BIOS 662 Fall 2018

One Sample Tests for Location

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Outline

- Small sample, normally distributed
- Large sample
- Nonparametric
  - Sign test
  - Wilcoxon signed rank test

Applications

- Cross-sectional study: Collect data to test a hypothesis about the mean or median of Y
- Paired data; examples:
  - Study of how a characteristic changes from before to after a treatment
  - Study of twins
  - Individually-matched case-control study

Small Sample, Normal Distribution

- Consider a small sample, $Y_1, \ldots, Y_n$, iid with $Y_i \sim N(\mu, \sigma^2)$,
- From the previous set of notes, test statistic:

$$T = \frac{\bar{Y} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

- For two-sided alternative $H_A : \mu \neq \mu_0$, critical region:

$$C_\alpha = \{t : |t| > t_{n-1,1-\alpha/2}\}$$

- For one-sided alternative $H_A : \mu > \mu_0$, critical region:

$$C_\alpha = \{t : t > t_{n-1,1-\alpha}\}$$

Small Sample, Normal Distribution

- P-value: Probability of obtaining a test statistic as extreme or more extreme than the one observed from the sample

- For two-sided alternative $H_A : \mu \neq \mu_0$,

$$p = \Pr[T \leq -|t|] + \Pr[T \geq |t|]$$

where $T \sim t_{n-1}$. Equivalently

$$p = 2\Pr[T \leq -|t|] = 2\Pr[T \geq |t|]$$

- For one-sided alternative $H_A : \mu > \mu_0$,

$$p = \Pr[T \geq t]$$

SIDS Example

- Example: Text page 281; problem 8.2
- Investigators are interested in whether babies that die of SIDS have different birthweight than babies who do not die of SIDS.
- A study of 22 dizygotic twins compared the birthweight of the baby who died with the baby who did not die.
- Let our random variable Y be the weight of the SIDS baby (twin) minus the weight of non-SIDS baby (twin)

$$H_0 : \mu_{\text{diff}} = 0$$

$$H_A : \mu_{\text{diff}} \neq 0$$

SIDS Example cont.

- Critical region at $\alpha = 0.05$

$$C_{0.05} = \{t : |t| > t_{21,0.975} = 2.08\}$$

- $\bar{y} = 0.1818, \ \ s = 369.57, \ \ n = 22$

$$t = \frac{\bar{y}}{s/\sqrt{n}} = 0.0023$$

- P-value

$$p = 2 \times \Pr[T \leq -0.0023] = 0.9982$$

Example: Using R

```
> t.test(sids.diffs)

        One Sample t-test

data:  sids.diffs
t = 0.0023, df = 21, p-value = 0.9982
alternative hypothesis: true mean is not equal to 0


> t.test(sids.diffs,alternative="less")

        One Sample t-test

data:  sids.diffs
t = 0.0023, df = 21, p-value = 0.5009
alternative hypothesis: true mean is less than 0
```

Example: Using SAS

```
proc ttest;
   var diff;
```

```
                         Statistics

                    Lower CL              Upper CL
Variable       N        Mean      Mean        Mean

diff          22      -163.7    0.1818      164.04


                 T-Tests

Variable       DF     t Value     Pr > |t|

diff           21        0.00       0.9982
```

Small Sample

- t test assumptions:
  - Observations are independent
  - Sample is from the normal distribution

Large Sample

- For large sample, using the normal approximation (CLT)

$$\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$$

  and Slutsky's theorem

$$Z = \frac{\bar{Y} - \mu}{s/\sqrt{n}} \quad \text{is approximately} \quad N(0,1)$$

- Approximation improves as $n \to \infty$
- Note: Ys do not need to be normally distributed

Large Sample Example

- Example: Iron deficiency
- Iron deficiency anemia is an important nutritional health issue.
- A study was conducted of 63 boys aged 9-11 from families with income below the poverty level.
- The mean daily iron intake in the U.S. population is known to be 14.45 mg.
- Question: Is iron intake in boys associated with family income?

Large Sample Example cont.

- Conduct large sample test

$$H_0 : \mu = 14.45; \;\; H_A : \mu \neq 14.45$$

$$C_{0.05} = \{z : |z| > 1.96\}$$

$$\bar{y} = 12.5; \;\; s^2 = 22.5625; \;\; n = 63$$

$$z = \frac{12.5 - 14.45}{\sqrt{22.5626/63}} = -3.26$$

- Reject H_0
- Question: What is $2\Phi(-3.26)$?

```
> 2*pnorm(-3.26)
[1] 0.001114122
```

Testing/Estimation: Large Sample

- Testing $H_0 : \mu = \mu_0$ versus $H_A : \mu \neq \mu_0$

$$C_\alpha = \{z : |z| > z_{1-\alpha/2}\}$$

where

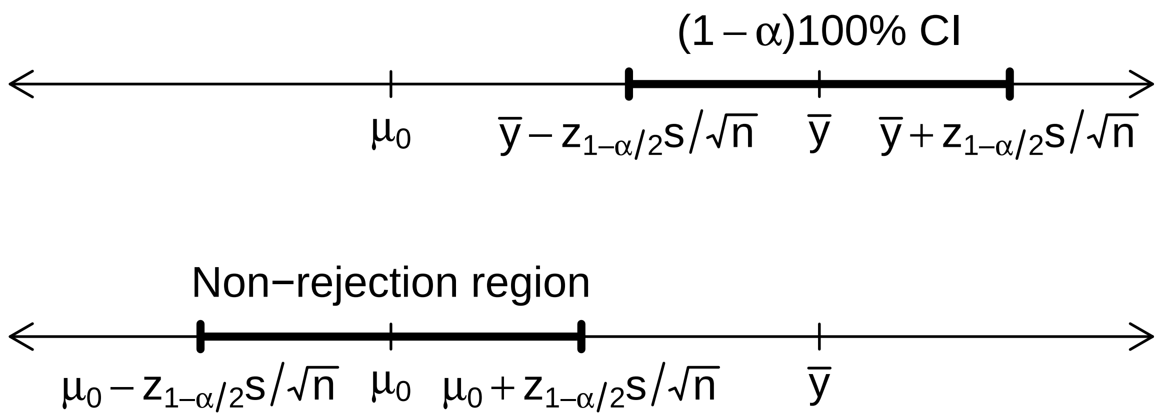
$$z = \frac{\bar{y} - \mu_0}{s/\sqrt{n}}$$

- Estimation: confidence interval for μ

$$\bar{y} \pm z_{1-\alpha/2}\frac{s}{\sqrt{n}}$$

- Can show: Reject H_0 if and only if the CI excludes μ_0

Testing/Estimation: Large Sample



Testing/Estimation: Large Sample

- Theorem: Reject H_0 if and only if CI excludes μ_0
- Sketch of proof: Suppose CI excludes μ_0, that is,

$$\mu_0 \notin [\bar{y} - z_{1-\alpha/2}\frac{s}{\sqrt{n}},\ \bar{y} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}]$$

  Without loss of generality, assume

$$\mu_0 < \bar{y} - z_{1-\alpha/2}\frac{s}{\sqrt{n}},$$

  This implies

$$z_{1-\alpha/2} < \frac{\bar{y} - \mu_0}{s/\sqrt{n}} \equiv z$$

  implying $z \in C_\alpha$

Testing/Estimation

- Equivalent; two sides of the same coin
- See text Section 4.7
- Emphasize testing beforehand (to inform power, sample size calculations) and estimation afterward
- Or, test first and then use the point estimate and confidence interval to describe the results

Small Sample, Non-normal

- Transformation, bootstrap
- Nonparametric tests:
  - Sign test
  - Wilcoxon signed rank test
- Read text sections 8.1–8.5

Sign Test

- Suppose $Y_1, \ldots, Y_n$ are iid continuous from F with median $\zeta_{0.5}$
- Hypotheses

$$H_0 : \zeta_{0.5} = \zeta_{0.5,0} \text{ (median = specified value)}$$
$$H_A : \zeta_{0.5} \neq \zeta_{0.5,0}$$

- If the true median is $\zeta_{0.5,0}$ and F is continuous,

$$\Pr[Y < \zeta_{0.5,0}] = \Pr[Y > \zeta_{0.5,0}] = 0.5$$

for any randomly selected observation Y

Sign Test

- Let R be the number of observations $> \zeta_{0.5,0}$
- Under H_0

$$R \sim \text{Binomial}(n, 0.5)$$

$$\Pr[R \leq r] = \sum_{i=0}^{r} \binom{n}{i} \left(\frac{1}{2}\right)^{i} \left(1 - \frac{1}{2}\right)^{n-i}$$

$$= \frac{1}{2^n} \sum_{i=0}^{r} \binom{n}{i}$$

Sign Test

- Critical region

$$C_\alpha = \{r : r \le r_{\alpha/2} \ \text{ or } \ r \ge r_{1-\alpha/2}\}$$

where $r_{\alpha/2}$ and $r_{1-\alpha/2}$ are such that

$$\Pr[R \le r_{\alpha/2} \mid H_0] + \Pr[R \ge r_{1-\alpha/2} \mid H_0] \le \alpha$$

Sign Test

- Because Binomial$(n, 0.5)$ is symmetric, $r_{1-\alpha/2} = n - r_{\alpha/2}$
- Thus need $r_{\alpha/2}$ such that

$$\Pr[R \leq r_{\alpha/2} \mid H_0] = \frac{1}{2^n} \sum_{i=0}^{r_{\alpha/2}} \binom{n}{i} \leq \frac{\alpha}{2}$$

- Choose largest $r_{\alpha/2}$ such that this inequality holds
- For $n = 10$, the critical region for a 2-sided test with $\alpha = 0.05$ is $C_{0.05} = \{0, 1, 9, 10\}$

Sign Test

- CDF for Binomial($n = 10, \pi = 0.5$)

r	Cumulative Probability	$2 \cdot \Pr(R \leq r)$
0	0.0010	0.0020
1	0.0107	0.0214
2	0.0547	0.1094
3	0.1719	0.3437
4	0.3770	0.7539
5	0.6231	
6	0.8282	
7	0.9453	
8	0.9893	
9	0.9990	
10	1.0000	

Sign Test Example

Calcium supplementation in African-American men

	treatment	before	after	diff
1.	calcium	107	100	-7
2.	calcium	110	114	4
3.	calcium	123	105	-18
4.	calcium	129	112	-17
5.	calcium	112	115	3
6.	calcium	111	116	5
7.	calcium	107	106	-1
8.	calcium	112	102	-10
9.	calcium	136	125	-11
10.	calcium	102	104	2

Sign Test Example cont.

- Testing

$$H_0 : \zeta_{0.5} = 0 \text{ vs. } H_A : \zeta_{0.5} \neq 0$$

- For $n = 10$ and $\alpha = 0.05$, $C_{0.05} = \{0, 1, 9, 10\}$
- $r = 4$ is not in $C_{0.05}$
- Hence do not reject H_0
- P-value

$$2 \times \left\{ \frac{1}{2^{10}} \sum_{i=0}^{4} \binom{10}{i} \right\} = 0.754$$

$$= 1 - \frac{1}{2^{10}} \binom{10}{5} = 1 - 0.246$$

Sign Test Example cont.

- R code

```
> 2*sum(dbinom(0:4,10,0.5))
[1] 0.7539063

> 2*pbinom(4,10,0.5)
[1] 0.7539063

> # First need to install the package BSDA
> # Basic Statistics and Data Analysis
> library("BSDA")
> SIGN.test(diff)

        One-sample Sign-Test

data:  diff
s = 4, p-value = 0.7539
alternative hypothesis: true median is not equal to 0
```

Sign Test Example cont.

- SAS proc univariate output:

```
                     The UNIVARIATE Procedure
                         Variable:  diff

                             Moments

N                          10    Sum Weights                 10
Mean                       -5    Sum Observations           -50
Std Deviation      8.74325137    Variance            76.4444444
Skewness           -0.3378852    Kurtosis            -1.5550482
Uncorrected SS            938    Corrected SS               688
Coeff Variation    -174.86503    Std Error Mean      2.76485885

                     Tests for Location: Mu0=0

        Test           -Statistic-    -----p Value------

        Student's t    t  -1.80841    Pr > |t|    0.1040
        Sign           M        -1    Pr >= |M|   0.7539
        Signed Rank    S     -13.5    Pr >= |S|   0.1934
```

Sign Test: Comments

- Typically $\zeta_{0.5,0} = 0$
- Alternative formulation of the null

$$\Pr[Y < \zeta_{0.5,0}] = \Pr[Y > \zeta_{0.5,0}]$$

- For example, if comparing difference in outcome for a new drug versus control, the null says the probability the new drug is better than the control is the same as the probability the new drug is worse than the control
- Delete any observations $= \zeta_{0.5,0}$ and reduce n by 1 for each

Sign Test: Large Samples

- If n is large, we can use a version of the CLT for the sign test
- Recall, for $R \sim \text{Binomial}(n, \pi)$

$$E(R) = n\pi, \quad \text{Var}(R) = n\pi(1-\pi)$$

- Thus

$$Z = \frac{R - n\pi}{\sqrt{n\pi(1-\pi)}}$$

  will be approximately $N(0, 1)$

- The approximation gets better as $n \to \infty$

Sign Test: Large Samples

- For the sign test, $H_0 : \pi = 0.5$
- Therefore we compute

$$Z = \frac{R - n/2}{\sqrt{n/4}}$$

- Critical region comes from $\Phi(z)$

Sign Test: Large Samples

- The normal approximation to the sign test works well for $n \geq 40$
- For $40 \leq n \leq 100$, the approximation is better using the following adjustment:

$$Z = \begin{cases} \frac{R-(n+1)/2}{\sqrt{n/4}} & \text{if } (R - n/2) > 1/2 \\ \\ \frac{R-n/2}{\sqrt{n/4}} & \text{if } |R - n/2| \leq 1/2 \\ \\ \frac{R-(n-1)/2}{\sqrt{n/4}} & \text{if } (R - n/2) < -1/2 \end{cases}$$

Sign Test Example

- A study was conducted to compare an automated machine for measuring blood pressure with measures made by a nurse using a standard mercury sphygmomanometer
- 100 people had their blood pressure measured using both techniques
- We use $\text{sign}(Y = \text{BP}_{\text{auto}} - \text{BP}_{\text{nurse}})$

$$H_0 : \Pr[Y < 0] = \Pr[Y > 0]$$

vs.

$$H_A : \Pr[Y < 0] \neq \Pr[Y > 0]$$

Sign Test Example cont.

- For the study:

$$r = 64$$

$\Rightarrow$

$$r - n/2 > 1/2$$

$\Rightarrow$

$$z = 13.5/5 = 2.7$$

- Reject H_0 and conclude that the automated machine is more likely to give a higher reading

Binomial Continuity Correction

- Suppose $R \sim \text{Binomial}(n, \pi)$ and $X \sim N(n\pi, n\pi(1-\pi))$
- By the CLT

$$\Pr[R \leq x] \approx \Pr[X \leq x]$$

- Continuity correction

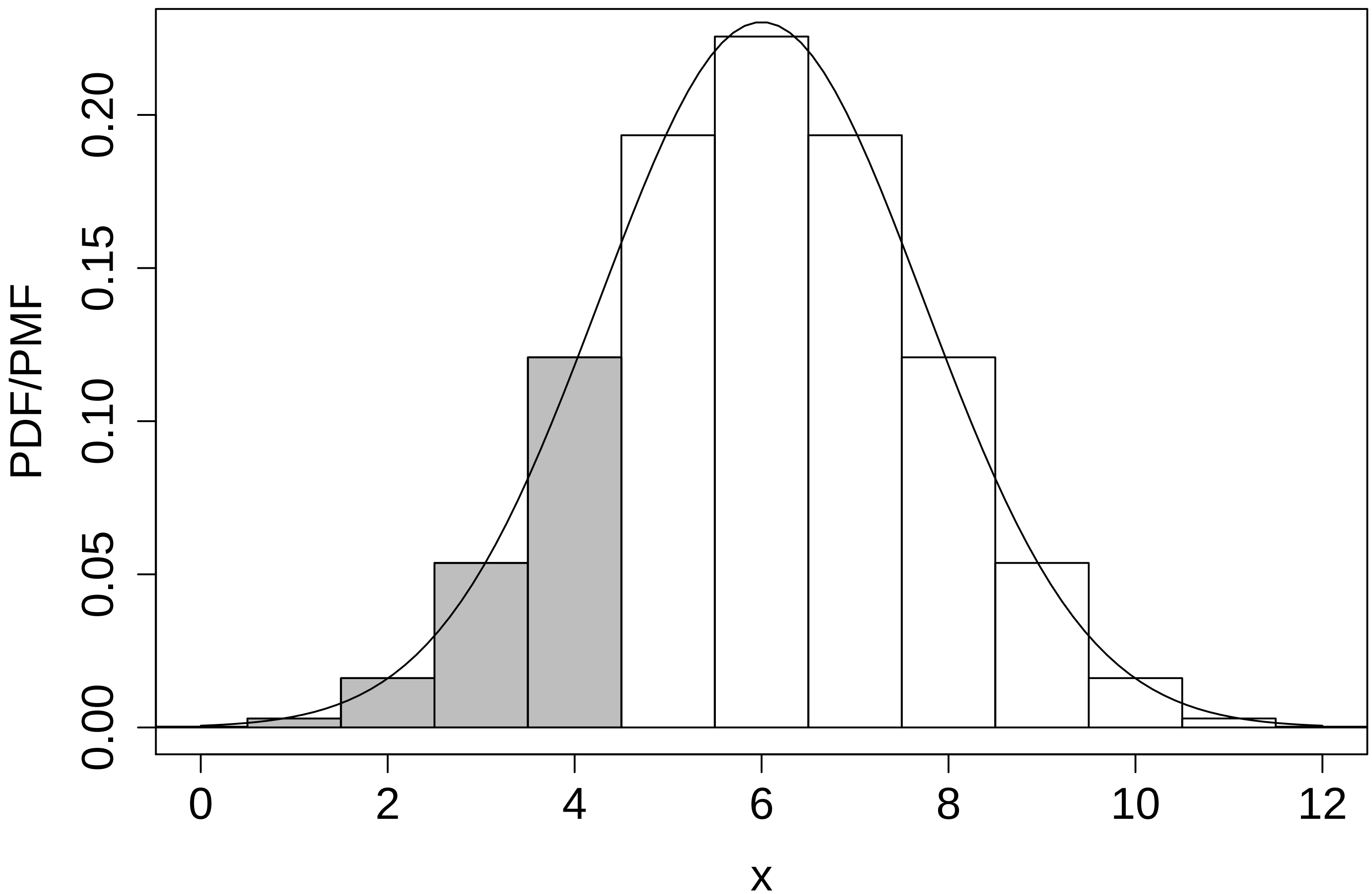
$$\Pr[R \leq x] \approx \Pr[X \leq x + 1/2]$$

- For example, if $n = 12$, $\pi = 0.5$, and $x = 4$

$$0.194 = \Pr[R \leq 4] \approx \Pr[X \leq 4 + 1/2] = 0.193$$

whereas $\Pr[X \leq 4] = 0.124$

Binomial Continuity Correction



Binomial Continuity Correction

- Likewise

$$\Pr[R \geq x] \approx \Pr[X \geq x - 1/2]$$

- Returning to the sign test where $\pi = 1/2$, if $r < \frac{n}{2} - \frac{1}{2}$, then the one-sided p-value will be equal to

$$\Pr[R \leq r] \approx \Pr[X \leq r + 1/2] = \Pr\left[Z \leq \frac{r + 1/2 - n/2}{\sqrt{n/4}}\right]$$

$$= \Pr\left[Z \leq \frac{r - (n-1)/2}{\sqrt{n/4}}\right]$$

Sign Test: Efficiency

- Definition 8.6. The *relative efficiency* of statistical procedure A compared to B is the ratio of the sample size needed for B to that of A in order for both procedures to have the same statistical power

- If sampling from a normal distribution

  - n small: the sign test is almost as efficient as the t-test
  - As n gets larger, the sign test becomes less efficient and the asymptotic relative efficiency (ARE) is $2/\pi = 0.64$

- If sampling from a non-normal distribution, the sign test can be more efficient

Wilcoxon Signed Rank Test

- Suppose $Y_1, Y_2, \ldots, Y_n$ are iid according to a symmetric distribution F with median $\zeta_{0.5}$
- Hypotheses

$$H_0 : \zeta_{0.5} = \zeta_{0.5,0}$$

vs.

$$H_A : \zeta_{0.5} \neq \zeta_{0.5,0}$$

Wilcoxon Signed Rank Test

- Delete any Y_i equal to $\zeta_{0.5,0}$ and adjust n
- Compute $Y_i' = Y_i - \zeta_{0.5,0}$
- Rank the $\left|Y_i'\right|$ from smallest to largest
- The statistic S^+ is the sum of the ranks of the observations with Y_i' positive
- S^- defined similarly
- Aside: SAS uses $S^+ - n(n+1)/4$

Wilcoxon Signed Rank Test Example

Example: Calcium supplementation in African-American men

	treatment	before	after	Y_i diff	$\|Y_i\|$ absol	rank	sign*rank
1.	calcium	107	100	-7	7	6	-6
2.	calcium	110	114	4	4	4	4
3.	calcium	123	105	-18	18	10	-10
4.	calcium	129	112	-17	17	9	-9
5.	calcium	112	115	3	3	3	3
6.	calcium	111	116	5	5	5	5
7.	calcium	107	106	-1	1	1	-1
8.	calcium	112	102	-10	10	7	-7
9.	calcium	136	125	-11	11	8	-8
10.	calcium	102	104	2	2	2	2

Wilcoxon Signed Rank Test Example cont.

- Table on the course's Sakai site gives critical values
- For $n = 10$, $C_{0.05} = \{S : S \leq 8\}$
- $S^+ = 4 + 3 + 5 + 2 = 14$; $S^- = 41$
- $S = \min\{S^+, S^-\} = 14$
- Therefore, do not reject H_0 : median is 0
- R code:

```
> x <- c(-7, 4, -18, -17, 3, 5, -1, -10, -11, 2)
> wilcox.test(x)

        Wilcoxon signed rank test

data:  x
V = 14, p-value = 0.1934
alternative hypothesis: true mu is not equal to 0
```

Wilcoxon Signed Rank Test

- Because

$$\sum_{i=1}^{n} i = n\left(\frac{n+1}{2}\right)$$

- It follows that

$$S^{+} + S^{-} = n\left(\frac{n+1}{2}\right)$$

- So only smaller values are tabulated

Wilcoxon Signed Rank Test

- How to compute null distribution of signed-rank test?
- Under the null, each ranked observation has probability $1/2$ of having positive sign
- The n signs are independent
- There are 2^n possible outcomes
- Thus each outcome occurs with probability $1/2^n$

Distribution of S^+ Under H_0

- Calculating the null distribution for $n = 4$; a + in the column indicates that the sign of the rank is positive

ranks				
1	2	3	4	S^+
$-$	$-$	$-$	$-$	0
$+$	$-$	$-$	$-$	1
$-$	$+$	$-$	$-$	2
$-$	$-$	$+$	$-$	3
$-$	$-$	$-$	$+$	4
$+$	$+$	$-$	$-$	3
$+$	$-$	$+$	$-$	4
$+$	$-$	$-$	$+$	5
$-$	$+$	$+$	$-$	5
$-$	$+$	$-$	$+$	6
$-$	$-$	$+$	$+$	7
$+$	$+$	$+$	$-$	6
$+$	$+$	$-$	$+$	7
$+$	$-$	$+$	$+$	8
$-$	$+$	$+$	$+$	9
$+$	$+$	$+$	$+$	10

Distribution of S^+ Under H_0

k	$\Pr[S^+ = k]$	$\Pr[S^+ \leq k]$
0	1/16	$1/16 = 0.0625$
1	1/16	$1/8 = 0.1250$
2	1/16	$3/16 = 0.1875$
3	2/16	$5/16 = 0.3125$
4	2/16	$7/16 = 0.4375$
5	2/16	$9/16 = 0.5625$
6	2/16	$11/16 = 0.6875$
7	2/16	$13/16 = 0.8125$
8	1/16	$7/8 = 0.8750$
9	1/16	$15/16 = 0.9375$
10	1/16	1

Table A.9: Distribution of S^+ Under H_0

n	k	$\Pr[S^+ \le k]$
4	0	$1/16 = 0.0625$
	1	$1/8 = 0.125$
	$\vdots$	$\vdots$
6	0	$1/2^6 = 0.015625$
	1	$1/32 = 0.03125$
	$\vdots$	$\vdots$
7	0	$1/2^7 = 0.0078125$
	1	$2/2^7 = 0.015625$
	2	$3/2^7 = 0.0234$
	3	$5/2^7 = 0.039$

- Bold face values denote critical values listed in Table A.9 of the text on page 834 for one sided $\alpha = 0.025$ and two-sided $\alpha = 0.05$. Is the critical value in the critical region?

Distribution of S^+ Under H_0

- Large sample distribution
- Can show

$$E(S^+) = \frac{n(n+1)}{4} \quad \text{and } \mathrm{Var}(S^+) = \frac{n(n+1)(2n+1)}{24}$$

- If $n \geq 15$,

$$Z = \frac{S^+ - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}} \sim N(0,1)$$

Wilcoxon Signed Rank Test: Ties

- If there are two or more observations with the same value of $|Y'|$, the observations are said to be *tied*
- For tied observations we assign the average rank or *midrank*
- Example: $\boldsymbol{Y} = \{23, 25, 45, 13, 23, 46\}$

  Midranks: $\{2.5, 4, 5, 1, 2.5, 6\}$

Wilcoxon Signed Rank Test: Ties

- Can show
$$E(S^+) = \frac{n(n+1)}{4}$$
- To accommodate ties, the variance is adjusted
$$\text{Var}(S^+) = \frac{n(n+1)(2n+1) - \frac{1}{2}\sum_{i=1}^{q} t_i(t_i-1)(t_i+1)}{24}$$
where q equals the number of sets of ties and t_i is the number of observations in the ith set
- For example on previous slide, $q = 1$ and $t_1 = 2$ such that
$$\text{Var}(S^+) = \frac{6(6+1)(2\cdot 6+1) - \frac{1}{2}\cdot 2\cdot 1\cdot 3}{24}$$

Wilcoxon Signed Rank Test: Ties

- If n is large, use the variance adjusted for ties in the normal approximation
- If n is small and there are ties, need to compute the null distribution from permutation principles, i.e., tables of critical values are not guaranteed to be correct in the presence of ties

Signed Rank Test: Example with Ties

- From Table 8.7, page 281

	SIDS	nonSIDS	Y'	|Y'|	rank	sgnrnk
1.	1474	2098	-624	624	21	-21
2.	3657	3119	538	538	19	19
3.	3005	3515	-510	510	18	-18
4.	2041	2126	-85	85	3	-3
5.	2325	2211	114	114	4	4
6.	2296	2750	-454	454	15	-15
7.	3430	3402	28	28	1	1
8.	3515	3232	283	283	9	9
9.	1956	1701	255	255	8	8
10.	2098	2410	-312	312	11	-11
11.	3204	2892	312	312	11	11
12.	2381	2608	-227	227	7	-7
13.	2892	2693	199	199	6	6
14.	2920	3232	-312	312	11	-11
15.	3005	3005	0	0		
16.	2268	2325	-57	57	2	-2
17.	3260	3686	-426	426	14	-14
18.	3260	2778	482	482	16.5	16.5
19.	2155	2552	-397	397	13	-13
20.	2835	2693	142	142	5	5
21.	2466	1899	567	567	20	20
22.	3232	3714	-482	482	16.5	-16.5

Signed Rank Test: Example with Ties cont.

- $S^+ = 99.5;\;\; E(S^+) = 21(22)/4 = 115.5$
- $q = 2;\;\; t_1 = 3;\;\; t_2 = 2$ so that

$$\text{Var}(S^+) = \frac{21(22)(43) - \frac{1}{2}[3(2)(4) + 2(1)(3)]}{24} = 827.15$$

- Thus

$$Z = \frac{99.5 - 115.5}{\sqrt{827.15}} = -0.5563$$

- Yielding $p = 2 \times \Phi(-0.5563) = 0.578$

Signed Rank Test: Example with Ties cont.

- R code:

```
> wilcox.test(sids.diffs,exact=F,correct=F)

        Wilcoxon signed rank test

data:  sids.diffs
V = 99.5, p-value = 0.578
alternative hypothesis: true mu is not equal to 0
```

- SAS uses a slightly different large sample approximation; proc univariate uses the exact version of the test when $n \leq 20$ and a large sample approximation when $n > 20$; here the large sample approximation yields $p = 0.5905$

Efficiency of Signed Rank Test

- If the Ys come from a normal distribution, the ARE of the signed rank test compared to the normal distribution test is $3/\pi = 0.955$

  (cf. Lehmann 1998, page 80)

- Gain in efficiency over the sign test is because of an additional assumption: symmetry

Signed rank test: H_A

- Suppose $n = 10$, $\alpha = 0.05$, $H_0 : \zeta_{0.5} = 0$
- For $H_A : \zeta_{0.5} \neq 0$, $S = \min\{S^+, S^-\}$

$$C_{0.05} = \{s : s \leq 8\}$$

- Equivalently

$$C_{0.05} = \{s^+ : s^+ \leq 8 \text{ or } s^+ \geq 47\}$$

or

$$C_{0.05} = \{s^- : s^- \leq 8 \text{ or } s^- \geq 47\}$$

Signed rank test: H_A

- For $H_A : \zeta_{0.5} < 0$, what is the rejection region?

$$C_{0.05} = \{s^+ : s^+ \leq 10\}$$

or

$$C_{0.05} = \{s^- : s^- \geq 45\}$$