# BIOS 660/BIOS 672 (3 Credits): Probability and Statistical Inference I

Jianwen Cai

https://sakai.unc.edu/portal/site/bios660-bios672-3-credits

Notes 1

### Sets

- Sets:
  - Sets are basic concepts in mathematics and probability.
  - Crudely defined as: "a collection of some elements".
  - Usually denoted with a capital letter (e.g. $A$, $B$, $S$)
- Special sets:
  - $\mathbb{N}$ = Natural numbers (1, 2, 3, 4, ...)
  - $\mathbb{Z}$ = Integers (0, +1, -1, +2, -2, ...)
  - $\mathbb{Q}$ = Rational numbers
  - $\mathbb{R}$ = Real numbers



### Common Notation

- $\emptyset$: empty set
- $w \in A$: $w$ is an element of the set $A$
- $\{w\} \subset A$ or $\{w\} \subseteq A$ : the set consisting of the singleton $w \in A$ is a subset of $A$
- $(a, b)$, where $a, b \in \mathbb{R}$, is the set of real numbers between (but not including) $a$ and $b$
- $[a, b]$, where $a, b \in \mathbb{R}$, is the set of real numbers between and including $a$ and $b$
- $[a, b)$, where $a, b \in \mathbb{R}$, is the set of real numbers between and including $a$ but not $b$
- $(a, b]$, where $a, b \in \mathbb{R}$, is the set of real numbers between and including $b$ but not $a$
- $\{w : \text{a statement}\}$ : the set of elements $w$ for which the statement holds. Example: the open interval $(a, b)$ can be defined as $\{w : a < w < b\}$.
- $A = B$ if $A$ and $B$ contain exactly the same elements (this can be shown by showing (1) $A \subset B$ and (2) $B \subset A$)

## The Sample Space:

- Need to define an "abstract space", often denoted by $\Omega$, as a non-empty set of all the elements concerned. These elements are called "points" and often denoted with lower case letter
- In probability we often use $\Omega$ to denote the **sample space**, which is the collection of all possible distinct realizations of a non-deterministic experiment. (Usually idealized)
- Choice of sample space is the first stem in formulating a probabilistic model for an experiment. Examples of sample spaces:

1. Draw a card from a deck of 52 cards: $\Omega = \{1, 2, \ldots, 52\}$, which is a **finite** sample space
2. Toss a coin until one gets two successive heads and record the number of tosses performed: $\Omega = \{2, 3, 4, \ldots, \infty\}$, which is a **countably infinite** sample space.
3. Two components in an electrical system record their failure times: $\Omega = \{(x, y) : x \geq 0, y \geq 0\}$, which is an **uncountably infinite** sample space.



## Events:

- An event is always a subset of $\Omega$, "events" .....
- ... but not all subsets are necessarily events!
- If $\Omega$ is countable (i.e. finite or countably infinite), then any subset of $\Omega$ is an event
- If $\Omega$ is uncountable, we cannot handle all possible subsets (not all are events). Instead, we restrict events to be a "well-behaved" class of subsets. More on this later.
- Individual points in $\Omega$ are called "simple events"

**Union**

**Definition**: For two sets $A$ and $B$, their **union** is denoted as $A \cup B = \{w : w \in A$ or $w \in B\}$



**Intersection**

**Definition**: For two sets $A$ and $B$, their **intersection** is denoted as $A \cap B = \{w : w \in A$ and $w \in B\}$

## Complementation

**Definition**: For a set $A$ in $\Omega$, the complement of $A$ (w.r.t. $\Omega$) is denoted by $A^c = \{w \in \Omega : w \notin A\}$

Note: One can show that under complementation, $\subset$ and $\supset$ are swapped.



## Difference

**Definition**: For two sets $A$ and $B$, their **difference** is denoted as
$A - B = \{w : w \in A, w \notin B\} = A \cap B^c$

## Symmetric Difference

**Definition**: For two sets $A$ and $B$, their **symmetric difference** is denoted as
$A \Delta B = (A - B) \cup (B - A) = \{w : w \in$ exactly one of $A$ and $B\}$



## Disjoint Union

**Definition**: Two sets $A$ and $B$ are called **disjoint** if $A \cap B = \phi$.

**Definition**: For two disjoint sets $A$ and $B$, their **disjoint union** is denoted as $A \cup B = A + B$

## Properties of the Union/Intersection

For sets $A_1$, $A_2$, $A_3$:

- Properties of Union:
  1. Associative: $(A_1 \cup A_2) \cup A_3 = A_1 \cup (A_2 \cup A_3)$
  2. Distributive: $(A_1 \cup A_2) \cap A_3 = (A_1 \cap A_3) \cup (A_1 \cap A_3)$
  3. Commutative: $A_1 \cup A_2 = A_2 \cup A_1$
- Properties of Intersection:
  1. Associative: $(A_1 \cap A_2) \cap A_3 = A_1 \cap (A_2 \cap A_3)$
  2. Distributive: $(A_1 \cap A_2) \cup A_3 = (A_1 \cup A_3) \cap (A_1 \cup A_3)$
  3. Commutative: $A_1 \cap A_2 = A_2 \cap A_1$



## Countable Unions/Intersections

- Finite sequence of sets: Let $\{A_1, \dots, A_k\}$ be a finite sequence of $k$ sets in $\Omega$. We define

$$\sup_{1 \le n \le k} A_n = \bigcup_{n=1}^{k} A_n = \{w : w \in A_n \text{ for some } 1 \le n \le k\}$$

$$\inf_{1 \le n \le k} A_n = \bigcap_{n=1}^{k} A_n = \{w : w \in A_n \text{ for any } 1 \le n \le k\}$$

- Countable sequence of sets: Let $\{A_n\}$ be an infinite sequence of sets in $\Omega$. We define

$$\sup_{n \ge 1} A_n = \bigcup_{n=1}^{\infty} A_n = \{w : w \in A_n \text{ for some } n\}$$

$$\inf_{n \ge 1} A_n = \bigcap_{n=1}^{\infty} A_n = \{w : w \in A_n \text{ for any } n \ge 1\}$$

## Uncountable Unions/Intersections

We extend the intersection over a set of integers to any arbitrary set:

**Definition:** For $\{A_t, t \in T\}$, where $T$ is a (possibly uncountable) index set, $\bigcup_{t \in T} A_t = \{w : w \in A_t$ for some $t \in T\}$. The definition for intersection is similar.



## DeMorgan's Rule:

**DeMorgan's Rule (for 2 sets):**

$$(A \cup B)^c = A^c \cap B^c$$

and

$$(A \cap B)^c = A^c \cup B^c$$

**DeMorgan's Rule (general):**

$$\left( \bigcup_{t \in T} A_t \right)^c = \bigcap_{t \in T} A_t^c$$

and

$$\left( \bigcap_{t \in T} A_t \right)^c = \bigcup_{t \in T} A_t^c$$

where $T$ is any index set (finite, countably infinite, uncountably infinite).

**Proof of DeMorgan's Rule (2 sets):**

$$(A \cap B)^c = A^c \cup B^c$$

**Proof:**



**Proof of DeMorgan's Rule (continued):**

The proof that $(A \cup B)^c = A^c \cap B^c$ is similar.

### Limsup and Liminf of Sets

- **Definition:** Let $\{A_n\}$ be a sequence of sets in $\Omega$. Define

$$A^* = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

and

$$A_* = \liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

these are the upper and lower limits of the sequence.
- **Theorem:** (a) $w \in \limsup_n A_n$ if and only if $w$ is in infinitely many of the $A_n$.
(b) $w \in \liminf_n A_n$ if and only if there is an $m$ such that $w \in A_n$ for all $n \geq m$ (i.e. $w$ is in all but the first $m$) [Proof omitted]
- **Theorem:** $\liminf_n A_n \subset \limsup_n A_n$ (proof for homework)



### Limit of Sets

If $\limsup A_n = \liminf A_n$ then we say that $\{A_n\}$ is convergent and write
$\lim_{n\to\infty} A_n = \limsup A_n = \liminf A_n$

Example: let $\Omega = [0, 1], A_n = [0, 1/n]$ if $n$ is even and $A_n = [1 - 1/n, 1)$ if $n$ is odd.

- Then by definition $A_* = \emptyset$ and $A^* = \{0\}$.
- Since $A_* \neq A^*$, then $\lim A_n$ does not exist.
- On the other hand, if we let $A_n = [0, 1/n]$ for all $n$, then $A_* = A^* = \{0\}$ and the limit exists.

## More Examples

What is $A_*$ and $A^*$ for:

- Example: let $\Omega = \{0, 1\}$, then consider the sequence $\{0\}$, $\{1\}$, $\{0\}$, $\{1\}$, $\{0\}$, $\dots$
- Example: let $\Omega = \mathbb{Z}$, Consider the sequence of sets: $\{0\}$, $\{0, 1\}$,$\{0\}$,$\{0, 1\}$,$\{0\}$,$\{0, 1\}$,$\{0\}$, $\dots$
- Example: let $\Omega = \mathbb{Z}$, Consider the sequence of sets: $\{0\}$, $\{0, 1\}$, $\{0, 1, 2\}$, $\{0, 1, 2, 3\}$, $\{0, 1, 2, 3, 4, \}$, $\dots$
- Example: let $\Omega = [0, 1], A_n = [0, 1/n]$ if $n$ is even and $A_n = [1 - 1/n, 1)$ if $n$ is odd.



## Monotonicity

- **Monotonicity:** A monotine sequence of sets is defined as:
  - $\{A_n\}$ is called monotone increasing iff $A_n \subset A_{n+1}$ for any $n$
  - $\{A_n\}$ is called monotone decreasing iff $A_n \supset A_{n+1}$ for any $n$
- **Theorem:** A monotone sequence of sets is convergent
  **Proof** for increasing sequence: