Problem 1

(a)

$$(X_1, X_2, X_3) \sim Multinomial(n, p = [(1-\theta)^2, 2\theta(1-\theta), \theta^2])$$

$$\ell(\theta|x) = \log\left(\frac{n!}{x_1!x_2!x_3!}[(1-\theta)^2]^{x_1} 2\theta(1-\theta)^{x_2}(\theta^2)^{x_3}\right)$$

$$= \log\left(\frac{n!}{x_1!x_2!x_3!}\right) + 2x_1\log(1-\theta) + x_2\log(2\theta(1-\theta)) + 2x_3\log(\theta)$$

$$= \log\left(\frac{n!}{x_1!x_2!x_3!}\right) + (2x_1+x_2)\log(1-\theta) + (2x_3+x_2)\log(\theta) + x_2\log(2)$$

$$\frac{\partial\ell}{\partial\theta} = -\frac{2x_1+x_2}{1-\theta} + \frac{2x_3+x_2}{\theta} = 0$$

$$\frac{2x_1+x_2}{1-\theta} = \frac{2x_3+x_2}{\theta}$$

$$\frac{1-\theta}{\theta} = \frac{2x_1+x_2}{2x_3+x_2}$$

$$\frac{1}{\theta} = \frac{2x_1+x_2}{2x_3+x_2} + 1$$

$$\theta = \frac{2x_3+x_2}{2x_3+2x_2+2x_1}$$

Since $x_1 + x_2 + x_3 = n$:

$$\hat{\theta} = \frac{2x_3+x_2}{2n}$$

$$\frac{\partial\ell}{\partial\theta^2}^2 = -\frac{2x_1+x_2}{(1-\theta)^2} - \frac{2x_3+x_2}{\theta^2}$$

Plugging in $\hat{\theta}$:

$$-\left(\frac{2x_3+x_2}{(1-(2x_1+x_2)/2n)^2} + \frac{2x_3+x_2}{((2x_3+x_2)/2n)^2}\right)$$

$$-\left(\frac{2x_3+x_2}{((2n-2x_1+x_2)/2n)^2} + \frac{2x_3+x_2}{((2x_3+x_2)/2n)^2}\right) < 0$$

Thus $\hat{\theta}$ is the MLE

$$E(\hat{\theta}) = E\left(\frac{2x_3+x_2}{2n}\right) = \frac{1}{2n}[2E(X_3) + E(X_2)]$$

$$= \frac{2n\theta^2 + 2n\theta(1-\theta)}{2n} = \theta^2 + \theta(1-\theta) = \theta^2 + \theta - \theta^2 = \theta$$

$$E(\hat{\theta}) = \theta, \text{ Thus } \hat{\theta} \text{ is unbiased}$$

(b)

$$
\begin{aligned}
\frac{\partial \ell}{\partial \theta} &= -\frac{2x_1 + x_2}{1-\theta} + \frac{2x_3 + x_2}{\theta} \\
\frac{\partial \ell}{\partial \theta^2}^2 &= -\left(\frac{2x_1 + x_2}{(1-\theta)^2} + \frac{2x_3 + x_2}{\theta^2}\right) \\
-E[(\frac{\partial^2}{\partial \theta^2} \log f(x|\theta))] &= -E\left(-\frac{2x_1 + x_2}{(1-\theta)^2} - \frac{2x_3 + x_2}{\theta^2}\right) \\
&= \left[\frac{1}{(1-\theta)^2} 2(E(X_1) + E(X_2)) + \frac{1}{\theta^2}(2E(X_3) + E(X_2))\right] \\
&= \left[\frac{1}{(1-\theta)^2}(2n(1-\theta^2) + 2n\theta(1-\theta)) + \frac{1}{\theta^2}(2n\theta^2 + 2n\theta(1-\theta))\right] \\
&= 2n\left(2 + \frac{\theta}{1-\theta} + \frac{1-\theta}{\theta}\right) = 2n\left(\frac{\theta}{1-\theta} + \frac{1}{\theta} + 1\right) \\
&= 2n\left(\frac{\theta^2 + (1-\theta) + \theta(1-\theta)}{\theta(1-\theta)}\right) \\
-E[(\frac{\partial^2}{\partial \theta^2} \log f(x|\theta))] &= \frac{2n}{\theta(1-\theta)} \\
CRLB &= \frac{1}{-E[(\frac{\partial^2}{\partial \theta^2} \log f(x|\theta))]} = \frac{\theta(1-\theta)}{2n}
\end{aligned}
$$

(c)

$$\text{Show } \sqrt{n}(X_3/n - \theta^2) \xrightarrow{d} N(0, \theta^2(1-\theta^2))$$

Since the outcome is either AA,Aa, or aa:

$$P(AA) + P(Aa) + P(aa) = 1$$

$$P(aa) = 1 - [P(AA) + P(Aa)] = \theta^2$$

$$\text{Let } A = aa$$

$$A \sim Bern(\theta^2)$$

$$X_3 = \sum_{i=1}^{n} a_i$$

X_3 can be written as a sum of n iid $Bern(\theta^2)$ rvs each with:

$$\mu = \theta^2 \quad \sigma^2 = \theta^2(1-\theta^2)$$

$$\text{Let } Z_n = \sqrt{n}(X_3/n - \theta^2)$$

$$\text{By CLT: } Z_n \xrightarrow{d} N(0, \theta^2(1-\theta^2))$$

$$\text{Find } \sigma^2 \text{ such that } \sqrt{n}(\sqrt{X_3/n} - \theta) \xrightarrow{d} N(0, \sigma^2)$$

Using delta method:

$$\sqrt{n}\{g(X_3/n) - g(\theta^2)\} \xrightarrow{d} N(0, \{g^{'}(\theta^2)\}^2\theta^2(1-\theta^2))$$

$$g(x) = x^{1/2} \quad g^{'}(x) = (1/2)x^{-1/2}$$

$$\sqrt{n}\{\sqrt{X_3/n} - \theta^2\} \xrightarrow{d} N(0, \{(1/2)(\theta^2)^{-1/2}\}^2\theta^2(1-\theta^2)) = N(0, (1-\theta^2)/4)$$

$$\sigma^2/n = (1-\theta^2)/4n$$

$$\frac{1-\theta^2}{4n} \ ? \ \frac{\theta(1-\theta)}{2n}$$

$$1-\theta^2 \ ? \ 2\theta(1-\theta)$$

$$1-\theta^2 = P(AA) + P(Aa)$$

$$2\theta(1-\theta) = P(Aa)$$

$$\text{Since } P(AA + P(Aa) > P(Aa)$$

$$1-\theta^2 > 2\theta(1-\theta)$$

$$\text{Thus } \frac{\sigma^2}{n} > \frac{\theta(1-\theta)}{2n}$$

Problem 2

(a)

T_1, T_2 are SS, U unbiased estimator of $\theta \quad V_1 = E(U|T_1)$

Two ways to solve:

1) Let $\tau(\theta) = \theta$

U is an unbiased estimator of $\tau(\theta)$ and T_1 is an SS for θ

Let $\phi(T_1) = E(U|T_1)$

Then $E(\phi(T_1)) = \tau\theta = \theta$

Thus V_1 is an unbiased estimator of $\tau(\theta)$ by Rao-Blackwell Thm

2) $E(V_1) = E(E(U|T_1)) = E(U) = \theta$

Thus V_1 is an unbiased estimator of θ

$V_2 = E(V_1|T_2)$

$E(V_2) = E(E(V_1|T_2)) = E(V_1) = \theta$

Thus V_2 is an unbiased estimator of θ

(b)

WTS: $Var(V_2) \leq Var(V_1)$

Let $\tau(\theta) = \theta$ and $\phi(T_2) = E(V_1|T_2)$

$V_2 = E(V_1|T_2) = \tau(\theta)$

By Rao-Blackwell: $Var(\phi(T_2)) \leq Var(V_1)$

Thus $Var(V_2) \leq Var(V_1)$

Problem 3

(a)

$$f(x|\beta) = \frac{1}{\beta} e^{-x/\beta} \quad x > 0,\ \beta > 0$$

$$\ell(\beta|x) = -n\log(\beta) - \beta^{-1}\sum_{i=1}^{n} x_i$$

$$\frac{\partial \ell}{\partial \beta} = \frac{-n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i = 0$$

$$\frac{n}{\beta} = \frac{1}{\beta^2}\sum_{i=1}^{n} x_i$$

$$\hat{\beta} = \bar{X}$$

$$\frac{\partial \ell}{\partial \beta^2}^2 = \frac{n}{\beta^2} - \frac{2}{\beta^3}\sum_{i=1}^{n} x_i$$

Plugging in $\hat{\beta}$:

$$\frac{n}{\bar{X}^2} - \frac{2}{\bar{X}^3} n\bar{X}$$

$$= \frac{n - 2n}{\bar{X}^2}$$

$$-\frac{n}{\bar{X}^2} < 0$$

Thus $\hat{\beta}$ is the MLE

$$\tau(\beta) = \beta^2 = \theta$$

By the invariance property, the MLE of θ is $\bar{X}^2$

$$L(\theta|x) = \prod_{i=1}^{n} \theta^{-1/2}\exp(-x_i(\theta)^{-1/2})$$

$$= \theta^{-n/2}\exp\left(-\theta^{-1/2}\sum_{i=1}^{n} x_i\right)$$

$$\ell(\theta|x) = (-n/2)\log(\theta) - \theta^{-1/2}\sum_{i=1}^{n} x_i$$

$$\frac{\partial \ell}{\partial \theta} = \frac{-n}{2\theta} + \frac{1}{2\theta^{3/2}}\sum_{i=1}^{n} x_i = 0$$

$$\frac{n}{2\theta} = \frac{1}{2\theta^{3/2}}\sum_{i=1}^{n} x_i$$

$$\frac{\theta^{3/2}}{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\theta^{1/2} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\hat{\theta} = \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2$$
$$\hat{\theta} = \bar{X}^2$$
$$\frac{\partial^2}{\partial\theta^2} = \frac{n}{2\theta^2} - \frac{3}{4\theta^{5/2}}\sum_{i=1}^{n} x_i$$

(b)

$$E(\bar{X}^2) = Var(\bar{X}) + E(\bar{X})^2$$
$$= \beta^2/n + \beta^2 = \frac{\beta^2(n+1)}{n}$$
$$\hat{\theta}^* = \frac{n}{n+1}\bar{X}^2$$

Since $\bar{X}$ is a CSS (from exponential family)

and $\hat{\theta}^* = \dfrac{n}{n+1}\bar{X}^2$ is an unbiased estimator for $\tau(\beta) = \beta^2$ and:

$$E\left(\frac{n}{n+1}\bar{X}^2|\bar{X}\right) = \frac{n}{n+1}\bar{X}^2$$

By Lehmann-Sheffe Thm, $\hat{\theta}^*$ is the UMVUE

(c)

$$Var(\hat{\theta}^*) = Var\left(\frac{n}{n+1}\bar{X}^2\right) \quad \text{Let } W = \sum_{i=1}^{n} x_i \quad W \sim Gamma(n, \beta)$$

$$= Var\left(\frac{n}{n+1}\frac{1}{n^2}W^2\right) = \left(\frac{1}{n(n+1)}\right)^2 Var(W^2)$$

$$= \frac{1}{n^2(n+1)^2}(E(W^4) - E(W^2)^2)$$

$$E(W^4) = \frac{\Gamma(n+4)}{\Gamma(n)}\beta^4 \quad E(W^2) = \frac{\Gamma(n+2)}{\Gamma(n)}\beta^2$$

$$Var(\hat{\theta}^*) = \frac{1}{n^2(n+1)^2}\left[\frac{\Gamma(n+4)}{\Gamma(n)}\beta^4 - \left(\frac{\Gamma(n+2)}{\Gamma(n)}\beta^2\right)^2\right]$$

$$= \frac{1}{n^2(n+1)^2}\beta^4\left[\frac{\Gamma(n+4)}{\Gamma(n)} - \frac{\Gamma(n+2)^2}{\Gamma(n)^2}\right]$$

$$= \frac{1}{n^2(n+1)^2}\beta^4\left(\frac{(n+3)(n+2)(n+1)n\Gamma(n)}{\Gamma(n)} - \frac{(n+1)^2(n^2)\Gamma(n)^2}{\Gamma(n)^2}\right)$$

$$= \beta^4\frac{(n+3)(n+2)(n+1)n - (n+1)^2(n^2)}{n^2(n+1)^2}$$

$$= \beta^4\left(\frac{(n+3)(n+2)}{n(n+1)} - 1\right)$$

$$Var(\hat{\theta}^*) = \beta^4\left(\frac{4n+6}{n(n+1)}\right)$$

$$\frac{\partial\ell}{\partial\beta^2}^2 = \frac{n}{\beta^2} - \frac{2}{\beta^3}\sum_{i=1}^{n} x_i$$

$$-E[\frac{\partial^2}{\partial\theta^2}\log(f(X_1|\theta))] = -E\left(\frac{n}{\beta^2} - \frac{2}{\beta^3}\sum_{i=1}^{n} x_i\right) = -\frac{n}{\beta^2} + \frac{2}{\beta^3}E\left(\sum_{i=1}^{n} x_i\right)$$

$$= -\frac{n}{\beta^2} + \frac{2}{\beta^3}n\beta = \frac{-n+2n}{\beta^2} = \frac{n}{\beta^2}$$

$$\text{Since } \theta = \beta^2 \text{ we have: } \frac{n}{\beta^4}$$

$$CRLB = 1/\frac{n}{\beta^4} = \frac{\beta^4}{n}$$

$$Var(\hat{\theta}^*) = \beta^4\left(\frac{4n+6}{n(n+1)}\right) = \left(\frac{6}{n} - \frac{2}{n+1}\right)\beta^4 \approx \frac{4\beta^4}{n} \geq \frac{\beta^4}{n}$$

$$\text{alternatively as } n \to \infty : Var(\hat{\theta}*) \approx \frac{4\beta^4}{n} \quad \frac{4\beta^4}{n} \geq \frac{\beta^4}{n}$$

Thus $Var(\hat{\theta}^*)$ never reaches the CRLB

(d)

$$L(\beta|x) = \beta^{-n} \exp\left(\beta^{-1} \sum_{i=1}^{n} x_i\right)$$
$$\lambda(x) = \frac{L(\theta_0|x)}{L(\bar{x}|x)}$$
$$= \frac{\beta_0^{-n} \exp\left(-\sum_{i=1}^{n} x_i/\beta_0\right)}{\bar{x}^{-n} \exp\left(-\sum_{i=1}^{n} x_i/\bar{x}\right)}$$
$$= \left(\frac{\beta_0}{\bar{x}}\right)^{-n} \frac{\exp\left(-\sum_{i=1}^{n} x_i/\beta_0\right)}{\exp\left(-n^{-1}\right)}$$
$$\lambda(x) = \left(\frac{\beta_0}{\bar{x}}\right)^{-n} \exp\left(-\sum_{i=1}^{n} x_i/\beta_0 + n\right)$$

(e)

$$R = \{x : \lambda(x) \leq c\}$$
$$R^* = \{x : \bar{x} \leq c_1^* \text{ or } \bar{x} \geq c_2^*\}$$
WTS: R is equivalent to R^*
$$\lambda(x) = \left(\frac{\beta_0}{\bar{x}}\right)^{-n} \exp\left(-\sum_{i=1}^{n} x_i/\beta_0 + n\right)$$
$$= \left(\frac{\bar{x}}{\beta_0}\right)^{n} \exp\left(-n\bar{x}/\beta_0 + n\right)$$
Let $y = \frac{\bar{x}}{\beta_0}$
$$\lambda(x) = y^n \exp(-ny + n)$$
$$R = \{x : y^n \exp(-ny + n) \leq c\}$$
$$R = \{x : \log(y) - y \leq \log(c)/n - 1\}$$
$\log(y) - y$ is a concave function of y, thus:
R is equivalent to R^*