Homework Assignment 2

Question 1

The center of A_1 is at $(-1, 0)$, A_2 is at $(1/2, 0)$, A_3 is at $(-1/3, 0)$, A_4 is at $(1/4, 0)$, and so on. We can see that the center gets closer and closer to $(0, 0)$ but oscillates between the negative and positive part of the x-axis. The circle gets closer and closer to the circle centered at the origin with radius 1 but never approaches it. If we denote that circle by C, consider the set $\{(x, y) : x^2 + y^2 < 1\}$, then for all points in that set, we can find an N large enough such that for all $n \geq N$, A_n contains (x, y). Therefore, $\liminf_n A_n = \{(x, y) : x^2 + y^2 < 1\}$.

Now consider the border of C in the set $\{(x, y) : x^2+y^2 = 1\}$, if we only consider the A_i's with odd i's, then their centers all lie on the negative part of the x-axis. Therefore, the left half border $\{(x, y) : x^2 + y^2 = 1 \text{ where } x < 0\}$ is in all such infinitely many A_i's. Similarly, the right half $\{(x, y) : x^2 + y^2 = 1 \text{ where } x > 0\}$ is also in infinitely many A_i's for even i's. Since the center never gets to the origin, (0,1) and (0,-1) is not in any circles. Therefore, $\limsup_n A_n = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } (x, y) \neq (0, 1) \text{ or } (0, -1)\}$

Question 2

Show $\mathcal{F}$ is a σ-field.

$\mathcal{F}$ is non-empty since we can find A in $\mathbb{R}$ such that A is countable. For example, A can be finite sets of integers.

Next, we want to show that $\mathcal{F}$ is closed under complement. If $A \in \mathcal{F}$ and A is countable, then $A^c \in \mathcal{F}$ because $(A^c)^c = A$ is countable. If $A \in \mathcal{F}$ and A^c is countable, then it follows that $A^c \in \mathcal{F}$. Therefore, $\mathcal{F}$ is closed under complement.

Finally, we need to show that $\mathcal{F}$ is closed under countable union. Assume $\cup_{n=1}^{\infty} A_n$ is uncountable. If all the A_n's are countable, then $\cup_{n=1}^{\infty} A_n$ is also countable, since countable union of countable sets is countable. If some of the A_n's are uncountable, which means that some of the A_n^c's are countable, then by De Morgan's law, $(\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty} A_n^c$, which is countable, since we have some A_n^c's that are countable. Therefore, $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$ for all $A \in \mathcal{F}$, which completes the proof.

Question 3

Let $A \in \mathcal{X}_1 \cap \mathcal{X}_2$ and A_n be a sequence of such A's.

Since $\mathcal{X}_1$ is a σ-field, we have $A^c \in \mathcal{X}_1$ and $\cup_{n=1}^{\infty} A_n \in \mathcal{X}_1$.

Since $\mathcal{X}_2$ is a σ-field, we have $A^c \in \mathcal{X}_2$ and $\cup_{n=1}^{\infty} A_n \in \mathcal{X}_2$.

It follows that $A^c \in \mathcal{X}_1 \cap \mathcal{X}_2$ and $\cup_{n=1}^{\infty} A_n \in \mathcal{X}_1 \cap \mathcal{X}_2$. $\mathcal{X}_1 \cap \mathcal{X}_2$ is non-empty since both $\mathcal{X}_1$ and $\mathcal{X}_2$ contain Ω. Therefore, $\mathcal{X}_1 \cap \mathcal{X}_2$ is a σ-field.

Question 4

Let $A \in \cap_{\mathcal{X} \in \mathbf{G}} \mathcal{X}$ and A_n be a sequence of such A's.

We know that $\cap_{\mathcal{X} \in \mathbf{G}} \mathcal{X}$ is not empty since $\Omega \subset \mathcal{X}$ for every $\mathcal{X} \in \mathbf{G}$.

It follows that $A^c \in \mathcal{X}$ and $\cup_{n=1}^{\infty} A_n \in \mathcal{X}$ for all $\mathcal{X} \in \mathbf{G}$, since all $\mathcal{X}$'s are σ-algebras, which means that $A^c \in \cap_{\mathcal{X} \in \mathbf{G}} \mathcal{X}$ and $\cup_{n=1}^{\infty} A_n \in \cap_{\mathcal{X} \in \mathbf{G}} \mathcal{X}$. Therefore, $\cap_{\mathcal{X} \in \mathbf{G}} \mathcal{X}$ is also a σ-algebra.

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5.

$X_1 \cup X_2 = \{\emptyset, A, B, A^c, B^c, \Omega\}$ is not a σ-field because it is not closed under countable union. Specifically there is no guarantee that the union of the sets unique to X_1, A and A^c, and the sets unique to X_2, B and B^c, will be in $X_1 \cup X_2$, for example $A \cup B$ or $A^c \cup B$. Thus $X_1 \cup X_2$ is not a σ-field.

6.

If $\{A_n\}$ is a decreasing sequence of sets $\implies$ $\{A_n^c\}$ is an increasing sequence of sets.

$\rightarrow lim_n P(A_n^c) = P(lim_n A_n^c)$ for an increasing sequence of sets as proved in lecture.

Since $\{A_n^c\}$ is an increasing sequence, $lim_n A_n^c = \cup_{n=1}^{\infty} A_n^c$.

Using DeMorgan's law, $\cup_{n=1}^{\infty} A_n^c = (\cap_{n=1}^{\infty} A_n)^c = lim_n A_n^c$

As discussed in class, $P(A) = 1 - P(A^c)$.

$\rightarrow P(lim_n A_n^c) = 1 - P((lim_n A_n^c)^c)$

Using the equality derived from Demorgan's law, $\rightarrow 1 - P((lim_n A_n^c)^c) = 1 - P(((\cap_{n=1}^{\infty} A_n)^c)^c) = 1 - P(\cap_{n=1}^{\infty} A_n)$.

As $\{A_n\}$ is a decreasing set, $\cap_{n=1}^{\infty} A_n = lim_n A_n \implies 1 - P(\cap_{n=1}^{\infty} A_n) = 1 - P(lim_n A_n)$

Thus $P(lim_n A_n^c) = 1 - P(lim_n A_n)$

For $lim_n P(A_n^c)$, again using the complement property given above, $lim_n P(A_n^c) = lim_n(1 - P(A_n))$

As the limit of the sum is equal to the sum of the limit, $\rightarrow lim_n(1 - P(A_n)) = lim_n 1 - lim_n P(A_n)$

Additionally, the limit of a constant is equal to the constant.

$\implies lim_n 1 - lim_n P(A_n) = 1 - lim_n P(A_n)$

Thus $lim_n P(A_n^c) = 1 - lim_n P(A_n)$

Going back to $lim_n P(A_n^c) = P(lim_n A_n^c)$, using our newfound results, we can now say $lim_n P(A_n^c) = P(lim_n A_n^c) \implies 1 - lim_n P(A_n) = 1 - P(lim_n A_n)$

subtracting 1 from both sides gives us $-lim_n P(A_n) = -P(lim_n A_n)$ and multiplying by -1 gives us $lim_n P(A_n) = P(lim_n A_n)$

Thus, for a decreasing sequence $\{A_n\}$, $lim_n P(A_n) = P(lim_n A_n)$.

7.

Let $F \cup G = A$

As denoted in lecture, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

So $P(E \cup F \cup G) = P(E \cup A) = P(E) + P(A) - P(E \cap A)$

$= P(E) + P(F \cup G) - P(E \cap (F \cup G))$

$= P(E) + P(F \cup G) - P((E \cap F) \cup (E \cap G))$ by Demorgan's law

Let $E \cap F = B$ and $E \cap G = C$

$= P(E) + P(F \cup G) - P(B \cup C)$

$= P(E) + P(F \cup G) - (P(B) + P(C) - P(B \cap C))$

$= P(E) + P(F \cup G) - (P(E \cap F) + P(E \cap G) - P((E \cap F) \cap (E \cap G)))$

$= P(E) + P(F \cup G) - (P(E \cap F) + P(E \cap G) - P(E \cap F \cap G))$

$= P(E) + P(F \cup G) - P(E \cap F) - P(E \cap G) + P(E \cap F \cap G)$

$= P(E) + P(F) + P(G) - P(F \cap G) - P(E \cap F) - P(E \cap G) + P(E \cap F \cap G)$

8.

a.

Axiom of Countable Additivity: If $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint, $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Axiom of Finite Additivity: If $A \in \mathcal{B}$ and $B \in \mathcal{B}$ are disjoint, $P(A \cup B) = P(A) + P(B)$

Assume the axiom of Countable Additivity to be true, i.e., that $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint. Then $P(\cup_{n=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Let $i, j \in \mathbb{N}$ s.t. $i \neq j$

For any A_i and A_j, let it be known that $P(A_i \cap A_j) = 0$

Let all $A_n = \emptyset \ \forall n \neq i, j$

Then $P(A_i \cup A_j) = P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n) = P(A_i) + P(A_j)$

Finally, let $A_i = A$ and $A_j = B$. Thus, we can say that by the axiom of Countable Additivity, $P(A \cup B) = P(A) + P(B)$, i.e. that the Axiom of Finite Additivity holds.

b.

Axiom of Continuity: If $A_n \downarrow \emptyset$, $P(A_n) \to 0$.

Assume $A_1, A_2, \ldots$ are all pairwise disjoint and that the axiom of Continuity and axiom of Finite Additivity hold.

$P(\cup_{n=1}^{\infty} A_n) = P(\cup_{n=1}^{i} A_n \cup \cup_{n=i+1}^{\infty} A_n)$ As the A_n are disjoint and the axiom of Finite Additivty is assumed true.

$= P(\cup_{n=1}^{i} A_n) + P(\cup_{n=i+1}^{\infty} A_n)$ Again by the axiom of Finite Additivity.

$$= \sum_{i=1}^{n} P(A_i) + P\left(\bigcup_{i=n+1}^{\infty} A_i\right) \qquad \text{(finite additivity)}$$

Now define $B_k = \bigcup_{i=k}^{\infty} A_i$. Note that $B_{k+1} \subset B_k$ and $B_k \to \phi$ as $k \to \infty$. (Otherwise the sum of the probabilities would be infinite.) Thus

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty}\left[\sum_{i=1}^{n} P(A_i) + P(B_{n+1})\right] = \sum_{i=1}^{\infty} P(A_i).$$

9. $P(E\cap F) = P(E) + P(F) - P(E\cup F)$

$\geq P(E) + P(F) - P(\Omega)$

$= 0.9 + 0.8 - 1 = 0.7$

Proof: From Probability calculus, we know $P(E\cup F) = P(E) + P(F) - P(E\cap F)$

And for any set A, $P(A) \leq P(\Omega) = 1$

Thus, $P(E\cap F) = P(E) + P(F) - P(E\cup F)$

$\geq P(E) + P(F) - P(\Omega)$

$= P(E) + P(F) - 1$

Therefore $P(E\cap F) \geq P(E) + P(F) - 1$ //

10. ① Let $n=1$, then L.H.S $= P(E_1\cap \cdots \cap E_n) = P(E_1)$

R.H.S $= P(E_1) + \cdots + P(E_n) - (n-1) = P(E_1) - (1-1) = P(E_1)$

Thus $P(E_1) \geq P(E_1)$ holds.

② Assume the result is true for any natural number k. That is when $n=k$,

$P(E_1\cap E_2\cap\cdots\cap E_k) \geq P(E_1) + P(E_2) + \cdots + P(E_k) - (k-1)$ ……(1)

From the Proof in #9., we know $P(E\cap F) \geq P(E) + P(F) - 1$ ……(2)

When $n=k+1$, we have

L.H.S. $= P(E_1\cap E_2\cap\cdots\cap E_k\cap E_{k+1}) = P[(E_1\cap E_2\cap\cdots\cap E_k)\cap E_{k+1}]$

$\geq P(E_1\cap E_2\cap\cdots\cap E_k) + P(E_{k+1}) - 1$ [Using (2)]

[Using (1)] $\geq P(E_1) + P(E_2) + \cdots + P(E_k) - (k-1) + P(E_{k+1}) - 1$

$= P(E_1) + P(E_2) + \cdots + P(E_k) + P(E_{k+1}) - (k+1-1) =$ R.H.S

Thus $P(E_1\cap E_2\cap\cdots\cap E_{k+1}) \geq P(E_1) + \cdots + P(E_{k+1}) - (k+1-1)$ is true

whenever $P(E_1\cap\cdots\cap E_k) \geq P(E_1) + \cdots + P(E_k) - (k-1)$ is true.

Hence by the principle of mathematical induction,

$P(E_1\cap\cdots\cap E_n) \geq P(E_1) + \cdots + P(E_n) - (n-1)$ is true

for all natural numbers n.

11. (a) $P = \frac{3}{5}$

(b) If an odd digit is selected at the first selection, then $P_o = \frac{3}{5}\times\frac{2}{4} = \frac{3}{10}$

If an even digit is selected at the first selection, then $P_e = \frac{2}{5}\times\frac{3}{4} = \frac{3}{10}$

Thus, $P = \frac{3}{10} + \frac{3}{10} = \frac{3}{5}$

(c) $P = \frac{3}{5}\times\frac{2}{4} = \frac{3}{10}$