(REML Example)

Solution

- $\text{cov}(Y_i, \bar{Y}) = \text{cov}\left(Y_i, \frac{1}{n}\sum_{j=1}^{n} Y_j\right)$

$$= \frac{1}{n}\text{cov}(Y_i, Y_i) + 0$$

$$= \frac{\sigma^2}{n}$$

- $\text{var}(R_i) = \text{var}(Y_i - \bar{Y})$

$$= \text{var}(Y_i) + \text{var}(\bar{Y}) - 2\,\text{cov}(Y_i, \bar{Y})$$

$$= \sigma^2 + \frac{\sigma^2}{n} - 2\frac{\sigma^2}{n}$$

$$= \sigma^2\left(1 - \frac{1}{n}\right)$$

- $\text{cov}(R_i, R_j) = \text{cov}(Y_i - \bar{Y}, Y_j - \bar{Y}) \qquad i \neq j$

$$= \text{cov}(Y_i, Y_j) - \text{cov}(Y_i, \bar{Y}) - \text{cov}(\bar{Y}, Y_j) + \text{cov}(\bar{Y}, \bar{Y})$$

$$= 0 - \frac{\sigma^2}{n} - \frac{\sigma^2}{n} + \frac{\sigma^2}{n}$$

$$= -\frac{\sigma^2}{n}$$

- $\Sigma = \text{cov}(\underset{\sim}{R}) = \sigma^2 I - \frac{\sigma^2}{n} J$,

Σ is an $m \times m$ matrix.

- $\Sigma = (a - b)\, I + b\, J$

where $a = \sigma^2\left(1 - \frac{1}{n}\right)$,

$$b = -\frac{\sigma^2}{n}$$

- $$\Sigma^{-1} = (a-b)^{-1}\left\{ I - \frac{b}{a+(m-1)b} J \right\}$$
$$= \cdots$$
$$= \frac{1}{\sigma^2}(I + J).$$

- $$\underset{\sim}{R}^T \Sigma^{-1} \underset{\sim}{R} = \frac{1}{\sigma^2} \underset{\sim}{R}^T (I + J) \underset{\sim}{R}$$
$$= \frac{1}{\sigma^2}\left\{ \sum_{i=1}^{m} R_i^2 + \left(\sum_{i=1}^{m} R_i \right)^2 \right\}.$$

- Note: $\underset{\sim}{R}^T J \underset{\sim}{R} = \underset{\sim}{R}^T \underset{\sim}{1}\, \underset{\sim}{1}^T \underset{\sim}{R} = (\underset{\sim}{R}^T \underset{\sim}{1})(\underset{\sim}{1}^T \underset{\sim}{R}) = \left(\sum_{i=1}^{m} R_i\right)^2$

- Note: The sums are from 1 to m, not 1 to n.
- $|\Sigma| = n^{-1} (\sigma^2)^m$.
- The multivariate normal pdf is

$$f_{\underset{\sim}{R}}(\underset{\sim}{r}) = (2\pi)^{-m/2} |\Sigma|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \underset{\sim}{r}^T \Sigma \underset{\sim}{r} \right\}$$

- Log-likelihood (REML)
$$\ell(\sigma^2; \underset{\sim}{r}) = -\frac{m}{2}\log\sigma^2 - \frac{t}{2\sigma^2} + \text{constant},$$

$$t := \sum_{i=1}^{m} r_i^2 + \left(\sum_{i=1}^{m} r_i\right)^2$$
$$= \sum_{1}^{m} r_i^2 + (-r_n)^2 \qquad \left[\sum_{1}^{n} r_i = 0\right]$$
$$= \sum_{i=1}^{n} r_i^2 \equiv RSS.$$

- $\Rightarrow \hat{\sigma}^2_{REML} = RSS/m = RSS/(n-1)$
Verify.
- $\hat{\sigma}^2_{MLE} = RSS/n$, different.