Cheynna Crowley
Bios 667 HW#1
Due 9/20/17

1)
Known $n_1 = 50$ $n_2 = 50$ $d = 3$

Test$_1$ = lead levels
Test 2 = differences from baseline
→ comment on each as a "treatment effect"
→ code: large sample, unequal variance two sample t-test

Test 1 – Lead levels
↳ data points = {lead0, lead1, lead4, lead6}
↳ two sample t-test is valid to test placebo vs. active trt (by definition) since independent obs. ✓

Test 2 – differences from baseline
↳ diff1 = lead1 – lead0
diff4 = lead4 – lead0
diff6 = lead6 – lead0
} datapoint = {diff1, diff4, diff6}

↳ Note: • different spacing between time points but all being compared to the same time point (baseline)
• baseline should be the same in both groups but can check

$\boxed{\text{Data}}$ Two Groups: Placebo and Active

For baseline (lead 0), $\mu_A = 26.54$ $\mu_P = 26.27$
$n_A = 50$ $n_P = 50$

↳ Two-Sample T-test
$H_0: \mu_A = \mu_P \Leftrightarrow \mu_A - \mu_P = 0$. Don't need all this
H_A: otw.

$T_{98} = 0.27$ p-value = 0.79

⇒ we cannot reject Ho and we conclude statistically equal means of lead0.

⇒ since baseline (lead0) is the same in placebo and active, two sample t-test is a valid test for differences from baseline. ✓

Question 2

$\mu = [\mu_{w0}\ \mu_{w1}\ \mu_{w4}\ \mu_{w6}] = [26.406\quad 19.091\quad 19.792\quad 22.204]$

$\mu_A = [\mu_{w0}\ \mu_{w1}\ \mu_{w4}\ \mu_{w6}] = [26.540\quad 13.522\quad 15.514\quad 20.762]$

$\mu_P = [\mu_{w0}\ \mu_{w1}\ \mu_{w4}\ \mu_{w6}] = [26.272\quad 24.660\quad 24.070\quad 23.646]$

$\boxed{\text{Active}}$ - Hotelling T^2 1 sample

Ho: there is a linear trend in the four elements of μ_A
H1: otw "no restriction at all"

$d = 4$
$n = 50$

Ho: $\mu_1 = \alpha$ "week 0"
$\mu_2 = \alpha + \beta$ "week 1"
$\mu_3 = \alpha + 4\beta$ "week 4"
$\mu_4 = \alpha + 6\beta$ "week 6"

$$\left.\begin{aligned} E[X_{i2} - X_{i1}] &= \beta \\ E[X_{i3} - X_{i2}] &= 3\beta \\ E[X_{i4} - X_{i3}] &= 2\beta \end{aligned}\right\} \Rightarrow \begin{aligned} Y_{i1} &= X_{i3} - 4X_{i2} + 3X_{i1} \\ Y_{i2} &= 3X_{i4} - 5X_{i3} + 2X_{i2} \end{aligned}$$

$Y = d - 2\times 1$
$= 4 - 2\times 1 = 2$

$F_{2,48} = 75.16$
p-value < 0.001

⇒ since p-value < 0.001, we reject the null and conclude there is not a linear trend in the four elements of μ_A. (active treatment group.

$\boxed{\text{Placebo}}$ - Hotelling T^2 1 sample

Ho: there is a linear trend in the four elements of μ_P
H1: otw "no restriction at all".

$d = 4$
$d = 50$

$F_{2,48} = 4.03$ p-value = 0.024

⇒ since the p-value = 0.024, we reject the null and conclude there is not a linear trend in the four elements of μ_P (placebo group).

2.1

$\boxed{\text{Active}}$ - Large one sample

H_0: there is a linear trend in the four elements of μ_A.

H_1: otw "no restriction"

$\chi^2_2 = 153.4504$

p-value < 0.001

⇒ since the p-value is less than alpha we reject Ho and conclude there is no linear trend in the four elements of μ_A.

$\boxed{\text{Placebo}}$ - Large one sample

H_0: there is a linear trend in the four elements of μ_P.

H_1: otw "no restrict"

$\chi^2_2 = 8.2232$

p-value = 0.016

⇒ since the p-value is less tha alpha, we reject Ho and conclude there is no linear trend in the four elements of μ_P.

3]. $X_{n\times m}$ w/ orthogonal columns and 1's in the first column. $\Rightarrow \langle X_{\cdot,i}, X_{\cdot,j} \rangle = 0$

$\hookrightarrow X^T X$ is diagonal

$\hookrightarrow$ orthogonal columns $\not\Leftrightarrow X^T = X^{-1}$

if W is positive definite $\Rightarrow$ all its eigenvalues are positive, symmetric, all pivots are positive.

Part 1 – what can we say about $\underset{m\times n}{X^T}\ \underset{n\times n}{W}\ \underset{n\times m}{X}$ let = some matrix M.

$$X = \begin{bmatrix} 1 & (x_{11}-\bar{x}_1) & \cdots \\ \vdots & & \\ 1 & (x_{11}-\bar{x}_1) & \end{bmatrix}_{n\times m} \quad W = \begin{bmatrix} d_1 & a & \\ a & d_2 & \\ & & d_i \end{bmatrix}_{n\times n}$$

$$X^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & & & \end{bmatrix}_{m\times n} \Rightarrow X^T W = \begin{bmatrix} & \end{bmatrix}_{m\times n}$$

- some new matrix
  - $\hookrightarrow$ could be transformed s.t. not diagonal since $X^T W$ is not the transpose of X where $X^T X$ is always a diagonal

$\rightarrow X^T X$ is semi positive definite

$X^T W X = M_{m\times m}$

$\rightarrow X^T W X$ is a square $m\times m$ matrix.

$\Rightarrow X^T W X$ isn't necessarily a diagonal matrix (possible but not always true)

part 2 – what can we say about $X^T W X$ if W has the form $W = aI + bJ$ where a and b are real numbers, I is idenity and J is a matrix of 1's

$$X^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times n}$$

$$X = \begin{bmatrix} 1 & \\ 1 & \end{bmatrix}_{n\times m}$$

$$W = a\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}_{n\times n} + \underbrace{b\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & & & 1 \end{bmatrix}_{n\times n}}$$

$X^T W = X^T(aI) + X^T(bJ)$ by distributive property

$$X^T W = \underbrace{\begin{bmatrix} 1 & \cdots & 1 \\ & & \end{bmatrix}_{m\times n} \begin{bmatrix} a & & \\ & a & \\ & & a \end{bmatrix}_{n\times m}}_{\text{scales } X^T} + \underbrace{\begin{bmatrix} 1 & \cdots & \\ & & \end{bmatrix}_{m\times n} \begin{bmatrix} b & b & \cdots & b \\ b & & & b \end{bmatrix}_{n\times}}_{\text{location parameter}}$$

$\Rightarrow X^T W$ is orthogonal rows but not necessarily 1's in the 1st row.

$\Rightarrow X^T W X$ is still a diagonal just change elements scaled by a and shifted by b.

Question 4

Part 1

→ In the one sample problem (~N) derive the MLE of the variance.

Let $X_i \sim$ Normal and assume the mean μ is known and the variance σ^2 is unknown

$$f(x_i; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$$L(\sigma^2, x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\ell(\sigma^2, x) = \log(L(\sigma^2))$$

$$\ell(\sigma^2, x) = \sum_{i=1}^{n} \left[-\frac{1}{2}\log(2\pi\sigma^2) - (x-\mu)^2/2\sigma^2 \right]$$

$$= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(x-\mu)^2}{2\sigma^2}$$

$$\frac{d}{d\,\ell(\sigma^2, x)} = -\frac{n}{2}\left(\frac{1}{\sigma^2}\right) - \frac{\sum_{i=1}^{n}(x-\mu)^2 (-1)(\sigma^2)^{-2}}{2}$$

set = 0

$$\frac{n}{2}\left(\frac{1}{\sigma^2}\right) = \frac{\sum_{i=1}^{n}(x-\mu)^2}{2}(\sigma^2)^{-2}$$

$$\hat{\sigma}^2_{MLE} = \frac{\sum_{i=1}^{n}(x-\mu)^2}{n}$$

Note: if μ is unknown, and given $\hat{\mu}_{MLE} = \bar{x}$ where $\bar{x} = \sum_{i=1}^{n} x_i / n$ then

- $\hat{\sigma}^2_{MLE} = \sum_{i=1}^{n}(x_i - \bar{x})^2 / n$
- $\hat{\sigma}^2_{MLE} = \sum_{i=1}^{n}(x_i - \hat{\mu})^2 / n$

Part 2

→ Derive the MLE based on the REML likelihood.

$$f_x(x=x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$\boxed{\text{note}}$ $\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu$

$$L(\sigma^2, x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\ell(\sigma^2, x) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2}$$

→ find likelihood for $\bar{x} = \sum x_i/n$

$$\bar{x} = \frac{1}{\sqrt{2\pi\sigma^2/n}} e^{-(\bar{x}-\mu)^2/2\sigma^2/n} = \sqrt{\frac{n}{2\pi\sigma^2}}\, e^{-n(\bar{x}-\mu)^2/2\sigma^2}$$

4 cont.

$\rightarrow$ log likelihood of $\bar{x}$

$$\ell(\sigma^2_{,}\bar{x}) = \frac{1}{2}\log(n) - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{n(\bar{x}-\mu)^2}{2\sigma^2}$$

then use this in $\ell(\sigma^2, x)$

recall,

$$\ell(\sigma^2, x) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2}$$

adding $\bar{x}$ log likelihood $\Rightarrow$

$$\Rightarrow -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(n) - \frac{1}{2}\log(\sigma^2) - \frac{n}{2}\log(\sigma^2)$$

$$- \frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \left[\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right]$$

$$\Rightarrow -\frac{n-1}{2}\log(2\pi) + \frac{1}{2}\log(n) - \frac{n-1}{2}\log(\sigma^2) - \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{2\sigma^2}$$

take derivative wrt σ^2

$$0 + 0 - \frac{n-1}{2}\frac{1}{\sigma^2} + \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{2}(\sigma^2)^{-2}$$ ✓

set $=0$

$$\Rightarrow \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{2}(\sigma^2)^{-2} = \frac{n-1}{2}\left(\frac{1}{\sigma^2}\right)$$

$$\Rightarrow \hat{\sigma}^2_{REML} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$$

conclusion: No, the $\hat{\sigma}^2_{MLE} \neq \hat{\sigma}^2_{REML}$. the REML takes into account the log-likelihood of $\bar{x} = \hat{\mu}_{MLE}$. $\hat{\sigma}^2_{MLE} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$ while $\hat{\sigma}^2_{REML} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$ ✓

check second derivatives -2

5) Question 5

Part 1: Test the null hypothesis that the mean is constant over time against an unrestricted alternative

Brock data contains 4 time points per subject [vocab1 vocab2 vocab3 vocab4]

μ_1 = mean vocab1
μ_2 = mean vocab2
μ_3 = mean vocab3
μ_4 = mean vocab4

$\mu_{diff_1} = \mu_2 - \mu_1$
$\mu_{diff_2} = \mu_3 - \mu_2$
$\mu_{diff_3} = \mu_4 - \mu_3$

Ho: the mean is constant over time
HA: otw.

Ho: "$\mu_{diff_1} = \mu_{diff_2} = \mu_{diff_3} = \beta$" or

$\mu_1 = \alpha$
$\mu_2 = \alpha + \beta$
$\mu_3 = (\alpha+\beta) + \beta$
$\mu_4 = (\alpha + \beta + \beta) + \beta$

$\Rightarrow$
$\mu_{diff_1} = \mu_2 - \mu_1 = \alpha + \beta - \alpha = \beta$
$\mu_{diff_2} = \mu_3 - \mu_2 = \alpha + 2\beta - (\alpha + \beta) = \beta$
$\mu_{diff_3} = \mu_4 - \mu_3 = \alpha + 3\beta - (\alpha + 2\beta) = \beta$

$F_{3,61} = 96.45$
p-value < 0.001

✓

$\Rightarrow$ since the p-value < 0.001 which is less than alpha of 0.05, we reject the null and conclude the mean is not constant over the four vocab time points in the brock data.

Part 2: which of the tests in the textbook is it?

- The way the data is set up implies MANOVA
- testing if the mean is constant over time a.k.a. no effect of time.

$\Rightarrow$ The test is called the multivariate test of time effect. (pg 38) ✗ Reported on page 43 −10