Problem 1

(a)

$$X_1, \ldots, X_{n_1} \sim N(\mu_1, \sigma^2)$$
$$E(c \sum_{i=1}^{n_1-1} (X_{i+1} - X_i)^2) = \sigma^2$$
$$(X_{i+1} - X_i) \sim N(\mu_1 - \mu_1, 2\sigma^2)$$
$$\text{Let } Z = \frac{(X_{i+1} - X_i)}{\sqrt{2}\sigma}$$
$$Z \sim N(0,1) \quad Z^2 \sim \chi^2_1$$
$$\sum_{i=1}^{n_1-1} Z^2 \sim \chi^2_{n_1-1}$$
$$E(\sum Z^2) = n_1 - 1$$
$$E(\sum_{i=1}^{n_1-1} (X_{i+1} - X_i)^2) = (n_1 - 1)2\sigma^2$$
$$\frac{1}{2(n_1-1)} E(\sum_{i=1}^{n_1-1} (X_{i+1} - X_i)^2) = \sigma^2$$
$$c = \frac{1}{2(n_1-1)}$$

(b)

$$Y \sim N(\mu_2, \sigma^2) \quad X \perp Y$$
$$S_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \quad S_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$
$$S_p^2 = aS_1^2 + (1-a)S_2^2$$
$$\text{WTS: } E(S_p^2) = \sigma^2$$
$$E(aS_1^2 + (1-a)S_2^2)$$
$$= \left(\frac{a}{n_1 - 1}\right)(n_1 - 1)\sigma^2 + \left(\frac{1-a}{n_2-1}\right)(n_2 - 1)\sigma^2$$
$$= \sigma^2(a + 1 - a) = \sigma^2$$

(c)

$$\text{Since } (n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$$
$$Var(\frac{n_1-1}{\sigma^2}S_1^2) = 2(n_1-1)$$
$$Var(\frac{n_2-1}{\sigma^2}S_2^2) = 2(n_2-1)$$
$$Var(S_1^2) = \frac{2(n_1-1)}{(n_1-1)^2}\sigma^4 = \frac{2\sigma^2}{n_1-1}$$
$$Var(S_2^2) = \frac{2\sigma^2}{n_2-1}$$
$$Var(S_p^2) = Var(aS_1^2 + (1-a)S_2^2)$$
$$= a^2 Var(S_1^2) + (1-a)^2 Var(S_2^2) + 2a(1-a)Cov(S_1^2, S_2^2)$$
$$= a^2\left(\frac{2\sigma^2}{n_1-1}\right) + (1-a)^2\left(\frac{2\sigma^2}{n_2-1}\right) + 0$$
$$= \sigma^4\left[2\left(\frac{a^2}{n_1-1} + \frac{(1-a)^2}{n_2-1}\right)\right]$$
$$Var(S_p^2) = \sigma^4 g(a)$$
$$g(a) = 2\left(\frac{a^2}{n_1-1} + \frac{(1-a)^2}{n_2-1}\right)$$
$$g'(a) = 2\left(\frac{2a}{n_1-1} - \frac{2(1-a)}{n_2-1}\right) = 0$$
$$4\left(\frac{a}{n_1-1} - \frac{(1-a)}{n_2-1}\right) = 0$$
$$\frac{a}{n_1-1} = \frac{1-a}{n_2-1}$$
$$\frac{1-a}{a} = \frac{n_2-1}{n_1-1}$$
$$a = \frac{n_1-1}{n_1+n_2-2}$$
$$g''(a) = 4\left(\frac{1}{n_1-1} + \frac{1}{n_2-1}\right) > 0$$
$$\text{Thus } a = \frac{n_1-1}{n_1+n_2-2}$$
$$\text{Let } P = \sum_{i=1}^{n}(X_i - \bar{X})^2 \quad Q = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$$
$$S_p^2 = \frac{\frac{n_1-1}{n_1+n_2-2}}{n_1-1}P + \frac{\frac{n_2-1}{n_1+n_2-2}}{n_2-1}Q = \frac{1}{n_1+n_2-2}(P+Q)$$

Alternatively:

$$S_p^2 = a\frac{1}{n_1-1}\sum_{i=1}^{n}(X_i-\bar{X})^2 + (1-a)\frac{1}{n_2-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2$$

$$\text{Let } P = \sum_{i=1}^{n}(X_i-\bar{X})^2 \quad Q = \sum_{i=1}^{n}(Y_i-\bar{Y})^2$$

$$S_p^2 = \frac{a}{n_1-1}P + \frac{1-a}{n_2-1}Q = \frac{1}{n_1+n_2-2}(P+Q)$$

We have the following two equations which are equal to each other:

$$1) \quad \frac{a}{n_1-1} = \frac{1}{n_1+n_2-2}$$

$$2) \quad \frac{1-a}{n_2-1} = \frac{1}{n_1+n_2-2}$$

$$\text{From 1)} \quad a = \frac{n_1-1}{n_1+n_2-2}$$

$$\text{From 2)} \quad 1-a = \frac{n_2-1}{n_1+n_2-2}$$

$$S_p^2 = \frac{\frac{n_1-1}{n_1+n_2-2}}{n_1-1}P + \frac{\frac{n_2-1}{n_1+n_2-2}}{n_2-1}Q = \frac{1}{n_1+n_2-2}(P+Q)$$

Problem 2

(a)

$$\tau(\theta) = e^{-\theta}$$
$$L(\theta|x) = \prod_{i=1}^{n} \left(\frac{1}{x_i!}\right) \theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}$$
$$\propto \ell(\theta|x) = \sum_{i=1}^{n} x_i \log(\theta) - n\theta$$
$$\frac{\partial \ell}{\partial \theta} = \frac{\sum_{i=1}^{n} x_i}{\theta} - n = 0$$
$$\hat{\theta}_{MLE} = \bar{x}$$
$$\frac{\partial \ell}{\partial \theta^2}^2 = \frac{-\sum_{i=1}^{n} x_i}{\theta^2} < 0$$
Thus $\hat{\theta}$ is the MLE

By invariance property:
$$\tau(\hat{\theta})_{MLE} = e^{-\hat{\theta}_{MLE}} = e^{-\bar{x}}$$

(b)

(c)

(d)