BIOS 660/BIOS 672 (3 Credits):
Probability and Statistical Inference I

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Independence

The random variables X and Y are said to be *independent* if for any two Borel sets A and B,

$$P(X \in A,\ Y \in B) = P(X \in A)\, P(Y \in B)$$

All events defined in terms of X are independent of all events defined in terms of Y.

Using the Kolmogorov axioms of probability, it can be shown that X and Y are independent if and only if $\forall(x, y)$ (*except possibly for sets of prob. 0*)

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

or in terms of *pmfs* (discrete) and *pdf*'s (continuous)

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

* Check previous examples.

Checking independence

1. A necessary condition for independence of X and Y is that their joint pdf/pmf has positive probability on a rectangular domain.
2. If the domain is rectangular, one can try to write the joint pdf/pmf as a product of functions of x and y only.

Lemma C-B 4.2.7: Let (X, Y) be a bivariate random vector with joint pdf or pmf $f(x, y)$. Then X and Y are independent if and only if there exist functions $g(x)$ and $h(y)$ such that for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$f(x, y) = g(x)h(y)$$

<u>Proof:</u>

Example

Two points are selected randomly on a line of length a so as to be on opposite sides of the mid-point of the line. Find the probability that the distance between them is less than $a/3$.

Solution:
Let X be the coordinate of a point selected randomly in $[0, a/2]$ and Y the coordinate of a point selected randomly in $[a/2, a]$. Assume X and Y are independent and uniform over its interval. The joint density is:

$$f_{X,Y}(x, y) = 4/a^2, \qquad 0 \le x \le a/2, \;\; a/2 \le y \le a$$

Hence, the solution is

$$P(Y - X < a/3) = \int_{a/6}^{a/2} \int_{a/2}^{a/3+x} \frac{4}{a^2} dy dx = 2/9$$

Example: Buffon's Needle

A table is ruled with lines distance 1 unit apart. A needle of length $L \le 1$ is thrown randomly on the table. What is the probability that the needle intersects a line?

Solution: Define two random variables:

- X: distance from low end of the needle to the nearest line above
- θ: angle from the vertical to the needle.

Example (cont.)

By "random", we assume X and θ are independent, and

$$X \sim U(0,1) \quad \text{and} \quad \theta \sim U[-\pi/2, \pi/2].$$

This means that

$$f_{X,\Theta}(x,\theta) = 1/\pi, \qquad 0 \le x \le 1, \quad -\pi/2 \le \theta \le \pi/2$$

For the needle to intersect a line, we need $X < L\ \cos(\theta)$. So,

$$\begin{aligned} P(\text{needle intersects a line}) &= P(X < L\cos(\theta)) \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{L\cos\theta} \frac{1}{\pi} dx d\theta \\ &= \frac{2L}{\pi} \end{aligned}$$

Expectations of Independent RVs

Theorem C-B 4.2.10 Let X and Y be independent rvs.

- For any $A \subset \mathcal{R}$ and $B \subset \mathcal{R}$,

  $$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

  i.e. the events $\{X \in A\}$ and $\{Y \in B\}$ are independent.
  (C-B write it as a theorem, we took it as definition.)

- Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then

  $$\mathsf{E}(g(X)h(Y)) = (\mathsf{E}g(X))(\mathsf{E}h(Y))$$

Example: X, Y indep.

$$\mathsf{E}(X^2Y^3) = (\mathsf{E}X^2)(\mathsf{E}Y^3)$$
$$\mathsf{E}(Y^2Y^3) \neq (\mathsf{E}Y^2)(\mathsf{E}Y^3)$$

Proof

$$\begin{aligned}
\mathsf{E}(g(X)h(Y)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_{X,Y}(x,y)dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dx\,dy \\
&= \left(\int_{-\infty}^{\infty} g(x)f_X(x)dx\right)\left(\int_{-\infty}^{\infty} h(y)f_Y(y)dy\right) \\
&= (\mathsf{E}\,g(X))(\mathsf{E}\,h(Y))
\end{aligned}$$

Bivariate Transformations

Functions of random vectors

Let (X, Y) be a bivariate rv with known distribution. Define (U, V) by

$$U = g_1(X, Y), \qquad V = g_2(X, Y)$$

Probability mapping: For any Borel set $B \subset \mathbb{R}^2$,

$$P[(U, V) \in B] = P[(X, Y) \in A]$$

where A is the inverse mapping of B, i.e.

$$A = \left\{(x, y) \in \mathbb{R}^2 : \quad (g_1(x, y), g_2(x, y)) \in B\right\}$$

The inverse is well defined even if the mapping is not bijective.

Example: Let $g_1(x, y) = x$, $g_2(x, y) = x^2 + y^2$.

Discrete RVs

Suppose that (X, Y) is a discrete rv, i.e. the pmf is positive on a countable set $\mathcal{A}$. Then (U, V) is also discrete and takes values on a countable set $\mathcal{B}$. Define

$$A_{uv} = \{(x, y) \in \mathcal{A} : \quad g_1(x, y) = u, \quad g_2(x, y) = v)\}$$

Then

$$f_{U,V}(u, v) = P(U = u, V = v) = \sum_{(x,y) \in A_{uv}} f_{X,Y}(x, y)$$

Sum of two independent Poissons

Let $X \sim Po(\lambda_1)$, $Y \sim Po(\lambda_2)$, independent, and define

$$U = X + Y, \qquad V = Y$$

- (X, Y) takes values in $\mathcal{A} = \{0, 1, 2, \ldots\}^2$.
- (U, V) takes values on $\mathcal{B} = \{(u, v) : \quad v = 0, 1, 2, \ldots, \quad u = v, v + 1, v + 2, \ldots\}$.
- For a particular (u, v), $A_{uv} = \{(x, y) \in \mathcal{A} : \quad x + y = u, \quad y = v\} = (u - v, u)$.

The joint pmf of U and V is

$$f_{U,V}(u, v) = f_{X,Y}(u - v, v) = \frac{e^{-\lambda_1} \lambda_1^{u-v}}{(u - v)!} \frac{e^{-\lambda_2} \lambda_2^{v}}{v!}$$

cont.

The distribution of $U = X + Y$ is the marginal

$$\begin{aligned} f_U(u) &= \sum_{v=0}^{u} \frac{e^{-\lambda_1}\ \lambda_1^{u-v}}{(u-v)!}\ \frac{e^{-\lambda_2}\ \lambda_2^{v}}{v!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{u!} \sum_{v=0}^{u} \left(\begin{array}{c} u \\ v \end{array} \right) \lambda_1^{u-v}\ \lambda_2^{v} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{u!} (\lambda_1+\lambda_2)^u \end{aligned}$$

We obtain that U is Poisson with parameter $\lambda = \lambda_1 + \lambda_2$.

Bivariate Transformations of Continuous RVs

Continuous RVs

Suppose (X, Y) is continuous and the joint transformation

$$u = g_1(x, y), \qquad v = g_2(x, y)$$

is one-to-one and differentiable. Define the inverse mapping

$$x = h_1(u, v), \qquad y = h_2(u, v)$$

Then

$$f_{UV}(u, v) = f_{XY}\big(h_1(u, v), h_2(u, v)\big)\ |J(u, v)|$$

where $J(u, v)$ is the Jacobian of the transformation $(x, y) \to (u, v)$ given by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right|$$

Rotation of a bivariate normal vector

Let $X \sim N(0,1)$, $Y \sim N(0,1)$, independent. Define the rotation

$$\begin{aligned} U &= X\cos\theta - Y\sin\theta \\ V &= X\sin\theta + Y\cos\theta \end{aligned}$$

for fixed θ. Then $U \sim N(0,1)$, $V \sim N(0,1)$, independent.

Proof: The range of (X,Y) is $\mathbb{R}^2$. The range of (U,V) is $\mathbb{R}^2$. Need the inverse transformation

$$\begin{aligned} X &= U\cos\theta + V\sin\theta \\ Y &= -U\sin\theta + V\cos\theta \end{aligned}$$

with Jacobian

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

cont.

The joint pdf of (X,Y) is

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}}e^{-y^2/2} = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$$

The joint pdf of (U,V) is

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{2\pi}e^{-[(u\cos\theta+v\sin\theta)^2+(-u\sin\theta+v\cos\theta)^2]/2} \ \cdot \ |1| \\ &= \frac{1}{2\pi}e^{-(u^2+v^2)/2} = \frac{1}{\sqrt{2\pi}}e^{-u^2/2} \cdot \frac{1}{\sqrt{2\pi}}e^{-v^2/2} \end{aligned}$$

so $U \sim N(0,1)$, $V \sim N(0,1)$, independent.

Functions of independent random variables

Theorem C-B 4.3.5: Let X and Y be independent rvs. Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be functions. Then the random variables $U = g(X)$ and $V = h(Y)$ are independent.

Proof:

Extensions of previous example

- Suppose $X \sim N(0, \sigma^2)$, $Y \sim N(0, \sigma^2)$, independent

$$\begin{aligned} U &= a(X \cos\theta - Y \sin\theta) \\ V &= a(X \sin\theta + Y \cos\theta) \end{aligned}$$

  Then $U \sim N(0, a^2\sigma^2)$, $V \sim N(0, a^2\sigma^2)$, independent.

- Above, take $\theta = \pi/4$, $a = \sqrt{2}$:

$$\begin{aligned} U &= \sqrt{2}(X/\sqrt{2} - Y/\sqrt{2}) = X - Y \\ V &= \sqrt{2}(X/\sqrt{2} + Y/\sqrt{2}) = X + Y \end{aligned}$$

  We get $U \sim N(0, 2\sigma^2)$, $V \sim N(0, 2\sigma^2)$, independent.

Ratio of two independent normals

Let $X \sim N(0,1)$, $Y \sim N(0,1)$, independent. The ratio X/Y has the Cauchy distribution.

Proof: Define the variables

$$U = X/Y, \qquad V = Y$$

with inverse

$$X = UV, \qquad Y = V$$

The Jacobian is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

The range of (X,Y) is $\mathbb{R}^2$. The range of (U,V) is $\mathbb{R}^2$.

cont.

The joint pdf of (X,Y) is

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

The joint pdf of (U,V) is

$$f_{UV}(u,v) = \frac{1}{2\pi} e^{-[(uv)^2+v^2]/2} \cdot |v| = \frac{|v|}{2\pi} e^{-(u^2+1)v^2/2}$$

The marginal of U is

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{UV}(u,v)\, dv = 2\int_0^{\infty} \frac{v}{2\pi} e^{-(u^2+1)v^2/2}\, dv \\ &= \frac{1}{\pi}\int_0^{\infty} e^{-(u^2+1)z}\, dz = \frac{1}{\pi(u^2+1)} \end{aligned}$$

Sum of Two Independent RVs

Suppose X and Y are independent. What is the distribution of $Z = X + Y$?

In general:

$$F_Z(z) = P(X + Y \leq z) = P(\{(x, y) \text{ such that } x + y \leq z\})$$

Various approaches:

- Bivariate transformation method (continuous and discrete)
- Discrete convolution

$$f_Z(z) = \sum_{x+y=z} f_X(x) f_Y(y) = \sum_x f_X(x) f_Y(z - x)$$

- Continuous convolution (C-B Section 5.2)
- Mgf/cf method (continuous and discrete)

Sum of two independent Poissons

Define X, Y to be two independent random variables having Poisson distributions with parameters $\lambda_i, \ \ i = 1, 2.$ Then:

$$f_{X,Y}(x, y) = \frac{e^{-\lambda_1} \ \lambda_1^x}{x!} \ \frac{e^{-\lambda_2} \ \lambda_2^y}{y!} \qquad x, y = 0, 1, 2, ...$$

The distribution of $S = X + Y$ is

$$\begin{aligned} f_S(s) &= \sum_{x=0}^{s} \frac{e^{-\lambda_1} \ \lambda_1^x}{x!} \ \frac{e^{-\lambda_2} \ \lambda_2^{s-x}}{(s-x)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{s!} \sum_{x=0}^{s} \binom{s}{x} \lambda_1^x \ \lambda_2^{s-x} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{s!} (\lambda_1 + \lambda_2)^s \end{aligned}$$

Thus S is Poisson with parameter $\lambda = \lambda_1 + \lambda_2$.

Characteristic Function

Theorem 4.2.12 Let X and Y be independent rvs with characteristic functions $\phi_X(\cdot)$ and $\phi_Y(\cdot)$, respectively. Then the characteristic function of $Z = X + Y$ is

$$\phi_Z(\theta) = \phi_X(\theta)\,\phi_Y(\theta)$$

Proof:

$$\begin{aligned}
\phi_Z(\theta) &= \mathsf{E}\exp(iZ\theta) &&= \mathsf{E}\exp\left[i(X+Y)\theta\right] \\
&= \mathsf{E}\exp(iX\theta)\exp(iY\theta) &&= \mathsf{E}\exp(iX\theta)\mathsf{E}\exp(iY\theta) \\
&= \phi_X(\theta)\,\phi_Y(\theta)
\end{aligned}$$

Corollary If X and Y independent and $Z = X - Y$,

$$\phi_Z(\theta) = \phi_X(\theta)\,\phi_Y(-\theta)$$

Sum of two independent Poissons

Suppose $X \sim \text{Poisson}(\lambda_X)$ and $Y \sim \text{Poisson}(\lambda_Y)$ and put $Z = X + Y$. Then, $Z \sim \text{Poisson}(\lambda_X + \lambda_Y)$.

Proof:

$$\begin{aligned}
\phi_Z(\theta) &= \exp\left[\lambda_X\left(e^{\theta}-1\right)\right]\,\exp\left[\lambda_Y\left(e^{\theta}-1\right)\right] \\
&= \exp\left[\left(\lambda_X+\lambda_Y\right)\left(e^{\theta}-1\right)\right]
\end{aligned}$$

Sum of Two Independent Normals

Suppose $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ and X and Y are independent and $Z = X + Y$ then

$$Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

Proof

$$\begin{aligned} \phi_Z(\theta) &= \exp\left(i\,\mu_x\theta - \frac{1}{2}\sigma_x^2\theta^2\right)\,\exp\left(i\,\mu_y\theta - \frac{1}{2}\sigma_y^2\theta^2\right) \\ &= \exp\left[i\,(\mu_x + \mu_y)\theta - \frac{1}{2}(\sigma_x^2 + \sigma_y^2)\theta^2\right] \end{aligned}$$

Sum of two independent gammas

Suppose $X \sim \Gamma(\alpha_y, \beta)$ and independently $Y \sim \Gamma(\alpha_y, \beta)$ and put $Z = X + Y$. Then, $Z \sim \Gamma((\alpha_x + \alpha_y), \beta)$

Proof:

$$\begin{aligned} \phi_Z(\theta) &= \left(\frac{1}{1-\beta\theta}\right)^{\alpha_x}\left(\frac{1}{1-\beta\theta}\right)^{\alpha_y} \\ &= \left(\frac{1}{1-\beta\theta}\right)^{\alpha_x+\alpha_y} \end{aligned}$$

Remember that

- If $\alpha = 1$ we have an exponential with parameter β.
- If $\alpha = n/2$ and $\beta = 2$, we have a $\chi^2(n)$ (with n d.f.). The above result states that $\chi^2(n_1) + \chi^2(n_2) = \chi^2(n_1 + n_2)$.