HW #7

1.) $$M_X(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} c$$

$$= 1 + c\left[\sum_{n=0}^{\infty} \frac{t^n}{n!} - 1\right]$$

$$= 1 + c(e^t - 1)$$

$$M_X(t) = ce^t + 1 - c$$

Which is the MGF for Bernoulli(c)

2.) $$E[X^n] = \frac{2^n}{n+1} \quad \forall\ n \geq 1$$

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n (2^n)}{n!(n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(2t)^n}{(n+1)!}$$

$$= \frac{1}{2t} \sum_{n=0}^{\infty} \frac{(2t)^{n+1}}{(n+1)!} \qquad \text{Let } i = n+1$$

$$= \frac{1}{2t} \sum_{i=1}^{\infty} \frac{(2t)^i}{i!}$$

$$= \frac{1}{2t}\left[\sum_{i=0}^{\infty} \frac{(2t)^i}{i!} - 1\right]$$

$$= \frac{1}{2t}\left[e^{2t} - 1\right] \qquad M_X(t) = \begin{cases} \frac{e^{2t}-1}{2t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

$M_X(0) = 1$ due to MGF restriction

$X \sim Unif(0,2)$

3) a.) Let $Y = -X$ $\quad g^{-1}(y) = -y$

$$\left|\frac{d\,g^{-1}(y)}{dy}\right| = |-1| = 1$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d\,g^{-1}(y)}{dy}\right|$$

$$= f_X(-y)$$

$$= f_X(y)$$

Therefore X & Y have the same distribution.

b.) $M_X(0+\varepsilon) = M_X(\varepsilon) = \int_{-\infty}^{\infty} e^{\varepsilon x} f_X(x)\,dx$

$$= \int_{0}^{\infty} e^{\varepsilon x} f_X(x)\,dx + \int_{-\infty}^{0} e^{\varepsilon x} f_X(x)\,dx$$

Changing integration $\quad = \int_{-\infty}^{0} e^{-\varepsilon x} f_X(-x)\,dx + \int_{0}^{\infty} e^{-\varepsilon x} f_X(-x)\,dx$

$$= \int_{-\infty}^{\infty} e^{-\varepsilon x} f_X(-x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\varepsilon x} f_X(x)\,dx \quad f_X \text{ Symmetric}$$

$$= M_X(-\varepsilon)$$

$= M_X(0-\varepsilon)$ $\quad$ Therefore $M_X(t)$ is symmetric about 0.

Question 4

There does not exist a distribution in which $M_X(t) = t/(1-t)$, $|t| < 1$

Since $M_X(t) = E(e^{tX})$,

then $M_X(0) = E(e^{0X})$

$= E(1)$

$= 1$

However, when $t=0$, $t/(1-t) = 0/(1-0) = 0 \neq 1$

Therefore, there does not exist such an mgf.

Question 5

$\frac{d}{dt} S(t)\big|_{t=0} = \frac{d}{dt}\left(\log M_X(t)\right)\big|_{t=0}$

$= \frac{M_X'(t)}{M_X(t)}\big|_{t=0}$

$= \left(\frac{d}{dt} M_X(t)\big|_{t=0}\right) / \left(M_X(t)\big|_{t=0}\right)$

$= (E(X)) / E(e^0)$

$= E(X)/1$

$= E(X)$

$\frac{d^2}{dt^2} S(t)\big|_{t=0} = \frac{d}{dt}\left(\frac{M_X'(t)}{M_X(t)}\big|_{t=0}\right)$

$= \left(\frac{M_X''(t)}{M_X(t)} - \frac{(M_X'(t))^2}{(M_X(t))^2}\right)\big|_{t=0}$

$= E(X^2)/1 - (E(X)/1)^2$

$= E(X^2) - (E(X))^2$

$= Var(X)$.

Question 6

(a) $M_X(t) = E(e^{tX})$

$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} \cdot e^{tx}$

$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$

$= e^{-\lambda} e^{\lambda e^t}$

$= e^{\lambda e^t - \lambda}$

$= e^{\lambda(e^t - 1)}$

$E(X) = \frac{d}{dt} e^{\lambda(e^t-1)}\big|_{t=0} = \lambda e^t e^{\lambda(e^t-1)}\big|_{t=0} = \lambda(1)(1) = \lambda$

$$E(X^2) = \frac{d}{dt}\, \lambda e^t e^{\lambda(e^t-1)} \Big|_{t=0}$$
$$= \lambda e^t e^{\lambda(e^t-1)} + (\lambda e^t)^2 e^{\lambda(e^t-1)} \Big|_{t=0}$$
$$= \lambda(1)(1) + (\lambda(1))^2(1)$$
$$= \lambda + \lambda^2$$

Therefore, $Var(X) = E(X^2) - (E(X))^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$

(b) $M_X(t) = E(e^{tX})$
$$= \sum_{x=0}^{\infty} e^{tx} p(1-p)^x$$
$$= p \sum_{x=0}^{\infty} (e^t(1-p))^x$$
$$= (p)\left(\frac{1}{1-e^t(1-p)}\right)$$
$$= \frac{p}{1-e^t(1-p)}$$

$$E(X) = \frac{d}{dt} \frac{p}{1-e^t(1-p)} \Big|_{t=0} = \frac{-p}{(1-(1-p)e^t)^2}(-(1-p)e^t) \Big|_{t=0}$$
$$= \frac{-p}{(1-1+p)^2}(-1+p) = -\frac{1}{p}(-(1-p)) = \frac{1-p}{p}$$

$$E(X^2) = \frac{d}{dt}\left(\frac{-p}{(1-(1-p)e^t)^2}(-(1-p)e^t)\right) \Big|_{t=0}$$
$$= \frac{(1-(1-p)e^t)^2(p(1-p)e^t) + p(1-p)e^t\, 2(1-(1-p)e^t)(1-p)e^t}{(1-(1-p)e^t)^4} \Big|_{t=0}$$
$$= \frac{p(1-p) + 2(1-p)^2}{p^2}$$

therefore, $Var(X) = \frac{p(1-p)+2(1-p)^2}{p^2} - \frac{(1-p)^2}{p^2} = \frac{p(1-p)+(1-p)^2}{p^2} = \frac{1-p}{p^2}$

(c) $M_X(t) = E(e^{tX})$
$$= \int_{-\infty}^{\infty} \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\,\sigma} e^{tx}\, dx$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)} e^{tx}\, dx$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{(x-(\mu+\sigma^2 t))^2 - (2\mu\sigma^2 t + (\sigma^2 t)^2}\, dx$$
$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$E(X) = \frac{d}{dt} e^{\mu t + \frac{\sigma^2 t^2}{2}} \Big|_{t=0} = (\mu + \sigma^2 t) e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0}.$$
$$= (\mu + 0)e^0 = \mu$$

$$E(X^2) = \frac{d}{dt} (\mu + \sigma^2 t) e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0}$$
$$= (\mu + \sigma^2 t)^2 e^{\mu t + \sigma^2 t^2/2} + e^{\mu t + \sigma^2 t/2} \cdot \sigma^2 \Big|_{t=0}$$
$$= (\mu + 0)^2 e^0 + e^0 \sigma^2 = \mu^2 + \sigma^2$$

therefore, $Var(X) = E(X^2) - (E(X))^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

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7\. C&B 2.36

$\because f(x) = \frac{1}{\sqrt{2\pi}x} \cdot e^{-\frac{1}{2}(\ln x)^2}, \quad 0 \le x < \infty$

$\therefore$ MGF: $E(e^{tX}) = \int_0^{+\infty} f(x) \cdot e^{tx} dx = \int_0^{+\infty} \frac{1}{\sqrt{2\pi} \cdot x} \cdot e^{tx - \frac{1}{2}(\ln x)^2} dx$

$\because \lim_{x\to\infty} \frac{tx - \frac{1}{2}(\ln x)^2}{tx}$ is in $\frac{\infty}{\infty}$ form

$\therefore \lim_{x\to\infty} \frac{tx - \frac{1}{2}(\ln x)^2}{tx} = \lim_{x\to\infty} \frac{t - (\ln x)\cdot\frac{1}{x}}{t}$ by L'Hospital Rules.

$\because \lim_{x\to\infty} (\ln x)\cdot\frac{1}{x}$ is in $\frac{\infty}{\infty}$ form, $\therefore \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

$\therefore \lim_{x\to\infty} \frac{tx - \frac{1}{2}(\ln x)^2}{tx} = \lim_{x\to\infty} \frac{t - (\frac{\ln x}{x})}{t} = \frac{t}{t} = 1$

$\therefore \lim_{x\to\infty} tx - \frac{1}{2}(\ln x)^2 = \lim_{x\to\infty} tx = \infty$

$\therefore \forall k>0, \exists c$, c is constant. $\int_k^{+\infty} \frac{1}{\sqrt{2\pi}x} \cdot e^{tx - \frac{1}{2}(\ln x)^2} dx \geq c \cdot \int_k^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} = \frac{1}{\sqrt{2\pi}} \ln x \Big|_k^{\infty} = \infty$

$\therefore \int_k^{+\infty} \frac{1}{\sqrt{2\pi}\cdot x} \cdot e^{-\frac{1}{2}(\ln x)^2} \cdot d\, e^{tx} dx = \infty$, does not exist.

8\. C&B 2.38

(a). $\because f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad x = 0, 1, 2, \cdots$

$\therefore$ MGF: $E(e^{tX}) = \sum_{x=0}^{\infty} f(x) \cdot e^{tx} = \sum_{x=0}^{\infty} \binom{r+x-1}{x} \cdot p^r (1-p)^x \cdot e^{tx}$

$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} \cdot p^r [(1-p)\cdot e^t]^x$

$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} \cdot (1-(1-p)\cdot e^t)^r ((1-p)e^t)^x \cdot \left(\frac{p}{1-(1-p)e^t}\right)^r$

$= \left(\frac{p}{1-(1-p)e^t}\right)^r \cdot \left[\sum_{x=0}^{\infty} \binom{r+x-1}{x} \cdot (1-(1-p)e^t)^r ((1-p)e^t)^x\right]$

$= \left(\frac{p}{1-(1-p)e^t}\right)^r \cdot 1 = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad \frac{p}{1-(1-p)e^t} < 1 \Rightarrow t > 0$

$\therefore$ MGF of X is $M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t > 0$

(b). $\because Y = 2pX \quad \therefore$ MGF of Y is $M_Y(t) = E(e^{tY}) = E(e^{2ptX})$

$\therefore M_Y(t) = E(e^{2ptX}) = \sum_{x=0}^{\infty} f(x) \cdot e^{2ptx} = \sum_{x=0}^{\infty} \binom{r+x-1}{x} \cdot p^r (1-p)^x \cdot e^{2ptx}$

$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} \cdot p^r \cdot ((1-p)\cdot e^{2pt})^x = \sum_{x=0}^{\infty} \binom{r+x-1}{x} \cdot (1-(1-p)e^{2pt})^r ((1-p)\cdot e^{2pt})^x \left(\frac{p}{1-(1-p)e^{2pt}}\right)^r$

$= \left(\frac{p}{1-(1-p)e^{2pt}}\right)^r \cdot \left[\sum_{x=0}^{\infty} \binom{r+x-1}{x} \cdot (1-(1-p)e^{2pt})^r ((1-p)e^{2pt})^x\right]$

$= \left(\frac{p}{1-(1-p)e^{2pt}}\right)^r \cdot 1 = \left(\frac{p}{1-(1-p)e^{2pt}}\right)^r$ 이고

8. (b) continued

$\because$ MGF of Y is $M_Y(t) = \left(\frac{p}{1-(1-p)e^{2pt}}\right)^r$

$\therefore \lim_{p\to 0} M_Y(t) = \lim_{p\to 0}\left(\frac{p}{1-(1-p)e^{2pt}}\right)^r$

$\because \lim_{p\to 0}\frac{p}{1-(1-p)e^{2pt}}$ is in $\frac{0}{0}$ form. by L'Hospital Rules,

$$\lim_{p\to 0}\frac{p}{1-(1-p)e^{2pt}} = \lim_{p\to 0}\frac{1}{e^{2pt}+(p-1)\cdot 2t\cdot e^{2pt}} = \frac{1}{1-2t}$$

$\therefore \lim_{p\to 0} M_Y(t) = \left(\frac{1}{1-2t}\right)^r$,

$\because M_Y(t) = E(e^{ty}) > 0$, $\therefore \frac{1}{1-2t} > 0$, $0<t<\frac{1}{2}$

$\therefore \lim_{p\to 0} M_Y(t) = \left(\frac{1}{1-2t}\right)^r$, $0<t<\frac{1}{2}$

9. (a) $\because X\sim ber(p)$ $\therefore P(X=x) = p^x\cdot(1-p)^{1-x}$, $x\in\{0,1\}$, $0\le p\le 1$

$\therefore \phi_X(t) = E(e^{itX}) = \sum_{x=0}^{1} P(X=x)\cdot e^{itx}$

$= p\cdot e^{it\cdot 1} + (1-p)\cdot e^{it\cdot 0} = p\cdot e^{it} + 1 - p$

$\therefore$ Let $q=1-p$, $\phi_X(t) = q + p\cdot e^{it}$, $0\le p\le 1, q=1-p$.

(b) $\because X\sim bin(n,p)$ $\therefore P(X=x) = \binom{n}{x}\cdot p^x\cdot q^{n-x}$, $x=0,1,2,\cdots,n$, $q=1-p$, $0\le 1\le p$

$\therefore \phi_X(t) = E(e^{itX}) = \sum_{x=0}^{n} P(X=x)\cdot e^{itx}$

$= \sum_{x=0}^{n}\binom{n}{x}\cdot p^x q^{n-x}\cdot e^{itx} = \sum_{x=0}^{n}\binom{n}{x}\cdot(p\cdot e^{it})^x\cdot q^{n-x} = (p\cdot e^{it}+q)^n$

$\therefore \phi_X(t) = (q+pe^{it})^n$

(c) $\because X\sim Geo(p)$ $\therefore P(X=x) = (1-p)^x\cdot p$, $x=0,1,2,3,\cdots$

$\therefore \phi_X(t) = \sum_{x=0}^{\infty} P(X=x)\cdot e^{itx} = \sum_{x=1}^{\infty}(1-p)^x\cdot p\cdot e^{itx}$

$= \sum_{x=0}^{\infty}((1-p)\cdot e^{it})^x\cdot p = \sum_{x=0}^{\infty}((1-p)e^{it})^x\cdot(1-(1-p)e^{it})\cdot\frac{p}{1-(1-p)e^{2t}}$

$= \frac{p}{1-(1-p)e^{it}}\cdot\left[\sum_{x=0}^{\infty}((1-p)e^{it})^{x+1}\cdot(1-(1-p)e^{it})^1\right]$

$= \frac{p}{1-qe^{it}}\cdot 1$, where $q=1-p$

$\therefore \phi_X(t) = \frac{p}{1-qe^{it}}$

9. (d) $\because X \sim Pois(m)$, $\therefore P(X=x) = \dfrac{e^{-m} \cdot m^{x}}{x!}$, $x = 0, 1, 2, \cdots$

$\therefore \phi_X(t) = E(e^{itX}) = \sum_{x=0}^{\infty} P(X=x) \cdot e^{itx}$

$= \sum_{x=0}^{\infty} \dfrac{e^{-m} \cdot m^{x}}{x!} \cdot e^{itx} = \sum_{x=0}^{\infty} \dfrac{e^{-m} \cdot (m \cdot e^{it})^{x}}{x!}$

$= \sum_{x=0}^{\infty} \dfrac{e^{-(me^{it})} (m \cdot e^{it})^{x}}{x!} \cdot e^{m(e^{it}-1)}$

$= e^{m(e^{it}-1)} \cdot \left[\sum_{x=0}^{\infty} \dfrac{e^{-(me^{it})} \cdot (me^{it})^{x}}{x!} \right] = e^{m(e^{it}-1)} \cdot 1$

$= e^{m(e^{it}-1)}$

$\therefore \phi_X(t) = e^{m(e^{it}-1)}$