BIOS 660/BIOS 672 (3 Credits): Probability and Statistical Inference I — Solutions
Midterm #2, Tuesday, October 30, 2018

1. [9 *pts*] Let X be the random variable following an exponential distribution with parameter $\lambda > 0$ with the density function $f_X(x) = \lambda e^{-\lambda x}$, for $x > 0$, and $f_X(x) = 0$, elsewhere.

   (a) Derive the distribution of $Y = 10 - e^{-\lambda X}$.

   Solution: There are several ways to answer this question. The following is one possible approach.
   As $x \to 0$, we have $y \to 9$. As $x \to +\infty$, we have $y \to 10$. Since $0 < x < +\infty$, we have $9 < y < 10$.

$$
\begin{aligned}
X &= -\frac{1}{\lambda}\ln(10 - Y) \\
\frac{\mathrm{dx}}{\mathrm{dy}} &= \left(-\frac{1}{\lambda}\right)\frac{(-1)}{10 - y} \\
&= \frac{1}{\lambda(10 - y)}
\end{aligned}
$$

$$
f_Y(y) = \lambda e^{-\lambda(-\frac{1}{\lambda}\ln(10-y))}\frac{1}{\lambda(10-y)} = \lambda e^{\ln(10-y)}\frac{1}{\lambda(10-y)} = \frac{10-y}{10-y} = 1, \quad \text{for } 9 < y < 10
$$

   Therefore, Y follows Unif(9,10).

   Another approach: You could also recognize that the CDF function for the exponential distribution is $F(x) = 1 - e^{-\lambda x}$. Therefore $Z = F(X) = 1 - e^{-\lambda X}$ follows Unif(0,1). Note that $Y = 9 + (1 - e^{-\lambda X}) = 9 + Z$, therefore Y is just Z shifted to the right 9 units and therefore follows Unif(9,10).

   (b) Derive the moment generating function of Y.

   Solution:

$$
M_Y(t) = \mathrm{E}(e^{tY}) = \begin{cases} \int_9^{10} e^{ty}dy = \frac{1}{t}e^{ty}|_9^{10} = \frac{1}{t}(e^{10t} - e^{9t}), & t \neq 0 \\ \int_9^{10} dy = 1, & t = 0 \end{cases}
$$

   (c) Let $X_1, X_2, \ldots, X_{10}$ be a random sample from the above exponential distribution with parameter $\lambda > 0$ (X_i's $(i = 1, 2, \ldots, 10)$ are independent). Find the density function of $Z = \min\{X_1,, X_2, \ldots, X_{10}\}$. (Hint: $P(Z > z) = P(X_1 > z, \ldots, X_{10} > z)$).

   Solution:

$$
\begin{aligned}
P(Z > z) &= P(X_1 > z, \ldots X_{10} > z) \\
\text{(because of independence and identically distributed)} &= [P(X_1 > z)]^{10} = \begin{cases} [e^{-\lambda z}]^{10}, & z > 0 \\ 1, & z \leq 0 \end{cases}
\end{aligned}
$$

$$
f_Z(z) = \begin{cases} 10\lambda e^{-10\lambda z}, & z > 0 \\ 0, & \text{elsewhere} \end{cases}
$$

2. [5 *pts*] You own a bakery and are making cupcakes for a party. Unfortunately, the order is too large and you need to hire a second worker to help. This worker is not very good at making cupcakes and thus each cupcake only has an 80% chance of being accepted. Assume that each cupcake is made independently.

   (a) If the worker bakes 90 cupcakes, what is the expected number of unacceptable cupcakes?

   Solution: Let X denote the number of acceptable cakes. Then X can be modeled through a Bin(n=90,p=0.8).
   For binomial distribution, $\mathrm{E}(X) = np$. Therefore the expected number of acceptable cupcakes=90*0.8=72. Hence the expected number of unacceptable cupcakes=90-72=18.

   (b) If the worker bakes 90 cupcakes, what is the probability that you get at least 85 acceptable cupcakes? You do not need to calculate the final number.

   Solution: $\sum_{x=85}^{90} P(X = x) = \sum_{x=85}^{90} \binom{90}{x} 0.8^x 0.2^{90-x}$.

   (c) What is the expected number of cupcakes the worker needs to bake in order to have 90 acceptable cupcakes.

   Solution: Let Y denote the number of cakes that is needed in order to have 90 acceptable cupcakes. Then $Y \sim NegBin(s = 90, p = 0.8)$.
   For the negative binomial distribution, $\mathrm{E}(Y) = s/p$. Therefore the expected number of cupcakes the worker needs to bake in order to have 90 acceptable cupcakes= $90/0.8 = 112.5 \approx 113$.

3. [3 *pts*] Assuming all integrals exist, find $\min_b \mathrm{E}[(e^X - b)^2]$ expressed in the simplest form.

Solution: There are a couple of ways to answer this question. The following is one approach.
Let $g(b) = \mathrm{E}[(e^X - b)^2] = \mathrm{E}(e^{2X}) - 2b\mathrm{E}(e^X) + b^2$. Then $g'(b) = -2\mathrm{E}(e^X) + 2b$ and $g''(b) = 2 > 0$. Therefore $g(b)$ reaches the minimum at b^*, where b^* satisfies $g'(b^*) = -2\mathrm{E}(e^X) + 2b^* = 0$. Solving for b^*, we have $b^* = \mathrm{E}(e^X)$. Therefore,

$$\min_b \mathrm{E}[(e^X - b)^2] = \mathrm{E}[(e^X - \mathrm{E}(e^X))^2] = \mathrm{Var}(e^X).$$

4. [13 *pts*] A person arrives at the side of a single-lane, one-way street as a car passes by. Consider this time as time 0. There are no traffic lights and no pedestrian lanes in the nearby area. He stands there watching cars pass by. The number of cars to pass by within a period of time can be modeled using a Poisson distribution. Suppose the cars pass by the person at a rate of 4.5 cars per minute. Assume that the cars pass by instantaneously.

(a) Find the probability that no cars pass by him within a 2-minute window.

Solution: Let the random variable X be the number of cars to pass in 2 minutes. Then $X \sim pois(\lambda)$ where $\lambda = 2 * 4.5 = 9$. Then $P(X = 0) = \lambda^0 e^{-\lambda}/0! = e^{-9} = 0.00012$.

(b) If the person had to guess how many cars would pass by in the next 5 minutes, what number should he guess? State your reasoning.

Solution: Let the random variable X be the number of cars to pass in 5 minutes. Then $X \sim pois(\lambda)$ where $\lambda = 5 * 4.5 = 22.5 = \mathrm{E}(X)$. Since we cannot have half of a car, we need to choose an integer around the mean. Because $\mathrm{P}(X = 23) = 22.5^{23} e^{-22.5}/23!$ and $\mathrm{P}(X = 22) = 22.5^{22} e^{-22.5}/22!$, this gives us $\frac{\mathrm{P}(X=23)}{\mathrm{P}(X=22)} = 22.5/23 < 1$, hence $\mathrm{P}(X = 23) < \mathrm{P}(X = 22)$. So the person should guess 22 cars would pass by in the next 5 minutes.

(c) What is the **distribution** for the time until the first car passes by him (consider time as continuous)? What is the **expected time** until the first car passes by him?

Solution: Since the number of cars passing a point follows a Poisson distribution, then we know that the time between events follows an exponential distribution with rate $\lambda = 4.5$. The mean for exponential distribution with parameter λ is $1/\lambda = 1/4.5 = 0.222$. Hence the expected time until the first car passes him is 0.222 minutes or 13.3 seconds.

(d) Suppose that the person decides to wait until no car has gone by for 10 seconds before attempting to cross the street. What is the probability mass function (PMF) for the number of cars that pass by him until he will start crossing?

Solution: We will look at the times between cars. If the gap between two subsequent cars is greater than 10 seconds, then the person will start crossing. Thus, we can model this as a geometric random variable Y which is the number of gaps before the first gap that is longer than 10 seconds:

$$P(Y = y) = (1 - p)^y p$$

where p is the probability of success, in this case the probability that a gap is longer than 10 seconds. To find p we again use the fact that we know each gap time follows an exponential distribution with parameter $\lambda = 4.5$ such that

$$p = P(T > 10 \text{ seconds}) = P(T > 10/60 \text{ minutes}) = e^{-4.5*10/60} = 0.47.$$

Thus we have that

$$P(Y = y) = (0.53)^y 0.47 \text{ for y} = 0,\ 1,\ 2,\ \ldots \tag{1}$$

Note that Y characterizes the number of gaps until the first gap longer than 10 seconds which exactly corresponds to the number of cars that pass before the person can cross. Therefore the pmf for the number of cars that pass by him until he will start crossing has the same form as (1) above.