1. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution with pdf

$$f_X(x) = \frac{1}{\theta}, \quad 0 < \theta < \infty.$$

Let $X_{(1)}, \ldots, X_{(n)}$ be order statistics, and define a new random variable $Y_i = X_{(i)}/X_{(i+1)}$ for $i = 1, \ldots, n-1$ and $Y_n = X_{(n)}$.

(a) Find the inverse function for $X_{(1)}, \ldots, X_{(n)}$, as a function of $y_1, y_2, \ldots, y_n$, and show that the Jacobian is $y_2 y_3^2 \cdots y_n^{n-1}$.

Solution: The inverse function is $X_{(i)} = \prod_{j=i}^n Y_j$. The Jacobian matrix is upper triangular $n \times n$ matrix. Hence the determinant of the matrix is the product of diagonal elements, which is $y_2 y_3^2 \cdots y_n^{n-1}$.

(b) Find the joint pdf of $Y_1, \ldots, Y_n$, and show that $Y_1, \ldots, Y_n$ are mutually independent.

Solution: The joint pdf of $Y_1, \ldots, Y_n$ can be derived from the join pdf of $X_{(1)}, \ldots, X_{(n)}$ using transformation method. The joint pdf of new random variables is

$$\begin{aligned} f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) &= f_{X_{(1)},\ldots,X_{(n)}}(\prod_{j=1}^n y_j, \ldots, y_n)|J| \\ &= \frac{n!}{\theta^n} y_2 y_3^2 \cdots y_n^{n-1}, \end{aligned}$$

which can be factorized into individual functions of $y_1, \ldots, y_n$. Hence, we can claim $Y_1, \ldots, Y_n$ are mutually independent.

(c) Find the distribution of Y_1, and show that Y_n/θ follows a distribution with pdf

$$f(y) = ny^{n-1},$$

which is Beta$(n, 1)$.

Solution: Since there is no y_1 term in the join pdf and the range of Y_1 is between 0 and 1, one can know the Y_1 follows a uniform distribution with pdf $f_{Y_1}(y_1) = 1$. The range of $Y_2, \ldots, Y_{n-1}$ are the same as Y_1. By a standardization to a pdf, one can see, for $i = 2, \ldots, (n-1)$, $f_{Y_i}(y_i) = iy_i^{i-1}$, $0 < y_i < 1$. One can see $f_{Y_n}(y_n) = ny_n^{n-1}/\theta^n$. A transformation method can be used to get the distribution of $Z = Y_n/\theta$ as

$$f_Z(z) = nz^{n-1}, \quad 0 < z < \theta,$$

which is Beta$(n, 1)$. One can also use the pdf derivation formula for the pdf of $Y_n = X_{(n)}$, a maximum order statistic.

2. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two independent random samples from a normal distribution with mean μ and variance σ^2. Let $\bar{X}$ and $\bar{Y}$ denote the sample mean of X and Y, respectively.

   (a) Derive the distribution of $\bar{X} - \bar{Y}$.

   Solution: By the property of a normal distribution, $\bar{X} \sim N(\mu, \sigma^2/n)$, $\bar{Y} \sim N(\mu, \sigma^2/n)$, and $\bar{X} - \bar{Y} \sim N(0, 2\sigma^2/n)$.

   (b) Show that $\lim_{n\to\infty} P(|\bar{X} - \bar{Y}| > \sigma) = 0$.

   Solution: One can see

$$\begin{aligned}
P(|\bar{X} - \bar{Y}| > \sigma) &= 1 - P(|\bar{X} - \bar{Y}| \leq \sigma) \\
&= 1 - P(-\sigma < \bar{X} - \bar{Y} < \sigma) \\
&= 1 - P(-\frac{\sigma}{\sqrt{2}\sigma/\sqrt{n}} < \frac{\bar{X} - \bar{Y}}{\sqrt{2}\sigma/\sqrt{n}} < \frac{\sigma}{\sqrt{2}\sigma/\sqrt{n}}) \\
&= 1 - \Phi(\sqrt{n/2}) + \Phi(-\sqrt{n/2}).
\end{aligned}$$

   When $n \to \infty$, $P(|\bar{X} - \bar{Y}| > \sigma) \to 0$ since $\Phi(\sqrt{n/2}) \to 1$ and $\Phi(-\sqrt{n/2}) \to 0$.

3. Suppose that X_1 and X_2 are independent and follow Poisson distributions with mean λ_1 and λ_2, respectively.

   (a) Show that, given $X_1 + X_2 = n$, X_1 follows a binomial distribution with mean

$$E(X_1 | X_1 + X_2 = n) = \frac{n\lambda_1}{\lambda_1 + \lambda_2}.$$

Solution: The conditional distribution of X_1 is

$$\begin{aligned} P(X_1 = x|X_1 + X_2 = n) &= \frac{P(X_1 = x, X_1 + X_2 = n)}{P(X_1 + X_2 = n)} \\ &= \frac{P(X_1 = x)P(X_2 = n - x)}{P(X_1 + X_2 = n)} \\ &= \left(\frac{\lambda_1^x e^{-\lambda_1}}{x!} \frac{\lambda_2^{n-x} e^{-\lambda_2}}{(n-x)!} \right) \Big/ \left(\frac{(\lambda_1 + \lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!} \right) \\ &= \binom{n}{x} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^x \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n-x}, \end{aligned}$$

which is a binomial distribution with mean $E(X_1|X_1+X_2 = n) = n\lambda_1/(\lambda_1+\lambda_2)$.

(b) Argue that whether or not $\sum_{i=1}^{2} X_i$ is a sufficient statistic of λ_1.

Solution: No. It is not a sufficient statistic since the conditional distribution given $X_1 + X_2 = n$ still contains the information about λ_1.

4. When information is *unobservable*, one common approach in biostatistics is to use the so-called *latent* variable. For example, it is almost impossible to determine the true level of radioactive iodine exposure in association with the occurrence of thyroid cancer, one may assume that the level of exposure X follows a Gamma distribution with $\alpha = 1$ and some value of β, i.e., the pdf of X is

$$f_X(x) = \frac{x^{\beta-1} e^{-x}}{\Gamma(\beta)}, \quad x > 0, \quad \beta > 1,$$

and, given $X = x$, the time to the occurrence of thyroid cancer T follows an exponential distribution with pdf

$$f_T(t|X = x) = \theta x e^{-\theta x t}, \quad t > 0, \quad \theta > 0.$$

That means, the mean occurrence time of the thyroid cancer is $E(T|X = x) = (\theta x)^{-1}$, which indicates that a higher level of radioactive iodine exposure is associated with a shorter time of cancer occurrence.

(a) However, since X is not observable, one cannot use the conditional distribution to make inferences on θ. A marginal pdf $f_T(t)$, which integrates out the latent variable X from the joint pdf $f_{T,X}(t, x)$, is of more practical use. Derive $f_T(t)$ here.

Solution: One can see

$$\begin{aligned} f_T(t) &= \int_0^\infty f_T(t|X=x) f_X(x) dx \\ &= \frac{\theta\Gamma(\beta+1)}{\Gamma(\beta)(\theta t+1)^{\beta+1}} \int_0^\infty \frac{1}{\Gamma(\beta+1)(\theta t+1)^{-(\beta+1)}} x^\beta e^{-(\theta t+1)x} dx \\ &= \frac{\theta\beta}{(\theta t+1)^{\beta+1}}. \end{aligned}$$

(b) Show that

$$E(T) = \frac{1}{\theta(\beta-1)}.$$

Solution: By double expectation,

$$E(T) = E_X E_{T|X}(T|X=x) = E_X(\theta^{-1}X^{-1}) = \frac{1}{\theta(\beta-1)},$$

where

$$\begin{aligned} E_X(X^{-1}) &= \int_0^\infty x^{-1} \frac{x^{\beta-1}e^{-x}}{\Gamma(\beta)} \\ &= \frac{\Gamma(\beta-1)}{\Gamma(\beta)} \int_0^\infty \frac{1}{\Gamma(\beta-1)} x^{\beta-2} e^{-x} dx \\ &= \frac{1}{\beta-1}. \end{aligned}$$

One may also use the definition of expectation to derive the result, where

$$\begin{aligned} E(T) &= \int_0^\infty t \frac{\theta\beta}{(\theta t+1)^{\beta+1}} dt \\ &= -\int_0^\infty t d(\theta t+1)^{-\beta} \\ &= -t(\theta t+1)^{-\beta}|_0^\infty + \int_0^\infty (\theta t+1)^{-\beta} dt \\ &= \frac{1}{\theta(1-\beta)} (\theta t+1)^{-\beta+1}|_0^\infty \\ &= \frac{1}{\theta(\beta-1)}. \end{aligned}$$

(c) Assuming that one has a good estimator $\hat{\theta}$ for θ with

$$\sqrt{n}(\hat{\theta}-\theta) \to_d N(0, 2\theta^2),$$

provided that $\beta = 2$. To derive the large sample property of $\gamma = E(T)$, one may use a continuous function g to transform θ with $\gamma = g(\theta)$, and then obtain

$$\sqrt{n}(g(\hat{\theta}) - \gamma) \rightarrow_d N(0, \sigma^2).$$

Identify the function g and derive the variance σ^2.

Solution: The g function satisfies $g(\theta) = \gamma = E(T) = 1/\theta$, provided that $\beta = 2$. According to the delta method,

$$\sigma^2 = \{g'(\theta)\}^2 2\theta^2 = \theta^{-4} 2\theta^2 = 2\theta^{-2}.$$