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A Unified Approach to Mixed Linear Models

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The mixed model equations as presented by C. R. Henderson offers the base for a methodology that provides flexibility of fitting models with various fixed and random elements with the possible assumption of correlation among random effects. The advantage of teaching analysis of variance applications from this methodology is presented. Particular emphasis is placed upon the relationship between choice of estimable function and inference space.

KEY WORDS: Best linear unbiased prediction; Inference space; Mixed model equations; Realized values; Standard errors.

1. INTRODUCTION

The analysis of data from experiments with mixed models is often not well understood by experimenters. Some of the lack of understanding, even by individuals who have completed a "full" complement of statistical methods courses, may be attributable to the way in which this subject has been presented in these classes. Consider, for example, a simple case in which a researcher wants to estimate the standard error of a treatment mean from a randomized complete block design. Many standard textbooks either fail to outline a procedure for estimation or report one that is inconsistent with the inference space. Likewise, many commonly used software packages do not return a calculated standard error or report one that is inconsistent with the model assumption.

Another presently unresolved issue concerns the expected mean squares for mixed models. The following quotation from Wilk and Kempthorne (1955, p. 1145) indicates the difficulty that statisticians have had and continue to have on this subject.

> We feel that any mathematical assumptions employed in the analysis of natural phenomena must have an explicit, recognizable, relationship to the physical situation. In particular, if the analysis of variance is to be generally useful in the interpretation of experimental data it is necessary that its meaning and justification should transcend the set of arbitrary assumptions which are usually put forth.
>
> Largely because explicit and objective methods for obtaining the appropriate model for a given experiment are not generally used, one finds that certain rules and results, concerning expectations of mean squares and the choice of error terms, given by Mood (1950), Hald (1952), Mentzer (1953), and Scheffe (1954) are contradicted by Kempthorne (1952), Anderson and Bancroft (1952), Tukey (1949), Cornfield (1953), and Villars (1951).

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The differences among proposed models have existed for years and the amount of confusion has continued to grow. The dominance of certain textbooks, which give expected mean squares for mixed models assuming that interactions of fixed and random effects sum to 0 over the fixed effect levels, has left a legacy in the minds of many users that these are the "CORRECT" expectations.

Even though other authors (Hocking 1985; Kirk 1982; Mood 1950; Searle 1971) have presented a different set of expected mean squares based on dropping the requirement concerning the sum of interaction effects, many consider the former as "CORRECT." When SAS (SAS Institute Inc. 1988) began to report the latter model form, many users began to inquire as to which model is correct.

The choice of model can have dramatic consequences. If users are not familiar with the inference space implicit in each model, severe misuse may occur. For example, if a quantitative geneticist estimates the variance component of a random variable by equating the expected mean squares implied by these models and subsequently estimates heritability, drastic differences may be observed between the two models. The dilemma is further complicated if the data are unbalanced. Most variance component estimation procedures used for unbalanced data are based on the assumption of independent interaction effects; how valid is this "standard operating procedure" in those cases when the former model would have been chosen if the data were balanced?

The mathematical relationship between these models was well documented by Mendenhall (1968), Searle (1971), and Hocking (1973, 1985). However, none of these articles addressed the interpretation of these models with respect to an implied inference space. Denoting the relationship mathematically is not particularly satisfying to those interested in a conceptual view; practitioners consistently report that the statistical literature to date falls well short of providing them with adequate guidelines for making an informed choice of model.

Since many, if not most, analyses of experimental data involve some mixed model aspect, there is an apparent need for teaching consistent analysis procedures directed toward appropriate model selection and interpretation based on the physical situation. The needed approach to linear models should be general enough to allow various models, various error structures, methods for balanced and unbalanced data, etcetera, to be treated as variations on a common theme. It should teach students tools for selecting the appropriate variation on the theme. Finally, the approach should be sufficiently tractable so that practitioners may make use of it without undue computational or theoretical heroics.

The purpose of this article is to demonstrate the application of mixed model procedures (MMP), developed by Goldberger (1962), Henderson (1950, 1963, 1973, 1984a), and Harville (1976a,b), to common statistical inference tasks for simple experimental designs that give rise to mixed models. Estimable functions in the mixed model sense will be constructed for these examples and their relationship to the implied inference space will be developed. This provides a basis for appropriately choosing from among the various model and inference alternatives. Since MMP may be applied to any linear model, balanced or unbalanced, with a general covariance structure, this demonstration will illustrate how application of MMP in this fashion leads to a unified approach to linear models. The approach can be used by teachers to present complex issues in linear models to students and by consultants to extend these presently underused methods to researchers.

2. MIXED LINEAR MODEL METHODOLOGY

The mixed linear model is given in matrix form, using the notation of boldfaced print for all matrices. Henderson's notation and results yield the following:

$$\mathbf{y} = \mathbf{XB} + \mathbf{ZU} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\mathbf{y}$ is a $m \times 1$ vector of measured responses, $\mathbf{X}$ is an $m \times p$ known matrix with the rank of $\mathbf{X} \leq m, p$, $\mathbf{B}$ is a $p \times 1$ vector of fixed effects that are unknown, $\mathbf{Z}$ is an $m \times q$ incidence matrix, $\mathbf{U}$ is a $q \times 1$ vector of random effects with $\mathrm{E}(\mathbf{U}) = \mathbf{0}$, $\boldsymbol{\epsilon}$ is an $m \times 1$ vector with $\mathrm{E}(\boldsymbol{\epsilon}) = \mathbf{0}$, and

$$\mathrm{var}\begin{bmatrix}\mathbf{U}\\ \boldsymbol{\epsilon}\end{bmatrix} = \begin{bmatrix}\mathbf{G} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}.$$

For this model we have $\mathrm{E}(\mathbf{y}) = \mathbf{XB}$ and $\mathrm{var}(\mathbf{y}) = \mathbf{ZGZ}' + \mathbf{R}$. One estimation procedure that may be used is generalized least squares for obtaining the best linear unbiased estimator (BLUE) of $\mathbf{B}$. The problem that exists is that the normal equations for generalized least squares require the inverse of var($\mathbf{y}$), which is m by m and often nondiagonal. Henderson (1984a) showed that the solution to the mixed model equations

$$\begin{bmatrix}\mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z}\\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{b}\\ \mathbf{u}\end{bmatrix} \doteq \begin{bmatrix}\mathbf{X}'\mathbf{R}^{-1}\mathbf{y}\\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y}\end{bmatrix} \tag{2}$$

gives a solution for which the estimate $\mathbf{b}$ of the fixed effects is equal to that obtained by generalized least squares. The solution to this equation requires finding a g-inverse of a $p + q$-by-$p + q$ matrix, which is easier than finding the inverse of an m-by-m matrix. Calculational efficiencies (Graham 1981; Kaplan 1983) exist for both types of models and are dependent on the structure of $\mathbf{G}$ and $\mathbf{R}$. For problems with a large number of observations the coefficient matrix in Equation 2 may be built as the data are read by the computer software if $\mathbf{R}$ and $\mathbf{G}$ are block-diagonal matrices and easily invertible. In all of our applications, these two matrices are nonsingular. Henderson reasoned, later supported mathematically by Harville, that one can formulate the random portion of this model into one of estimation of realized values of random variables, the random variables being the elements of the $\mathbf{U}$ vector. This technique was originally called "best linear unbiased prediction" (BLUP) by Henderson.

Under the assumption of normality of the $\boldsymbol{\varepsilon}$ and $\mathbf{U}$ and writing any g-inverse, denoted by the superscript minus, of the coefficient matrix of Equation (2) as

$$\begin{bmatrix}\mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z}\\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1}\end{bmatrix}^{-} = \begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}\\ \mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix} = \mathbf{C}, \tag{3}$$

some of the important properties of the BLUP solution of the mixed model solution, as discussed by Henderson (1984a), may be summarized as follows.

1. In the class of linear unbiased predictors, BLUP maximizes the correlation between $\mathbf{U}$ and $\mathbf{u}$.
2. $\mathbf{K}'\mathbf{b}$ is BLUE of the set of estimable linear functions, $\mathbf{K}'\mathbf{B}$.
3. $\mathrm{E}(\mathbf{U} \mid \mathbf{u}) = \mathbf{u}$.
4. $\mathbf{u}$ is unique.
5. $\mathbf{K}'\mathbf{b} + \mathbf{M}'\mathbf{u}$ is BLUP of $\mathbf{K}'\mathbf{B} + \mathbf{M}'\mathbf{U}$ provided that $\mathbf{K}'\mathbf{B}$ is estimable.
6. $\mathrm{var}(\mathbf{K}'\mathbf{b}) = \mathbf{K}'\mathbf{C}_{11}\mathbf{K}$.
7. $\mathrm{var}(\mathbf{K}'\mathbf{b} + \mathbf{M}'\mathbf{u}) = \mathbf{K}'\mathbf{C}_{11}\mathbf{K} + \mathbf{M}'(\mathbf{G} - \mathbf{C}_{22})\mathbf{M}$.
8. $\mathrm{var}(\mathbf{K}'\mathbf{b} + \mathbf{M}'\mathbf{u} - \mathbf{K}'\mathbf{B} - \mathbf{M}'\mathbf{U}) = (\mathbf{K}'\mathbf{M}')\mathbf{C}(\mathbf{K}'\mathbf{M}')'$.
9. $\mathrm{cov}(\mathbf{K}'\mathbf{b}, \mathbf{u}') = \mathbf{0}$.
10. $\mathrm{cov}(\mathbf{K}'\mathbf{b}, \mathbf{U}') = -\mathbf{K}'\mathbf{C}_{12}$.
11. $\mathrm{cov}(\mathbf{K}'\mathbf{b}, \mathbf{U}' - \mathbf{u}') = -\mathbf{K}'\mathbf{C}_{12}$.
12. $\mathrm{var}(\mathbf{u}) = \mathrm{cov}(\mathbf{u}, \mathbf{U}') = \mathbf{G} - \mathbf{C}_{22}$.
13. $\mathrm{var}(\mathbf{u} - \mathbf{U}) = \mathbf{C}_{22}$.

The foregoing mixed model equations, their solutions, and the properties make possible a broad set of applications with BLUE and BLUP solutions. An important aspect of this discussion is that $\mathbf{K}'\mathbf{b}$ is a generalized least squares estimate of $\mathbf{K}'\mathbf{B}$ even in the case of unbalanced data as long as $\mathbf{K}'\mathbf{B}$ is estimable. Thus, under the foregoing assumptions, $\mathrm{var}(\mathbf{K}'\mathbf{b}) = \mathbf{K}'\mathbf{C}_{11}\mathbf{K} = \mathbf{K}'(\mathbf{X}'(\mathrm{var}(\mathbf{y}))^{-1}\mathbf{X})^{-}\mathbf{K}$, as shown by Henderson (1984a). In cases where $\mathbf{G}$ is singular the matrix $\mathbf{C}$ will not yield sampling variances directly. Harville (1976a) and Henderson (1984a) discussed this situation and gave expressions that yield the sampling variances as a function of $\mathbf{C}$.

Some examples of problems that may be addressed with the model of Equation (1) are all of the problems addressed with the conventional linear model with only fixed effects; problems with a known $\mathbf{G}$ structure (animal selection) where all elements may differ from 0; cases where $\mathbf{G}$ and $\mathbf{R}$ are known diagonal matrices; some combination of $\mathbf{G}$ and $\mathbf{R}$ being block-diagonal, which allows a multivariate-type analysis with known variance–covariance matrix; recovery of interblock information; and covariance problems associated with all of the preceding types. Henderson (1984b) also illustrated variance component estimation procedures that allow the use of these techniques in the absence of known variances and covariances.

Kackar and Harville (1984) showed that the prediction error variance given in Property 8 above tends to be underestimated when variance component estimates made from the data are substituted into this expression. They proposed the use of a correction term that decreases the bias in the estimated variance of a predictable function, that is, one that satisfies Property 5. Jeske and Harville (1988) gave a procedure for computing degrees of freedom for variances of predictable functions that in turn may be used for forming prediction intervals. McLean and Sanders (1988) illustrated the reliability of prediction intervals for various predictable functions when used in a two-way mixed model. The effect of using the correction term as proposed by Kackar and Harville is noticeable whenever the predictable function involves specific random effects. Whenever the random effects are not used or averaged out and the emphasis is on the fixed effects, then the substitution of the estimated variance for the population variance without correction does not appear to introduce much of a problem.

Henderson (1984a) and Harville (1976a) discussed the differences that result when a singular **G** matrix is required. Additional steps are required to compute variances of estimable functions. In addition, restricted maximum likelihood (REML) estimation procedures are different. Since the majority of the applications may be analyzed with a nonsingular **G** matrix, the additional complexity of using a singular **G** will not be discussed here.

3. COMPLETELY RANDOMIZED DESIGN EXAMPLE

The majority of the problems that need to be addressed for the mixed model applications may be illustrated with a two-factor example. This example will briefly illustrate the technique of using the MMP under the assumptions of Equation 1. One major strength of this technique is that the analysis may be done with unbalanced as well as balanced data. The example given here will be balanced to show comparisons of MMP with the more familiar conventional models. The results for an unbalanced example will be given in the next section. The data given in Table 1 are from an industrial experiment with machines (F) considered as a fixed factor and operators (R) taken to be a random factor.

This illustration was selected as it easily presents the possibility of the interaction effects being precisely determined when the operator is selected, as would be the case if the interaction is dependent on certain physical characteristics of the operator. It is just as easy to visualize a situation in which the outcome of the interaction is highly dependent on the mental state of the operator at the time of the experiment. This would quickly negate the assumption that the interaction terms sum to 0 as the interaction effects would truly be random variables depending on the mental state of the operator at the time of the experiment. These illustrations allow one to envision cases for which estimates are desired for specific operators while regarding the operator by machine interaction as an aspect of random variation. Note that these various conditions will have a strong impact on the standard errors when the appropriate inference is made.

For calculational purposes the operator and machine by operator interaction (FR) are placed into Equation (1) as independent random variables each with homogeneous variances, as shown below. This assumption, however, may be altered to comply with the assumption that the interaction terms sum to 0 by the selection of the coefficients in the predictable function. Thus with this standard assumption we can write $\mathbf{R} = \sigma^2\mathbf{I}(12)$, where $\mathbf{I}(\mathbf{n})$ is an n-by-n identity matrix and

$$\mathbf{G} = \begin{bmatrix} \sigma_R^2\mathbf{I}(3) & \mathbf{0} \\ \mathbf{0} & \sigma_{FR}^2\mathbf{I}(6) \end{bmatrix}.$$

The expression in Equation 1 can also be written as

$$\mathbf{y} = (\mathbf{X}\ \mathbf{Z})(\mathbf{B}\ \mathbf{U})' + \boldsymbol{\epsilon},$$

and for the data given in Table 1 this expression would appear as shown in Figure 1. The matrix **X** is 12 by 3 and **Z** is 12 by 9. The number of rows of **X** and **Z** is equal to the number of observations, and the number of columns of each is controlled by the number of effects in the (**B U**) vector.

To apply the MMP approach to these data it is necessary to either know the variance components described previously or obtain an estimate of the variance component using a variance estimation procedure, such as REML. For this first illustration we will use REML estimated variance components for operators, interaction, and error, which are .1073, .0510, and .0485, respectively. After inclusion into Equation (2), then the following solution vector is obtained: $\mathbf{b}' = (51.96\ \ -1.01\ \ 0)$ and $\mathbf{u}' = (.2295\ \ .0851\ \ -.3146\ \ .120\ \ -.131\ \ .011\ \ -.011\ \ .172\ \ -.161)$. Harville (1977) referred to the estimates of the random effects as shrinkage estimators.

Observe that in the aforementioned solution the three estimates of the operator effects sum to 0, as do each of the two groups of three interaction effects. The majority of all applications will have this characteristic for the random effects; it will always be true for nonsingular **G** matrices. This phenomenon will allow the experimenter to select a desired inference space for a given estimator and then obtain the appropriate standard error. Inference for this type of example would either be to the specific operators used in the experiment or to the population of all operators. Infer-

Table 1. Data for Machine Operator Example

	Operator (R)			
	1	2	3	Mean
Machine (F)				
1	51.43	50.93	50.47	
	51.28	50.75	50.83	50.948
2	51.91	52.26	51.58	
	52.43	52.33	51.23	51.957
Mean	51.762	51.568	51.028	51.452

$$\begin{bmatrix} 51.43 \\ 51.28 \\ 50.93 \\ 50.75 \\ 50.47 \\ 50.83 \\ 51.91 \\ 52.43 \\ 52.26 \\ 52.33 \\ 51.58 \\ 51.23 \end{bmatrix} = \begin{bmatrix} 1 & 1\ 0 & 1\ 0\ 0 & 1\ 0\ 0\ 0\ 0\ 0 \\ 1 & 1\ 0 & 1\ 0\ 0 & 1\ 0\ 0\ 0\ 0\ 0 \\ 1 & 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0\ 0\ 0\ 0 \\ 1 & 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0\ 0\ 0\ 0 \\ 1 & 1\ 0 & 0\ 0\ 1 & 0\ 0\ 1\ 0\ 0\ 0 \\ 1 & 1\ 0 & 0\ 0\ 1 & 0\ 0\ 1\ 0\ 0\ 0 \\ 1 & 0\ 1 & 1\ 0\ 0 & 0\ 0\ 0\ 1\ 0\ 0 \\ 1 & 0\ 1 & 1\ 0\ 0 & 0\ 0\ 0\ 1\ 0\ 0 \\ 1 & 0\ 1 & 1\ 0\ 0 & 0\ 0\ 0\ 0\ 1\ 0 \\ 1 & 0\ 1 & 1\ 0\ 0 & 0\ 0\ 0\ 0\ 1\ 0 \\ 1 & 0\ 1 & 1\ 0\ 0 & 0\ 0\ 0\ 0\ 0\ 1 \\ 1 & 0\ 1 & 1\ 0\ 0 & 0\ 0\ 0\ 0\ 0\ 1 \end{bmatrix} \begin{bmatrix} \mu \\ F_1 \\ F_2 \\ R_1 \\ R_2 \\ R_3 \\ FR_{11} \\ FR_{12} \\ FR_{13} \\ FR_{21} \\ FR_{22} \\ FR_{23} \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \\ \epsilon_{10} \\ \epsilon_{11} \\ \epsilon_{12} \end{bmatrix}$$

Figure 1. Equation (1) for Data in Table 1.

ence to specific levels of random effects will henceforth be referred to as "the narrow inference space," whereas inference to the entire population of random effects will be referred to as "the broad inference space." The latter space would naturally require larger standard errors for any estimable function. In some cases, illustrated subsequently, inference can be narrow to certain random effects (e.g., operators) and broad to others (e.g., operator-by-machine interaction); this will be called "intermediate inference space."

The researcher has control over the inference space by merely selecting zeros or ones for use in the estimable function. In addition, these constants must be selected, thus forcing the researcher to consider which inference space is desired. Examples of this and possible interpretations are given in the comparison of Functions 1, 3, 4, 5, and 15 in Table 2.

The solution vector given previously may be used with Property 8 of Section 2 to obtain estimates of specific linear combinations of the effects and their associated standard errors. The functions given in Table 2 will be used for illustration purposes. The function column of the table contains the function number and a brief indication of the functional effect being estimated. The last three columns of the table contain the estimated value of the function, the associated standard error, and the appropriate inference space, described as narrow (N), intermediate (I), or broad (B). The first three elements of each ($\mathbf{K}'\ \mathbf{M}'$) vector are the elements of $\mathbf{K}'$ and are the multipliers of the fixed effects, the next three elements are the multipliers of the operator effects, and the last six elements are associated with the interaction terms in the order as set up in the data analysis. Note that in all cases, $\mathbf{K}'\mathbf{b}$ is an estimable function as described by Searle (1971).

Functions 1 and 2 estimate the mean of Machines 1 and 2, respectively. These effects are averaged over the three operators that were used in the experiment and also averaged over the appropriate interaction terms. If these operators were chosen at random from a population of operators, then the standard error for Functions 1 and 2 represent the uncertainty of the estimates of the marginal means for Machines 1 and 2 assuming that additional operators are not selected to use the machines. That is, these same operators once chosen are to be used to operate these machines in the future and their effects, including the interaction, do not change. For balanced data, this is equivalent to the fixed effect model and the inference space is narrow.

The standard error for Function 3 assumes that inference is being made for the fixed effect marginal means in relationship to the entire population of random effects including random interaction effects. This assumption would imply that for this problem there is a large population of operators and that there is a population of interaction terms that affect our experiment and these effects are independent of the random main effects. Again, this condition could very easily occur if the interaction is a function of the operator's mental state, which may change frequently. Another illustration would be in agricultural experiments where varieties are considered as fixed and years or locations are considered as random. The interaction among fixed and random effects could vary as a function of environmental effects that are close to impossible to quantify. This is often the space to which experimenters want to draw inference in mixed model problems. Such cases are all examples of the broad inference space.

The standard errors and associated inference space for Functions 4 and 5 have not been used by many experimenters in the past. These two types of estimable functions give more insight into the various inference spaces that are available to the experimenter. Function 4 allows one to make inference to the three specific levels of the random effect and allows the interaction to be considered as a random effect. For example, this might occur if the experimenter desires to make predictions or infer results applicable to the specific set of operators observed in the experiment, but regards their interactions with machines as random phenomena in the manner described previously. Alternatively, an agricultural researcher might want to apply inference for particular varieties and locations, but regards their interaction as an expression of experimental error. Function 5 will allow inference to the population of random main effects

Table 2. Estimable Functions for Completely Randomized Example

	(K' M')														
							FR								
Function	μ	F_1	F_2	R_1	R_2	R_3	11	12	13	21	22	23	($\mathbf{K}'\mathbf{b} + \mathbf{M}'\mathbf{u}$)	Standard error	Inference space*
1. *F* mean	(3	3	0	1	1	1	1	1	1	0	0	0)/3	50.95	.0899	N
2. *F* mean	(3	0	3	1	1	1	0	0	0	1	1	1)/3	51.96	.0899	N
3. *F* mean	(3	3	0	0	0	0	0	0	0	0	0	0)/3	50.95	.2467	B
4. *F* mean	(3	3	0	1	1	1	0	0	0	0	0	0)/3	50.95	.1584	I
5. *F* mean	(3	3	0	0	0	0	1	1	1	0	0	0)/3	50.95	.2095	I
6. *F* difference	(0	3	−3	0	0	0	1	1	1	−1	−1	−1)/3	−1.01	.1272	N
7. *F* difference	(0	1	−1	0	0	0	0	0	0	0	0	0)	−1.01	.2240	B
8. *R* mean	(2	1	1	2	0	0	1	0	0	1	0	0)/2	51.74	.1070	N
9. *R* mean	(2	1	1	0	2	0	0	1	0	0	1	0)/2	51.56	.1070	N
10. *R* difference	(0	0	0	2	−2	0	1	−1	0	1	−1	0)/2	.179	.1491	N
11. *FR* mean	(1	1	0	1	0	0	1	0	0	0	0	0)	51.30	.1448	N
12. *FR* mean	(1	1	0	0	1	0	0	1	0	0	0	0)	50.90	.1448	N
13. *R* difference	(0	0	0	1	−1	0	1	−1	0	0	0	0)	.396	.1967	N
14. *R* difference	(0	0	0	1	−1	0	0	0	0	0	0	0)	.144	.2361	B
15. *F* mean	(6	6	0	2	2	2	1	1	1	1	1	1)/6	50.95	.1288	I

*Inference space: B, broad; I, intermediate; N, narrow.

and treat the interaction effects as being quantities that would remain constant in repeated application of the experiment with various operators. Function 5, for example, could be used if the interactions effects between operators and machines were primarily created by physical characteristics that classify the operators into a small number of groups, say left- and right-handed operators, and this fact was not preplanned. Inference can then be made to a broad body of operators that possess the stratification structure that gave rise to these interaction effects. The standard errors for these two functions are both larger than that for Function 1 but not as large as that for Function 3. All of these are examples of intermediate inference spaces.

Functions 6 and 7 give the same estimate of the difference between Machines 1 and 2. The standard error of Function 6 considers the inference space to be either fixed or random for operators but the interactions must be fixed. The inference space for Function 7 is much larger, as the interactions are considered as random effects and the operators may be considered as random or fixed. The coefficients in Function 6 may be formed by subtracting Function 2 from Function 1 or taking the difference of two functions like Function 5. Function 7 may be formed by starting with Function 3 or Function 4 and subtracting a similar set of coefficients for estimating the effect of Machine 2.

Functions 8 and 9 estimate the mean of Operators 1 and 2, respectively. These estimates are averaged over both machines and take into consideration the interaction terms. These estimates are shrinkage estimates and are different from those obtained by ordinary least squares procedures when one pretends that operators are fixed effects. Henderson (1984a) referred to these values as BLUP, and Harville (1985) used the expression estimates of realized values of the random effects. Function 10, determined from the difference of Functions 8 and 9, gives the shrinkage estimate of the difference between Operators 1 and 2 along with the associated standard error.

Functions 11 and 12 represent estimates of Operators 1 and 2, respectively, on Machine 1. Function 13 is obtained by subtracting Function 12 from Function 11. Thus the interpretation is that it gives the estimate of the difference of Operator 1 minus Operator 2 for Machine 1, a narrow inference space. Note that this standard error is larger than for Function 10, where the effects were averaged over both machines, which is a narrow inference space but with more information. Function 14 would be obtained from the difference of two functions similar to Functions 8 and 9 except that they would have coefficients of 0 for all interaction effects. Thus Function 14 would be estimating the difference between Operators 1 and 2, averaged over Machines 1 and 2. This corresponds to prediction of operator differences over a population of interaction effects of which those observed are merely random selected representatives. Note that the standard error for Function 14 is much larger than for Function 10.

To date, most users of the MMP approach have been interested in the evaluation of estimable functions and the associated standard errors. For experimenters who are primarily interested in a global test on differences among treatment effects, the following computing procedure produces a test statistic to test the hypothesis of no machine effects, $\mathbf{H}'\mathbf{B} = \mathbf{0}$. Here the matrix $\mathbf{H}'$ may be the $\mathbf{K}'$ portion of the vector given in Function 7. This matrix will have one linearly independent row in this example and will have an additional independent vector for every additional fixed effect, and the degrees of freedom for treatments will be equal to the number of columns of **H.** Since we are interested in testing the treatment effects, the $\mathbf{b}'$ portion of the solution vector is used. Then the expression

$$\mathbf{b}'\mathbf{H}(\mathbf{H}'\mathbf{C}_{11}\mathbf{H})^{-1}\mathbf{H}'\mathbf{b} = 20.26 \tag{4}$$

will be distributed as chi squared with degrees of freedom equal to the number of independent columns of $\mathbf{H}$ if the variance components are known and $\mathbf{y}$ is multivariate normal (Henderson 1984a).

When the variance components in Equation (4) have to be estimated, then this value divided by the rank of $\mathbf{H}$ will be distributed approximately as F. The degrees of freedom for the numerator of F would be the rank of $\mathbf{H}$, and the degrees of freedom for the denominator may be approximated by using the Satterthwaite procedure as discussed by Jeske and Harville (1988). For the balanced case the degrees of freedom for the denominator will be that of the interaction mean square in the conventional analysis of variance (ANOVA) table. Preliminary work with the Jeske and Harville (1988) procedures indicates that in the case of missing cells the degrees of freedom for the denominator decrease about 1 for each missing cell (McLean and Sanders 1988). The practical use of this procedure is dependent on software development.

4. COMPARISON OF MMP TO CONVENTIONAL ANALYSES FOR THE COMPLETELY RANDOMIZED DESIGN

How does the MMP approach discussed earlier compare with procedures currently covered in elementary textbooks? The two competing models, with two different sets of expected mean squares for one model, will be discussed. A comparison of these procedures with MMP will be made via standard errors of marginal means and differences of these means. To make a valid comparison of the two approaches we must consider the balanced case, as the material given in elementary- to intermediate-level textbooks does not cover the unbalanced case. As shown in Section 5, the calculations required for the analysis of the unbalanced case will be the same as for the balanced case.

The following mathematical model is the most readily found in current textbooks, for example, Anderson and McLean (1974).

$$Y_{ijk} = \mu + F_i + R_j + FR_{ij} + \varepsilon_{(ij)k},$$
$$i = 1, \ldots, f;\ j = 1, \ldots, r;\ k = 1, \ldots, n. \tag{5}$$

The assumptions on the parameters are that the sum of the F_i effects and the sum of the FR_{ij} effects over i are equal to 0, and

$$R_j \sim N(0, \sigma_R^2),\ FR_{ij} \sim N(0, \sigma_{FR}^2),\ \varepsilon_{(ij)k} \sim N(0, \sigma^2),$$

with all random effects being pairwise independent. The

Table 3. ANOVA Table for Equation (5)

Source	df	Mean squares	F	Expected mean squares
F	$f-1$	MSF	MSF/MSFR	$\sigma^2 + n\sigma^2_{FR} + rn\Phi(F)$
R	$r-1$	MSR	MSR/MSE	$\sigma^2 + nf\sigma^2_R$
FR	$(f-1)(r-1)$	MSFR	MSFR/MSE	$\sigma^2 + n\sigma^2_{FR}$
Error	$fr(n-1)$	MSE		σ^2
Total	$frn-1$			

NOTE: *MSF, MSR, MSFR, MSE* are mean squares for the corresponding source effect.

ANOVA table normally associated with this model is given in Table 3.

The expected mean squares in the ANOVA presented in Steel and Torrie (1980) for Equation (5) contains a finite correction term. This term results from the actual covariance that exists, and is discussed subsequently, among the interaction effects within each level of the random effect. This finite correction term is mentioned by several authors but omitted by most, as it does not influence hypothesis tests on the fixed effects. When one is interested in variance component estimation, however, this difference may be important. The expected mean squares for this set of assumptions are given in Table 4.

An alternative model for this experiment, as written in Searle (1971) or SAS (SAS Institute Inc. 1988), is as follows:

$$Y_{ijk} = \mu + F_i + R_j + FR_{ij} + \varepsilon_{(ij)k},$$

$$i = 1, \ldots, f; j = 1, \ldots, r; k = 1, \ldots, n, \quad (6)$$

and

$$R_j \sim N(0, \sigma^2_R),\ FR_{ij} \sim N(0, \sigma^2_{FR}),\ \varepsilon_{(ij)k} \sim N(0, \sigma^2),$$

with all random effects being pairwise independent. The ANOVA table for this model is given in Table 5.

As Searle (1971) and Hocking (1985) indicated, the terms in Equations (5) and (6) have different meanings. The major difference is that in Equation (5) the fixed effect-by-random effect interaction terms are assumed to sum to 0 over the levels of the fixed effect, whereas in Equation (6) this assumption is not made. Searle (1971) defined explicitly the exact algebraic difference among these models and the expected mean squares in Tables 4 and 5. The use of Equation (5) implies the presence of correlation among the interaction parameters within a given level of the random effect. Specifically,

$$\operatorname{cov}(FR(5)_{ik}, FR(5)_{i'j}) = -\sigma^2_{FR(5)}/(f-1), \quad (7)$$

where $FR(5)$ indicates the interaction term from Equation (5). Using this type of notation for terms from each of the two models, Searle showed the equivalence among Tables 4 and 5 by using the following equalities.

1. $FR(5)_{ij} = FR(6)_{ij} - \overline{\mathrm{FR}}(6)_{.j}$, where $\overline{FR}(6)_{.j}$ is the mean of the $FR(6)_{ij}$ for a given level of j.
2. $R(5)_j = R(6)_j + \overline{FR}(6)_{.j}$.

The data given in Table 1 will be used to compare the results of the foregoing models with that of MMP. The basic ANOVA table for these data is given in Table 6 with F values computed based on the expected mean squares of Tables 3, 4, and 5. The expected mean squares in Tables 4 and 5 are easily derived, whereas those in Table 3 are approximations to those in Table 4.

The standard error of the difference between two machine means may be derived by using either Equation (5) or (6). In the case of Equation (5), using the finite correction factor, the variance of the difference would be

$$(2/rn)(\sigma^2 + n(f/(f-1))\sigma^2_{FR(5)}).$$

For Equation (6) the corresponding variance is

$$(2/rn)(\sigma^2 + n\sigma^2_{FR(6)}).$$

In both cases the variance of the difference of two treatment means would be estimated by $(2/rn)$ times the interaction mean square. This same estimate is normally used when doing an analysis in agreement with Table 3. The derivation of this variance, when using Equation (5) and Table 3, requires the incorrect assumption that the covariance among all interaction terms is 0. Thus in all three cases the accepted standard error of the difference among two treatment means is .2240 with 2 df.

The variance of a single machine mean must also include the random component due to operator. For both models it is accepted that the variance of a machine mean would be given by

$$(1/rn)(\sigma^2 + n\sigma^2_{FR} + n\sigma^2_R).$$

Table 4. ANOVA Table for Equation (5) With Finite Correction

Source	df	Mean squares	F	Expected mean squares
F	$f-1$	MSF	MSF/MSFR	$\sigma^2 + n(f/(f-1))\sigma^2_{FR} + rn\Phi(F)$
R	$r-1$	MSR	MSR/MSE	$\sigma^2 + nf\sigma^2_R$
FR	$(f-1)(r-1)$	MSFR	MSFR/MSE	$\sigma^2 + n(f/(f-1))\sigma^2_{FR}$
Error	$fr(n-1)$	MSE		σ^2
Total	$frn-1$			

NOTE: *MSF, MSR, MSFR, MSE* are mean squares for the corresponding source effect.

Table 5. ANOVA Table for Equation (6)

Source	df	Mean squares	F	Expected mean squares
F	$f - 1$	MSF	$MSF/MSFR$	$\sigma^2 + n\sigma^2_{FR} + rn\Phi(F)$
R	$r - 1$	MSR	MSR/MSE	$\sigma^2 + n\sigma^2_{FR} + nf\sigma^2_R$
FR	$(f - 1)(r - 1)$	$MSFR$	$MSFR/MSE$	$\sigma^2 + n\sigma^2_{FR}$
Error	$fr(n - 1)$	MSE		σ^2
Total	$frn - 1$			

NOTE: *MSF*, *MSR*, *MSFR*, *MSE* are mean squares for the corresponding source effect.

In all three cases this expression is not directly estimable from the ANOVA table; however, the operator and machine-by-operator variance components may be estimated and then used to compute the standard error. The results of estimating the three different variance components by the various models is illustrated in Table 7. Table 8 illustrates the effect of model assumption on the standard errors of marginal means and differences among marginal means.

If both machines and operators are considered fixed, and their interaction is considered random, then the implied variance of the estimate of the marginal mean for Machine 1 is

$$(1/6)(\sigma^2 + 2\sigma^2_{FR}).$$

Using the expected mean squares for Equation (6) as given in Table 5, an estimated standard error is .1584, which is identical to the computed standard error of Function 4.

If variance component estimates obtained from Equation (5) and Table 4 were used in Property 8 of Section 2 to evaluate the standard error of Function 4, the value .1288 would be obtained. Note that this is identical to the standard error for Function 15 using estimates from Equation (6) and Table 5. What is the relationship between these two standard errors? If Function 4 is defined in terms of the parameters of Equation (5), it has the form $\mu + F_1 + \overline{R}(5)_.$. Using the relationship between $R(5)_j$ and $R(6)_j$ given by Searle (1971) (see above), this is equivalent to Function 15 using Equation (6) parameters, that is, $\mu + F_1 + \overline{R}(6)_. + \overline{FR}(6)_{..}$. Thus their inference space and standard errors are identical. This illustrates how the solutions from one version of the mixed model can readily be used to obtain estimates intended for another version using suitably defined estimable functions. This can have very important practical consequences. In this case, Equation (5) requires the use of a singular **G**, because of the relationship in Equation (7), whereas Equation (6) does not.

In the previous discussion Equation (1) was used with diagonal **R** and **G** matrices, which implies that all random components of the model are uncorrelated and have no constraints. Thus MMP has the same assumptions as Equation (6), and this explains the agreement between the two procedures shown in Tables 7 and 8 on page 61. It has been demonstrated, however, that if the researcher did not feel that Equation (6) was appropriate, then estimable functions, like Function 15, could be formed. Thus the calculations done with a standard model could be used to do estimation under an alternative set of model assumptions, such as those found in Equation (5). The assumptions of Equation (5) may be used in MMP with a singular **G**, but this requires a deviation (Henderson 1984a, p. 48) from Equation (2), which is structured with a nonsingular **G**. MMP also has the flexibility to treat the least squares means of the fixed effect as if both factors were fixed, to allow inference across all operators and associated interactions and the possible use of some combination of the inference space associated with either of the two random effects.

5. AN EXAMPLE WITH UNBALANCED DATA

One of the major strengths of MMP is that the procedure works in exactly the same fashion for unbalanced data as it does for balanced data. As an illustration, consider the data given in Table 1 with the observation value of 52.33 in the Operator 2 and Machine 2 cell deleted. Equation (1) would then appear as shown in Figure 2. The use of an REML estimation procedure yields the estimates of .11532, .01985, and .06046 for the variance components for operators, interaction, and error, respectively.

The REML estimates were then used in Equation (2) directly to obtain estimates of $\mathbf{b}' = (51.908\ -.9593\ 0)$ and $\mathbf{u}' = (.2748\ .0541\ -.3289\ .0523\ -.0644\ .0121\ -.0049\ .0738\ -.0689)$. The structure of **G** is exactly the same as in the balanced case, and **R** is now 11 by 11. The same estimable functions were evaluated as in the balanced case, and those results are given in Table 9. It should be noticed that all of the values given in Table 9 are relatively close to the values given in Table 2 and all interpretations are the same.

Table 6. ANOVA Table for Machine Operator Data

Source	df	Sum of Squares	Mean squares	F (Table 4)	F (Table 5)
F	1	3.050208	3.050208	20.26	20.26
R	2	1.159800	.579900	11.95	3.85
FR	2	.301067	.150533	3.10	3.10
Error	6	.291150	.048525		
Total	11	4.802225			

Table 7. Variance Component Estimates

Component	Equation (5), Table 3	Equation (5), Table 4 (finite correlation)	Equation (6), Table 5	MMP–REML
Operators (R)	.1328	.1328	.1073	.1073
FR interaction	.0510	.0255	.0510	.0510
Error	.0485	.0485	.0485	.0485
Total	.2323	.2068	.2068	.2068
Operator/Total	.5717	.6422	.5189	.5189

If an entire cell is missing in an analysis, the impact on estimability differs from conventional fixed linear model theory. Henderson (1984a) showed that BLUP's can be obtained for random effects even when corresponding cells are missing. Thus missing cells will not affect estimability unless they are missing for an entire level of a fixed effect. For example, suppose that both observations for Operator 2, Machine 2 are lost. Under conventional fixed linear model theory, Functions 2, 6, 9, 10, and 15 would be affected. Under mixed model theory, however, a BLUP can be obtained for FR_{22}, and thus all of these functions can still be estimated.

6. MULTIVARIATE APPROACH

The data for the balanced case given in Section 3 could be viewed as a multivariate-type experiment since the observations on both machines were generated by the same operators. This approach may be visualized by considering the correlation among machines that exists as a result of differences among operators and the associated interaction between machines and operators. A plot of the cell means of the data in Table 1 helps visualize the correlation among machines and is shown in Figure 3. The sample variance of these cell means for Machine 1 and Machine 2 and the corresponding covariance are .1331, .2322, and .1073, respectively. Note also that the covariance between the two machines is the same as the variance component due to operators as estimated by Table 6 and MMP–REML. The sum of the variance components for operators and interaction as determined from Table 6 and MMP–REML is .1583. This is equal to the average of the two sample variances less the estimated variance of a cell mean and may be computed as $.5^*(.1331 + .2322) - (.0485/2) = .1583$. Anderson and Bancroft (1952, p. 341) hinted about the possible use of intraclass correlations to resolve some of the model difficulties that exist. Scheffe (1959) described the relationship shown here. Hocking (1985) has done considerable work in relating covariances of the observations to the variance components of various models. He also gave an excellent comparison of maximum likelihood, REML, and the analysis of means procedures for variance component estimation. The estimation of variance components via covariances appears to have computational advantages over the more traditional procedures. We proceed to show that there is an alternative way of using MMP that permits the use of this concept of covariances for data analysis.

For this illustration consider Equation (1). Take $\mathbf{X}$ to be a 12-by-2 matrix to denote the two machine effects. The overall mean has been included with the machine effects, thus reducing the size of the parameter vector by 1. The matrix $\mathbf{Z}$ is taken to be 12 by 6, one column for each operator and machine combination. This then allows the consideration of the covariance structure between machines for each operator. Thus the matrix $\mathbf{G}$ will be block-diagonal, consisting of three 2-by-2 blocks. One method of analysis would be to assume homogeneous variance and covariance structure for all operators. This assumption indicates that all three 2-by-2 blocks of $\mathbf{G}$ are the same. If prior estimates of these variance and covariance elements are available, they can be used in place of those estimated by the current data. This last fact may motivate experimenters to make use of past data and thus give stronger results. The REML procedure described previously was used to obtain the two components required to form $\mathbf{G}$ and the one component for the diagonal elements of $\mathbf{R}$. The estimate of the diagonal element of $\mathbf{G}$ is .1583, and the off-diagonal element is .1073, with the diagonal elements of $\mathbf{R}$ estimated as .0485. All of these values were discussed previously. After obtaining these estimates the solution to the mixed model equations gives (50.95 51.96 .350 .219 −.046 .257 −.304 −.475) as the estimates of ($\mathbf{B}'$ $\mathbf{U}'$). Elements 3, 5, and 7 of this vector represent the estimates of the realized values of the effects of Operators 1, 2, and 3 on Machine 1. Elements 4, 6, and 8 are for the same operators on Machine 2. Note that each of these two sets of three elements sum to 0, allowing us to estimate the effect for Machine 1 by using the vector (3 0 1 0 1 0 1 0)/3 or the vector (1 0 0 0 0 0 0 0) and obtain a standard error of .0899 or .2467 using Property 8 of Section 2. The estimate of the Machine 1 mean is 50.95, and these standard errors are exactly the same as those obtained using Equation (6). The first standard error would be used for confidence intervals involving inference to these three specific operators. The second standard error would apply to the Machine 1 mean when inference applies to all operators in the population sampled. It was mentioned previously that one could estimate separate variance compo-

Table 8. Standard Errors

Effect	Table 3	Table 4	Table 5	MMP–broad	MMP–narrow
Machine mean	.2634	.2467	.2467	.2467, Function 4	.0899, Function 1
Difference	.2240	.2240	.2240	.2240, Function 6	.1272, Function 5

$$
\begin{bmatrix} 51.43 \\ 51.28 \\ 50.93 \\ 50.75 \\ 50.47 \\ 50.83 \\ 51.91 \\ 52.43 \\ 52.26 \\ 51.58 \\ 51.23 \end{bmatrix}
=
\begin{bmatrix}
1 & 1\,0 & 1\,0\,0 & 1\,0\,0\,0\,0\,0 \\
1 & 1\,0 & 1\,0\,0 & 1\,0\,0\,0\,0\,0 \\
1 & 1\,0 & 0\,1\,0 & 0\,1\,0\,0\,0\,0 \\
1 & 1\,0 & 0\,1\,0 & 0\,1\,0\,0\,0\,0 \\
1 & 1\,0 & 0\,0\,1 & 0\,0\,1\,0\,0\,0 \\
1 & 1\,0 & 0\,0\,1 & 0\,0\,1\,0\,0\,0 \\
1 & 0\,1 & 1\,0\,0 & 0\,0\,0\,1\,0\,0 \\
1 & 0\,1 & 1\,0\,0 & 0\,0\,0\,1\,0\,0 \\
1 & 0\,1 & 1\,0\,0 & 0\,0\,0\,0\,1\,0 \\
1 & 0\,1 & 1\,0\,0 & 0\,0\,0\,0\,0\,1 \\
1 & 0\,1 & 1\,0\,0 & 0\,0\,0\,0\,0\,1
\end{bmatrix}
\begin{bmatrix} \mu \\ F_1 \\ F_2 \\ R_1 \\ R_2 \\ R_3 \\ FR_{11} \\ FR_{12} \\ FR_{13} \\ FR_{21} \\ FR_{22} \\ FR_{23} \end{bmatrix}
+
\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \\ \epsilon_{10} \\ \epsilon_{11} \end{bmatrix}
$$

Figure 2. Equation 1 for Data in Table 1 With One Data Point Missing.

nents for the diagonal elements of the 2-by-2 blocks in the matrix **G** by using the same REML algorithm. The estimated components for Machines 1 and 2 were .1088 and .2079, respectively. Note that the average of these is that obtained in the previous part of this example. The estimates of the other two variance components were the same, and the estimate of the (**B′ U′**) vector was (50.95 51.96 .317 .232 −.031 .267 −.286 −.500), which differs in the estimates of the random effects when compared with that obtained when using the pooled estimate of variance. The corresponding standard errors for Machine 1 are .0899 and .2106. The first standard error did not change, since it does not depend on the variation among operators, but the second standard error is now smaller, as there is less apparent variation among operators with Machine 1 as compared with Machine 2.

Henderson (1984a) has done considerable work with multiple trait models, which may be considered as a multivariate-type problem in that there is more than one response variable. By altering the structure of **G** and **R** it is possible to present excellent multivariate models on a strong parametric foundation. The **R** matrix can handle the various covariances and variances for the various response variables, and **G** can still be structured to accommodate the relationships among the random effects. The strong parametric modeling procedures that are available will require less sampling units than conventional multivariate procedure. In addition, most multivariate software programs require complete records of data for inclusion into the data set, whereas incomplete records may be used in the MMP approach.

7. CONCLUSIONS

This material has been written as a general appeal to statisticians to teach the concepts of linear models, including fixed, random, and mixed models, in a fashion that will be consistent throughout the course of instruction. Historically, much effort has been put into material that requires balanced designs, with an equal number of observations in every cell. In the mixed model case the estimation (prediction) of random effects has been ignored in elementary-level courses. We feel that now is the time to make use of current potential of existing computer hardware and theoretical knowledge about mixed linear models to give the practitioner guidance that reflects developments of this decade. We would like to recommend the use of Equation (6) as the default for the analysis of mixed linear models. This recommendation is based on the following facts.

1. The standard errors for fixed effect marginal means and differences among such means are the same for Equation (6) and Equation (5), using the finite correction.
2. The MMP corresponds to Equation (6) and offers the extension to estimation (prediction) of the random effects, ranking of the random effects, and analysis of unbalanced data.
3. There is a wealth of literature supporting the use of variance component estimation, based on Equation (6).
4. Where relevant, estimable functions can be constructed to permit inference implied by Equation (5), using MMP methods (e.g., Function 15).
5. Perhaps most important, Equation (6) provides a far more tractable approach to the estimation of variance components—and, more generally, the working with the mixed model equations—than does Equation (5). On the other hand, in cases in which the desired inference space is defined in terms of Equation (5), the estimators can easily be ob-

Table 9. Estimable Functions for Completely Randomized Example

	(K′ M′)														
							FR								
Function	μ	F_1	F_2	R_1	R_2	R_3	11	12	13	21	22	23	($\mathbf{K'b} + \mathbf{M'u}$)	Standard error	Inference space*
1. *F* mean	(3	3	0	1	1	1	1	1	1	0	0	0)/3	50.95	.1004	N
2. *F* mean	(3	0	3	1	1	1	0	0	0	1	1	1)/3	51.91	.1131	N
3. *F* mean	(3	3	0	0	0	0	0	0	0	0	0	0)/3	50.95	.2347	B
4. *F* mean	(3	3	0	1	1	1	0	0	0	0	0	0)/3	50.95	.1292	I
5. *F* mean	(3	3	0	0	0	0	1	1	1	0	0	0)/3	50.95	.2202	I
6. *F* difference	(0	3	−3	0	0	0	1	1	1	−1	−1	−1)/3	−.96	.1512	N
7. *F* difference	(0	1	−1	0	0	0	0	0	0	0	0	0)	−.96	.1900	B
8. *R* mean	(2	1	1	2	0	0	1	0	0	1	0	0)/2	51.73	.1185	N
9. *R* mean	(2	1	1	0	2	0	0	1	0	0	1	0)/2	51.49	.1389	N
10. *R* difference	(0	0	0	2	−2	0	1	−1	0	1	−1	0)/2	.240	.1784	N
11. *FR* mean	(1	1	0	1	0	0	1	0	0	0	0	0)	51.28	.1524	N
12. *FR* mean	(1	1	0	0	1	0	0	1	0	0	0	0)	50.94	.1540	N
13. *R* difference	(0	0	0	1	−1	0	1	−1	0	0	0	0)	.337	.2011	N
14. *R* difference	(0	0	0	1	−1	0	0	0	0	0	0	0)	.221	.2128	B
15. *F* mean	(6	6	0	2	2	2	1	1	1	1	1	1)/6	50.95	.1178	I

*Inference space: B, broad; I, intermediate; N, narrow.

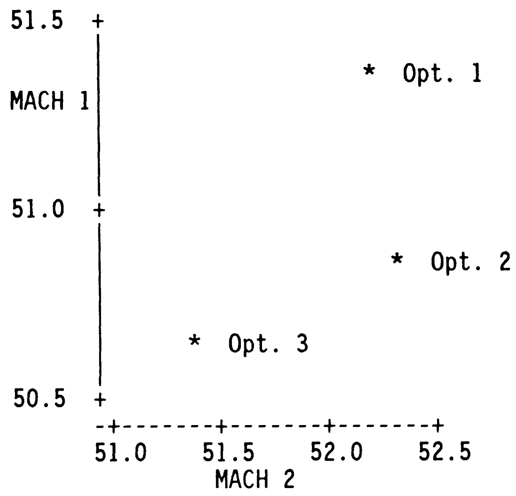

Figure 3. Plot of Cell Means of Table 6.

tained via Equation (6) (e.g., Function 15)—more easily, in fact, than by working directly with Equation (5)! This is of enormous consequence to software developers and data analysts as well as students of linear models.

In addition, the authors are especially strong in discouraging the use of Equation (5) without the finite correction, as shown in Table 3. Doing so produces variance component estimates that do not relate consistently to either model assumption and thus do not sum to a meaningful total variance (see Table 7). Consequently, application of these variance estimates to the calculation of various standard errors results in erroneous values. For example, as shown in Table 8, it produces estimates of standard errors that are incorrect, and could be drastically so whenever used with a small number of levels of the fixed effects.

The MMP approach to analysis is favored, as it is based on Equation (6) and combines the benefits of generalized least squares estimation of fixed effects and shrinkage estimation of realized values of random effects. The ability to select a $(\mathbf{K}'\ \mathbf{M}')$ vector that relates directly to the inference space is invaluable to the experimenter both as a tool and as a stimulus to careful thought. A further advantage of the MMP approach is that it allows the experimenter to have a consistent alternative when faced with the unbalanced case. The one major drawback that exists with Henderson's techniques is that there are no existing canned computer programs to do the analysis. Programs for this type of analysis are easily written in matrix language software modules like SAS/IML, APL, and Gauss.

In addition to these simple examples, one may apply the MMP to more complicated designs such as split plots. For the analysis of data from a split plot design the selection of the correct standard error for estimates of means and differences between means is much more difficult with the conventional ANOVA table approach, whereas the MMP approach gives the appropriate estimates of standard error as long as estimable functions of $K'B$ are used and the experimenter selects the appropriate inference space.

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HARRY O. POSTEN, Section Editor

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Statistical Computer Teaching Package: ISP

Available from Lincoln Systems Corporation, P.O. Box 391, Westford, MA 01886, (617) 692-3910. By Spyros Makridakis and Robert L. Winkler. For IBM–PC and compatible computers and VAX systems. $40 for academic and $150 for corporate use on IBM–PC, $1,500 for academic and $4,500 for corporate use on VAX. Requires 384K RAM with high-resolution graphics and 256K otherwise. PC– or MS–DOS version 2.0 or higher. In English and French. Copy protected.

Can computer-based programs help instructors teach and students learn introductory statistics? If the answer is a qualified "yes, to some extent" and instruction time is a scarce resource, then each instructor should determine if the net gain from using these programs is of practical significance. Recent research and our experience do not show that great gains can be had with instructional software, although it might prove useful for "solitary" study.

Where research and our own experience are in general agreement is that small class size, rapid feedback, regular tests and assignments, and joint student work projects are significantly more effective in producing practical improvements in student performance. As to solitary study, "Invariably, the researchers reported, 'students who study in small groups do better than students studying alone' " (Harvard University 1990). These results are for the average performance; some students will do well on their own. And the nature of the subject matter and the particular mode of self-study may play a role.

In our own experience we have found solitary study most useful in reinforcing knowledge in areas in which some learning has already taken place. We view computer-based instructional software as falling into the same category as the hard-copy study guides that we have frequently assigned. While waiting for objective research results on whether solitary computer-based study in statistics is more effective and efficient than hard-copy study, we suggest that each instructor make their own evaluation. Our evaluation of a program we recently examined follows.

ISP (Interactive Statistical Programs) is a self-instructional software program with some specific features, which we describe to help you decide if you want to try it. ISP has a digest of standard statistical textbook materials, but without any equations. It has a hierarchical modular structure, and the text is in a module reached directly from the main menu.

Also accessible from the main menu are the major functional modules: basic statistics, distributions, plotting and displaying, regression and analysis of variance, forecasting, and games and simulations. For utility models, we have transformations, calculations, a DOS shell, and file management. Each of these main menu models has submodules and so forth, in the menu-driven tradition, which students and faculty both will find easy to use.

Assuming that you have entered data, you can choose the modular program of your choice. For example, if you enter regression, then you choose variables from an automatic listing on the regression menu and automatically get the standard outputs. You can get the residuals and predicted point estimates. The inquisitive student can modify the data set and rerun the regression.

Bayesian estimators are an uncommon feature in introductory statistical software (and courses). ISP discusses and illustrates Bayesian estimates and finds Bayesian confidence intervals for the mean.

For a beginner, ISP could be a satisfactory first statistical software program. It is a limited program, but students (and instructors) can use it to analyze their own data sets. This would save an instructor (and the students) from having to work with another program.

You enter data in a spreadsheet-like format (similar to NCSS and STATISTIX, for example). The maximum number of columns is 50 and the maximum number of rows is 1,000, but the overall limit on number of items is 16,000. The editor is about as easy to use as you can get.

The self-teaching modules give you first the textbook material on the subject—generally in the form of summaries that are useful as refreshers. Then you can get examples and instructions on the use of ISP in that context.

There are self-teaching modules for all of the major headings of basic statistics and a module of exercises on all covered subjects. You can choose multiple choice or computational exercises, and when your answers are wrong, you are given the right answer. Your scores are cumulated and reported when you exit from the program.