Ann Marie Weideman

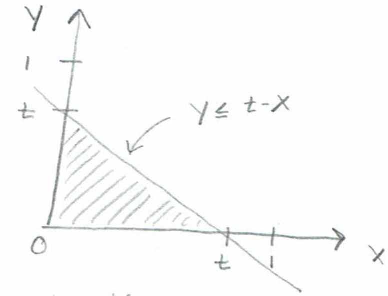
MS Exam 2017

1. Given $X \sim \text{Unif}(0,1)$, $Y \sim \exp(1)$ where $X \perp\!\!\!\perp Y$.

Define $T = X + Y$.

a) Given a constant $t \in (0,1)$, derive an explicit expression for $P(T \le t)$.

$$P(T \le t) = P(X+Y \le t) = P(Y \le t - X)$$



$$= \int_{x=0}^{t} \int_{y=0}^{t-x} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{x=0}^{t} \int_{y=0}^{t-x} e^{-y}\,dy\,dx$$ (since $X \perp\!\!\!\perp Y$, can write joint pdf as the product of the pdf of X & the pdf of Y)

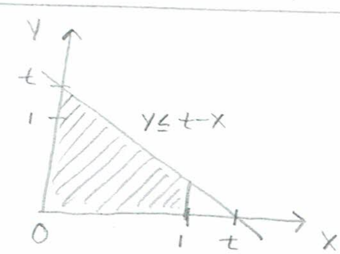
$$= \int_{x=0}^{t} -e^{-y}\Big|_0^{t-x} dx = \int_{x=0}^{t} \left(-e^{-(t-x)} + 1\right) dx = -e^{-(t-x)} + x \Big|_0^t$$

$$= \left(-e^{-(t-t)} + t\right) - \left(-e^{-(t-0)} + 0\right) = \boxed{-1 + t + e^{-t}, \quad t \in (0,1)}$$ ✓

b) Given a constant $t \in (1,\infty)$, derive an explicit expression for $P(T \le t)$.

$$P(T \le t) = P(X+Y \le t) = P(Y \le t - X)$$

$$= \int_{x=0}^{1} \int_{y=0}^{t-x} e^{-y}\,dy\,dx$$



$$= \int_{x=0}^{1} \left(-e^{-(t-x)} + 1\right) dx$$

$$= -e^{-(t-x)} + x \Big|_0^1 = \left(-e^{-(t-1)} + 1\right) - \left(-e^{-(t-0)} + 0\right) = \boxed{-e^{-t+1} + 1 + e^{-t}, \quad t \in (1,\infty)}$$ ✓

next pg. →

Ann Marie Weideman
MS Exam 2017

1. c) Find $E[T]$, $Var[T]$, and $Corr(X,T)$

Find $E[T] = E[X+Y] = E[X] + E[Y] = 1/2 + 1 = \boxed{1.5} = \boxed{3/2}$

Find $Var[T] = Var[X+Y] = Var[X] + Var[Y] + \underbrace{2(0)}_{X \perp\!\!\!\perp Y} = \frac{1}{12} + 1 = \boxed{\frac{13}{12}}$

Find $Corr(X,T) = \frac{Cov(X,T)}{\sqrt{Var(X)}\sqrt{Var(T)}} = \frac{E[X \cdot T] - E[X]E[T]}{\sqrt{Var(X)}\sqrt{Var(T)}}$ $*$

(Hard method)

Where $E[X \cdot T] = E[X(X+Y)] = E[X^2 + XY] = E[X^2] + E[XY]$

$= E[X^2] + \underbrace{E[X]E[Y]}_{\text{since } X \perp\!\!\!\perp Y} = \underbrace{Var[X] + E[X]^2}_{E[X^2]} + E[X]E[Y] = \frac{1}{12} + \frac{1}{4} + \frac{1}{2}(1)$

$= \frac{1}{12} + \frac{3}{12} + \frac{6}{12} = \frac{10}{12} \div \frac{2}{2} = \frac{5}{6}$

Then, $* = \frac{\frac{5}{6} - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{\sqrt{1/12}\sqrt{13/12}} = \frac{\frac{10}{12} - \frac{9}{12}}{\sqrt{13/12}} = \frac{\frac{1}{12}}{\frac{\sqrt{13}}{12}} = \frac{1}{\cancel{12}} \cdot \frac{\cancel{12}}{\sqrt{13}} = \boxed{\frac{1}{\sqrt{13}}}$

Note: I'm an idiot; would have been much easier to find $Corr(X,T)$ using

Find $Corr(X,T) = \frac{Cov(X,T)}{\sqrt{Var(X)}\sqrt{Var(T)}} = \frac{Cov(X, X+Y)}{\sqrt{Var(X)}\sqrt{Var(T)}} = \frac{\overbrace{Cov(X,X)}^{Var(X)} + \overbrace{Cov(X,Y)}^{= 0 \text{ by independence}}}{\sqrt{Var(X)}\sqrt{Var(T)}}$

(Easy Method)

$= \frac{Var(X)}{\sqrt{Var(X)}\sqrt{Var(T)}} = \frac{\frac{1}{12}}{\sqrt{1/12}\sqrt{13/12}} = \frac{\frac{1}{12}}{\sqrt{13}/12} = \boxed{\frac{1}{\sqrt{13}}}$

Ann Marie W.
MS Exam 2017

d) Define $W = 13X - T$. Find $Cov(T, W)$. Are T and W indep.? Justify.

$$\ulcorner \quad Cov(T,W) = Cov(T, 13X-T) = Cov(T, 13X) - \underbrace{Cov(T,T)}_{Var(T)}$$

$$= Cov(X+Y, 13X) - Var(X+Y) = Cov(X, 13X) + \underbrace{Cov(Y, 13X)}_{=0 \text{ since } X \perp\!\!\!\perp Y} - \underbrace{Var(X) - Var(Y)}_{\text{no covariance b/c } X \perp\!\!\!\perp Y}$$

$$= 13\underbrace{Var(X)}_{1/12} + \underbrace{0}_{X \perp\!\!\!\perp Y} - 1/12 - 1 = \frac{13}{12} - \frac{1}{12} - 1 = \frac{12}{12} - 1 = 0$$

$Cov(T,W) = 0 \not\Rightarrow T \perp\!\!\!\perp W$ unless T & W are jointly bivariate normal

Given $\begin{cases} W = 13X - T \\ T = X + Y \end{cases} \Rightarrow \begin{cases} X = \frac{W+T}{13} \\ Y = T - X = T - \left(\frac{W+T}{13}\right) = \frac{12T - W}{13} \end{cases}$

$$\Rightarrow |J| = \begin{vmatrix} \frac{\partial X}{\partial W} & \frac{\partial X}{\partial T} \\ \frac{\partial Y}{\partial W} & \frac{\partial Y}{\partial T} \end{vmatrix} = \begin{vmatrix} 1/13 & 1/13 \\ -1/13 & 12/13 \end{vmatrix} = \left| \frac{12}{13^2} + \frac{1}{13^2} \right| = \left| \frac{13}{13^2} \right| = \frac{1}{13}$$

Then, since $f_{X,Y}(x,y) = e^{-y}$, $0 \le y < \infty$ (since $X \perp\!\!\!\perp Y$, can mult. pdfs)

$$\Rightarrow f_{X,Y}\left(\frac{W+T}{13}, \frac{12T-W}{13}\right) = \frac{1}{13} e^{\frac{-12T+W}{13}}, \quad \begin{matrix} 0 \le T < \infty \\ -\infty < W \le 13 \end{matrix}$$

Since the joint pdf of W and T can be factored into separate functions of T & W as such, $\frac{1}{13} e^{\frac{-12T}{13}} \cdot e^{\frac{W}{13}}$, $\begin{matrix} 0 \le T < \infty \\ -\infty < W \le 13 \end{matrix}$,

then $W \perp\!\!\!\perp T$. $\lrcorner$

independent

This is wrong. Should be NOT indep.

Derive the marginals & you will see that their product does NOT equal the joint.

AMW
MS Exam 2017

e) Find constants a and b $\ni E[a+bT-X]=0$ and $Var(a+bT-X)$ is minimized.

$E[a+bT-X]=0 \Rightarrow E[a]+bE[T]-E[X]=0$

$\Rightarrow a+b(3/2)-1/2=0 \Rightarrow a=-\frac{3}{2}b+\frac{1}{2}$

$$Var(a+bT-X) = b^2 \underbrace{Var(T)}_{13/12} + \underbrace{Var(X)}_{1/12} - 2b\underbrace{Cov(T,X)}$$

$= Cov(X+Y, X)$
$= Cov(X,X) + \cancel{Cov(X,Y)}^{0}$
$= Var(X) = 1/12$

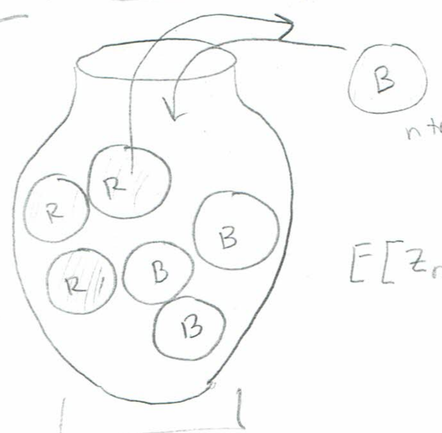
$= \frac{13}{12}b^2 - \frac{2}{12}b + \frac{1}{12} \stackrel{set}{=} 0$

$\Rightarrow 13b^2 - 2b + 1 = 0$. Take 1st derivative to find minimum.

$26b - 2 = 0 \Rightarrow \boxed{b = 1/13}$ ✓

Then $a = -\frac{3}{12}\left(\frac{1}{13}\right) + \frac{1}{2} = \frac{-3}{26} + \frac{13}{26} = \frac{10}{26} \div \frac{2}{2} = \boxed{\frac{5}{13}} = a$ ✓

f) Not writing this one out; too long.



6 balls

$$\begin{array}{c|c}
R & B \\
3 & 3 \\
\vdots & \vdots \\
Z_n & 6-Z_n
\end{array}$$

n trials

$$E[Z_{n+1}|Z_n] = \underbrace{(Z_n-1)}_{\substack{\#\text{ red remaining}\\ \text{if } Z_{n+1} \text{ is red}}}\underbrace{(Z_n/6)}_{P(red)} + \underbrace{(Z_n+1)}_{\substack{\#\text{ red remaining if}\\ Z_{n+1} \text{ is blue}}}\underbrace{\left(1-\frac{Z_n}{6}\right)}_{P(blue)}$$

$= \cancel{Z_n^2/6} - Z_n/6 + Z_n - \cancel{Z_n^2/6} + 1 - Z_n/6$ ✓

$= -\frac{2Z_n}{6} + Z_n + 1 = \frac{4Z_n}{6} + 1 = \frac{2}{3}Z_n + 1$

Then, $E[Z_{n+1}] = E\left[E[Z_{n+1}|Z_n]\right] = E\left[\frac{2}{3}Z_n + 1\right] = \frac{2}{3}E[Z_n] + 1$

Then, since $Z_1 = 3 \Rightarrow E[Z_2] = \frac{2}{3}(3)+1 = 3$, $E[Z_3] = \frac{2}{3}(3)+1 = 3$,

$\ldots E[Z_n] = \frac{2}{3}(3)+1 = 3.$

Ann Marie W.
MS Exam 2017

2. Given $f(y|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} \exp(-y/\beta)$, $y>0, \alpha>0, \beta>0$

where $y_i \overset{iid}{\sim}$ gamma (α,β)

a) Assume α Known. Derive the MLE of $\hat{\beta}$ and show that $\hat{\beta}$ is an unbiased estimator of β.

$$L(\beta|y) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1}\exp(-y/\beta) = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} y_i^{\alpha-1}\right)\exp\left(-\sum_{i=1}^{n} y_i/\beta\right)$$

$$\Rightarrow \ell(\beta|y) = -n\log(\Gamma(\alpha)\beta^\alpha) + \sum_{i=1}^{n}(\alpha-1)\log(y_i) - \sum_{i=1}^{n} y_i/\beta$$

$$= -n\log(\Gamma(\alpha)) - n\log(\beta^\alpha) + \sum_{i=1}^{n}(\alpha-1)\log(y_i) - \sum_{i=1}^{n} y_i/\beta$$

$$\Rightarrow \frac{\partial \ell}{\partial \beta} = \underbrace{\frac{-n\alpha\beta^{\alpha-1}}{\beta^\alpha}}_{= \frac{-n\alpha}{\beta}} + \sum_{i=1}^{n} y_i/\beta^2 \overset{set}{=} 0$$

✓

$$\Rightarrow \frac{n\alpha}{\beta} = \sum_{i=1}^{n} y_i/\beta^2 \Rightarrow \boxed{\hat{\beta} = \frac{1}{n\alpha}\sum_{i=1}^{n} y_i} = \boxed{\frac{1}{\alpha}\bar{Y}}$$

Know that $\hat{\beta}$ occurs @ a global max b/c:

$$\left.\frac{\partial^2 \ell}{\partial \beta^2}\right|_{\beta=\hat{\beta}} = \left.\frac{n\alpha}{\beta^2} - \frac{2\sum_{i=1}^{n} y_i}{\beta^3}\right|_{\beta=\hat{\beta}} = \frac{n\alpha}{\left(\frac{1}{n\alpha}\sum_{i=1}^{n} y_i\right)^2} - \frac{2\sum_{i=1}^{n} y_i}{\left(\frac{1}{n\alpha}\sum_{i=1}^{n} y_i\right)^3} = \frac{n^3\alpha^3}{\left(\sum_{i=1}^{n} y_i\right)^2} - \frac{2n^3\alpha^3}{\left(\sum_{i=1}^{n} y_i\right)^2}$$

$$= \frac{-n^3\alpha^3}{\left(\sum_{i=1}^{n} y_i\right)^2} < 0.$$

$$E[\hat{\beta}] = E\left[\frac{1}{n\alpha}\sum_{i=1}^{n} y_i\right] = \frac{1}{n\alpha}\sum_{i=1}^{n} E[y_i] = \frac{1}{\cancel{n\alpha}}(\cancel{n\alpha}\beta) = \beta$$

due to independence

✓

Since $E[\hat{\beta}] = \beta \Rightarrow \hat{\beta}$ is an unbiased estimator of β.

Ann Marie W
MS Exam 2017

2. b) Derive the MLE of $S(t) = P(Y > t)$, given that $\alpha = 1$.

For $\alpha = 1$, $f(y|\beta) = \frac{1}{\beta} \exp(-y/\beta)$

$$\Rightarrow F(y|\beta) = \int_0^y \frac{1}{\beta} \exp\left(-\frac{t}{\beta}\right) dt = \frac{1}{\beta} \cdot \frac{-\beta}{1} \exp\left(-\frac{t}{\beta}\right)\Big|_0^y = -\exp\left(-\frac{t}{\beta}\right)\Big|_0^y$$

$$= 1 - \exp(-y/\beta)$$

$$\Rightarrow S(t) = P(Y > t) = 1 - P(Y \leq t) = 1 - \left(1 - \exp\left(-\frac{t}{\beta}\right)\right) = \exp\left(-\frac{t}{\beta}\right)$$

Then, by the invariance property of the MLE,

have $\hat{S}(t) = \exp\left(-\frac{t}{\hat{\beta}}\right) = \exp\left(\frac{-\alpha t}{\bar{Y}}\right)$

$\Vert \leftarrow \alpha = 1$

$\boxed{\exp\left(-\frac{t}{\bar{Y}}\right)}$, $t > 0$, $y > 0$

Ann Marie W
MS Exam 2017

2\. c) Let $V_1 = \begin{cases} 1, & Y_1 > t \\ 0, & \text{else} \end{cases}$. Show that V_1 is an unbiased estimator of $S(t)$.

$$E[V_1] = E[I(Y_1 > t)] = P(Y_1 > t) = S(t)$$

Since $E[V_1] = S(t)$, then V_1 is an unbiased estimator of $S(t)$.

2 d) Fix $\alpha = 1$. Show that the conditional pdf of Y_1 given $U = \sum_{i=1}^{n} Y_i$ is

$$f_{Y_1|U}(y_1|u) = \begin{cases} \frac{n-1}{u^{n-1}}(u-y_1)^{n-2}, & 0 < y_1 < u \\ 0, & \text{else} \end{cases}$$

Note: $Y_1, \dots, Y_n$ $\perp\!\!\!\perp$ and $\sim$ gamma$(1, \beta)$

$$f_{Y_1|U}(y_1|u) = \frac{f_{Y_1,U}(y_1,u)}{f_U(u)} = \frac{f_{Y_1, U-Y_1}(y_1, u-y_1)}{f_U(u)}$$

$$= \frac{\left(\frac{1}{\Gamma(1)\beta^1}\cancel{\exp(-y_1/\beta)}\right)\left(\frac{1}{\Gamma(n-1)\beta^{n-1}}(u-y_1)^{n-2}\cancel{\exp(-(u-y_1)/\beta)}\right)}{\left(\frac{1}{\Gamma(n)\beta^n}u^{n-1}\cancel{\exp(-u/\beta)}\right)}$$

$$= \frac{1}{\Gamma(n-1)\cancel{\beta^n}} \cdot \frac{\Gamma(n)\cancel{\beta^n}}{1} \frac{1}{u^{n-1}}(u-y_1)^{n-2}$$

$$= \frac{(n-1)!}{(n-2)!}\frac{1}{u^{n-1}}(u-y_1)^{n-2} = \begin{cases} \frac{n-1}{u^{n-1}}(u-y_1)^{n-2}, & 0 < y_1 < u \\ 0, & \text{else} \end{cases}$$

Ann Marie Weideman

MS Exam 2017

2 e)

i) Show that $E[V_1|U] = \left(1 - \frac{t}{u}\right)^{n-1} I(u>t)$

$\ulcorner$ Know $E[V_1|U] = E[I(Y_1>t)|U] = P[Y_1|U] = \int_0^u f_{Y_1|U}(y_1|u)\,dy_1$

$= \left[\int_t^u f_{Y_1|U}(y_1|u)\,dy_1\right] I(Y_1>t)$

$= \left[\int_t^u \frac{n-1}{u^{n-1}}(u-y_1)^{n-2}\,dy_1\right] I(Y_1>t)$

$= \left[\frac{\cancel{(n-1)}}{u^{n-1}} \cdot \frac{-1}{\cancel{(n-1)}} (u-y_1)^{n-1}\Big|_t^u\right] I(Y_1>t)$

Note: Need final expectation in terms of U. Since $I(Y_1>t) \Rightarrow I\left(\sum_{i=1}^{n} Y_i > t\right) = I(U>t)$

$= \left[\cancelto{0}{\frac{-1}{u^{n-1}}(u-u)^{n-1}} + \frac{1}{u^{n-1}}(u-t)^{n-1}\right] I(U>t)$

$= \left(\frac{u-t}{u}\right)^{n-1} I(u>t) = \left(1-\frac{t}{u}\right)^{n-1} I(U>t)$ ✓ $\lrcorner$

ii) <u>Now, want to show that $E[V_1|U]$ is the UMVUE</u>

I have no clue what to do here

$\ulcorner$ Have $E[E(V_1|U)] = E[V_1] = P(Y_1>t) = S(t)$

$\Rightarrow E(V_1|U)$ is an unbiased estimator of $S(t)$.

Also, from part i), know that $E(V_1|U) = \left(1-\frac{t}{u}\right)^{n-1} I(U>t)$

We know that if $E_\theta(g(T))=0 \Rightarrow P_\theta(g(T)=0)=1$, $\forall \theta \in \Theta$, then T is a complete and sufficient statistic.

$g(T) = E(V_1|U)$ here. Then, $E[E(V_1|U)] = 0 \Rightarrow E[V_1]=0 \Rightarrow I(U>t)=0$.

Then, $P(g(T)) = P(E(V_1|U)) = P\Big(\underbrace{\left(1-\frac{t}{u}\right)^{n-1} \cdot I(u>t)}_{=0}\Big) = 1$.

Thus, by Lehmann Scheffe, $E(V_1|U)$ is the UMVUE. $\lrcorner$

3. Given $X_1, \dots, X_n \overset{iid}{\sim} F$ with EDF defined as $F_n(x)$.

a) Let $Y_i = I(X_i \le x)$. EDF can be written as $\frac{1}{n}\sum_{i=1}^{n} Y_i$

Show $F_n(x)$ is a consistent estimator of $F(x)$.

$$\lim_{n\to\infty} E\{F_n(x)\} = \lim_{n\to\infty}\left[E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} E(Y_i)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} E(I(X_i \le x))\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} P(X_i \le x)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} F(x)\right] = \lim_{n\to\infty}\left[\frac{1}{\cancel{n}}(\cancel{n}\, F(x))\right] = F(x).$$

$$\lim_{n\to\infty} Var\{F_n(x)\} = \lim_{n\to\infty}\left[Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)\right] = \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} Var(I(X_i \le x))\right] = \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} P(X_i \le x)(1 - P(X_i \le x))\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^{\cancel{2}}}(\cancel{n}\, F(x)(1-F(x)))\right] = 0$$

blows up w/ limit

AMW

MS Exam 2017

3.b) Given a specific $x \in A := \{t : 0 < F(t) < 1\}$, describe the asymptotic distribution of $F_n(x)$ when $n \to \infty$, and derive an approximate 95% CI for $F(x)$ when n is large.

From part a) Know that $E\{F_n(x)\} = F(x)$ and $Var\{F_n(x)\} = \frac{F(x)(1-F(x))}{n}$

Then, by CLT, the asymptotic dist. of $F_n(x)$ when $n \to \infty$ is:

$$\boxed{\sqrt{n}\,(F_n(x) - F(x)) \xrightarrow{d} N(0, F(x)(1-F(x)))}$$

Also, want 95% CI $= \left(-Z_{\alpha/2} \le \frac{F_n(x) - F(x)}{\sqrt{\frac{F(x)(1-F(x))}{n}}} \le Z_{\alpha/2}\right)$

$$\approx \Bigg(\underbrace{-Z_{\alpha/2} \le \frac{F_n(x) - F(x)}{\sqrt{\frac{F_n(x)(1-F_n(x))}{n}}}}_{\text{holds by Slutsky's}} \le Z_{\alpha/2}\Bigg) = \left(-Z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}} \le F_n(x) - F(x) \le Z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}}\right)$$

$$= \left(-Z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}} - F_n(x) \le -F(x) \le Z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}} - F_n(x)\right)$$

$$= \left(F_n(x) - Z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}} \le F(x) \le F_n(x) + Z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}}\right)$$

$$\Rightarrow \boxed{95\%\ CI(F(x)) = \left(F_n(x) - Z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}},\ F_n(x) + Z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}}\right)}$$

where $Z_{\alpha/2}$ denotes the upper $\alpha/2$ - quantile of the std normal dist.

Ann Marie Weideman
MS Exam 2017

3 c) Test $H_0: F(x)=0.5$ vs. $H_1: F(x) \neq 0.5$.

Find the LRT statistic and its distribution under H_0.

$\left(\text{Note: when } p=0.5 \quad P(X=x \mid p) = 0.5^x(1-0.5)^{1-x} = 0.5^x(0.5)^{1-x} = 0.5 \right)$

$H_0: \underbrace{F(x)}_{\text{P(survival) @ a specific concentration}} = 0.5$ vs. $H_1: F(x) \neq 0.5$. Let P(survival) $= p$.

Then, can model each $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(p)$.

Then, $$L(p \mid x) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}$$

$$\Rightarrow \ell(p \mid x) = \sum_{i=1}^n x_i \log(p) + \left(n - \sum_{i=1}^n x_i\right)\log(1-p)$$

$$\Rightarrow \frac{\partial \ell}{\partial p} = \frac{\sum_{i=1}^n x_i}{p} - \frac{\left(n - \sum_{i=1}^n x_i\right)}{1-p} \overset{set}{=} 0$$

$$\Rightarrow \frac{1-p}{p} = \frac{\left(n-\sum_{i=1}^n x_i\right)}{\sum_{i=1}^n x_i} \Rightarrow \frac{1}{p} \cancel{-1} = \frac{n}{\sum_{i=1}^n x_i} \cancel{-1} \Rightarrow \hat{p} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$$

Note that $\hat{p}$ occurs @ a global max since,

$$\left.\frac{\partial^2 \ell}{\partial p^2}\right|_{p=\hat{p}} = \left. \frac{-\sum_{i=1}^n x_i}{p^2} + \frac{2\left(n-\sum_{i=1}^n x_i\right)}{(1-p)^2}\right|_{p=\hat{p}} \quad \frac{-n\bar{x}}{\bar{x}^2} + \frac{2(n-n\bar{x})}{(1-\bar{x})^2}$$

$$= \frac{-n}{\bar{x}} + \frac{2n\cancel{(1-\bar{x})}}{(1-\bar{x})^{\cancel{2}}} = \frac{-n}{\bar{x}} + \frac{2n}{(1-\bar{x})} = \frac{-n(1-\bar{x})}{\bar{x}(1-\bar{x})} + \frac{2n\bar{x}}{\bar{x}(1-\bar{x})}$$

$$= \frac{-n+n\bar{x}+2n\bar{x}}{\bar{x}(1-\bar{x})} = \frac{-n+n\bar{x}}{\bar{x}(1-\bar{x})} = \frac{n(-1+\bar{x})}{\bar{x}(1-\bar{x})} = \frac{-n}{\bar{x}} < 0.$$

Then, LRT test statistic is: $$\lambda(x) = \frac{\sup_{p=0.5} L(p \mid x)}{\sup_{0<p\le 1} L(p \mid x)} = \frac{L(0.5)}{L(\hat{p})} = \frac{\prod_{i=1}^n (0.5)^{x_i}(0.5)^{1-x_i}}{\prod_{i=1}^n (\bar{x})^{x_i}(1-\bar{x})^{1-x_i}}$$

$$= \frac{(0.5)^n}{\bar{x}^{\sum_{i=1}^n x_i}(1-\bar{x})^{n-\sum x_i}} = \frac{(0.5)^n}{\bar{x}^{n\bar{x}}(1-\bar{x})^{n-n\bar{x}}} = \boxed{\left(\frac{0.5}{\bar{x}^{\bar{x}}(1-\bar{x})^{1-\bar{x}}}\right)^n} \Big\} \text{ LRT statistic}$$

As $n \to \infty$, $$-2\log(\lambda(x)) = -2\log\left(\frac{0.5}{\bar{x}^{\bar{x}}(1-\bar{x})^{1-\bar{x}}}\right)^n = -2n\left[\log(0.5) - \log(\bar{x}^{\bar{x}}) - \log\left[(1-\bar{x})^{1-\bar{x}}\right]\right]$$

$$= \boxed{-2n\left[\log(0.5) - \bar{x}\log(\bar{x}) - (1-\bar{x})\log(1-\bar{x})\right] \xrightarrow{d} \chi^2_1}$$

Asymptotic dist. is chi-squared w/ 1 df

3 d) and e) to ask Dr. Q this Wed.

Will upload after.