1. For $i = 1, 2, \ldots, n$, suppose that the conditional distribution of Y_i given $X = x_i$ is normal with mean $E(Y_i|X = x_i) = \alpha + \beta x_i$ and with variance $V(Y_i|X = x_i) = \sigma^2$, **where σ^2 has a known value**. Assume that (x_i, Y_i), $i = 1, 2, \ldots, n$, constitute a set of mutually independent pairs of data (i.e., given the fixed x_i, the Y_i constitutes a set of mutually independent random variables).

   (a) With β fixed, show that the maximum likelihood estimator (MLE) of α is $\tilde{\alpha} = \bar{Y} - \beta\bar{x}$, where $\bar{Y} = n^{-1}\sum_{i=1}^n Y_i$ and $\bar{x} = n^{-1}\sum_{i=1}^n x_i$, and that $\tilde{\alpha}$ is also a method of moment estimator.

   Solution: With β fixed, the likelihood function is

$$L(\alpha) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(Y_i - \alpha - \beta x_i)^2}{2\sigma^2}\right\}$$
$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (Y_i - \alpha - \beta x_i)^2\right\}.$$

   The log-likelihood function $\ell(\alpha)$ hence is proportional to

$$-\frac{1}{2\sigma^2}\sum_{i=1}^n (Y_i - \alpha - \beta x_i)^2.$$

   To find the maximizer of the function, one take the first derivative and set it to zero,

$$\frac{\partial}{\partial\alpha}\ell(\alpha) = \frac{1}{\sigma^2}\sum_{i=1}^n (Y_i - \alpha - \beta x_i) = 0.$$

   One can get

$$\tilde{\alpha} = \frac{1}{n}\sum_{i=1}^n (Y_i - \beta x_i) = \bar{Y} - \beta\bar{x}.$$

   It is obvious that $\partial^2\ell(\alpha)/\partial\alpha^2 = -n/\sigma^2 < 0$. Therefore, we can claim $\hat{\alpha}$ is the global maximizer.

   By match the first population mean $E(n^{-1}\sum_{i=1}^n Y_i)$ to first sample mean $n^{-1}\sum_{i=1}^n Y_i$, one can easily find $\tilde{\alpha}$ is also a method of moment estimator since $E(n^{-1}\sum_{i=1}^n Y_i) = \alpha + \beta\bar{x}$.

   (b) With α fixed at $\tilde{\alpha} = \bar{Y} - \beta\bar{x}$, show that the maximum likelihood estimator (MLE) of β is

$$\hat{\beta} = \sum_{i=1}^n (x_i - \bar{x})Y_i / \sum_{i=1}^n (x_i - \bar{x})^2.$$

Solution: With α fixed at $\hat{\alpha} = \bar{Y} - \beta\bar{x}$, the likelihood function becomes

$$L(\beta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\{(Y_i - \bar{Y}) - \beta(x_i - \bar{x})\}^2\right],$$

and the log-likelihood function $\ell(\beta)$ is proportional to

$$-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\{(Y_i - \bar{Y}) - \beta(x_i - \bar{x})\}^2.$$

To find the maximizer of the function, one take the first derivative and set it to zero,

$$\frac{\partial}{\partial\beta}\ell(\beta) = \frac{1}{\sigma^2}\sum_{i=1}^{n}\{(Y_i - \bar{Y}) - \beta(x_i - \bar{x})\}(x_i - \bar{x}) = 0.$$

One can get

$$\hat{\beta} = \sum_{i=1}^{n}(Y_i - \bar{Y})(x_i - \bar{x}) / \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i - \bar{x})Y_i / \sum_{i=1}^{n}(x_i - \bar{x})^2.$$

It is obvious that $\partial^2\ell(\beta)/\partial\beta^2 = -\sum_{i=1}^{n}(x_i - \bar{x})^2/\sigma^2 < 0$. Therefore, we can claim $\hat{\beta}$ is the global maximizer.

(c) Show that $\hat{\beta}$ is an unbiased estimator of β, and derive an explicit expression for $V(\hat{\beta})$.

Solution: One can see

$$\begin{aligned} E(\hat{\beta}) &= \sum_{i=1}^{n}(x_i - \bar{x})E(Y_i) / \sum_{i=1}^{n}(x_i - \bar{x})^2 \\ &= \sum_{i=1}^{n}(x_i - \bar{x})(\alpha + \beta x_i) / \sum_{i=1}^{n}(x_i - \bar{x})^2 \\ &= \beta\sum_{i=1}^{n}(x_i - \bar{x})x_i / \sum_{i=1}^{n}(x_i - \bar{x})^2 = \beta, \end{aligned}$$

which shows $\hat{\beta}$ is an unbiased estimator of β. To derive the variance of $\hat{\beta}$, one can have

$$\begin{aligned} V(\hat{\beta}) &= \sum_{i=1}^{n}(x_i - \bar{x})^2 V(Y_i) / \left\{\sum_{i=1}^{n}(x_i - \bar{x})^2\right\}^2 \\ &= \sigma^2 / \sum_{i=1}^{n}(x_i - \bar{x})^2. \end{aligned}$$

(d) Using the fact that $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x}$ and $\hat{\beta}$ simultaneously maximize the likelihood function, find the maximum likelihood estimator (MLE) of $\theta = \alpha + \beta x_0$, where x_0 is a known fixed constant.

Solution: Using the invariance property of MLE, the MLE of θ is

$$\hat{\theta} = \hat{\alpha} + \hat{\beta}x_0 = \bar{Y} + \hat{\beta}(x_0 - \bar{x}).$$

(e) Let $\hat{\theta} = \bar{Y} + \hat{\beta}(x_0 - \bar{x})$. Show that $\hat{\theta}$ is an unbiased estimator of θ, and derive an explicit expression for $V(\hat{\theta})$.

Solution: One have

$$E(\hat{\theta}) = E(\bar{Y}) + E(\hat{\beta})(x_0 - \bar{x}) = \frac{1}{n}\sum_{i=1}^{n}(\alpha + \beta x_i) + \beta(x_0 - \bar{x}) = \alpha + \beta x_0 = \theta,$$

which shows $\hat{\theta}$ is an unbiased estimator of θ. For the variance of $\hat{\theta}$, one have

$$\begin{aligned} V(\hat{\theta}) &= V(\bar{Y}) + V(\hat{\beta})(x_0 - \bar{x})^2 + 2(x_0 - \bar{x})\text{Cov}(\bar{Y}, \hat{\beta}) \\ &= \sigma^2/n + \sigma^2(x_0 - \bar{x})^2 / \sum_{i=1}^{n}(x_i - \bar{x})^2, \end{aligned}$$

where $\text{Cov}(\bar{Y}, \hat{\beta}) = 0$. To show $\text{Cov}(\bar{Y}, \hat{\beta}) = 0$, one have

$$\begin{aligned} \text{Cov}(\bar{Y}, \hat{\beta}) &= \text{Cov}(\bar{Y}, \sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y}) / \sum_{i=1}^{n}(x_i - \bar{x})^2) \\ &= \sum_{i=1}^{n}(x_i - \bar{x})\text{Cov}(\bar{Y}, Y_i - \bar{Y}) / \sum_{i=1}^{n}(x_i - \bar{x})^2 = 0, \end{aligned}$$

since $\text{Cov}(\bar{Y}, Y_i - \bar{Y}) = \text{Cov}(\bar{Y}, Y_i) - \sigma^2/n = \sigma^2/n - \sigma^2/n = 0$.

2. Let $X_1, \ldots, X_n$ be a random sample from the probability density function

$$f_X(x|\theta) = \theta x^{\theta - 1}, \quad 0 < x < 1, \quad \theta > 0.$$

(a) Show that the expectation of score function is 0, i.e., $E(\frac{\partial}{\partial\theta}\log f(X|\theta)) = 0$.

Solution: Since

$$\frac{\partial}{\partial\theta}\log f(X|\theta) = \theta^{-1} + \log(X),$$

one have

$$E\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right) = \theta^{-1} - \theta^{-1} = 0,$$

where $Y = -\log(X)$ has pdf

$$f_Y(y) = f_X(e^{-y})e^{-y} = \theta e^{-y(\theta-1)}e^{-y} = \theta e^{-\theta y},$$

which is exponential distribution with $E(Y) = E(-\log(X)) = \theta^{-1}$.

(b) Derive the explicit expression for the Cramer-Rao lower bound (CRLB) for unbiased estimators of $\tau(\theta) = \theta^{-1}$.

Solution: Since

$$\frac{\partial}{\partial\theta}\log f(x_1|\theta) = \theta^{-1} + \log(x_1)$$

and

$$\frac{\partial^2}{\partial\theta^2}\log f(x_1|\theta) = (-1)\theta^{-2},$$

the CRLB equals θ^{-2}/n.

(c) Find the uniformly minimum variance unbiased estimator (UMVUE) for $\tau(\theta) = \theta^{-1}$, and justify whether its variance reaches the CRLB.

Solution: Re-write the pdf as the exponential family

$$f_X(x|\theta) = x^{-1}\theta\exp\{\theta\log(x)\} = h(x)c(\theta)\exp\{w(\theta)t(x)\},$$

where $h(x) = x^{-1}$, $c(\theta) = \theta$, $w(\theta) = -\theta$, and $t(x) = -\log(x)$. By the property of the exponential family, we know that $T = -\sum_{i=1}^n \log(X_i)$ is a complete sufficient statistic.

Since

$$E(T/n) = E(-\sum_{i=1}^{n}\log(X_i)/n) = \sum_{i=1}^{n} E(-\log(X_i))/n = \theta^{-1},$$

we know T/n is an unbiased estimator of θ^{-1}. Since T is a complete and sufficient statistic, we can conclude that T/n, as a function of the complete and sufficient statistic, is the UMVUE by Lehmann-Sheffe Theorem.

Since the variance of T/n is

$$V(T/n) = \sum_{i=1}^{n} V(-\log(X_i))/n^2 = \theta^{-2}/n,$$

we can see it reaches the CRLB.

(d) Derive the likelihood ratio test (LRT) statistic $\lambda(x)$ for $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, and show that the rejection region $R = \{x : \lambda(x) \leq c\}$ is equivalent to $R = \{x : T \geq c_1^* \text{ or } T \leq c_2^*\}$, where $T = -\sum_{i=1}^{n} \log(X_i)$.

[Hint: $t^n e^{-ct}$ is a concave function of t if the constant $c > 0$.]

Solution: One can write the likelihood function as

$$L(\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1},$$

and show that the MLE of θ is $\hat{\theta} = -n/\sum_{i=1}^{n} \log(X_i) = (T/n)^{-1}$. The likelihood ratio statistic hence can be written as

$$\lambda(x) = \frac{\sup_{\theta\in\Theta_0} L(\theta)}{\sup_{\theta\in\Theta} L(\theta)} = \frac{L(\theta_0)}{L(\hat{\theta})}.$$

It is not easy to see the relationship between $\lambda(x)$ and T.

Actually, it may be easier to work on the distribution of $Y = -\log(X)$ with pdf $f_Y(y) = \theta e^{-\theta y}$ and

$$L(\theta) = \theta^n e^{-\theta \sum_{i=1}^{n} y_i} = \theta^n e^{-\theta t}.$$

The likelihood ratio statistic becomes

$$\lambda(y) = \frac{\sup_{\theta\in\Theta_0} L(\theta)}{\sup_{\theta\in\Theta} L(\theta)} = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_0^n e^{-\theta_0 t}}{(t/n)^{-n} e^{-n}} = c_1 t^n e^{-\theta_0 t},$$

where $\hat{\theta} = (t/n)^{-1}$ and c_1 is some constant. Since $t^n e^{-\theta_0 t}$ is a concave function of t, one know that $R = \{x : \lambda(x) \leq c\}$ is equivalent to $R = \{x : T \geq c_1^* \text{ or } T \leq c_2^*\}$.

(e) Find c_1^* and c_2^* in (d) with test size α.

Solution: Since $T = \sum_{i=1}^{n} Y_i$, where Y_i follows exponential distribution with mean θ^{-1}. One would know T follows Gamma(n, θ^{-1}). Hence, to satisfy the type-I error rate α, one have

$$\alpha = \sup_{\theta \in \Theta_0} P(T \geq c_1^* \ \text{ or } \ T \leq c_2^*) = P(T \geq c_1^* \ \text{ or } \ T \leq c_2^* | \theta = \theta_0).$$

One can divide α into halves $(\alpha/2)$, or one can assign α_1 and α_2 for

$$\alpha_1 = P(T \geq c_1^* | \theta = \theta_0),$$

$$\alpha_2 = P(T \leq c_2^* | \theta = \theta_0),$$

and $\alpha = \alpha_1 + \alpha_2$. Since under the null hypothesis $\theta = \theta_0$, we know T follows Gamma(n, θ_0^{-1}). We can choose $c_1^* = \Gamma_{n, \theta_0^{-1}, 1-\alpha_1}$, as the $(1 - \alpha_1)$th quantile of Gamma(n, θ_0^{-1}), and $c_2^* = \Gamma_{n, \theta_0^{-1}, \alpha_2}$.