BIOS 660. Homework 11.

1. C&B 4.62

$\because E(g(X)) = \sum_x P(X=x)\cdot g(x)$

$\because a+bx$ is a line tangent to $g(x)$ at $x = E(X)$

$\therefore \; g(E(X)) = a + b\cdot E(X) = a + b\cdot\sum_x P(X=x)\cdot x = \sum_x P(X=x)\cdot(a+bx)$

$\therefore E(g(X)) - g(E(X)) = \sum_x P(X=x)\cdot[g(x) - (a+bx)]$

$= \sum_{x\neq E(X)} P(X=x)\cdot[g(x) - (a+bx)] + P(X=E(X))[g(E(X)) - (a+b E(X))]$

$= \sum_{x\neq E(X)} P(X=x)\cdot[g(x) - (a+bx)] + 0$

$\because g(x) > a+bx$ except $x = E(X)$

$\therefore$ If $E(g(X)) - g(E(X)) \leq 0 \Rightarrow E(g(X)) \leq g(E(X))$

$\sum_{x\neq E(X)} P(X=x)\cdot[g(x) - (a+bx)] \leq 0$

$\because \cancel{P(X=x) \geq 0}, \quad g(x) - (a+bx) > 0 \quad$ for $x \neq E(X)$

$\therefore$ ~~If $E(g(X)) \leq g(E(X))$~~, $\quad P(X=x) = 0$ for $\forall x \neq E(X)$

~~then $P(X=E(X))$~~ $= 1 - \sum_{x\neq E(X)} P(X=x) = 1 - 0 = 1$

If $P(X=E(X)) \neq 1$, then $\exists x'$ that $P(X=x') > 0, \quad x' \neq E(X)$

$P(X=x')\cdot[g(x') - (a+bx')] > 0$

$\therefore E(g(X)) - g(E(X)) = \sum_{x\neq E(X)} P(X=x)\cdot[g(x) - (a+bx)]$

$\geq P(X=x')\cdot[g(x') - (a+bx')] > 0$

$\therefore \; E(g(X)) > g(E(X))$.

To sum up, $E(g(X)) > g(E(X))$ unless $P(X = E(X)) = 1$.

2. (a)

$\because f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}, \quad \begin{matrix}-\infty<x<+\infty\\ -\infty<y<+\infty\end{matrix}$

$\therefore f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\}\cdot\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\}$

$\therefore f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$

$= \frac{\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\cdot\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\cdot\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\}dy$

$= \frac{\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\}}{\sqrt{2\pi}\cdot\sigma_X}\cdot\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}(\sqrt{1-\rho^2})}\cdot\exp\left\{-\frac{\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2}{2(\sqrt{1-\rho^2})^2}\right\}d\left(\frac{y-\mu_Y}{\sigma_Y}\right)$

$= \frac{\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\}}{\sqrt{2\pi}\cdot\sigma_X}\cdot 1 = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left\{-\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2}\right\}, \quad -\infty<x<+\infty$

$\therefore$ Marginal distribution of X is $X \sim N(\mu_X, \sigma_X^2)$

1

2. (b). $\because X \sim N(\mu_X, \sigma_X^2)$ $\quad \therefore f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot \exp\left\{\frac{-(x-\mu_X)^2}{2\sigma_X^2}\right\} = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot \exp\left\{-\frac{1}{2}\left(\right.\right.$

$\because f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left\{-\frac{1}{2(1-\rho^2)} \cdot \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$

$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left\{-\frac{1}{2(1-\rho^2)} \cdot \left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\} \cdot \exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right.\right.$

$\therefore f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\sqrt{2\pi}\cdot\sigma_X}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\}$

$= \frac{1}{\sqrt{2\pi}\cdot(\sigma_Y\sqrt{1-\rho^2})} \cdot \exp\left\{-\frac{1}{2\cdot[\sigma_Y^2(1-\rho^2)]}\left[(y-\mu_Y) - \frac{\rho\cdot\sigma_Y}{\sigma_X}\cdot(x-\mu_X)\right]^2\right\}$

$= \frac{1}{\sqrt{2\pi}\,(\sigma_Y\sqrt{1-\rho^2})} \cdot \exp\left\{-\frac{1}{2\,(\sigma_Y\sqrt{1-\rho^2})^2} \cdot \left[y - \left(\mu_Y + \rho\cdot\left(\frac{\sigma_Y}{\sigma_X}\right)\cdot(x-\mu_X)\right)\right]^2\right\}$

$-\infty < y < +\infty$

$\therefore Y|X \sim N\left[\mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)\cdot(x-\mu_X),\ \sigma_Y^2(1-\rho^2)\right]$

(c) Let $U = \frac{X-\mu_X}{\sigma_X}$, $V = \frac{Y-\mu_Y}{\sigma_Y}$

$\therefore X = U\cdot\sigma_X + \mu_X$, $\quad Y = V\cdot\sigma_Y + \mu_Y$

$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sigma_X & 0 \\ 0 & \sigma_Y \end{vmatrix} = \sigma_X\sigma_Y > 0 \quad |J| = \sigma_X\sigma_Y$

$\therefore f_{U,V}(u,v) = f_{X,Y}(\mu_X+\sigma_X u,\ \mu_Y+\sigma_Y\cdot v)\cdot|J|$

$= \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2(1-\rho^2)}\cdot[u^2-2\rho uv+v^2]\right)$

$\because$ Joint MGF of (U,V) is

$M_{U,V}(t_1,t_2) = E(e^{t_1U+t_2V}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{t_1u+t_2v} \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2(1-\rho^2)}[u^2-2\rho uv+v^2]\right) du\,dv$

$= \int_{-\infty}^{+\infty} \frac{e^{t_1u}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2(1-\rho^2)}[u^2-2\rho uv+v^2-2(1-\rho^2)vt_2]\right) dv\,du$

$= \int_{-\infty}^{+\infty} \frac{e^{t_1u}}{\sqrt{2\pi}}\,du \left\{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}[(v-(1-\rho^2)t_2)-\rho u]^2\right) dv\right\} \cdot \exp\left(-\frac{1}{2}u^2 + \rho ut_2 + \frac{1-\rho^2}{2}t_2^2\right)$

$= \int_{-\infty}^{+\infty} \frac{e^{t_1u}}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{1}{2}(u^2 - 2\rho ut_2 - (1-\rho^2)\cdot t_2^2)\right\} \cdot du$

$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{1}{2}(u^2-2\rho ut_2-(1-\rho^2)t_2^2-2t_1u)\right\} du$

$= \exp\left\{\frac{1}{2}(t_1^2+2\rho t_1t_2+t_2^2)\right\} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[(u-t_1)-\rho t_2]^2\right\} du$

$= \exp\left\{\frac{1}{2}(t_1^2+2\rho t_1t_2+t_2^2)\right\}.$

$\therefore$ Joint MGF of (U,V) is $\quad M_{U,V}(t_1,t_2) = \exp\left\{\frac{1}{2}(t_1^2+2\rho t_1t_2+t_2^2)\right\}.$

2. (c) continued

$\because M_{U,V}(t_1,t_2) = \exp\{\frac{1}{2}(t_1^2 + 2\rho t_1 t_2 + t_2^2)\}$.

$X = U\cdot\sigma_X + \mu_X$, $\quad Y = V\cdot\sigma_Y + \mu_Y$

$\therefore M_{X,Y}(\theta_1,\theta_2) = E(e^{\theta_1(\mu_X+\sigma_X U) + \theta_2(\mu_Y + \sigma_Y\cdot V)})$

$= e^{\theta_1\mu_X + \theta_2\mu_Y}\cdot M_{U,V}(\theta_1\sigma_X, \theta_2\sigma_Y)$

$= e^{\theta_1\mu_X + \theta_2\mu_Y}\cdot \exp\{\frac{1}{2}(\sigma_X^2\cdot\theta_1^2 + 2\rho\,\theta_1\theta_2\sigma_X\sigma_Y + \theta_2^2\cdot\sigma_Y^2)\}$

$= \exp\{\theta_1\mu_X + \theta_2\mu_Y + \frac{1}{2}(\theta_1^2\sigma_X^2 + 2\rho\theta_1\theta_2\sigma_X\sigma_Y + \theta_2^2\sigma_Y^2)\}$.

$\therefore M_{aX+bY}(\theta) = M_{X,Y}(a\theta, b\theta)$

$= \exp\{a\theta\mu_X + b\theta\cdot\mu_Y + \frac{1}{2}(\theta^2 a^2\sigma_X^2 + 2\rho\theta^2 ab\,\sigma_X\sigma_Y + \theta^2\cdot b^2\sigma_Y^2)\}$

$= \exp\{(a\mu_X + b\mu_Y)\cdot\theta + \frac{1}{2}(a^2\sigma_X^2 + 2\rho ab\cdot\sigma_X\sigma_Y + b^2\cdot\sigma_Y^2)\cdot\theta^2\}$.

Which is the MGF of $N(a\mu_X + b\mu_Y,\ a^2\sigma_X^2 + 2\rho ab\,\sigma_X\sigma_Y + b^2\cdot\sigma_Y^2)$

$\therefore aX + bY \sim N(a\mu_X + b\mu_Y,\ a^2\sigma_X^2 + b^2\cdot\sigma_Y^2 + 2\rho ab\,\sigma_X\sigma_Y)$

3. MGF of $[X, Y]'$ is $\psi_{X,Y}(t,u) = \exp\{2t + 3u + t^2 + atu + 2u^2\}$

(a) $\because \psi_{X,Y}(t,u) = E(e^{tX+uY}) = \exp\{2t + 3u + t^2 + atu + 2u^2\}$

$\therefore \psi_X(t) = E(e^{tX}) = E(e^{tX+0\cdot Y}) = \psi_{X,Y}(t,0) = \exp\{2t + t^2\}$

$\psi_Y(u) = E(e^{uY}) = E(e^{0\cdot X + u\cdot Y}) = \psi_{X,Y}(0,u) = \exp\{3u + 2u^2\}$

$\therefore E(X) = \frac{\partial}{\partial t}\cdot\psi_X(t)\Big|_{t=0} = (2+2t)\cdot\exp\{2t+t^2\}\Big|_{t=0} = 2$

$E(X^2) = \frac{\partial^2}{\partial t^2}\cdot\psi_X(t)\Big|_{t=0} = 2\cdot\exp\{2t+t^2\} + (2+2t)^2\exp\{2t+t^2\}\Big|_{t=0} = 6$

$E(Y) = \frac{\partial}{\partial u}\cdot\psi_Y(u)\Big|_{u=0} = (3+4u)\cdot\exp\{3u+2u^2\}\Big|_{u=0} = 3$

$E(Y^2) = \frac{\partial^2}{\partial u^2}\psi_Y(u)\Big|_{u=0} = 4\cdot\exp\{3u+2u^2\} + (3+4u)^2\exp\{3u+2u^2\}\Big|_{u=0} = 13$

$\because E(XY) = \frac{\partial^2}{\partial t\partial u}\cdot\psi_{X,Y}(t,u)\Big|_{t=0,u=0} = \frac{\partial}{\partial u}\cdot(2 + 2t + au)\cdot\exp\{2t+3u+t^2+atu+2u^2\}\Big|_{t=0,u=0}$

$= \Big(a + (2+2t+au)\cdot(3+4u+at)\Big)\exp\{2t+3u+t^2+atu+2u^2\}\Big|_{t=0,u=0}$

$= a + 6$

$\therefore E(X+2Y) = E(X) + 2E(Y) = 8$, $\quad E(2X-Y) = 2E(X) - E(Y) = 1$.

$\because E((X+2Y)\cdot(2X-Y)) = E(2X^2 + 3XY - 2Y^2) = 2E(X^2) + 3E(XY) - 2E(Y^2)$.

$= 2\times 6 + 3(a+6) - 2\times 13 = 4 + 3a$

$\therefore$ If $X+2Y$, $2X-Y$ are independent, $E((X+2Y)(2X-Y)) = E(X+2Y)\cdot E(2X-Y)$

$\therefore 4 + 3a = 8\times 1 = 8 \Rightarrow \quad \therefore a = \frac{4}{3}$

3. (b) From (a), we know $a = \frac{4}{3}$.

$\therefore \psi_{X,Y}(t,u) = \exp\{2t + 3u + t^2 + \frac{4}{3}tu + 2u^2\}$

Let $U = X+2Y$, $V = 2X - Y$. U and V are independent

$\therefore$ MGF of U is $\psi_U(t) = E(e^{Ut}) = E(e^{tX+2tY}) = \psi_{X,Y}(t, 2t)$

$$= \exp\{2t + 3(2t) + t^2 + \frac{4}{3}t(2t) + 2(2t)^2\}$$

$$= \exp\{8t + \frac{35}{3}t^2\} = \exp\{8t + \frac{1}{2}(\frac{70}{3})t^2\}$$

$\therefore U \sim N(8, \frac{70}{3})$

$\therefore$ MGF of V is $\psi_V(t) = E(e^{Vt}) = E(e^{2tX - tY}) = \psi_{X,Y}(2t, -t)$

$$= \exp\{2(2t) + 3(-t) + (2t)^2 + \frac{4}{3}(2t)\cdot(-t) + 2(-t)^2\}$$

$$= \exp\{t + \frac{10}{3}t^2\} = \exp\{t + \frac{1}{2}\cdot(\frac{20}{3})t^2\}$$

$\therefore V \sim N(1, \frac{20}{3})$

$\because U, V$ are independent $\therefore (U-V) \sim N(8-1, \frac{70}{3} + \frac{20}{3})$.

$\therefore U - V \sim N(7, 30)$

$\therefore P(X+2Y < 2X - Y) = P(U < V) = P((U-V) < 0)$

$$= \phi\left(\frac{0-7}{\sqrt{30}}\right) = 1 - \phi\left(\frac{7}{\sqrt{30}}\right)$$

$$\approx 1 - 0.8997 \approx 0.1003$$

Question 4

(a) $Y_1 = X_1 - 3X_2 + 2$ and $Y_2 = 2X_1 - X_2 - 1$

then, $X_1 = \frac{3}{5}Y_2 - \frac{1}{5}Y_1 + 1$ and $X_2 = \frac{1}{5}Y_2 - \frac{2}{5}Y_1 + 1$

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} -\frac{1}{5} & \frac{3}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{vmatrix} = -\frac{1}{25} + \frac{6}{25} = \frac{1}{5}$$

Since X_1 and X_2 are independent $N(0,1)$ random variables. Then

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$
$$= \frac{1}{\sqrt{2\pi}} \exp(-x_1^2/2) \cdot \frac{1}{\sqrt{2\pi}} \exp(-x_2^2/2)$$
$$= \frac{1}{2\pi} \exp\left(-\frac{(x_1^2 + x_2^2)}{2}\right)$$

Then the joint distribution of Y_1 and Y_2 is:

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi} \exp\left(-\frac{(\frac{3}{5}y_2 - \frac{1}{5}y_1 + 1)^2 + (-\frac{2}{5}y_1 + \frac{1}{5}y_2 + 1)^2}{2}\right) \cdot \frac{1}{5}$$
$$= \frac{1}{10\pi} \exp\left(\frac{-y_1^2 - 2y_2^2 + 2y_1y_2 - 8y_2 + 6y_1 - 10}{10}\right)$$
$$= \frac{1}{2\pi\sqrt{10}\sqrt{5}\sqrt{1-1/2}} \exp\left(-\frac{1}{2(1-\frac{1}{2})}\left(\left(\frac{x-2}{\sqrt{10}}\right)^2 - 2\frac{1}{\sqrt{2}}\left(\frac{x-2}{\sqrt{10}}\right)\left(\frac{y+1}{\sqrt{5}}\right) + \left(\frac{y+1}{\sqrt{5}}\right)^2\right)\right)$$

therefore, Y_1 and Y_2 are bivariate normal with $\mu_{Y_1} = 2$, $\mu_{Y_2} = -1$, $\sigma_{Y_1} = \sqrt{10}$, $\sigma_{Y_2} = \sqrt{5}$, $\rho = 1/\sqrt{2}$

then the joint density of $Y = (Y_1, Y_2)$ is

$$f_Y(Y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}(Y-\mu)^T \Sigma^{-1} (Y-\mu)\right),$$

where $\mu = \begin{pmatrix} \mu_{Y_1} \\ \mu_{Y_2} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\Sigma = \begin{pmatrix} \sigma_{Y_1}^2 & \rho\sigma_{Y_1}\sigma_{Y_2} \\ \rho\sigma_{Y_1}\sigma_{Y_2} & \sigma_{Y_2}^2 \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix}$

$$f_Y(Y) = \frac{1}{10\pi} \exp\left(-\frac{1}{2}\left(Y - \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)^T \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \left(Y - \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)\right)$$

(b) For bivariate normal, the conditional distribution is normal:

$$Y_1 | Y_2 = y \sim N\left(\mu_{Y_1} + \rho(\sigma_{Y_1}/\sigma_{Y_2})(y - \mu_{Y_2}),\ \sigma^2_{Y_1}(1-\rho^2)\right)$$

$$\mu_{Y_1} + \rho(\sigma_{Y_1}/\sigma_{Y_2})(y - \mu_{Y_2})$$
$$= 2 + \tfrac{1}{\sqrt{2}}(\sqrt{10}/\sqrt{5})(y+1)$$
$$= y+3$$

$$\sigma^2_{Y_1}(1-\rho^2)$$
$$= (\sqrt{10})^2\left(1-\left(\tfrac{1}{\sqrt{2}}\right)^2\right)$$
$$= 5.$$

Therefore, $Y_1 | Y_2 = y \sim N(y+3, 5)$

Question 5 (CB 5.24)

$X_1, \cdots, X_n$ follow $f_X(x) = 1/\theta$ for $0 < x < \theta$ and $F_X(x) = \frac{x}{\theta}$ for $0 < x < \theta$. Then, the joint distribution of $X_{(1)}$ and $X_{(n)}$ is

$$f_{X_{(1)}, X_{(n)}}(v, u) = \frac{n!}{(n-2)!} \frac{1}{\theta^2} \left(\frac{u}{\theta} - \frac{v}{\theta}\right)^{n-2} = \frac{n(n-1)}{\theta^n}(u-v)^{n-2}, \quad 0 < v < u < \theta$$

Then, let $Y = X_{(1)}/X_{(n)}$ and $Z = X_{(n)}$;

$X_{(1)} = YZ$, $X_{(n)} = Z$

$$J(y,z) = \begin{vmatrix} \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ z & y \end{vmatrix} = -z$$

then $f_{Y,Z}(y,z) = \frac{n(n-1)}{\theta^n}(z - yz)^{n-2} z = \frac{n(n-1)}{\theta^n}(1-y)^{n-2} z^{n-1}$, $0 < y < 1$, $0 < z < \theta$.

The joint distribution can be written as the product of a function of Z and a function of Y. Thus, Y and Z are independent.

Question 6

X_1, X_2 follow geom(p). Then $f_X(x) = p(1-p)^{x-1}$, $x = 1, 2, \cdots$ and $F_X(x) = 1 - (1-p)^x$, $x = 1, 2, \cdots$ Then, the joint distribution of $X_{(1)}$ and $X_{(2)}$ is

$$f_{X_{(1)}, X_{(2)}}(u,v) = \frac{2!}{0!0!0!} p(1-p)^{u-1} p(1-p)^{v-1} = 2p^2(1-p)^{u+v-2}, \quad u \le v \text{ and } u,v = 1, 2, \cdots$$

Then, let $Y = X_{(1)}$ and $Z = X_{(2)} - X_{(1)}$

$X_{(1)} = Y$, $X_{(2)} = Z + Y$

$$J(y,z) = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

then, $f_{Y,Z}(y,z) = 2p^2(1-p)^{y+z+y-2} = 2p^2(1-p)^{2y-2}(1-p)^z$, $y = 1, 2, \cdots$, $z = 0, 1, 2, \cdots$

the joint distribution can be written as the product of a function of Y and a function of Z. Thus Y and Z are independent.