Qaqish Review 6/26/19

Completeness

$Y_1, \dots, Y_n$ iid $N(\mu, \mu^2)$, $\mu > 0$

$$\prod_{i=1}^{n} \frac{e^{-\frac{1}{2\mu^2}(y_i-\mu)^2}}{\sqrt{2\pi\mu^2}}$$

$$= \exp\left\{ \sum_{i=1}^{n} \left[-\frac{1}{2}\log \mu^2 - \frac{y_i^2 - 2y_i\mu + \mu^2}{2\mu^2} \right] \right\} c$$

$$= \exp\left\{ \left(\sum_{i=1}^{n} y_i^2\right)\left(-\frac{1}{2\mu^2}\right) + \left(\sum_{i=1}^{n} y_i\right)\left(\frac{1}{\mu}\right) - \frac{n}{2} - \frac{n}{2}\log \mu^2 \right\} c$$

$$T_1(y) \cdot w_1(\mu) + T_2(y) \cdot w_2(\mu) + K(\mu) \} \cdot c$$

MSS is $\left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} Y_i^2 \right)$

Complete? Have to look at parameter space

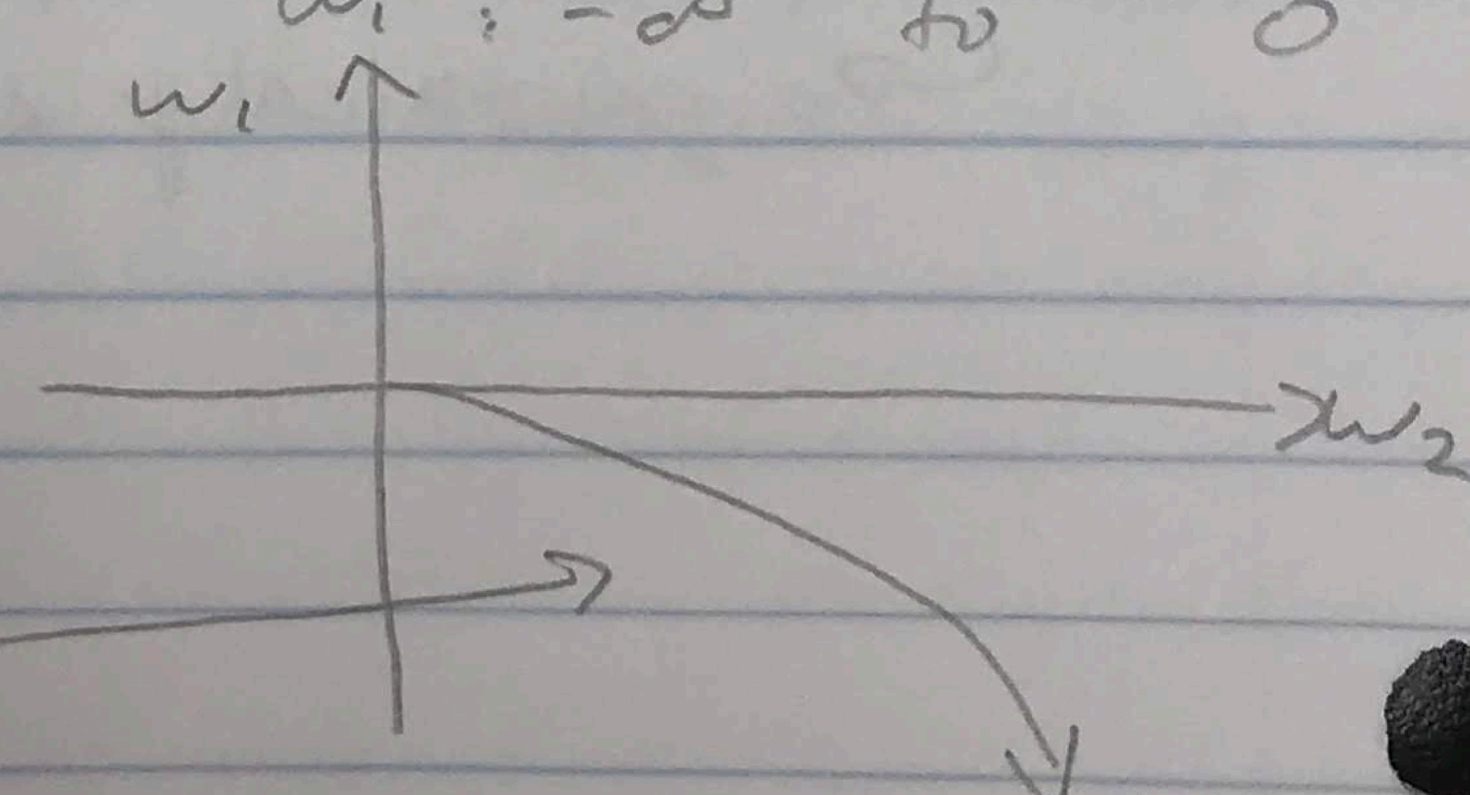
Need to look at $w_1(\mu)$, $w_2(\mu)$ and parameter space $\mu > 0$:

$(w_1(\mu), w_2(\mu)) \quad \mu > 0$

$\left(-\frac{1}{2\mu^2}, \frac{1}{\mu}\right)$

$\left(-\frac{1}{2}\{w_2(\mu)\}^2, w_2(\mu)\right)$

μ : 0 to ∞
w_2 : ∞ to 0
w_1 : $-\infty$ to 0



w_1 and w_2 only take values on this line

It doesn't contain an open set in $\mathbb{R}^2$ (which would be a disk or a square). So, we don't know if its complete or not.

	E	var
$\sum_{i=1}^n Y_i$	$n\mu$	$n\mu^2$

$\sum_{i=1}^n Y_i^2 \quad 2n\mu^2$

apply linearity of expectation

$$E[Y_i^2] = (EY_i)^2 + var(Y_i)$$
$$= \mu^2 + \mu^2 = 2\mu^2$$

$\left(\sum_{i=1}^n Y_i\right)^2 \quad \underline{E}$

$n^2\mu^2 + n\mu^2 = (n^2+n)\mu^2$

$$E\left(\frac{\sum_{i=1}^n Y_i^2}{2n}\right) = \mu^2 \qquad E\left(\frac{\left(\sum_{i=1}^n Y_i\right)^2}{n^2+n}\right) = \mu^2$$

$n > 1$

$$E\left[\frac{\left(\sum_{i=1}^n Y_i\right)^2}{n^2+n} - \frac{\sum_{i=1}^n Y_i^2}{2n}\right] = 0 \quad \forall \mu > 0$$

So, this shows that this MSS is not complete.

The only things that can be complete are MSS. Things that are not minimal can't be complete.

Suppose I have 1 observation, $N(\mu, \mu^2)$
$\mu > 0$

Find an exact 95% CI for μ

$$\frac{Y-\mu}{\mu} \sim N(0,1)$$

Answer: use a pivot

?

$$P\left(-1.96 < \frac{Y-\mu}{\mu} < 1.96\right) = 0.95$$

get μ by itself?

Always keep pivots in mind.
Pivots are not statistics, b/c it involves an unknown parameter

$Y \sim Exp(\mu)$

$$\frac{Y}{\mu} \sim Exp(1)$$

pivot

$\boxed{2017}$ 3

3a) CLT

b) $F_n(x) \to$ avg of Bernoulli's, use CLT

c) a particular choice of value $F'(x)$

x given, so $p = F(x)$

$H_0: p = \frac{1}{2}$ vs $H_1: p \neq \frac{1}{2}$

$Y_1, \dots, Y_n$ iid $Ber(p)$

$$L(p|y) = p^{\sum y_i}(1-p)^{n - \sum y_i}$$

$$= p^{n\bar{y}}(1-p)^{n(1-\bar{y})}$$

$$= \{p^{\bar{y}}(1-p)^{1-\bar{y}}\}^n$$

$$\hat{p} = \bar{y}$$

$$L(\hat{p}|y) = \{\bar{y}^{\bar{y}}(1-\bar{y})^{1-\bar{y}}\}^n$$ ← largest possible value

Under H_0: $L(\frac{1}{2}|y) = \{\frac{1}{2}^{\bar{y}}(\frac{1}{2})^{1-\bar{y}}\}^n = \{\frac{1}{2}\}^n$

switched from how we did it

$$\frac{L(\hat{p}|y)}{L(\frac{1}{2}|y)} = \{2\bar{y}^{\bar{y}}(1-\bar{y})^{1-\bar{y}}\}^n$$

$$\log(\,) = n\{\log 2 + \bar{y}\log\bar{y} + (1-\bar{y})\log(1-\bar{y})\}$$

function of $\bar{y}$, a transform of $\bar{y}$: $g(\bar{y})$

remember it is χ^2 and the df is the number of restrictions

χ^2_1 (b/c one constraint)

$g(\bar{y})$

CLT: $\sqrt{n}(\bar{y} - p) \xrightarrow{d} N(0, p(1-p))$, then use delta method

as $n \to \infty$

3c) but problem:

$g'(\frac{1}{2}) = 0$

so go to next derivative

$g''(\frac{1}{2}) \neq 0$

$n(\bar{y} - p)^2 \longrightarrow (?)\ \chi^2_1$

Different number of contraints:

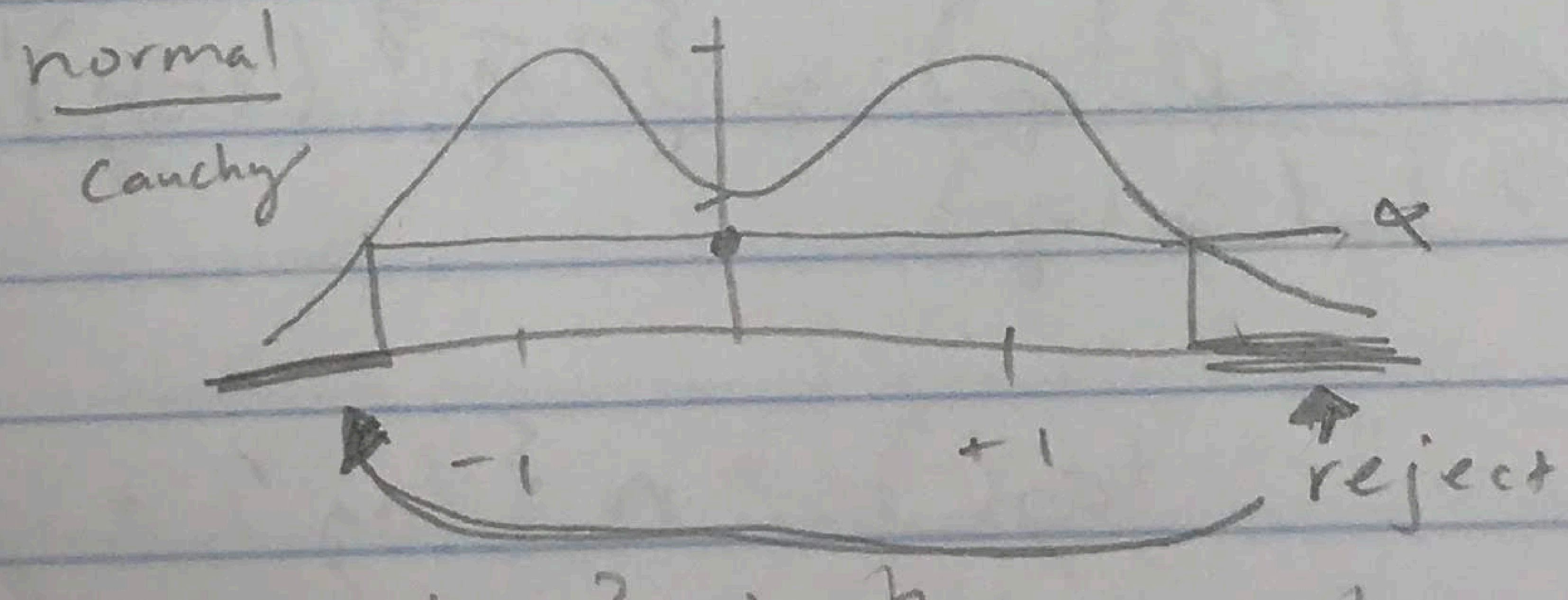
$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} \longrightarrow \theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$$

$$\begin{pmatrix} \theta_1 - \theta_2 \\ \theta_1 - \theta_3 \\ \theta_1 - \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\leftarrow$ 3 restrictions, so LRT is χ^2_3

d) $G(x) \sim N$, $H(x) \sim$ Cauchy

likelihood ratio: $\dfrac{G(x)}{H(x)}$



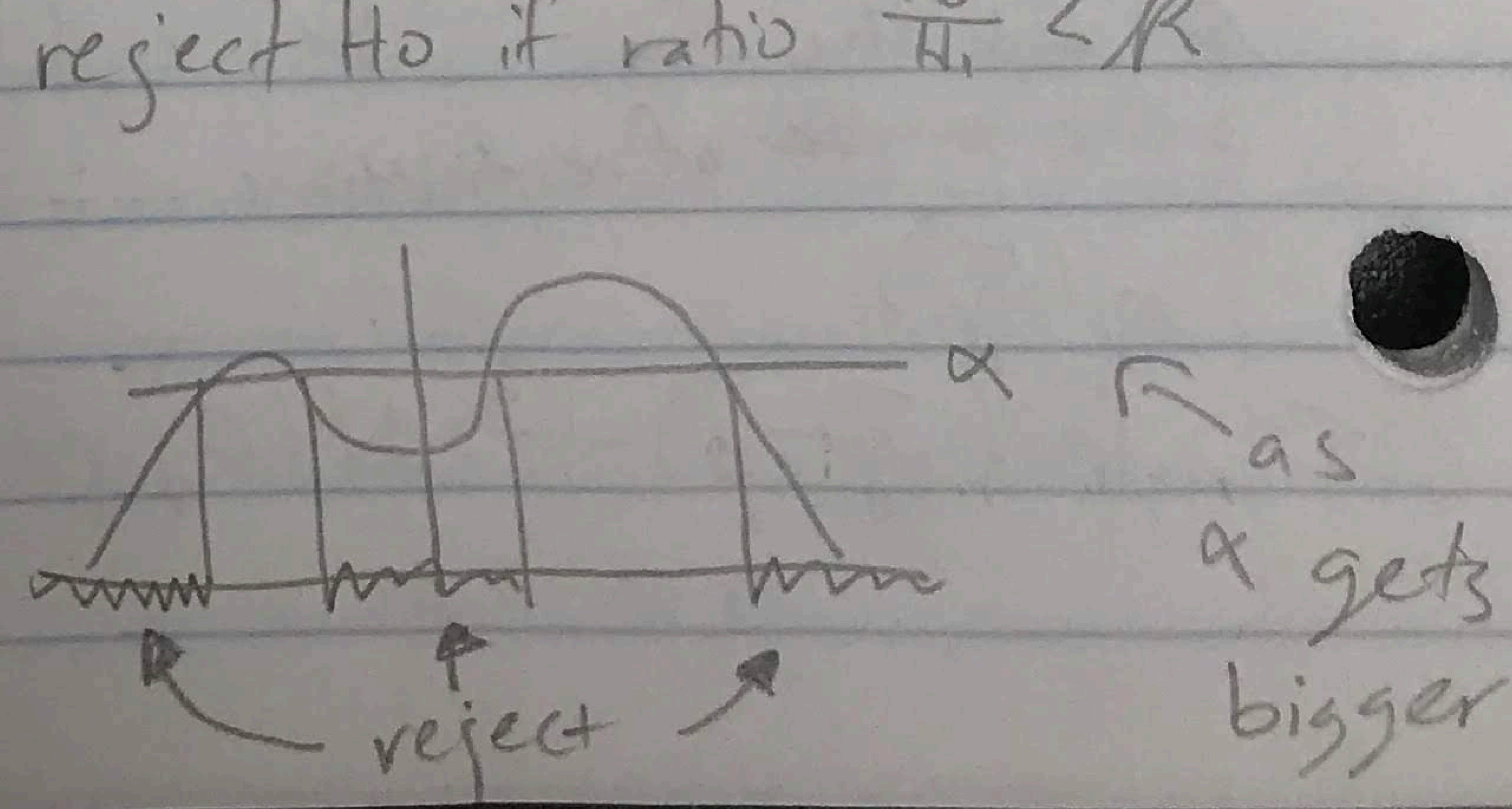
What's the critical region? is the question

H_0: normal

H_1: Cauchy

So reject Ho if ratio $\frac{H_0}{H_1} < k$

for small α, rejects if value is too large or too small, for large α reject if x too large, too small or close to zero

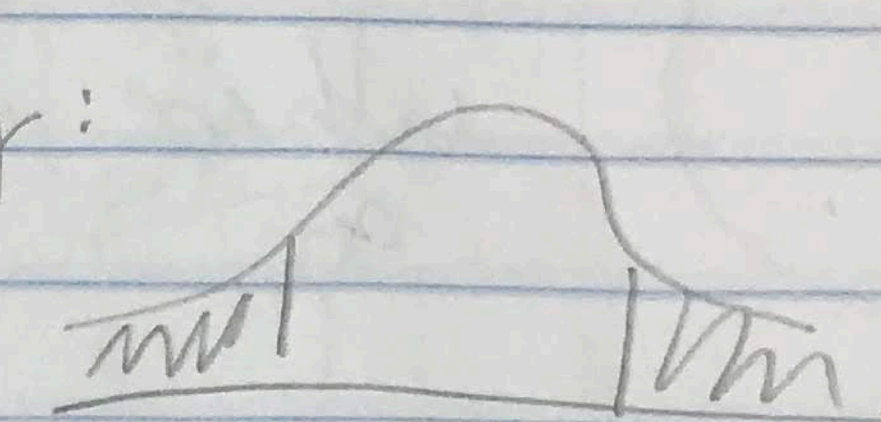


d) can define critical region in terms of x or the ratio

e) power: prob of rejecting H_0 given H_1 is true

? $P(|X| > \Phi^{-1}(1-\frac{\alpha}{2}) \mid X \sim \text{Cauchy})$

Cauchy:



Need Cauchy CDF

↳ do integral, has tan()

Same:

$x^2 > \{\Phi^{-1}(\ )\}^2$

$\sim F(1,1)$

$1 - $ CDF of $F(1,1)$

$\boxed{2016}$

3) a) X is a_1 or a_2 or a_3 or a_4

if $X = a_1$, MLE of θ is θ_2

if $X = a_2$, MLE is θ_1

$= a_3$ θ_3

$= a_4$ θ_2

b) LRT is function of X value you'll observe

	a_1	a_2	a_3	a_4	
MLE	θ_2	θ_1	θ_3	θ_2	
H_1: $L(MLE)$	0.4	0.4	0.5	0.3	← max value of likelihood
H_0: $L(\theta_1)$	0.3	0.4	0.1	0.2	
ratio $\frac{H_1}{H_0}$:	$4/3 = 1\frac{1}{3}$	1	5	$3/2 = 1\frac{1}{2}$	reject null if $\frac{H_1}{H_0} \geq k$

reject if ratio $\frac{H_1}{H_0} \geq 5$

i.e if $x = a_3$, otherwise fail to reject.

3b) if $\alpha = 0.3$

0.1

Values : 5 $1\frac{1}{2}$ | $1\frac{1}{3}$ 1

?

where you draw your line is based on α

look at value of likelihood ratio statistic not x.

You can achieve other alphas but that test is not going to be UMP.

Looking w/in a row : pmf
Looking w/in column: likelihood

c) What we did (?)
d) Point vs Point

reject

	a_1	a_2	a_3	a_4
	0.3	0.4	0.1	0.2
	0.4	0.1	0.2	0.3
$\frac{L(\theta_1)}{L(\theta_2)}$ ratio:	3/4	4	1/2	2/3

sort them: 4 3/4 2/3 1/2

Probability under H_0: 0.4 .3 0.2 $\boxed{0.1}$

reject if too small

want type I error 0.1

reject if this ratio if $\frac{1}{2}$ or less

UMP b/c point vs. point, relies on N-P Lemma

e) θ_1 against θ_2 UMP is what we got
the question is

θ_1 against θ_3 :

	a_1	a_2	a_3	a_4
$\frac{L(\theta_1)}{L(\theta_3)} =$	$\frac{3}{2}$	4	$\frac{1}{5}$ or 1	1

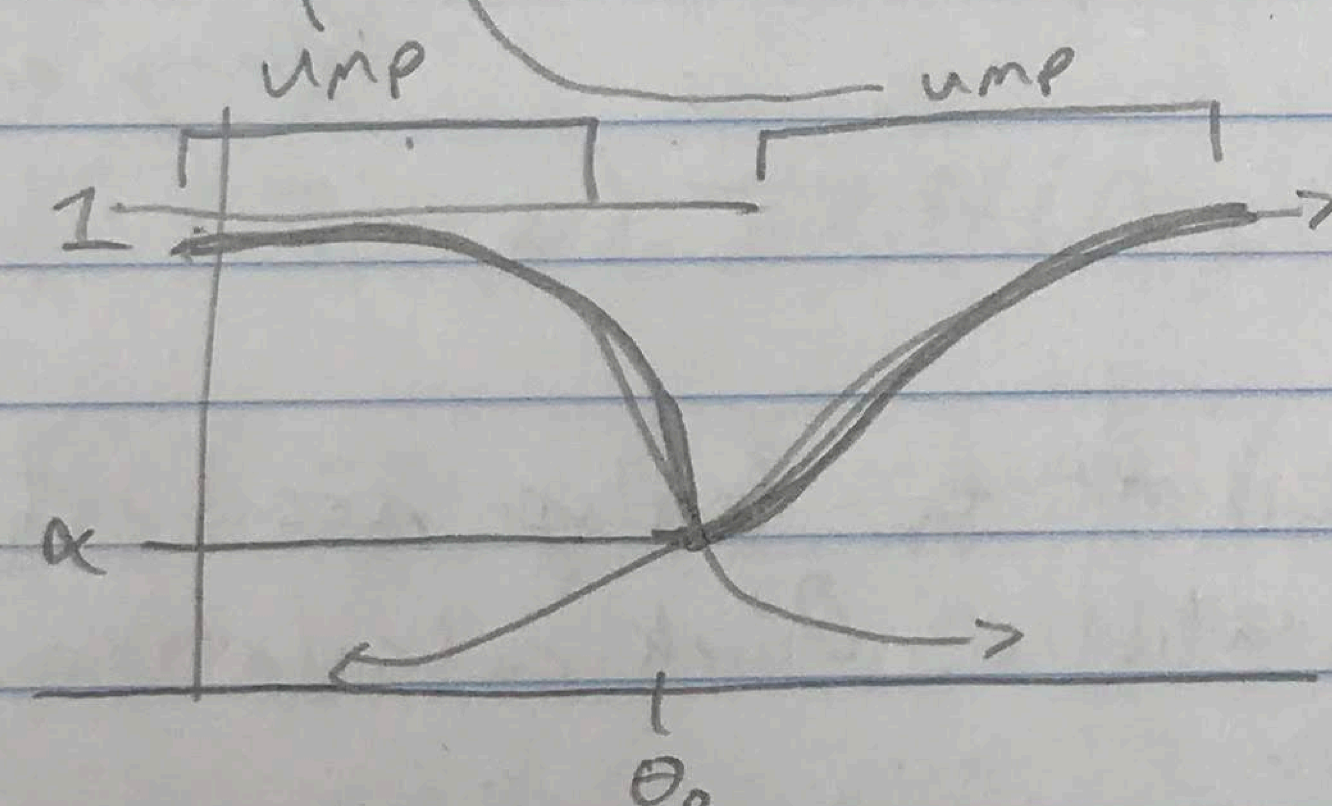
Probability under H_0 :

Against θ_2 and against θ_3, this is UMP,
if $x = a_3$

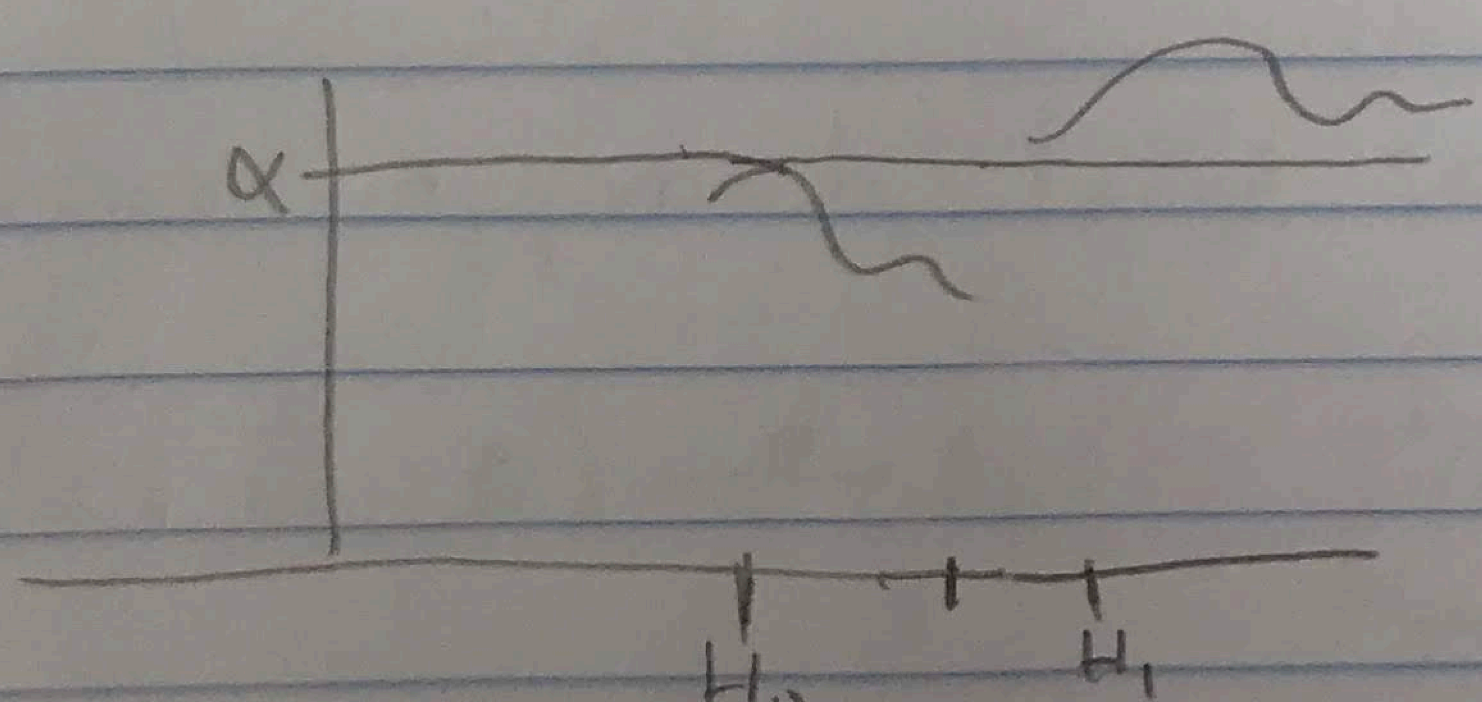
$\theta = \theta_0$ vs $\theta > \theta_0$

$\theta = \theta_0$ vs $\theta < \theta_0$

$\theta = \theta_0$ vs $\theta \neq \theta_0$



Unbiased test: any test as a size, probability of rejecting H_0 when H_0 true, which is α. Unbiaset test: power curve never dips below α. (when H_1 is true)



prefer this to be larger than the second. → P(reject Ho | Ho False)

P(reject Ho | Ho True)

This is true of unbiased tests.

2015

3e) For LRT just need likelihood function

Likelihood:

$L_1(\theta | \underline{x}) \cdot L_2(\beta | y)$ ← maximize them separately under alternative

H_1: $\hat{\theta} = \frac{1}{\bar{x}}$, $\hat{\beta} = \frac{1}{\bar{y}}$ ← MLE separately

$\theta = \beta$ One sample w/size $2n$ → H_0: X's and Y's are all iid, so can look at all of them together:

$$\hat{\theta} = \hat{\beta} = \frac{2n}{n\bar{X} + n\bar{Y}}$$

(1/mean) reciprocal of the mean of the combined sample b/c H_0 is they have the same distribution $exp(\theta)$ or $exp(\beta)$ (they are same under H_0)

$X_1, \ldots, X_n \sim Exp(\theta)$ iid

$Y_1, \ldots, Y_n \sim Exp(\beta)$ iid

Score: take derivative, only need H_0. Calculate variance of score statistic. Quick calculation

7/3/19 Review

$\boxed{2015}$ 3) a) $\hat{\theta} = \frac{1}{\bar{x}}$

b) $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \_\_\_)$

know it's Normal b/c

$(T - \mu)$

CLT $\rightarrow$ $\sqrt{n}(\bar{x} - \frac{1}{\theta}) \xrightarrow{d} N(0, \frac{1}{\theta^2})$

$\mu = EX = \frac{1}{\theta}$

$\sigma^2 = Var X = \frac{1}{\theta^2} = \mu^2$

$\sqrt{n}(\frac{1}{\bar{x}} - \theta) \xrightarrow{d} N(0, \frac{1}{\theta^2} \cdot \_\_)$ ← Delta Method

$g(\bar{x}) = \frac{1}{\bar{x}}$, $g(t) = \frac{1}{t}$

$g'(t) = -1/t^2$ substitute in μ for t

$T = \bar{X}$

$\sqrt{n}(T - \mu) \xrightarrow{N} N(0, \mu^2)$ ← this is our starting point

$\sqrt{n}(\frac{1}{\bar{x}} - \frac{1}{\mu}) \xrightarrow{N} N(0, \mu^2 \cdot (-\frac{1}{\mu^2})^2)$

$\equiv N(0, \frac{1}{\mu^2})$

$\equiv N(0, \theta^2)$

Can only use b/c $\hat{\theta}$ is function of $\bar{X}$, and have CLT w/ $\bar{X}$

Other approach: large sample properties of MLE

$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \frac{1}{i(\theta)})$ $\underbrace{\text{as } n \to \infty}_{\text{always}}$

information for 1 (or avg of all)

$\boxed{2015}$ 3 b)

$\theta e^{-\theta x_i}$ (when iid, just do 1 observation)

$\ell(\theta) = \log\theta - \theta x_i$

$\frac{\partial \ell}{\partial \theta} = \frac{1}{\theta} - x_i \longrightarrow$ get var $\rightarrow i(\theta) = \frac{1}{\theta^2}$

2^{nd} method is easier + more general

CRLB: just $\frac{1}{i(\theta)}$, so <u>asymptotically</u>

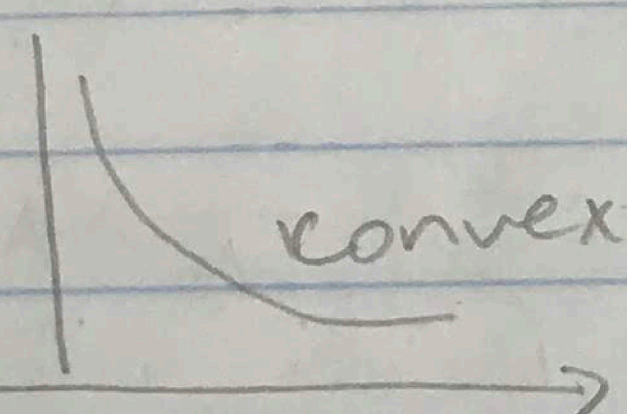
it does achieve the CRLB.

$\rightarrow$ Is $\hat{\theta}$ biased or unbiased?

$E(\bar{x}) = \frac{1}{\theta}$

$E[\frac{1}{\bar{x}}] \stackrel{?}{=} \theta$ $\leftarrow$ not linear function, so not equal

$\bar{x}$ is positive



<u>Jensen's</u>

if g convex, $E[g(x)] > g(E[x])$

so

$E[\hat{\theta}] = E[\frac{1}{\bar{x}}] > \frac{1}{E\bar{x}} = \frac{1}{1/\theta} = \theta$

On exam, don't try to figure out exact expected value. Waste of time.

Side: $Y \sim$ Gamma, $E[\frac{1}{Y^r}]$ $r = 1, 2, 3, \ldots \rightarrow$ rearrange

$\boxed{2015}$ 1) d-f

d) Use induction for these kinds of problems

Assuming relationship is true for n, show it is true for $n+1$

and show it's true for 1 (EX_i)

e) hard to find prob distribution marginally, but can condition on X_n

Step	R	B	Total (fixed)
0	a	b	$a+b$
n	X_n	$a+b-X_n$	

$$\underbrace{P(\text{red}}_{\text{event } R}|X_n) = \frac{X_n}{a+b} \quad \leftarrow \text{linear function of } X_n$$

double expectation:

prob is $E[I(R)|X_n]$

$$EE[I(R)|X_n] = E[I(R)] = P(R)$$

$$\downarrow$$

$$E[X_n/a+b]$$

f) $P(Y = a+b) = 1$ $\longrightarrow$ means Y is a constant equal to $a+b$ which is total

Given $\varepsilon > 0$

$P(|X_n - (a+b)| > \varepsilon) \longrightarrow 0$ as $n \to \infty$

(X_n is only integer, $a+b$ is integer, ε can be 0.1

$\to P(X_n = a+b) \longrightarrow 1$

2015 1f)

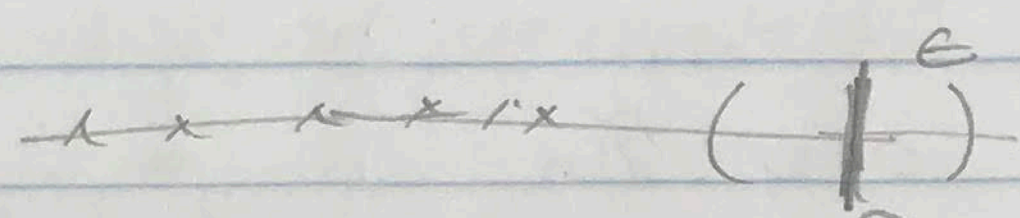
prob $\overbrace{P(R_n)}^{\text{sequence}} \longrightarrow 1$ as $n \to \infty$

Given $\epsilon > 0$ $\exists$ (there is) an integer m such that

$$|P(R_n) - 1| < \epsilon \quad \forall n > m$$

only values $P(R_n)$ takes $\longrightarrow P(R_n) \in \{\frac{0}{a+b}, \frac{1}{a+b}, \dots, \frac{a+b-1}{a+b}, 1\}$

1) mathematical convergence from e)



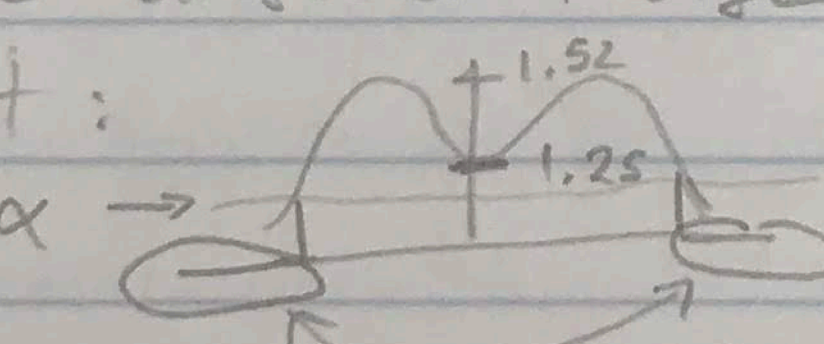
after m, every single point will be in the ϵ interval

→ in this problem, the numbers are fixed and if you're within 1/2 of $\frac{1}{a+b}$, then you have to be at 1.

2017 3d, e)

d) Give answer: use N-P Lemma

Plot:

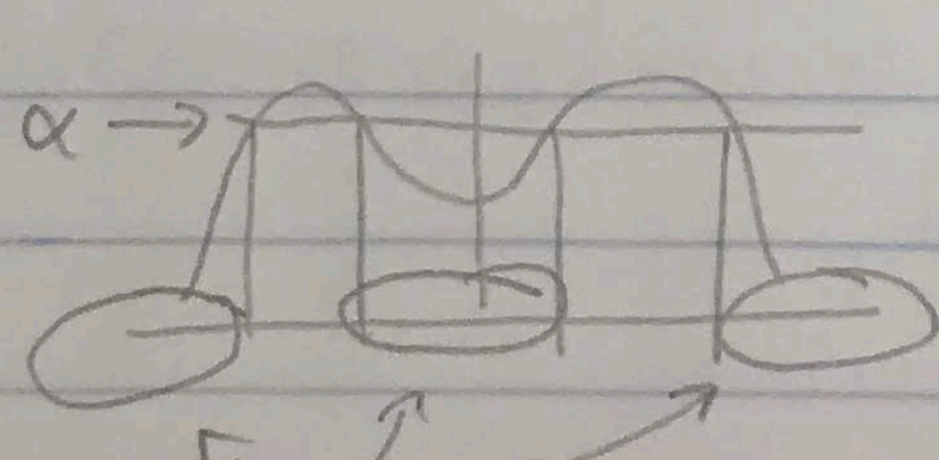


$R: \{x_1 < -c\} \cup \{x_1 > c\}$

$P(x_1 \in R \mid H_0) = \alpha$

union of these two areas is your rejection region

α is always under the null, $N(0,1)$ so find critical region here



$\alpha: 0.63$ or 0.32

Rejection Region

e) Just use the definition of power

$\boxed{2017}$ 3 c)

Y_i's → Bernoulli's

$Y_1, \dots, Y_n$ iid Bern(θ)

$H_0: \theta = \frac{1}{2}$ $\quad$ $H_1: \theta \neq \frac{1}{2}$

$\hat{\theta} = \bar{Y}$

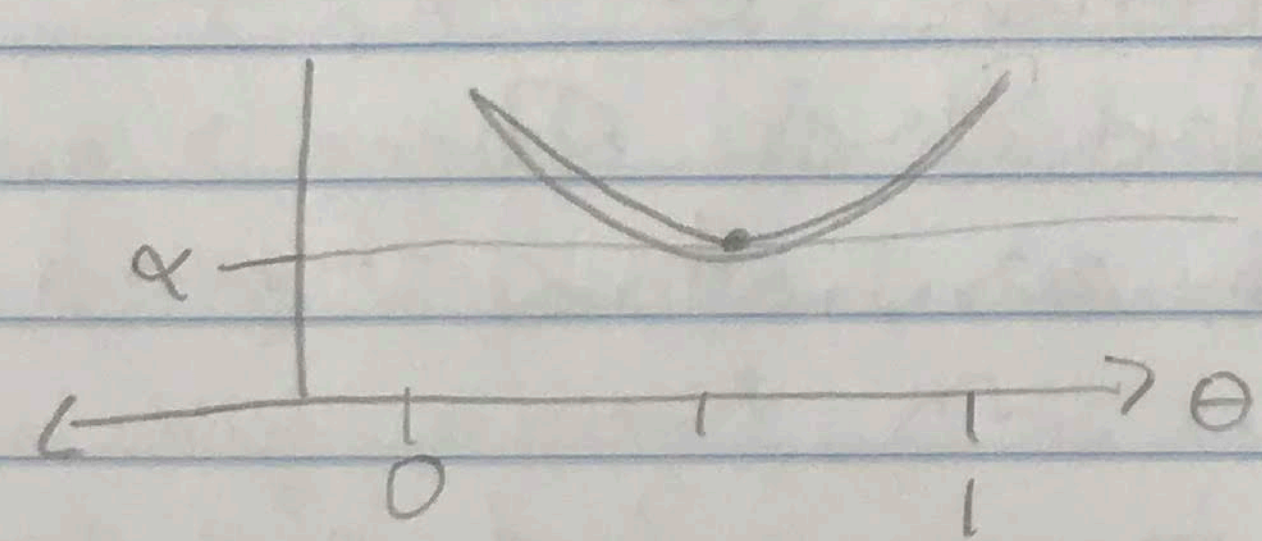
LRT: $\frac{H_0}{H_1} = \left(\frac{\left(\frac{1}{2}\right)}{\left\{ \bar{Y}^{\bar{Y}} (1-\bar{Y})^{1-\bar{Y}} \right\}} \right)^n$

take log, -2

asymptotic is easier

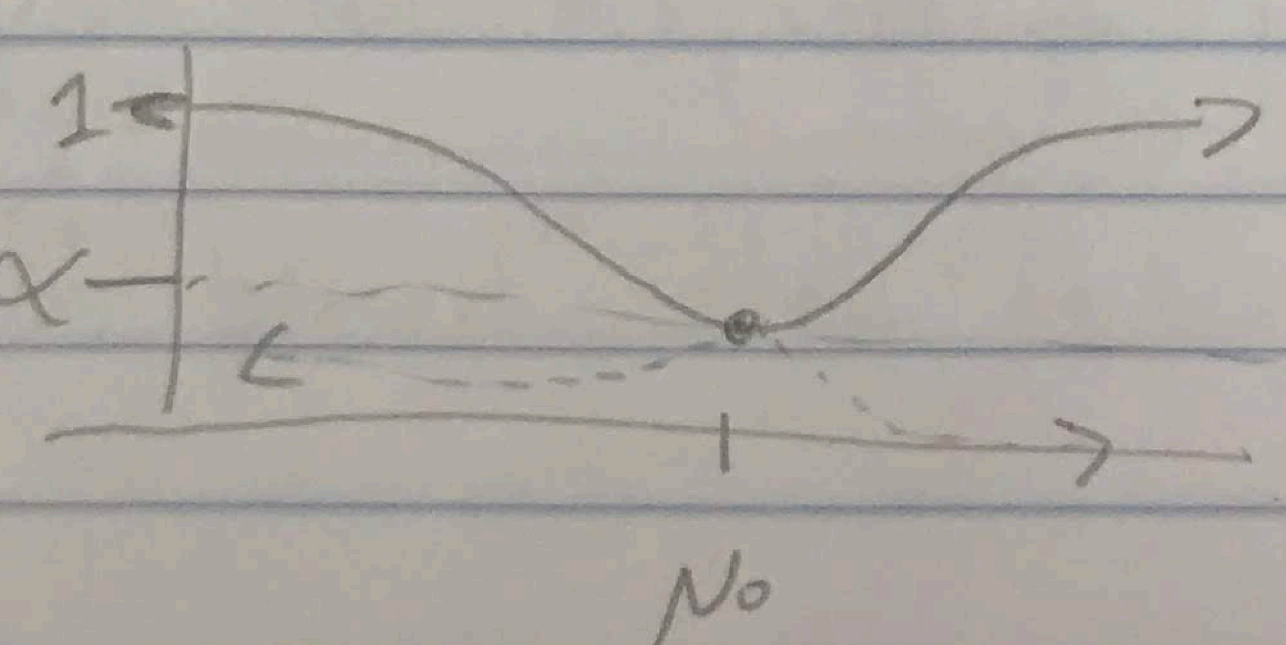
3e) Can't draw power function b/c simple vs simple

→ Sketching Power Function



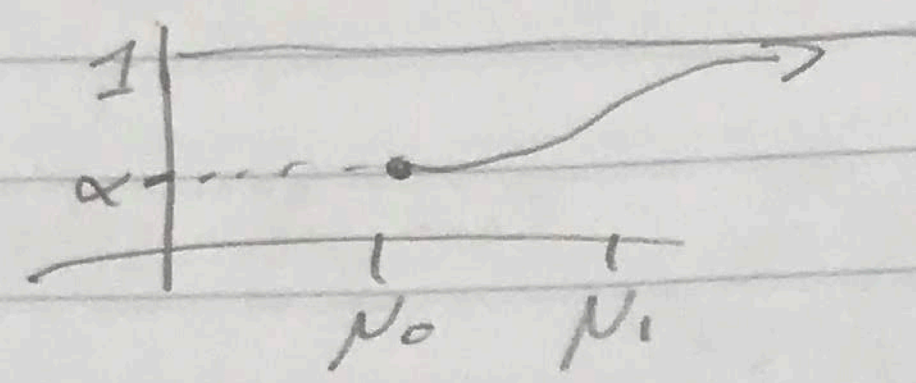
$N(\mu, 1)$

$H_0: \mu = \mu_0$ $\quad$ $H_1: \mu \neq \mu_0$ $\quad$ Z statistic



Want to guarantee α is no larger than a certain place

One-sided Test:



$\boxed{2015}$ 4a)

this transformation preserves the order $\rightarrow Y_i = \dfrac{X_i - \theta}{a} \longrightarrow X_i = aY_i + \theta$

b) Derive pdf of X_n, calculate expected value, it is a linear

$$\underbrace{Y_{(n)}}_{\sim \text{Beta}} = \frac{X_{(n)} - \theta}{a} \longrightarrow X_{(n)} = aY_{(n)} - \theta$$

, but can also find CDF, derivative

$E(Y_{(n)})$

CRLB $\longrightarrow$ b/c log likelihood is not differentiable anywhere $\theta = X_i$?
CRLB requires derivatives. There is an indicator function, so not differentiable

For MLE's always imagine what plot looks like.

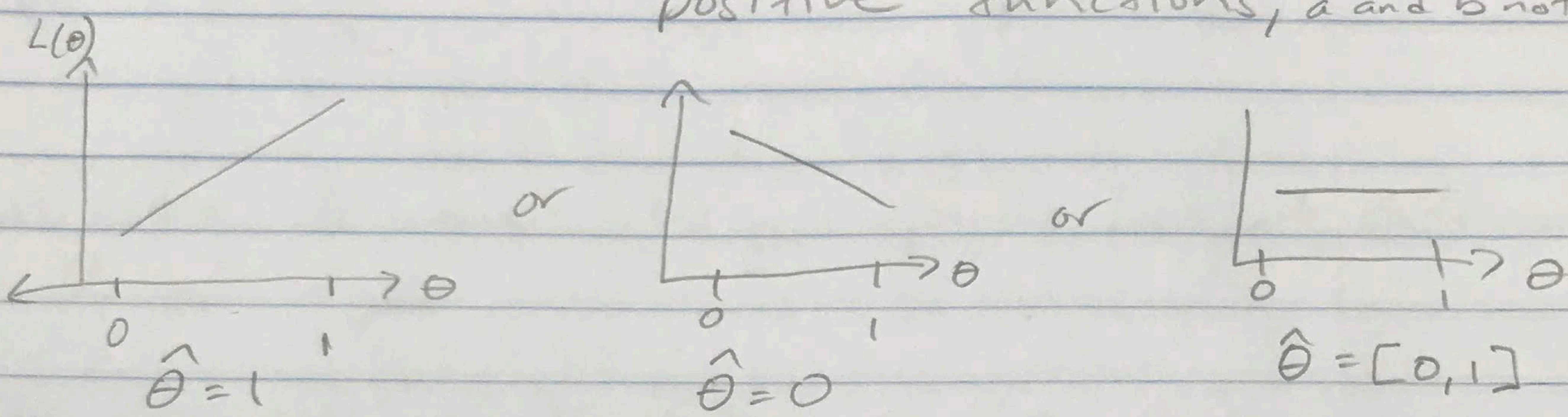
There was a problem where the density looked like:

$$f_X(x|\theta) = \theta \cdot a(x) + (1-\theta)\, b(x) = \theta \cdot a(x) + b(x) - \theta b(x)$$

$$\theta \in [0, 1]$$

a is given, only one observation

Find mle: $(a(x)$ is pdf, so is $b(x)$, both positive functions, a and b not same$)$



$$f_X(x|\theta) = \theta\{a(x) - b(x)\} + b(x)$$

↑ linear function of θ

slope is $\{a(x) - b(x)\}$

↳ if positive, then $\hat{\theta} = 1$

if negative, then $\hat{\theta} = 0$

→ if 0, then $\hat{\theta} = [0, 1]$

doesn't happen b/c X is continuous so probability of it being a particular number is zero, so don't need to worry about this case

Don't be mechanical. Open up your mind and understand the problem the way it is.

$\boxed{\text{Try to find this problem}}$

$\hat{\theta}$ biased? How to fix bias?

1 observation, $Y \sim Bern(\theta)$

MLE of $\theta = \bar{Y} = Y \longrightarrow$ could be 0 or 1

MLE Binomial: $\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}$