Dagish Review

$\boxed{2014}$ (1e)

1e) $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}$, $E\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \underset{\sim}{\mu}$, iid

if X, Y are bivariate normal, $\bar{X}, \bar{Y}$ are bivariate normal

$$cov\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \Sigma_{2\times 2} = \begin{bmatrix} var(X_i) & cov(X_i, Y_i) \\ & var(Y_i) \end{bmatrix}$$

$$\boxed{\begin{aligned} &E\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \underset{\sim}{\mu} \\ &Cov\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \frac{1}{n}\Sigma = \begin{bmatrix} \sigma_{11}/n & \sigma_{12}/n \\ & \sigma_{22}/n \end{bmatrix} \end{aligned}}$$

$$\frac{\sigma_{12}/n}{\sqrt{\frac{\sigma_{11}}{n} \cdot \frac{\sigma_{22}}{n}}} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{12}}} = corr(X_i, Y_i)$$ ← not necessary

New Notation:

$$\begin{pmatrix} X_1 \\ \vdots \\ X_K \end{pmatrix} \sim MVN(\underset{K\times 1}{\underset{\sim}{\mu}}, \underset{K\times K}{\Sigma})$$

multivariate normal

Single: $\dfrac{\exp\{-(x_i-\mu_i)^2/2\sigma_{ii}\}}{\sqrt{2\pi\sigma_{ii}}}$

if iid multiple: (product) $\dfrac{\exp\{-\sum_{i=1}^{K}(x_i-\mu_i)^2/2\sigma_{ii}\}}{(2\pi)^{K/2}\left(\prod_{i=1}^{K}\sigma_{ii}\right)^{1/2}}$

notation: $|\Sigma|^{K/2}$

Determinant of Σ ^ $K/2$

General sigma, not i.i.d (Σ is not diagonal)

$(\underset{\sim}{x}-\underset{\sim}{\mu})^T \Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})$ in exponent

$$\frac{\exp\{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})^T \Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})}{(2\pi)^{k/2}|\Sigma|^{k/2}}$$

$\boxed{2014}$

2a)

$Y|X \sim \text{Poisson}(X)$

$E[\text{common colds} \mid X] = X$, $X \sim U(0,2)$

Y

- if any RV shows up on right side for expectation + probability, it must be conditional on that RV

$\cancel{Y \sim \text{Poisson}(X)}$ ← total nonsense!! It must be $\boxed{Y|X}$ (unless little x is known constant)

g) $E[X \mid \underbrace{Y=0}_{\text{no common colds}}]$ Y, discrete, X continuous

d) $E[Y-X] \overset{?}{=} 0$ ← if unbiased

$= E[E[Y-X|X]] = E[X-X] = 0$

Prediction mean squared error: $E(Y-X)^2 =$

$E(Y-X)^2 = \text{Bias}^2 + \text{var}(Y) = (E(Y-X))^2 + \text{var}(Y)$

$= 0 + 4/3$

$\text{Var}(Y) = 1 + \frac{1}{3} = \frac{4}{3}$ ← conditional var

$\boxed{2014}$

2e) $E(X-a-bY)=0$, $Var(X-a-bY) \leftarrow$ min.

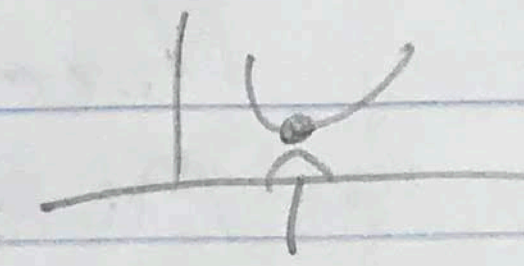
$\hookrightarrow = E(X) - a - bE[Y]$

$= 1 - a - b(1) = 0 = \boxed{a = 1-b}$

$E[Y] = EE(Y|X) = E(X) = 1$

$Var(X-a-bY) = Var(X-bY)$ (constant doesn't matter)

$= \underbrace{Var(X)}_{\text{out of our control}} + \underbrace{b^2 Var(Y) - 2b\cdot cov(X,Y)}_{\text{so minimize this}}$

$b^2 Var(Y) - 2b\,cov(X,Y) \longrightarrow$ 

Derivative: $\frac{\partial}{\partial b} = 2b\,Var(Y) - 2cov(X,Y) = 0$

$\Rightarrow \boxed{b = \frac{cov(X,Y)}{Var(Y)} \quad a = 1-b}$

$\boxed{b = \frac{1/3}{4/3} = \frac{1}{4}, \quad a = 3/4}$

look up

$cov(X,Y) =$ apply double expectation: $1/3$
(it's the same as the variance conditional)

$T = \frac{3}{4} + \frac{1}{4}Y \leftarrow$ new predictor

$E[T] = \frac{3}{4} + \frac{1}{4} = 1 = EX$

$Var(T-X) = Var(\frac{1}{4}Y - X) = \underbrace{\frac{1}{16}\cdot\frac{4}{3}}_{Var Y} + \underbrace{\frac{1}{3}}_{Var X} - \frac{1}{2}\underbrace{\left(\frac{1}{3}\right)}_{\text{covariance}} = \frac{1}{4}$ smaller than original

This is linear regression (apparently)

Predict X from Y

from 663: $E[X] + \frac{cov(X,Y)}{var(Y)}(Y - EY)$
(notation is flipped)

$$= 1 + \frac{1/3}{4/3}(Y-1)$$

$$= 1/4\, Y + 3/4$$

best linear unbiased predictor (linear in Y)

Normal, σ^2, S^2 is unbiased
MLE is $\frac{n-1}{n}S^2$, there is an $a+bS^2$
that has smaller MSE but is biased

2014 2a) $Y|X \sim Poisson(X)$, $X \sim U(0,2)$
Find: $E[X|Y=0]$
no obvious way to do this. Find pdf of X for $Y=0$

$$f_{X|Y}(x|y=0) = \frac{f_{X,Y}(x,0)}{f_Y(0)}$$

$$= \frac{\frac{1}{2} \cdot e^{-x}}{\int_0^2 \frac{1}{2} \cdot e^{-x}\,dx} \qquad \boxed{0 < x < 2}$$

density of x

density of y given X, at $y=0$

marginal of y: joint density integrated over x

$$= \frac{e^{-x}/2}{\frac{1}{2}[-e^{-x}]_0^2} = \frac{\frac{1}{2}e^{-x}}{\frac{1}{2}[e^{-x}]_2^0} = \frac{e^{-x}}{1-e^{-2}} = f_{X|Y}(x,y),$$
$x \in (0,2)$

$$E[X|Y=0] = \int_0^2 \frac{xe^{-x}}{1-e^{-2}}\,dx = \frac{[-xe^{-x}(x+1)]_0^2}{1-e^{-2}}$$

2g) $\int x^n e^{-x}\,dx$ ← occurs a lot in statistics, so he just knew the formula

$$f_X(x) \cdot f_{Y|X}(0|x) = \frac{1}{2} \cdot e^{-x}$$

$\boxed{2014}$ 4) $X_1, \ldots X_n \sim \text{iid} \sim N(0, \sigma^2)$

$H_0: \sigma = \sigma_0$ vs $H_1: \sigma \neq \sigma_0$

$$\delta(\underset{\sim}{X}) = 1 \text{ if } \sum_{i=1}^{n} X_i^2 < c_1 \text{ or } > c_2$$

d) N-P lemma to get UMP

sum of squares will be larger than

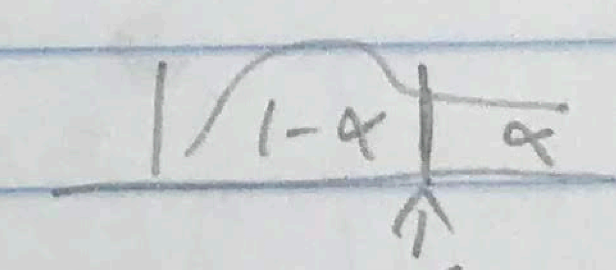
N-P lemma: H_0 $\sigma = \sigma_0$ vs $H_1: \sigma = \sigma_1 > \sigma_0$

↳ gives you the most powerful test for H_0 vs H_1 for its size α. (when H_1 is one point)

Rejection Rule: Reject H_0 if $\sum_{i=1}^{n} X_i^2 > c$

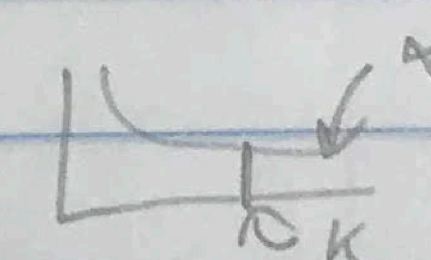
chose c on alpha and the distribution of the quantity under H_0

here: $\sigma_0^2 \chi^2_n$:



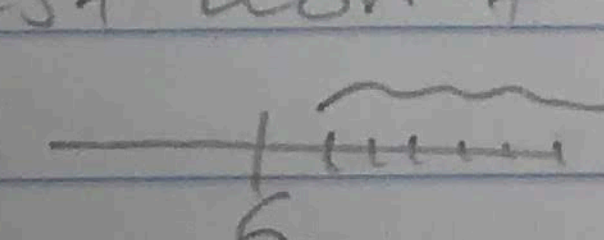
Another test: reject H_0 if $X_1^2 > K$

$\sigma_0^2 \chi_1^2$

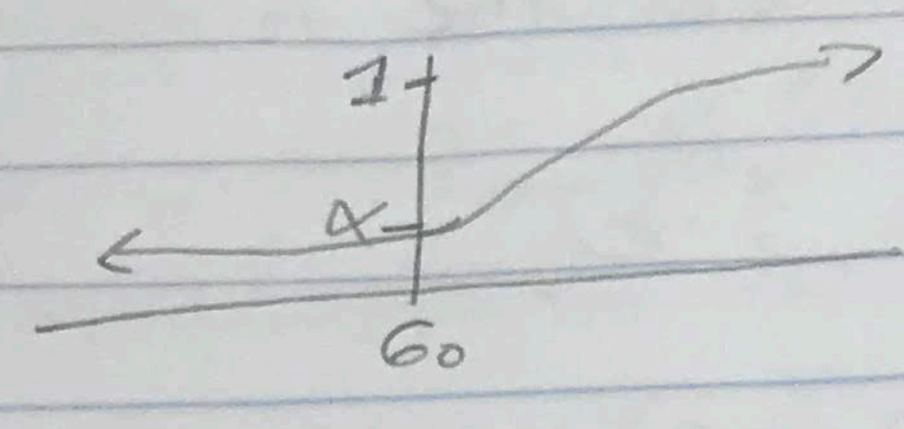


but not most powerful

The next step to uniformly most powerful. B/c the rejection region doesn't depend on σ_1 at all, only on σ_0, and so the test won't change as long as the σ_1 is $> \sigma_0$



$\boxed{2014}$ 4e) ~~When $H_0 : \sigma = \sigma_0$ H_1 $\sigma \neq \sigma_1$~~



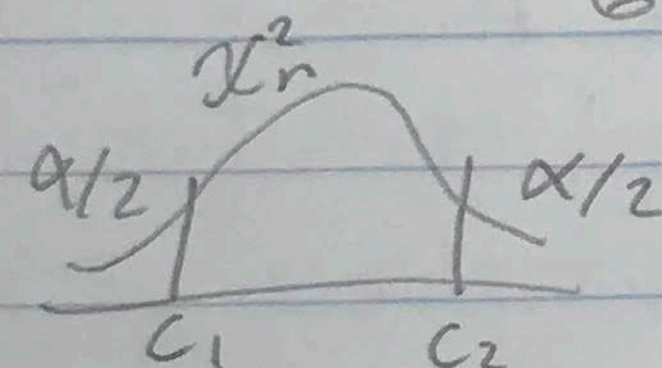
$$\beta(\sigma^2) = P(\delta = 1 \mid \sigma^2)$$
$$= P\left(c_1 < \sum_{i=1}^{n} x_i^2 < c_2 \mid \sigma^2 \right)$$

get c_1 and c_2 from χ^2_n

Beginning: $H_0 : \sigma = \sigma_0$ H_1 $\sigma \neq \sigma_0$

$$\frac{\sum x_i^2}{\sigma^2} \sim \chi^2_n$$

Chi-squared, n



Inverse CDF

$$c_1 = \sigma_0^2 F^{-1}_{\chi^2_n}(\alpha/2), \quad c_2 = \sigma_0^2 F^{-1}_{\chi^2_n}(1 - \alpha/2)$$

$$= P\left(\frac{c_1}{\sigma^2} < \underbrace{\frac{\sum x_i^2}{\sigma^2}}_{\chi^2_n} < \frac{c_2}{\sigma^2} \right)$$

σ^2 is true value

$$= 1 - P\left(\frac{c_1}{\sigma^2} < \chi^2_n < \frac{c_2}{\sigma^2} \right)$$

c_1, c_2 are fixed quantities

Prob of rejection →

$$= 1 - P\left(\frac{\sigma_0^2}{\sigma^2} F^{-1}_{\chi^2_n}(\alpha/2) < \chi^2_n < \frac{c_2}{\sigma^2} \right)$$

~~Do~~ over

4e) $H_0: \sigma = \sigma_0$ vs $H_1: \sigma > \sigma_0$

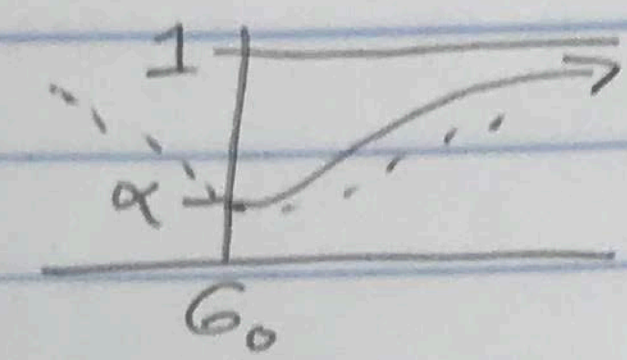
$\delta = 1 - I(c_1 < \sum x_i^2 < c_2)$

UMP: Reject if $\sum x_i^2 > k$

$P(\sum x_i^2 > k \mid H_0) = \alpha$

If hypothesis is $\sigma = \sigma_1$

- Part d) —
- Part a) ----



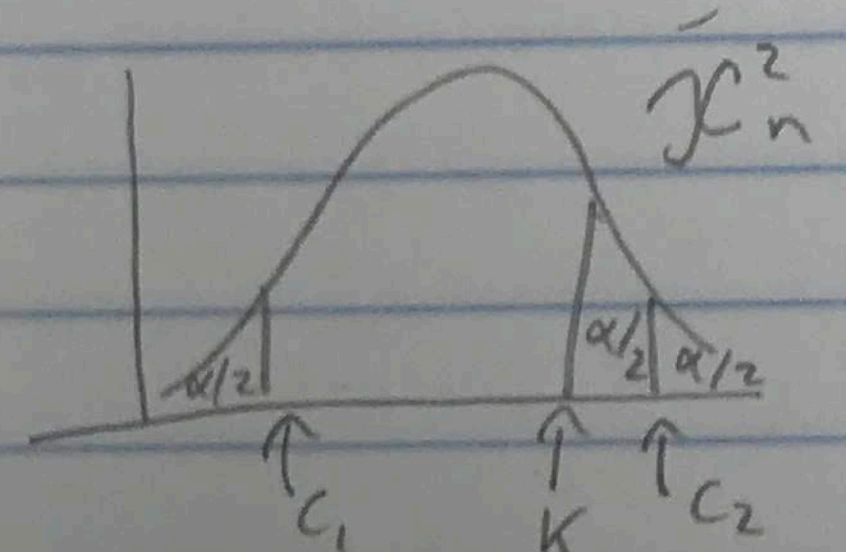
- Power function for δ:

$P(\sum X_i^2 < c_1$ or $\sum x_i^2 > c_2 \mid \sigma^2)$

- Power for UMP:

$P(\sum x_i^2 > k)$

Under H_0, both will give 5%



$\leftarrow \dfrac{\sum X_i^2}{\sigma_0^2}$ under H_0

$\dfrac{\sum x_i^2}{\sigma_0^2}$ under alternative is not χ^2_n, need: $\underbrace{\dfrac{\sigma_1^2}{\sigma_0^2}}$ $\boxed{\dfrac{\sum X_i^2}{\sigma_1^2}}$

$\uparrow \chi^2_n$

This is >1, so it's shifting to the right (stretching the distribution)