Problem 1

(a)

$$f_X(x) = \frac{1}{\theta}e^{-x/\theta} \quad 0 < x < \infty,\ 0 < \theta < \infty$$

$$U = \frac{X_1}{\sum_{i=1}^n X_i} \quad V = \sum_{i=1}^n X_i \quad Y_1 = \sum_{i=2}^n X_i$$

$$U = \frac{X_1}{X_1 + Y_1} \quad V = X_1 + Y_1$$

$$X_1 = UV \quad Y_1 = V(1-U)$$

$$J = \begin{bmatrix} v & -v \\ u & 1-u \end{bmatrix} = |v(1-u) + uv| = |v|$$

(b)

$$f_X(x) = \frac{1}{\theta}e^{-x/\theta} \quad 0 < x < \infty,\ 0 < \theta < \infty$$

$$f_{Y_1}(y) = \frac{1}{\Gamma(n-1)}\theta^{n-1}y^{n-2}e^{-y/\theta} \quad 0 < y < \infty$$

$$0 < v < \infty \quad 0 < u < 1$$

$$f_{X_1,Y_1}(x,y) = \frac{1}{\Gamma(n-1)\theta^{n-1}}y^{n-2}e^{-y/\theta}\frac{1}{\theta}e^{-x/\theta}$$

$$f_{X_1,Y_1}(x,y) = \frac{1}{\Gamma(n-1)\theta^n}y^{n-2}e^{-(x+y)/\theta}$$

$$f_{U,V}(u,v) = f_{X_1,Y_1}(uv, v-uv)|v| = \frac{1}{\Gamma(n-1)\theta^n}(v(1-u))^{n-2}e^{-(uv+(v-uv))/\theta}|v|$$

$$f_{U,V}(u,v) = \frac{1}{\Gamma(n-1)\theta^n}(1-u)^{n-2}v^{n-1}e^{-v/\theta} \quad 0 < v < \infty,\ 0 < u < 1$$

(c)

$$f_{U,V}(u,v) = \frac{1}{\Gamma(n-1)\theta^n}(1-u)^{n-2}v^{n-1}e^{-v/\theta} \quad 0 < v < \infty,\ 0 < u < 1$$

$$f_{U,V}(u,v) = \left[\frac{1}{\Gamma(n)\theta^n}v^{n-1}e^{-v/\theta}\right]\left[\frac{\Gamma(n)}{\Gamma(n-1)}(1-u)^{n-2}\right]$$

Since $f_{U,V}(u,v)$ is factored into $f_U(u)f_V(v)$ U and V are independent

$$f_{U,V}(u,v) = f_U(u)f_V(v)$$

$$f_V(v) = \frac{1}{\Gamma(n)\theta^n}v^{n-1}e^{-v/\theta} \quad 0 < v < \infty$$

$$V \sim gamma(n,\theta)$$

$$f_U(u) = \frac{\Gamma([n-1]+1)}{\Gamma(n-1)}(1-u)^{([n-1]+1)} \quad 0 < u < 1$$

$$U \sim beta(1, n-1)$$

(d)

$$T(X) = V = \sum_{i=1}^{n} X_i$$

$$f(x|\theta) = \frac{1}{\theta}e^{-x/\theta} \quad 0 < x < \infty,\ 0 < \theta < \infty$$

V can be written as an exponential family in the form:

$$f(x|\theta) = h(x)c(\theta)\exp\left(w(\theta)t(x)\right)$$

$$h(x) = I(0 < x < \infty) \quad c(\theta) = 1/\theta \quad w(\theta) = -1/\theta \quad t(x) = x$$

$$\sum_{i=1}^{n} t_i(x_i) = \sum_{i=1}^{n} X_i$$

Thus $T(X) = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for θ

$$U = X_1 / \sum_{i=1}^{n} X_i$$

$$f_U(u) = \frac{\Gamma([n-1]+1)}{\Gamma(n-1)}(1-u)^{([n-1]+1)} \quad 0 < u < 1$$

U is independent from θ thus it does not depend on θ

Therefore U is an ancillary statistic of θ

(e)

$$E\{\delta(X_1)|\sum_{i=1}^{n} X_i = t\} = P(X_1 > c|\sum_{i=1}^{n} X_i = t)$$
$$= P\left(\frac{X_1}{\sum_{i=1}^{n} X_i} > \frac{c}{t}|\sum_{i=1}^{n} X_i = t\right)$$
$$= P\left(U > \frac{c}{t}|V = t\right)$$

From part d we know that U is an ancillary statistic of θ
and V is complete and sufficient
thus U and V are independent by Basu's Theorem giving us:

$$= P\left(U > \frac{c}{t}\right)$$
$$= \int_{c/t}^{1} (n-1)(1-x)^{n-2}\, \mathrm{d}x$$
$$= \Big|_{c/t}^{1} \frac{-(n-1)}{(n-1)}(1-x)^{n-1}$$
$$= (1 - c/t)^{n-1}$$

$$E(X_1|\sum_{i=1}^{n} X_i = t)$$
$$E\left(\frac{X_1}{\sum_{i=1}^{n} X_i}t|\sum_{i=1}^{n} X_i = t\right)$$
$$= E(Ut|V = t)$$

Using Basu's theorem, U and V are independent
Thus we have:

$$E(Ut|V = t) = E(Ut) = tE(U)$$
$$E(U) = \frac{1}{1+n-1} = \frac{1}{n} \text{ Thus:}$$
$$tE(U) = \frac{t}{n}$$

Problem 2

(a)

$$X_1, \dots, X_n \sim Bern(\theta_1) \quad 0 < \theta_1 < 1$$
$$Y_1, \dots, Y_n \sim Bern(\theta_2) \quad 0 < \theta_2 < 1$$
$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$
$$\text{Let } W_n = \sqrt{n}(\bar{X} - \theta_1)$$
$$\text{Let } Z_n = \sqrt{n}(\bar{Y} - \theta_2)$$
$$\text{Using CLT we have:}$$
$$W_n \xrightarrow{d} N(0, \theta_1(1-\theta_1))$$
$$Z_n \xrightarrow{d} N(0, \theta_2(1-\theta_2))$$

(b)

$$\gamma_1 = \frac{\theta_1}{1-\theta_1} \quad \gamma_2 = \frac{\theta_2}{1-\theta_2}$$
$$\log(\hat{\gamma_1}) = \log\left(\frac{\bar{X}}{1-\bar{X}}\right) \quad \log(\hat{\gamma_2}) = \log\left(\frac{\bar{Y}}{1-\bar{Y}}\right)$$
$$\log(\gamma_1) = \log\left(\frac{\theta_1}{1-\theta_1}\right) = \log(\theta_1) - \log(1-\theta_1)$$
$$\log(\gamma_2) = \log\left(\frac{\theta_2}{1-\theta_2}\right) = \log(\theta_2) - \log(1-\theta_2)$$
$$\text{Delta Method:}$$
$$\sqrt{n}(\bar{X} - \theta_1) \xrightarrow{d} N(0, \theta_1(1-\theta_1))$$
$$\sqrt{n}\left[\log\left(\frac{\bar{X}}{1-\bar{X}}\right) - \log\left(\frac{\theta_1}{1-\theta_1}\right)\right] \xrightarrow{d} N(0, \{g^{'}(\theta_1)\}^2\theta_1(1-\theta_1))$$
$$g(\theta_1) = \log\left(\frac{\theta_1}{1-\theta_1}\right)$$
$$g^{'}(\theta_1) = \frac{1}{\theta_1} + \frac{1}{1-\theta_1} = \frac{1}{\theta_1(1-\theta_1)}$$

$$\sqrt{n}\left[\log\left(\frac{\bar{X}}{1-\bar{X}}\right)-\log\left(\frac{\theta_1}{1-\theta_1}\right)\right]\xrightarrow{d}N\left(0,\left[\frac{1}{\theta_1(1-\theta_1)}\right]^2\theta_1(1-\theta_1)\right)$$
$$\sqrt{n}[\log(\hat{\gamma_1})-\log(\gamma_1)]\xrightarrow{d}N\left(0,\frac{1}{\theta_1(1-\theta_1)}\right)$$
$$\sqrt{n}[\log(\hat{\gamma_2})-\log(\gamma_2)]\xrightarrow{d}N\left(0,\frac{1}{\theta_2(1-\theta_2)}\right)$$

(c)

$$\text{WTS: }\sqrt{n}\{[\log(\hat{\gamma_1})-\log(\hat{\gamma_2})]-[\log(\gamma_1)-\log(\gamma_2)]\}\xrightarrow{d}N(0,\sigma^2)$$
Given $X \perp Y$
$$OR=\frac{\theta_1/(1-\theta_1)}{\theta_2/(1-\theta_2)}$$
$$\log(OR)=\log\left(\frac{\theta_1/(1-\theta_1)}{\theta_2/(1-\theta_2)}\right)$$
$$=\log\left(\frac{\theta_1}{1-\theta_1}\right)-\log\left(\frac{\theta_2}{1-\theta_2}\right)$$
$$=\log(\gamma_1)-\log(\gamma_2)$$
$$\log\left(\frac{\hat{\gamma_1}}{\hat{\gamma_2}}\right)=\log(\hat{\gamma_1})-\log(\hat{\gamma_2})$$
$$\sqrt{n}\{[\log(\hat{\gamma_1})-\log(\hat{\gamma_2})]-[\log(\gamma_1)-\log(\gamma_2)]\}$$
$$=\sqrt{n}\{[\log(\hat{\gamma_1})-\log(\gamma_1)]-[\log(\hat{\gamma_2})-\log(\gamma_2)]\}$$
From part b we know:
$$\sqrt{n}[\log(\hat{\gamma_1})-\log(\gamma_1)]\xrightarrow{d}N\left(0,\frac{1}{\theta_1(1-\theta_1)}\right)$$
$$\sqrt{n}[\log(\hat{\gamma_2})-\log(\gamma_2)]\xrightarrow{d}N\left(0,\frac{1}{\theta_2(1-\theta_2)}\right)$$
Since $X \perp Y$
$$\sqrt{n}\{[\log(\hat{\gamma_1})-\log(\gamma_1)]-[\log(\hat{\gamma_2})-\log(\gamma_2)]\}\xrightarrow{d}N(0,\sigma^2)$$
$$\sigma^2=\frac{1}{\theta_1(1-\theta_1)}+\frac{1}{\theta_2(1-\theta_2)}$$
The negative goes away because variance is squared:
$$X-Y=X+(-1)Y=N(0,\sigma^2)+N(0,(-1)^2\sigma^2)=N(0,\sigma^2)+N(0,\sigma^2)$$
$\mu=0$ for both and $0-0=0$

Thus we have $N(\mu_1-\mu_2,\sigma_1^2+\sigma_2^2)$

Problem 3

(a)

$$f_X(x) = 1/\theta \quad 0 < x < \theta$$
$$F_X(x) = \frac{x}{\theta} \quad 0 < x < \theta$$
$$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(X_1 \le x, \ldots, X_n \le x) = \{F(x)\}^n$$
$$F_{X_{(n)}}(x) = \{F(x)\}^n = \left(\frac{x}{\theta}\right)^n$$
$$\text{Thus } P(X_{(n)} \le x) = \left(\frac{x}{\theta}\right)^n$$

$$\begin{aligned}
P(|X_{(n)} - \theta| \le \epsilon) &= P(-\epsilon \le X_{(n)} - \theta \le \epsilon) \\
&= P(-\epsilon + \theta \le X_{(n)} \le \epsilon + \theta) \\
&= P(-\epsilon + \theta \le X_{(n)} \le \theta) \text{ since } x < \theta \\
&= F_{X_{(n)}}(\theta) - F_{X_{(n)}}(-\epsilon + \theta) \\
&= (\theta/\theta)^n - \left(\frac{\theta - \epsilon}{\theta}\right)^n \\
&= 1 - (1 - \epsilon/\theta)^n
\end{aligned}$$

$$\text{WTS: } X_{(n)} \xrightarrow{p} \theta \text{ which is the same as } \lim_{n\to\infty} P(|X_{(n)} - \theta| < \epsilon) = 1$$
$$\lim_{n\to\infty} P(|X_{(n)} - \theta| < \epsilon) = \lim_{n\to\infty} 1 - (1 - \epsilon/\theta)^n$$
$$(1 - \epsilon/\theta)^\infty \to 0 \text{ giving us:}$$
$$= 1 - 0 = 1$$
$$\text{Thus } X_{(n)} \xrightarrow{p} \theta$$

(b)

$$\text{WTS: } Z_n = n(\theta - X_{(n)}) \xrightarrow{d} exp(\theta)$$

$$\text{Using the fact that } \lim_{n\to\infty}(1-x/n)^n = e^{-x} \text{ for some } x \in (0,n)$$

$$F_{X_{(n)}}(x) = \left(\frac{x}{\theta}\right)^n$$

$$F_{Z_n}(z) = P(Z_n \le z)$$

$$= P(n(\theta - X_{(n)}) \le z) = P(\theta - X_{(n)} \le z/n)$$

$$= P(X_{(n)} \ge \theta - z/n)$$

$$= 1 - P(X_{(n)} \le \theta - z/n)$$

$$= 1 - P(X_1 \le \theta - z/n, \dots, X_n \le \theta - z/n)$$

$$= 1 - P(X_1 \le \theta - z/n) \cdots P(X_n \le \theta - z/n)$$

$$= 1 - \{F(\theta - z/n)\}^n$$

$$= 1 - \left(\frac{\theta - z/n}{\theta}\right)^n$$

$$= 1 - \left(1 - \frac{(z/\theta)}{n}\right)^n$$

$$\text{Since } \lim_{n\to\infty}(1-x/n)^n = e^{-x} \text{ where } x = z/\theta \text{ we have:}$$

$$\lim_{n\to\infty} 1 - \left(1 - \frac{(z/\theta)}{n}\right)^n = 1 - e^{-(z/\theta)}$$

$$\text{Which is } exp(\theta)$$

$$\text{Thus } Z_n \xrightarrow{d} exp(\theta)$$

(c)

$$\text{WTS: } Y_n = n\{1 - F_X(X_{(n)})\} \xrightarrow{d} exp(1) = 1 - e^{-y}$$

$$F_{Y_n}(y) = P(Y_n \le y)$$

$$= P(n(1 - F_X(X_{(n)})) \le y)$$

$$= P(F_X(X_{(n)}) \ge 1 - y/n)$$

$$= 1 - P(F_X(X_{(n)}) \le 1 - y/n)$$

$$= 1 - P(X_{(n)} \le F_X^{-1}(1 - y/n))$$

$$= 1 - P(X_1 \le F_X^{-1}(1-y/n), \dots, X_n \le F_X^{-1}(1-y/n))$$

$$= 1 - P(F_X(X_1) \le 1 - y/n, \dots, F_X(X_n) \le 1 - y/n)$$

$$= 1 - P(X_1/\theta \le 1 - y/n, \dots, X_n/\theta \le 1 - y/n)$$

$$= 1 - P(X_1 \leq \theta(1 - y/n), \ldots, X_n \leq \theta(1 - y/n))$$
$$1 - \{P(X_1 \leq \theta(1 - y/n))\}^n$$
$$= 1 - \{F_X(\theta(1 - y/n))\}^n$$
$$= 1 - \left(\frac{\theta(1 - y/n)}{\theta}\right)^n$$
$$= 1 - (1 - y/n)^n$$
$$\lim_{n\to\infty} 1 - (1 - y/n)^n = 1 - e^{-y}$$
$$\text{Thus } Y_n \xrightarrow{d} exp(1)$$

$$\text{Alternatively: } Y_n = n(1 - F_X(X_{(n)})) = n(1 - X_{(n)}/\theta)$$
$$F_{Y_n}(y) = P(Y_n \leq y)$$
$$= P(n(1 - X_{(n)}/\theta) \leq y)$$
$$= P(X_{(n)} \geq \theta(1 - y/n))$$
$$= 1 - P(X_{(n)} \leq \theta(1 - y/n))$$
$$= 1 - P(X_1 \leq \theta(1 - y/n)) \cdots P(X_n \leq \theta(1 - y/n))$$
$$1 - \{F_X(\theta(1 - y/n))\}^n$$
$$= 1 - (1 - y/n)^n$$
$$\lim_{n\to\infty} 1 - (1 - y/n)^n = 1 - e^{-y}$$
$$\text{Thus } Y_n \xrightarrow{d} exp(1)$$