Identity of Likelihood Score Statistic to Residual Score Statistic

For logistic regression, let

$\boldsymbol{X}_A = [\mathbf{1}, \boldsymbol{X}]$ denote primary model

$\boldsymbol{X}_E = [\mathbf{1}, \boldsymbol{X}, \boldsymbol{W}]$ denote expanded model

Assume both models have full rank

Let $\widehat{\boldsymbol{\beta}}_A$ denote mle for $\boldsymbol{X}_A$

Let $\widetilde{\boldsymbol{\beta}}_E = (\widetilde{\boldsymbol{\beta}}'_A, \widetilde{\boldsymbol{\beta}}'_W)'$ denote mle for $\boldsymbol{X}_E$

Let $\overline{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}'_A, \mathbf{0}')'$ denote restriction of $\widetilde{\boldsymbol{\beta}}_E$ to

have $\widehat{\boldsymbol{\beta}}_W = \mathbf{0}$ relative to hypothesis $\boldsymbol{\beta}_W = \mathbf{0}$

Let $\boldsymbol{U}(\boldsymbol{\beta}_E) = \frac{\delta lnL}{\delta \boldsymbol{\beta}_E} = \boldsymbol{X}'_E\{\boldsymbol{y} - \boldsymbol{D_n}\boldsymbol{\pi}(\boldsymbol{\beta}_E)\}$

Let $\boldsymbol{I}(\boldsymbol{\beta}_E) = -\frac{\delta^2 lnL}{\delta \boldsymbol{\beta}_E \delta \boldsymbol{\beta}'_E} = \boldsymbol{X}'_E \boldsymbol{D}_{\boldsymbol{V}(\boldsymbol{\beta}_E)} \boldsymbol{X}_E$

$\boldsymbol{V}(\boldsymbol{\beta}_E) = \{n_i \pi_i(\boldsymbol{\beta}_E)(1 - \pi_i(\boldsymbol{\beta}_E))\}$ as vector

$$\pi_i(\boldsymbol{\beta}_E) = \frac{exp(\boldsymbol{X}'_{iA}\boldsymbol{\beta}_A + \boldsymbol{W}'_i\boldsymbol{\beta}_W)}{1 + exp(\boldsymbol{X}'_{iA}\boldsymbol{\beta}_A + \boldsymbol{W}'_i\boldsymbol{\beta}_W)}$$

$$Q_S = [\boldsymbol{U}(\overline{\boldsymbol{B}})]'[\boldsymbol{I}(\overline{\boldsymbol{B}})]^{-1}[U(\overline{\boldsymbol{B}})]$$

$$\pi_i(\overline{\boldsymbol{\beta}}) = \frac{exp(\boldsymbol{X}'_{iA}\widehat{\boldsymbol{\beta}}_A)}{1 + exp(\boldsymbol{X}'_{iA}\widehat{\boldsymbol{B}}_A)} = \pi_i(\widehat{\boldsymbol{\beta}}_A)$$

$$\boldsymbol{U}(\overline{\boldsymbol{\beta}}) = \begin{bmatrix} \boldsymbol{X}'_A \\ \boldsymbol{W}' \end{bmatrix} \left[\boldsymbol{y} - \boldsymbol{D}_n\boldsymbol{\pi}(\widehat{\boldsymbol{\beta}}_A)\right] = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{W}'[\boldsymbol{y} - \boldsymbol{D}_n\boldsymbol{\pi}(\widehat{\boldsymbol{\beta}}_A)] \end{bmatrix} \text{ since } \boldsymbol{X}'_A\boldsymbol{y} = \boldsymbol{X}'_A\boldsymbol{D_n}\boldsymbol{\pi}(\widehat{\boldsymbol{\beta}}_A)$$

$$\boldsymbol{I}(\overline{\boldsymbol{\beta}}) = \begin{bmatrix} \boldsymbol{X}'_A \\ \boldsymbol{W}' \end{bmatrix} \boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{\beta}}_A)} \left[\boldsymbol{X}_A, \boldsymbol{W}\right] = \begin{bmatrix} \boldsymbol{X}'_A\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{\beta}}_A)}\boldsymbol{X}_A, & \boldsymbol{X}'_A\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{\beta}}_A)}\boldsymbol{W} \\ \boldsymbol{W}'\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{\beta}}_A)}\boldsymbol{X}_A, & \boldsymbol{W}'\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{\beta}}_A)}\widehat{\boldsymbol{W}} \end{bmatrix}$$

$$\begin{aligned} Q_S &= [\boldsymbol{U}(\overline{\boldsymbol{\beta}})]'[\boldsymbol{I}(\overline{\boldsymbol{\beta}})]^{-1}[\boldsymbol{U}(\overline{\boldsymbol{\beta}})] \\ &= [\mathbf{0}', *]\begin{bmatrix} * & * \\ * & * \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{0} \\ * \end{bmatrix} \end{aligned}$$

$$= \left[\boldsymbol{y} - \boldsymbol{D}_n\boldsymbol{\pi}(\widehat{\boldsymbol{B}}_A)\right]'\boldsymbol{W}\left\{\boldsymbol{W}'\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{B}}_A)}\boldsymbol{W} - \boldsymbol{W}'\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{B}}_A)}\boldsymbol{X}_A(\boldsymbol{X}'_A\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{B}}_A)}\boldsymbol{X}_A)^{-1}\boldsymbol{X}_A\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{B}}_A)}\boldsymbol{W}\right\}^{-1}\boldsymbol{W}'\left[\boldsymbol{y} - \boldsymbol{D}_n\boldsymbol{\pi}(\widehat{\boldsymbol{B}}_A)\right]$$

$$= \left[\boldsymbol{y} - \boldsymbol{D}_n\boldsymbol{\pi}(\widehat{\boldsymbol{B}}_A)\right]'\boldsymbol{W}\left\{\boldsymbol{W}'\left[\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{B}}_A)} - \boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{B}}_A)}\boldsymbol{X}_A(\boldsymbol{X}'_A\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{B}}_A)}\boldsymbol{X}_A)^{-1}\boldsymbol{X}_A\boldsymbol{D}_{\boldsymbol{V}(\widehat{\boldsymbol{B}}_A)}\right]\boldsymbol{W}\right\}^{-1}\boldsymbol{W}'\left[\boldsymbol{y} - \boldsymbol{D}_n\boldsymbol{\pi}(\widehat{\boldsymbol{B}}_A)\right]$$