2. Let $X \sim \text{Poisson}(\lambda)$. Show that the PMF $p_X(k)$ increases monotonically with k up to the point where k reaches the largest integer not exceeding λ and after that point decreases monotonically with k. What's def. of the point where k reaches the largest integer not exceeding λ?

$P_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ Consider two consecutive poisson prob. $P_X(k)$ and $P_X(k+1)$.

Then the ratio $\frac{P_X(k+1)}{P_X(k)} = \frac{\frac{\lambda^{k+1}}{(k+1)!}\cancel{e^{-\lambda}}}{\frac{\lambda^k}{k!}\cancel{e^{-\lambda}}} = \frac{\lambda}{k+1}$

$\frac{\lambda}{k+1} > 1$ when $k+1 < \lambda$ => increase monotonically up to largest integer $k < \lambda - 1$

$\frac{\lambda}{k+1} < 1$ when $k+1 > \lambda$ => decrease monotonically after the integer $k > \lambda - 1$

The def. of the point where k reaches the largest integer not exceeding λ is the number of events (occurrences) with the highest probability, which is the mode.

3. P(Pedestrian wait exactly 4 secs before starting to cross)?

Individual time units $X_1, X_2, \ldots \sim \text{Bernoulli}(p)$ where p is prob. of car passing

Pedestrian can only cross if no car is to pass during next 3 seconds

Let A = car passes, B = car does not pass.

$P(A) = p$ $P(B) = 1-p$

Then desired prob. = $P(\underbrace{\_\,\_\,\_}_{\text{at least one A}}ABBB)$

P(at least one A in first 3 secs) = 1 − P(all B in first 3 secs) = $1-(1-p)^3$

so $P(\underbrace{\_\,\_\,\_}_{\text{at least one A}}ABBB) = [1-(1-p)^3]\,p\,(1-p)^3$

4. Let X = # effective cases   Total of 100 cases

$P(X \geq 85 \mid$ new and old drugs are equally effective$)$?

So assume new and old drugs are equally effective $\Rightarrow P_{old} = P_{new} = 0.8$

$$P(X \geq 85 \mid p = 0.8) = \sum_{x=85}^{100} 0.8^{x} (1-0.8)^{100-x} = 0.1285$$

Since the prob. of observing 85 patients or more assuming old drug is as effective as the new drug is not very small, so we do not have enough evidence to say that new drug is superior than the old drug.

7. a) P(4 cocaine then 2 noncocaine) =

$$\frac{(\text{\# ways to select 4 cocaine})(\text{\# ways to select 2 noncocaine})}{(\text{\# ways to select 4 from } N+M)(\text{\# ways to select 2 from the rest})}$$

$$= \frac{\binom{N}{4}\binom{M}{2}}{\binom{N+M}{4}\binom{N+M-4}{2}}$$

b) Let $P(M,N) = \frac{\binom{N}{4}\binom{M}{2}}{\binom{N+M}{4}\binom{N+M-4}{2}}$

$$= \frac{\frac{N!M!}{(N-4)!\,4!\,(M-2)!\,2!}}{\frac{(N+M)!\,(N+M-4)!}{(N+M-4)!\,4!\,(N+M-6)!\,2!}}$$

$$= \frac{N!\,M!\,(N+M-6)!}{(N-4)!\,(M-2)!\,(M+N)!}$$

Then $P(M-1,N+1) = \frac{(N+1)!\,(M-1)!}{(N-3)!\,(M-3)!} \cdot \frac{(N+M-6)!}{(M+N)!}$ Second term is constant

$$= \frac{(N+1)(M-2)}{(N-3)(M)} P(M,N)$$

If $\frac{N+1}{N-3} > \frac{M}{M-2} \rightarrow P(M-1,N+1) > P(M,N)$

If $\frac{N+1}{N-3} < \frac{M}{M-2} \rightarrow P(M-1,N+1) < P(M,N)$

Let $f(x) = (N+1)(M-2) - M(N-3) = MN - 2N + M - 2 - MN + 3M = 1982 - 6N$

Set $1982 - 6N = 0 \rightarrow N = \frac{991}{3}$, But it must be integer, so $N = 331$

$M = 165$. This is the point it changes from an increasing function to a decreasing one $\rightarrow$ maximum, Prob? −1 let $n=331$ and $m=165$

then $PMF = \dfrac{\binom{331}{4}\binom{165}{2}}{\binom{496}{4}\binom{492}{2}} = .0220816809$

9.) If $X \sim NegBin(r, p)$

$$M_X(t) = \left(\frac{p}{1-e^t(1-p)}\right)^r$$

$$= \left(\frac{1 - e^t(1-p) - 1 + e^t(1-p) + p}{1 - e^t(1-p)}\right)^r$$

$$= \left(1 + \frac{p - 1 + e^t(1-p)}{1 - e^t(1-p)}\right)^r$$

$$= \left(1 + \frac{r(1-p)(e^t-1)}{r(1 - e^t(1-p))}\right)^r$$

$$= \left(1 + \frac{r(1-p)(e^t-1)}{r - e^t r(1-p)}\right)^r$$

$$\frac{r(1-p)(e^t-1)}{r - e^t r(1-p)} \longrightarrow \frac{\lambda(e^t-1)}{r - e^t\lambda} \quad \text{as } r(1-p) \to \lambda$$

$$M_X(t) = \left(1 + \frac{\lambda(e^t-1)}{r - e^t\lambda}\right)^r$$

$$\xrightarrow[r\to\infty]{} e^{\lambda(e^t-1)}$$

Note: $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$

$\lim_{n\to\infty} \left(1 + \frac{a}{n+c}\right)^n = e^a$

Which is the MGF for the Poisson distribution

10. a) $\Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t}\,dt$

let $u = t^\alpha$ $dv = e^{-t}$
$du = \alpha t^{\alpha-1}$ $v = -e^{-t}$

$= \left[t^\alpha e^{-t}\right]_0^\infty + \int_0^\infty \alpha t^{\alpha-1} e^{-t}\,dt$

$= 0 + \int_0^\infty \alpha t^{\alpha-1} e^{-t}\,dt$

$= \alpha\,\Gamma(\alpha)$

b) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt$

Let $t = \frac{x^2}{2}$, $dt = x dx$

$= \int_0^\infty \left(\frac{x^2}{2}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} x dx$

$= \int_0^\infty \frac{1}{\sqrt{\frac{x^2}{2}}} e^{-\frac{x^2}{2}} x dx$

$= \int_0^\infty \sqrt{2} e^{-\frac{x^2}{2}} dx$

But we know the standard normal distribution:

$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$

$\rightarrow \int_0^\infty e^{-\frac{x^2}{2}} dx = \frac{\sqrt{2\pi}}{2}$

So, $\Gamma\left(\frac{1}{2}\right) = \sqrt{2}\left(\frac{\sqrt{2\pi}}{2}\right) = \sqrt{\pi}$

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11. Let X be the number of occurrences that 6 appear

$\therefore X \sim Bin(1000, \frac{1}{6})$, $n = 1000$, $p = \frac{1}{6}$

$\therefore E(X) = np = \frac{1000}{6}$, $D(X) = np(1-p) = 1000 \times \frac{1}{6} \times \frac{5}{6} = \frac{5000}{6}$

$\therefore$ By Central Limit Theorem, $X \sim N(\frac{1000}{6}, \frac{5000}{36})$

$$\therefore P(150 \le X \le 200) \approx \phi\left(\frac{200 - E(X)}{\sqrt{D(X)}}\right) - \phi\left(\frac{150 - E(X)}{\sqrt{D(X)}}\right)$$

$$= \phi\left(\frac{200 - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}}\right) - \phi\left(\frac{150 - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}}\right)$$

$$= \phi(2\sqrt{2}) - \phi(-\sqrt{2})$$

$$\approx 0.9977 - (1 - 0.9207) = 0.9184$$

$\therefore$ The approximated probability that 6 will appear between 150 and 200 times is 0.9184.

14. $\because X \sim Exp(\lambda)$ $\therefore f_X(x) = \lambda \cdot e^{-\lambda x}$, $x \ge 0$

Let $Y = cX$, $c > 0$ $\therefore \frac{dx}{dy} = \frac{1}{c}$, $x = \frac{1}{c}y$

$$\therefore f_Y(y) = f_X(x = \tfrac{y}{c}) \cdot \left|\frac{dx}{dy}\right| = \lambda \cdot e^{-\lambda(\frac{1}{c}y)} \cdot \left|\frac{1}{c}\right| = \frac{\lambda}{c} \cdot e^{-\frac{\lambda}{c} \cdot y}, \quad y \ge 0$$

$\because f_Y(y) = \frac{\lambda}{c} e^{-\frac{\lambda}{c}y}$, $y \ge 0$ $\therefore Y \sim Exp(\frac{\lambda}{c})$

$\therefore cX \sim Exp(\frac{\lambda}{c})$.

15. $\because f_X(s) = \frac{\beta}{\alpha} \cdot \left(\frac{s-\nu}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{s-\nu}{\alpha}\right)^\beta\right\}$ for $s > \nu$

$\therefore F_X(s) = 1 - \exp\left[-\left(\frac{s-\nu}{\alpha}\right)^\beta\right]$, $s > \nu$.

$\therefore 1 - F(x) = \exp\left[-\left(\frac{x-\nu}{\alpha}\right)^\beta\right]$, $x > \nu$

$\therefore$ Let $\nu = 0$, $\log((1-F(x))^{-1}) = -\log(1-F(x)) = \left(\frac{x-\nu}{\alpha}\right)^\beta$

$$\log(\log((1-F(x))^{-1})) = \log\left(\left(\frac{x-\nu}{\alpha}\right)^\beta\right) = \beta \cdot \log \cdot \frac{x-\nu}{\alpha}$$

$$= \beta \cdot \log \frac{x-0}{\alpha} \quad \text{if } \nu = 0$$

$\therefore \log(\log((1-F(x))^{-1})) = \beta \cdot \log \frac{x}{\alpha} = \beta \cdot \log x - \beta \cdot \log \alpha$

$\therefore \log(\log((1-F(x))^{-1})) = \beta \cdot (\log x) - (\beta \log \alpha)$

thus, $\log(\log((1-F(x))^{-1})) = \beta \cdot \log x - \beta \cdot \log \alpha$ against $\log x$ is a straight line with slope β

15. continued.

$\because \nu = 0$, then $f_X(s) = \frac{\beta}{\alpha}\left(\frac{s}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{s}{\alpha}\right)^{\beta}\right\}$ for $s > 0$.

$\therefore F(x) = 1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]$, $x > 0$.

$\therefore P(X \le \alpha) = F_X(\alpha) = 1 - \exp\left[-\left(\frac{\alpha}{\alpha}\right)^{\beta}\right]$

$= 1 - e^{-1} \approx 0.632$

$\therefore$ Approximately 63.2% of all observations from such a distribution will be less than α.