HW 11. Group 1.

4.15

If $X \sim \text{Poisson}(\theta)$, $Y \sim \text{Possion}(\lambda)$ and they are independent

$Z = X+Y \sim \text{Poisson}(\theta+\lambda)$.

the conditional PMF of X, given Z is $P(X \mid Z)$.

$$P(X \mid Z) = \frac{P(X, Y = Z-X)}{P(Z)} = \frac{P_X(X) \cdot P_Y(Z-X)}{P_Z(Z)}$$

$$= \frac{\dfrac{e^{-\theta} \cdot \theta^X}{X!} \cdot \dfrac{e^{-\lambda} \cdot \lambda^{Z-X}}{(Z-X)!}}{\dfrac{e^{-(\lambda+\theta)} \cdot (\lambda+\theta)^Z}{Z!}}$$

$$= \frac{Z!}{X!\,(Z-X)!} \cdot \left(\frac{\theta}{\lambda+\theta}\right)^X \cdot \left(\frac{\lambda}{\lambda+\theta}\right)^{Z-X} \qquad \text{if } p = \frac{\theta}{\lambda+\theta}.$$

$$= \binom{Z}{X} \cdot p^X (1-p)^{Z-X}. \qquad , X = 0, \cdots, Z$$

RV, $X \mid X+Y$ follows a binomial distribution with $Z = X+Y$ trials and success probability $\frac{\theta}{\lambda+\theta}$.

The conditional PMF of $Y \mid X+Y$ is similar to above proof.

$$P(Y \mid Z) = \frac{P(Y, X = Z-Y)}{P(Z)} = \frac{P_Y(Y) P_X(Z-Y)}{P_Z(Z)} = \binom{Z}{Y} \cdot (1-p)^Y \cdot p^{Z-Y} \qquad Y = 0, \cdots, Z.$$

$\therefore$ the distribution of Y given $X+Y=Z$ is also a binomial distribution with total Z trials and successful probability $\frac{\lambda}{\lambda+\theta}$.

4.16.

(a) ① if $V = X - Y < 0$, $U = X$, for $V = -1, -2, -3, \cdots$

Then $X = U$, $Y = U - V$

$\therefore f(u, v) = P(X = u, Y = u - v)$

$= p(1-p)^{u-1} \cdot p(1-p)^{u-v-1}$

$= p^2(1-p)^{2u-v-2}$, $u = 1, 2, \cdots Z^+$

$v = -1, -2, \cdots Z^-$

② if $V = X - Y > 0$, $U = Y$, for $V = Z^+$

Then $Y = U$, $X = U + V$

$f(u, v) = P(X = u + v, Y = u)$

$= p^2(1-p)^{2u+v-2}$ $u = 1, 2, 3, \cdots Z^+$

$v = 1, 2, 3, \cdots Z^+$

③ if $V = X - Y = 0$, $U = X$ for $V = 0$

$f(u, v) = P(U = X = Y, V = 0) = p^2(1-p)^{2u-2}$

$u = 1, 2, 3, \cdots, Z^+$

$\therefore f_{U,V}(u, v) = p^2(1-p)^{2u-2} \cdot (1-p)^{|v|}$

for $u = 1, 2, 3, \cdots$

$v = 0, \pm 1, \pm 2, \pm 3, \cdots$

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(b). $Z = \frac{X}{X+Y}$, $X, Y \overset{iid}{\sim}$ geometric distribution with p.

so $0 \le Z \le 1$ and the range of Z is all fraction of $\frac{r}{s}$ where $0 < r \le s$, and, r, s are positive integer, $\frac{r}{s}$ is in the reduced form

if $\frac{X}{X+Y} = \frac{r}{s}$, then the value of (X, Y) are pairs of $(ir, i(s-r))$ with $i = 1, 2, 3, \cdots$.

$$\therefore P(Z = \frac{r}{s}) = \sum_{i=1}^{\infty} P(X = ir, Y = i(s-r))$$

$$= \sum_{i=1}^{\infty} p(1-p)^{ir-1} \cdot p(1-p)^{i(s-r)-1}$$

$$= \sum_{i=1}^{\infty} p^2 (1-p)^{-2} \cdot (1-p)^{is}$$

$$= p^2 (1-p)^{-2} \cdot \frac{(1-p)^s}{1-(1-p)^s}$$

$$= \frac{p^2 (1-p)^{s-2}}{1-(1-p)^s}$$

(c). $P(X, Z = X+Y) = P(X, Y = Z-X)$

$$= p^2 (1-p)^{X-1} (1-p)^{Z-X-1}$$

$$= p^2 (1-p)^{Z-2},$$

with $0 \le X \le Z$, X, Z are positive integer.

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(a). $Y_1 = X_1^2 + X_2^2$

$Y_2 = \frac{X_1}{\sqrt{Y_1}}$

From the equation above, it's obvious to notice that the sign of X_2 is not determined by Y_1, Y_2.

so the support of (X_1, X_2) could be divided into

$A_0 = \{x_1 \in R, X_2 = 0\}$, $A_1 = \{X_1 \in R, x_2 < 0\}$, $A_2 = \{x_1 \in R, X_2 > 0\}$

The support of B is $\{Y_1 > 0, Y_2 \in (-1, 1)\}$.

① In the inverse transformation from B to A_1

$X_1 = \sqrt{Y_1}\, Y_2 \qquad X_2 = -\sqrt{Y_1(1-Y_2^2)}$

$$|J| = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1}, & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1}, & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} \frac{Y_2}{2\sqrt{Y_1}}, & \sqrt{Y_1} \\ \frac{\sqrt{1-Y_2^2}}{2\sqrt{Y_1}}, & \frac{-\sqrt{Y_1}\,Y_2}{\sqrt{1-Y_2^2}} \end{vmatrix} = \frac{1}{2\sqrt{1-Y_2^2}}.$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(\sqrt{y_1}\,y_2, -\sqrt{y_1(1-y_2^2)}) \cdot |J|$$

$$= \frac{1}{2\pi\sigma^2} \exp\left(-\left[\frac{y_1 y_2^2 + y_1^2(1-y_2^2)}{2\sigma^2}\right]\right) \cdot \frac{1}{2\sqrt{1-y_2^2}}.$$

$$= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{y_1^2}{2\sigma^2}\right) \cdot \frac{1}{2\sqrt{1-y_2^2}}.$$

② In the inverse transformation from B to A_2.

$X = \sqrt{Y_1}\; Y_2 \qquad X_2 = \sqrt{Y_1(1-Y_2^2)}$

$$\therefore |J| = \begin{vmatrix} \frac{Y_2}{2\sqrt{Y_1}}, & \sqrt{Y_1} \\ -\sqrt{Y_1(1-Y_2^2)}, & \frac{\sqrt{Y_1}\,Y_2}{\sqrt{1-Y_2^2}} \end{vmatrix} = \frac{1}{2\sqrt{1-Y_2^2}}$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(\sqrt{y_1}\cdot y_2\ \sqrt{y_1(1-y_2^2)}) \cdot |J| = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{y_1^2}{2\sigma^2}\right)\frac{1}{2\sqrt{1-y_2^2}}.$$

The $f(y_1, y_2)$ is the sum of above two part ① ②.

$$\therefore f_{Y_1,Y_2}(y_1, y_2) = \left[\frac{1}{2\pi\sigma^2}\exp\left(-\frac{y_1^2}{2\sigma^2}\right)\right] \cdot \frac{1}{\sqrt{1-y_2^2}}$$

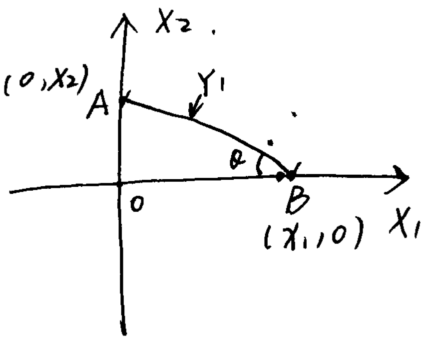
(b). From the joint pdf of Y_1, Y_2, we could notice $f_{Y_1,Y_2}(y_1,y_2) = g_1(y_1) \cdot g_2(y_2)$

$$g_1(y_1) = \left[\frac{1}{2\pi\sigma^2}\exp\left(-\frac{y_1^2}{2\sigma^2}\right)\right].$$

$$g_2(y_2) = \frac{1}{\sqrt{1-y_2^2}}$$

$\therefore$ y_1 and y_2 are independent by Definition (4.2.1).

If we plot the (X_1, X_2) in x-axis and y-axis.



Y_1 is the square of distance between A.B.

Y_2 is $\cos\theta$ of triangle ABO.

For a triangle, the angle of ABO is independent of the length of one side of it.

So Y_1 is independent of Y_2.

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4. Exercise 4.23: Find the distribution of XY by transforming then integrating out V.

$X \sim \text{beta}(\alpha, \beta)$ $\quad Y \sim \text{beta}(\alpha+\beta, \gamma)$, X, Y independent

a. Let $U = XY$ $\quad V = Y$.

Then $Y = V$ and $X = \frac{U}{Y} = \frac{U}{V}$

$$f_{X,Y}(x,y) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \left(\frac{1}{B(\alpha+\beta,\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1} \right)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1}$$

$$J(u,v) = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v} \qquad 0<u<v<1$$

$$f_{U,V}(u,v) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{u}{v}\right)^{\alpha-1} \left(1-\frac{u}{v}\right)^{\beta-1} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} v^{\alpha+\beta-1}(1-v)^{\gamma-1} \left(\frac{1}{v}\right)$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 v^{\beta-1}(1-v)^{\gamma-1} \left(\frac{v-u}{v}\right)^{\beta-1} dv$$

Change variables: let $y = \frac{v-u}{1-u}$ then $dy = \frac{1}{1-u} dv$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_0^1 y^{\beta-1}(1-y)^{\gamma-1} dy$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \qquad 0<u<1$$

So $U = XY \sim \text{beta}(\alpha, \beta+\gamma)$

b. Let $U = XY$ and $V = \frac{X}{Y}$. Then, $V = \frac{X}{\left(\frac{U}{X}\right)} = \frac{X^2}{U}$ so

$X = \sqrt{VU}$ and $Y = \frac{U}{X} = \sqrt{\frac{U}{V}}$.

$$J(u,v) = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(vu)^{-\frac{1}{2}}v & \frac{1}{2}(vu)^{-\frac{1}{2}}u \\ \frac{1}{2}(vu)^{-\frac{1}{2}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \end{vmatrix}$$

$$= \left(\tfrac{1}{2}v^{\frac{1}{2}}u^{-\frac{1}{2}}\right)\left(-\tfrac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}}\right) - \left(\tfrac{1}{2}v^{-\frac{1}{2}}u^{-\frac{1}{2}}\right)\left(\tfrac{1}{2}v^{-\frac{1}{2}}u^{\frac{1}{2}}\right)$$

$$= -\tfrac{1}{4}v^{-1} - \tfrac{1}{4}v^{-1} = \frac{-1}{2v}$$

$$f_{U,V}(u,v) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sqrt{vu}^{\,\alpha-1}(1-\sqrt{vu})^{\beta-1}\left(\sqrt{\frac{u}{v}}\right)^{\alpha+\beta-1}$$

$$\left(1-\sqrt{\frac{u}{v}}\right)^{\gamma-1}\left|\frac{1}{2v}\right| \qquad 0<u<v<\tfrac{1}{u}, \quad 0<u<1$$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}$$

$$\int_u^{\frac{1}{u}} \left(\frac{1-\sqrt{uv}}{1-u}\right)^{\beta-1}\left(\frac{1-\sqrt{u/v}}{1-u}\right)^{\gamma-1}\left(\frac{\sqrt{u/v}^{\,\beta}}{2v(1-u)}\right) dv$$

change variables: let $z = \frac{\sqrt{u/v} - u}{1-u}$ then $u = v \to z = 1$, $v = \frac{1}{u} \to z = 0$

$$dz = \frac{-\sqrt{u/v}}{2(1-u)v}\, dv$$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}\int_0^1 z^{\beta-1}(1-z)^{\gamma-1}\, dz$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \qquad 0<u<1$$

$U = XY \sim \text{beta}(\alpha, \beta+\gamma)$

5. Exercise 4.30: $Y|X=x \sim n(x, x^2)$ and marginal of $X \sim unif(0,1)$

a. Find $E(Y)$, $Var(Y)$, $Cov(X,Y)$.

Finding $E(Y)$:

$E(Y) = E(E(Y|X)) = E(X) = \boxed{\frac{1}{2}}$

Finding $Var(Y)$:

$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$

we know $Var(Y|X) = X^2$ so $E(Var(Y|X)) = \int_0^1 x^2 f_X(x)\,dx$

$= \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{1}{3}$

and $Var(E(Y|X)) = Var(X) = \frac{1}{12}$

so $Var(Y) = \frac{1}{12} + \frac{1}{3} = \boxed{\frac{5}{12}}$

Finding $Cov(X,Y)$:

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$E(XY) = E[E(XY|X)] = E[X\,E(Y|X)] = E[X^2] = \frac{1}{3}$

$Cov(X,Y) = \frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \boxed{\frac{1}{12}}$

b. Prove $\frac{Y}{X}$ and X are independent.

$$f_{X,Y}(x,y) = \frac{1}{x\sqrt{2\pi}} e^{\frac{-(y-x)^2}{2x^2}} \quad -\infty < y < \infty \quad 0 \leq x \leq 1$$

let $U = \frac{Y}{X}$ and $V = X$ then $Y = UX = UV$

$$J(u,v) = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = v$$

$$f_{U,V}(u,v) = \frac{1}{v\sqrt{2\pi}} e^{\frac{-(uv-v)^2}{2v^2}} \quad -\infty < u < \infty \quad 0 \leq v \leq 1$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-1)^2}$$

$$f_U(u) = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-1)^2}\,dv = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-1)^2} \quad -\infty < u < \infty$$

$$f_V(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u-1)^2}\, du = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = 1$$

then $f_U(u) \cdot f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-1)^2}\,(1)$

which is equal to $f_{U,V}(u,v)$

So, $\frac{Y}{X}$ and X are independent because $f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$

6. Exercise 4.33: X_i = # eggs laid for insect $i = 1, 2, \ldots$, X_i are iid, $H = X_1 + \ldots + H_n$ = total eggs laid, $n \sim \text{Poisson}(\lambda)$, $p = (X_i = t) = \frac{-1}{\log(p)} \frac{(1-p)^t}{t}$, $H \mid n = X_1 + \ldots + X_n$

Show marginal distribution of H is negative bin (r, p) where $r = \frac{-\lambda}{\log(p)}$.

$$E(e^{Ht}) = E(E(e^{Ht} \mid n)) = E(E(e^{(X_1+\ldots+X_n)t} \mid n)) = E([E(e^{X_i t} \mid n)]^n)$$

$$E(e^{X_i t}) = \sum_{X_i=1}^{\infty} e^{X_i t} \frac{-1}{\log(p)} \frac{(1-p)^{X_i}}{X_i} = \frac{-1}{\log(p)} \sum_{X_i=1}^{\infty} \frac{(e^t(1-p))^{X_i}}{X_i}$$

$$= \frac{-1}{\log(p)} (-\log(1 - e^t(1-p))) = \underbrace{\frac{\log(1 - e^t(1-p))}{\log(p)}}_{\text{mgf}}$$

$$E\left(\frac{\log(1-e^t(1-p))}{\log(p)}\right)^n = \sum_{n=0}^{\infty} \left(\frac{\log(1-e^t(1-p))}{\log(p)}\right)^n \frac{e^{-\lambda}\lambda^n}{n!}$$

since $n \sim \text{Poisson}(\lambda)$

$$= e^{-\lambda} e^{\frac{\lambda \log(1-e^t(1-p))}{\log(p)}} \underbrace{\sum_{n=0}^{\infty} \frac{e^{-\frac{\lambda \log(1-e^t(1-p))}{\log(p)}} \left(\frac{\lambda \log(1-e^t(1-p))}{\log(p)}\right)^n}{n!}}_{\text{pmf of Poisson}\left(\frac{\lambda \log(1-e^t(1-p))}{\log(p)}\right)}$$

so $E(e^{Ht}) = e^{-\lambda} e^{\frac{\lambda \log(1-e^t(1-p))}{\log(p)}}$

$$= e^{-\lambda} \left[e^{\log(1-e^t(1-p))}\right]^{\lambda/\log(p)}$$

$$= \left[e^{\log(p)}\right]^{-\lambda/\log(p)} \left(\frac{1}{1-e^t(1-p)}\right)^{-\lambda/\log(p)}$$

$$= \left(\frac{p}{1-e^t(1-p)}\right)^{-\lambda/\log(p)}$$

This is the mgf for the negative binomial distribution with $r = \frac{-\lambda}{\log(p)}$ and $p = p$.

✓

7. Casella and Berger, 4.43

i) $Cov(X_1 + X_2, X_2 + X_3) = E\big((X_1 + X_2)(X_2 + X_3)\big) - E(X_1 + X_2)E(X_2 + X_3)$

$= E(X_1X_2 + X_1X_3 + X_2X_2 + X_2X_3) - \big(E(X_1) + E(X_2)\big)\big(E(X_2) + E(X_3)\big)$

$= E(X_1X_2) + E(X_1X_3) + E(X_2X_2) + E(X_2X_3) - (2\mu)(2\mu)$

$= E(X_1X_2) + E(X_1X_3) + E(X_2X_2) + E(X_2X_3) - 4\mu^2$

And since they are pairwise uncorrelated, then $Cov(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j) = 0$ which means that $E(X_iX_j) = E(X_i)E(X_j) = \mu^2$ for $i \neq j$. Also, $Var(X_i) = E(X_i^2) - \mu^2$, which means that $E(X_i^2) = \sigma^2 + \mu^2$. So,

$E(X_1X_2) + E(X_1X_3) + E(X_2X_2) + E(X_2X_3) - 4\mu^2$

$= 3\mu^2 + \sigma^2 + \mu^2 - 4\mu^2 = \sigma^2.$

ii) $Cov(X_1 + X_2, X_1 - X_2) = E\big((X_1 + X_2)(X_1 - X_2)\big) - E(X_1 + X_2)E(X_1 - X_2)$

$= E(X_1X_1 - X_1X_2 + X_1X_2 - X_2X_2) - \big(E(X_1) + E(X_2)\big)\big(E(X_1) - E(X_2)\big)$

$= E(X_1^2) - E(X_2^2) - (2\mu)(\mu - \mu)$

$= \sigma^2 + \mu^2 - \sigma^2 + \mu^2 - 0 = 0.$

8. Casella and Berger, 4.50

Since $f_{X,Y}(x,y) = \frac{1}{2\pi(1-p^2)^{\frac{1}{2}}}\exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right)$, the text and our notes gives us that $X \sim N(0,1)$

and $Y \sim N(0,1)$. Also, that $Y|X \sim N(\rho X, 1-\rho^2)$ and $X|Y \sim N(\rho Y, 1-\rho^2)$.

So $Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{E(XY)-E(X)E(Y)}{(1)(1)} = E(XY) - (0)(0) = E(XY) = E\big(E(XY|X)\big)$

$= E(XE(Y|X)) = E(X\rho X) = \rho E(X^2) = \rho.$

Note that $E(E(XY|X)) = E(XE(Y|X))$ since X is given inside the expectation. Also, since $Var(X) = E(X^2) - E(X)^2 = 1$ then $E(X^2) = 1 - 0^2 = 1$. The same applies for $E(Y^2) = 1$.

Furthermore, since $E(XY) = \rho$, and $Var(XY) = E\big(Var(XY|X)\big) + Var\big(E(XY|X)\big)$

$= E\big(X^2Var(Y|X)\big) + Var\big(XE(Y|X)\big) = E\big(X^2(1-\rho^2)\big) + Var(XpX) = (1-\rho^2)E(X^2) + Var(pX^2)$

$= (1-\rho^2)(1) + p^2Var(X^2) = (1-\rho^2) + p^2Var(X^2)$, then:

$$Corr(X^2, Y^2) = \frac{Cov(X^2, Y^2)}{\sigma_{X^2}\sigma_{Y^2}} = \frac{E((XY)^2) - E(X^2)E(Y^2)}{\sigma_{X^2}\sigma_{Y^2}} = \frac{E((XY)^2) - (1)(1)}{\sigma_{X^2}\sigma_{Y^2}} = \frac{E((XY)^2) - 1}{\sigma_{X^2}\sigma_{Y^2}}$$

Since X and Y come from the same distribution, we know that $\sigma_{X^2} = \sigma_{Y^2}$ which means that $\sigma_{X^2}\sigma_{Y^2} = \sigma_{X^2}^2 = Var(X^2)$. Also, we know that $Var(XY) = E((XY)^2) - E(XY)^2$ so $E((XY)^2) = Var(XY) + E(XY)^2 = (1-\rho^2) + p^2Var(X^2) + \rho^2$. So,

$$\frac{E((XY)^2) - 1}{\sigma_{X^2}\sigma_{Y^2}} = \frac{(1-\rho^2) + p^2Var(X^2) + \rho^2 - 1}{Var(X^2)} = \frac{p^2Var(X^2)}{Var(X^2)} = p^2.$$

Q9. 4.38

(a) $Cov(X,Y) = Cov(X, E(Y|X))$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$= E(E(XY|X)) - E(X)E(E(Y|X))$

$= E(XE(Y|X)) - E(X)E(E(Y|X))$

$= Cov(X, E(Y|X))$

$Cov(X, Y - E(Y|X)) = Cov(X,Y) - Cov(X, E(Y|X)) = 0$ by part (a)
Therefore X and $Y - E(Y|X)$ are uncorrelated.

$Var(Y - E(Y|X)) = E(Var(Y|X))$

$$\begin{aligned} Var(Y - E(Y|X)) &= Var(E[Y - E(Y|X) \mid X]) + E[Var[Y - E(Y|X) \mid X]] \\ &= Var(E(Y|X) - E(Y|X)) + E[Var[Y - E(Y|X) \mid X]] \\ &= Var(0) + E[Var(Y|X)] \\ &= E[Var(Y|X)] \end{aligned}$$

Since $E[Y|X]$ is constant with respect to $Y|X$.