BIOS 660

09/11/2018 Problem Set #3 Q 1~3

[redacted]

(a) Can all points be equally likely

1. Let Ω be a sample space with countably infinite disjoint events corresponding to countably infinite points.
Assume the events $E_1, E_2, \cdots$ have the same probability.
By the property of probability measure,
$P(E_1 \cup E_2 \cup \cdots) = \sum_{i=1}^{\infty} P(E_i)$.

Case 1: $P(E_i) = 0$ for all $i \in N$. Then $\sum_{i=1}^{\infty} P(E_i) = 0$.
This contradicts the property of probability measure that $\sum_{i=1}^{\infty} P(E_i)$ (for all $E_i \in \Omega$) $= 1$. ✓

Case 2: $P(E_i) > 0$ for all $i \in N$. Then $\sum_{i=1}^{\infty} P(E_i) = \infty$.
This contradicts the property of probability measure that $\sum_{i=1}^{\infty} P(E_i) = 1$. ✓

Thus, $P(E_i)$ for $i \in N$ cannot all be equal and all points cannot be equally likely.

(b) can all points have positive probability

Yes., A example would be

Let Ω be a sample space countably infinite disjoint events $E_1, E_2, \ldots$ corresponding to countably infinite points.

Let $(a_n) = \left(\frac{1}{2^n}\right)$ be an infinite sequence.

Each $E_i, i \in N$ corresponds to the i^{th} term in a_n.

Thus, $P(E_i) = \left(\frac{1}{2^i}\right) > 0$. So each point has positive probability. Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, it also satisfy the probability measure of $\sum_{n=1}^{\infty} P(E_i) = 1 = P(N)$.

Thus, all points can have positive probability and satisfy the properties of probability measure.

2.

a) exacting two given names:

$\underset{A-Z}{26} \times \underset{A-Z}{26} \times \underset{A-Z}{26} = 17576$ initials ✓

b) at most two given names:

Assuming that each person has to have at least one given name.

$\underbrace{26\times26}_{\text{one given name}} + \underbrace{26\times26\times26}_{\text{two given names}} = 18252$ initials. ✓

c) at most three given names:

Same assumption as part b)

$\underbrace{26\times26}_{\text{one given name}} + \underbrace{26\times26\times26}_{\text{two given names}} + \underbrace{26\times26\times26\times26}_{\text{three given names}} = 475228$ initials. ✓

3. There n! total number of ways to order n digits.

(a) There are 2! ways of ordering the two digits.
There are n-1 spots to place the two digits
and n-2 spots for the rest of the digits after
placing 1 and 2.
Thus, there are $2! \times \binom{n-1}{1} \times (n-2)!$ orderings.
This gives us the probability of $\frac{2 \times (n-1)(n-2)!}{n!}$
$= \frac{2 \times (n-1)}{n \times (n-1)} = \frac{2}{n}$.

(b) There are 3! ways of ordering the three digits.
There are n-2 spots to place the three digits
and n-3 spots to place the rest of the digits
after placing 1, 2, and 3.
Thus there are $3! \times \binom{n-2}{1} \times (n-3)!$ orderings.
This gives us the probability of $\frac{3! \times (n-2) \times (n-3)!}{n!}$
$= \frac{6 \times (n-2)}{(n)(n-1)(n-2)} = \frac{6}{n^2-n}$.

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BIOS 660 - HW #3 - (Q4-Q6)

(4) Need to find all combinations of two people, where order does not count.

$$\binom{20}{2} = \frac{20!}{2!(20-18)!} = \frac{20 \times 19}{2} = \boxed{190 \text{ handshakes}}$$

(5) Need to find combinations of 7 from 10 people, and order does matter.

$$(10 P 7) = \frac{10!}{(10-7)!} = \boxed{604{,}800 \text{ distinct results}}$$

(6) (a) For the urns and balls problem, we know that the amount of possibilities (assuming some of the urns can be left empty), is:

$$\frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{r-1} = \binom{11}{3} = \frac{11!}{3!(8)!} = \boxed{165 \text{ ways}}$$

(b) For urns & balls problem, we know that the amt of possibilities, assuming each urn has at least one ball, is:

$$\binom{n-1}{r-1} = \binom{7}{3} = \boxed{35 \text{ ways}}$$

7. (a) If an investment must be made in each opportunity.

∵ The minimal investments are 2, 2, 3, and 4 thousand dollars for the 4 possible opportunities and we have 20 thousand dollars in total, the problem can be simplifed as investing

$20-(2+2+3+4)=9$ thousand dollars on the 4 opportunities as x_1, x_2, x_3, x_4 respectively.

$$x_1+x_2+x_3+x_4=9 \quad \text{and} \quad x_i \geq 0, \quad i=1,2,3,4$$

$$\therefore \binom{9+4-1}{4-1} = \binom{12}{3} = \frac{12!}{3!9!} = 220$$

220 different investment strategies are available.

(b) investments must be made in at least 3 of the 4 opportunities.

① If we make investments on all the 4 opportunities, the result will be the same as in (a) that there exist 220 different strategies

② If we don't make investment on opportunity 1, the problem is the same as investing $20-(2+3+4)=11$ thousand dollars on the 3 opportunities x_2, x_3, x_4 respectively:

$$x_2+x_3+x_4=11 \quad \text{and} \quad x_i \geq 0, \quad i=2,3,4$$

$$\therefore \binom{11+3-1}{3-1} = \binom{13}{2} = \frac{13!}{2!11!} = 78$$ strategies are available.

③ If we don't make investment on opportunity 2, because the minimal investment of opportunity 2 is the same as opportunity 1, so as is shown in ②, $\binom{11+3-1}{3-1} = \binom{13}{2} = 78$ strategies are available

④ If we don't make investment on opportunity 3, the problem is the same as investing $20-(2+2+4)=12$ thousand dollars on the 3 opportunities x_1, x_2, x_4 respectively

$$x_1+x_2+x_4=12 \quad \text{and} \quad x_i \geq 0, \quad i=1,2,4$$

$$\therefore \binom{12+3-1}{3-1} = \binom{14}{2} = \frac{14!}{2!12!} = 91$$ strategies are available

⑤ If we don't make investment on opportunity 4, the problem is the same as investing $20-(2+2+3)=13$ thousand dollars on the 3 opportunities x_1, x_2, x_3 respectively.

$$x_1+x_2+x_3=13 \quad \text{and} \quad x_i \geq 0, \quad i=1,2,3$$

$\therefore \binom{13+3-1}{3-1} = \binom{15}{2} = 105$ different strategies are available.

7. (b) continued

$\therefore$ In summary, if investments must be made in at least 3 of the opportunities,

$$\binom{9+4-1}{4-1} + 2\times\binom{11+3-1}{3-1} + \binom{12+3-1}{3-1} + \binom{13+3-1}{3-1}$$

$$= \binom{12}{3} + 2\times\binom{13}{2} + \binom{14}{2} + \binom{15}{2}$$

$$= 220 + 2\times 78 + 91 + 105$$

✓ $$= 572$$

572 different investment strategies are available.

8. ① Let $n = n_0 = 1$

$(a+b)^{n_0} = (a+b)^1 = a+b$

$\sum_{r=0}^{n_0} \binom{n_0}{r} \cdot a^r b^{n_0 - r} = \binom{1}{0} a^0 b^{1-0} + \binom{1}{1} \cdot a^1 \cdot b^0 = a+b$

$\therefore$ $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$ holds for $n = n_0 = 1$ and $n = n_0' = 0$

② Let $n > 2$; $(n-1) \geq 1$

Assume $(a+b)^{n-1} = \sum_{r=0}^{n-1} \binom{n-1}{r} a^r b^{(n-1)-r}$ holds

$\therefore$ $(a+b)^n = (a+b) \cdot (a+b)^{n-1} = (a+b) \cdot \left[\sum_{r=0}^{n-1} \binom{n-1}{r} a^r b^{(n-1)-r}\right]$

$$= \sum_{r=0}^{n-1} \binom{n-1}{r} \cdot a^{r+1} \cdot b^{(n-1)-r} + \sum_{r=0}^{n-1} \binom{n-1}{r} \cdot a^r \cdot b^{(n-1)-r+1}$$

$$= \sum_{r=1}^{n} \binom{n-1}{r-1} \cdot a^r \cdot b^{n-r} + \sum_{r=0}^{n-1} \binom{n-1}{r} \cdot a^r \cdot b^{n-r}$$

✓ $$= \left[\binom{n-1}{n-1} \cdot a^n \cdot b^0 + \sum_{r=1}^{n-1} \binom{n-1}{r-1} a^r b^{n-r}\right] + \left[\binom{n-1}{0} \cdot a^0 b^n + \sum_{r=1}^{n-1} \binom{n-1}{r} \cdot a^r b^{n-r}\right]$$

$$= a^n b^0 + a^0 b^n + \sum_{r=1}^{n-1} \left\{\left[\binom{n-1}{r-1} + \binom{n-1}{r}\right] \cdot a^r b^{n-r}\right\}$$

$$= a^n b^0 + a^0 b^n + \sum_{r=1}^{n-1} \binom{n}{r} a^r b^{n-r}$$

$$= \binom{n}{0} a^0 b^{n-0} + \sum_{r=1}^{n-1} \binom{n}{r} a^r b^{n-r} + \binom{n}{n} a^n \cdot b^{n-n}$$

$$= \sum_{r=0}^{n} a^r b^{n-r}$$

$\therefore$ $(a+b)^n = \sum_{r=0}^{n} a^r b^{n-r}$ holds for $n = 0, 1, 2, \cdots$

9. $\because \binom{n}{r-1} = \dfrac{n!}{(r-1)!\,(n-(r-1))!} = \dfrac{n!}{(r-1)!\,(n+1-r)!}$

$\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$

$$\therefore \binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!\,(n+1-r)!} + \frac{n!}{r!\,(n-r)!}$$

$$= \frac{r\cdot(n!) + (n+1-r)\cdot(n!)}{r!\,(n+1-r)!}$$

$$= \frac{[r+(n+1-r)]\cdot n!}{r!\,(n+1-r)!}$$

$$= \frac{(n+1)\cdot(n!)}{r!\,(n+1-r)!} = \frac{(n+1)!}{r!\,((n+1)-r)!} = \binom{n+1}{r}$$

$$\therefore \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$