1. According to Hardy-Weinberg law, if gene frequencies are in equilibrium, the chance of observing genotypes AA, Aa, aa in a population equals $(1-\theta)^2$, $2\theta(1-\theta)$, and θ^2, respectively. Let X_1, X_2, and X_3 denote the counts observed for blood type M, MN, and N, respectively, out of sample size n. Assuming Hardy–Weinberg law holds, one may assume X_1, X_2 and X_3 follow a multinomial distribution with probability density function
$$f(x_1, x_2, x_3|\theta) = \frac{n!}{x_1!x_2!x_3!}(1-\theta)^{2x_1}\{2\theta(1-\theta)\}^{x_2}\theta^{2x_3},$$
and $E(X_1) = n(1-\theta)^2$, $E(X_2) = 2n\theta(1-\theta)$, and $E(X_3) = n\theta^2$.

   (a) Find the maximum likelihood estimator of θ and comment on whether it is unbiased.

   (b) Find the Cramér–Rao lower bound (CRLB) for any unbiased estimator of θ.

   (c) Naively, one may use an unbiased estimator X_3/n to estimate θ^2. Show that
$$\sqrt{n}(X_3/n - \theta^2) \to_d N(0, \theta^2(1-\theta^2)),$$
and find σ^2 such that
$$\sqrt{n}(\sqrt{X_3/n} - \theta) \to_d N(0, \sigma^2).$$
Compare σ^2/n to the CRLB in (b) and comment on which one is smaller.

2. Let T_1 and T_2 be sufficient statistics for θ, and suppose that U be an unbiased estimator of θ. Let
$$V_1 = E(U|T_1),$$
$$V_2 = E(V_1|T_2).$$

   (a) Show that both V_1 and V_2 are unbiased estimators of θ.

   (b) Show that $\text{Var}(V_2) \leq \text{Var}(V_1)$.

3. In a certain laboratory experiment, the time X (in milliseconds) for a certain clotting agent to show an observable effect is assumed to have an exponential distribution
$$f(x|\beta) = \frac{1}{\beta}\exp(-x/\beta), \quad x > 0, \quad \beta > 0.$$
It is of interest to make statistical inferences about the unknown parameter $\theta = \beta^2$, which is the variance of X.

   (a) Develop an explicit expression for MLE $\hat{\theta}$ of θ.

(b) Find the uniformly minimum variance unbiased estimator (UMVUE) $\hat{\theta}^*$ of θ.

(c) Comment on whether the variance of $\hat{\theta}^*$ reaches CRLB.

(d) Derive the likelihood ratio test statistic $\lambda(\boldsymbol{x})$ of $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$.

(e) Show that the rejection region $R = \{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\}$ is equivalent to $R^* = \{\boldsymbol{x} : \bar{x} \leq c_1^* \text{ or } \bar{x} \geq c_2^*\}$.

Hint 1: If a random variable W follows Gamma(n, β), then, for $r > -n$,

$$E(W^r) = \frac{\Gamma(n+r)}{\Gamma(n)}\beta^r.$$

Hint 2: A function $g(y) = y^n \exp(-y)$ is a quadratic function of y.