# BIOS 660/BIOS 672 (3 Credits): Probability and Statistical Inference I

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Notes 4

### Combinations (Subpopulations)

How many ways can we select $r$ objects from $n$ paying no attention to order? Continue to suppose that the $n$ items are distinct.

| | |
|---|---|
| $n = 4$ | $A, B, C, D$ |
| $r = 2$ | Possible ordered samples:<br>$(A, B), (A, C), (A, D), (B, C), (B, D), (C, D)$<br>$(B, A), (C, A), (D, A), (C, B), (D, B), (D, C)$ |
| $n!/(n-r)!$ | $= 4!/2! = 4 \cdot 3 = 12$ |

6 = No. of ways of choosing 2 objects from among 4 objects with no ordering... Why?

There are 4*3 = 12 ways of choosing ordered pairs. Then there are 2! = 2*1 orderings that we don't really care about, so 12/2 = 6.



### Combinations (2)

No. of ordered samples = No. of Combinations $\times$ No. of ways of permuting each ordered sample.

$$\begin{aligned}
\frac{n!}{(n-r)!} &= c \cdot r! \\
c &= \frac{n!}{(n-r)!r!} = \binom{n}{r} = \binom{n}{n-r} \\
\binom{n}{r} &= \text{No. of ways of choosing } r \text{ objects from among } n \\
&= \text{we say ``}n\text{ choose }r\text{''}
\end{aligned}$$

**Theorem:** A population of $n$ elements possesses $\binom{n}{r}$ different combinations (subpopulations) of size $r \le n$, i.e. $\binom{n}{r}$ represents the number of possible combinations of $n$ objects taken $r$ at a time.

## Conventions

<u>Conventions</u>:

$$0! = 1; \left( \begin{array}{c} n \\ 0 \end{array} \right) = 1 \; ; \quad \left( \begin{array}{c} n \\ r \end{array} \right) = 0 \qquad \text{if } \; r > n, \; r < 0$$



## Examples

<u>Applications of $\left( \begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!}$</u>

- From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?
- How many different poker hands with 5 cards are possible?
- What is the *probability* of 4 aces in a poker hand with 5 cards?

## Examples(2)

- What is the *probability* of 4 of a kind (e.x. 4 Kings, 4 Queens, etc)?
- What is the *probability* of (exactly) 3 of a kind with no other pairs?
  **Size of Event:**



## Binomial Theorem

Suppose $n$ is a positive integer. The Binomial Theorem says

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

Proof: Homework, by induction.

Note that if $a+b=1$ one has

$$\sum_{r=0}^{n} \binom{n}{r} a^r (1-a)^{n-r} = 1,$$

and that if $a=b=1$,

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Note: the left hand side is the number of all possible subsets from $\{X_1, \ldots, X_n\}$. The above equality tells us that the number of all possible subsets from $\{X_1, \ldots, X_n\}$ is $2^n$, which motivates the notation $2^\Omega$ for indicating power set .

## Pascal's Triangle (1)

Pascal's triangle is an illustration of the following result:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Proof: Homework.



## Pascal's Triangle (2)

Pascal's triangle

$$\begin{array}{ccccccc}
 & & & \binom{0}{0}=1 & & & \\
 & & \binom{1}{0}=1 & & \binom{1}{1}=1 & & \\
 & \binom{2}{0}=1 & & \binom{2}{1}=2 & & \binom{2}{2}=1 & \\
\binom{3}{0}=1 & & \binom{3}{1}=3 & & \binom{3}{2}=3 & & \binom{3}{3}=1
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

## Partitions

- Beyond a single subpopulation...
- **Theorem:** Let $r_1, r_2, \cdots, r_k$ be nonnegative integers such that

$$r_1 + r_2 + \cdots + r_k = n$$

  The number of ways in which a population of $n$ elements can be divided into $k$ ordered groups (partitioned into $k$-subpopulations) of which the first contains $r_1$ elements, the second $r_2$ elements, etc., is

$$\frac{n!}{r_1!r_2!\cdots r_k!}$$

- Example: Ten children are being divided into an A team and a B team of 5 each. How many different divisions are possible?



## Occupancy Problem

Suppose we have $n$ indistinguishable balls which we wish to place in $r$ distinguishable urns. How many different outcomes are possible?

- We can describe the outcome by a vector $(x_1, \ldots, x_r)$ where $x_j \geq 0$ is the number of balls in the $j$th urn and $\sum_{j=1}^{r} x_j = n$. We are therefore looking for the number of possible vectors.
- We can further think of this as taking $n$ indistinguishable items and dividing them into $r$ groups.
- First consider the situation that each urn should have at least 1 ball. In this situation, we are dealing with $x_j > 0$ ($j = 1, \cdots, r$). There are $n - 1$ spaces between the objects, and we select $r - 1$ of them to choose our dividing points. Therefore, there are $\binom{n-1}{r-1}$ different partitions.
- Now suppose that some of the urns are allowed to be empty. In this case we can imagine that we have $n$ objects plus $r - 1$ dividers. The number of possible partitions is therefore the number of (non-redundant) orderings of these $n + r - 1$ items which is equal to $(n + r - 1)!/[n!(r - 1)!] = \binom{n+r-1}{r-1}$

## Occupancy: Results

- **Theorem:** there are $\binom{n-1}{r-1}$ distinct positive integer valued vectors $(x_1, \ldots, x_r)$ satisfying

$$x_1 + \cdots + x_r = n, \quad x_j > 0, j = 1, \ldots, r$$

- **Theorem:** there are $\binom{n+r-1}{r-1}$ distinct nonnegative integer-valued vectors $(x_1, \ldots, x_r)$ satisfying

$$x_1 + \cdots + x_r = n, \quad x_j \geq 0, j = 1, \ldots, r$$



## Examples

1. How many distinct nonnegative integer-valued solutions of $x_1 + x_2 = 3$ are possible?

2. The state health department has 20 cases of flu vaccine to distribute among 4 possible hospitals. If all of the vaccine must distributed, how many different ways can the 20 cases be distributed if not every hospital has to receive some? What if the department wants to have the flexibility to hold some in reserve and not every hospital has to receive some?

## Stirling's Approximation

- **Stirling's Formula:**

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$