Qaqish Review 6/19/19

$\boxed{2017}$ 2c)

- write down pdf of $Y_1$ to $Y_n$
  Gamma, $\alpha = 1 \longrightarrow$ exponential:

$$f_Y(y \mid \alpha = 1, \beta) = \frac{e^{-y/\beta}}{\beta} \qquad y > 0, \ \alpha = 1, \beta > 0$$

b/c exp fam + can't do ratio

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y} \mid \alpha = 1, \beta) = \beta^{-n} e^{-\sum_{i=1}^{n} y_i / \beta}$$

min suff. stat.

$\hookrightarrow$ DATA $\underset{\sim}{y}$

- all you need for likelihood is minimal sufficient statistic (up to a proportionality constant)
- same for log likelihood plus a constant

exponential family: function of data times function of parameter

$$L(\beta \mid \underset{\sim}{y}) = \exp\{T(\underset{\sim}{y}) w(\beta) - K(\beta)\}$$

Domain: $\beta > 0$ important to write b/c of open set stuff

$$= \exp\left\{-\frac{u}{\beta} - n \log \beta\right\} \qquad u = \sum_{i=1}^{n} Y_i$$

The density of $\underset{\sim}{y}$ (a vector) their joint distribution belongs to a 1 parameter exponential family.

$U$ is complete because $\{w(\beta), \beta > 0\} = (-\infty, 0)$ in $\mathbb{R}^1$
$w(\beta) = -1/\beta \to (-\infty, 0)$

because the range of $w(\beta)$ contains an open set in $\mathbb{R}^1$, e.g. $(-2, 1)$

Can't just say that since its exponential family it is CSS. Need the extra condition that it contains an open set $\rightarrow$ show it.

If you just say "b/c it's exponential family" this shows lack of understanding of completeness. Completeness is a mathematical concept, not statistical. Example:

$X \sim \text{Poisson}(\theta)$
$Y \sim \text{Poisson}(\theta^2)$ } independent $\quad \theta > 0$

what is min sufficient stat?

$$f_{X,Y}(x,y|\theta) = \frac{e^{-\theta}\theta^{x}}{x!} \frac{e^{-\theta^2}\theta^{2y}}{y!}$$

$$= \frac{e^{-\theta-\theta^2}\theta^{x+2y}}{x!\,y!}$$

$$= \exp\{\overbrace{(x+2y)}\overbrace{\log\theta}^{w(\theta)} - \overbrace{(\theta+\theta^2)}^{K(\theta)} + c(x,y)\}$$

$\hookrightarrow$ minimal Suff stat is $(x+2y)$, the range of $\theta$ is $(-\infty, \infty)$, so also complete

$(x, y)$ is a sufficient, but not minimal sufficient statistic. It is two dimensional. Is it complete?

Can we find a function of x and y that has expectation zero?

$x \rightarrow E(x) = \theta \qquad x^2 \rightarrow E(X^2) = \theta + \theta^2$

$y \rightarrow E(y) = \theta^2$

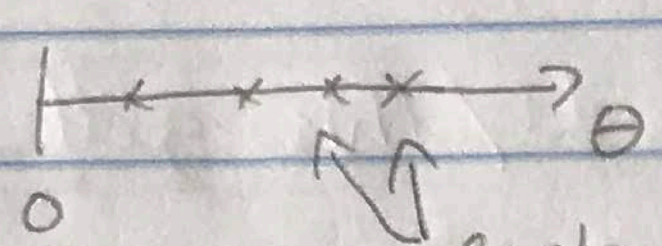
$$E[X^2 - Y - X] = 0 \quad \forall \theta > 0$$

So, not complete.

Completeness is a property for each statistic.
This is a shorthand :

Completeness is a property of a family (set) of distributions.

$\{ \text{Poisson}(\theta),\ \theta > 0 \}$ ← a set of distributions




each point represents one theta on real number line. Each is a different distribution

Think about the family of distributions, not the statistic

Only minimal sufficient statistic can be complete.

$Y \sim Bin(2, \theta)$

$Y \in \{0, 1, 2\}$

$\theta = \{ \frac{1}{4}, \frac{1}{2} \}$ ← only two points

| $y$ | $P(Y=y)$ |
|---|---|
| 0 | $(1-\theta)^2$ |
| 1 | $2\theta(1-\theta)$ |
| 2 | $\theta^2$ |
| | 1 |

| $y$ | $P(Y=y)$ | $\theta = \frac{1}{4}$ | $\theta = \frac{1}{2}$ |
|---|---|---|---|
| 0 | $(1-\theta)^2$ | $9/16$ | $1/4 = 4/16$ |
| 1 | $2\theta(1-\theta)$ | $6/16$ | $2/4 = 8/16$ |
| 2 | $\theta^2$ | $1/16$ | $1/4 = 4/16$ |
| | | 1 | 1 |

Y is not complete

Y only takes on three values

proportional to
$\theta = \frac{1}{4}$ : 9 6 1
$\theta = \frac{1}{2}$ : 4 8 4

$$\rightarrow \begin{bmatrix} 9 & 6 & 1 \\ 4 & 8 & 4 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

this parameter space is not complete $\rightarrow$ $\Theta = \{ \frac{1}{4}, \frac{1}{2} \}$

There are infinite solutions for $f_1, f_2, f_3$

If you set $f_1 = 1$, then:

$$\begin{bmatrix} 6 & 1 \\ 8 & 4 \end{bmatrix} \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} -9 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \leftarrow \text{w/ this get } E(\quad) = 0$$

functions that are not zero, where no matter what the true parameter is, the expectation is zero.

Change problem to: $\Theta = \{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \}$, will not find $f_1, f_2, f_3$ where $E(\ )$ is zero

Or, if $\Theta = \{ \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \}$

This parameter space is complete

| $y$ | $\theta = 3/4$ |
|---|---|
| 0 | 1/16 |
| 1 | 6/16 |
| 2 | 9/16 |

$P(Y=0) = P_0$
$P(Y=1) = P_1$
$P(Y=2) = P_2$

Expectation: $5 \cdot P_0$
$3 \cdot P_1$
$+ 7 \cdot P_2$
$\overline{\quad 0 \quad}$

you cannot solve this system! $\rightarrow$

$$\begin{bmatrix} 9 & 6 & 1 \\ 4 & 8 & 4 \\ 1 & 6 & 9 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So complete Binomial has property that any three different values of $\Theta$ will

- This parameter space doesn't contain an open set, but it is still complete

$E[I(Y_1 > t)]$

obvious estimator is
the indicator of the event

$E[\underbrace{I(A)}] = P(A)$

R.V., unbiased estimator of probability of event A

parameter

- completeness doesn't impact ability to estimate

What you are really talking about is the family of distributions that that statistic belongs to

UMVUE's doesn't always exist
↳ had one in previous exam

If you can't apply Lemon-Scheffe, UMVUE might still exist.

To find UMVUE:

usually a lot of work

- have statistic, $T(Y)$, and $E[T(Y)] = \theta$ unbiased
  - then, compute $var(T(Y))$
    - compute CRLB, it is UMVUE if the same
-

$(X, Y)$, $Y_1 \dots Y_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ known

$\theta$

$\Phi\left(\frac{-\mu}{\sigma}\right) = \theta = P\left(Z \leq \frac{-\mu}{\sigma}\right) = P(Z\sigma + \mu \leq 0) = P(Y_1 \leq 0)$

CDF of standard normal

$\sim N(\mu, \sigma^2) \rightarrow$ distribution of $Y_1$ to $Y_n$

$V_1 = I(Y_1 \leq 0)$, so $V_1$ is unbiased estimator of $\theta$.

$\bar{Y}$ is MSS (min suff stat), $-\infty < \mu < \infty$

what is the bridge between $Y_1$ and $\bar{Y} \rightarrow$ Rao Blackwell Theory

$E[V_1 \mid \bar{Y}] = g(\bar{Y})$, $E(g(\bar{Y})) = \theta$

$\boxed{2016}$

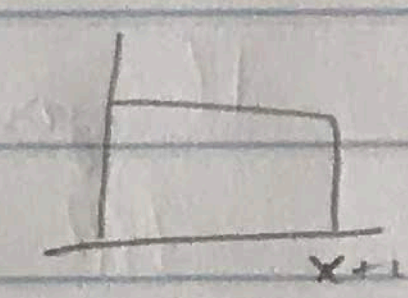
1) $\underbrace{0000}\,1\,\underbrace{000}\,1$ ← 2nd succes

\# of Zeros = $X$ $\qquad \binom{x+1}{1}$

a) $P(X=x) = (x+1)p^2q^x \quad , x = 0, 1, 2 \ldots$

$Y \mid X = x \sim U(0, x+1)$

Y is continuous: $U(0,1)$, etc.



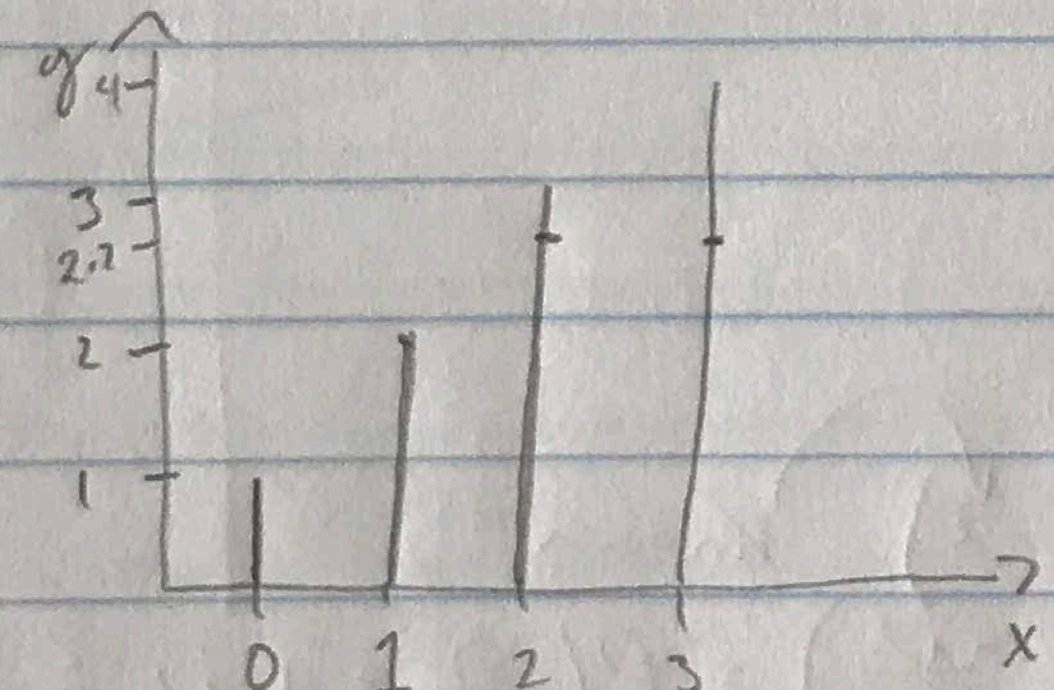
$f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x)$

$= (x+1)p^2q^x \cdot \frac{I(0<y<x+1)}{x+1}$

not done until you write down the range ↳ bivariate

$\{(x,y) : x \in \{0, 1, \ldots\}, 0 < y < x+1\}$

Draw the support:



lines in $\mathbb{R}^2$

if $y = 2.7$ then $x$ can be $2, 3, \ldots$

b) $f_Y(y) = \sum_{x \in A(y)} f_{X,Y}(x,y) \qquad A(y) = \{\lfloor y \rfloor, \lfloor y \rfloor + 1, \ldots\}$

$y^* = \lfloor y \rfloor$

$= \sum_{x=y^*}^{x=\infty} \cancel{(x+1)}p^2q^x \frac{I(0<y<x+1)}{\cancel{x+1}}$

$= p^2 \sum_{y^*}^{\infty} q^x \quad$ ← geometric summation

$= \frac{p^2 q^{y^*}}{1-q} = pq^{y^*}, \quad 0 < y < \infty$ ★

↳ $= pq^{\lfloor y \rfloor}$

1c) $E[Y]$ → Don't get integral of $f_Y(y)$ → will get the right answer but too much work

$$E[Y \mid X] = \frac{X+1}{2}$$

$$E[Y] = E\left[\frac{X+1}{2}\right] = \frac{\frac{2q}{p}+1}{2}$$

d) $Cov(X, Y) = E[Cov(X, Y \mid X)] + Cov(E[Y \mid X], E[X \mid X])$

$= E[0] + Cov(\frac{X+1}{2}, X)$

$= \frac{1}{2} Var(X)$

e) Guess → Cov is zero

f) Cov is zero, but need to check.
Easiest to check ranges

$$T = 2Y - X$$

↓

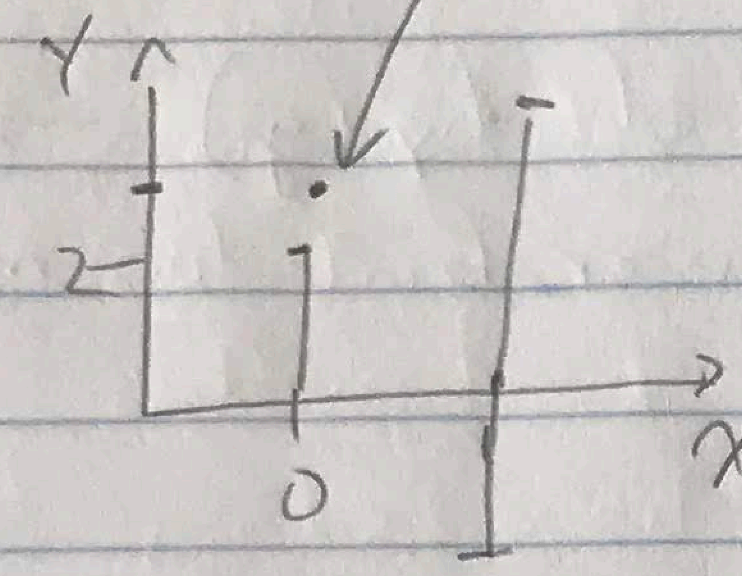
range: $(0, X+1)$
range $2Y$: $(0, 2X+2)$, now $-X$

| $X$ | $Y$ | $2Y$ | $T$ |
|---|---|---|---|
| 0 | $(0,1)$ | $(0,2)$ | $(0,2)$ |
| 1 | | $(0,4)$ | $(-1,3)$ |

↪ They are not independent

The range of $T$ depends on $X$

The conditional range of $T$ given $X = x$ depends on $X$

1f) 

0 at the point, the value of the joint is not equal to the product of the marginals