1 3.17

$$EX^v = \int_0^\infty x^v \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$
$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{(v+\alpha)-1} e^{-x/\beta} dx$$
$$= \frac{\Gamma(\alpha+v)\beta^{\alpha+v}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \frac{x^{(v+\alpha)-1}e^{-x/\beta}}{\Gamma(\alpha+v)\beta^{\alpha+v}} dx$$
$$= \frac{\Gamma(\alpha+v)\beta^{\alpha+v}}{\Gamma(\alpha)\beta^\alpha} = \frac{\Gamma(\alpha+v)\beta^v}{\Gamma(\alpha)}$$

2 3.18

Recall from Homework 7 that the MGF of a negative binomial random variable, Y, with parameters r and p, is

$$M_Y(t) = (\frac{p}{1-(1-p)e^t})^r$$

Let $Z = pY$. Thus,

$$M_Z(t) = M_pY(t) = M_Y(pt) = (\frac{p}{1-(1-p)e^{pt}})^r$$

Observe that by L'Hospital's Rule,

$$lim_{p\to 0}(\frac{p}{1-(1-p)e^{pt}}) = lim_{p\to 0}(\frac{p}{1-e^{pt}+pe^{pt}}) = lim_{p\to 0}(\frac{1}{-te^{pt}+pte^{pt}+e^{pt}}) = \frac{1}{1-t}$$

This implies that as $p \to 0$, the MGF of pY where Y is a negative binomial random variables converges to

$$(\frac{1}{1-t})^r$$

The MGF of a gamma random variable with parameters α and β is

$$(\frac{1}{1-\beta t})$$

Thus, as $p \to 0$, the MGF of pY where Y is a negative binomial random variables converges to the MGF of a gamma random variable with $\alpha = r$ and $\beta = 1$.

3 3.20

a Find the mean and variance of X.

Mean:

$$\begin{aligned} EX &= \int_0^\infty x\frac{2}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= -\frac{2}{\sqrt{2\pi}}[e^{-x^2/2}]_0^\infty \\ &= -\frac{2}{\sqrt{2\pi}}[0-1] \\ &= \frac{2}{\sqrt{2\pi}} \end{aligned}$$

Variance:

$$EX^2 = \int_0^\infty x^2\frac{2}{\sqrt{2\pi}}e^{-x^2/2}dx$$

Let $u = \frac{x^2}{2}$, which implies $du = xdx$, which implies $dx = \frac{1}{\sqrt{2u}}du$. Thus,

$$\begin{aligned} EX^2 &= 2\int_0^\infty (2u)\frac{e^{-u}}{\sqrt{2\pi}}\frac{1}{\sqrt{2u}}du \\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty u^{1/2}e^{-u}du \end{aligned}$$

Observe that the integral is $\Gamma(\frac{3}{2})$. Therefore,

$$EX^2 = \frac{2}{\sqrt{\pi}}\Gamma(\frac{3}{2})$$

Recall that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Thus,

$$EX^2 = \frac{2}{\sqrt{\pi}}\frac{1}{2}\Gamma(\frac{1}{2})$$

Recall that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Therefore,

$$EX^2 = \frac{2}{\sqrt{\pi}}\frac{1}{2}\sqrt{\pi} = 1$$

Thus,

$$VarX = E(X^2) - (E(X))^2 = 1 - (\frac{2}{\sqrt{2\pi}})^2 = 1 - \frac{4}{2\pi} = \frac{\pi - 2}{\pi}$$

(b) Let $Y = X^2$ and $X = \sqrt{Y}$ and $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$.

$$f_Y(y) = f_X(g_x(y))|\frac{dx}{dy}| = \frac{1}{\sqrt{2}\sqrt{\pi}} \cdot y^{\frac{1}{2}-1}e^{-\frac{y}{2}}$$

Let $\Gamma(\frac{1}{2}) = \sqrt{\pi}$:

$$f_Y(y) = \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} y^{\frac{1}{2}-1}e^{-\frac{y}{2}}$$

Therefore, in our case $Y \sim Gamma((1/2), 2)$.

HW 9

4. C-B 3.25

$$h_T(t) = \lim_{\delta \to 0} \frac{p(t \le T \le t+\delta \mid T \ge t)}{\delta}$$

$$= \lim_{\delta \to 0} \frac{p(t \le T \le t+\delta)}{\delta \cdot p(T \ge t)} = \lim_{\delta \to 0} \frac{F(t+\delta) - F(t)}{\delta\, p(T \ge t)}$$

$$\overset{\text{L'Hopital}}{=} \lim_{\delta \to 0} \frac{f(t+\delta)}{p(T \ge t)} = \frac{f_T(t)}{p(T \ge t)} = \frac{f_T(t)}{1-F(t)}$$

✓

$$\frac{f_T(t)}{1-F_T(t)} = \frac{1}{1-F_T(t)} \frac{dF_T(t)}{dt} = \frac{-1}{1-F_T(t)} \frac{d(1-F_T(t))}{dt}$$

$$= -\frac{d(\ln(1-F_T(t)))}{dt}$$

5. C-B 3.26

a) $T \sim \exp(\beta)$

$$f_T(t) = \frac{1}{\beta} e^{-\frac{t}{\beta}}$$

$$F_T(t) = 1 - e^{-\frac{t}{\beta}}$$

$$h_T(t) = \frac{f_T(t)}{1-F_T(t)} = \frac{\frac{1}{\beta} e^{-\frac{t}{\beta}}}{1-(1-e^{-t/\beta})} = \frac{\frac{1}{\beta} e^{-\frac{t}{\beta}}}{e^{-t/\beta}} = \frac{1}{\beta} \qquad (t>0).$$

✓

b) $T \sim \text{Weibull}(\gamma, \beta)$

$$f_T(t) = (\gamma/\beta)\, t^{\gamma-1} e^{-(t^\gamma/\beta)}$$

$$F_T(t) = \int_0^T f_T(t)\,dt = -e^{-(t^\gamma/\beta)}\Big|_0^T = 1 - e^{-(t^\gamma/\beta)}$$

✓

$$h_T(t) = \frac{f_T(t)}{1-F_T(t)} = \frac{(\gamma/\beta)\, t^{\gamma-1} e^{-(t^\gamma/\beta)}}{1-(1-e^{-(t^\gamma/\beta)})} = \frac{(\gamma/\beta)\, t^{\gamma-1} e^{-(t^\gamma/\beta)}}{e^{-(t^\gamma/\beta)}} = (\gamma/\beta)\, t^{\gamma-1} \qquad (t>0)$$

(c) $T \sim \text{logistic}(\mu, \beta)$

$$F_T(t) = \frac{1}{1+e^{-(t-\mu)/\beta}}$$

$$f_T(t) = F_T'(t) = -\frac{1}{(1+e^{-(t-\mu)/\beta})^2} \cdot e^{-\frac{(t-\mu)}{\beta}} \left(-\frac{1}{\beta}\right)$$

✓

$$h_T(t) = \frac{f_T(t)}{1-F_T(t)} = F_T(t)(1-F_T(t))\frac{1}{\beta} \Big/ (1-F_T(t)) = (1/\beta)\, F_T(t)$$

6.

a. Uniform(a, b) has pdf $f_X(x) = \frac{1}{b-a}$, which is constant. Thus for all $c \in (a, b)$, we have for $y \leq x \leq c$, $f(c) \geq f(x) \geq f(y)$ since $f(c) = \frac{1}{b-a} = f(x) = f(y)$. Similarly for $x, y \geq f(c)$, $f(x) = f(y) = f(c)$. Since c is arbitrary, all points are the mode and Uniform(a,b) is unimodal.

6b Consider a gamma distribution with parameter values $\alpha = 1/2$, $\beta = 1$. Then the pdf is

$$f_X(x) = \frac{1}{\Gamma(1/2)} x^{(-1/2)} \exp(-x)$$

The derivative is

$$\frac{d}{dx} f_X(x) = -\frac{\exp(-x)(2x+1)}{2x^{(3/2)}\Gamma(1/2)}$$

This is always negative in the range $x \in (0, \infty)$. Furthermore the right hand limit as x goes to 0 of $f_X(x)$ is ∞. Therefore we cannot define $f_X(x)$ on the range $[0, \infty)$ without introducing a discontinuity. For every $\epsilon_1 > 0$ there exists an ϵ_2 such that $0 < \epsilon_2 < \epsilon_1$ which means $f_X(\epsilon_2) > f_X(\epsilon_1)$. Therefore there is no value $a \in (0, \infty)$ that satisfies the condition from 2.27, and the gamma is not unimodal.

C. The Normal distribution is unimodal. Take $f_X(x) = \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-(x-\mu)^2/2\sigma^2}$, ignoring constants,

$$\frac{d}{dx}[f(x)] = -\frac{(x-\mu)}{\sigma^2} e^{-(x-\mu)^2/\sigma^2},$$ which

is positive (and thus $f(x)$ is increasing) for $x < \mu$, and negative ($f(x)$ decreasing) for $x > \mu$, 0 at $x = \mu$. By the first derivative test, $\Rightarrow \mu$ is a absolute maximum of $f(x)$, and thus, the mode.

(d).

$X \sim beta(\alpha, \beta)$

$$f(x) = \frac{1}{B(\alpha,\beta)} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1}, \quad x \in (0,1), \ \alpha>0, \ \beta>0.$$

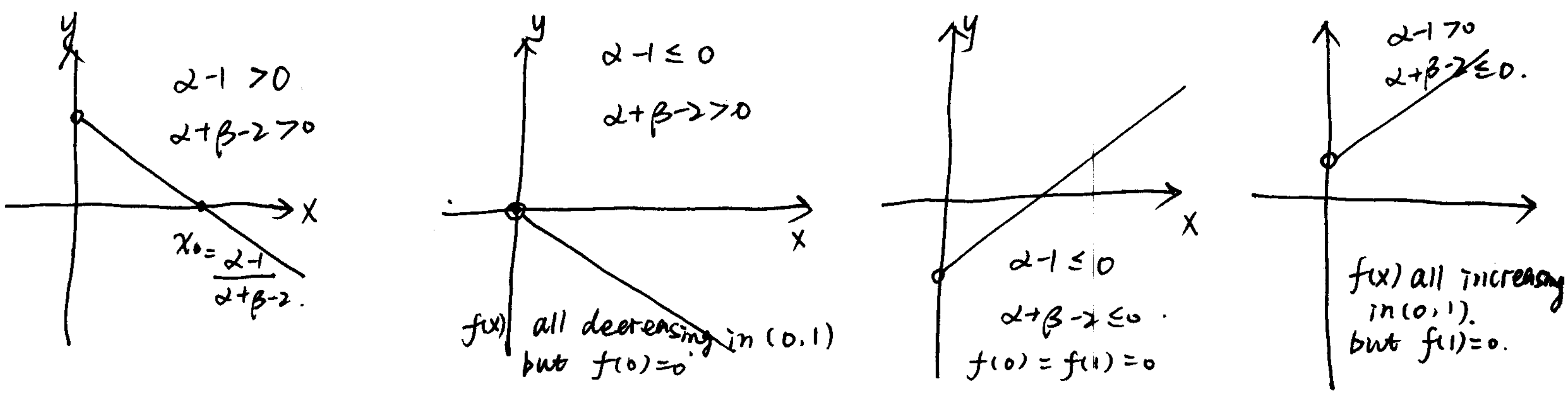
$$\frac{df(x)}{dx} = \frac{1}{B(\alpha,\beta)} \cdot x^{\alpha-2}(1-x)^{\beta-2}\left[(\alpha-1)-(\alpha+\beta-2)\cdot x\right].$$

✓

if $\alpha \in (0,1]$, then beta(α,β) has no mode (just

the positive sign of $f'(x)$ is dependent on $(\alpha-1)-(\alpha+\beta-2)x = g(x)$

There are 4 situation for this linear function. $g(x)$



$\therefore$ only when $\alpha-1>0$, $\alpha+\beta-2>0$, $\frac{\alpha-1}{\alpha+\beta-2} \in (0,1) \iff \beta>1$

which is $\alpha>1$, $\beta>1$. Then x_0 is the mode of $f(x)$

and $f(x)$ is unimodal.

BIOS 660 Homework 9 Question 7-9

7. C&B 3.32.

(a) In form (3.4.7), $f(x|\eta) = h(x)\cdot c^{*}(\eta)\cdot \exp\left(\sum_{i=1}^{k}\eta_i \cdot t_i(x)\right)$.

$\because$ If $f(x|\theta) = h(x)\cdot c(\theta)\cdot \exp\left(\sum_{i=1}^{k} w_i(\theta)\cdot t_i(x)\right)$

then $E\left(\sum_{i=1}^{k}\frac{\partial w_i(\theta)}{\partial \theta_j}\cdot t_i(X)\right) = -\frac{\partial}{\partial\theta_j}\log c(\theta)$

$Var\left(\sum_{i=1}^{k}\frac{\partial w_i(\theta)}{\partial\theta_j}\cdot t_i(X)\right) = -\frac{\partial^2}{\partial\theta_j^2}\cdot\log c(\theta) - E\left(\sum_{i=1}^{k}\frac{\partial^2 w_i(\theta)}{\partial\theta_j^2}\cdot t_i(X)\right)$

$\because w_i(\eta) = \eta_i$, $\therefore \frac{\partial w_i(\eta)}{\eta_i} = 1$, $\frac{\partial w_i(\eta)}{\eta_j} = 0$ if $i\neq j$. $\therefore \frac{\partial^2 w_i(\eta)}{\partial\eta_j} = \frac{\partial^2 w_i(\eta)}{\partial\eta_i} = 0$ if $i\neq j$

$\therefore E\left(\sum_{i=1}^{k}\frac{\partial w_i(\eta)}{\partial\eta_j}\cdot t_i(x)\right) = E(1\cdot t_j(x)) = -\frac{\partial}{\partial\eta_j}\cdot\log c^{*}(\eta)$

$\therefore E(t_j(X)) = -\frac{\partial}{\partial\eta_j}\cdot\log c^{*}(\eta)$.

$\therefore Var\left(\sum_{i=1}^{k}\frac{\partial w_i(\eta)}{\partial\eta_j}\cdot t_i(x)\right) = Var(1\cdot t_j(x)) = Var(t_j(X))$

$= -\frac{\partial^2}{\partial\eta_j^2}\cdot\log c^{*}(\eta) - E\left(\sum_{i=1}^{k}\frac{\partial^2\eta_i}{\partial\eta_j^2}\cdot t_i(x)\right)$

$= -\frac{\partial^2}{\partial\eta_j^2}\cdot\log c^{*}(\eta) - 0$.

$\therefore Var(t_j(x)) = -\frac{\partial^2}{\partial\eta_j^2}\cdot\log c^{*}(\eta)$.

(b) Let $X\sim gamma(a,b)$,

then $f_X(x|a,b) = \frac{1}{\Gamma(a)\cdot b^a}\cdot x^{a-1}\cdot e^{-\frac{x}{b}}$, $x\in(0,\infty)$, $a>0$, $b>0$.

$\therefore f_X(x|a,b) = \frac{1}{\Gamma(a)\cdot b^a}\left(1_{(0,\infty)}(x)\right)\exp\left((a-1)\cdot\ln x - \frac{1}{b}\cdot x\right)$

$\therefore \eta_1 = a-1$, $t_1(x) = \ln x$,

$\eta_2 = -\frac{1}{b}$, $t_2(x) = x$.

$c^{*}(\eta) = \frac{1}{\Gamma(a)\cdot b^a} = \frac{(-\eta_2)^a}{\Gamma(\eta_1+1)}$

$\therefore E(t_2(x)) = E(X) = -\frac{\partial}{\partial\eta_2}\cdot\log c^{*}(\eta) = -\frac{\Gamma(\eta_1+1)}{(-\eta_2)^a}\cdot a\cdot(-\eta_2)^{a-1}\cdot\frac{1}{\Gamma(\eta_1+1)}$

$= \frac{\Gamma(\eta_1+1)}{-\eta_2}\cdot a\cdot\frac{1}{\Gamma(\eta_1+1)} = -a\eta_2^{-1} = ab$

$Var(t_2(x)) = Var(X) = -\frac{\partial^2}{\partial\eta_2^2}\cdot\log c^{*}(\eta) = -\frac{\partial(a\eta_2^{-1})}{\partial\eta_2} = a\cdot\eta_2^{-2} = ab^2$

$\therefore$ For $X\sim gamma(a,b)$, $E(X) = ab$, $Var(X) = ab^2$

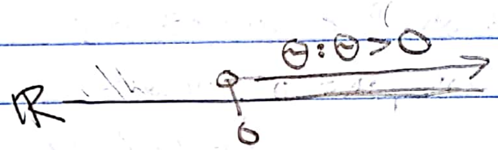
8. a) $N(\Theta, \Theta)$

(3.33) i) Suppose $Y \sim N(\mu = \Theta, \sigma^2 = \Theta)$, $\Theta \in (0, \infty)$. Then:

$$f_Y(x) = \frac{1}{\sqrt{2\pi\Theta}} \exp\left(-\frac{(x-\Theta)^2}{2\Theta}\right) 1(x \in \mathbb{R})$$
$$= \frac{1}{\sqrt{2\pi\Theta}} \exp\left(-\frac{\Theta^2}{2\Theta}\right) \exp\left(-\frac{x^2}{2\Theta} + \frac{2x\Theta}{2\Theta}\right) 1(x \in \mathbb{R})$$
$$= \frac{1}{\sqrt{2\pi\Theta}} \exp\left(-\frac{\Theta}{2}\right) \exp\left(-\frac{x^2}{2\Theta} + x\right) 1(x \in \mathbb{R})$$
$$= \frac{1}{\sqrt{2\pi\Theta}} \exp\left(-\frac{\Theta}{2}\right) \exp(x) \exp\left(-\frac{x^2}{2\Theta}\right) 1(x \in \mathbb{R})$$
$$= h(x) c(\Theta) \exp(w_1(\Theta) t_1(x)),\ x \in \mathbb{R}, \text{ where}$$

$h(x) = \exp(x) 1(x \in \mathbb{R})$, $c(\Theta) = \frac{1}{\sqrt{2\pi\Theta}} \exp\left(-\frac{\Theta}{2}\right)$, $w_1(\Theta) = \frac{1}{2\Theta}$, $t_1(x) = -x^2$. Thus $N(\Theta, \Theta)$ an exponential family.

ii) The parameter space is $\Theta \in \{\Theta : \Theta > 0\}$ which lies on the positive real line

iii) 

Sketch of parameter space

b) $N(\Theta, a\Theta^2)$

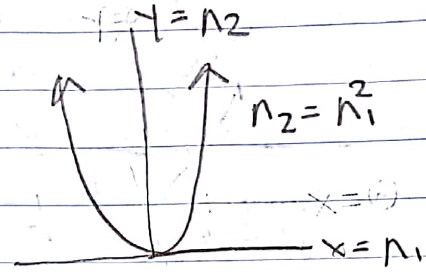
i) Suppose $Y \sim N(\mu = \Theta, \sigma^2 = a\Theta^2)$, $\Theta \in (0, \infty)$. Then:

$$f_Y(x) = \frac{1}{\sqrt{2\pi a}\,\sigma} \exp\left(-\frac{(x-\Theta)^2}{2a\Theta^2}\right) 1(x \in \mathbb{R})$$
$$= \frac{1}{\sqrt{2\pi a}\,\sigma} \exp\left(-\frac{\Theta^2}{2a\Theta^2}\right) \exp\left(-\frac{x^2}{2a\Theta^2} + \frac{2x\Theta}{2a\Theta^2}\right) 1(x \in \mathbb{R})$$
$$= \frac{1}{\sqrt{2\pi a}\,\sigma} \exp\left(-\frac{1}{2a}\right) \exp\left(-\frac{x^2}{2a\Theta^2} + \frac{x}{a\Theta}\right) 1(x \in \mathbb{R})$$
$$= h(x) c(\Theta) \exp\left(\textstyle\sum_{i=1} w_i(\Theta) t_i(x)\right), \text{ where}$$

$h(x) = 1(x \in \mathbb{R})$, $c(\Theta) = \frac{1}{\sqrt{2\pi a}\,\sigma} \exp\left(-\frac{1}{2a}\right)$, $w_1(\Theta) = \frac{1}{\Theta^2}$, $t_1(x) = -\frac{x^2}{2a}$, $w_2(\Theta) = \frac{1}{\Theta}$, $t_2(x) = \frac{x}{a}$

Thus $N(\Theta, a\Theta^2)$ is an exponential family iii & iii)

Let $\eta_1 = w_2(\theta) = \frac{1}{\theta}$ and $w_1(\theta) = \frac{1}{\theta^2} = \eta_1^2$. The parameter space is $(\eta_1, \eta_2) \in \{(\eta_1, \eta_1^2) : \eta_1 \in \mathbb{R}\}$, a parabola.



c) We first show that gamma(α,β), $(\alpha,\beta) \in (0,\infty)\times(0,\infty)$ is an exponential family

For $X \sim$ gamma(α,β):

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp[-x/\beta]\, 1(x>0)$$
$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp[\log(x^{\alpha-1})] \exp[-x/\beta]\, 1(x>0)$$
$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp\left[-\frac{1}{\beta}x + (\alpha-1)\log(x)\right] 1(x>0)$$
$$= h(x)c(\alpha,\beta)\exp\left(\textstyle\sum_{i=1}^2 w_i(\alpha,\beta)t_i(x)\right),$$ where

$h(x) = 1(x>0)$, $c(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$, $w_1(\alpha,\beta) = 1/\beta$, $t_1(x) = -x$, $w_2(\alpha,\beta) = \alpha - 1$, $t_2(x) = \log(x)$

For $Y \sim$ gamma$(\alpha = \alpha, \beta = 1/\alpha)$, $\alpha > 0$:

$$f_X(x) = \frac{1}{\Gamma(\alpha)(1/\alpha)^\alpha} \exp[-\alpha x + (\alpha-1)\log(x)]\, 1(x>0)$$
$$= h(x)c(\alpha)\exp\left(\textstyle\sum_{i=1}^2 w_i(\alpha)t_i(x)\right),\quad h(x) = 1(x>0)$$

$c(\alpha) = \frac{\alpha^\alpha}{\Gamma(\alpha)}$, $w_1(\alpha) = \alpha$, $w_2(\alpha) = \alpha - 1$, $t_1(x) = -x$

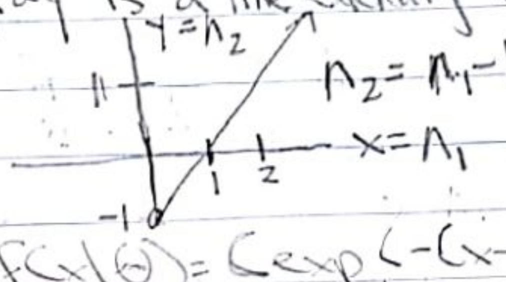
$t_2(x) = \log(x)$

Thus gamma$(\alpha, 1/\alpha)$ is an exponential family

[as is gamma(α,β), $(\alpha,\beta) \in (0,\infty)\times(0,\infty)$]

8. c) (continued)

Let $\eta_1 = w_1(\alpha) = \alpha$, $w_2(\alpha) = \alpha - 1 = \eta_2$. The parameter space is $(\eta_1, \eta_2) \in \{(\eta_1, \eta_1 - 1) : \eta_1 > 0\}$, a ray (a ray is a line extending in one direction)



d) i) Suppose $f(x|\theta) = C\exp(-(x-\theta)^4)$ is a pdf of an rv X where C is a normalizing constant, $\theta \in \mathbb{R}$. By the binomial theorem, $\exp(-(x-\theta)^4) = \exp(-\sum_{n=0}^{4} \binom{4}{n} x^n (-\theta)^{4-n})$. Further, as C is a normalizing constant, $C = [\int_{-\infty}^{\infty} \exp(-\sum_{n=0}^{4} \binom{4}{n} y^n (-\theta)^{4-n}) dy]^{-1}$ (C does not depend on x). Thus:

$$f(x|\theta) = C\exp(-(x-\theta)^4)\mathbb{1}(x \in \mathbb{R})$$
$$= C\exp(-\sum_{n=0}^{4} \binom{4}{n} x^n (-\theta)^{4-n})\mathbb{1}(x \in \mathbb{R})$$
$$= C\exp(-[\binom{4}{0}(-\theta)^4 + \binom{4}{1}x^1(-\theta)^3 + \binom{4}{2}x^2(-\theta)^2$$
$$+ \binom{4}{3}x^3(-\theta)^1 + \binom{4}{4}x^4])\mathbb{1}(x \in \mathbb{R})$$
$$= C\exp(-\theta^4)\exp(-x^4)\exp(-\binom{4}{1}x^1(-\theta)^3$$
$$-\binom{4}{2}x^2(-\theta)^2 - \binom{4}{3}x^3(-\theta)^1)\mathbb{1}(x \in \mathbb{R})$$
$$= h(x)c(\theta)\exp(\sum_{i=1}^{3} w_i(\theta)t_i(x)), \text{ where}$$

$h(x) = \mathbb{1}(x \in \mathbb{R})\exp(-x^4)$, $c(\theta) = C\exp(-\theta^4)$, $w_i(\theta) = (\theta)^{4-i}$, $t_i(x) = -\binom{4}{i}x^i(-1)^{4-i}$, $i \in \{1,2,3\}$

ii) The parameter space is $(\theta, \theta^2, \theta^3) \in \{(\theta, \theta^2, \theta^3) : \theta \in \mathbb{R}\}$, a curve on a three-dimensional plane. Attached is a sketch with the associated R code. θ^2 is the y-axis, θ^3 z-axis, θ x-axis

9. C&B 3.47

$\because Z \sim N(0,1) \qquad \therefore f_Z(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty$

$\therefore P(|Z| \geq t) = 2P(Z \geq t) = 2\int_t^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} \cdot dz$

$= 2 \cdot \int_t^{\infty} \frac{z^2+1}{z^2+1} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} dz = \sqrt{\frac{2}{\pi}} \cdot \left[\int_t^{\infty} \frac{1}{z^2+1} \cdot e^{-\frac{z^2}{2}} dz + \int_t^{\infty} \frac{z^2}{1+z^2} \cdot e^{-\frac{z^2}{2}} dz \right]$

$\because \int_t^{\infty} \frac{z^2}{1+z^2} \cdot e^{-\frac{z^2}{2}} dz = \frac{-z}{1+z^2} \cdot e^{-\frac{z^2}{2}} \Big|_t^{\infty} + \int_t^{\infty} \frac{1+z^2-2z^2}{(1+z^2)^2} \cdot e^{-\frac{z^2}{2}} dz$

$= \frac{t}{t^2+1} \cdot e^{-\frac{t^2}{2}} + \int_t^{\infty} \frac{1-z^2}{(1+z^2)^2} \cdot e^{-\frac{z^2}{2}} dz$

$\therefore P(|Z| \geq t) = \sqrt{\frac{2}{\pi}} \cdot \left(\frac{t}{t^2+1} \cdot e^{-\frac{t^2}{2}} + \int_t^{\infty} \left(\frac{1-z^2}{(1+z^2)^2} + \frac{1}{z^2+1} \right) \cdot e^{-\frac{z^2}{2}} dz \right)$

$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{t}{t^2+1} \cdot e^{-\frac{t^2}{2}} + \int_t^{\infty} \frac{2}{(1+z^2)^2} \cdot e^{-\frac{z^2}{2}} dz \right)$

$\because \frac{2}{(1+z^2)^2} > 0 \qquad \therefore \int_t^{\infty} \frac{2}{(1+z^2)^2} \cdot e^{-\frac{z^2}{2}} dz \geq 0$

$\therefore P(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \cdot \frac{t}{t^2+1} \cdot e^{-\frac{t^2}{2}}$