Hotelling's T^2

Model

- **Let $X_1, \ldots, X_n$ be iid $d \times 1$ random vectors each distributed as $\mathbf{N}_d(0, \Sigma)$, $\Sigma > 0$ (i.e. Σ is positive definite). It is assumed throughout that Σ is positive definite, and that $n - 1 \geq d$.**
- **MLE's and their distributions:**

  The MLE's are
  $$\hat{\mu} = \bar{X} = (1/n) \sum_{i=1}^{n} X_i,$$
  $$\hat{\Sigma} = Q/n,$$
  where $Q := \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})^\top$.
- **Note that the usual sample covariance matrix is $S = Q/(n-1)$. The sample mean $\bar{X} \sim \mathbf{N}_d(\mu, \Sigma/n)$, and is independent of $Q = (n-1)S$.**

Hotelling's T^2

- $$T^2 := n(\bar{X} - \mu)^\top S^{-1}(\bar{X} - \mu) \sim T^2_{d,n-1},$$

  and

  $$n(\bar{X} - \mu)^\top \Sigma^{-1}(\bar{X} - \mu) \sim \chi^2_d.$$

Hypothesis testing, the one-sample problem

- $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$.
- **Under** H_0,

$$T_0^2 := n(\bar{X} - \mu_0)^\top S^{-1}(\bar{X} - \mu_0) \sim T^2_{d,n-1}, \qquad F_0 := \frac{n-d}{(n-1)d} T_0^2 \sim F_{d,n-d}.$$

- **For a size-α test, reject H_0 if F_0 exceeds the $1 - \alpha$ quantile of the $F_{d,n-d}$ distribution.**
- T_0^2 **is the likelihood-ratio statistic for H_0 against H_1.**
- **The case $d = 1$?**

Idea: Reduction to univariate

- **Linear combinations**
- **If** H_0 **is true then** $H_a : a^\top \mu = a^\top \mu_0$ **will be true for all** $a \in R^d$**.**
- H_a **can be tested by a univariate one-sample t-test (of** $H_a : \mathbf{E}[a^\top X_i] = a^\top \mu_0$**, since** $a^\top X_i$ **and** $a^\top \mu_0$ **are scalars).**
- **The "most-significant" univariate test (the largest possible** $|t|$**) is obtained by** $a \propto S^{-1}(\bar{X} - \mu_0)$**.**
- **Since we maximized over all possible linear combinations** a**, the usual** t_{n-1} **is not valid if** $d > 1$**. The correct distribution under** H_0 **is Hotelling's** T^2**.**
- **Power:**
  If H_0 **is false then** $F_0 \sim F_{d,n-d,\delta}$ **where** $\delta = n(\mu - \mu_0)^\top \Sigma^{-1}(\mu - \mu_0)$**. Power of the test is** $\mathrm{pr}\{F_{d,n-d,\delta} > F_{d,n-d,0,1-\alpha}\}$**.**
- **Note that the non-centrality parameter is proportional to sample size** n**. This fact will be used later in sample size calculations.**

Idea: Using Contrasts

- **Linear combinations**
- **Suppose that H_0 states that the means $\mu_1, \mu_2, ..., \mu_d$ follow a linear trend, while the alternative, H_1, places no restrictions at all.**
- **H_0 does not specify what the linear trend is (intercept and slope unspecified).**
- **H_0 can be written as**

$$\begin{aligned}
\mu_1 &= \alpha, \\
\mu_2 &= \alpha + \beta, \\
\mu_3 &= \alpha + 2\beta, \\
\mu_4 &= \alpha + 3\beta, \\
\cdots &.
\end{aligned}$$

for some unspecified (α, β).

- $\mathbf{E}[X_{i2} - X_{i1}] = \beta$
  $\mathbf{E}[X_{i3} - X_{i2}] = \beta$
  $\mathbf{E}[X_{i4} - X_{i3}] = \beta$
  $\ldots$

- **Define**
  $Y_{i1} := X_{i3} - 2X_{i2} + X_{i1}$
  $Y_{i2} := X_{i4} - 2X_{i3} + X_{i2}$
  $Y_{i3} := X_{i5} - 2X_{i4} + X_{i3}$

  Y **is** $(d-2) \times 1$**.**

- **If** H_0 **is true then** $H_0^* : \mathbf{E}[Y] = 0$ **will be true.**

- **If** H_0^* **is true then** H_0 **will be true.**

- $H_0^* = H_0$**.**
  H_0 **can be tested by testing** H_0^***.**

- **Hotelling's** T^2 **can be used.**

- **Converted a "linear trend" problem to a "one-sample problem"**

- **Profile analysis**

The two-sample problem

- **Hypothesis testing, the two-sample problem with a common covariance matrix:**
- **The two independent samples are:**

  $X_1, \ldots, X_n$ **are iid** $d \times 1$ **random vectors each distributed as** $\mathbf{N}_d(\mu_1, \Sigma)$**,**

  $Y_1, \ldots, X_m$ **are iid** $d \times 1$ **random vectors each distributed as** $\mathbf{N}_d(\mu_2, \Sigma)$**,**
  $\Sigma > 0$**.**

- **Define**
  $\theta = \mu_1 - \mu_2$**,**
  $\hat{\mu}_1 = \bar{X} = (1/n) \sum_{i=1}^{n} X_i$**,**
  $\hat{\mu}_2 = \bar{Y} = (1/m) \sum_{i=1}^{m} Y_i$**,**
  $Q_1 := \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})^\top$**,**
  $Q_2 := \sum_{i=1}^{m} (Y_i - \bar{Y})(Y_i - \bar{Y})^\top$**,**
  $Q = Q_1 + Q_2$**,**
  $S_p = Q/(n + m - 2)$**.**

- **Distributions:**
  $\bar{X}, \bar{Y}, Q_1, Q_2$ **are mutually independent.**
  $\bar{X} \sim \mathbf{N}_d(\mu_1, \Sigma/n)$**,**
  $\bar{Y} \sim \mathbf{N}_d(\mu_2, \Sigma/m)$**,**
  $\bar{X} - \bar{Y} \sim \mathbf{N}_d(\theta, (1/n + 1/m)\Sigma)$**.**
-
$$T^2 = (\frac{1}{n} + \frac{1}{m})^{-1}(\bar{X} - \bar{Y} - \theta)^\top S_p^{-1}(\bar{X} - \bar{Y} - \theta) \sim T^2_{d,n+m-2},$$
$$T^2 \frac{n+m-d-1}{(n+m-2)d} \sim F_{d,n+m-d-1}.$$
- **To test** $H_0 : \theta = 0$ **against** $H_1 : \theta \neq 0$**, the test statistic** T_0^2 **is obtained as** T^2 **above with** θ **replaced by** 0**, and** $F_0 = T_0^2(n+m-d-1)/\{(n+m-2)d\}$**. For a size-**$\alpha$ **test, reject** H_0 **if** F_0 **exceeds the** $1-\alpha$ **quantile of the** $F_{d,n+m-d-1}$ **distribution.**

Strengths

- **Exact in small samples**
- **Easy to apply.**

Weaknesses

- **Rely on normality and equal variance**
- **Missing data not easily handled**
- **All subjects must be observed at the same time points**

Non-normality, unequal variance

- **Large sample tests, but need large samples**
- **Missing data not easily handled**
- **All subjects must be observed at the same time points**
- **Still need more general approaches to ...**
- **allow unequal number of observations per subject**
- **allow different observation times for different subject**

Regression models (marginal)

- **Based on normal theory**
- **Non-normal data**
- **In *both* of the above, we'll rely on large-sample methods (standard errors, confidence intervals, hypothesis test)**

Likelihood methods