1. Let $X_1, \ldots, X_{n_1}$ be a random sample of size n_1 from $N(\mu_1, \sigma^2)$.

   (a) Find a constant c such that $c\sum_{i=1}^{n_1-1}(X_{i+1} - X_i)^2$ is an unbiased estimator of σ^2.

   (b) Let $Y_1, \ldots, Y_{n_2}$ be another random sample of size n_2 from $N(\mu_2, \sigma^2)$. Assuming X and Y are mutually independent, show that $S_p^2 = aS_1^2 + (1-a)S_2^2$ is an unbiased estimator of σ^2, where

   $$S_1^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}(X_i - \bar{X})^2 \quad \text{and} \quad S_2^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}(Y_i - \bar{Y})^2,$$

   with a constant $a \in [0, 1]$.

   (c) Find a such that $\text{var}(S_p^2)$ is minimized and the estimator becomes

   $$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left\{\sum_{i=1}^{n_1}(X_i - \bar{X})^2 + \sum_{i=1}^{n_2}(Y_i - \bar{Y})^2\right\}.$$

   [Hint]: Show that $\text{var}(S_p^2) = g(a)\sigma^4$ and find a such that $g'(a) = 0$ and $g''(a) > 0$.

2. For women in a certain high-risk population, suppose that the number of lifetime events of domestic violence involving emergency room treatments is assumed to have a Poisson distribution

   $$f_X(x|\theta) = \theta^x e^{-\theta}/x!, \quad x = 0, 1, \ldots, \quad \theta > 0.$$

   Let $X_1, \ldots, X_n$ be a random sample chosen for the high-risk population.

   (a) Find the maximum likelihood estimator (MLE) of the parameter $\tau(\theta)$, where

   $$\tau(\theta) = P(X_1 = 0) = e^{-\theta}.$$

   (b) Find Crámer-Rao Lower Bound (CRLB) for every unbiased estimator of the parameter $\tau(\theta)$.

   (c) Let $Y = \sum_{i=1}^{n} X_i$. Show that $T = (1 - 1/n)^Y$ is an unbiased estimator of $\tau(\theta)$ using the fact that $Y \sim \text{Poisson}(n\theta)$. Derive the variance of T and compare the variance to the CRLB.

(d) Show that T is the uniformly minimum variance unbiased estimator (UMVUE) of $\tau(\theta)$.

3. Assume that the distribution of wages (in thousands of dollars) in a large U.S. city follows a Pareto distribution with pdf

$$f_Y(y) = \theta\gamma^\theta y^{-(\theta+1)}, \quad 0 < \gamma < y < \infty, \quad 0 < \theta < \infty,$$

where γ is a unknown parameter but θ is assumed a known constant. Let $Y_1, \cdots, Y_n$ be a random sample from $f_Y(y)$. To test the null hypothesis $H_0 : \gamma \leq \gamma_0$ versus the alternative hypothesis $H_1 : \gamma > \gamma_0$, one intend to develop a likelihood ratio test (LRT) to conclude whether the minimum wage is smaller than some value γ_0.

(a) Derive the likelihood ratio test statistic $\lambda(\boldsymbol{y})$ and show that the rejection region $R = \{\boldsymbol{y} : \lambda(\boldsymbol{y}) \leq c\}$ is equivalent to $R^* = \{\boldsymbol{y} : y_{(1)} \geq c^*\}$, where $y_{(1)}$ is the minimum order statistic.