1. Let $X_1, \ldots, X_n$ be a random sample of size n from a distribution with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & 0 < x < 1, \quad 0 < \theta < \infty \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that the maximum likelihood estimator of θ is $\hat{\theta} = -n/\sum_{i=1}^n \log X_i$.

   The likelihood function is

$$L(\theta) = \prod_{i=1}^n \theta x_i^{\theta-1}.$$

   Taking the first derivative to $\ell(\theta) = \log L(\theta)$, one has

$$\frac{\partial}{\partial\theta}\ell(\theta) = n\frac{1}{\theta} + \sum_{i=1}^n \log x_i.$$

   Setting it to zero, one has $\hat{\theta} = -n/\sum_{i=1}^n \log X_i$. This is indeed global maximum since

$$\frac{\partial^2}{\partial\theta^2}\ell(\theta) = -n\theta^{-2} < 0.$$

   (b) Let $Y_i = -\log X_i$. Show that the distribution of Y_i is $\text{Exp}(\theta^{-1})$. Hence, $\sum_{i=1}^n Y_i$ follows $\text{Gamma}(n, \theta^{-1})$.

   [The pdf of $\text{Exp}(\beta)$ is $f(y) = \frac{1}{\beta}e^{-y/\beta}$, $0 < y < \infty$.]

   Using the transformation method, the pdf of Y is

$$f_Y(y) = \theta e^{-y(\theta-1)}e^{-y} = \theta e^{-\theta y}, \quad 0 < y < \infty.$$

   Hence, $Y_i = -\log X_i$ follows an exponential distribution with mean θ^{-1}, and $\sum_{i=1}^n Y_i$ follows $\text{Gamma}(n, \theta^{-1})$.

   (c) If a random variable W follows $\text{Gamma}(n, \theta^{-1})$, then, for $r > -n$,

$$E(W^r) = \frac{\Gamma(n+r)}{\Gamma(n)}\theta^{-r}.$$

   Find $E(\hat{\theta})$ and comment on if $\hat{\theta}$ is an unbiased estimator.

   The expectation of $\hat{\theta}$ is

$$E(\hat{\theta}) = nE\{(\sum_{i=1}^n Y_i)^{-1}\} = n\frac{\Gamma(n-1)}{\Gamma(n)}\theta = \frac{n}{n-1}\theta.$$

   The estimator is biased for θ.

(d) Find the the Crámer-Rao lower bound for every unbiased estimator and comment on if the variance of $\hat{\theta}$ reach the lower bound.

The denominator for CRLB is

$$E\left\{-\frac{\partial^2}{\partial\theta^2}\ell(\theta)\right\} = n\theta^{-2}.$$

The CRLB is θ^2/n. One has

$$E(\hat{\theta}^2) = n^2 E\{(\sum_{i=1}^n Y_i)^{-2}\} = n^2\frac{\Gamma(n-2)}{\Gamma(n)}\theta^2 = \frac{n^2}{(n-1)(n-2)}\theta^2.$$

The variance of $\hat{\theta}$ is

$$\begin{aligned} Var(\hat{\theta}) &= E(\hat{\theta}^2) - E(\hat{\theta})^2 \\ &= \frac{n^2}{(n-1)(n-2)}\theta^2 - \frac{n^2}{(n-1)^2}\theta^2 \\ &= \frac{n^2}{(n-1)^2(n-2)}\theta^2, \end{aligned}$$

which does not reach the CRLB.

2. Let $X_1, \ldots, X_n$ be a random sample from Bernoulli distribution with probability of success $\theta \in (0,1)$.

(a) Find the method of moment estimator for $\tau(\theta) = Var(X_1) = \theta(1-\theta)$.

The method of moment estimator for θ is $\bar{X}$. One may claim the method of moment estimator for $\tau(\theta)$ is $\bar{X}(1-\bar{X})$. One may also rewrite

$$Var(X_1) = E(X_1^2) - E(X_1)^2,$$

and matching it with sample moments. Using this approach, the method of moment estimator is

$$\frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2.$$

(b) Show that $\bar{X}(1-\bar{X})/(n-1)$ is the UMVUE of $Var(\bar{X}) = \theta(1-\theta)/n$.
One can show that

$$\begin{aligned} E(\bar{X}(1-\bar{X})) &= E(\bar{X}) - E(\bar{X}^2) \\ &= \theta - \frac{\theta(1-\theta)}{n} - \theta^2 \\ &= \theta(1-\theta)\left(1-\frac{1}{n}\right), \end{aligned}$$

which gives $E\{\bar{X}(1-\bar{X})/(n-1)\} = \theta(1-\theta)/n$. Since $\bar{X}(1-\bar{X})/(n-1)$ is an unbiased estimator and $\bar{X}$ is complete and sufficient, $\bar{X}(1-\bar{X})/(n-1)$ is the UMVUE.

3. Let $X_1, \ldots, X_n$ be a random sample of size $n > 2$ from a normal distribution with mean 0 and variance $\sigma^2 > 0$. To test the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$:

   (a) Find the maximum likelihood estimator of σ^2 under the overall parameter space $\Theta = \Theta^0 \cup \Theta^c = (0, \infty)$.

   The likelihood function for σ^2 is

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x_i^2}{2\sigma^2}\right) = (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right).$$

   Taking the first derivative of the log-likelihood function, one has

$$\frac{\partial}{\partial\sigma^2}\ell(\sigma^2) = -\frac{n}{2}(\sigma^2)^{-1} + \frac{1}{2}(\sigma^2)^{-2}\sum_{i=1}^{n} x_i^2.$$

   Setting $(\partial/\partial\sigma^2)\ell(\sigma^2) = 0$, one has the MLE of σ^2 as

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

   This is a global maximizer since

$$\frac{\partial^2}{\partial(\sigma^2)^2}\ell(\sigma^2)|_{\sigma^2=\hat{\sigma}^2} = \frac{n}{2}(\hat{\sigma}^2)^{-2} - (\hat{\sigma}^2)^{-3}\sum_{i=1}^{n} x_i^2 < 0.$$

   (b) Show that the likelihood ratio test statistic $\lambda(\boldsymbol{x}) = c_1 t^{n/2}\exp(-c_2 t)$, where $t = \sum_{i=1}^n x_i^2$ and c_1 and c_2 are functions of σ_0^2 and n (constants).

   The MLE of σ^2 is $\hat{\sigma}^2$ under the overall parameter space. Hence, the denominator of $\lambda(\boldsymbol{x})$ is

$$L(\hat{\sigma}^2) = (2\pi)^{-n/2}(\hat{\sigma}^2)^{-n/2}\exp\left(-\frac{n}{2}\right).$$

   Meanwhile, the numerator of $\lambda(\boldsymbol{x})$ can be written as

$$L(\sigma_0^2) = (2\pi)^{-n/2}(\sigma_0^2)^{-n/2}\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma_0^2}\right).$$

Therefore, the likelihood ratio test statistic $\lambda(\boldsymbol{x})$ can be written as

$$\begin{aligned}\lambda(\boldsymbol{x}) = \frac{L(\sigma_0^2)}{L(\hat{\sigma}^2)} &= \left(\frac{\sigma_0^2}{\hat{\sigma}^2}\right)^{-n/2} \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma_0^2} + \frac{n}{2}\right) \\ &= c_1 t^{n/2} \exp(-c_2 t)\end{aligned}$$

(c) By the likelihood ratio test, one rejects H_0 if $\delta(x) = 1$, where

$$\delta(x) = \begin{cases} 1 & \text{if } \lambda(x) \le c \\ 0 & \text{if } \lambda(x) > c. \end{cases}$$

Show that, equivalently, one can use the following rejection region:

$$\delta(x) = \begin{cases} 1 & \text{if } t \ge c_1^* \text{ or } t \le c_2^* \\ 0 & \text{if otherwise,} \end{cases}$$

given that $\lambda(\boldsymbol{x})$ is a concave function of $t = \sum_{i=1}^n x_i^2$.

(d) Following (c), find c_1^* and c_2^* such that the type I error probability of the test equals α.

Using the result in (c), one has

$$\begin{aligned}\alpha &= P(T \ge c_1^* | \sigma^2 = \sigma_0^2) + P(T \le c_2^* | \sigma^2 = \sigma_0^2) \\ &= P\left(\frac{T}{\sigma_0^2} \ge \frac{c_1^*}{\sigma_0^2}\right) + P\left(\frac{T}{\sigma_0^2} \le \frac{c_2^*}{\sigma_0^2}\right).\end{aligned}$$

Since T/σ_0^2 follows χ^2 distribution with degree of freedom n, one can assign $c_1^* = \sigma_0^2 \chi^2_{n,1-\alpha/2}$ and $c_2^* = \sigma_0^2 \chi^2_{n,\alpha/2}$, or any $c_1^* = \sigma_0^2 \chi^2_{n,1-\alpha_1}$ and $c_2^* = \sigma_0^2 \chi^2_{n,\alpha_2}$ such that $\alpha_1 + \alpha_2 = \alpha$. Even more precisely, one should consider $c_1^* = \sigma_0^2 \chi^2_{n,1-\alpha_1}$ and $c_2^* = \sigma_0^2 \chi^2_{n,\alpha_2}$ such that $\alpha_1 + \alpha_2 = \alpha$ and $c_1 c_1^{*n/2} \exp(-c_2 c_1^*) = c_1 c_2^{*n/2} \exp(-c_2 c_2^*)$ (the $\lambda(x)$ has the same height c at $t = c_1^*$ and $t = c_2^*$).