Problem 1

(a)

$$\text{Reduced row echelon form of } \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus all of the columns of $\boldsymbol{A}$ are linearly independent since they are not multiples of each other.

$$\text{Reduced row echelon form of } \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus all of the columns of $\boldsymbol{B}$ are linearly independent since they are not multiples of each other.

(b)

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \left| \begin{bmatrix} 2-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix} \right| = 0$$

$$\lambda^2 - \lambda + 6 = 0$$

$$\text{eigenvalues } \lambda = 4.732 \text{ and } \lambda = 1.268$$

$$\lambda = 4.732 \text{ normalized eigenvector } = (-.3437, -.9391)^{'}$$

$$\lambda = 1.268 \text{ normalized eigenvector } = (-.8069, .5907)^{'}$$

Problem 2

(a)

$$\text{Let } Y = AX$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X \sim N(0, \Sigma)$$

$$Y \sim N(\mu^*, \Sigma^*)$$

$$\mu^* = A\mu = 0$$

$$\Sigma^* = A\sigma A^{'}$$

$$= \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & .6 \\ 0 & 2 & .5 \\ .6 & .5 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 25.6$$
$$Y \sim N(0, 25.6)$$

(b)

$$\text{Partition } \mu \text{ as } \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$
$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\mu^* = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_3 - \mu_2)$$
$$= 0 + \begin{bmatrix} .6 \\ .5 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} (3 - 0)$$
$$\mu^* = \begin{bmatrix} 1.8 \\ 1.5 \end{bmatrix}$$
$$\text{Partition } \Sigma \text{ as } \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$
$$\Sigma = \left[\begin{array}{cc|c} 2 & 0 & .6 \\ 0 & 2 & .5 \\ \hline .6 & .5 & 1 \end{array} \right]$$
$$\Sigma^* = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} .6 \\ .5 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} .6 & .5 \end{bmatrix}$$
$$\Sigma^* = \begin{bmatrix} 1.64 & -.3 \\ -.3 & 1.75 \end{bmatrix}$$
$$(x_1, x_2 | x_3 = 3) \sim N(\mu^*, \Sigma^*)$$

(c)

$$cov(ax+by, cw+dv) = ac\,cov(x,w) + ad\,cov(x,v) + bc\,cov(y,w) + bd\,cov(y,v)$$
$$cov((x_1 + 2x_2), 3x_2 + x_3)$$
$$= 3cov(x_1, x_2) + cov(x_1, x_3) + 6cov(x_2, x_2) + 2cov(x_2, x_3)$$
$$= 3(0) + .6 + 6(2) + 2(.5)$$
$$= 13.6$$

Problem 3

$$\mu^* = E(Y) = E(a_1x_1 + a_2x_2 + \dots a_kx_k)$$
$$= a_1E(x_1) + \cdots + a_kE(x_k)$$
$$\mu^* = \sum_{i=1}^{k} a_i\mu_i$$
$$\Sigma^* = Var(Y) = Var(a_1x_1 + a_2x_2 + \dots a_kx_k)$$
$$= a_1^2Var(x_1) + \cdots + a_k^2Var(x_k) + 2\left(\sum_{j>i}^{k}\sum_{i=1}^{k-1} a_ia_j(\Sigma_{i,j})\right)$$
$$\Sigma^* = \sum_{i=1}^{k} a_i^2\sigma_i^2 + 2\sum_{j>i}^{k}\sum_{i=1}^{k-1} a_ia_j(\Sigma_{i,j})$$
$$Y \sim N(\mu^*, \Sigma^*)$$

Problem 4

(a)

$$y = X\beta + \epsilon$$
$$E(y) = X\beta$$
$$E(\hat{\beta_w}) = E[(X^{'}V^{-1}X)^{-1}X^{'}V^{-1}y]$$
Since V is a matrix of constants and X is given we have:
$$= (X^{'}V^{-1}X)^{-1}X^{'}V^{-1}E(y)$$
$$= (X^{'}V^{-1}X)^{-1}X^{'}V^{-1}X\beta$$
$$= X^{-1}(V^{-1})^{-1}X^{'-1}X^{'}V^{-1}X\beta$$
$$= X^{-1}VIV^{-1}X\beta$$
$$= X^{-1}IX\beta$$
$$= I\beta$$
$$E(\hat{\beta_w}) = \beta$$

(b)

$$cov(y) = cov(X\beta + \epsilon) = cov(\epsilon) = \sigma^2 V$$
$$cov(\hat{\beta}_w) = cov[(X^{'}V^{-1}X)^{-1}X^{'}V^{-1}y]$$
$$= [(X^{'}V^{-1}X)^{-1}X^{'}V^{-1}]cov(y)[(X^{'}V^{-1}X)^{-1}X^{'}V^{-1}]^{'}$$
$$= X^{-1}(V^{-1})^{-1}X^{'-1}X^{'}V^{-1}\sigma^2 V[X^{-1}(V^{-1})^{-1}X^{'-1}X^{'}V^{-1}]^{'}$$
$$= \sigma^2 X^{-1}VX^{'-1}X^{'}V^{-1}VV^{-1'}XX^{-1}V^{'}X^{-1'}$$
$$= \sigma^2 X^{-1}VIIV^{-1'}IV^{'}X^{-1'}$$
$$= \sigma^2 X^{-1}VV^{-1'}V^{'}X^{-1'}$$
$$= \sigma^2 X^{-1}VX^{-1'}$$
$$cov(\hat{\beta}_w) = \sigma^2(X^{'}V^{-1}X)^{-1}$$

(c)

Assuming normality of residuals:

$$\hat{\beta_w} \sim N(\beta, \sigma^2(X^{'}V^{-1}X)^{-1})$$

(d)

With this choice of V, you are dividing by the sample size
which gives you the variance of the distribution