BIOS663 Homework 1
Due Wednesday, Feb 6 in class

To report a test, provide H_0, the test statistic, the degrees of freedom, the p-value, the decision (accept vs. reject H_0), and an interpretation of the decision in terms of the subject matter.

1. (a) Prove or dis-prove (with details) that

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 8 & -6 \\ 4 & 1 & 7 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 8 & 0 \\ 4 & 1 & -2 \end{bmatrix}$$

   have linearly independent columns, respectively.

   (b) Find the eigenvalues and eigenvectors of

$$\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$$

2. Suppose

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim N(0, \mathbf{\Sigma}) \text{ where } \mathbf{\Sigma} = \begin{bmatrix} 2 & 0 & 0.6 \\ 0 & 2 & 0.5 \\ 0.6 & 0.5 & 1 \end{bmatrix}$$

   (a) Derive the distribution of $3x_1 + x_2 + x_3$.

   (b) Derive the distribution of $(x_1, x_2 \mid x_3 = 3)$.

   (c) Calculate $Cov(x_1 + 2x_2, 3x_2 + x_3)$.

3. Suppose $X_1, \ldots, X_k$ are multivariate normally distributed with $X_i \sim N_n(\mu_i, \Sigma_i)$, $i = 1, \ldots, k$. Further, let $Cov(X_i, X_j) = \Sigma_{ij} (i \neq j)$. Suppose $a_1, \ldots, a_k$ are scalars and define $Y = a_1 X_1 + \ldots + a_k X_k$. Find the distribution of Y.

4. *Weighted least squares* is a modification of standard regression analysis that may be used for a set of data when the assumption of variance homogeneity does not hold. (Assume the responses are independent.) If the ith response is an average of m_i equally variable observations, then $\text{Var}(y_i) = \frac{\sigma^2}{m_i}$. In this case, we have the model $\mathbf{y}_{n\times 1} = \mathbf{X}_{n\times p}\, \boldsymbol{\beta}_{p\times 1} + \boldsymbol{\varepsilon}_{n\times 1}$, where $E(\boldsymbol{\varepsilon}) = \mathbf{0}$, $\text{Cov}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{V}$, and

$$\mathbf{V} = \begin{bmatrix} \frac{1}{m_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{m_2} & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{1}{m_n} \end{bmatrix}.$$

   The fixed and known positive definite matrix $\mathbf{V}_{n\times n}$ has rank n. The weighted least squares estimator of $\boldsymbol{\beta}$ is given by

$$\widehat{\boldsymbol{\beta}}_W = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}.$$

(a) Derive the expectation of $\widehat{\boldsymbol{\beta}}_W$, $E[\widehat{\boldsymbol{\beta}}_W]$.

(b) Derive the covariance matrix of $\widehat{\boldsymbol{\beta}}_W$, $\text{Cov}(\widehat{\boldsymbol{\beta}}_W)$.

(c) Find the exact distribution of $\widehat{\boldsymbol{\beta}}_W$. If it is necessary to make any reasonable further assumptions in order to find the distribution of $\widehat{\boldsymbol{\beta}}_W$, provide them.

(d) Explain why this particular choice of $\mathbf{V}$ makes sense when our responses are averages.