Old quals link: bios.unc.edu/distrib/exam/ms

General Advice From Qaqish

- 3 problems 6 hours
- Read question for about 30 minutes, important that you fully understand question before you start working. Read and reread
- Don't jump to quick conclusions, find a shorter way to solve problem. Don't make kneejerk assumptions
- Conditional Distribution- use double expectation to find $E(X), Var(X), Cov$
- Just because you can't do part a does not mean you can't do parts b and c, etc
- For biased/unbiased estimator questions answer yes/no, don't automatically find $E(X)$
  ex: $\bar{X}^2$ est of $\mu^2 \quad \mu = E(X)$
  Jensen's inequality: if $g(x)$ convex then $E[g(x)] > g(E[X])$
  $g(x) = X^2 \Rightarrow$ convex
  $E(X^2) > (E(X))^2 = \mu^2$ Thus biased

2017 Quals Problem 1

(a) Given a constant $t \in (0, 1)$ derive an explicit expression for $P(T \leq t)$.

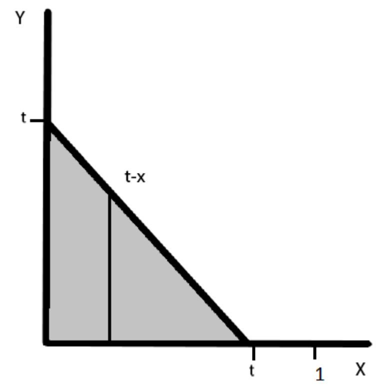
(c) Find E$[T]$, Var(T) and Corr(X, T).

(e) Find constants a and b such that E$[a + bT - X] = 0$ and Var$(a + bT - X)$ is minimized.

(f) An urn contains 6 balls; 3 red and 3 blue. A "step" is defined as drawing a ball at random from the urn, and replacing it by a ball of the other color (taken from another urn). That is, if the ball drawn is red, it is replaced with a blue ball; if the ball drawn is blue, it is replaced with a red ball. The number of balls in the urn remains equal to 6 after each step. Let the random variable Z_n denote the number of red balls in the urn after n steps (the initial number is $Z_0 = 3$). Prove that E$[Z_n] = 3$ for all $n \geq 1$. Hint: E$[Z_{n+1}|Z_n]$.

(a)

$$f_Y(y) = e^{-y} \quad y > 0$$
$$x \in (0,1) \quad y \in (0,\infty)$$
$$T = X + Y$$
$$\text{given } t \in (0,1)$$
$$P(X + Y \leq t)$$
$$\{X + Y \leq t\} \text{ (event)}$$
$$= \{y \leq t - x\} \text{ (plot)}$$



integrate x from 0 to t

for each x, integrate y from 0 to t-x

$$\int_{x=0}^{x=t} \int_{y=0}^{y=t-x} f_{X,Y}(x,y)\; dy\; dx \qquad f_{X,Y}(x,y) = e^{-y}$$
$$\int_0^t \int_0^{t-x} e^{-y}\; dy\; dx$$
$$\Rightarrow \int_0^t 1 - e^{-(t-x)}\mathrm{d}x \Rightarrow t - 1 + e^{-t}$$
$$P(T \leq t) = t - 1 + e^{-t}$$

(c)

$$\text{Find } E(T) \quad Var(T) \quad Corr(T)$$
$$E(T) = E(X) + E(Y) = 1/2 + 1 = 3/2$$
$$Var(T) = Var(X) + Var(Y) + Cov(X,Y)$$

$$= 1/12 + 1 + 0 \quad (X \perp Y)$$
$$Var(T) = 13/12$$
$$T = X + Y$$
$$Cov(X,T) = Cov(X, X+Y) \Rightarrow Cov(X,X) + Cov(X,Y)$$
$$= Var(X) + 0 = 1/12$$
$$Cov(X,T) = 1/12$$
$$Corr(X,T) = \frac{Cov(X,T)}{\sqrt{Var(X)Var(T)}} = \frac{1/12}{\sqrt{(1/12)(13/12)}} = \frac{1}{\sqrt{13}}$$

(e)

find a and b s.t. $E(a + bT - X) = 0$ and $Var(a + bT - X)$ is minimized

Since $E(T) = 1.5 \quad E(X) = .5$ we have:

$$E(a + bT - X) = a + b(1.5) - .5$$
$$b = \frac{.5 - a}{1.5}$$
$$Var(a + bT - X) = 0 + Var(bT) + Var(X) - 2Cov(bt, X)$$
$$= b^2(13/12) + 1/12 - 2bCov(T, X)$$
$$= b^2(13/12) + 1/12 - 2b/12 \Rightarrow (1/12)(13b^2 - 2b + 1) \text{ (convex)}$$
$$\frac{d}{db} = (1/12)(26b - 2) = 0 \text{ (minimizing)}$$
$$\Rightarrow b = 13$$

Plugging back into $b = \frac{.5 - a}{1.5}$ we get:

$$13 * 1.5 = .5 - a$$
$$\Rightarrow a = -19$$

Thus $a = -19, b = 13$

(f)

Red	$Blue$
3	3
Z_n	$6 - Z_n$

$$P(\text{Red Ball}|Z_n \text{ currently}) = Z_n/6$$
$$P(\text{Blue}|Z_n) = 1 - Z_n/6$$

Draw	Red	Blue
Red	$Z_n - 1$	$6 - Z_n + 1$
Blue	$Z_n + 1$	$6 - Z_n - 1$

$$P(Z_{n+1} = Z_n - 1|Z_n) = Z_n/6$$
$$P(Z_{n+1} = Z_n + 1|Z_n) = 1 - Z_n/6$$
$$E(Z_{n+1}|Z_n) = (Z_n - 1)Z_n/6 + (Z_n + 1)(1 - Z_n/6)$$
$$= -Z_n/6 + Z_n + 1 - Z_n/6$$
$$E(Z_n + 1|Z_n) = 1 + (2/3)Z_n$$
$$E(Z_1|Z_0 = 3) = 3$$
$$E(Z_2|Z_1) = 1 + (2/3)Z_1$$
$$E(E(Z_{n+1}|Z_n)) = E(Z_n + 1)$$
$$E(Z_{n+1}) = 1 + (2/3)E(Z_n)$$

In the form of: $a = 1 + (2/3)a \Rightarrow (1/3)a = 1 \Rightarrow a = 3$