } \lim_{n\to\infty} P\left(|X_n - a| \leq \epsilon\right) = 1$$
$$\text{We have } \lim_{n\to\infty} P\left(|X_n - a| \leq \frac{a\epsilon}{1+\epsilon}\right) = 1$$
$$\text{Therefore } \lim_{n\to\infty} P(|a/X_n - 1| \leq \epsilon) = 1$$
$$\text{Thus } a/X_n \xrightarrow{p} 1$$

(b)

$$\text{WTS: } \frac{\sigma}{S_n} \xrightarrow{p} 1$$
$$\text{Using the result from the first part of a:}$$
$$X_n \xrightarrow{p} a \text{ means that } \sqrt{X_n} \xrightarrow{p} \sqrt{a}$$
$$S_n = \sqrt{S_n^2}$$
$$\sqrt{S_n^2} \xrightarrow{p} \sqrt{\sigma^2}$$
$$S_n \xrightarrow{p} \sigma$$
$$\text{Using the result from the second part of a:}$$
$$X_n \xrightarrow{p} a \text{ means that } a/X_n \xrightarrow{p} 1$$
$$\text{Since } S_n \xrightarrow{p} \sigma$$
$$\frac{\sigma}{S_n} \xrightarrow{p} 1$$

Problem 3

$$f_Y(y) = \theta\gamma^\theta y^{-(\theta+1)} \quad 0 < \gamma < y < \infty,\ 2 < \theta < \infty$$
$$F_Y(y) = 1 - \left(\frac{\gamma}{y}\right)^\theta$$
$$f_{Y_{(1)}}(y) = n(\theta\gamma^\theta y^{-(\theta+1)}) \left[1 - \left(1 - \left(\frac{\gamma}{y}\right)^\theta\right)\right]^{n-1}$$
$$f_{Y_{(1)}}(y) = n\theta\gamma^{\theta n} y^{-(\theta n+1)} \quad 0 < \gamma < y < \infty,\ 2 < \theta < \infty$$
$$EY_{(1)} = \int_\gamma^\infty y(n\theta\gamma^{\theta n} y^{-(\theta n+1)})\ dy$$
$$= n\theta\gamma^{\theta n} \int_\gamma^\infty y^{-\theta n}\ dy$$
$$= n\theta\gamma^{\theta n} \Big|_\gamma^\infty \frac{1}{-\theta n + 1} y^{-\theta n+1}$$
$$= \frac{n\theta\gamma^{\theta n}}{-\theta n + 1}(0 - \gamma^{-\theta n+1})$$
$$EY_{(1)} = \frac{-n\theta\gamma}{-\theta n + 1}$$
$$\lim_{n\to\infty} EY_{(1)} = \lim_{n\to\infty} \frac{-n\theta\gamma}{-\theta n + 1}$$
$$= \lim_{n\to\infty} \frac{-\theta\gamma}{-\theta + 1/n}$$
$$= \frac{-\theta\gamma}{-\theta + 0} = \gamma$$
$$\lim_{n\to\infty} EY_{(1)} = \gamma$$
$$Var(Y_{(1)}) = E(Y_{(1)}^2) - (EY_{(1)})^2$$
$$E(Y_{(1)}^2) = \int_\gamma^\infty y^2(n\theta\gamma^{\theta n} y^{-(\theta n+1)})\ dy$$
$$= n\theta\gamma^{\theta n} \int_\gamma^\infty y^{-\theta n+1}\ dy$$
$$= n\theta\gamma^{\theta n} \Big|_\gamma^\infty \frac{1}{-\theta n + 2} y^{-\theta n+2}$$
$$= \frac{n\theta\gamma^{\theta n}}{-\theta n + 2}(0 - \gamma^{-\theta n+2})$$
$$E(Y_{(1)}^2) = \frac{-n\theta\gamma^2}{-\theta n + 2}$$

$$\lim_{n\to\infty} Var(Y_{(1)}) = \lim_{n\to\infty} E(Y_{(1)}^2) - \lim_{n\to\infty} (EY_{(1)})^2$$
$$\lim_{n\to\infty} E(Y_{(1)}^2) = \lim_{n\to\infty} \frac{-n\theta\gamma^2}{-\theta n+2}$$
$$= \lim_{n\to\infty} \frac{-\theta\gamma^2}{-\theta+(2/n)}$$
$$= \frac{-\theta\gamma^2}{-\theta+0} = \gamma^2$$
$$\lim_{n\to\infty} E(Y_{(1)}^2) = \gamma^2$$
$$\lim_{n\to\infty} E(Y_{(1)})^2 = \lim_{n\to\infty} \left(\frac{-n\theta\gamma}{-\theta n+1}\right)^2$$
$$= \lim_{n\to\infty} \frac{n^2\theta^2\gamma^2}{\theta^2 n^2 - 2\theta^2 n + 1}$$
$$= \lim_{n\to\infty} \frac{\theta^2\gamma^2}{\theta^2 - 2\theta^2(1/n) + 1/n^2}$$
$$= \frac{\theta^2\gamma^2}{\theta^2 - 0 + 0} = \gamma^2$$
$$\lim_{n\to\infty} E(Y_{(1)})^2 = \gamma^2$$
$$\lim_{n\to\infty} Var(Y_{(1)}) = \gamma^2 - \gamma^2 = 0$$

Thus $Y_{(1)}$ is a consistent estimator of γ

$$Y_{(1)} \xrightarrow{p} \gamma$$

Problem 4

(a)

$$X_1, \ldots, X_n \text{are iid random variables } \sim Pois(\mu) \quad \mu > 0$$
$$E(X_i) = \mu$$
$$\bar{X_n} = n^{-1}\sum_{i=1}^{n} X_i$$
$$\sqrt{n}(\bar{X_n} - EX_1) \xrightarrow{d} N(0, \sigma^2)$$

Since the mean and the variance of poisson rv are equal:

$$\sigma^2 = \mu \quad \sigma = \sqrt{\mu}$$

Since we have iid random variables, a finite mean and variance:

$$\frac{\sqrt{n}(\bar{X_n} - \mu)}{\sqrt{\mu}} \xrightarrow{d} N(0,1)$$

(b)

$$T_n = \sqrt{n}(\bar{X_n} - \mu)$$
$$T_n \xrightarrow{d} N(0, \mu)$$
$$\text{asymptotic variance } = \mu$$

(c)

$$\text{Want to find } h(\bar{X_n}) \text{ such that } h(\bar{X_n})T_n \xrightarrow{d} N(0,1)$$
$$T_n = \sqrt{n}(\bar{X_n} - \mu) \xrightarrow{d} N(0, \mu)$$
$$\frac{\sqrt{n}(\bar{X_n} - \mu)}{\sqrt{\mu}} \xrightarrow{d} N(0,1) \text{ (CLT)}$$
$$\text{Since Xs are iid, } EX_i = \mu \text{, variance is finite:}$$
$$\bar{X_n} \xrightarrow{p} \mu \text{ (WLLN)}$$
$$\frac{\sqrt{\mu}}{\sqrt{\bar{X_n}}} \xrightarrow{p} 1$$
$$\frac{\sqrt{\mu}}{\sqrt{\bar{X_n}}} \frac{\sqrt{n}(\bar{X_n} - \mu)}{\sqrt{\mu}} \xrightarrow{d} N(0,1) \text{ (Slutsky's Thm)}$$
$$\frac{\sqrt{n}(\bar{X_n} - \mu)}{\sqrt{\bar{\bar{X_n}}}} \xrightarrow{d} N(0,1)$$
$$g(\bar{X_n}) \xrightarrow{p} g(\mu)$$
$$g(x) = \frac{1}{\sqrt{x}}$$
$$h(\bar{X_n}) = \frac{1}{\sqrt{\bar{X_n}}}$$

(d)

$$95\% \text{ CI for } \mu:$$
$$\text{Since } \frac{\bar{X} - \mu)}{\sqrt{\bar{X_n}}} \xrightarrow{d} N(0,1)$$

$$
\begin{aligned}
.95 &\approx P\left(-1.96 < \frac{\sqrt{n}(\bar{X}-\mu)}{\sqrt{\bar{X}_n}} < 1.96\right)\\
&= P\left(\bar{X} + \frac{1.96\sqrt{\bar{X}_n}}{\sqrt{n}} > \mu > \bar{X} - \frac{1.96\sqrt{\bar{X}_n}}{\sqrt{n}}\right)\\
&= P\left(\bar{X} - \frac{1.96\sqrt{\bar{X}_n}}{\sqrt{n}} < \mu < \bar{X} + \frac{1.96\sqrt{\bar{X}_n}}{\sqrt{n}}\right)\\
&\text{Since } \frac{\bar{X}-\mu)}{\sqrt{\mu}} \xrightarrow{d} N(0,1)\\
&\approx P\left(-1.96 < \frac{\bar{X}-\mu)}{\sqrt{\mu}} < 1.96\right)
\end{aligned}
$$

(e)

$$
\begin{aligned}
\sqrt{n}(g(\bar{X}_n) - g(\mu)) &\xrightarrow{d} N(0,1)\\
\sqrt{n}(g(\bar{X}_n) - g(\mu)) &\xrightarrow{d} N(0,[g^{'}(\mu)]^2\mu)\\
[g^{'}(\mu)]^2\mu &= 1\\
[g^{'}(\mu)]^2 &= \frac{1}{\mu}\\
g^{'}(\mu) &= \frac{1}{\sqrt{\mu}} = \mu^{-1/2}\\
g(\mu) &= 2\mu^{1/2}
\end{aligned}
$$

Problem 5

(a)

$$
\begin{aligned}
X_1,\ldots,X_n &\sim N(\mu,1)\\
U_n &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i-\mu)\\
&\text{Want to find limiting distribution for } U_n\\
\sqrt{n}(\bar{X}_n-\mu) &\xrightarrow{d} N(0,1) \text{ (CLT)}\\
&= \sqrt{n}\left(\left[1/n\sum_{i=1}^{n}X_i\right]-\mu\right)
\end{aligned}
$$

$$= \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} X_i \right) - \sqrt{n}\mu$$
$$= \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} (X_i - \mu) \right) = U_n$$
$$\text{Thus } U_n \xrightarrow{d} N(0,1)$$

(b)

$$X_1, \dots, X_n \sim N(\mu, 1)$$
$$Var(X_i) = 1$$
$$V_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$$
$$\text{WTS: } V_n \xrightarrow{p} 1$$
$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} E(X_1) \text{ (WLLN)}$$
$$\frac{1}{n} \sum_{i=1}^{n} X_i^2 \xrightarrow{p} E(X_i^2)$$
$$\text{If } Y_i = X_i^2 \text{ then } \frac{1}{n} \sum_{i=1}^{n} Y_i \xrightarrow{p} E(Y_i) = E(X_i^2)$$
$$\text{Let } Y_i = X_i - \mu$$
$$V_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2 = \frac{1}{n} \sum_{i=1}^{n} Y_i^2 \xrightarrow{p} E(Y_i^2) = E[(X_1 - \mu)^2] = Var(X_i) = 1$$
$$V_n \xrightarrow{p} 1$$

(c)

$$W_n = U_n / V_n$$
$$U_n \xrightarrow{d} N(0,1)$$
$$V_n \xrightarrow{p} 1$$
$$U_n / V_n \xrightarrow{d} N(0,1)/1 = N(0,1) \text{ (Slutsky's Thm)}$$

(d)

$$\sqrt{n}(\bar{X}^2 - \mu^2) \xrightarrow{d} ?$$
$$\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$$
$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N(0,1)$$
$$\sqrt{n}(g(\bar{X_n}) - g(\mu)) \xrightarrow{d} N(0, [g^{'}(\mu)]^2\sigma^2)$$
$$g(\mu) = \mu^2$$
$$g^{'}(\mu) = 2\mu$$
$$N(0, [g^{'}(\mu)]^2\sigma^2) = N(0, [2\mu]^2(1)) = N(0, 4\mu^2)$$
$$\sqrt{n}(\bar{X}^2 - \mu^2) \xrightarrow{d} N(0, 4\mu^2)$$

(e)

$$X_1, \ldots, X_n \sim N(\mu, 1)$$
$$\bar{X} \sim N(\mu, 1/n)$$
$$\text{Exact 95\% CI}$$
$$.95 = P\left(-1.96 \leq \frac{\bar{X} - \mu}{\sqrt{1/n}} \leq 1.96\right)$$
$$= P\left(\bar{X} - 1.96\frac{1}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{1}{\sqrt{n}}\right)$$
$$.95 = P\left(L^2 \leq \mu^2 \leq U^2\right)$$
$$= P\left(\left[\bar{X} - 1.96\frac{1}{\sqrt{n}}\right]^2 \leq \mu^2 \leq \left[\bar{X} + 1.96\frac{1}{\sqrt{n}}\right]^2\right)$$