1. (a) Flipping a coin 4 times creates a finite sample space with $2^4 = 16$ elements. Each element consists of 4 coin flips and the result of each coin flip is denoted by H or T. An example element is [HHHH]. Thus, the sample space may be represented as

$$S = \{[HHHH], [HHHT], [HHTH], \ldots, [TTTT]\}$$

(b) The number of damaged leaves is a countably infinite sample space which must be greater than or equal to 0.

$$S = \{0, 1, 2, \ldots, \infty\}$$

(c) The lifetime of a lightbulb is an uncountably infinite sample space, with an interval beginning at an including 0 hours.

$$S = \{x : x \geq 0\}$$

(d) The weight of a rat must be greater than 0 units of mass (e.g. grams). The sample space is uncountably infinite.

$$S = \{w : w > 0\}$$

(e) The proportion of defectives in a shipment is a countably infinite sample space because it is a rational number representable by $\frac{m}{n}$, with m being the number of defectives and n being the number of items in the shipment.

$$S = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$$

2. (a) $S = \{(x, y) : x \in \{0, 1\}, y \in \{g, f, s\}\}$

(b) $A = \{(0, s), (1, s)\}$

(c) $B = \{(0, g), (0, f), (0, s)\}$

(d) $B^c \cup A = \{(1, g), (1, f), (1, s), (0, s)\}$

3. (a) $A^c = \{x : 0 < x \leq 0.5\}$

(b) A^c *with respect to* $\mathbb{R}^2 = \{(x, y) : x^2 + y^2 \geq 2\}$

Therefore

A^c *with respect to* $\Omega = \{(x, y) : x^2 + y^2 \geq 2, |x| + |y| \leq 2\}$

(c) $\cap_{n=1}^{\infty} B_n = \{x :\in (0, 0)\} = \emptyset$

Therefore

$A^c = \mathbb{R}^1$

BIOS 660 Homework 01

Question 4 and 5

Question 4

This proof has two parts:

(1) If $A \Delta B = C$, then $A = B \Delta C$

First we need to prove that $A \subset B \Delta C$:

Let $x \in A$, then $x \in (A \cap B) \cup (A - B)$. Then $x \in (A \cap B)$ or $x \in (A - B)$

If $x \in (A \cap B)$, then $x \in B$ and $x \notin C$, so $x \in B \Delta C$.

If $x \in (A - B)$, then $x \in A$ and $x \notin B$, so $x \in B \Delta C$.

Thus $A \subset B \Delta C$.

Second, we need to prove that $B \Delta C \subset A$:

let $x \in B \Delta C$, then $x \in (B - C) \cup (C - B)$, according to the definition of symmetric difference. Then $x \in (B - C)$ or $x \in (C - B)$

If $x \in (B - C)$, then $x \in B$, and $x \notin C$, so $x \in B$ and $x \notin A \Delta B$.

Then $x \in A \cap B$, so $x \in A$

If $x \in (C - B)$. then $x \in C$ and $x \notin B$, so $x \in A \Delta B$ and $x \notin B$.

Thus $x \in A$

Therefore, if $A \Delta B = C$, then $A = B \Delta C$

(2) If $A = B \Delta C$, then $A \Delta B = C$

First we need to prove that $A \Delta B \subset C$:

Let $x \in A \Delta B$, then $x \in (A - B) \cup (B - A)$, according to the definition of symmetric difference. Then $x \in (A - B)$ or $x \in (B - A)$

If $x \in (A - B)$, then $x \in A$ and $x \notin B$, so $x \in B \Delta C$ and $x \notin B$.

Thus $x \in C$

If $x \in (B - A)$, then $x \in B$ and $x \notin A$, so $x \in B$ and $x \notin B \Delta C$

Then $x \in B \cap C$ and $x \in C$.

Second, we need to prove that $C \subset A \Delta B$.

let $x \in C$, then $x \in (B \cap C) \cup (C - B)$. Then, $x \in (B \cap C)$ or $x \in (C - B)$. If $x \in (B \cap C)$, then $x \in B$ but $x \notin A$, so $x \in A \Delta B$.

If $x \in (C - B)$, then $x \in A$ and $x \notin B$, so $x \in A \Delta B$

Thus $C \subset A \Delta B$.

Therefore, if $A = B \Delta C$, then $A \Delta B = C$

Therefore, $A \Delta B = C$ if and only if $A = B \Delta C$, for any three sets, A, B, and C

Question 5

(a) (1) Proof of $(\bigcup_\alpha A_\alpha)^c \subset \bigcap_\alpha A_\alpha^c$:

Let $x \in (\bigcup_\alpha A_\alpha)^c$, then $x \notin \bigcup_\alpha A_\alpha$. Then, $x \in A_\alpha^c$ for all $\alpha \in \Gamma$. Thus $x \in \bigcap_\alpha A_\alpha^c$ according to the definition of intersection. ✓

(2) Proof of $\bigcap_\alpha A_\alpha^c \subset (\bigcup_\alpha A_\alpha)^c$

Let $x \in \bigcap_\alpha A_\alpha^c$, then $x \in A_\alpha^c$ for all $\alpha \in \Gamma$. Then, according to the definition of complement, $x \notin A_\alpha$ for all $\alpha \in \Gamma$. Thus $x \notin \bigcup_\alpha A_\alpha$, and $x \in (\bigcup_\alpha A_\alpha)^c$

Since we have proved that $(\bigcup_\alpha A_\alpha)^c \subset \bigcap_\alpha A_\alpha^c$ and $\bigcap_\alpha A_\alpha^c \subset (\bigcup_\alpha A_\alpha)^c$, we have $(\bigcup_\alpha A_\alpha)^c = \bigcap_\alpha A_\alpha^c$ ✓

(b) (1) Proof of $(\bigcap_\alpha A_\alpha)^c \subset \bigcup_\alpha A_\alpha^c$:

Let $x \in (\bigcap_\alpha A_\alpha)^c$, then $x \notin \bigcap_\alpha A_\alpha$. Then, there exist some $\alpha \in \Gamma$ such that $x \notin A_\alpha$. Then $x \in A_\alpha^c$ for some $\alpha \in \Gamma$. Thus $x \in \bigcup_\alpha A_\alpha^c$ according to the definition of union. ✓

(2) Proof of $\bigcup_\alpha A_\alpha^c \subset (\bigcap_\alpha A_\alpha)^c$

Let $x \in \bigcup_\alpha A_\alpha^c$, then $x \in A_\alpha^c$ for some $\alpha \in \Gamma$. Then, $x \notin A_\alpha$ for some $\alpha \in \Gamma$. Therefore $x \notin \bigcap_\alpha A_\alpha$. In other words, $x \in (\bigcap_\alpha A_\alpha)^c$

Since we have proved that $(\bigcap_\alpha A_\alpha)^c \subset \bigcup_\alpha A_\alpha^c$ and $\bigcup_\alpha A_\alpha^c \subset (\bigcap_\alpha A_\alpha)^c$, we have $(\bigcap_\alpha A_\alpha)^c \subset \bigcup_\alpha A_\alpha^c$ ✓

9/9

Question 6

Show $(\limsup_n A_n) \cap (\limsup_n B_n) \supset \limsup_n (A_n \cap B_n)$ and $(\limsup_n A_n) \cup (\limsup_n B_n) = \limsup_n (A_n \cup B_n)$

(a) Show $(\limsup_n A_n) \cap (\limsup_n B_n) \supset \limsup_n (A_n \cap B_n)$.

Let $x \in \limsup_n (A_n \cap B_n)$, then x is in infinitely many of $A_n \cap B_n$, which means that x is in infintely many of A_n and infinitely many of B_n. Therefore, $x \in \limsup_n A_n$ and $x \in \limsup_n B_n$, which implies $x \in (\limsup_n A_n) \cap (\limsup_n B_n)$, and $(\limsup_n A_n) \cap (\limsup_n B_n) \supset \limsup_n (A_n \cap B_n)$.

(b) Show $(\limsup_n A_n) \cup (\limsup_n B_n) = \limsup_n (A_n \cup B_n)$

Let $x \in \limsup_n (A_n \cup B_n)$, then x is in infinitely many of $A_n \cup B_n$, which means that x is in infinitely many of A_n or infinitely many of B_n. Therefore, $x \in \limsup_n A_n$ or $x \in \limsup_n B_n$, which is the same as $x \in (\limsup_n A_n) \cup (\limsup_n B_n)$. Therefore, $\limsup_n (A_n \cup B_n) \subset (\limsup_n A_n) \cup (\limsup_n B_n)$.

Let $x \in (\limsup_n A_n) \cup (\limsup_n B_n)$, then $x \in \limsup_n A_n$ or $x \in \limsup_n B_n$, which says x is in infinitely many of A_n or infinitely many of B_n. Thus, x is in infinitely many of $A_n \cup B_n$ and $x \in \limsup_n (A_n \cup B_n)$. Therefore, $\limsup_n (A_n \cup B_n) \supset (\limsup_n A_n) \cup (\limsup_n B_n)$ and we conclude $(\limsup_n A_n) \cup (\limsup_n B_n) = \limsup_n (A_n \cup B_n)$.

Question 7

Show $\liminf_n A_n \subset \limsup_n A_n$.

If $x \in \liminf_n A_n$, then there is an m such that $x \in A_n$ for all $n \geq m$, which means that $x \notin A_n$ for finitely many $n < m$. Therefore, x is in infinitely many of A_n, which implies that $x \in \limsup_n A_n$. We conclude that $\liminf_n A_n \subset \limsup_n A_n$.