BIOS 660 HW6. Question 1-5

1. $\because P(K=k)=\frac{1}{2n+1}$, where $k\in[-n,n]$ and k is an integer

$\because X=a^{|K|}$ $\therefore X=a^i$, $i=0,1,2,\cdots n$

If $i=0$, $X=a^i=1$, $P(X=a^0)=P(X=1)=P(K=0)=\frac{1}{2n+1}$

If $i\neq 0$, $P(X=a^i)=P(|K|=i)=P(K=-i)+P(K=i)=\frac{2}{2n+1}$

$\therefore$ PMF of X is $P(X=x)=\begin{cases}\frac{1}{2n+1} & \text{if } x=0\\ \frac{2}{2n+1} & \text{if } x=a^i,\ i=1,2,3,\cdots,n\\ 0 & \text{otherwise}\end{cases}$

$\because Y=\log(X)$, $\therefore X=e^Y$

$\because x=a^i$, $i=0,1,2,\cdots,n$ $\therefore Y=\log(a^i)=i\cdot\log(a)$ $i=0,1,2,\cdots,n$

$\therefore P(Y=y)=P(Y=i\log(a))=P(\log(X)=i\cdot\log(a))=P(X=a^i)$

$\therefore$ PMF of Y is $P(Y=y)=\begin{cases}\frac{1}{2n+1} & \text{if } y=0\\ \frac{2}{2n+1} & \text{if } y=i\cdot\log a \quad i=1,2,3,\cdots,n\\ 0 & \text{otherwise}\end{cases}$

2. $P_X(x)=\begin{cases}\frac{x^2}{a} & \text{if } x=-3,-2,-1,0,1,2,3\\ 0 & \text{otherwise}\end{cases}$

(a) $\because \sum_x P_X(x)=\sum_{x=-3}^{3}\frac{x^2}{a}=\frac{1}{a}((-3)^2+(-2)^2+(-1)^2+0^2+1^2+2^2+3^2)=\frac{28}{a}=1$

$\therefore a=28$

$\therefore E(X)=\sum_x P_X(x)\cdot x=\sum_{x=-3}^{3}(\frac{x^2}{28})\cdot x=\frac{1}{28}((-3)^3+(-2)^3+(-1)^3+0^3+1^3+2^3+3^3)=0$

$\therefore E(X)=0$

(b). $\because E(X)=0$ $\therefore Z=(X-E(X))^2=(X-0)^2=X^2$

$\therefore$ if $z=0$, $P(Z=0)=P(X^2=0)=P(X=0)=\frac{0^2}{28}=0$

if $z=1,4,9$, $P(Z=z)=P(X^2=z)=P(X=-\sqrt{z})+P(X=\sqrt{z})=\frac{(-\sqrt{z})^2+(\sqrt{z})^2}{28}=\frac{z}{14}$

$\therefore$ PMF of Z is $P_Z(z)=\begin{cases}\frac{z}{14} & \text{if } z=1,4,9\\ 0 & \text{otherwise.}\end{cases}$

(c). $\because Z=(X-E(X))^2$ $\therefore Var(X)=E((X-E(X))^2)=E(Z)$

$\because E(Z)=\sum_z P_Z(z)\cdot z=1\cdot\frac{1}{14}+4\cdot\frac{4}{14}+9\cdot\frac{9}{14}=\frac{98}{14}=7$

$\therefore Var(X)=E(Z)=7$

(d). $\because E(X)=0$ $\therefore Var(X)=\sum_x(X-E(X))^2\cdot P_X(x)=\sum_x X^2\cdot P_X(x)$

$=\sum_{x=-3}^{3}x^2\cdot(\frac{x^2}{28})=\frac{1}{28}((-3)^4+(-2)^4+(-1)^4+0+1^4+2^4+3^4)$

$=\frac{1}{28}\cdot(81+16+1+0+1+16+81)=7$

$\therefore Var(X)=7.$

3. Since x takes values, the power of 2 in the interval $[2^a, 2^b]$ which are $[2^a, 2^{a+1}, 2^{a+2}, \cdots, 2^b]$ with the equal probability. So there are $b-a+1$ points, and probability is $\frac{1}{b-a+1}$

$$\therefore f(x) = P\{X(w) = x\} = \begin{cases} \frac{1}{b-a+1} & \text{for } x \text{ in } [2^a, 2^{a+1}, \cdots, 2^b] \\ 0 & \text{, otherwise} \end{cases}$$

$$E(X) = \sum_x P(x) X = \frac{1}{b-a+1} \sum_x x = \frac{1}{b-a+1} \cdot \frac{2^a(1-2^{b-a+1})}{1-2} = \frac{2^{b+1}-2^a}{b-a+1}$$

$$E(X^2) = \sum_x X^2 P(x) = \frac{1}{b-a+1} \sum_x x^2 = \frac{1}{b-a+1} \frac{2^{2a}(1-4^{b-a+1})}{1-4} = \frac{2^{2b+2}-2^{2a}}{3(b-a+1)}$$

$$\therefore Var(X) = E(X^2) - E^2(x) = \frac{2^{2b+2}-2^{2a}}{3(b-a+1)} - \frac{(2^{b+1}-2^a)^2}{(b-a+1)^2}$$

$$= \frac{(b-a-2)2^{2b+2} - (b-a+4)2^{2a} + 3 \cdot 2^{a+b+2}}{3(b-a+1)^2}$$

4 let X be random variable indicating the number of candy bars you need to eat to find a ticket.

So $P(X=x) = p(1-p)^{x-1} = pq^{x-1}$, $x=1,2,3,\cdots$, $q=1-p$ $p,q>0$

$= 0$, otherwise.

$\therefore E(X) = \sum_{x=1}^{\infty} x p q^{x-1} = p \sum_{x=1}^{\infty} x q^{x-1} = p \sum_{x=1}^{\infty} \frac{dq^x}{dq} = p \frac{d}{dq}\left(\sum_{x=1}^{\infty} q^x\right)$

$= p \cdot \frac{d}{dq}\left(\frac{q}{1-q}\right)$

$= p \frac{d}{dq}\left(\frac{1}{1-q} - 1\right)$

$= \frac{p}{(1-q)^2} = \frac{1}{p}$

$\because Var(X) = E(X^2) - E^2(x)$

$= E(X(X-1)) + E(X) - E^2(X)$

$= \sum_{x=1}^{\infty} x(x-1) p q^{x-1} + \frac{1}{p} - \frac{1}{p^2}$

$= pq \sum_{x=1}^{\infty} x(x-1) q^{x-2} + \frac{1}{p} - \frac{1}{p^2}$

$= pq \sum_{x=1}^{\infty} \frac{d^2 q^x}{dq^2} + \frac{p-1}{p^2}$

$= pq \frac{d^2}{dq^2} \sum_{x=1}^{\infty} q^x + \frac{p-1}{p^2}$

$= pq \cdot \frac{d^2}{dq^2}\left(\frac{1}{1-q} - 1\right) + \frac{p-1}{p^2}$

$= \frac{2pq}{(1-q)^3} + \frac{p-1}{p^2}$

$= \frac{2q+p-1}{p^2}$

$= \frac{1-p}{p^2}$

5. Let X be the number of tosses until the first tail appears.
Let H represent 'head', T represents 'tail'.
$\because$ Each toss is independent from others.
$\therefore$ For each toss, $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$.

$P(X=x) = P(H)^{x-1} \cdot P(T)$, $x = 1, 2, 3, 4, \cdots$

$\therefore P(X=x) = (\frac{1}{2})^{x-1} \cdot \frac{1}{2} = (\frac{1}{2})^x$, $x = 1, 2, 3, 4, \cdots$

$$\therefore E(2^X) = \sum_x \cdot P_X(x) \cdot 2^x$$
$$= \sum_{x=1}^{\infty} \cdot (\frac{1}{2})^x \cdot 2^x = \sum_{x=1}^{\infty} 1 = \infty$$

The expected amount will be received is infinite.
Though the expected payoff of this game is infinite, but I am willing to only pay 5 bucks to play the game considering that the expected utility of this game may not be unbounded and may be small.

BIOS 660 – HW #6 – (Q6–Q10)

(6) CB 2.2

a) $Y = X^2$, $f_X(x) = 1$, $x \in (0,1)$

$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$f_Y(y) = F_X'(\sqrt{y}) \cdot \frac{1}{2}y^{-1/2} - F_X'(-\sqrt{y}) \cdot (\frac{-1}{2}y^{-1/2})$

$= \frac{1}{2}y^{-1/2}[f_X(\sqrt{y}) + f_X(-\sqrt{y})]$

$= \frac{1}{2}y^{-1/2}[1 + 0] = \boxed{\begin{cases} \frac{1}{2}y^{-1/2} & \text{for } y \in (0,1) \\ 0, & \text{else} \end{cases}}$

b) $Y = -\log(X)$, $f_X(x) = \frac{(n+m+1)!}{n!\,m!} \cdot x^n (1-x)^m$, ($m, n > 0$ int, $0 < x < 1$)

$Y = g(X)$ is decreasing...

$F_Y(y) = P(g(x) \le y) = P(X > g^{-1}(y)) = 1 - F_X(g^{-1}(y))$

(negative)

$f_Y(y) = -F_X'(g^{-1}(y)) \cdot \frac{d}{dy}[g^{-1}(y)] = -f_X(g^{-1}(y)) \cdot \frac{d}{dy}(x)$

$= f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$

Know: $Y = -\log(X) \Rightarrow -Y = \log(X) \Rightarrow \underline{X = e^{-Y}}$

$\underline{dx/dy = -e^{-Y}}$, and $-\log(0) =$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = f_X(x) \cdot \left|\frac{dx}{dy}\right| = f_X(e^{-y}) \cdot |-e^{-y}|$

$= \frac{(n+m+1)!}{n!\,m!}(e^{-y})^n(1-e^{-y})^m(e^{-y})$

$= \boxed{\frac{(n+m+1)!}{n!\,m!} e^{-y(n+1)} \cdot (1-e^{-y})^m \text{ for } y \in (0,\infty)}$

(and 0, else)

c) $Y = e^x$, $f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2}$, $x \in (0,\infty)$, $\sigma^2 > 0$

Know: $f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right|$

$x \in (0,\infty)$

$Y = e^X \Rightarrow X = \ln(Y)$, $\frac{dx}{dy} = 1/Y$. $\Rightarrow y \in (1,\infty)$

$$f_Y(y) = f_X(\ln(Y)) \left|1/Y\right|$$

$$= \frac{1}{\sigma^2}(\ln(y)) e^{-(\ln(y)/\sigma)^2/2} \cdot \left|\frac{1}{Y}\right|, \quad y \in (1,\infty)$$

$$= \boxed{\begin{cases} \dfrac{\ln(y)\, e^{-(\ln(y)/\sigma)^2/2}}{y\sigma^2}, & y \in (1,\infty) \\ 0, & \text{else} \end{cases}}$$

prove pdf = 1 ?

too!

⑦ CB 2.6

a) $f_X(x) = \frac{1}{2} e^{-|x|}$, $x \in (-\infty,\infty)$, $Y = |X|^3 \Rightarrow \begin{cases} Y = X^3 \text{ on } (0,\infty) \\ Y = -X^3 \text{ on } (-\infty,0) \end{cases}$

$\Rightarrow X = \sqrt[3]{Y}$ on $(0,\infty)$, $X = \sqrt[3]{-y}$ on $(-\infty,0)$

$$f_Y(y) = f_X(y^{1/3}) \cdot \left(\tfrac{1}{3} y^{-2/3}\right) + f_X((-y)^{1/3}) \cdot \tfrac{1}{3}(y)^{-2/3}(-1)$$

$$= \tfrac{1}{2} e^{-y^{1/3}}\left(\tfrac{1}{3} y^{-2/3}\right) + \tfrac{1}{2} e^{-y^{1/3}}\left(\tfrac{1}{3} y^{-2/3}\right)$$

$$= \boxed{\begin{cases} \dfrac{1}{3} y^{-2/3} \cdot e^{-y^{1/3}}, & 0 \le y < \infty \\ 0, & \text{else} \end{cases}}$$

$$\frac{1}{3}\int_0^\infty y^{-2/3} \cdot e^{-y^{1/3}}\, dy \qquad u = -y^{1/3}, \quad du = -\tfrac{1}{3} y^{-2/3}$$

$$= \tfrac{1}{3}(-3)\int e^u\, du = -\left[e^u\right]_0^{-\infty} = -[0-1] = \boxed{1}$$

b) $f_X(x) = \frac{3}{8}(x+1)^2, x \in (-1,1), Y = 1 - X^2, Y \in [0,1)$

$X = \sqrt{1-y}, \ x \in (0,1)$

$X = -\sqrt{1-y}, x \in (-1,0)$

$A_1 = (0,1), A_2 = (-1,0)$

$$f_Y(y) = \underbrace{f_X(g^{-1}(y))\left|\tfrac{dx}{dy}\right|}_{1^{st} \text{ interval}} + \underbrace{f_X(g^{-1}(y))\left|\tfrac{dx}{dy}\right|}_{2^{nd} \text{ interval}}$$

$$= f_X(-\sqrt{1-y})\left|\tfrac{-1}{2}(1-y)^{-1/2}(-1)\right| + f_X(\sqrt{1-y})\left|\tfrac{1}{2}(1-y)^{-1/2}(-1)\right|$$

$$= \frac{3}{8}\left[-\sqrt{1-y}+1\right]^2\left(\frac{1}{2\sqrt{1-y}}\right) + \frac{3}{8}\left[\sqrt{1-y}+1\right]^2\left(\frac{1}{2\sqrt{1-y}}\right)$$

$$= \frac{3}{16}\left(1-\sqrt{1-y}\right)^2\left(\frac{1}{\sqrt{1-y}}\right) + \frac{3}{16}\left(\sqrt{1-y}+1\right)^2\left(\frac{1}{\sqrt{1-y}}\right)$$

$$= \frac{3}{16\sqrt{1-y}}\left[1-\cancel{2\sqrt{1-y}}+(1-y)+(1-y)+\cancel{2\sqrt{1-y}}+1\right]$$

$$= \frac{3(4-2y)}{16\sqrt{1-y}} = \boxed{\begin{cases} \dfrac{3(2-y)}{8\sqrt{1-y}}, & y \in (0,1) \\ 0, & \text{else} \end{cases}}$$

$$\frac{3(1+1-y)}{8\sqrt{1-y}} = \frac{3}{8\sqrt{1-y}} + \frac{3(1-y)}{8(1-y)^{1/2}} = \frac{3}{8}(1-y)^{-1/2} + \frac{3}{8}(1-y)^{1/2}$$

$$\frac{3}{8}\int(1-y)^{-1/2}dy + \frac{3}{8}\int(1-y)^{1/2}dy$$

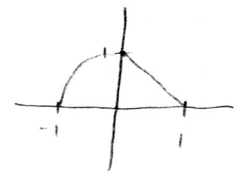
$$= \frac{-3}{8}\left[\frac{u^{1/2}}{1/2}\right]_0^1 + \frac{-3}{8}\left[\frac{u^{3/2}}{3/2}\right]_0^1 = \frac{-3}{4}\left[\sqrt{1-y}\right]_0^1 + \frac{1}{4}\left[(\sqrt{1-y})^3\right]_0^1$$

$$= -\tfrac{3}{4}(0-1) + \tfrac{-1}{4}(0-1) = \tfrac{3}{4} + \tfrac{1}{4} = \boxed{1}$$

c) $f_X(x) = \frac{3}{8}(x+1)^2,\ x \in (-1,1) \Rightarrow \left\{ \begin{array}{l} Y = 1 - X^2,\ X \le 0 \\ Y = 1 - X,\ X > 0 \end{array} \right\}$ $\begin{array}{l} \Rightarrow X = -\sqrt{1-Y} \ \checkmark \\ \Rightarrow X = 1 - Y \end{array}$

$A_1 = (-1,0),\ A_2 = (0,1)$

$$f_Y(y) = \underbrace{f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right|}_{1^{st}\ \text{interval}} + \underbrace{f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right|}_{2^{nd}\ \text{interval}}$$



$$= f_X(-\sqrt{1-y}) \left| -\tfrac{1}{2}(1-y)^{-1/2}(-1) \right| + f_X(1-y)\left|-1\right|$$

$$= \frac{3}{8}\left[1 - \sqrt{1-y}\right]^2 \left(\frac{1}{2\sqrt{1-y}}\right) + \frac{3}{8}\left[2-y\right]^2$$

$$= \boxed{\left\{ \begin{array}{ll} \frac{3}{16\sqrt{1-y}}\left(1-\sqrt{1-y}\right)^2 + \frac{3}{8}(2-y)^2, & y \in (0,1) \\ 0, & \text{else} \end{array} \right\}} \ \checkmark$$

$$\frac{3}{16}\int_0^1 \frac{(1-\sqrt{1-y})^2}{\sqrt{1-y}}\,dy + \frac{3}{8}\int_0^1 (4 - 4y + y^2)\,dy$$

$u = 1 - \sqrt{1-y}$

$du = \frac{-1}{2}(1-y)^{-1/2}(-1)\,dy$

$$= \frac{3}{16}(2)\int u^2\,du + \frac{3}{8}\int_0^1 4\,dy - \frac{3}{2}\int_0^1 y\,dy + \frac{3}{8}\int y^2\,dy$$

$$= \frac{3}{8}\left(\frac{1}{3}\right)\left[(1-\sqrt{1-y})^3\right]_0^1 + \frac{3}{2}\left[y\right]_0^1 - \frac{3}{4}\left[y^2\right]_0^1 + \frac{1}{8}\left[y^3\right]_0^1$$

$$= \frac{1}{8}(1) + \frac{3}{2}(1) - \frac{3}{4}(1) + \frac{1}{8}(1) = \boxed{1} \ \checkmark$$

(8) CB 2.9

X s.t. $f_X(x) = \begin{cases} \frac{x-1}{2}, & 1 < x < 3 \\ 0, & \text{else} \end{cases}$

Find $Y = u(X)$ s.t. $Y \sim \text{Uniform}(0,1)$.

$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right|$

Based on Theorem 2.1.10, let $Y = F_X(X)$.

$F_X(x) = \frac{x^2}{4} - \frac{x}{2} + c$

$F_X(3) = 3^2/4 - 3/2 + c = 1$

$\frac{9}{4} - \frac{6}{4} + c = 1 \Rightarrow c = 1/4$

$$\Rightarrow \boxed{u(X) = \begin{cases} \frac{x^2}{4} - \frac{x}{2} + \frac{1}{4}, & x \in (1,3) \\ 0, & x < 1 \\ 1, & x > 3 \end{cases}}$$

case 1 y-values

case 2 y-values

9.) Let x_y be a jump point in $F_X(x)$.
Let $A_y = \{x : F_X(x) \le y\} = (-\infty, x_y)$ Note: A_y can't be closed $(-\infty, x_y]$

$$F_Y(y) = P(Y \le y) = P(F_X(X) \le y) = P(X \in A_y)$$

$$P(X \in A_y) = P(X \in (-\infty, x_y)) = \lim_{x \to x_y^-} P(x \in (-\infty, x]) = \lim_{x \to x_y^-} F_X(x) \le y$$

The last inequality is because $F_X(x) \le y \quad \forall x \in A_y$

To show $\exists y$ s.t. $F_Y(y) < y$
Let y' be "jumped" over by F_X. s.t.

$$F_X(x_y - \varepsilon) < y' < F_X(x_y) \Rightarrow \lim_{x \to x_y^-} F_X(x) < y' < F_X(x_y)$$

$$A_{y'} = (-\infty, x_y)$$

Then $F_Y(y') = \lim_{x \to x_y^-} F_X(x) < y'$

This directly proves b and can prove a
by $P(Y > y) = 1 - P(Y \le y) \ge 1 - y$ for all y $0 < y < 1$
$P(Y > y) = 1 - P(Y \le y) > 1 - y$ for some y $0 < y < 1$

is not a jump point.

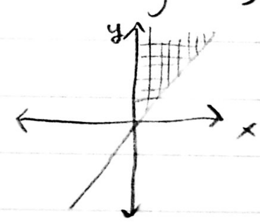
10. a) $\int_0^\infty (1-F_X(x))\,dx$

$= \int_0^\infty \left(1-\int_0^x f_X(y)\,dy\right) dx$

$= \int_0^\infty \int_x^\infty f_X(y)\,dy\,dx$ since $f_X(y)$ is pdf

$= \int_0^\infty \int_0^y f_X(y)\,dx\,dy$ change integration order (diagram).

$= \int_0^\infty y f_X(y)\,dy = E(X)$.



b) $\sum_{k=0}^{\infty} (1-F_X(k))$

$= \sum_{k=0}^{\infty} \left(1-\sum_{l=0}^{k} f_X(l)\right)$

$= \sum_{k=0}^{\infty} \sum_{l=k+1}^{\infty} f_X(l)$ since $f_X(l)$ is pmf

$= \sum_{l=0}^{\infty} \sum_{k=0}^{l} f_X(l)$ similiar to above.

$= \sum_{l=0}^{\infty} l f_X(l) = E(X)$ This is the discrete version of part a)

11. (a) $f(x) = 3x^2 \quad , \; 0<x<1$

$$\int_0^m 3x^2\,dx = \int_m^1 3x^2\,dx = \frac{1}{2}$$

$$\Rightarrow \quad x^3\Big|_0^m = x^3\Big|_m^1 = \frac{1}{2}$$

$$\Rightarrow \quad m^3 - 0 = 1 - m^3 = \frac{1}{2}$$

$$\Rightarrow \quad m^3 = \frac{1}{2} \quad \Rightarrow \quad m = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

(b) $f(x) = \frac{1}{\pi(1+x^2)} \quad , \; -\infty < x < \infty$

$$\int_{-\infty}^m \frac{1}{\pi}\cdot\frac{1}{1+x^2}\,dx = \int_m^\infty \frac{1}{\pi}\cdot\frac{1}{1+x^2}\,dx = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{\pi}\arctan x\Big|_{-\infty}^m = \frac{1}{\pi}\arctan x\Big|_m^\infty = \frac{1}{2}$$

$$\Rightarrow \quad \frac{\arctan m}{\pi} - \frac{1}{2} = \frac{1}{2} - \frac{\arctan m}{\pi} = \frac{1}{2}$$

$$\Rightarrow \quad m = 0$$

12. $$E|X-a| = \int_{-\infty}^\infty f_X(x)\,|x-a|\,dx$$

$$= \int_{-\infty}^a f_X(x)\,|x-a|\,dx + \int_a^\infty f_X(x)\,|x-a|\,dx$$

$$= -\int_{-\infty}^a f_X(x)(x-a)\,dx + \int_a^\infty f_X(x)(x-a)\,dx$$

$$\frac{dE|X-a|}{da} = \int_{-\infty}^a f_X(x)\,dx - \int_a^\infty f_X(x)\,dx \qquad \frac{d^2E|X-a|}{da^2} = 2f(a) > 0$$

Thus, when $\frac{dE|X-a|}{da} = 0$, $E|X-a|$ is min

$\because \int_{-\infty}^a f_X(x)\,dx + \int_a^\infty f_X(x)\,dx = \int_{-\infty}^\infty f_X(x)\,dx = 1$

$\therefore$ The solution of $\int_{-\infty}^a f_X(x)\,dx - \int_a^\infty f_X(x)\,dx = 0$ is

$a =$ median

That is $\min_a E|X-a| = E|X-m|$

13. (a) The uniform $(0,1)$ pdf is symmetric about $\frac{1}{2}$

$$f\left(\tfrac{1}{2}+\epsilon\right) = f\left(\tfrac{1}{2}-\epsilon\right) = \begin{cases} 1, & \text{if } 0<\epsilon<\frac{1}{2} \\ 0, & \text{if } \frac{1}{2}\le\epsilon<\infty \end{cases}$$

The standard normal pdf is symmetric about 0. $f(0+\epsilon) = f(0$

The Cauchy pdf is also symmetric about 0. $f(0+\epsilon) = f(0$

(b) $$\int_{-\infty}^a f(x)\,dx = \int_{-\infty}^0 f(a-\epsilon)\,d\epsilon$$

$$\int_a^\infty f(x)\,dx = \int_0^\infty f(a+\epsilon)\,d\epsilon$$

$\because f(a-\epsilon) = f(a+\epsilon) \quad \therefore \int_{-\infty}^0 f(a-\epsilon)\,d\epsilon = \int_0^\infty f(a+\epsilon)\,d\epsilon$

$\because \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^\infty f(x)\,dx = 1$

$\therefore \int_{-\infty}^{a} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx = \frac{1}{2}$

$\therefore$ $a =$ midian

(c) $EX - a = E(X-a) = \int_{-\infty}^{\infty} f(x)(x-a)\,dx$

$$= \int_{-\infty}^{a} f(x)(x-a)\,dx + \int_{a}^{\infty} f(x)(x-a)\,dx$$

$$= \int_{0}^{\infty} f(a-\epsilon)(\epsilon)\,d\epsilon + \int_{0}^{\infty} f(a+\epsilon)\,\epsilon\,d\epsilon$$

$f(a-\epsilon) = f(a+\epsilon)$ $\quad = \int_{0}^{\infty} f(a+\epsilon)\,\epsilon\,d\epsilon - \int_{0}^{\infty} f(a-\epsilon)\,\epsilon\,d\epsilon = 0$

$\therefore EX - a = 0 \qquad EX = a$

(d) for $a>0$, let $a > \epsilon$

$$f(a-\epsilon) = e^{-(a-\epsilon)} \geq e^{-(a+\epsilon)} = f(a+\epsilon)$$

Thus $f(x)$ is not symmetric about $a>0$

for $a<0$, let $a > -\epsilon$

$$f(a-\epsilon) = \underbrace{e^{-(a-\epsilon)}}_{=0} < e^{-(a+\epsilon)} = f(a+\epsilon)$$

Thus $f(x)$ is not symmetric about $a<0$

Therefore $f(x)$ is not symmetric about any a.

(e) $\int_{0}^{m} e^{-x}\,dx = \frac{1}{2}$

$\Rightarrow -e^{-x}\big|_{0}^{m} = -e^{-m} - (-1) = \frac{1}{2}$

$\Rightarrow -e^{-m} = -\frac{1}{2} \Rightarrow e^{-m} = \frac{1}{2} \Rightarrow m = \log 2$

$EX = \int_{0}^{\infty} e^{-x}\cdot x\,dx = \int_{0}^{\infty} -x\,de^{-x} = -x\cdot e^{-x}\big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x}\,dx$

$$= \left[-x\cdot e^{-x} - e^{-x}\right]\Big|_{0}^{\infty} = -e^{-x}(x+1)\Big|_{0}^{\infty}$$

$$= 1$$

$\therefore m = \log 2 < 1 = EX$ The median is less than the mean.

14. (a) The standard normal pdf with the unique mode 0.

(b) The uniform $(0,1)$ pdf with the mode $\frac{1}{2}$ which is not unique.

(c) Prove by contradiction. Let a be the symmetric point and b be the mode.

If b is the unique mode. Then assume $a \neq b$. That is $a = b+\epsilon > b$ for ϵ. $f(b) \geq f(b+\epsilon) \geq f(b+2\epsilon) \Rightarrow f(a-\epsilon) \geq f(a) \geq f(a+\epsilon)$

Which contradicts a is the symmetric point. Thus, $a = b$.

If b is not the unique mode. Then assume $a \notin (x_1, x_2)$ where any $b \in (x_1, x_2)$ is a mode. That is $a = b+\epsilon > b$ for ϵ and $f(b) \geq f(b+\epsilon) \geq f(b+2\epsilon)$

$\Rightarrow f(a-\epsilon) \geq f(a) \geq f(a+\epsilon)$ which contradicts a is the symmetric point.

Thus, $a \in (x_1, x_2)$ is a mode.

Therefore, if $f(x)$ is both symmetric and unimodal, then the point of symmetric is a mode.

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(d) $f(x) = e^{-x}$, $x \geq 0$ is decreasing in the interval $[0, +\infty)$

Thus $f(x)_{max} = f(0)$ satisfies that $f(0) > f(x) > f(y)$ for any $0 < x < y$.

Thus $f(x)$ is unimodal with a mode 0.