Problem 1

(a)

$$X_1, \dots, X_n \sim U(0, \theta)$$
$$f_X(x) = \frac{1}{\theta} \quad 0 < \theta < \infty$$
$$X_{(1)}, \dots, X_{(n)} \text{ are order statistics}$$
$$Y_i = \frac{X_i}{X_{i+1}} \text{ for } i = 1, \dots, n-1$$
$$Y_n = X_{(n)}$$
$$Y_1 = \frac{X_{(1)}}{X_{(2)}} \quad Y_{n-1} = \frac{X_{(n-1)}}{X_{(n)}}$$
$$Y_n = X_{(n)}$$
$$X_{(n)} = Y_n \quad X_{(n-1)} = Y_n Y_{n-1} \ \cdots \ X_1 = Y_1 Y_2 \cdots Y_n$$
$$X_{(i)} = \prod_{j=i}^{n} Y_j$$
$$J = \begin{bmatrix} \frac{dx_1}{dy_1} & \dots & \frac{dx_1}{dy_n} \\ \frac{dx_2}{dy_1} & \dots & \dots \\ \dots & \dots & \dots \\ \frac{dx_n}{dy_1} & \dots & \frac{dx_n}{dy_n} \end{bmatrix}$$
$$= \begin{bmatrix} y_2 \cdots y_n & \dots & y_1 \cdots y_{n-1} \\ 0 & y_3 \cdots y_n & y_2 \cdots y_{n-1} \\ \dots & \dots & \dots \\ 0 & \dots & 1 \end{bmatrix}$$

J is an upper triangular matrix thus its determinant is the product of its main diagonal

$$J = |y_2 y_3^2 \dots y_n^{n-1}| = y_2 y_3^2 \dots y_n^{n-1}$$

(b)

$$f_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = f_{X_{(1)},\dots,X_{(n)}}\Big(\prod_{j=1}^{n}(y_j, \dots, y_n)\Big)|J|$$
$$= \frac{n!}{\theta^n}(y_2 y_3^2 \dots y_n^{n-1})$$

This can be factored into functions of y_i for $i = 1, \dots, n$

$$\text{Thus the } Ys \text{ are mutually independent}$$

(c)

$$\text{Since there is no } y_1 \text{ term:}$$
$$f_{Y_1}(y_1) = 1 \quad 0 < y_1 < 1$$
$$f_{Y_i}(y_i) = iy_i^{i-1} \quad 0 < y_i < 1$$
$$f_{Y_n}(y_n) = n\frac{y_n^{n-1}}{\theta^n}$$
$$F_{Y_n}(y_n) = \frac{y_n^n}{\theta^n} \quad 0 < y < \theta$$
$$\text{Let } Z_n = Y_n/\theta$$
$$F_Z(z) = P(Z \leq z) = P(Y_n \leq Z\theta) = F_y(z\theta)$$
$$F_y(z\theta) = \frac{(z\theta)^n}{\theta^n} = z^n = F_Z(z)$$
$$f_Z(z) = nz^{n-1} \quad 0 < z < 1$$
$$Beta(n,1) = \frac{\Gamma(n+1)}{\Gamma(n)\Gamma(1)} z^{n-1}(1-z)^{1-1} \quad 0 < z < 1$$
$$= \frac{n!}{n-1!} z^{n-1} = nz^{n-1}$$
$$Z \sim Beta(n,1)$$

Problem 2

(a)

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$
$$\text{Since } X_1, \ldots, X_n \text{ and } Y_1, \ldots, Y_n \text{ are random samples from } N(\mu, \sigma^2):$$
$$\bar{X} \sim N(\mu, \sigma^2/n) \quad \bar{Y} \sim N(\mu, \sigma^2/n)$$
$$\bar{X} - \bar{Y} \sim N(\mu - \mu, \sigma^2/n + (-\sigma)^2/n) = N(0, 2\sigma^2/n)$$

(b)

$$\text{WTS: } \lim_{n\to\infty} P(|\bar{X}-\bar{Y}|>\sigma)=0$$
$$P(|\bar{X}-\bar{Y}|>\sigma)=1-P(|\bar{X}-\bar{Y}|\leq\sigma)$$
$$=1-P(-\sigma\leq\bar{X}-\bar{Y}\leq\sigma)$$
$$=1-P\left(\frac{-\sigma}{\sqrt{2}\sigma/\sqrt{n}}\leq\frac{\bar{X}-\bar{Y}}{\sqrt{2}\sigma/\sqrt{n}}\leq\frac{\sigma}{\sqrt{2}\sigma/\sqrt{n}}\right)$$
$$1-P\left(-\sqrt{n/2}\leq\frac{\bar{X}-\bar{Y}}{\sqrt{2}\sigma/\sqrt{n}}\leq\sqrt{n/2}\right)$$
$$=1-(\Phi(\sqrt{n/2})-\Phi(-\sqrt{n/2}))$$
$$=1-\Phi(\sqrt{n/2})+\Phi(-\sqrt{n/2}))$$
$$\lim_{n\to\infty}1-\Phi(\sqrt{n/2})+\Phi(-\sqrt{n/2})$$
$$\lim_{n\to\infty}\Phi(\sqrt{n/2})=\Phi(\infty)=1\quad\lim_{n\to\infty}\Phi(-\sqrt{n/2})=\Phi(-\infty)=0$$
$$=1-1+0=0$$
$$\text{Thus } \lim_{n\to\infty}P(|\bar{X}-\bar{Y}|>\sigma)=0$$

Problem 3

(a)

$$X_1\sim pois(\lambda_1)\quad X_2\sim pois(\lambda_2)$$
$$X_1 \perp X_2$$
$$\text{WTS: } X_1|X_1+X_2\sim bin(n,p) \text{ where}$$
$$E(X_1|X_2+X_2=n)=\frac{n\lambda_1}{\lambda_1+\lambda_2}$$
$$p=\frac{\lambda_1}{\lambda_1+\lambda_2}$$
$$(X_1+X_2=n)\sim pois(n|\lambda_1+\lambda_2)$$
$$=\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}$$
$$f_{X_1,X_2}(x_1,x_2)=\frac{e^{-(\lambda_1+\lambda_2)}\lambda_1^{x_1}}{x_1!}\frac{\lambda_2^{x_2}}{x_2!}$$
$$P(X_1=x|X_1+X_2=n)=\frac{P(X_1=x,X_1+X_2=n)}{P(X_1+X_2=n)}$$

$$= \frac{P(X_1 = x)P(X_2 = n - x)}{P(X_1 + X_2 = n)}$$
$$= \frac{e^{-(\lambda_1+\lambda_2)}\lambda_1^x\lambda_2^{n-x}}{x!(n-x)!} / \frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}$$
$$= \binom{n}{x}\frac{e^{-(\lambda_1+\lambda_2)}\lambda_1^x\lambda_2^{n-x}}{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}$$
$$= \binom{n}{x}\frac{\lambda_1^x\lambda_2^{n-x}}{(\lambda_1+\lambda_2)^n}$$
$$= \binom{n}{x}\frac{\lambda_1^x\lambda_2^{n-x}}{(\lambda_1+\lambda_2)^n}\frac{(\lambda_1+\lambda_2)^{n-x}}{(\lambda_1+\lambda_2)^{n-x}}$$
$$= \binom{n}{x}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x$$
$$\text{Which is } bin(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$$
$$\text{Thus } E(X_1|X_2 + X_2 = n) = \frac{n\lambda_1}{\lambda_1+\lambda_2}$$

(b)

$$\text{WTS: whether } T(X) = \sum_{i=1}^{2} X_i = X_1 + X_2 \text{ is an SS for } \lambda_1$$
$$P(X = x, T(X) = t) = \frac{P(X = x, X_1 + X_2 = t)}{P(X_1 + X_t}$$
$$= \frac{P(X_1 = x, X_2 = x_2)}{P(X_1 + X_2 = t)}$$
$$= \frac{\lambda_1^{x_1}e^{-\lambda_1}\lambda_2^{x_2}e^{-\lambda_2}}{x_1!x_2!} / \frac{(\lambda_1+\lambda_2)^t e^{-(\lambda_1+\lambda_2)}}{t!}$$
$$= \frac{t!}{x_1!x_2!}\frac{\lambda_1^{x_1}\lambda_2^{x_2}}{(\lambda_1+\lambda_2)^t}$$
$$\text{This includes } \lambda_1$$
$$\text{Thus } (X_1 + X_2) \text{ not } \perp \lambda_1$$
$$X_1 + X_2 \text{ not SS of } \lambda_1$$

Problem 4

(a)

$$
\begin{aligned}
f_X(x) &= \frac{x^{\beta-1}e^{-x}}{\Gamma(\beta)} \quad x>0, \quad \beta>1 \\
f_T(t|X=x) &= \theta x e^{-\theta x t} \quad t>0, \quad \theta>0 \\
f_T(t) &= \int_0^\infty f_{T,X}(t,x)\ \mathrm{d}x \\
&= \int_0^\infty f_T(t|X=x) f_X(x)\ \mathrm{d}x \\
&= \int_0^\infty \theta x e^{-\theta x t}\frac{x^{\beta-1}e^{-x}}{\Gamma(\beta)}\ \mathrm{d}x \\
&= \frac{\theta}{\Gamma(\beta)}\int_0^\infty x^{\beta}e^{-(\theta t+1)x}\ \mathrm{d}x
\end{aligned}
$$

Want to get integral in the form of $gamma(\beta+1, 1/(\theta t+1))$

$$
\begin{aligned}
&= \frac{\Gamma(\beta+1)\theta}{\Gamma(\beta)(\theta t+1)^{\beta+1}}\int_0^\infty \frac{(\theta t+1)^{\beta+1}}{\Gamma(\beta+1)}x^{\beta}e^{-(\theta t+1)x}\ \mathrm{d}x \\
&= \frac{\Gamma(\beta+1)\theta}{\Gamma(\beta)(\theta t+1)^{\beta+1}} \\
f_T(t) &= \frac{\beta\theta}{(\theta t+1)^{\beta+1}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\text{WTS: } E(T) &= \frac{1}{\theta(\beta-1)} \\
E(T) &= E(E(T|X=x)) \\
E(T|X=x) &= 1/(\theta X) \text{ giving us:} \\
&= E(1/(\theta X)) = \frac{1}{\theta}E(X^{-1}) \\
E(X^{-1}) &= \int_0^\infty x^{-1}f_X(x)\ \mathrm{d}x \\
&= \int_0^\infty \frac{x^{\beta-2}e^{-x}}{\Gamma(\beta)}\ \mathrm{d}x \\
&= \frac{\Gamma(\beta-1)}{\Gamma(\beta)}\int_0^\infty \frac{x^{\beta-2}e^{-x}}{\Gamma(\beta-1)}\ \mathrm{d}x \\
&= \frac{\Gamma(\beta-1)}{\Gamma(\beta)} \\
&= \frac{(\beta-2)!}{(\beta-1)!}
\end{aligned}
$$

$$E(T) = \frac{1}{\theta} E(X^{-1}) = \frac{1}{\theta(\beta - 1)}$$

(c)

$$\text{Given } \beta = 2 \quad E(T) = \frac{1}{\theta(2-1)} = \frac{1}{\theta}$$

$$\text{and } \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, 2\theta^2) \quad T_n = \hat{\theta}$$

$$\text{Using delta method: } \sqrt{n}\{g(T_n) - g(\theta)\} \xrightarrow{d} N(0, \{g^{'}(\theta)\}^2 \sigma^2)$$

$$g(\theta) = \frac{1}{\theta} \quad g^{'}(\theta) = -\theta^{-2}$$

$$\sqrt{n}(1/\hat{\theta} - 1/\theta) \xrightarrow{d} N(0, \theta^{-4} 2\theta^2) = N(0, 2\theta^{-2})$$

$$\sigma^2 = 2\theta^{-2}$$