① Determine the number of vectors $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ s.t. $x_i = 0$ or 1 and $\sum_{i=1}^{n} x_i \geq k$

$n \geq k$

if k ones: $\binom{n}{k}$ ways

if $k+1$ ones: $\binom{n}{k+1}$ ways

if $k+2$ ones: $\binom{n}{k+2}$ ways

$\vdots$

if n ones: $\binom{n}{n}$ ways

sum all of these together: $\sum_{i=k}^{n} \binom{n}{i}$ ways

② Show that for $n>0$, $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$

binomial expansion: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

let $a=1$ so that $(1+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^k$

let $b=-1$ so that $\underbrace{(1-1)^n}_{=0} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$

this shows that $\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0$ for $n>0$

(a)

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

(b)

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$$

(c)

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}}$$

(d)

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$$

(e)

$$\frac{\binom{13}{1}\binom{48}{1}}{\binom{52}{5}}$$

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4. $\binom{52}{2}$ selections total. Assuming the rules are as follows: 10,J,Q,K are all 10 points, and Aces are 11 points. (Since two cards have to form 21, I assume the Ace will be 11 otherwise it's impossible). That being the case, the problem depends on first getting an Ace, then the 10, Jack, Queen, and King of any other suits. There are 4 aces, and 16 cards, making 64 two card hands that form 21.

$$P(Blackjack) = \frac{64}{\binom{52}{2}}$$

5. There are 8! arrangements in a row. There are also 4 couples, so whenever a couple sits together we'll call that event A_i where $i = 1, 2, 3, 4$. From this many arrangements, we need to subtract the moments where a couple is sitting together, so calculate $P(\bigcup_{i=1}^{4} A_i)$ and subtract from 1.
$P(A_i)$ = probability of ith couple sitting next to each other. Treat as a unit, and there will be $7! * 2$ arrangements. (7 units, 2 ways to arrange that couple)
$P(A_i \cap A_j)$ = probability of 2 couples sitting together. Treat both couples as units $= 6! * 2 * 2$ arrangements. Out of the 4 couples, there are $\binom{4}{2}$ ways to pick 2 of them.
$P(A_i \cap A_j \cap A_k)$ = probability of 3 couples. $= 5! * 2 * 2 * 2$, with $4 choose 3$ ways to pick them.
$P(A_i \cap A_j \cap A_k \cap A_l) = 4! * 2 * 2 * 2 * 2$
When we add together $\sum_{i=1}^{4} P(A_i)$ we'll have to subtract the intersections for $P(A_i \cap A_j)$. Then we'll have to add in $P(A_i \cap A_j \cap A_k)$, then subtract $P(A_i \cap A_j \cap A_k \cap A_l)$. I think we proved this in a previous homework where we extended the basic principle of $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to more terms. So:

$$P(\bigcup_{i=1}^{4} A_i) = \sum_{i=1}^{4} P(A_i) - \binom{4}{2} P(A_i \cap A_j) + \binom{4}{3} P(A_i \cap A_j \cap A_k) - P(A_i \cap A_j \cap A_k \cap A_l)$$

where $1 \leq i \neq j \neq k \neq l \leq 4$

$$= 4 * \frac{7! * 2}{8!} - 6 * \frac{6! * 2^2}{8!} + 4 * \frac{5! * 2^3}{8!} - \frac{4! * 2^4}{8!} = 1 - \frac{3}{7} + \frac{2}{21} - \frac{1}{105} = .657$$

Subtract this from 1, and we have .343, the probability of NO couples.

6. If m balls fall into the first compartment, the remaining $N-1$ compartments receive $n-m$ balls. The first compartment receives $\binom{n}{m}$ balls, while the remaining balls each have $N-1$ containers. Therefore the number of arrangements for the $n-m$ balls is $(N-1)^{n-m}$. Therefore:

$$P(mballsfirstcompartment) = \frac{\binom{n}{m}(N-1)^{n-m}}{N^n}$$

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7 What is the probability that both groups will have the same number of men?

There are

$$\frac{12!}{6!6!} = 924$$

ways of splitting 12 people into 2 groups of size 6 each. There are

$$\frac{6!}{3!3!} = 20$$

ways of splitting the 6 men into 2 groups of size 3 each, and also

$$\frac{6!}{3!3!} = 20$$

ways of splitting the 6 women into 2 groups of size 3 each. Therefore, the probability that both groups will have the same number of men

$$\frac{20 \times 20}{924} = 0.4329$$

8 What is the probability that both neighboring places are empty?

Since one of the r cars that still remain in the parking lot out of the original N cars in the lot is the car of the owner in question, there are

$$\binom{N-1}{r-1} = \frac{(N-1)!}{(N-1-r+1)!(r-1)!}$$

ways of having $r-1$ cars remain (or in other words, $r-1$ parking spots filled), out of the original $N-1$ cars/spots. We want to find the number of ways that both neighboring places are empty. In this case, we want to fix that both neighboring spots are empty, such that $r-1$ cars must occupy the other $N-3$ spots remain (not including the spot of the owner's car, and not the spots beside the owner's car). Thus we have

$$\binom{N-3}{r-1} = \frac{(N-3)!}{(N-3-r+1)!(r-1)!}$$

ways of having the $r-1$ cars in these $N-3$ spots. Therefore, the probability that both neighboring places are empty is

$$\frac{\binom{N-3}{r-1}}{\binom{N-1}{r-1}} = \frac{(N-3)!}{(N-r-2)!(r-1)!} \times \frac{(N-r)!(r-1)!}{(N-1)!} = \frac{(N-r)(N-r-1)}{(N-2)(N-1)}$$

9 What is the probability that the opposite face is tails?

$$P(H|Coin1) = 1, P(H|Coin2) = 0, P(H|Coin3) = \frac{1}{2}$$

$$P(H) = P(H|Coin1)P(Coin1)+P(H|Coin2)P(Coin2)+P(H|Coin3)P(Coin3)$$

$$= 1 \times \frac{1}{3} + 0 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3}$$

$$= \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$P(Coin3|H) = \frac{P(Coin3 \cap H)}{P(H)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$