Answers to Selected Problems

CHAPTER 1

1. 67,600,000; 19,656,000 **2.** 1296 **4.** 24; 4 **5.** 144; 18 **6.** 2401 **7.** 720; 72; 144; 72 **8.** 120; 1260; 34,650 **9.** 27,720 **10.** 40,320; 10,080; 1152; 2880; 384 **11.** 720; 72; 144 **12.** 24,300,000; 17,100,720 **13.** 190 **14.** 2,598,960 **16.** 42; 94 **17.** 604,800 **18.** 600 **19.** 896; 1000; 910 **20.** 36; 26 **21.** 35 **22.** 18 **23.** 48 **25.** $52!/(13!)^4$ **27.** 27,720 **28.** 65,536; 2520 **29.** 12,600; 945 **30.** 564,480 **31.** 165; 35 **32.** 1287; 14,112 **33.** 220; 572

CHAPTER 2

9. 74 **10.** .4; .1 **11.** 70; 2 **12.** .5; .32; 149/198 **13.** 20,000; 12,000; 11,000; 68,000; 10,000 **14.** 1.057 **15.** .0020; .4226; .0475; .0211; .00024 **17.** 9.10947×10^{-6} **18.** .048 **19.** 5/18 **20.** .9052 **22.** $(n+1)/2^n$ **23.** 5/12 **25.** .4 **26.** .492929 **27.** .0888; .2477; .1243; .2099 **30.** 1/18; 1/6; 1/2 **31.** 2/9; 1/9 **33.** 70/323 **36.** .0045; .0588 **37.** .0833; .5 **38.** 4 **39.** .48 **40.** 1/64; 21/64; 36/64; 6/64 **41.** .5177 **44.** .3; .2; .1 **46.** 5 **48.** 1.0604×10^{-3} **49.** .4329 **50.** 2.6084×10^{-6} **52.** .09145; .4268 **53.** 12/35 **54.** .0511 **55.** .2198; .0343

CHAPTER 3

1. 1/3
2. 1/6; 1/5; 1/4; 1/3; 1/2; 1 **3.** .339 **5.** 6/91 **6.** 1/2 **7.** 2/3
8. 1/2 **9.** 7/11 **10.** .22 **11.** 1/17; 1/33 **12.** .504; .3629 **14.** 35/768; 210/768 **15.** .4848 **16.** .9835 **17.** .0792; .264 **18.** .331; .383; .286; 48.62 **19.** 44.29; 41.18 **20.** .4; 1/26 **21.** .496; 3/14; 9/62 **22.** 5/9; 1/6; 5/54 **23.** 4/9; 1/2 **24.** 1/3; 1/2 **26.** 20/21; 40/41 **28.** 3/128; 29/1536 **29.** .0893 **30.** 7/12; 3/5 **33.** .76, 49/76 **34.** 27/31 **35.** .62, 10/19 **36.** 1/2 **37.** 1/3; 1/5; 1 **38.** 12/37 **39.** 46/185 **40.** 3/13; 5/13; 5/52; 15/52 **41.** 43/459 **42.** 34.48 **43.** 4/9 **45.** 1/11 **48.** 2/3 **50.** 17.5; 38/165; 17/33 **51.** .65; 56/65; 8/65; 1/65; 14/35; 12/35; 9/35 **52.** .11; 16/89; 12/27; 3/5; 9/25 **55.** 9 **57.** (c) 2/3 **60.** 2/3; 1/3; 3/4 **61.** 1/6; 3/20 **65.** 9/13; 1/2 **69.** 9; 9; 18; 110; 4; 4; 8; 120 all over 128 **70.** 1/9; 1/18 **71.** 38/64; 13/64; 13/64 **73.** 1/16; 1/32; 5/16; 1/4; 31/32 **74.** 9/19 **75.** 3/4, 7/12 **78.** $p^2/(1 - 2p + 2p^2)$ **79.** .5550 **81.** .9530 **83.** .5; .6; .8 **84.** 9/19; 6/19; 4/19; 7/15; 53/165; 7/33 **89.** 97/142; 15/26; 33/102

CHAPTER 4

1. $p(4) = 6/91$; $p(2) = 8/91$; $p(1) = 32/91$; $p(0) = 1/91$; $p(-1) = 16/91$; $p(-2) = 28/91$ **4.** (a) 1/2; 5/18; 5/36; 5/84; 5/252; 1/252; 0; 0; 0; 0 **5.** $n - 2i$;

$i = 0, \ldots, n$ **6.** $p(3) = p(-3) = 1/8$; $p(1) = p(-1) = 3/8$ **12.** $p(4) = 1/16$; $p(3) = 1/8$; $p(2) = 1/16$; $p(0) = 1/2$; $p(-i) = p(i)$; $p(0) = 1$ **13.** $p(0) = .28$; $p(500) = .27$, $p(1000) = .315$; $p(1500) = .09$; $p(2000) = .045$ **14.** $p(0) = 1/2$; $p(1) = 1/6$; $p(2) = 1/12$; $p(3) = 1/20$; $p(4) = 1/5$ **17.** 1/4; 1/6; 1/12; 1/2 **19.** 1/2; 1/10; 1/5; 1/10; 1/10 **20.** .5918; no; $-.108$ **21.** 39.28; 37 **24.** $p = 11/18$; maximum $= 23/72$ **25.** .46, 1.3 **26.** 11/2; 17/5 **27.** $A(p + 1/10)$ **28.** 3/5 **31.** p^* **32.** $11 - 10(.9)^{10}$ **33.** 3 **35.** $-.067$; 1.089 **37.** 82.2; 84.5 **39.** 3/8 **40.** 11/243 **42.** $p \geq 1/2$ **45.** 3 **50.** 1/10; 1/10 **51.** $e^{-.2}$; $1 - 1.2e^{-.2}$ **53.** $1 - e^{-.6}$; $1 - e^{-219.18}$ **56.** 253 **57.** .5768; .6070 **59.** .3935; .3033; .0902 **60.** .8886 **61.** .4082 **63.** .0821; .2424 **65.** .3935; .2293; .3935 **66.** $2/(2n - 1)$; $2/(2n - 2)$; e^{-1} **67.** $2/n$; $(2n - 3)/(n - 1)^2$; e^{-2} **68.** $e^{-10e^{-5}}$ **70.** $p + (1 - p)e^{-\lambda t}$ **71.** .1500; .1012 **73.** 5.8125 **74.** 32/243; 4864/6561; 160/729; 160/729 **78.** $18(17)^{n-1}/(35)^n$ **81.** 3/10; 5/6; 75/138 **82.** .3439 **83.** 1.5

CHAPTER 5

2. $3.5e^{-5/2}$ **3.** no; no **4.** 1/2 **5.** $1 - (.01)^{1/5}$ **6.** $4, 0, \infty$ **7.** 3/5; 6/5 **8.** 2 **10.** 2/3; 2/3 **11.** 2/5 **13.** 2/3; 1/3 **15.** .7977; .6827; .3695; .9522; .1587 **16.** $(.9938)^{10}$ **18.** 22.66 **19.** 14.56 **20.** .9994; .75; .977 **22.** 9.5; .0019 **23.** .9258; .1762 **26.** .0606; .0525 **28.** .8363 **29.** .9993 **32.** e^{-1}; $e^{-1/2}$ **34.** e^{-1}; $1/3$ **38.** 3/5 **40.** $1/y$

CHAPTER 6

2. (a) 14/39; 10/39; 10/39; 5/39 (b) 84; 70; 70; 70; 40; 40; 40; 15 all divided by 429 **3.** 15/26; 5/26; 5/26; 1/26 **4.** 25/169; 40/169; 40/169; 64/169 **7.** $p(i, j) = p^2(1 - p)^{i+j}$ **8.** $c = 1/8$; $E[X] = 0$ **9.** $(12x^2 + 6x)/7$; $15/56$; $.8625$; $5/7$; $8/7$ **10.** 1/2; $1 - e^{-a}$ **11.** .1458 **12.** $39.3e^{-5}$ **13.** 1/6; 1/2 **15.** $\pi/4$ **16.** $n(1/2)^{n-1}$ **17.** 1/3 **18.** 7/9 **19.** 1/2 **21.** 2/5; 2/5 **22.** no; 1/3 **23.** 1/2; 2/3; 1/20; 1/18 **25.** $e^{-1}/i!$ **28.** $\frac{1}{2}e^{-t}$; $1-3e^{-2}$ **29.** .0326 **30.** .3772; .2061 **31.** .0829; .3766 **32.** e^{-2}; $1 - 3e^{-2}$ **35.** 5/13; 8/13 **36.** 1/6; 5/6; 1/4; 3/4 **41.** $(y + 1)^2 xe^{-x(y+1)}$; xe^{-xy}; e^{-x} **42.** $1/2 + 3y/(4x) - y^3/(4x^3)$ **46.** $(1 - 2d/L)^3$ **47.** .79297 **48.** $1 - e^{-5\lambda a}$; $(1 - e^{-\lambda a})^5$ **52.** r/π **53.** r **56.** (a) $u/(\nu + 1)^2$

CHAPTER 7

1. 52.5/12 **2.** 324; 199.6 **3.** 1/2; 1/4; 0 **4.** 1/6; 1/4; 1/2 **5.** 3/2 **6.** 35 **7.** .9; 4.9; 4.2 **8.** $(1 - (1 - p)^N)/p$ **10.** .6; 0 **11.** $2(n - 1)p(1 - p)$ **12.** $(3n^2 - n)/(4n - 2)$, $3n^2/(4n - 2)$ **14.** $m/(1 - p)$ **15.** 1/2 **18.** 4 **21.** .9301; 87.5755 **22.** 14.7 **23.** 147/110 **26.** $n/(n + 1)$; $1/(n + 1)$ **29.** $\frac{437}{35}$; 12; 4; $\frac{123}{35}$ **31.** 175/6 **33.** 14 **34.** 20/19; 360/361 **35.** 21.2; 18.929; 49.214 **36.** $-n/36$ **37.** 0 **38.** 1/8 **41.** 6; 112/33 **42.** 100/19; 16,200/6137; 10/19; 3240/6137 **45.** 1/2; 0 **47.** $1/(n - 1)$ **48.** 6; 7; 5.8192 **49.** 6.06 **50.** $2y^2$ **51.** $y^3/4$ **53.** 12 **54.** 8 **56.** $N(1 - e^{-10/N})$ **57.** 12.5 **63.** $-96/145$ **65.** 5.16 **66.** 218 **67.** $x[1 + (2p - 1)^2]^n$ **69.** 1/2; 1/16; 2/81 **70.** 1/2, 1/3 **72.** $1/i$; $[i(i + 1)]^{-1}$; ∞ **73.** μ; $1 + \sigma^2$; yes; σ^2 **79.** .176; .141

CHAPTER 8

1. $\geq 19/20$ **2.** 15/17; $\geq 3/4$; ≥ 10 **3.** ≥ 3 **4.** $\leq 4/3$; .8428 **5.** .1416 **6.** .9431 **7.** .3085 **8.** .6932 **9.** $(327)^2$ **10.** 117 **11.** $\geq .057$ **13.** .0162; .0003; .2514; .2514 **14.** $n \geq 23$ **16.** .013; .018; .691 **18.** $\leq .2$ **23.** .769; .357; .4267; .1093; .112184

CHAPTER 9

1. 1/9; 5/9 **3.** .9735; .9098; .7358; .5578 **10.** (b) 1/6 **14.** 2.585; .5417; 3.1267 **15.** 5.5098

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Solutions to Self-Test Problems and Exercises

CHAPTER 1

1.1. **(a)** There are 4! different orderings of the letters C, D, E, F. For each of these orderings, we can obtain an ordering with A and B next to each other by inserting A and B, either in the order A, B or in the order B, A, in any of 5 places, namely, either before the first letter of the permutation of C, D, E, F, or between the first and second, and so on. Hence, there are $2 \cdot 5 \cdot 4! = 240$ arrangements. Another way of solving this problem is to imagine that B is glued to the back of A. Then there are 5! orderings in which A is immediately before B. Since there are also 5! orderings in which B is immediately before A, we again obtain a total of $2 \cdot 5! = 240$ different arrangements.

(b) There are $6! = 720$ possible arrangements, and since there are as many with A before B as with B before A, there are 360 arrangements.

(c) Of the 720 possible arrangements, there are as many that have A before B before C as have any of the 3! possible orderings of A, B, and C. Hence, there are $720/6 = 120$ possible orderings.

(d) Of the 360 arrangements that have A before B, half will have C before D and half D before C. Hence, there are 180 arrangements having A before B and C before D.

(e) Gluing B to the back of A and D to the back of C yields $4! = 24$ different orderings in which B immediately follows A and D immediately follows C. Since the order of A and B and of C and D can be reversed, there are $4 \cdot 24 = 96$ different arrangements.

(f) There are 5! orderings in which E is last. Hence, there are $6! - 5! = 600$ orderings in which E is not last.

1.2. 3! 4! 3! 3!, since there are 3! possible orderings of countries and then the countrymen must be ordered.

1.3. **(a)** $10 \cdot 9 \cdot 8 = 720$

(b) $8 \cdot 7 \cdot 6 + 2 \cdot 3 \cdot 8 \cdot 7 = 672$. The result of part (b) follows because there are $8 \cdot 7 \cdot 6$ choices not including A or B and there are $3 \cdot 8 \cdot 7$ choices in which a specified one of A and B, but not the other, serves. The latter follows because the serving member of the pair can be assigned to any of the 3 offices, the next position can then be filled by any of the other 8 people, and the final position by any of the remaining 7.

(c) $8 \cdot 7 \cdot 6 + 3 \cdot 2 \cdot 8 = 384$.

(d) $3 \cdot 9 \cdot 8 = 216$.

(e) $9 \cdot 8 \cdot 7 + 9 \cdot 8 = 576$.

1.4. **(a)** $\binom{10}{7}$

(b) $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2}$

1.5. $\binom{7}{3,2,2} = 210$

1.6. There are $\binom{7}{3} = 35$ choices of the three places for the letters. For each choice, there are $(26)^3(10)^4$ different license plates. Hence, altogether there are $35 \cdot (26)^3 \cdot (10)^4$ different plates.

1.7. Any choice of r of the n items is equivalent to a choice of $n - r$, namely, those items not selected.

1.8. **(a)** $10 \cdot 9 \cdot 9 \cdots 9 = 10 \cdot 9^{n-1}$

(b) $\binom{n}{i} 9^{n-i}$, since there are $\binom{n}{i}$ choices of the i places to put the zeroes and then each of the other $n - i$ positions can be any of the digits $1, \ldots, 9$.

1.9. **(a)** $\binom{3n}{3}$

(b) $3\binom{n}{3}$

(c) $\binom{3}{1}\binom{2}{1}\binom{n}{2}\binom{n}{1} = 3n^2(n - 1)$

(d) n^3

(e) $\binom{3n}{3} = 3\binom{n}{3} + 3n^2(n - 1) + n^3$

1.10. There are $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ numbers in which no digit is repeated. There are $\binom{5}{2} \cdot 8 \cdot 7 \cdot 6$ numbers in which only one specified digit appears twice, so there are $9\binom{5}{2} \cdot 8 \cdot 7 \cdot 6$ numbers in which only a single digit appears twice. There are $7 \cdot \frac{5!}{2!2!}$ numbers in which two specified digits appear twice, so there are $\binom{9}{2} 7 \cdot \frac{5!}{2!2!}$ numbers in which two digits appear twice. Thus, the answer is

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 + 9\binom{5}{2} \cdot 8 \cdot 7 \cdot 6 + \binom{9}{2} 7 \cdot \frac{5!}{2!2!}$$

1.11. **(a)** We can regard this as a seven-stage experiment. First choose the 6 married couples that have a representative in the group, and then select one of the members of each of these couples. By the generalized basic principle of counting, there are $\binom{10}{6}2^6$ different choices.

(b) First select the 6 married couples that have a representative in the group, and then select the 3 of those couples that are to contribute a man. Hence, there are $\binom{10}{6}\binom{6}{3} = \frac{10!}{4!3!3!}$ different choices. Another way to solve this is to first select 3 men and then select 3 women not related to the selected men. This shows that there are $\binom{10}{3}\binom{7}{3} = \frac{10!}{3!3!4!}$ different choices.

1.12. $\binom{8}{3}\binom{7}{3} + \binom{8}{4}\binom{7}{2} = 3430$. The first term gives the number of committees that have 3 women and 3 men; the second gives the number that have 4 women and 2 men.

1.13. (number of solutions of $x_1 + \cdots + x_5 = 4$)(number of solutions of $x_1 + \cdots + x_5 = 5$)(number of solutions of $x_1 + \cdots + x_5 = 6$) $= \binom{8}{4}\binom{9}{4}\binom{10}{4}$.

1.14. Since there are $\binom{j-1}{n-1}$ positive vectors whose sum is j, there must be $\sum_{j=n}^{k}\binom{j-1}{n-1}$ such vectors. But $\binom{j-1}{n-1}$ is the number of subsets of size n from the set of numbers $\{1,\ldots,k\}$ in which j is the largest element in the subset. Consequently, $\sum_{j=n}^{k}\binom{j-1}{n-1}$ is just the total number of subsets of size n from a set of size k, showing that the preceding answer is equal to $\binom{k}{n}$.

1.15. Let us first determine the number of different results in which k people pass. Because there are $\binom{n}{k}$ different groups of size k and $k!$ possible orderings of their scores, it follows that there are $\binom{n}{k}k!$ possible results in which k people pass. Consequently, there are $\sum_{k=0}^{n}\binom{n}{k}k!$ possible results.

1.16. The number of subsets of size 4 is $\binom{20}{4} = 4845$. Because the number of these that contain none of the first five elements is $\binom{15}{4} = 1365$, the number that contain at least one is 3480. Another way to solve this problem is to note that there are $\binom{5}{i}\binom{15}{4-i}$ that contain exactly i of the first five elements and sum this for $i = 1, 2, 3, 4$.

1.17. Multiplying both sides by 2, we must show that

$$n(n-1) = k(k-1) + 2k(n-k) + (n-k)(n-k-1)$$

This follows because the right side is equal to

$$k^2(1-2+1) + k(-1+2n-n-n+1) + n(n-1)$$

For a combinatorial argument, consider a group of n items and a subgroup of k of the n items. Then $\binom{k}{2}$ is the number of subsets of size 2 that contain 2 items from the subgroup of size k, $k(n-k)$ is the number that contain 1 item from the subgroup, and $\binom{n-k}{2}$ is the number that contain 0 items from the subgroup. Adding these terms gives the total number of subgroups of size 2, namely, $\binom{n}{2}$.

1.18. There are 3 choices that can be made from families consisting of a single parent and 1 child; there are $3 \cdot 1 \cdot 2 = 6$ choices that can be made from families consisting of a single parent and 2 children; there are $5 \cdot 2 \cdot 1 = 10$ choices that can be made from families consisting of 2 parents and a single child; there are $7 \cdot 2 \cdot 2 = 28$ choices that can be made from families consisting of 2 parents and 2 children; there are $6 \cdot 2 \cdot 3 = 36$ choices that can be made from families consisting of 2 parents and 3 children. Hence, there are 80 possible choices.

1.19. First choose the 3 positions for the digits, and then put in the letters and digits. Thus, there are $\binom{8}{3} \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 10 \cdot 9 \cdot 8$ different plates. If the digits must be consecutive, then there are 6 possible positions for the digits, showing that there are now $6 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 10 \cdot 9 \cdot 8$ different plates.

CHAPTER 2

2.1. **(a)** $2 \cdot 3 \cdot 4 = 24$
(b) $2 \cdot 3 = 6$
(c) $3 \cdot 4 = 12$
(d) $AB = \{(c, \text{pasta}, i), (c, \text{rice}, i), (c, \text{potatoes}, i)\}$
(e) 8
(f) $ABC = \{(c, \text{rice}, i)\}$

2.2. Let A be the event that a suit is purchased, B be the event that a shirt is purchased, and C be the event that a tie is purchased. Then

$$P(A \cup B \cup C) = .22 + .30 + .28 - .11 - .14 - .10 + .06 = .51$$

(a) $1 - .51 = .49$
(b) The probability that two or more items are purchased is

$$P(AB \cup AC \cup BC) = .11 + .14 + .10 - .06 - .06 - .06 + .06 = .23$$

Hence, the probability that exactly 1 item is purchased is $.51 - .23 = .28$.

2.3. By symmetry, the 14th card is equally likely to be any of the 52 cards; thus, the probability is 4/52. A more formal argument is to count the number of the 52! outcomes for which the 14th card is an ace. This yields

$$p = \frac{4 \cdot 51 \cdot 50 \cdots 2 \cdot 1}{(52)!} = \frac{4}{52}$$

Letting A be the event that the first ace occurs on the 14th card, we have

$$P(A) = \frac{48 \cdot 47 \cdots 36 \cdot 4}{52 \cdot 51 \cdots 40 \cdot 39} = .0312$$

2.4. Let D denote the event that the minimum temperature is 70 degrees. Then

$$P(A \cup B) = P(A) + P(B) - P(AB) = .7 - P(AB)$$
$$P(C \cup D) = P(C) + P(D) - P(CD) = .2 + P(D) - P(DC)$$

Since $A \cup B = C \cup D$ and $AB = CD$, subtracting one of the preceding equations from the other yields

$$0 = .5 - P(D)$$

or $P(D) = .5$.

2.5. **(a)** $\dfrac{52 \cdot 48 \cdot 44 \cdot 40}{52 \cdot 51 \cdot 50 \cdot 49} = .6761$

(b) $\dfrac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49} = .1055$

2.6. Let R be the event that both balls are red, and let B be the event that both are black. Then

$$P(R \cup B) = P(R) + P(B) = \frac{3 \cdot 4}{6 \cdot 10} + \frac{3 \cdot 6}{6 \cdot 10} = 1/2$$

2.7. **(a)** $\dfrac{1}{\binom{40}{8}} = 1.3 \times 10^{-8}$

(b) $\dfrac{\binom{8}{7}\binom{32}{1}}{\binom{40}{8}} = 3.3 \times 10^{-6}$

(c) $\dfrac{\binom{8}{6}\binom{32}{2}}{\binom{40}{8}} + 1.3 \times 10^{-8} + 3.3 \times 10^{-6} = 1.8 \times 10^{-4}$

2.8. **(a)** $\dfrac{3 \cdot 4 \cdot 4 \cdot 3}{\binom{14}{4}} = .1439$

(b) $\dfrac{\binom{4}{2}\binom{4}{2}}{\binom{14}{4}} = .0360$

(c) $\dfrac{\binom{8}{4}}{\binom{14}{4}} = .0699$

2.9. Let $S = \bigcup_{i=1}^{n} A_i$, and consider the experiment of randomly choosing an element of S. Then $P(A) = N(A)/N(S)$, and the results follow from Propositions 4.3 and 4.4.

2.10. Since there are $5! = 120$ outcomes in which the position of horse number 1 is specified, it follows that $N(A) = 360$. Similarly, $N(B) = 120$, and $N(AB) = 2 \cdot 4! = 48$. Hence, from Self-Test Problem 9, we obtain $N(A \cup B) = 432$.

2.11. One way to solve this problem is to start with the complementary probability that at least one suit does not appear. Let A_i, $i = 1, 2, 3, 4$, be the event that no cards from suit i appear. Then

$$P\left(\bigcup_{i=1}^{4} A_i\right) = \sum_i P(A_i) - \sum_j \sum_{i:i<j} P(A_i A_j) + \cdots - P(A_1 A_2 A_3 A_4)$$

$$= 4\frac{\binom{39}{5}}{\binom{52}{5}} - \binom{4}{2}\frac{\binom{26}{5}}{\binom{52}{5}} + \binom{4}{3}\frac{\binom{13}{5}}{\binom{52}{5}}$$

$$= 4\frac{\binom{39}{5}}{\binom{52}{5}} - 6\frac{\binom{26}{5}}{\binom{52}{5}} + 4\frac{\binom{13}{5}}{\binom{52}{5}}$$

The desired probability is then 1 minus the preceding. Another way to solve is to let A be the event that all 4 suits are represented, and then use

$$P(A) = P(n,n,n,n,o) + P(n,n,n,o,n) + P(n,n,o,n,n) + P(n,o,n,n,n)$$

where $P(n, n, n, o, n)$, for instance, is the probability that the first card is from a new suit, the second is from a new suit, the third is from a new suit, the fourth is from an old suit (that is, one which has already appeared) and the fifth is from a new suit. This gives

$$\begin{aligned} P(A) &= \frac{52 \cdot 39 \cdot 26 \cdot 13 \cdot 48 + 52 \cdot 39 \cdot 26 \cdot 36 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \\ &\quad + \frac{52 \cdot 39 \cdot 24 \cdot 26 \cdot 13 + 52 \cdot 12 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \\ &= \frac{52 \cdot 39 \cdot 26 \cdot 13(48 + 36 + 24 + 12)}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \\ &= .2637 \end{aligned}$$

2.12. There are $(10)!/2^5$ different divisions of the 10 players into a first roommate pair, a second roommate pair, and so on. Hence, there are $(10)!/(5!2^5)$ divisions into 5 roommate pairs. There are $\binom{6}{2}\binom{4}{2}$ ways of choosing the frontcourt and backcourt players to be in the mixed roommate pairs and then 2 ways of pairing them up. As there is then 1 way to pair up the remaining two backcourt players and $4!/(2!2^2) = 3$ ways of making two roommate pairs from the remaining four frontcourt players, the desired probability is

$$P\{2 \text{ mixed pairs}\} = \frac{\binom{6}{2}\binom{4}{2}(2)(3)}{(10)!/(5!2^5)} = .5714$$

2.13. Let R denote the event that letter R is repeated; similarly, define the events E and V. Then

$$P\{\text{same letter}\} = P(R) + P(E) + P(V) = \frac{2}{7}\frac{1}{8} + \frac{3}{7}\frac{1}{8} + \frac{1}{7}\frac{1}{8} = \frac{3}{28}$$

2.14. Let $B_1 = A_1, B_i = A_i \left(\bigcup_{j=1}^{i-1} A_j \right)^c, i > 1$. Then

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) &= P\left(\bigcup_{i=1}^{\infty} B_i\right) \\ &= \sum_{i=1}^{\infty} P(B_i) \\ &\le \sum_{i=1}^{\infty} P(A_i) \end{aligned}$$

where the final equality uses the fact that the B_i are mutually exclusive. The inequality then follows, since $B_i \subset A_i$.

2.15.

$$\begin{aligned} P\left(\bigcap_{i=1}^{\infty} A_i\right) &= 1 - P\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right) \\ &= 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right) \\ &\ge 1 - \sum_{i=1}^{\infty} P(A_i^c) \\ &= 1 \end{aligned}$$

2.16. The number of partitions for which {1} is a subset is equal to the number of partitions of the remaining $n - 1$ elements into $k - 1$ nonempty subsets, namely, $T_{k-1}(n - 1)$. Because there are $T_k(n - 1)$ partitions of $\{2, \ldots, n - 1\}$ into k nonempty subsets and then a choice of k of them in which to place element 1, it follows that there are $kT_k(n - 1)$ partitions for which {1} is not a subset. Hence, the result follows.

2.17. Let R, W, B denote, respectively, the events that there are no red, no white, and no blue balls chosen. Then

$$\begin{aligned} P(R \cup W \cup B) &= P(R) + P(W) + P(B) - P(RW) - P(RB) \\ &\quad - P(WB) + P(RWB) \\ &= \frac{\binom{13}{5}}{\binom{18}{5}} + \frac{\binom{12}{5}}{\binom{18}{5}} + \frac{\binom{11}{5}}{\binom{18}{5}} - \frac{\binom{7}{5}}{\binom{18}{5}} - \frac{\binom{6}{5}}{\binom{18}{5}} \\ &\quad - \frac{\binom{5}{5}}{\binom{18}{5}} \\ &\approx 0.2933 \end{aligned}$$

Thus, the probability that all colors appear in the chosen subset is approximately $1 - 0.2933 = 0.7067$.

2.18. **(a)** $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13} = \frac{2}{221}$

(b) Because there are 9 nonblue balls, the probability is $\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13} = \frac{9}{442}$.

(c) Because there are 3! possible orderings of the different colors and all possibilities for the final 3 balls are equally likely, the probability is $\frac{3! \cdot 4 \cdot 8 \cdot 5}{17 \cdot 16 \cdot 15} = \frac{4}{17}$.

(d) The probability that the red balls are in a specified 4 spots is $\frac{4 \cdot 3 \cdot 2 \cdot 1}{17 \cdot 16 \cdot 15 \cdot 14}$. Because there are 14 possible locations of the red balls where they are all together, the probability is $\frac{14 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{17 \cdot 16 \cdot 15 \cdot 14} = \frac{1}{170}$.

2.19. **(a)** The probability that the 10 cards consist of 4 spades, 3 hearts, 2 diamonds, and 1 club is $\frac{\binom{13}{4}\binom{13}{3}\binom{13}{2}\binom{13}{1}}{\binom{52}{10}}$. Because there are 4! possible choices of the suits to have 4, 3, 2, and 1 cards, respectively, it follows that the probability is $\frac{24\binom{13}{4}\binom{13}{3}\binom{13}{2}\binom{13}{1}}{\binom{52}{10}}$.

(b) Because there are $\binom{4}{2} = 6$ choices of the two suits that are to have 3 cards and then 2 choices for the suit to have 4 cards, the probability is $\frac{12\binom{13}{3}\binom{13}{3}\binom{13}{4}}{\binom{52}{10}}$.

2.20. All the red balls are removed before all the blue ones if and only if the very last ball removed is blue. Because all 30 balls are equally likely to be the last ball removed, the probability is 10/30.

CHAPTER 3

3.1. **(a)** $P(\text{no aces}) = \binom{35}{13} \Big/ \binom{39}{13}$

(b) $1 - P(\text{no aces}) - \dfrac{4\binom{35}{12}}{\binom{39}{13}}$

(c) $P(i \text{ aces}) = \dfrac{\binom{3}{i}\binom{36}{13 - i}}{\binom{39}{13}}$

3.2. Let L_i denote the event that the life of the battery is greater than $10,000 \times i$ miles.

(a) $P(L_2|L_1) = P(L_1L_2)/P(L_1) = P(L_2)/P(L_1) = 1/2$

(b) $P(L_3|L_1) = P(L_1L_3)/P(L_1) = P(L_3)/P(L_1) = 1/8$

3.3. Put 1 white and 0 black balls in urn one, and the remaining 9 white and 10 black balls in urn two.

3.4. Let T be the event that the transferred ball is white, and let W be the event that a white ball is drawn from urn B. Then

$$P(T|W) = \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|T^c)P(T^c)} = \frac{(2/7)(2/3)}{(2/7)(2/3) + (1/7)(1/3)} = 4/5$$

3.5. **(a)** $\frac{r}{r+w}$, because each of the $r + w$ balls is equally likely to be the ith ball removed.

(b), (c)

$$P(R_j|R_i) = \frac{P(R_iR_j)}{P(R_i)} = \frac{\frac{\binom{r}{2}}{\binom{r+w}{2}}}{\frac{r}{r+w}} = \frac{r-1}{r+w-1}$$

A simpler argument is to note that, for $i \neq j$, given that the ith removal is a red ball, the jth removal is equally likely to be any of the remaining $r + w - 1$ balls, of which $r - 1$ are red.

3.6. Let B_i denote the event that ball i is black, and let $R_i = B_i^c$. Then

$$P(B_1|R_2) = \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)} = \frac{[r/[(b+r+c)][b/(b+r)]}{[r/(b+r+c)][b/(b+r)] + [(r+c)/(b+r+c)][r/(b+r)]} = \frac{b}{b+r+c}$$

3.7. Let B denote the event that both cards are aces.

(a)

$$P\{B|\text{yes to ace of spades}\} = \frac{P\{B, \text{yes to ace of spades}\}}{P\{\text{yes to ace of spades}\}} = \frac{\binom{1}{1}\binom{3}{1}}{\binom{52}{2}} \Big/ \frac{\binom{1}{1}\binom{51}{1}}{\binom{52}{2}} = 3/51$$

(b) Since the second card is equally likely to be any of the remaining 51, of which 3 are aces, we see that the answer in this situation is also 3/51.

(c) Because we can always interchange which card is considered first and which is considered second, the result should be the same as in part (b). A more formal argument is as follows:

$$\begin{aligned}
P\{B|\text{second is ace}\} &= \frac{P\{B, \text{second is ace}\}}{P\{\text{second is ace}\}} \\
&= \frac{P(B)}{P(B) + P\{\text{first is not ace, second is ace}\}} \\
&= \frac{(4/52)(3/51)}{(4/52)(3/51) + (48/52)(4/51)} \\
&= 3/51
\end{aligned}$$

(d)

$$\begin{aligned}
P\{B|\text{at least one}\} &= \frac{P(B)}{P\{\text{at least one}\}} \\
&= \frac{(4/52)(3/51)}{1 - (48/52)(47/51)} \\
&= 1/33
\end{aligned}$$

3.8.

$$\frac{P(H|E)}{P(G|E)} = \frac{P(HE)}{P(GE)} = \frac{P(H)P(E|H)}{P(G)P(E|G)}$$

Hypothesis H is 1.5 times as likely.

3.9. Let A denote the event that the plant is alive and let W be the event that it was watered.

(a)

$$\begin{aligned}
P(A) &= P(A|W)P(W) + P(A|W^c)P(W^c) \\
&= (.85)(.9) + (.2)(.1) = .785
\end{aligned}$$

(b)

$$\begin{aligned}
P(W^c|A^c) &= \frac{P(A^c|W^c)P(W^c)}{P(A^c)} \\
&= \frac{(.8)(.1)}{.215} = \frac{16}{43}
\end{aligned}$$

3.10. (a) $1 - P(\text{no red balls}) = 1 - \dfrac{\binom{22}{6}}{\binom{30}{6}}$

(b) Given that no red balls are chosen, the 6 chosen are equally likely to be any of the 22 nonred balls. Thus,

$$P(2 \text{ green}|\text{no red}) = \frac{\binom{10}{2}\binom{12}{4}}{\binom{22}{6}}$$

3.11. Let W be the event that the battery works, and let C and D denote the events that the battery is a type C and that it is a type D battery, respectively.

(a) $P(W) = P(W|C)P(C) + P(W|D)P(D) = .7(8/14) + .4(6/14) = 4/7$

(b)

$$P(C|W^c) = \frac{P(CW^c)}{P(W^c)} = \frac{P(W^c|C)P(C)}{3/7} = \frac{.3(8/14)}{3/7} = .4$$

3.12. Let L_i be the event that Maria likes book $i, i = 1, 2$. Then

$$P(L_2|L_1^c) = \frac{P(L_1^c L_2)}{P(L_1^c)} = \frac{P(L_1^c L_2)}{.4}$$

Using that L_2 is the union of the mutually exclusive events $L_1 L_2$ and $L_1^c L_2$, we see that

$$.5 = P(L_2) = P(L_1 L_2) + P(L_1^c L_2) = .4 + P(L_1^c L_2)$$

Thus,

$$P\left(L_2|L_1^c\right) = \frac{.1}{.4} = .25$$

3.13. **(a)** This is the probability that the last ball removed is blue. Because each of the 30 balls is equally likely to be the last one removed, the probability is 1/3.

(b) This is the probability that the last red or blue ball to be removed is a blue ball. Because it is equally likely to be any of the 30 red or blue balls, the probability that it is blue is 1/3.

(c) Let B_1, R_2, G_3 denote, respectively, the events that the first color removed is blue, the second is red, and the third is green. Then

$$P(B_1 R_2 G_3) = P(G_3)P(R_2|G_3)P(B_1|R_2 G_3) = \frac{8}{38}\frac{20}{30} = \frac{8}{57}$$

where $P(G_3)$ is just the probability that the very last ball is green and $P(R_2|G_3)$ is computed by noting that, given that the last ball is green, each of the 20 red and 10 blue balls is equally likely to be the last of that group to be removed, so the probability that it is one of the red balls is 20/30. (Of course, $P(B_1|R_2 G_3) = 1$.)

(d) $P(B_1) = P(B_1 G_2 R_3) + P(B_1 R_2 G_3) = \frac{20}{38}\frac{8}{18} + \frac{8}{57} = \frac{64}{171}$

3.14. Let H be the event that the coin lands heads, let T_h be the event that B is told that the coin landed heads, let F be the event that A forgets the result of the toss, and let C be the event that B is told the correct result. Then

(a)

$$\begin{aligned} P(T_h) &= P(T_h|F)P(F) + P(T_h|F^c)P(F^c) \\ &= (.5)(.4) + P(H)(.6) \\ &= .68 \end{aligned}$$

(b)

$$\begin{aligned} P(C) &= P(C|F)P(F) + P(C|F^c)P(F^c) \\ &= (.5)(.4) + 1(.6) = .80 \end{aligned}$$

(c)

$$P(H|T_h) = \frac{P(HT_h)}{P(T_h)}$$

Now,

$$\begin{aligned}P(HT_h) &= P(HT_h|F)P(F) + P(HT_h|F^c)P(F^c)\\ &= P(H|F)P(T_h|HF)P(F) + P(H)P(F^c)\\ &= (.8)(.5)(.4) + (.8)(.6) = .64\end{aligned}$$

giving the result $P(H|T_h) = .64/.68 = 16/17$.

3.15. Since the black rat has a brown sibling, we can conclude that both of its parents have one black and one brown gene.

(a)

$$P(2\text{ black}|\text{at least one}) = \frac{P(2)}{P(\text{at least one})} = \frac{1/4}{3/4} = \frac{1}{3}$$

(b) Let F be the event that all 5 offspring are black, let B_2 be the event that the black rat has 2 black genes, and let B_1 be the event that it has 1 black and 1 brown gene. Then

$$\begin{aligned}P(B_2|F) &= \frac{P(F|B_2)P(B_2)}{P(F|B_2)P(B_2) + P(F|B_1)P(B_1)}\\ &= \frac{(1)(1/3)}{(1)(1/3) + (1/2)^5(2/3)} = \frac{16}{17}\end{aligned}$$

3.16. Let F be the event that a current flows from A to B, and let C_i be the event that relay i closes. Then

$$P(F) = P(F|C_1)p_1 + P(F|C_1^c)(1 - p_1)$$

Now,

$$\begin{aligned}P(F|C_1) &= P(C_4 \cup C_2C_5)\\ &= P(C_4) + P(C_2C_5) - P(C_4C_2C_5)\\ &= p_4 + p_2p_5 - p_4p_2p_5\end{aligned}$$

Also,

$$\begin{aligned}P(F|C_1^c) &= P(C_2C_5 \cup C_2C_3C_4)\\ &= p_2p_5 + p_2p_3p_4 - p_2p_3p_4p_5\end{aligned}$$

Hence, for part (a), we obtain

$$P(F) = p_1(p_4 + p_2p_5 - p_4p_2p_5) + (1 - p_1)p_2(p_5 + p_3p_4 - p_3p_4p_5)$$

For part (b), let $q_i = 1 - p_i$. Then

$$\begin{aligned}P(C_3|F) &= P(F|C_3)P(C_3)/P(F)\\ &= p_3[1 - P(C_1^cC_2^c \cup C_4^cC_5^c)]/P(F)\\ &= p_3(1 - q_1q_2 - q_4q_5 + q_1q_2q_4q_5)/P(F)\end{aligned}$$

3.17. Let A be the event that component 1 is working, and let F be the event that the system functions.

(a)

$$P(A|F) = \frac{P(AF)}{P(F)} = \frac{P(A)}{P(F)} = \frac{1/2}{1 - (1/2)^2} = \frac{2}{3}$$

where $P(F)$ was computed by noting that it is equal to 1 minus the probability that components 1 and 2 are both failed.

(b)

$$P(A|F) = \frac{P(AF)}{P(F)} = \frac{P(F|A)P(A)}{P(F)} = \frac{(3/4)(1/2)}{(1/2)^3 + 3(1/2)^3} = \frac{3}{4}$$

where $P(F)$ was computed by noting that it is equal to the probability that all 3 components work plus the three probabilities relating to exactly 2 of the components working.

3.18. If we assume that the outcomes of the successive spins are independent, then the conditional probability of the next outcome is unchanged by the result that the previous 10 spins landed on black.

3.19. Condition on the outcome of the initial tosses:

$$\begin{aligned} P(A \text{ odd}) = {} & P_1(1 - P_2)(1 - P_3) + (1 - P_1)P_2P_3 + P_1P_2P_3(A \text{ odd}) \\ & + (1 - P_1)(1 - P_2)(1 - P_3)P(A \text{ odd}) \end{aligned}$$

so,

$$P(A \text{ odd}) = \frac{P_1(1 - P_2)(1 - P_3) + (1 - P_1)P_2P_3}{P_1 + P_2 + P_3 - P_1P_2 - P_1P_3 - P_2P_3}$$

3.20. Let A and B be the events that the first trial is larger and that the second is larger, respectively. Also, let E be the event that the results of the trials are equal. Then

$$1 = P(A) + P(B) + P(E)$$

But, by symmetry, $P(A) = P(B)$: thus,

$$P(B) = \frac{1 - P(E)}{2} = \frac{1 - \sum_{i=1}^{n} p_i^2}{2}$$

Another way of solving the problem is to note that

$$\begin{aligned} P(B) &= \sum_i \sum_{j>i} P\{\text{first trial results in } i, \text{ second trial results in } j\} \\ &= \sum_i \sum_{j>i} p_i p_j \end{aligned}$$

To see that the two expressions derived for $P(B)$ are equal, observe that

$$\begin{aligned} 1 &= \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \\ &= \sum_i \sum_j p_i p_j \\ &= \sum_i p_i^2 + \sum_i \sum_{j \neq i} p_i p_j \\ &= \sum_i p_i^2 + 2 \sum_i \sum_{j>i} p_i p_j \end{aligned}$$

3.21. Let $E = \{A \text{ gets more heads than } B\}$; then

$$\begin{aligned} P(E) &= P(E|A \text{ leads after both flip } n)P(A \text{ leads after both flip } n) \\ &\quad + P(E|\text{ even after both flip } n)P(\text{ even after both flip } n) \\ &\quad + P(E|B \text{ leads after both flip } n)P(B \text{ leads after both flip } n) \\ &= P(A \text{ leads}) + \frac{1}{2}P(\text{ even}) \end{aligned}$$

Now, by symmetry,

$$\begin{aligned} P(A \text{ leads}) &= P(B \text{ leads}) \\ &= \frac{1 - P(\text{even})}{2} \end{aligned}$$

Hence,

$$P(E) = \frac{1}{2}$$

3.22. **(a)** Not true: In rolling 2 dice, let $E = \{\text{sum is } 7\}$, $F = \{\text{1st die does not land on } 4\}$, and $G = \{\text{2nd die does not land on } 3\}$. Then

$$P(E|F \cup G) = \frac{P\{7, \text{not } (4,3)\}}{P\{\text{not } (4,3)\}} = \frac{5/36}{35/36} = 5/35 \neq P(E)$$

(b)

$$\begin{aligned} P(E(F \cup G)) &= P(EF \cup EG) \\ &= P(EF) + P(EG) \qquad \text{since } EFG = \emptyset \\ &= P(E)[P(F) + P(G)] \\ &= P(E)P(F \cup G) \qquad \text{since } FG = \emptyset \end{aligned}$$

(c)

$$\begin{aligned} P(G|EF) &= \frac{P(EFG)}{P(EF)} \\ &= \frac{P(E)P(FG)}{P(EF)} \qquad \text{since } E \text{ is independent of } FG \\ &= \frac{P(E)P(F)P(G)}{P(E)P(F)} \qquad \text{by independence} \\ &= P(G). \end{aligned}$$

3.23. **(a)** necessarily false; if they were mutually exclusive, then we would have

$$0 = P(AB) \neq P(A)P(B)$$

(b) necessarily false; if they were independent, then we would have

$$P(AB) = P(A)P(B) > 0$$

(c) necessarily false; if they were mutually exclusive, then we would have

$$P(A \cup B) = P(A) + P(B) = 1.2$$

(d) possibly true

3.24. The probabilities in parts (a), (b), and (c) are .5, $(.8)^3 = .512$, and $(.9)^7 \approx .4783$, respectively.

3.25. Let $D_i, i = 1, 2$, denote the event that radio i is defective. Also, let A and B be the events that the radios were produced at factory A and at factory B, respectively. Then

$$\begin{aligned}
P(D_2|D_1) &= \frac{P(D_1D_2)}{P(D_1)} \\
&= \frac{P(D_1D_2|A)P(A) + P(D_1D_2|B)P(B)}{P(D_1|A)P(A) + P(D_1|B)P(B)} \\
&= \frac{(.05)^2(1/2) + (.01)^2(1/2)}{(.05)(1/2) + (.01)(1/2)} \\
&= 13/300
\end{aligned}$$

3.26. We are given that $P(AB) = P(B)$ and must show that this implies that $P(B^cA^c) = P(A^c)$. One way is as follows:

$$\begin{aligned}
P(B^cA^c) &= P((A \cup B)^c) \\
&= 1 - P(A \cup B) \\
&= 1 - P(A) - P(B) + P(AB) \\
&= 1 - P(A) \\
&= P(A^c)
\end{aligned}$$

3.27. The result is true for $n = 0$. With A_i denoting the event that there are i red balls in the urn after stage n, assume that

$$P(A_i) = \frac{1}{n + 1}, \qquad i = 1, \ldots, n + 1$$

Now let $B_j, j = 1, \ldots, n + 2$, denote the event that there are j red balls in the urn after stage $n + 1$. Then

$$\begin{aligned}
P(B_j) &= \sum_{i=1}^{n+1} P(B_j|A_i)P(A_i) \\
&= \frac{1}{n + 1}\sum_{i=1}^{n+1} P(B_j|A_i) \\
&= \frac{1}{n + 1}[P(B_j|A_{j-1}) + P(B_j|A_j)]
\end{aligned}$$

Because there are $n + 2$ balls in the urn after stage n, it follows that $P(B_j|A_{j-1})$ is the probability that a red ball is chosen when $j - 1$ of the $n + 2$ balls in the urn are red and $P(B_j|A_j)$ is the probability that a red ball is not chosen when j of the $n + 2$ balls in the urn are red. Consequently,

$$P(B_j|A_{j-1}) = \frac{j - 1}{n + 2}, \qquad P(B_j|A_j) = \frac{n + 2 - j}{n + 2}$$

Substituting these results into the equation for $P(B_j)$ gives

$$P(B_j) = \frac{1}{n+1}\left[\frac{j-1}{n+2} + \frac{n+2-j}{n+2}\right] = \frac{1}{n+2}$$

This completes the induction proof.

3.28. If A_i is the event that player i receives an ace, then

$$P(A_i) = 1 - \frac{\binom{2n-2}{n}}{\binom{2n}{n}} = 1 - \frac{1}{2}\frac{n-1}{2n-1} = \frac{3n-1}{4n-2}$$

By arbitrarily numbering the aces and noting that the player who does not receive ace number one will receive n of the remaining $2n - 1$ cards, we see that

$$P(A_1A_2) = \frac{n}{2n-1}$$

Therefore,

$$P(A_2^c|A_1) = 1 - P(A_2|A_1) = 1 - \frac{P(A_1A_2)}{P(A_1)} = \frac{n-1}{3n-1}$$

We may regard the card division outcome as the result of two trials, where trial $i, i = 1, 2$, is said to be a success if ace number i goes to the first player. Because the locations of the two aces become independent as n goes to infinity, with each one being equally likely to be given to either player, it follows that the trials become independent, each being a success with probability 1/2. Hence, in the limiting case where $n \to \infty$, the problem becomes one of determining the conditional probability that two heads result, given that at least one does, when two fair coins are flipped. Because $\frac{n-1}{3n-1}$ converges to 1/3, the answer agrees with that of Example 2b.

3.29. **(a)** For any permutation $i_1, \ldots, i_n$ of $1, 2, \ldots, n$, the probability that the successive types collected is $i_1, \ldots, i_n$ is $p_{i_1} \cdots p_{i_n} = \prod_{i=1}^n p_i$. Consequently, the desired probability is $n! \prod_{i=1}^n p_i$.

(b) For $i_1, \ldots, i_k$ all distinct,

$$P(E_{i_1} \cdots E_{i_k}) = \left(\frac{n-k}{n}\right)^n$$

which follows because there are no coupons of types $i_1, \ldots, i_k$ when each of the n independent selections is one of the other $n - k$ types. It now follows by the inclusion–exclusion identity that

$$P(\cup_{i=1}^n E_i) = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \left(\frac{n-k}{n}\right)^n$$

Because $1 - P(\cup_{i=1}^{n} E_i)$ is the probability that one of each type is obtained, by part (a) it is equal to $\frac{n!}{n^n}$. Substituting this into the preceding equation gives

$$1 - \frac{n!}{n^n} = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \left(\frac{n - k}{n}\right)^n$$

or

$$n! = n^n - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n - k)^n$$

or

$$n! = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n - k)^n$$

3.30.

$$P(E|E \cup F) = P(E|F(E \cup F))P(F|E \cup F) + P(E|F^c(E \cup F))P(F^c|E \cup F)$$

Using

$$F(E \cup F) = F \quad \text{and} \quad F^c(E \cup F) = F^c E$$

gives

$$\begin{aligned} P(E|E \cup F) &= P(E|F)P(F|E \cup F) + P(E|EF^c)P(F^c|E \cup F) \\ &= P(E|F)P(F|E \cup F) + P(F^c|E \cup F) \\ &\geq P(E|F)P(F|E \cup F) + P(E|F)P(F^c|E \cup F) \\ &= P(E|F) \end{aligned}$$

CHAPTER 4

4.1. Since the probabilities sum to 1, we must have $4P\{X = 3\} + .5 = 1$, implying that $P\{X = 0\} = .375, P\{X = 3\} = .125$. Hence, $E[X] = 1(.3) + 2(.2) + 3(.125) = 1.075$.

4.2. The relationship implies that $p_i = c^i p_0, i = 1, 2$, where $p_i = P\{X = i\}$. Because these probabilities sum to 1, it follows that

$$p_0(1 + c + c^2) = 1 \Rightarrow p_0 = \frac{1}{1 + c + c^2}$$

Hence,

$$E[X] = p_1 + 2p_2 = \frac{c + 2c^2}{1 + c + c^2}$$

4.3. Let X be the number of flips. Then the probability mass function of X is

$$p_2 = p^2 + (1 - p)^2, \quad p_3 = 1 - p_2 = 2p(1 - p)$$

Hence,

$$E[X] = 2p_2 + 3p_3 = 2p_2 + 3(1 - p_2) = 3 - p^2 - (1 - p)^2$$

4.4. The probability that a randomly chosen family will have i children is n_i/m. Thus,

$$E[X] = \sum_{i=1}^{r} in_i/m$$

Also, since there are in_i children in families having i children, it follows that the probability that a randomly chosen child is from a family with i children is $in_i/\sum_{i=1}^{r} in_i$. Therefore,

$$E[Y] = \frac{\sum_{i=1}^{r} i^2 n_i}{\sum_{i=1}^{r} in_i}$$

Thus, we must show that

$$\frac{\sum_{i=1}^{r} i^2 n_i}{\sum_{i=1}^{r} in_i} \geq \frac{\sum_{i=1}^{r} in_i}{\sum_{i=1}^{r} n_i}$$

or, equivalently, that

$$\sum_{j=1}^{r} n_j \sum_{i=1}^{r} i^2 n_i \geq \sum_{i=1}^{r} in_i \sum_{j=1}^{r} jn_j$$

or, equivalently, that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} i^2 n_i n_j \geq \sum_{i=1}^{r} \sum_{j=1}^{r} ij n_i n_j$$

But, for a fixed pair i, j, the coefficient of $n_i n_j$ in the left-side summation of the preceding inequality is $i^2 + j^2$, whereas its coefficient in the right-hand summation is $2ij$. Hence, it suffices to show that

$$i^2 + j^2 \geq 2ij$$

which follows because $(i - j)^2 \geq 0$.

4.5. Let $p = P\{X = 1\}$. Then $E[X] = p$ and $\mathrm{Var}(X) = p(1 - p)$, so

$$p = 3p(1 - p)$$

implying that $p = 2/3$. Hence, $P\{X = 0\} = 1/3$.

4.6. If you wager x on a bet that wins the amount wagered with probability p and loses that amount with probability $1 - p$, then your expected winnings are

$$xp - x(1 - p) = (2p - 1)x$$

which is positive (and increasing in x) if and only if $p > 1/2$. Thus, if $p \le 1/2$, one maximizes one's expected return by wagering 0, and if $p > 1/2$, one maximizes one's expected return by wagering the maximal possible bet. Therefore, if the information is that the .6 coin was chosen, then you should bet 10, and if the information is that the .3 coin was chosen, then you should bet 0. Hence, your expected payoff is

$$\frac{1}{2}(1.2 - 1)10 + \frac{1}{2}0 - C = 1 - C$$

Since your expected payoff is 0 without the information (because in this case the probability of winning is $\frac{1}{2}(.6) + \frac{1}{2}(.3) < 1/2$), it follows that if the information costs less than 1, then it pays to purchase it.

4.7. **(a)** If you turn over the red paper and observe the value x, then your expected return if you switch to the blue paper is

$$2x(1/2) + x/2(1/2) = 5x/4 > x$$

Thus, it would always be better to switch.

(b) Suppose the philanthropist writes the amount x on the red paper. Then the amount on the blue paper is either $2x$ or $x/2$. Note that if $x/2 \ge y$, then the amount on the blue paper will be at least y and will thus be accepted. Hence, in this case, the reward is equally likely to be either $2x$ or $x/2$, so

$$E[R_y(x)] = 5x/4, \quad \text{if } x/2 \ge y$$

If $x/2 < y \le 2x$, then the blue paper will be accepted if its value is $2x$ and rejected if it is $x/2$. Therefore,

$$E[R_y(x)] = 2x(1/2) + x(1/2) = 3x/2, \quad \text{if } x/2 < y \le 2x$$

Finally, if $2x < y$, then the blue paper will be rejected. Hence, in this case, the reward is x, so

$$R_y(x) = x, \quad \text{if } 2x < y$$

That is, we have shown that when the amount x is written on the red paper, the expected return under the y-policy is

$$E[R_y(x)] = \begin{cases} x & \text{if } x < y/2 \\ 3x/2 & \text{if } y/2 \le x < 2y \\ 5x/4 & \text{if } x \ge 2y \end{cases}$$

4.8. Suppose that n independent trials, each of which results in a success with probability p, are performed. Then the number of successes will be less than or equal to i if and only if the number of failures is greater than or equal to $n - i$. But since each trial is a failure with probability $1 - p$, it follows that the number of failures is a binomial random variable with parameters n and $1 - p$. Hence,

$$\begin{aligned}P\{\text{Bin}(n,p) \le i\} &= P\{\text{Bin }(n,1 - p) \ge n - i\} \\ &= 1 - P\{\text{Bin }(n,1 - p) \le n - i - 1\}\end{aligned}$$

The final equality follows from the fact that the probability that the number of failures is greater than or equal to $n - i$ is 1 minus the probability that it is less than $n - i$.

4.9. Since $E[X] = np, \text{Var}(X) = np(1 - p)$, we are given that $np = 6, np(1 - p) = 2.4$. Thus, $1 - p = .4$, or $p = .6, n = 10$. Hence,

$$P\{X = 5\} = \binom{10}{5}(.6)^5(.4)^5$$

4.10. Let $X_i, i = 1, \ldots, m$, denote the number on the ith ball drawn. Then

$$\begin{aligned}P\{X \le k\} &= P\{X_1 \le k, X_2 \le k, \ldots, X_m \le k\} \\ &= P\{X_1 \le k\}P\{X_2 \le k\}\cdots P\{X_m \le k\} \\ &= \left(\frac{k}{n}\right)^m\end{aligned}$$

Therefore,

$$P\{X = k\} = P\{X \le k\} - P\{X \le k - 1\} = \left(\frac{k}{n}\right)^m - \left(\frac{k - 1}{n}\right)^m$$

4.11. **(a)** Given that A wins the first game, it will win the series if, from then on, it wins 2 games before team B wins 3 games. Thus,

$$P\{A \text{ wins}|A \text{ wins first}\} = \sum_{i=2}^{4}\binom{4}{i}p^i(1 - p)^{4-i}$$

(b)

$$\begin{aligned}P\{A \text{ wins first}|A \text{ wins}\} &= \frac{P\{A \text{ wins}|A \text{ wins first}\}P\{A \text{ wins first}\}}{P\{A \text{ wins}\}} \\ &= \frac{\sum_{i=2}^{4}\binom{4}{i}p^{i+1}(1 - p)^{4-i}}{\sum_{i=3}^{5}\binom{5}{i}p^i(1 - p)^{5-i}}\end{aligned}$$

4.12. To obtain the solution, condition on whether the team wins this weekend:

$$.5\sum_{i=3}^{4}\binom{4}{i}(.4)^i(.6)^{4-i} + .5\sum_{i=3}^{4}\binom{4}{i}(.7)^i(.3)^{4-i}$$

4.13. Let C be the event that the jury makes the correct decision, and let F be the event that four of the judges agreed. Then

$$P(C) = \sum_{i=4}^{7} \binom{7}{i} (.7)^i (.3)^{7-i}$$

Also,

$$\begin{aligned} P(C|F) &= \frac{P(CF)}{P(F)} \\ &= \frac{\binom{7}{4} (.7)^4 (.3)^3}{\binom{7}{4} (.7)^4 (.3)^3 + \binom{7}{3} (.7)^3 (.3)^4} \\ &= .7 \end{aligned}$$

4.14. Assuming that the number of hurricanes can be approximated by a Poisson random variable, we obtain the solution

$$\sum_{i=0}^{3} e^{-5.2} (5.2)^i / i!$$

4.15.

$$\begin{aligned} E[Y] &= \sum_{i=1}^{\infty} iP\{X = i\}/P\{X > 0\} \\ &= E[X]/P\{X > 0\} \\ &= \frac{\lambda}{1 - e^{-\lambda}} \end{aligned}$$

4.16. (a) $1/n$

(b) Let D be the event that girl i and girl j choose different boys. Then

$$\begin{aligned} P(G_i G_j) &= P(G_i G_j | D) P(D) + P(G_i G_j | D^c) P(D^c) \\ &= (1/n)^2 (1 - 1/n) \\ &= \frac{n - 1}{n^3} \end{aligned}$$

Therefore,

$$P(G_i | G_j) = \frac{n - 1}{n^2}$$

(c), (d) Because, when n is large, $P(G_i|G_j)$ is small and nearly equal to $P(G_i)$, it follows from the Poisson paradigm that the number of couples is approximately Poisson distributed with mean $\sum_{i=1}^{n} P(G_i) = 1$. Hence, $P_0 \approx e^{-1}$ and $P_k \approx e^{-1}/k!$

(e) To determine the probability that a given set of k girls all are coupled, condition on whether or not D occurs, where D is the event that they all choose different boys. This gives

$$\begin{aligned}
P(G_{i_1}\cdots G_{i_k}) &= P(G_{i_1}\cdots G_{i_k}|D)P(D) + P(G_{i_1}\cdots G_{i_k}|D^c)P(D^c)\\
&= P(G_{i_1}\cdots G_{i_k}|D)P(D)\\
&= (1/n)^k\,\frac{n(n-1)\cdots(n-k+1)}{n^k}\\
&= \frac{n!}{(n-k)!n^{2k}}
\end{aligned}$$

Therefore,

$$\sum_{i_1<\ldots<i_k} P(G_{i_1}\cdots G_{i_k}) = \binom{n}{k}P(G_{i_1}\cdots G_{i_k}) = \frac{n!n!}{(n-k)!(n-k)!k!n^{2k}}$$

and the inclusion–exclusion identity yields

$$1 - P_0 = P(\cup_{i=1}^{n} G_i) = \sum_{k=1}^{n}(-1)^{k+1}\frac{n!n!}{(n-k)!(n-k)!k!n^{2k}}$$

4.17. **(a)** Because woman i is equally likely to be paired with any of the remaining $2n-1$ people, $P(W_i) = \frac{1}{2n-1}$

(b) Because, conditional on W_j, woman i is equally likely to be paired with any of $2n-3$ people, $P(W_i|W_j) = \frac{1}{2n-3}$

(c) When n is large, the number of wives paired with their husbands will approximately be Poisson with mean $\sum_{i=1}^{n} P(W_i) = \frac{n}{2n-1} \approx 1/2$. Therefore, the probability that there is no such pairing is approximately $e^{-1/2}$.

(d) It reduces to the match problem.

4.18. **(a)** $\binom{8}{3}(9/19)^3(10/19)^5(9/19) = \binom{8}{3}(9/19)^4(10/19)^5$

(b) If W is her final winnings and X is the number of bets she makes, then, since she would have won 4 bets and lost $X-4$ bets, it follows that

$$W = 20 - 5(X-4) = 40 - 5X$$

Hence,

$$E[W] = 40 - 5E[X] = 40 - 5[4/(9/19)] = -20/9$$

4.19. The probability that a round does not result in an "odd person" is equal to 1/4, the probability that all three coins land on the same side.

(a) $(1/4)^2(3/4) = 3/64$

(b) $(1/4)^4 = 1/256$

4.20. Let $q = 1 - p$. Then

$$
\begin{aligned}
E[1/X] &= \sum_{i=1}^{\infty} \frac{1}{i} q^{i-1} p \\
&= \frac{p}{q} \sum_{i=1}^{\infty} q^i / i \\
&= \frac{p}{q} \sum_{i=1}^{\infty} \int_0^q x^{i-1}\, dx \\
&= \frac{p}{q} \int_0^q \sum_{i=1}^{\infty} x^{i-1}\, dx \\
&= \frac{p}{q} \int_0^q \frac{1}{1 - x}\, dx \\
&= \frac{p}{q} \int_p^1 \frac{1}{y}\, dy \\
&= -\frac{p}{q} \log(p)
\end{aligned}
$$

4.21. Since $\frac{X-b}{a-b}$ will equal 1 with probability p or 0 with probability $1 - p$, it follows that it is a Bernoulli random variable with parameter p. Because the variance of such a Bernoulli random variable is $p(1 - p)$, we have

$$
p(1 - p) = \mathrm{Var}\left(\frac{X - b}{a - b}\right) = \frac{1}{(a - b)^2}\mathrm{Var}(X - b) = \frac{1}{(a - b)^2}\mathrm{Var}(X)
$$

Hence,

$$
\mathrm{Var}(X) = (a - b)^2 p(1 - p)
$$

4.22. Let X denote the number of games that you play and Y the number of games that you lose.

(a) After your fourth game, you will continue to play until you lose. Therefore, $X - 4$ is a geometric random variable with parameter $1 - p$, so

$$
E[X] = E[4 + (X - 4)] = 4 + E[X - 4] = 4 + \frac{1}{1 - p}
$$

(b) If we let Z denote the number of losses you have in the first 4 games, then Z is a binomial random variable with parameters 4 and $1 - p$. Because $Y = Z + 1$, we have

$$
E[Y] = E[Z + 1] = E[Z] + 1 = 4(1 - p) + 1
$$

4.23. A total of n white balls will be withdrawn before a total of m black balls if and only if there are at least n white balls in the first $n + m - 1$ withdrawals. (Compare with *the problem of the points*, Example 4j of Chapter 3.) With X equal to the number of white balls among the first $n + m - 1$ balls withdrawn, X is a hypergeometric random variable, and it follows that

$$P\{X \geq n\} = \sum_{i=n}^{n+m-1} P\{X = i\} = \sum_{i=n}^{n+m-1} \frac{\binom{N}{i}\binom{M}{n+m-1-i}}{\binom{N+M}{n+m-1}}$$

4.24. Because each ball independently goes into urn i with the same probability p_i, it follows that X_i is a binomial random variable with parameters $n = 10$, $p = p_i$.

First note that $X_i + X_j$ is the number of balls that go into either urn i or urn j. Then, because each of the 10 balls independently goes into one of these urns with probability $p_i + p_j$, it follows that $X_i + X_j$ is a binomial random variable with parameters 10 and $p_i + p_j$.

By the same logic, $X_1 + X_2 + X_3$ is a binomial random variable with parameters 10 and $p_1 + p_2 + p_3$. Therefore,

$$P\{X_1 + X_2 + X_3 = 7\} = \binom{10}{7}(p_1 + p_2 + p_3)^7(p_4 + p_5)^3$$

4.25. Let X_i equal 1 if person i has a match, and let it equal 0 otherwise. Then

$$X = \sum_{i=1}^{n} X_i$$

is the number of matches. Taking expectations gives

$$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{X_i = 1\} = \sum_{i=1}^{n} 1/n = 1$$

where the final equality follows because person i is equally likely to end up with any of the n hats.

To compute Var(X), we use Equation (9.1), which states that

$$E[X^2] = \sum_{i=1}^{n} E[X_i] + \sum_{i=1}^{n} \sum_{j \neq i} E[X_i X_j]$$

Now, for $i \neq j$,

$$E[X_i X_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1\}P\{X_j = 1|X_i = 1\} = \frac{1}{n}\frac{1}{n-1}$$

Hence,

$$\begin{aligned} E[X^2] &= 1 + \sum_{i=1}^{n} \sum_{j \neq i} \frac{1}{n(n-1)} \\ &= 1 + n(n-1)\frac{1}{n(n-1)} = 2 \end{aligned}$$

which yields

$$\text{Var}(X) = 2 - 1^2 = 1$$

4.26. With $q = 1 - p$, we have, on the one hand,

$$\begin{aligned} P(E) &= \sum_{i=1}^{\infty} P\{X = 2i\} \\ &= \sum_{i=1}^{\infty} pq^{2i-1} \\ &= pq\sum_{i=1}^{\infty} (q^2)^{i-1} \\ &= pq\,\frac{1}{1 - q^2} \\ &= \frac{pq}{(1 - q)(1 + q)} = \frac{q}{1 + q} \end{aligned}$$

On the other hand,

$$P(E) = P(E|X = 1)p + P(E|X > 1)q = qP(E|X > 1)$$

However, given that the first trial is not a success, the number of trials needed for a success is 1 plus the geometrically distributed number of additional trials required. Therefore,

$$P(E|X > 1) = P(X + 1 \text{ is even}) = P(E^c) = 1 - P(E)$$

which yields $P(E) = q/(1 + q)$.

CHAPTER 5

5.1. Let X be the number of minutes played.
(a) $P\{X > 15\} = 1 - P\{X \le 15\} = 1 - 5(.025) = .875$
(b) $P\{20 < X < 35\} = 10(.05) + 5(.025) = .625$
(c) $P\{X < 30\} = 10(.025) + 10(.05) = .75$
(d) $P\{X > 36\} = 4(.025) = .1$

5.2. **(a)** $1 = \int_0^1 cx^n dx = c/(n + 1) \Rightarrow c = n + 1$
(b) $P\{X > x\} = (n + 1)\int_x^1 x^n dx = x^{n+1}\Big|_x^1 = 1 - x^{n+1}$

5.3. First, let us find c by using

$$1 = \int_0^2 cx^4 dx = 32c/5 \Rightarrow c = 5/32$$

(a) $E[X] = \frac{5}{32}\int_0^2 x^5 dx = \frac{5}{32}\frac{64}{6} = 5/3$
(b) $E[X^2] = \frac{5}{32}\int_0^2 x^6 dx = \frac{5}{32}\frac{128}{7} = 20/7 \Rightarrow \text{Var}(X) = 20/7 - (5/3)^2 = 5/63$

5.4. Since

$$1 = \int_0^1 (ax + bx^2)dx = a/2 + b/3$$

$$.6 = \int_0^1 (ax^2 + bx^3)dx = a/3 + b/4$$

we obtain $a = 3.6, b = -2.4$. Hence,

(a) $P\{X < 1/2\} = \int_0^{1/2}(3.6x - 2.4x^2)dx = (1.8x^2 - .8x^3)\Big|_0^{1/2} = .35$

(b) $E[X^2] = \int_0^1(3.6x^3 - 2.4x^4)dx = .42 \Rightarrow \text{Var}(X) = .06$

5.5. For $i = 1, \ldots, n$,

$$\begin{aligned} P\{X = i\} &= P\{\text{Int}(nU) = i - 1\} \\ &= P\{i - 1 \le nU < i\} \\ &= P\left\{\frac{i-1}{n} \le U < \frac{i}{n}\right\} \\ &= 1/n \end{aligned}$$

5.6. If you bid $x, 70 \le x \le 140$, then you will either win the bid and make a profit of $x - 100$ with probability $(140 - x)/70$ or lose the bid and make a profit of 0 otherwise. Therefore, your expected profit if you bid x is

$$\frac{1}{70}(x - 100)(140 - x) = \frac{1}{70}(240x - x^2 - 14000)$$

Differentiating and setting the preceding equal to 0 gives

$$240 - 2x = 0$$

Therefore, you should bid 120 thousand dollars. Your expected profit will be 40/7 thousand dollars.

5.7. **(a)** $P\{U > .1\} = 9/10$

(b) $P\{U > .2|U > .1\} = P\{U > .2\}/P\{U > .1\} = 8/9$

(c) $P\{U > .3|U > .2, U > .1\} = P\{U > .3\}/P\{U > .2\} = 7/8$

(d) $P\{U > .3\} = 7/10$

The answer to part (d) could also have been obtained by multiplying the probabilities in parts (a), (b), and (c).

5.8. Let X be the test score, and let $Z = (X - 100)/15$. Note that Z is a standard normal random variable.

(a) $P\{X > 125\} = P\{Z > 25/15\} \approx .0478$

(b)

$$\begin{aligned} P\{90 < X < 110\} &= P\{-10/15 < Z < 10/15\} \\ &= P\{Z < 2/3\} - P\{Z < -2/3\} \\ &= P\{Z < 2/3\} - [1 - P\{Z < 2/3\}] \\ &\approx .4950 \end{aligned}$$

5.9. Let X be the travel time. We want to find x such that

$$P\{X > x\} = .05$$

which is equivalent to

$$P\left\{\frac{X - 40}{7} > \frac{x - 40}{7}\right\} = .05$$

That is, we need to find x such that

$$P\left\{Z > \frac{x - 40}{7}\right\} = .05$$

where Z is a standard normal random variable. But

$$P\{Z > 1.645\} = .05$$

Thus,

$$\frac{x - 40}{7} = 1.645 \quad \text{or} \quad x = 51.515$$

Therefore, you should leave no later than 8.485 minutes after 12 P.M.

5.10. Let X be the tire life in units of one thousand, and let $Z = (X - 34)/4$. Note that Z is a standard normal random variable.

(a) $P\{X > 40\} = P\{Z > 1.5\} \approx .0668$

(b) $P\{30 < X < 35\} = P\{-1 < Z < .25\} = P\{Z < .25\} - P\{Z > 1\} \approx .44$

(c)

$$\begin{aligned} P\{X > 40|X > 30\} &= P\{X > 40\}/P\{X > 30\} \\ &= P\{Z > 1.5\}/P\{Z > -1\} \approx .079 \end{aligned}$$

5.11. Let X be next year's rainfall and let $Z = (X - 40.2)/8.4$.

(a) $P\{X > 44\} = P\{Z > 3.8/8.4\} \approx P\{Z > .4524\} \approx .3255$

(b) $\binom{7}{3}(.3255)^3(.6745)^4$

5.12. Let M_i and W_i denote, respectively, the numbers of men and women in the samples that earn, in units of one thousand dollars, at least i per year. Also, let Z be a standard normal random variable.

(a)

$$\begin{aligned} P\{W_{25} \geq 70\} &= P\{W_{25} \geq 69.5\} \\ &= P\left\{\frac{W_{25} - 200(.34)}{\sqrt{200(.34)(.66)}} \geq \frac{69.5 - 200(.34)}{\sqrt{200(.34)(.66)}}\right\} \\ &\approx P\{Z \geq .2239\} \\ &\approx .4114 \end{aligned}$$

(b)

$$\begin{aligned} P\{M_{25} \leq 120\} &= P\{M_{25} \leq 120.5\} \\ &= P\left\{\frac{M_{25} - (200)(.587)}{\sqrt{(200)(.587)(.413)}} \leq \frac{120.5 - (200)(.587)}{\sqrt{(200)(.587)(.413)}}\right\} \\ &\approx P\{Z \leq .4452\} \\ &\approx .6719 \end{aligned}$$

(c)

$$
\begin{aligned}
P\{M_{20} \geq 150\} &= P\{M_{20} \geq 149.5\} \\
&= P\left\{\frac{M_{20} - (200)(.745)}{\sqrt{(200)(.745)(.255)}} \geq \frac{149.5 - (200)(.745)}{\sqrt{(200)(.745)(.255)}}\right\} \\
&\approx P\{Z \geq .0811\} \\
&\approx .4677 \\
P\{W_{20} \geq 100\} &= P\{W_{20} \geq 99.5\} \\
&= P\left\{\frac{W_{20} - (200)(.534)}{\sqrt{(200)(.534)(.466)}} \geq \frac{99.5 - (200)(.534)}{\sqrt{(200)(.534)(.466)}}\right\} \\
&\approx P\{Z \geq -1.0348\} \\
&= \approx .8496
\end{aligned}
$$

Hence,
$$P\{M_{20} \geq 150\}P\{W_{20} \geq 100\} \approx .3974$$

5.13. The lack of memory property of the exponential gives the result $e^{-4/5}$.

5.14. **(a)** $e^{-2^2} = e^{-4}$

(b) $F(3) - F(1) = e^{-1} - e^{-9}$

(c) $\lambda(t) = 2te^{-t^2}/e^{-t^2} = 2t$

(d) Let Z be a standard normal random variable. Use the identity $E[X] = \int_0^\infty P\{X > x\}\,dx$ to obtain

$$
\begin{aligned}
E[X] &= \int_0^\infty e^{-x^2}\,dx \\
&= 2^{-1/2}\int_0^\infty e^{-y^2/2}\,dy \\
&= 2^{-1/2}\sqrt{2\pi}P\{Z > 0\} \\
&= \sqrt{\pi}/2
\end{aligned}
$$

(e) Use the result of Theoretical Exercise 5 to obtain

$$E[X^2] = \int_0^\infty 2xe^{-x^2}\,dx = -e^{-x^2}\Big|_0^\infty = 1$$

Hence, $\text{Var}(X) = 1 - \pi/4$.

5.15. **(a)** $P\{X > 6\} = \exp\{-\int_0^6 \lambda(t)dt\} = e^{-3.45}$

(b)

$$
\begin{aligned}
P\{X < 8|X > 6\} &= 1 - P\{X > 8|X > 6\} \\
&= 1 - P\{X > 8\}/P\{X > 6\} \\
&= 1 - e^{-5.65}/e^{-3.45} \\
&\approx .8892
\end{aligned}
$$

5.16. For $x \geq 0$,

$$
\begin{aligned}
F_{1/X}(x) &= P\{1/X \leq x\} \\
&= P\{X \leq 0\} + P\{X \geq 1/x\} \\
&= 1/2 + 1 - F_X(1/x)
\end{aligned}
$$

Differentiation yields

$$\begin{aligned} f_{1/X}(x) &= x^{-2} f_X(1/x) \\ &= \frac{1}{x^2\pi(1 + (1/x)^2)} \\ &= f_X(x) \end{aligned}$$

The proof when $x < 0$ is similar.

5.17. If X denotes the number of the first n bets that you win, then the amount that you will be winning after n bets is

$$35X - (n - X) = 36X - n$$

Thus, we want to determine

$$p = P\{36X - n > 0\} = P\{X > n/36\}$$

when X is a binomial random variable with parameters n and $p = 1/38$.

(a) When $n = 34$,

$$\begin{aligned} p &= P\{X \geq 1\} \\ &= P\{X > .5\} \qquad \text{(the continuity correction)} \\ &= P\left\{\frac{X - 34/38}{\sqrt{34(1/38)(37/38)}} > \frac{.5 - 34/38}{\sqrt{34(1/38)(37/38)}}\right\} \\ &= P\left\{\frac{X - 34/38}{\sqrt{34(1/38)(37/38)}} > -.4229\right\} \\ &\approx \Phi(.4229) \\ &\approx .6638 \end{aligned}$$

(Because you will be ahead after 34 bets if you win at least 1 bet, the exact probability in this case is $1 - (37/38)^{34} = .5961$.)

(b) When $n = 1000$,

$$\begin{aligned} p &= P\{X > 27.5\} \\ &= P\left\{\frac{X - 1000/38}{\sqrt{1000(1/38)(37/38)}} > \frac{27.5 - 1000/38}{\sqrt{1000(1/38)(37/38)}}\right\} \\ &\approx 1 - \Phi(.2339) \\ &\approx .4075 \end{aligned}$$

The exact probability—namely, the probability that a binomial $n = 1000$, $p = 1/38$ random variable is greater than 27—is .3961.

(c) When $n = 100,000$,

$$\begin{aligned} p &= P\{X > 2777.5\} \\ &= P\left\{\frac{X - 100000/38}{\sqrt{100000(1/38)(37/38)}} > \frac{2777.5 - 100000/38}{\sqrt{100000(1/38)(37/38)}}\right\} \\ &\approx 1 - \Phi(2.883) \\ &\approx .0020 \end{aligned}$$

The exact probability in this case is .0021.

5.18. If X denotes the lifetime of the battery, then the desired probability, $P\{X > s + t|X > t\}$, can be determined as follows:

$$\begin{aligned} P\{X > s + t|X > t\} &= \frac{P\{X > s + t, X > t\}}{P\{X > t\}} \\ &= \frac{P\{X > s + t\}}{P\{X > t\}} \\ &= \frac{P\{X>s+t|\text{battery is type 1}\}p_1 + P\{X>s+t|\text{battery is type 2}\}p_2}{P\{X>t|\text{battery is type 1}\}p_1 + P\{X>t|\text{battery is type 2}\}p_2} \\ &= \frac{e^{-\lambda_1(s+t)}p_1 + e^{-\lambda_2(s+t)}p_2}{e^{-\lambda_1 t}p_1 + e^{-\lambda_2 t}p_2} \end{aligned}$$

Another approach is to directly condition on the type of battery and then use the lack-of-memory property of exponential random variables. That is, we could do the following:

$$\begin{aligned} P\{X > s + t|X > t\} &= P\{X > s + t|X > t, \text{type } 1\}P\{\text{type } 1|X > t\} \\ &\quad + P\{X > s + t|X > t, \text{type } 2\}P\{\text{type } 2|X > t\} \\ &= e^{-\lambda_1 s}P\{\text{type } 1|X > t\} + e^{-\lambda_2 s}P\{\text{type } 2|X > t\} \end{aligned}$$

Now for $i = 1, 2$, use

$$\begin{aligned} P\{\text{type } i|X > t\} &= \frac{P\{\text{type } i, X > t\}}{P\{X > t\}} \\ &= \frac{P\{X > t|\text{type } i\}p_i}{P\{X > t|\text{type } 1\}p_1 + P\{X > t|\text{type } 2\}p_2} \\ &= \frac{e^{-\lambda_i t}p_i}{e^{-\lambda_1 t}p_1 + e^{-\lambda_2 t}p_2} \end{aligned}$$

5.19. Let X_i be an exponential random variable with mean i, $i = 1, 2$.

(a) The value c should be such that $P\{X_1 > c\} = .05$. Therefore,

$$e^{-c} = .05 = 1/20$$

or $c = \log(20) = 2.996$.

(b)

$$P\{X_2 > c\} = e^{-c/2} = \frac{1}{\sqrt{20}} = .2236$$

5.20. (a)

$$\begin{aligned}
E[(Z - c)^+] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - c)^+ e^{-x^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{c}^{\infty} (x - c) e^{-x^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{c}^{\infty} x e^{-x^2/2}\, dx - \frac{1}{\sqrt{2\pi}} \int_{c}^{\infty} c\, e^{-x^2/2}\, dx \\
&= -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Big|_c^{\infty} - c(1 - \Phi(c)) \\
&= \frac{1}{\sqrt{2\pi}} e^{-c^2/2} - c(1 - \Phi(c))
\end{aligned}$$

(b) Using the fact that X has the same distribution as $\mu + \sigma Z$, where Z is a standard normal random variable, yields

$$\begin{aligned}
E[(X - c)^+] &= E[(\mu + \sigma Z - c)^+] \\
&= E\left[\left(\sigma\left(Z - \frac{c - \mu}{\sigma}\right)\right)^+\right] \\
&= E\left[\sigma (Z - \frac{c - \mu}{\sigma})^+\right] \\
&= \sigma\, E\left[\left(Z - \frac{c - \mu}{\sigma}\right)^+\right] \\
&= \sigma\left[\frac{1}{\sqrt{2\pi}} e^{-a^2/2} - a(1 - \Phi(a))\right]
\end{aligned}$$

where $a = \frac{c-\mu}{\sigma}$.

CHAPTER 6

6.1. (a) $3C + 6C = 1 \Rightarrow C = 1/9$

(b) Let $p(i,j) = P\{X = i, Y = j\}$. Then

$$p(1,1) = 4/9, p(1,0) = 2/9, P(0,1) = 1/9, p(0,0) = 2/9$$

(c) $\dfrac{(12)!}{2^6}(1/9)^6(2/9)^6$

(d) $\dfrac{(12)!}{(4!)^3}(1/3)^{12}$

(e) $\sum_{i=8}^{12} \binom{12}{i} (2/3)^i (1/3)^{12-i}$

6.2. (a) With $p_j = P\{XYZ = j\}$, we have

$$p_6 = p_2 = p_4 = p_{12} = 1/4$$

Hence,

$$E[XYZ] = (6 + 2 + 4 + 12)/4 = 6$$

(b) With $q_j = P\{XY + XZ + YZ = j\}$, we have

$$q_{11} = q_5 = q_8 = q_{16} = 1/4$$

Hence,

$$E[XY + XZ + YZ] = (11 + 5 + 8 + 16)/4 = 10$$

6.3. In this solution, we will make use of the identity

$$\int_0^\infty e^{-x}x^n\,dx = n!$$

which follows because $e^{-x}x^n/n!, x > 0$, is the density function of a gamma random variable with parameters $n + 1$ and λ and must thus integrate to 1.

(a)

$$\begin{aligned} 1 &= C\int_0^\infty e^{-y}\int_{-y}^{y}(y - x)\,dx\,dy \\ &= C\int_0^\infty e^{-y}2y^2\,dy = 4C \end{aligned}$$

Hence, $C = 1/4$.

(b) Since the joint density is nonzero only when $y > x$ and $y > -x$, we have, for $x > 0$,

$$\begin{aligned} f_X(x) &= \frac{1}{4}\int_x^\infty (y - x)e^{-y}\,dy \\ &= \frac{1}{4}\int_0^\infty ue^{-(x+u)}\,du \\ &= \frac{1}{4}e^{-x} \end{aligned}$$

For $x < 0$,

$$\begin{aligned} f_X(x) &= \frac{1}{4}\int_{-x}^\infty (y - x)e^{-y}\,dy \\ &= \frac{1}{4}[-ye^{-y} - e^{-y} + xe^{-y}]_{-x}^{\infty} \\ &= (-2xe^x + e^x)/4 \end{aligned}$$

(c) $f_Y(y) = \frac{1}{4}e^{-y}\int_{-y}^{y}(y - x)\,dx = \frac{1}{2}y^2e^{-y}$

(d)

$$\begin{aligned} E[X] &= \frac{1}{4}\left[\int_0^\infty xe^{-x}dx + \int_{-\infty}^{0}(-2x^2e^x + xe^x)\,dx\right] \\ &= \frac{1}{4}\left[1 - \int_0^\infty (2y^2e^{-y} + ye^{-y})\,dy\right] \\ &= \frac{1}{4}[1 - 4 - 1] = -1 \end{aligned}$$

(e) $E[Y] = \frac{1}{2}\int_0^\infty y^3e^{-y}\,dy = 3$

6.4. The multinomial random variables $X_i, i = 1, \ldots, r$, represent the numbers of each of the types of outcomes $1, \ldots, r$ that occur in n independent trials when each trial results in one of the outcomes $1, \ldots, r$ with respective probabilities $p_1, \ldots, p_r$. Now, say that a trial results in a category 1 outcome if that trial resulted in any of the outcome types $1, \ldots, r_1$; say that a trial results in a category 2 outcome if that trial resulted in any of the outcome types $r_1 + 1, \ldots, r_1 + r_2$; and so on. With these definitions, $Y_1, \ldots, Y_k$ represent the numbers of category 1 outcomes, category 2 outcomes, up to category k outcomes when n independent trials that each result in one of the categories $1, \ldots, k$ with respective probabilities $\sum_{j=r_{i-1}+1}^{r_{i-1}+r_i} p_j$, $i = 1, \ldots, k$, are performed. But by definition, such a vector has a multinomial distribution.

6.5. **(a)** Letting $p_j = P\{XYZ = j\}$, we have

$$p_1 = 1/8, \quad p_2 = 3/8, \quad p_4 = 3/8, \quad p_8 = 1/8$$

(b) Letting $p_j = P\{XY + XZ + YZ = j\}$, we have

$$p_3 = 1/8, \quad p_5 = 3/8, \quad p_8 = 3/8, \quad p_{12} = 1/8$$

(c) Letting $p_j = P\{X^2 + YZ = j\}$, we have

$$p_2 = 1/8, \quad p_3 = 1/4, \quad p_5 = 1/4, \quad p_6 = 1/4, \quad p_8 = 1/8$$

6.6. **(a)**

$$\begin{aligned}
1 &= \int_0^1 \int_1^5 (x/5 + cy)\, dy\, dx \\
&= \int_0^1 (4x/5 + 12c)\, dx \\
&= 12c + 2/5
\end{aligned}$$

Hence, $c = 1/20$.

(b) No, the density does not factor.

(c)

$$\begin{aligned}
P\{X + Y > 3\} &= \int_0^1 \int_{3-x}^5 (x/5 + y/20)\, dy\, dx \\
&= \int_0^1 [(2 + x)x/5 + 25/40 - (3 - x)^2/40]\, dx \\
&= 1/5 + 1/15 + 5/8 - 19/120 = 11/15
\end{aligned}$$

6.7. **(a)** Yes, the joint density function factors.

(b) $f_X(x) = x \int_0^2 y dy = 2x, \quad 0 < x < 1$

(c) $f_Y(y) = y \int_0^1 x dx = y/2, \quad 0 < y < 2$

(d)

$$\begin{aligned}
P\{X < x, Y < y\} &= P\{X < x\} P\{Y < y\} \\
&= \min(1, x^2) \min(1, y^2/4), \quad x > 0, y > 0
\end{aligned}$$

(e) $E[Y] = \int_0^2 y^2/2\, dy = 4/3$

(f)

$$P\{X + Y < 1\} = \int_0^1 x \int_0^{1-x} y\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 x(1 - x)^2\,dx = 1/24$$

6.8. Let T_i denote the time at which a shock type i, of $i = 1, 2, 3$, occurs. For $s > 0, t > 0$,

$$\begin{aligned} P\{X_1 > s, X_2 > t\} &= P\{T_1 > s, T_2 > t, T_3 > \max(s,t)\} \\ &= P\{T_1 > s\}P\{T_2 > t\}P\{T_3 > \max(s,t)\} \\ &= \exp\{-\lambda_1 s\}\exp\{-\lambda_2 t\}\exp\{-\lambda_3 \max(s,t)\} \\ &= \exp\{-(\lambda_1 s + \lambda_2 t + \lambda_3 \max(s,t))\} \end{aligned}$$

6.9. **(a)** No, advertisements on pages having many ads are less likely to be chosen than are ones on pages with few ads.

(b) $\frac{1}{m}\frac{n(i)}{n}$

(c) $\dfrac{\sum_{i=1}^{m} n(i)}{nm} = \overline{n}/n$, where $\overline{n} = \sum_{i=1}^{m} n(i)/m$

(d) $(1 - \overline{n}/n)^{k-1}\dfrac{1}{m}\dfrac{n(i)}{n}\dfrac{1}{n(i)} = (1 - \overline{n}/n)^{k-1}/(nm)$

(e) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{nm}(1 - \overline{n}/n)^{k-1} = \frac{1}{\overline{n}m}.$

(f) The number of iterations is geometric with mean $n\sqrt{n}$

6.10. **(a)** $P\{X = i\} = 1/m, \qquad i = 1, \ldots, m.$

(b) **Step 2.** Generate a uniform (0, 1) random variable U. If $U < n(X)/n$, go to step 3. Otherwise return to step 1.

Step 3. Generate a uniform (0, 1) random variable U, and select the element on page X in position $[n(X)U] + 1$.

6.11. Yes, they are independent. This can be easily seen by considering the equivalent question of whether X_N is independent of N. But this is indeed so, since knowing when the first random variable greater than c occurs does not affect the probability distribution of its value, which is the uniform distribution on $(c, 1)$.

6.12. Let p_i denote the probability of obtaining i points on a single throw of the dart. Then

$$\begin{aligned} p_{30} &= \pi/36 \\ p_{20} &= 4\pi/36 - p_{30} = \pi/12 \\ p_{10} &= 9\pi/36 - p_{20} - p_{30} = 5\pi/36 \\ p_0 &= 1 - p_{10} - p_{20} - p_{30} = 1 - \pi/4 \end{aligned}$$

(a) $\pi/12$

(b) $\pi/9$

(c) $1 - \pi/4$

(d) $\pi(30/36 + 20/12 + 50/36) = 35\pi/9$
(e) $(\pi/4)^2$
(f) $2(\pi/36)(1 - \pi/4) + 2(\pi/12)(5\pi/36)$

6.13. Let Z be a standard normal random variable.

(a)

$$P\left\{\sum_{i=1}^{4} X_i > 0\right\} = P\left\{\frac{\sum_{i=1}^{4} X_i - 6}{\sqrt{24}} > \frac{-6}{\sqrt{24}}\right\}$$
$$\approx P\{Z > -1.2247\} \approx .8897$$

(b)

$$P\left\{\sum_{i=1}^{4} X_i > 0 \middle| \sum_{i=1}^{2} X_i = -5\right\} = P\{X_3 + X_4 > 5\}$$
$$= P\left\{\frac{X_3 + X_4 - 3}{\sqrt{12}} > 2/\sqrt{12}\right\}$$
$$\approx P\{Z > .5774\} \approx .2818$$

(c)

$$p\left\{\sum_{i=1}^{4} X_i > 0 | X_1 = 5\right\} = P\{X_2 + X_3 + X_4 > -5\}$$
$$= P\left\{\frac{X_2 + X_3 + X_4 - 4.5}{\sqrt{18}} > -9.5/\sqrt{18}\right\}$$
$$\approx P\{Z > -2.239\} \approx .9874$$

6.14. In the following, C does not depend on n.

$$\begin{aligned} P\{N = n|X = x\} &= f_{X|N}(x|n)P\{N = n\}/f_X(x) \\ &= C\frac{1}{(n-1)!}(\lambda x)^{n-1}(1 - p)^{n-1} \\ &= C(\lambda(1 - p)x)^{n-1}/(n - 1)! \end{aligned}$$

which shows that, conditional on $X = x, N - 1$ is a Poisson random variable with mean $\lambda(1 - p)x$. That is,

$$\begin{aligned} P\{N = n|X = x\} &= P\{N - 1 = n - 1|X = x\} \\ &= e^{-\lambda(1-p)x}(\lambda(1 - p)x)^{n-1}/(n - 1)!, n \geq 1. \end{aligned}$$

6.15. (a) The Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

As the equations $u = x, v = x + y$ imply that $x = u, y = v - u$, we obtain

$$f_{U,V}(u, v) = f_{X,Y}(u, v - u) = 1, \quad 0 < u < 1, \quad 0 < v - u < 1$$

or, equivalently,

$$f_{U,V}(u, v) = 1, \quad \max(v - 1, 0) < u < \min(v, 1)$$

(b) For $0 < v < 1$,

$$f_V(v) = \int_0^v du = v$$

For $1 \le v \le 2$,

$$f_V(v) = \int_{v-1}^1 du = 2 - v$$

6.16. Let U be a uniform random variable on (7, 11). If you bid $x, 7 \le x \le 10$, you will be the high bidder with probability

$$(P\{U < x\})^3 = \left(P\left\{\frac{U - 7}{4} < \frac{x - 7}{4}\right\}\right)^3 = \left(\frac{x - 7}{4}\right)^3$$

Hence, your expected gain—call it $E[G(x)]$—if you bid x is

$$E[G(x)] = \frac{1}{4}(x - 7)^3(10 - x)$$

Calculus shows this is maximized when $x = 37/4$.

6.17. Let $i_1, i_2, \ldots, i_n$, be a permutation of $1, 2, \ldots, n$. Then

$$\begin{aligned} P\{X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} &= P\{X_1 = i_1\}P\{X_2 = i_2\} \cdots P\{X_n = i_n\} \\ &= p_{i_1}p_{i_2} \cdots p_{i_n} \\ &= p_1 p_2 \cdots p_n \end{aligned}$$

Therefore, the desired probability is $n!\, p_1 p_2 \cdots p_n$, which reduces to $\frac{n!}{n^n}$ when all $p_i = 1/n$.

6.18. **(a)** Because $\sum_{i=1}^n X_i = \sum_{i=1}^n Y_i$, it follows that $N = 2M$.

(b) Consider the $n - k$ coordinates whose Y-values are equal to 0, and call them the red coordinates. Because the k coordinates whose X-values are equal to 1 are equally likely to be any of the $\binom{n}{k}$ sets of k coordinates, it follows that the number of red coordinates among these k coordinates has the same distribution as the number of red balls chosen when one randomly chooses k of a set of n balls of which $n - k$ are red. Therefore, M is a hypergeometric random variable.

(c) $E[N] = E[2M] = 2E[M] = \frac{2k(n-k)}{n}$

(d) Using the formula for the variance of a hypergeometric given in Example 8j of Chapter 4, we obtain

$$\mathrm{Var}(N) = 4\,\mathrm{Var}(M) = 4\frac{n - k}{n - 1}k(1 - k/n)(k/n)$$

6.19. **(a)** First note that $S_n - S_k = \sum_{i=k+1}^n Z_i$ is a normal random variable with mean 0 and variance $n - k$ that is independent of S_k. Consequently, given that $S_k = y$, S_n is a normal random variable with mean y and variance $n - k$.

(b) Because the conditional density function of S_k given that $S_n = x$ is a density function whose argument is y, anything that does not depend on y can be regarded as a constant. (For instance, x is regarded as a fixed constant.) In the following, the quantities $C_i, i = 1, 2, 3, 4$ are all constants that do not depend on y:

$$
\begin{aligned}
f_{S_k|S_n}(y|x) &= \frac{f_{S_k,S_n}(y,x)}{f_{S_n}(x)} \\
&= C_1 f_{S_n|S_k}(x|y) f_{S_k}(y) \quad \left(\text{where } C_1 = \frac{1}{f_{S_n}(x)}\right) \\
&= C_1 \frac{1}{\sqrt{2\pi}\sqrt{n-k}} e^{-(x-y)^2/2(n-k)} \frac{1}{\sqrt{2\pi}\sqrt{k}} e^{-y^2/2k} \\
&= C_2 \exp\left\{-\frac{(x-y)^2}{2(n-k)} - \frac{y^2}{2k}\right\} \\
&= C_3 \exp\left\{\frac{2xy}{2(n-k)} - \frac{y^2}{2(n-k)} - \frac{y^2}{2k}\right\} \\
&= C_3 \exp\left\{-\frac{n}{2k(n-k)}\left(y^2 - 2\frac{k}{n}xy\right)\right\} \\
&= C_3 \exp\left\{-\frac{n}{2k(n-k)}\left[\left(y - \frac{k}{n}x\right)^2 - \left(\frac{k}{n}x\right)^2\right]\right\} \\
&= C_4 \exp\left\{-\frac{n}{2k(n-k)}\left(y - \frac{k}{n}x\right)^2\right\}
\end{aligned}
$$

But we recognize the preceding as the density function of a normal random variable with mean $\frac{k}{n}x$ and variance $\frac{k(n-k)}{n}$.

6.20. (a)

$$
\begin{aligned}
P\{X_6 > X_1 | X_1 = \max(X_1, \ldots, X_5)\} \\
&= \frac{P\{X_6 > X_1,\ X_1 = \max(X_1, \ldots, X_5)\}}{P\{X_1 = \max(X_1, \ldots, X_5)\}} \\
&= \frac{P\{X_6 = \max(X_1, \ldots, X_6),\ X_1 = \max(X_1, \ldots, X_5)\}}{1/5} \\
&= 5\,\frac{1}{6}\,\frac{1}{5} = \frac{1}{6}
\end{aligned}
$$

Thus, the probability that X_6 is the largest value is independent of which is the largest of the other five values. (Of course, this would not be true if the X_i had different distributions.)

(b) One way to solve this problem is to condition on whether $X_6 > X_1$. Now,

$$P\{X_6 > X_2 | X_1 = \max(X_1, \ldots, X_5), X_6 > X_1\} = 1$$

Also, by symmetry,

$$P\{X_6 > X_2 | X_1 = \max(X_1, \ldots, X_5), X_6 < X_1\} = \frac{1}{2}$$

From part (a),

$$P\{X_6 > X_1 | X_1 = \max(X_1, \ldots, X_5)\} = \frac{1}{6}$$

Thus, conditioning on whether $X_6 > X_1$ yields the result

$$P\{X_6 > X_2 | X_1 = \max(X_1, \ldots, X_5)\} = \frac{1}{6} + \frac{1}{2}\frac{5}{6} = \frac{7}{12}$$

CHAPTER 7

7.1. **(a)** $d = \sum_{i=1}^{m} 1/n(i)$

(b) $P\{X = i\} = P\{[mU] = i - 1\} = P\{i - 1 \le mU < i\} = 1/m, \quad i = 1, \ldots, m$

(c) $E\left[\frac{m}{n(X)}\right] = \sum_{i=1}^{m} \frac{m}{n(i)} P\{X = i\} = \sum_{i=1}^{m} \frac{m}{n(i)} \frac{1}{m} = d$

7.2. Let I_j equal 1 if the jth ball withdrawn is white and the $(j + 1)$st is black, and let I_j equal 0 otherwise. If X is the number of instances in which a white ball is immediately followed by a black one, then we may express X as

$$X = \sum_{j=1}^{n+m-1} I_j$$

Thus,

$$\begin{aligned}
E[X] &= \sum_{j=1}^{n+m-1} E[I_j] \\
&= \sum_{j=1}^{n+m-1} P\{j^{th} \text{ selection is white, } (j + 1)^{st} \text{ is black}\} \\
&= \sum_{j=1}^{n+m-1} P\{j^{th} \text{ selection is white}\} P\{j + 1)^{st} \text{ is black} | j^{th} \text{ is white}\} \\
&= \sum_{j=1}^{n+m-1} \frac{n}{n + m} \frac{m}{n + m - 1} \\
&= \frac{nm}{n + m}
\end{aligned}$$

The preceding used the fact that each of the $n + m$ balls is equally likely to be the jth one selected and, given that that selection is a white ball, each of the other $n + m - 1$ balls is equally likely to be the next ball chosen.

7.3. Arbitrarily number the couples, and then let I_j equal 1 if married couple number $j, j = 1, \ldots, 10$, is seated at the same table. Then, if X represents the number of married couples that are seated at the same table, we have

$$X = \sum_{j=1}^{10} I_j$$

so

$$E[X] = \sum_{j=1}^{10} E[I_j]$$

(a) To compute $E[I_j]$ in this case, consider wife number j. Since each of the $\binom{19}{3}$ groups of size 3 not including her is equally likely to be the remaining members of her table, it follows that the probability that her husband is at her table is

$$\frac{\binom{1}{1}\binom{18}{2}}{\binom{19}{3}} = \frac{3}{19}$$

Hence, $E[I_j] = 3/19$ and so

$$E[X] = 30/19$$

(b) In this case, since the 2 men at the table of wife j are equally likely to be any of the 10 men, it follows that the probability that one of them is her husband is 2/10, so

$$E[I_j] = 2/10 \quad \text{and} \quad E[X] = 2$$

7.4. From Example 2i, we know that the expected number of times that the die need be rolled until all sides have appeared at least once is $6(1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6) = 14.7$. Now, if we let X_i denote the total number of times that side i appears, then, since $\sum_{i=1}^{6} X_i$ is equal to the total number of rolls, we have

$$14.7 = E\left[\sum_{i=1}^{6} X_i\right] = \sum_{i=1}^{6} E[X_i]$$

But, by symmetry, $E[X_i]$ will be the same for all i, and thus it follows from the preceding that $E[X_1] = 14.7/6 = 2.45$.

7.5. Let I_j equal 1 if we win 1 when the jth red card to show is turned over, and let I_j equal 0 otherwise. (For instance, I_1 will equal 1 if the first card turned over is red.) Hence, if X is our total winnings, then

$$E[X] = E\left[\sum_{j=1}^{n} I_j\right] = \sum_{j=1}^{n} E[I_j]$$

Now, I_j will equal 1 if j red cards appear before j black cards. By symmetry, the probability of this event is equal to 1/2; therefore, $E[I_j] = 1/2$ and $E[X] = n/2$.

7.6. To see that $N \le n - 1 + I$, note that if all events occur, then both sides of the preceding inequality are equal to n, whereas if they do not all occur, then the inequality reduces to $N \le n - 1$, which is clearly true in this case. Taking expectations yields

$$E[N] \le n - 1 + E[I]$$

However, if we let I_i equal 1 if A_i occurs and 0 otherwise, then

$$E[N] = E\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} E[I_i] = \sum_{i=1}^{n} P(A_i)$$

Since $E[I] = P(A_1 \cdots A_n)$, the result follows.

7.7. Imagine that the values $1, 2, \ldots, n$ are lined up in their numerical order and that the k values selected are considered special. From Example 3e, the position of the first special value, equal to the smallest value chosen, has mean $1 + \dfrac{n - k}{k + 1} = \dfrac{n + 1}{k + 1}$.

For a more formal argument, note that $X \ge j$ if none of the $j - 1$ smallest values are chosen. Hence,

$$P\{X \ge j\} = \frac{\dbinom{n - j + 1}{k}}{\dbinom{n}{k}} = \frac{\dbinom{n - k}{j - 1}}{\dbinom{n}{j - 1}}$$

which shows that X has the same distribution as the random variable of Example 3e (with the notational change that the total number of balls is now n and the number of special balls is k).

7.8. Let X denote the number of families that depart after the Sanchez family leaves. Arbitrarily number all the $N - 1$ non-Sanchez families, and let I_r, $1 \le r \le N - 1$, equal 1 if family r departs after the Sanchez family does. Then

$$X = \sum_{r=1}^{N-1} I_r$$

Taking expectations gives

$$E[X] = \sum_{r=1}^{N-1} P\{\text{family } r \text{ departs after the Sanchez family}\}$$

Now consider any non-Sanchez family that checked in k pieces of luggage. Because each of the $k + j$ pieces of luggage checked in either by this family or by the Sanchez family is equally likely to be the last of these $k + j$ to appear, the probability that this family departs after the Sanchez family is $\frac{k}{k+j}$. Because the number of non-Sanchez families who checked in k pieces of luggage is n_k when $k \ne j$, or $n_j - 1$ when $k = j$, we obtain

$$E[X] = \sum_{k} \frac{kn_k}{k + j} - \frac{1}{2}$$

7.9. Let the neighborhood of any point on the rim be the arc starting at that point and extending for a length 1. Consider a uniformly chosen point on the rim of the circle—that is, the probability that this point lies on a specified arc of length x is $\frac{x}{2\pi}$—and let X denote the number of points that lie in its neighborhood. With I_j defined to equal 1 if item number j is in the neighborhood of the random point and to equal 0 otherwise, we have

$$X = \sum_{j=1}^{19} I_j$$

Taking expectations gives

$$E[X] = \sum_{j=1}^{19} P\{\text{item } j \text{ lies in the neighborhood of the random point}\}$$

But because item j will lie in its neighborhood if the random point is located on the arc of length 1 going from item j in the counterclockwise direction, it follows that

$$P\{\text{item } j \text{ lies in the neighborhood of the random point}\} = \frac{1}{2\pi}$$

Hence,

$$E[X] = \frac{19}{2\pi} > 3$$

Because $E[X] > 3$, at least one of the possible values of X must exceed 3, proving the result.

7.10. If $g(x) = x^{1/2}$, then

$$g'(x) = \frac{1}{2}x^{-1/2}, \quad g''(x) = -\frac{1}{4}x^{-3/2}$$

so the Taylor series expansion of $\sqrt{x}$ about λ gives

$$\sqrt{X} \approx \sqrt{\lambda} + \frac{1}{2}\lambda^{-1/2}(X - \lambda) - \frac{1}{8}\lambda^{-3/2}(X - \lambda)^2$$

Taking expectations yields

$$\begin{aligned} E[\sqrt{X}] &\approx \sqrt{\lambda} + \frac{1}{2}\lambda^{-1/2}E[X - \lambda] - \frac{1}{8}\lambda^{-3/2}E[(X - \lambda)^2] \\ &= \sqrt{\lambda} - \frac{1}{8}\lambda^{-3/2}\lambda \\ &= \sqrt{\lambda} - \frac{1}{8}\lambda^{-1/2} \end{aligned}$$

Hence,

$$\begin{aligned}\text{Var}(\sqrt{X}) &= E[X] - (E[\sqrt{X}])^2 \\ &\approx \lambda - \left(\sqrt{\lambda} - \frac{1}{8}\lambda^{-1/2}\right)^2 \\ &= 1/4 - \frac{1}{64\lambda} \\ &\approx 1/4\end{aligned}$$

7.11. Number the tables so that tables 1, 2, and 3 are the ones with four seats and tables 4, 5, 6, and 7 are the ones with two seats. Also, number the women, and let $X_{i,j}$ equal 1 if woman i is seated with her husband at table j. Note that

$$E[X_{i,j}] = \frac{\binom{2}{2}\binom{18}{2}}{\binom{20}{4}} = \frac{3}{95}, \quad j = 1, 2, 3$$

and

$$E[X_{i,j}] = \frac{1}{\binom{20}{2}} = \frac{1}{190}, \quad j = 4, 5, 6, 7$$

Now, X denotes the number of married couples that are seated at the same table, we have

$$\begin{aligned}E[X] &= E\left[\sum_{i=1}^{10}\sum_{j=1}^{7} X_{i,j}\right] \\ &= \sum_{i=1}^{22}\sum_{j=1}^{3} E[X_{i,j}] + \sum_{i=1}^{19}\sum_{j=4}^{7} E[X_{i,j}]\end{aligned}$$

7.12. Let X_i equal 1 if individual i does not recruit anyone, and let X_i equal 0 otherwise. Then

$$\begin{aligned}E[X_i] &= P\{i \text{ does not recruit any of } i+1, i+2, \ldots, n\} \\ &= \frac{i-1}{i}\frac{i}{i+1}\cdots\frac{n-2}{n-1} \\ &= \frac{i-1}{n-1}\end{aligned}$$

Hence,

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\frac{i-1}{n-1} = \frac{n}{2}$$

From the preceding we also obtain

$$\mathrm{Var}(X_i) = \frac{i-1}{n-1}\left(1 - \frac{i-1}{n-1}\right) = \frac{(i-1)(n-i)}{(n-1)^2}$$

Now, for $i < j$,

$$\begin{aligned}E[X_iX_j] &= \frac{i-1}{i}\cdots\frac{j-2}{j-1}\frac{j-2}{j}\frac{j-1}{j+1}\cdots\frac{n-3}{n-1} \\ &= \frac{(i-1)(j-2)}{(n-2)(n-1)}\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Cov}(X_i, X_j) &= \frac{(i-1)(j-2)}{(n-2)(n-1)} - \frac{i-1}{n-1}\frac{j-1}{n-1} \\ &= \frac{(i-1)(j-n)}{(n-2)(n-1)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n}\mathrm{Var}(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mathrm{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n}\frac{(i-1)(n-i)}{(n-1)^2} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{(i-1)(j-n)}{(n-2)(n-1)^2} \\ &= \frac{1}{(n-1)^2}\sum_{i=1}^{n}(i-1)(n-i) \\ &\quad - \frac{1}{(n-2)(n-1)^2}\sum_{i=1}^{n-1}(i-1)(n-i)(n-i-1)\end{aligned}$$

7.13. Let X_i equal 1 if the ith triple consists of one of each type of player. Then

$$E[X_i] = \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{\binom{9}{3}} = \frac{2}{7}$$

Hence, for part (a), we obtain

$$E\left[\sum_{i=1}^{3} X_i\right] = 6/7$$

It follows from the preceding that

$$\mathrm{Var}(X_i) = (2/7)(1 - 2/7) = 10/49$$

Also, for $i \neq j$,

$$\begin{aligned}
E[X_iX_j] &= P\{X_i = 1, X_j = 1\} \\
&= P\{X_i = 1\}P\{X_j = 1|X_i = 1\} \\
&= \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{\binom{9}{3}} \frac{\binom{1}{1}\binom{2}{1}\binom{3}{1}}{\binom{6}{3}} \\
&= 6/70
\end{aligned}$$

Hence, for part (b), we obtain

$$\begin{aligned}
\text{Var}\left(\sum_{i=1}^{3} X_i\right) &= \sum_{i=1}^{3} \text{Var}(X_i) + 2\sum\sum_{j>1} \text{Cov}(X_i, X_j) \\
&= 30/49 + 2\binom{3}{2}\left(\frac{6}{70} - \frac{4}{49}\right) \\
&= \frac{312}{490}
\end{aligned}$$

7.14. Let X_i, $i = 1, \ldots, 13$, equal 1 if the ith card is an ace and let X_i be 0 otherwise. Let Y_j equal 1 if the jth card is a spade and let $i, j = 1, \ldots, 13$, be 0 otherwise. Now,

$$\begin{aligned}
\text{Cov}(X, Y) &= \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(X_i, Y_j)
\end{aligned}$$

However, X_i is clearly independent of Y_j because knowing the suit of a particular card gives no information about whether it is an ace and thus cannot affect the probability that another specified card is an ace. More formally, let $A_{i,s}, A_{i,h}, A_{i,d}, A_{i,c}$ be the events, respectively, that card i is a spade, a heart, a diamond, and a club. Then

$$\begin{aligned}
P\{Y_j = 1\} = \frac{1}{4}(&P\{Y_j = 1|A_{i,s}\} + P\{Y_j = 1|A_{i,h}\} \\
&+ P\{Y_j = 1|A_{i,d}\} + P\{Y_j = 1|A_{i,c}\})
\end{aligned}$$

But, by symmetry, we have

$$P\{Y_j = 1|A_{i,s}\} = P\{Y_j = 1|A_{i,h}\} = P\{Y_j = 1|A_{i,d}\} = P\{Y_j = 1|A_{i,c}\}$$

Therefore,

$$P\{Y_j = 1\} = P\{Y_j = 1|A_{i,s}\}$$

As the preceding implies that

$$P\{Y_j = 1\} = P\{Y_j = 1|A_{i,s}^c\}$$

we see that Y_j and X_i are independent. Hence, $\text{Cov}(X_i, Y_j) = 0$, and thus $\text{Cov}(X, Y) = 0$.

The random variables X and Y, although uncorrelated, are not independent. This follows, for instance, from the fact that

$$P\{Y = 13|X = 4\} = 0 \neq P\{Y = 13\}$$

7.15. (a) Your expected gain without any information is 0.

(b) You should predict heads if $p > 1/2$ and tails otherwise.

(c) Conditioning on V, the value of the coin, gives

$$\begin{aligned}
E[\text{Gain}] &= \int_0^1 E[\text{Gain}|V = p]\,dp \\
&= \int_0^{1/2} [1(1 - p) - 1(p)]\,dp + \int_{1/2}^1 [1(p) - 1(1 - p)]\,dp \\
&= 1/2
\end{aligned}$$

7.16. Given that the name chosen appears in $n(X)$ different positions on the list, since each of these positions is equally likely to be the one chosen, it follows that

$$E[I|n(X)] = P\{I = 1|n(X)\} = 1/n(X)$$

Hence,

$$E[I] = E[1/n(X)]$$

Thus, $E[mI] = E[m/n(X)] = d$.

7.17. Letting X_i equal 1 if a collision occurs when the ith item is placed, and letting it equal 0 otherwise, we can express the total number of collisions X as

$$X = \sum_{i=1}^{m} X_i$$

Therefore,

$$E[X] = \sum_{i=1}^{m} E[X_i]$$

To determine $E[X_i]$, condition on the cell in which it is placed.

$$\begin{aligned}
E[X_i] &= \sum_j E[X_i|\text{ placed in cell} j]p_j \\
&= \sum_j P\{i \text{ causes collision}|\text{placed in cell } j]p_j \\
&= \sum_j [1 - (1 - p_j)^{i-1}]p_j \\
&= 1 - \sum_j (1 - p_j)^{i-1} p_j
\end{aligned}$$

The next to last equality used the fact that, conditional on item i being placed in cell j, item i will cause a collision if any of the preceding $i - 1$ items were put in cell j. Thus,

$$E[X] = m - \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - p_j)^{i-1} p_j$$

Interchanging the order of the summations gives

$$E[X] = m - n + \sum_{j=1}^{n} (1 - p_j)^m$$

Looking at the result shows that we could have derived it more easily by taking expectations of both sides of the identity

$$\text{number of nonempty cells} = m - X$$

The expected number of nonempty cells is then found by defining an indicator variable for each cell, equal to 1 if that cell is nonempty and to 0 otherwise, and then taking the expectation of the sum of these indicator variables.

7.18. Let L denote the length of the initial run. Conditioning on the first value gives

$$E[L] = E[L|\text{first value is one}]\frac{n}{n + m} + E[L|\text{first value is zero}]\frac{m}{n + m}$$

Now, if the first value is one, then the length of the run will be the position of the first zero when considering the remaining $n + m - 1$ values, of which $n - 1$ are ones and m are zeroes. (For instance, if the initial value of the remaining $n + m - 1$ is zero, then $L = 1$.) As a similar result is true given that the first value is a zero, we obtain from the preceding, upon using the result from Example 3e, that

$$\begin{aligned} E[L] &= \frac{n + m}{m + 1}\frac{n}{n + m} + \frac{n + m}{n + 1}\frac{m}{n + m} \\ &= \frac{n}{m + 1} + \frac{m}{n + 1} \end{aligned}$$

7.19. Let X be the number of flips needed for both boxes to become empty, and let Y denote the number of heads in the first $n + m$ flips. Then

$$\begin{aligned} E[X] &= \sum_{i=0}^{n+m} E[X|Y = i]P\{Y = i\} \\ &= \sum_{i=0}^{n+m} E[X|Y = i]\binom{n + m}{i} p^i(1 - p)^{n+m-i} \end{aligned}$$

Now, if the number of heads in the first $n + m$ flips is $i, i \le n$, then the number of additional flips is the number of flips needed to obtain an additional $n - i$ heads. Similarly, if the number of heads in the first $n + m$ flips is $i, i > n$, then, because there would have been a total of $n + m - i < m$ tails, the number of additional flips is the number needed to obtain an additional $i - n$ heads. Since the number of flips needed for j outcomes of a particular type is a negative binomial random variable whose mean is j divided by the probability of that outcome, we obtain

$$E[X] = \sum_{i=0}^{n} \frac{n-i}{p} \binom{n+m}{i} p^i (1-p)^{n+m-i} + \sum_{i=n+1}^{n+m} \frac{i-n}{1-p} \binom{n+m}{i} p^i (1-p)^{n+m-i}$$

7.20. Taking expectations of both sides of the identity given in the hint yields

$$\begin{aligned} E[X^n] &= E\left[n \int_0^\infty x^{n-1} I_X(x)\,dx\right] \\ &= n \int_0^\infty E[x^{n-1} I_X(x)]\,dx \\ &= n \int_0^\infty x^{n-1} E[I_X(x)]\,dx \\ &= n \int_0^\infty x^{n-1} \overline{F}(x)\,dx \end{aligned}$$

Taking the expectation inside the integral sign is justified because all the random variables $I_X(x), 0 < x < \infty$, are nonnegative.

7.21. Consider a random permutation $I_1, \ldots, I_n$ that is equally likely to be any of the $n!$ permutations. Then

$$\begin{aligned} E[a_{I_j} a_{I_{j+1}}] &= \sum_k E[a_{I_j} a_{I_{j+1}} | I_j = k] P\{I_j = k\} \\ &= \frac{1}{n} \sum_k a_k E[a_{I_{j+1}} | I_j = k] \\ &= \frac{1}{n} \sum_k a_k \sum_i a_i P\{I_{j+1} = i | I_j = k\} \\ &= \frac{1}{n(n-1)} \sum_k a_k \sum_{i \neq k} a_i \\ &= \frac{1}{n(n-1)} \sum_k a_k (-a_k) \\ &< 0 \end{aligned}$$

where the final equality followed from the assumption that $\sum_{i=1}^n a_i = 0$. Since the preceding shows that

$$E\left[\sum_{j=1}^{n} a_{I_j} a_{I_{j+1}}\right] < 0$$

it follows that there must be some permutation $i_1, \ldots, i_n$ for which

$$\sum_{j=1}^{n} a_{i_j} a_{i_{j+1}} < 0$$

7.22. **(a)** $E[X] = \lambda_1 + \lambda_2, \quad E[X] = \lambda_2 + \lambda_3$

(b)

$$\begin{aligned}\text{Cov}(X, Y) &= \text{Cov}(X_1 + X_2, X_2 + X_3)\\ &= \text{Cov}(X_1, X_2 + X_3) + \text{Cov}(X_2, X_2 + X_3)\\ &= \text{Cov}(X_2, X_2)\\ &= \text{Var}(X_2)\\ &= \lambda_2\end{aligned}$$

(c) Conditioning on X_2 gives

$$\begin{aligned}P\{X = i, Y = j\} &= \sum_k P\{X = i, Y = j|X_2 = k\}P\{X_2 = k\}\\ &= \sum_k P\{X_1 = i - k, X_3 = j - k|X_2 = k\}e^{-\lambda_2}\lambda_2^k/k!\\ &= \sum_k P\{X_1 = i - k, X_3 = j - k\}e^{-\lambda_2}\lambda_2^k/k!\\ &= \sum_k P\{X_1 = i - k\}P\{X_3 = j - k\}e^{-\lambda_2}\lambda_2^k/k!\\ &= \sum_{k=0}^{\min(i,j)} e^{-\lambda_1}\frac{\lambda_1^{i-k}}{(i - k)!}e^{-\lambda_3}\frac{\lambda_3^{j-k}}{(j - k)!}e^{-\lambda_2}\frac{\lambda_2^k}{k!}\end{aligned}$$

7.23.

$$\begin{aligned}\text{Corr}\left(\sum_i X_i, \sum_j Y_j\right) &= \frac{\text{Cov}(\sum_i X_i, \sum_j Y_j)}{\sqrt{\text{Var}(\sum_i X_i)\text{Var}(\sum_j Y_j)}}\\ &= \frac{\sum_i \sum_j \text{Cov}(X_i, Y_j)}{\sqrt{n\sigma_x^2 n\sigma_y^2}}\\ &= \frac{\sum_i \text{Cov}(X_i, Y_i) + \sum_i \sum_{j \neq i} \text{Cov}(X_i, Y_j)}{n\sigma_x\sigma_y}\\ &= \frac{n\rho\sigma_x\sigma_y}{n\sigma_x\sigma_y}\\ &= \rho\end{aligned}$$

where the next to last equality used the fact that $\text{Cov}(X_i, Y_i) = \rho\sigma_x\sigma_y$

7.24. Let X_i equal 1 if the ith card chosen is an ace, and let it equal 0 otherwise. Because

$$X = \sum_{i=1}^{3} X_i$$

and $E[X_i] = P\{X_i = 1\} = 1/13$, it follows that $E[X] = 3/13$. But, with A being the event that the ace of spades is chosen, we have

$$\begin{aligned}
E[X] &= E[X|A]P(A) + E[X|A^c]P(A^c) \\
&= E[X|A]\frac{3}{52} + E[X|A^c]\frac{49}{52} \\
&= E[X|A]\frac{3}{52} + \frac{49}{52}E\left[\sum_{i=1}^{3} X_i|A^c\right] \\
&= E[X|A]\frac{3}{52} + \frac{49}{52}\sum_{i=1}^{3} E[X_i|A^c] \\
&= E[X|A]\frac{3}{52} + \frac{49}{52}3\,\frac{3}{51}
\end{aligned}$$

Using that $E[X] = 3/13$ gives the result

$$E[X|A] = \frac{52}{3}\left(\frac{3}{13} - \frac{49}{52}\frac{3}{17}\right) = \frac{19}{17} = 1.1176$$

Similarly, letting L be the event that at least one ace is chosen, we have

$$\begin{aligned}
E[X] &= E[X|L]P(L) + E[X|L^c]P(L^c) \\
&= E[X|L]P(L) \\
&= E[X|L]\left(1 - \frac{48 \cdot 47 \cdot 46}{52 \cdot 51 \cdot 50}\right)
\end{aligned}$$

Thus,

$$E[X|L] = \frac{3/13}{1 - \frac{48 \cdot 47 \cdot 46}{52 \cdot 51 \cdot 50}} \approx 1.0616$$

Another way to solve this problem is to number the four aces, with the ace of spades having number 1, and then let Y_i equal 1 if ace number i is chosen and 0 otherwise. Then

$$\begin{aligned}
E[X|A] &= E\left[\sum_{i=1}^{4} Y_i|Y_1 = 1\right] \\
&= 1 + \sum_{i=2}^{4} E[Y_i|Y_1 = 1] \\
&= 1 + 3 \cdot \frac{2}{51} = 19/17
\end{aligned}$$

where we used that the fact given that the ace of spades is chosen the other two cards are equally likely to be any pair of the remaining 51 cards; so the conditional probability that any specified card (not equal to the ace of spades) is chosen is $2/51$. Also,

$$E[X|L] = E\left[\sum_{i=1}^{4} Y_i|L\right] = \sum_{i=1}^{4} E[Y_i|L] = 4P\{Y_1 = 1|L\}$$

Because

$$P\{Y_1 = 1|L\} = P(A|L) = \frac{P(AL)}{P(L)} = \frac{P(A)}{P(L)} = \frac{3/52}{1 - \frac{48 \cdot 47 \cdot 46}{52 \cdot 51 \cdot 50}}$$

we obtain the same answer as before.

7.25. **(a)** $E[I|X = x] = P\{Z < X|X = x\} = P\{Z < x|X = x\} = P\{Z < x\} = \Phi(x)$

(b) It follows from part (a) that $E[I|X] = \Phi(X)$. Therefore,

$$E[I] = E[E[I|X]] = E[\Phi(X)]$$

The result now follows because $E[I] = P\{I = 1\} = P\{Z < X\}$.

(c) Since $X - Z$ is normal with mean μ and variance 2, we have

$$\begin{aligned}
P\{X > Z\} &= P\{X - Z > 0\} \\
&= P\left\{\frac{X - Z - \mu}{2} > \frac{-\mu}{2}\right\} \\
&= 1 - \Phi\left(\frac{-\mu}{2}\right) \\
&= \Phi\left(\frac{\mu}{2}\right)
\end{aligned}$$

7.26. Let N be the number of heads in the first $n + m - 1$ flips. Let $M = \max(X, Y)$ be the number of flips needed to amass at least n heads and at least m tails. Conditioning on N gives

$$\begin{aligned}
E[M] &= \sum_i E[M|N = i]P\{N = i\} \\
&= \sum_{i=0}^{n-1} E[M|N = i]P\{N = i\} + \sum_{i=n}^{n+m-1} E[M|N = i]P\{N = i\}
\end{aligned}$$

Now, suppose we are given that there are a total of i heads in the first $n+m-1$ trials. If $i < n$, then we have already obtained at least m tails, so the additional number of flips needed is equal to the number needed for an additional $n - i$ heads; similarly, if $i \geq n$, then we have already obtained at least n heads, so the additional number of flips needed is equal to the number needed for an additional $m - (n + m - 1 - i)$ tails. Consequently, we have

$$\begin{aligned}
E[M] &= \sum_{i=0}^{n-1}\left(n + m - 1 + \frac{n - i}{p}\right)P\{N = i\} \\
&\quad + \sum_{i=n}^{n+m-1}\left(n + m - 1 + \frac{i + 1 - n}{1 - p}\right)P\{N = i\} \\
&= n + m - 1 + \sum_{i=0}^{n-1}\frac{n - i}{p}\binom{n + m - 1}{i}p^i(1 - p)^{n+m-1-i} \\
&\quad + \sum_{i=n}^{n+m-1}\frac{i + 1 - n}{1 - p}\binom{n + m - 1}{i}p^i(1 - p)^{n+m-1-i}
\end{aligned}$$

The expected number of flips to obtain either n heads or m tails, $E[\min(X, Y)]$, is now given by

$$E[\min(X, Y)] = E[X + Y - M] = \frac{n}{p} + \frac{m}{1 - p} - E[M]$$

7.27. This is just the expected time to collect $n - 1$ of the n types of coupons in Example 2i. By the results of that example the solution is

$$1 + \frac{n}{n-1} + \frac{n}{n-2} + \ldots + \frac{n}{2}$$

7.28. With $q = 1 - p$,

$$E[X] = \sum_{i=1}^{\infty} P\{X \geq i\} = \sum_{i=1}^{n} P\{X \geq i\} = \sum_{i=1}^{n} q^{i-1} = \frac{1 - q^n}{p}$$

7.29.

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = P(X = 1, Y = 1) - P(X = 1)P(Y = 1)$$

Hence,

$$\text{Cov}(X, Y) = 0 \quad \Leftrightarrow \quad P(X = 1, Y = 1) = P(X = 1)P(Y = 1)$$

Because

$$\text{Cov}(X, Y) = \text{Cov}(1 - X, 1 - Y) = -\text{Cov}(1 - X, Y) = -\text{Cov}(X, 1 - Y)$$

the preceding shows that all of the following are equivalent when X and Y are Bernoulli:

1. $\text{Cov}(X, Y) = 0$
2. $P(X = 1, Y = 1) = P(X = 1)P(Y = 1)$
3. $P(1 - X = 1, 1 - Y = 1) = P(1 - X = 1)P(1 - Y = 1)$
4. $P(1 - X = 1, Y = 1) = P(1 - X = 1)P(Y = 1)$
5. $P(X = 1, 1 - Y = 1) = P(X = 1)P(1 - Y = 1)$

7.30. Number the individuals, and let $X_{i,j}$ equal 1 if the jth individual who has hat size i chooses a hat of that size, and let $X_{i,j}$ equal 0 otherwise. Then the number of individuals who choose a hat of their size is

$$X = \sum_{i=1}^{r} \sum_{j=1}^{n_i} X_{i,j}$$

Hence,

$$E[X] = \sum_{i=1}^{r} \sum_{j=1}^{n_i} E[X_{i,j}] = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \frac{h_i}{n} = \frac{1}{n} \sum_{i=1}^{r} h_i n_i$$

CHAPTER 8

8.1. Let X denote the number of sales made next week, and note that X is integral. From Markov's inequality, we obtain the following:

(a) $P\{X > 18\} = P\{X \geq 19\} \leq \dfrac{E[X]}{19} = 16/19$

(b) $P\{X > 25\} = P\{X \geq 26\} \leq \dfrac{E[X]}{26} = 16/26$

8.2. **(a)**

$$\begin{aligned} P\{10 \le X \le 22\} &= P\{|X - 16| \le 6\} \\ &= P\{|X - \mu| \le 6\} \\ &= 1 - P\{|X - \mu| > 6\} \\ &\ge 1 - 9/36 = 3/4 \end{aligned}$$

(b) $P\{X \ge 19\} = P\{X - 16 \ge 3\} \le \dfrac{9}{9 + 9} = 1/2$

In part (a), we used Chebyshev's inequality; in part (b), we used its one-sided version. (See Proposition 5.1.)

8.3. First note that $E[X - Y] = 0$ and

$$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y) = 28$$

Using Chebyshev's inequality in part (a) and the one-sided version in parts (b) and (c) gives the following results:

(a) $P\{|X - Y| > 15\} \le 28/225$

(b) $P\{X - Y > 15\} \le \dfrac{28}{28 + 225} = 28/253$

(c) $P\{Y - X > 15\} \le \dfrac{28}{28 + 225} = 28/253$

8.4. If X is the number produced at factory A and Y the number produced at factory B, then

$$E[Y - X] = -2, \quad \text{Var}(Y - X) = 36 + 9 = 45$$

$$P\{Y - X > 0\} = P\{Y - X \ge 1\} = P\{Y - X + 2 \ge 3\} \le \frac{45}{45 + 9} = 45/54$$

8.5. Note first that

$$E[X_i] = \int_0^1 2x^2\, dx = 2/3$$

Now use the strong law of large numbers to obtain

$$\begin{aligned} r &= \lim_{n\to\infty} \frac{n}{S_n} \\ &= \lim_{n\to\infty} \frac{1}{S_n/n} \\ &= \frac{1}{\lim_{n\to\infty} S_n/n} \\ &= 1/(2/3) = 3/2 \end{aligned}$$

8.6. Because $E[X_i] = 2/3$ and

$$E[X_i^2] = \int_0^1 2x^3\, dx = 1/2$$

we have $\mathrm{Var}(X_i) = 1/2 - (2/3)^2 = 1/18$. Thus, if there are n components on hand, then

$$\begin{aligned}
P\{S_n \geq 35\} &= P\{S_n \geq 34.5\} \qquad \text{(the continuity correction)} \\
&= P\left\{\frac{S_n - 2n/3}{\sqrt{n/18}} \leq \frac{34.5 - 2n/3}{\sqrt{n/18}}\right\} \\
&\approx P\left\{Z \geq \frac{34.5 - 2n/3}{\sqrt{n/18}}\right\}
\end{aligned}$$

where Z is a standard normal random variable. Since

$$P\{Z > -1.284\} = P\{Z < 1.284\} \approx .90$$

we see that n should be chosen so that

$$(34.5 - 2n/3) \approx -1.284\sqrt{n/18}$$

A numerical computation gives the result $n = 55$.

8.7. If X is the time required to service a machine, then

$$E[X] = .2 + .3 = .5$$

Also, since the variance of an exponential random variable is equal to the square of its mean, we have

$$\mathrm{Var}(X) = (.2)^2 + (.3)^2 = .13$$

Therefore, with X_i being the time required to service job $i, i = 1, \ldots, 20$, and Z being a standard normal random variable, it follows that

$$\begin{aligned}
P\{X_1 + \cdots + X_{20} < 8\} &= P\left\{\frac{X_1 + \cdots + X_{20} - 10}{\sqrt{2.6}} < \frac{8 - 10}{\sqrt{2.6}}\right\} \\
&\approx P\{Z < -1.24035\} \\
&\approx .1074
\end{aligned}$$

8.8. Note first that if X is the gambler's winnings on a single bet, then

$$\begin{aligned}
E[X] = -.7 - .4 + 1 = -.1, E[X^2] = .7 + .8 + 10 = 11.5 \\
\rightarrow \mathrm{Var}(X) = 11.49
\end{aligned}$$

Therefore, with Z having a standard normal distribution,

$$\begin{aligned}
P\{X_1 + \cdots + X_{100} \leq -.5\} &= P\left\{\frac{X_1 + \cdots + X_{100} + 10}{\sqrt{1149}} \leq \frac{-.5 + 10}{\sqrt{1149}}\right\} \\
&\approx P\{Z \leq .2803\} \\
&\approx .6104
\end{aligned}$$

8.9. Using the notation of Problem 7, we have

$$P\{X_1 + \cdots + X_{20} < t\} = P\left\{\frac{X_1 + \cdots + X_{20} - 10}{\sqrt{2.6}} < \frac{t - 10}{\sqrt{2.6}}\right\}$$
$$\approx P\left\{Z < \frac{t - 10}{\sqrt{2.6}}\right\}$$

Now, $P\{Z < 1.645\} \approx .95$, so t should be such that

$$\frac{t - 10}{\sqrt{2.6}} \approx 1.645$$

which yields $t \approx 12.65$.

8.10. If the claim were true, then, by the central limit theorem, the average nicotine content (call it X) would approximately have a normal distribution with mean 2.2 and standard deviation .03. Thus, the probability that it would be as high as 3.1 is

$$P\{X > 3.1\} = P\left\{\frac{X - 2.2}{\sqrt{.03}} > \frac{3.1 - 2.2}{\sqrt{.03}}\right\}$$
$$\approx P\{Z > 5.196\}$$
$$\approx 0$$

where Z is a standard normal random variable.

8.11. (a) If we arbitrarily number the batteries and let X_i denote the life of battery i, $i = 1, \ldots, 40$, then the X_i are independent and identically distributed random variables. To compute the mean and variance of the life of, say, battery 1, we condition on its type. Letting I equal 1 if battery 1 is type A and letting it equal 0 if it is type B, we have

$$E[X_1|I = 1] = 50, \qquad E[X_1|I = 0] = 30$$

yielding

$$E[X_1] = 50P\{I = 1\} + 30P\{I = 0\} = 50(1/2) + 30(1/2) = 40$$

In addition, using the fact that $E[W^2] = (E[W])^2 + \text{Var}(W)$, we have

$$E[X_1^2|I = 1] = (50)^2 + (15)^2 = 2725, \qquad E[X_1^2|I = 0] = (30)^2 + 6^2 = 936$$

yielding

$$E[X_1^2] = (2725)(1/2) + (936)(1/2) = 1830.5$$

Thus, $X_1, \ldots, X_{40}$ are independent and identically distributed random variables having mean 40 and variance $1830.5 - 1600 = 230.5$. Hence, with $S = \sum_{i=1}^{40} X_i$, we have

$$E[S] = 40(40) = 1600, \qquad \text{Var}(S) = 40(230.5) = 9220$$

and the central limit theorem yields

$$\begin{aligned} P\{S > 1700\} &= P\left\{\frac{S - 1600}{\sqrt{9220}} > \frac{1700 - 1600}{\sqrt{9220}}\right\} \\ &\approx P\{Z > 1.041\} \\ &= 1 - \Phi(1.041) = .149 \end{aligned}$$

(b) For this part, let S_A be the total life of all the type A batteries and let S_B be the total life of all the type B batteries. Then, by the central limit theorem, S_A has approximately a normal distribution with mean $20(50) = 1000$ and variance $20(225) = 4500$, and S_B has approximately a normal distribution with mean $20(30) = 600$ and variance $20(36) = 720$. Because the sum of independent normal random variables is also a normal random variable, it follows that $S_A + S_B$ is approximately normal with mean 1600 and variance 5220. Consequently, with $S = S_A + S_B$,

$$\begin{aligned} P\{S > 1700\} &= P\left\{\frac{S - 1600}{\sqrt{5220}} > \frac{1700 - 1600}{\sqrt{5220}}\right\} \\ &\approx P\{Z > 1.384\} \\ &= 1 - \Phi(1.384) = .084 \end{aligned}$$

8.12. Let N denote the number of doctors who volunteer. Conditional on the event $N = i$, the number of patients seen is distributed as the sum of i independent Poisson random variables with common mean 30. Because the sum of independent Poisson random variables is also a Poisson random variable, it follows that the conditional distribution of X given that $N = i$ is Poisson with mean $30i$. Therefore,

$$E[X|N] = 30N \qquad \text{Var}(X|N) = 30N$$

As a result,

$$E[X] = E[E[X|N]] = 30E[N] = 90$$

Also, by the conditional variance formula,

$$\text{Var}(X) = E[\text{Var}(X|N)] + \text{Var}(E[X|N]) = 30E[N] + (30)^2\text{Var}(N)$$

Because

$$\text{Var}(N) = \frac{1}{3}(2^2 + 3^2 + 4^2) - 9 = 2/3$$

we obtain $\text{Var}(X) = 690$.

To approximate $P\{X > 65\}$, we would not be justified in assuming that the distribution of X is approximately that of a normal random variable with mean 90 and variance 690. What we do know, however, is that

$$P\{X > 65\} = \sum_{i=2}^{4} P\{X > 65|N = i\}P\{N = i\} = \frac{1}{3}\sum_{i=2}^{4}\overline{P}_i(65)$$

where $\overline{P}_i(65)$ is the probability that a Poisson random variable with mean $30i$ is greater than 65. That is,

$$\overline{P}_i(65) = 1 - \sum_{j=0}^{65} e^{-30i}(30i)^j/j!$$

Because a Poisson random variable with mean $30i$ has the same distribution as does the sum of $30i$ independent Poisson random variables with mean 1, it follows from the central limit theorem that its distribution is approximately normal with mean and variance equal to $30i$. Consequently, with X_i being a Poisson random variable with mean $30i$ and Z being a standard normal random variable, we can approximate $\overline{P}_i(65)$ as follows:

$$\begin{aligned}\overline{P}_i(65) &= P\{X > 65\}\\ &= P\{X \geq 65.5\}\\ &= P\left\{\frac{X - 30i}{\sqrt{30i}} \geq \frac{65.5 - 30i}{\sqrt{30i}}\right\}\\ &\approx P\left\{Z \geq \frac{65.5 - 30i}{\sqrt{30i}}\right\}\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{P}_2(65) &\approx P\{Z \geq .7100\} \approx .2389\\ \overline{P}_3(65) &\approx P\{Z \geq -2.583\} \approx .9951\\ \overline{P}_4(65) &\approx P\{Z \geq -4.975\} \approx 1\end{aligned}$$

leading to the result

$$P\{X > 65\} \approx .7447$$

If we would have mistakenly assumed that X was approximately normal, we would have obtained the approximate answer .8244. (The exact probability is .7440.)

8.13. Take logarithms and then apply the strong law of large numbers to obtain

$$\log\left[\left(\prod_{i=1}^{n} X_i\right)^{1/n}\right] = \frac{1}{n}\sum_{i=1}^{n}\log(X_i) \rightarrow E[\log(X_i)]$$

Therefore,

$$\left(\prod_{i=1}^{n} X_i\right)^{1/n} \rightarrow e^{E[\log(X_i)]}$$

CHAPTER 9

9.1. From axiom (iii), it follows that the number of events that occur between times 8 and 10 has the same distribution as the number of events that occur by time 2 and thus is a Poisson random variable with mean 6. Hence, we obtain the following solutions for parts (a) and (b):

(a) $P\{N(10) - N(8) = 0\} = e^{-6}$
(b) $E[N(10) - N(8)] = 6$

(c) It follows from axioms (ii) and (iii) that, from any point in time onward, the process of events occurring is a Poisson process with rate λ. Hence, the expected time of the fifth event after 2 P.M. is $2 + E[S_5] = 2 + 5/3$. That is, the expected time of this event is 3:40 P.M.

9.2. (a)

$$\begin{aligned}
P\{N(1/3) = 2|N(1) = 2\} \\
&= \frac{P\{N(1/3) = 2, N(1) = 2\}}{P\{N(1) = 2\}} \\
&= \frac{P\{N(1/3) = 2, N(1) - N(1/3) = 0\}}{P\{N(1) = 2\}} \\
&= \frac{P\{N(1/3) = 2\}P\{N(1) - N(1/3) = 0\}}{P\{N(1) = 2\}} \quad \text{(by axiom } (ii)) \\
&= \frac{P\{N(1/3) = 2\}P\{N(2/3) = 0\}}{P\{N(1) = 2\}} \quad \text{(by axiom } (iii)) \\
&= \frac{e^{-\lambda/3}(\lambda/3)^2/2!e^{-2\lambda/3}}{e^{-\lambda}\lambda^2/2!} \\
&= 1/9
\end{aligned}$$

(b)

$$\begin{aligned}
P\{N(1/2) \geq 1|N(1) = 2\} &= 1 - P\{N(1/2) = 0|N(1) = 2\} \\
&= 1 - \frac{P\{N(1/2) = 0, N(1) = 2\}}{P\{N(1) = 2\}} \\
&= 1 - \frac{P\{N(1/2) = 0, N(1) - N(1/2) = 2\}}{P\{N(1) = 2\}} \\
&= 1 - \frac{P\{N(1/2) = 0\}P\{N(1) - N(1/2) = 2\}}{P\{N(1) = 2\}} \\
&= 1 - \frac{P\{N(1/2) = 0\}P\{N(1/2) = 2\}}{P\{N(1) = 2\}} \\
&= 1 - \frac{e^{-\lambda/2}e^{-\lambda/2}(\lambda/2)^2/2!}{e^{-\lambda}\lambda^2/2!} \\
&= 1 - 1/4 = 3/4
\end{aligned}$$

9.3. Fix a point on the road and let X_n equal 0 if the nth vehicle to pass is a car and let it equal 1 if it is a truck, $n \geq 1$. We now suppose that the sequence $X_n, n \geq 1$, is a Markov chain with transition probabilities

$$P_{0,0} = 5/6, \quad P_{0,1} = 1/6, \quad P_{1,0} = 4/5, \quad P_{1,1} = 1/5$$

Then the long-run proportion of times is the solution of

$$\begin{aligned}
\pi_0 &= \pi_0(5/6) + \pi_1(4/5) \\
\pi_1 &= \pi_0(1/6) + \pi_1(1/5) \\
\pi_0 + \pi_1 &= 1
\end{aligned}$$

Solving the preceding equations gives

$$\pi_0 = 24/29 \qquad \pi_1 = 5/29$$

Thus, $2400/29 \approx 83$ percent of the vehicles on the road are cars.

9.4. The successive weather classifications constitute a Markov chain. If the states are 0 for rainy, 1 for sunny, and 2 for overcast, then the transition probability matrix is as follows:

$$\mathbf{P} = \begin{matrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{matrix}$$

The long-run proportions satisfy

$$\begin{aligned} \pi_0 &= \pi_1(1/3) + \pi_2(1/3) \\ \pi_1 &= \pi_0(1/2) + \pi_1(1/3) + \pi_2(1/3) \\ \pi_2 &= \pi_0(1/2) + \pi_1(1/3) + \pi_2(1/3) \\ 1 &= \pi_0 + \pi_1 + \pi_2 \end{aligned}$$

The solution of the preceding system of equations is

$$\pi_0 = 1/4, \quad \pi_1 = 3/8, \quad \pi_2 = 3/8$$

Hence, three-eighths of the days are sunny and one-fourth are rainy.

9.5. **(a)** A direct computation yields

$$H(X)/H(Y) \approx 1.06$$

(b) Both random variables take on two of their values with the same probabilities .35 and .05. The difference is that if they do not take on either of those values, then X, but not Y, is equally likely to take on any of its three remaining possible values. Hence, from Theoretical Exercise 13, we would expect the result of part (a).

CHAPTER 10

10.1. **(a)**

$$1 = C\int_0^1 e^x dx \Rightarrow C = 1/(e - 1)$$

(b)

$$F(x) = C\int_0^x e^y dy = \frac{e^x - 1}{e - 1}, \quad 0 \le x \le 1$$

Hence, if we let $X = F^{-1}(U)$, then

$$U = \frac{e^X - 1}{e - 1}$$

or

$$X = \log(U(e - 1) + 1)$$

Thus, we can simulate the random variable X by generating a random number U and then setting $X = \log(U(e - 1) + 1)$.

10.2. Use the acceptance–rejection method with $g(x) = 1, 0 < x < 1$. Calculus shows that the maximum value of $f(x)/g(x)$ occurs at a value of $x, 0 < x < 1$, such that

$$2x - 6x^2 + 4x^3 = 0$$

or, equivalently, when

$$4x^2 - 6x + 2 = (4x - 2)(x - 1) = 0$$

The maximum thus occurs when $x = 1/2$, and it follows that

$$C = \max f(x)/g(x) = 30(1/4 - 2/8 + 1/16) = 15/8$$

Hence, the algorithm is as follows:

Step 1. Generate a random number U_1.
Step 2. Generate a random number U_2.
Step 3. If $U_2 \leq 16(U_1^2 - 2U_1^3 + U_1^4)$, set $X = U_1$; else return to Step 1.

10.3. It is most efficient to check the higher probability values first, as in the following algorithm:

Step 1. Generate a random number U.
Step 2. If $U \leq .35$, set $X = 3$ and stop.
Step 3. If $U \leq .65$, set $X = 4$ and stop.
Step 4. If $U \leq .85$, set $X = 2$ and stop.
Step 5. $X = 1$.

10.4. $2\mu - X$

10.5. **(a)** Generate $2n$ independent exponential random variables with mean 1, $X_i, Y_i, i = 1, \ldots, n$, and then use the estimator $\sum_{i=1}^{n} e^{X_i Y_i}/n$.

(b) We can use XY as a control variate to obtain an estimator of the type

$$\sum_{i=1}^{n}(e^{X_i Y_i} + cX_i Y_i)/n$$

Another possibility would be to use $XY + X^2 Y^2/2$ as the control variate and so obtain an estimator of the type

$$\sum_{i=1}^{n}(e^{X_i Y_i} + c[X_i Y_i + X_i^2 Y_i^2/2 - 1/2])/n$$

The motivation behind the preceding formula is based on the fact that the first three terms of the MacLaurin series expansion of e^{xy} are $1 + xy + (x^2 y^2)/2$.