BIOS 660/BIOS 672 (3 Credits):
Probability and Statistical Inference I

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Notes 2

Intro: More on Events

- **Casella & Berger Definition**: An event is any collection of possible outcomes of an experiment, that is, any subset of Ω (including Ω itself). $\leftarrow$ not strictly true!
  Previously:
  - An event is necessarily a collection of possible outcomes of a random experiment (a set of elements in Ω, i.e. a subset of Ω)
  - For discrete (finite and countable) sample space, the set of possible events is the power set (2^Ω), i.e. any subset of Ω.
  - For continuous sample spaces (uncountably infinite spaces), the set of possible events is NOT the power set, there are subsets that are not considered events.
- Strictly speaking, **events** are the subsets of the sample space for which a probability is defined.

Intro: On Measures and Probability

A **measure** is a set function that assigns a number $\mu(A)$ to each set A <u>in a certain class of sets</u>.

Examples:

- Length
- Area
- Volume
- Probability

Some structure must be imposed on the class of sets in which the set function μ is defined (i.e. cannot take probability of any set in 2^Ω).

Classes of Sets

- **Definition:** A **class** is a collection of sets (set of sets) that satisfy some conditions. Usually denoted with script characters ($\mathcal{S}$, $\mathcal{X}$, $\mathcal{A}$, etc.)
- **Definition:** A class of sets $\mathcal{X}$ is closed under an operation (e.g. union, intersection, etc.) if when performed on any members of $\mathcal{X}$ yields a set which also belongs to the class.
- Example: $\mathcal{X} = \{\emptyset, A, B, C, A \cup B, A \cup C, B \cup C, A \cup B \cup C\}$ is closed under Union operation.

Fields

Definition: A class $\mathcal{X}$ of sets in Ω is called a **field** if

1. $\mathcal{X}$ is non-empty
2. $\mathcal{X}$ is closed under finite union
3. $\mathcal{X}$ is closed under complementation.

Examples:

- $\mathcal{X} = \{\emptyset, \Omega\}$ (trivial class)
- $\mathcal{X}$ = all subsets of Ω
- Let $\Omega = (-\infty, \infty)$, $\mathcal{X}_3$ = class of all finite intervals (a, b) where $a, b \in \mathbb{R}$.

Some Properties

Properties of a field $\mathcal{X}$:

1. $\mathcal{X}$ is also closed under finite intersection
2. $\emptyset \in \mathcal{X}$ and $\Omega \in \mathcal{X}$

σ-Fields

- **Definition:** A class $\mathcal{X}$ of sets is a σ**-field** if
  1. $\mathcal{X}$ is non-empty
  2. $\mathcal{X}$ is closed under countable unions
  3. $\mathcal{X}$ is closed under complementation
- Examples: (1) $\mathcal{X} = \{\emptyset, \Omega\}$ (2) $\mathcal{X}$ = all subsets of Ω are σ-fields
- Obviously, a σ-field is necessarily a field, but the converse does not hold: Consider class of all finite sets and sets whose complement is finite.
- **Theorem:** $\mathcal{X}_1$ and $\mathcal{X}_2$ are σ-fields, then $\mathcal{X}_1 \cap \mathcal{X}_2 = \{A : A \in \mathcal{X}_1$ and $\mathcal{X}_2\}$ is also a σ- field.

Example

Let $\Omega = \mathbb{R}$ and $\mathcal{A} = \{A : A$ or A^c is finite$\}$. Show that $\mathcal{A}$ is a field, but that $\mathcal{A}$ is not a σ-field. Note that $\mathcal{A}$ is the finite-cofinite field.

- Let $A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}, \cdots,$
- Then $\bigcup_{n=1}^{\infty} A_n = \mathbb{N} \notin \Omega$.

σ-field Generated by a Class of Sets

- **Theorem:** Given a class of sets, $\mathcal{S}$, not necessarily a σ-field, there is a minimum σ-field (denoted $\sigma(\mathcal{S})$ containing) it, i.e. $\sigma(\mathcal{S})$ is a class of sets that:
  - is a σ-field
  - it contains $\mathcal{S}$: if $A \in \mathcal{S}$, then it is also in $\sigma(\mathcal{S})$.
  - if $\mathcal{X}$ is another σ-field that contains $\mathcal{S}$, then $\mathcal{X}$ contains $\sigma(\mathcal{S})$.

  $\sigma(\mathcal{S})$ is also called the **σ-field generated by** $\mathcal{S}$.
- $\sigma(\mathcal{S})$ also defined as the intersection of all σ-fields that contain $\mathcal{S}$
- Properties:
  - $\sigma(\mathcal{S})$ is itself a σ-field
  - $\mathcal{S} \subset \sigma(\mathcal{S})$
  - $\mathcal{S}_1 \subset \mathcal{S}_2$ implies $\sigma(\mathcal{S}_1) \subset \sigma(\mathcal{S}_2)$
  - if $\mathcal{S}$ is itself a σ-field, then $\sigma(\mathcal{S}) = \mathcal{S}$

Borel σ-fields ($\mathcal{B}$)

- Let $\Omega = \mathbb{R} = (-\infty, \infty)$. We have 4 types of finite intervals:
  1. $\mathcal{S}_1 = \{[a, b) : a < b, a, b \in \mathbb{R}\}$,
  2. $\mathcal{S}_2 = \{(a, b) : a < b, a, b \in \mathbb{R}\}$,
  3. $\mathcal{S}_3 = \{(a, b] : a < b, a, b \in \mathbb{R}\}$,
  4. $\mathcal{S}_4 = \{[a, b] : a < b, a, b \in \mathbb{R}\}$
- Let $\mathcal{S} = \bigcup_{i=1}^4 \mathcal{S}_i$ = a class of all finite intervals. Note that $\mathcal{S}$ is neither a field nor a σ-field.
- We extend $\mathcal{S}$ to a σ-field through the following definition:

  Definition: The σ-field generated by $\mathcal{S}$ is called the **Borel σ-field on** $\mathbb{R}$ and is denoted by $\mathcal{B} = \sigma(\mathcal{S})$. Any set in $\mathcal{B}$ is called a **Borel Set**.
- Important: Do not try to characterize $\mathcal{B}$.
- We will return to this σ-field when we start talking about Random Variables.

Events... Again!

- Why do we care about σ-fields?
  They are key to understanding the definition of an event.
- **Definition:** A **measurable space** is a set Ω endowed with a σ-field $\mathcal{F}$ of subsets of Ω denoted by the pair $(\Omega, \mathcal{F})$.
- Sets in $\mathcal{F}$ are defined as **events**!
- In this class, we will often deal with the triplet $(\Omega, \mathcal{F}, P)$ where P is a probability measure on the space $(\Omega, \mathcal{F})$.
- For discrete (finite and countably infinite) Ω, $\mathcal{F} = 2^\Omega$, and for uncountable Ω, $\mathcal{F} = \mathcal{B}$.

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Notes 2.1

Examples for Sample Space and σ-fields

Definition: A σ-field on a set Ω is a collection of subsets of Ω that includes the empty subset, is closed under complement, and is closed under countable unions and countable intersections.

Examples for Sample Space and σ-fields:

1. **Experiment**: Toss a coin with a "Head" and a "Tail".
   Sample space Ω: {Head, Tail}
   σ-field on Ω: $\{\emptyset, \{\text{Head}\}, \{\text{Tail}\}, \{\text{Head, Tail}\}\}$.

2. **Experiment**: In a health survey among cancer patients, we ask each patient to report their perceived quality of life, categorized as {Very Good, Good, Poor}.
   Sample space Ω: {Very Good, Good, Poor}
   σ-field on Ω: $\{\emptyset, \{\text{Very Good}\}, \{\text{Good}\}, \{\text{Poor}\}, \{\text{Very Good, Good}\}, \{\text{Very Good, Poor}\}, \{\text{Good, Poor}\}, \{\text{Very Good, Good, Poor}\}\}$.

Examples (cont.)

3. **Experiment**: Toss 2 coins and record the results from both tosses.
   Sample space Ω: {HH, HT, TH, TT}
   σ-field on Ω: { $\emptyset$, {HH}, {HT}, {TH},{HT}, {HH, HT}, {HH, TH}, {HH, TT}, {HT, TH}, {HT, TT},{TH, TT},{HH, HT, TH}, {HH, HT, TT}, {HH, TH, TT}, {HT, TH, TT}, {HH, HT, TH, TT} }.

4. **Experiment**: Toss 2 coins and record the number of heads.
   Sample Space Ω:{0, 1, 2}
   σ-field on Ω: { $\emptyset$, {0},{1},{2},{0,1},{0,2},{1,2},{0,1,2} }.