BIOS 660 Homework 10. Question 1-3.

1. $X \sim \chi^2_m$, $Y \sim \chi^2_n$, $Z = \frac{X/m}{Y/n}$

Let $U = X/m$, $V = Y/n$, $Z = \frac{X/m}{Y/n} = \frac{U}{V}$

$\because X \sim \chi^2_m$, $Y \sim \chi^2_n$

$\therefore f_X(x) = \frac{(\frac{x}{2})^{\frac{m}{2}-1} e^{-\frac{x}{2}}}{2\Gamma(m/2)}$, $x \geq 0$

$f_Y(y) = \frac{(\frac{y}{2})^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{2\Gamma(n/2)}$, $x \geq 0$

$\because u = \frac{x}{m}$, $\therefore u > 0$, $\frac{dx}{du} = m > 0$, $x = mu$

$\therefore f_U(u) = f_X(mu) \cdot \left|\frac{dx}{du}\right| = \frac{(\frac{mu}{2})^{\frac{m}{2}-1} e^{-\frac{mu}{2}}}{2\Gamma(m/2)} \cdot m$, $u \geq 0$

$\because V = \frac{Y}{n}$, $\therefore v \geq 0$, $\frac{dy}{dv} = n > 0$, $y = n \cdot v$

$\therefore f_V(v) = f_Y(nv) \cdot \left|\frac{dy}{dv}\right| = \frac{(\frac{nv}{2})^{\frac{n}{2}-1} e^{-\frac{nv}{2}}}{2\Gamma(n/2)} \cdot n$, $v \geq 0$

$\because Z = \frac{X/m}{Y/n} = \frac{U}{V} \geq 0$, $U \geq 0$, $V \geq 0$

$\therefore F_Z(z) = P(Z \leq z) = P(\frac{U}{V} \leq z) = \iint_{\frac{u}{v} \leq z} f_{U,V}(u,v)\, du dv$

$= \int_0^{+\infty} (\int_0^{zv} f_{U,V}(u,v) du) \cdot dv$

$\therefore f_Z(z) = \frac{d}{dz} F_Z(z) = \int_0^{+\infty} f_{U,V}(zv, v) \cdot v\, dv$

$\because X, Y$ are independent. $\therefore U = \frac{X}{m}$, $V = \frac{Y}{n}$ are independent

$\therefore f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$ $\therefore f_{U,V}(zv, v) = f_U(zv) \cdot f_V(v)$

$\therefore f_Z(z) = \int_0^{+\infty} f_U(zv) f_V(v) \cdot v \cdot dv$

$= \int_0^{+\infty} \frac{(\frac{mzv}{2})^{\frac{m}{2}-1} e^{-\frac{mzv}{2}}}{2\Gamma(\frac{m}{2})} \cdot m \cdot \frac{(\frac{nv}{2})^{\frac{n}{2}-1} e^{-\frac{nv}{2}}}{2\Gamma(\frac{n}{2})} \cdot n \cdot v\, dv$

$= \frac{mn}{4\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \cdot (\frac{mz}{2})^{\frac{m}{2}-1} \cdot (\frac{n}{2})^{\frac{n}{2}-1} \int_0^{+\infty} v \cdot v^{\frac{m}{2}+\frac{n}{2}-2} \cdot e^{-\frac{mz+n}{2} \cdot v} dv$

Let $W = \frac{mz+n}{2} V$. $V = \frac{2W}{mz+n}$, $\frac{dv}{dw} = \frac{2}{mz+n}$, $w > 0$

$\therefore \int_0^{+\infty} v^{\frac{m}{2}+\frac{n}{2}-1} e^{-\frac{mz+n}{2} \cdot v} dv = \int_0^{+\infty} e^{-w} w^{\frac{m}{2}+\frac{n}{2}-1} (\frac{2}{mz+n})^{\frac{m}{2}+\frac{n}{2}} dw$

$= (\frac{2}{mz+n})^{\frac{m}{2}+\frac{n}{2}} \cdot \int_0^{+\infty} e^{-w} \cdot w^{\frac{m}{2}+\frac{n}{2}-1} dw$

$= (\frac{2}{mz+n})^{\frac{m}{2}+\frac{n}{2}} \cdot \Gamma(\frac{m}{2}+\frac{n}{2})$

$\therefore f_Z(z) = \frac{mn}{4 \cdot \Gamma(\frac{m}{2}) \cdot \Gamma(\frac{n}{2})} \cdot (\frac{mz}{2})^{\frac{m}{2}-1} \cdot (\frac{n}{2})^{\frac{n}{2}-1} \cdot (\frac{2}{mz+n})^{\frac{m}{2}+\frac{n}{2}} \cdot \Gamma(\frac{m}{2}+\frac{n}{2})$

1. Continued.

$$\because f_Z(z) = \frac{mn}{4\Gamma(\frac{m}{2})\cdot\Gamma(\frac{n}{2})}\cdot\left(\frac{mz}{2}\right)^{\frac{m}{2}-1}\left(\frac{n}{2}\right)^{\frac{n}{2}-1}\left(\frac{2}{mz+n}\right)^{\frac{m}{2}+\frac{n}{2}}\cdot\Gamma\left(\frac{m}{2}+\frac{n}{2}\right)$$

$$\therefore f_Z(z) = \frac{\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\cdot\Gamma(\frac{n}{2})}\cdot\frac{m\cdot n\cdot(mz)^{\frac{m}{2}-1}\, n^{\frac{n}{2}-1}}{(mz+n)^{\frac{m}{2}+\frac{n}{2}}}$$

$$f_Z(z) = \frac{\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\cdot\Gamma(\frac{n}{2})}\cdot\frac{(\frac{m}{n})\cdot(\frac{m}{n}\cdot z)^{\frac{m}{2}-1}\cdot n^{\frac{m}{2}+\frac{n}{2}}}{(mz+n)^{\frac{m}{2}+\frac{n}{2}}}$$

$$= \frac{\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\cdot\Gamma(\frac{n}{2})}\cdot\frac{(\frac{m}{n})\cdot(\frac{m}{n}\cdot z)^{\frac{m}{2}-1}}{(\frac{m}{n}\cdot z+1)^{\frac{m}{2}+\frac{n}{2}}}$$

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$$\therefore f_Z(z) = \frac{(\frac{m}{n})\cdot\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\cdot\Gamma(\frac{n}{2})}\cdot\frac{(\frac{m}{n}\cdot z)^{\frac{m}{2}-1}}{(\frac{m}{n}\cdot z+1)^{\frac{m+n}{2}}}, \qquad 0\le z<\infty.$$

which is the pdf of distribution $F(m,n)$

$$\therefore Z = \frac{X/m}{Y/n} \sim F(m,n) \quad \text{if } X\sim\chi^2_m,\ Y\sim\chi^2_n,\ X, Y \text{ independent}$$

2. $\because f_X(x) = \begin{cases} \frac{2}{2e-5}\, x_1^2 x_2\cdot e^{x_1x_2x_3}, & 0<x_1,x_2,x_3<1 \\ 0 & \text{otherwise} \end{cases}$

Let $Y_1 = X_1X_2X_3,\ Y_2 = X_1X_2,\ Y_3 = X_1,\ 0<Y_1<Y_2<Y_3<1,\ Y=(Y_1,Y_2,Y_3)'$

$$\therefore X_1 = Y_3,\quad X_2 = \frac{Y_2}{X_1} = \frac{Y_2}{Y_3},\quad X_3 = \frac{Y_1}{X_1X_2} = \frac{Y_1\cdot Y_3}{Y_3Y_2} = \frac{Y_1}{Y_2}$$

$$\therefore x_1 = y_3,\quad x_2 = \frac{y_2}{y_3},\quad x_3 = \frac{y_1}{y_2}.$$

$$\therefore J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{y_3} & \frac{-y_2}{y_3^2} \\ \frac{1}{y_2} & -\frac{y_1}{y_2^2} & 0 \end{vmatrix} = -\frac{1}{y_2y_3}$$

$$\because 0<y_1<y_2<y_3<1 \quad \therefore |J| = \left|-\frac{1}{y_2y_3}\right| = \frac{1}{y_2y_3}$$

$$\therefore f_Y(y) = \frac{2}{2e-5}\cdot y_3^2\cdot\left(\frac{y_2}{y_3}\right)\cdot e^{y_1}\cdot|J|;\quad 0<y_1<y_2<y_3<1$$

$$f_Y(y) = \frac{2}{2e-5}\cdot e^{y_1},\quad 0<y_1<y_2<y_3<1$$

$$\therefore f_{Y_1}(y_1) = \int_{y_1}^{1}\int_{y_1}^{y_3} f_Y(y)\cdot dy_2\, dy_3, \qquad Y=(Y_1,Y_2,Y_3)'$$

$$\therefore f_{Y_1}(y_1) = \int_{y_1}^{1} dy_3\int_{y_1}^{y_3} dy_2\cdot\frac{2}{2e-5}\cdot e^{y_1} = \frac{2}{2e-5}\cdot\int_{y_1}^{1} e^{y_1}(y_3-y_1)\,dy_3$$

$$\therefore f_{Y_1}(y_1) = \frac{2}{2e-5}\cdot\left(\frac{1}{2}y_3^2 e^{y_1} - y_1y_3\cdot e^{y_1}\right)\Big|_{y_1}^{1} = \frac{2}{2e-5}\cdot e^{y_1}\left(\frac{1}{2}-y_1+\frac{1}{2}y_1^2\right)$$

$0<y_1<1$

2. To sum up, $f_{Y_1}(y_1) = \frac{2}{2e-5} \cdot e^{y_1}(\frac{1}{2} - y_1 + \frac{1}{2}y_1^2)$, $0<y_1<1$.

where $Y_1 = X_1 \cdot X_2 \cdot X_3$.

3. $\because X \sim Exp(1)$, $Y \sim Exp(1)$, X, Y are independent

$\therefore f_X(x) = e^{-x}$, $x \geq 0$; $f_Y(y) = e^{-y}$, $y \geq 0$;

$\therefore f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)}$

Let $Z = X+Y$, $z>0$

$\because F_Z(z) = P(Z \leq z) = \iint_{X+Y\leq z} f_{X,Y}(x,y) \cdot dxdy = \int_0^z dx \cdot \int_0^{z-x} dy \cdot e^{-(x+y)}$

$= \int_0^z (e^{-x}(-e^{-y})\Big|_0^{z-x}) \cdot dx = \int_0^z -e^{-z} + e^{-x} dx$

$= -x \cdot e^{-z} - e^{-x}\Big|_0^z = 1 - z \cdot e^{-z} - e^{-z}$

$\therefore f_Z(z) = -(e^{-z} - ze^{-z}) + e^{-z} = z \cdot e^{-z}$, $z>0$ $\quad \therefore Z \sim Gamma(2,1)$

$\therefore f_{X|Z}(x|z) = \frac{f_{X,Y}(x, z-x)}{f_Z(z)} = \frac{e^{-z}}{z \cdot e^{-z}} = \frac{1}{z}$, $z>0$.

$\therefore$ When $Z = X+Y = c$, $z=c$. $f_{X|X+Y=c}(x|x+y=c) = \frac{1}{c}$

$\because 0 \leq (X | X+Y=c) \leq c$

$\therefore X|X+Y=c \sim U(0,c)$

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4. a) Given $X = \{-2, -1, 0, 1, 2, 3, 4\}$ and $Y = \{X-1, X, X+1\}$

$\Rightarrow (X,Y) = \{(-2,-3), (-2,-2), (-2,-1), (-1,-2), (-1,-1), (-1,0), \dots\}$

	-3	-2	-1	0	1	2	3	4	5	Sum
-2	1/21	1/21	1/21	0	0	0	0	0	0	3/21 = 1/7
-1	0	1/21	1/21	1/21	0	0	0	0	0	3/21 = 1/7
0	0	0	1/21	1/21	1/21	0	0	0	0	3/21 = 1/7
1	0	0	0	1/21	1/21	1/21	0	0	0	3/21 = 1/7
2	0	0	0	0	1/21	1/21	1/21	0	0	3/21 = 1/7
3	0	0	0	0	0	1/21	1/21	1/21	0	3/21 = 1/7
4	0	0	0	0	0	0	1/21	1/21	1/21	3/21 = 1/7
Sum	1/21	2/21	3/21	3/21	3/21	3/21	3/21	2/21	1/21	1

Joint PMF: $P_{X,Y}(x,y) = \begin{cases} 1/21, & -2 \le x \le 4,\ -1+x \le y \le 1+x \\ 0, & \text{else} \end{cases}$

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Marginal PMFs:

$$\boxed{P_X(x) = \begin{cases} 1/7, & -2 \le x \le 4 \\ 0, & \text{else} \end{cases} \qquad P_Y(y) = \begin{cases} 1/21, & y = -3, 5 \\ 2/21, & y = -2, 4 \\ 3/21, & -1 \le y \le 3 \\ 0, & \text{else} \end{cases}}$$

Means: $E[X] = \sum_{x=-2}^{4} x\,P(x) = \frac{1}{7}(-2 + -1 + 0 + 1 + 2 + 3 + 4) = \frac{7}{7} = \boxed{1}$

$$E[Y] = \sum_{y=-3}^{5} y\,P(y) = \frac{1}{21}(-3+5) + \frac{2}{21}(-2+4) + \frac{3}{21}(-1+0+1+2+3)$$

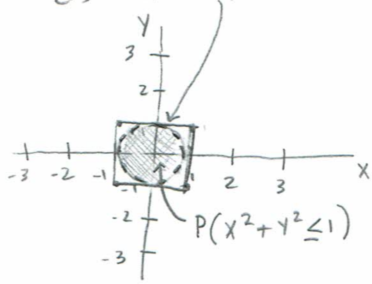
$$= \frac{2}{21} + \frac{4}{21} + \frac{15}{21} = \frac{21}{21} = \boxed{1}$$

b) Mean of trader's profit $= E(100X + 200Y) = 100E[X] + 200E[Y]$

$$= 100(1) + 200(1) = \boxed{300}$$

5. (CB, 4.1) A random point (X,Y) is distributed uniformly on a square with vertices $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$. That is, the joint pdf is $f(x,y) = \frac{1}{4}$ on the square. Determine the probabilities of the following events.

a) $X^2 + Y^2 < 1$

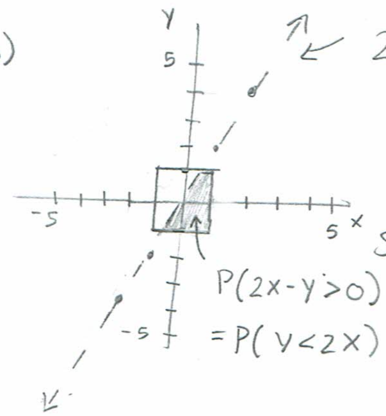


Area square $= 2 \times 2 = 4$

Area circle $= \pi(1)^2 = \pi$

Since points uniformly distributed, $P(X^2+Y^2 \le 1) = \frac{\text{Area circle}}{\text{Area square}} = \boxed{\frac{\pi}{4}}$

b) $2X - Y > 0 \Rightarrow Y < 2X$



Area Square $= 2 \times 2 = 4$

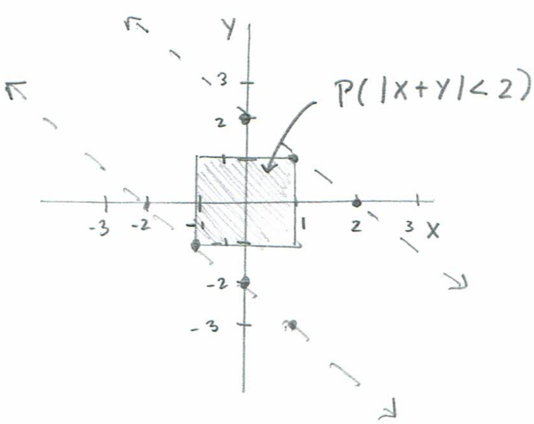
Area half square $= 2$

Since points uniformly distributed, $P(2X-Y>0) = P(Y<2X) =$

$\frac{\text{Area half square}}{\text{Area Square}} = \frac{2}{4} = \boxed{\frac{1}{2}}$

c) $|X+Y| < 2 \Rightarrow X+Y < 2$ and $X+Y > -2$

$\Rightarrow Y < 2 - X$ and $Y > -X - 2$



Since points uniformly distributed, and since area between parallel lines completely overlaps with area in circle, $\boxed{P(|X+Y|<2) = 1}$

6. (CB, 4.4) A pdf is defined by:

$$f(x,y) = \begin{cases} c(x+2y), & \text{if } 0<y<1 \text{ and } 0<x<2 \\ 0, & \text{else} \end{cases}$$

a) Find the value of C.

$$\int_0^1\int_0^2 c(x+2y)\,dx\,dy = c\int_0^1 \left(\tfrac{1}{2}x^2+2xy\right)\Big|_0^2 dy = c\int_0^1 \left(\tfrac{1}{2}(2)^2+2(2)y\right)dy$$

$$= c\int_0^1 (2+4y)\,dy = c(2y+2y^2)\Big|_0^1 = c(2+2) = 4c$$

Then, $4c = 1 \Rightarrow \boxed{c = 1/4}$

b) Find the marginal distribution of X.

$$f_X(x) = c\int_0^1 (x+2y)\,dy = c(xy+y^2)\Big|_0^1 = c(x(1)+1) = cx+c = \tfrac{1}{4}x+\tfrac{1}{4}$$
$$= \tfrac{1}{4}(x+1)$$

$$\therefore \boxed{f_X(x) = \begin{cases} \tfrac{1}{4}(x+1), & 0<x<2 \\ 0, & \text{else} \end{cases}}$$

c) Find the joint cdf of X and Y.

Joint pdf: $F_{XY}(x,y) = P(X\le x, Y\le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(v,u)\,dv\,du$

For $x\le 0, y\le 0$: $F_{XY}(x,y) = 0$

For $0<x<2,\ 0<y<1$: $F_{XY}(x,y) = c\int_0^x\int_0^y (u+2v)\,dv\,du = c\int_0^x (uv+v^2)\Big|_0^y du$

$$= c\int_0^x (uy+y^2)\,du = c\left(\tfrac{1}{2}u^2y+y^2u\right)\Big|_0^x = c\left(\tfrac{1}{2}x^2y+xy^2\right)$$

$$= \tfrac{1}{4}\left(\tfrac{1}{2}x^2y+xy^2\right) = \underline{\tfrac{1}{8}x^2y+\tfrac{1}{4}xy^2}$$

For $2\le x,\ 0<y<1$:

$$F_X(x,y) = c\int_0^2\int_0^y (u+2v)\,dv\,du = c\int_0^2 (uv+v^2)\Big|_0^y du = c\int_0^2 (uy+y^2)\,du$$

$$= c\left(\tfrac{1}{2}u^2y+uy^2\right)\Big|_0^2 = c\left(\tfrac{1}{2}(4)y+2y^2\right) = \tfrac{1}{2}y+\tfrac{1}{2}y^2 = \underline{\tfrac{1}{2}(y^2+y)}$$

For $0<x<2,\ 1\le y$:

$$F_X(x,y) = c\int_0^x\int_0^1 (u+2v)\,dv\,du = c\int_0^x (uv+v^2)\Big|_0^1 du = c\int_0^x (u+1)\,du = c\left(\tfrac{1}{2}u^2+u\right)\Big|_0^x$$

$$= c\left(\tfrac{1}{2}x^2+x\right) = \underline{\tfrac{1}{8}x^2+\tfrac{1}{4}x}$$

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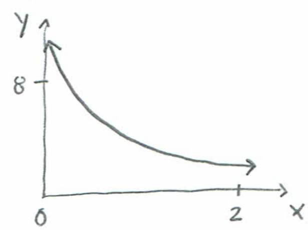
6. c) cont'd

For $2 \le x,\ 1 \le y$: $F_{XY}(x,y) = 1$

Thus, $\boxed{F_{XY}(x,y) = \begin{cases} 0, & x \le 0,\ y \le 0 \\ \frac{1}{8}x^2y + \frac{1}{4}xy^2, & 0 < x < 2,\ 0 < y < 1 \\ \frac{1}{2}(y^2+y), & 2 \le x,\ 0 < y < 1 \\ \frac{1}{8}x^2 + \frac{1}{4}x, & 0 < x < 2,\ 1 \le y \\ 1, & 2 \le x,\ 1 \le y \end{cases}}$

d) Find the pdf of the RV $Z = \dfrac{9}{(x+1)^2}$

Monotone on $0 < x < 2$, so $f_Z(z) = f_X(g^{-1}(z)) \left| \frac{d}{dx} g^{-1}(z) \right|$



g^{-1}: $z = \dfrac{9}{(x+1)^2} \Rightarrow x = \dfrac{9}{(z+1)^2} \Rightarrow \sqrt{x} = \dfrac{3}{(z+1)}$ ($\sqrt{x}$ pos. b/c $0<x<2$)

$\Rightarrow \dfrac{3}{\sqrt{x}} = z+1 \Rightarrow z = \dfrac{3}{\sqrt{x}} - 1 \Rightarrow g^{-1} = 3z^{-1/2} - 1$

$f_Z(z) = \left[\frac{1}{4}(3z^{-1/2} - 1) + \frac{1}{4}\right] \left| -\frac{3}{2} z^{-3/2} \right|$

$= \left(\frac{3}{4}z^{-1/2}\right)\left(\frac{3}{2}z^{-3/2}\right) = \boxed{\frac{9}{8}z^{-2},\quad 1 < z < 9}$

$= \dfrac{9}{8z^2}$

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7. (a)

$$P(X > \sqrt{Y}) = \int_0^1 \int_{\sqrt{y}}^1 (x+y)dxdy$$

$$= \int_0^1 [(\frac{1}{2}x^2 + yx|_{x=\sqrt{y}}^1)]dy = \int_0^1 [\frac{1}{2} + y - (\frac{1}{2}y + y^{3/2})]dy$$

$$= \int_0^1 (\frac{1}{2} + \frac{1}{2}y - y^{3/2})dy$$

$$= (\frac{1}{2}y + \frac{1}{4}y^2 - \frac{2}{5}y^{5/2})|_{y=0}^1$$

$$= \frac{7}{20}$$

(b)

$$P(X^2 < Y < X) = \int_0^1 \int_{x^2}^x 2x\ dx$$

$$= \int_0^1 (2xy|_{y=x^2}^x)dx = \int_0^1 (2x^2 - 2x^3)dx$$

$$= (\frac{2}{3}x^3 - \frac{1}{2}x^4)|_{x=0}^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

8.

$$X \sim U(0,30) \qquad Y \sim U(40,50)$$

Because X and Y are independent, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

$$f_{X,Y}(x,y) = \frac{1}{30} \cdot \frac{1}{10} = \frac{1}{300} \qquad 0 < x < 30 \quad 40 < y < 50$$

$$P(X+Y < 60) = P(X < 60 - Y) = \int_{40}^{50} \int_0^{60-y} \frac{1}{300} dxdy$$

$$= \frac{1}{300} \int_{40}^{50} (60-y)dy = \frac{1}{300}[(60y - \frac{1}{2}y^2)|_{y=40}^{50}]$$

$$= \frac{1}{300}[3000 - 1250 - (2400 - 800)] = \frac{1}{300}(1750 - 1600) = \frac{1}{300}(150)$$

$$P(X+Y < 60) = \frac{1}{2}$$

9.

$$P(a \le X \le b, c \le Y \le d) = P(X \le b, c \le Y \le d) - P(X \le a, c \le Y \le d)$$

$$= P(X \le b, Y \le d) - P(X \le b, Y \le c) - [P(X \le a, Y \le d) - P(X \le a, Y \le c)]$$

$$= P(X \le b, Y \le d) - P(X \le b, Y \le c) - P(X \le a, Y \le d) + P(X \le a, Y \le c)$$

$$= F_{X,Y}(b, d) - F_{X,Y}(b, c) - F_{X,Y}(a, d) + F_{X,Y}(a, c)$$

$$= F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c)$$

$$= F_X(b)[F_Y(d) - F_Y(c)] - F_X(a)[F_Y(d) - F_Y(c)]$$

$$= [F_X(b) - F_X(a)][F_Y(d) - F_Y(c)]$$

$$= [P(X \le b) - P(X \le a)][P(Y \le d) - P(Y \le c)]$$

$$= P(a \le X \le b)P(c \le Y \le d)$$

Discrete Case: X, Y both discrete.

Let a^- be the largest value less than a that X has mass at.

Let c^- be the largest value less than c that Y has mass at

$$P(a \le X \le b, c \le Y \le d) = P(a^- < X \le b, c^- < Y \le d)$$

$$= F_X(b)F_Y(d) - F_X(a^-)F_Y(d) - F_X(b)F_Y(c^-) + F_X(a^-)F_Y(c^-)$$

$$= F_X(b)[F_Y(d) - F_Y(c^-)] - F_X(a^-)[F_Y(d) - F_Y(c^-)]$$

$$= (F_X(b) - F_X(a^-))(F_Y(d) - F_Y(c^-))$$

$$= P(a^- < X \le b)P(c^- < Y \le d)$$

$$= P(a \le X \le b)P(c \le Y \le d)$$

If One of X, Y is continuous and one is discrete, proof would follow similar steps.