1. Let $X_1, \dots, X_n$ be a random sample of size n from a distribution with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & 0 < x < 1, \quad 0 < \theta < \infty \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that the maximum likelihood estimator of θ is $\hat{\theta} = -n / \sum_{i=1}^n \log X_i$.

   (b) Let $Y_i = -\log X_i$. Show that the distribution of Y_i is $\text{Exp}(\theta^{-1})$. Hence, $\sum_{i=1}^n Y_i$ follows $\text{Gamma}(n, \theta^{-1})$.

   [The pdf of $\text{Exp}(\beta)$ is $f(y) = \frac{1}{\beta} e^{-y/\beta}$, $0 < y < \infty$.]

   (c) If a random variable W follows $\text{Gamma}(n, \theta^{-1})$, then, for $r > -n$,

$$E(W^r) = \frac{\Gamma(n+r)}{\Gamma(n)} \theta^{-r}.$$

   Find $E(\hat{\theta})$ and comment on if $\hat{\theta}$ is an unbiased estimator.

   (d) Find the Crámer-Rao lower bound for every unbiased estimator and comment on if the variance of $\hat{\theta}$ reach the lower bound.

2. Let $X_1, \dots, X_n$ be a random sample from Bernoulli distribution with probability of success $\theta \in (0, 1)$.

   (a) Find the method of moment estimator for $\tau(\theta) = Var(X_1) = \theta(1-\theta)$.

   (b) Show that $\bar{X}(1-\bar{X})/(n-1)$ is the UMVUE of $Var(\bar{X}) = \theta(1-\theta)/n$.

3. Let $X_1, \dots, X_n$ be a random sample of size $n > 2$ from a normal distribution with mean 0 and variance $\sigma^2 > 0$. To test the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$:

   (a) Find the maximum likelihood estimator of σ^2 under the overall parameter space $\Theta = \Theta^0 \cup \Theta^c = (0, \infty)$.

   (b) Show that the likelihood ratio test statistic $\lambda(\boldsymbol{x}) = c_1 t^{n/2} \exp(-c_2 t)$, where $t = \sum_{i=1}^n x_i^2$ and c_1 and c_2 are functions of σ_0^2 and n (constants).

(c) By the likelihood ratio test, one rejects H_0 if $\delta(x) = 1$, where

$$\delta(x) = \begin{cases} 1 & \text{if } \lambda(x) \leq c \\ 0 & \text{if } \lambda(x) > c. \end{cases}$$

Show that, equivalently, one can use the following rejection region:

$$\delta(x) = \begin{cases} 1 & \text{if } t \geq c_1^* \text{ or } t \leq c_2^* \\ 0 & \text{if otherwise,} \end{cases}$$

given that $\lambda(\boldsymbol{x})$ is a concave function of $t = \sum_{i=1}^n x_i^2$.

(d) Following (c), find c_1^* and c_2^* such that the type I error probability of the test equals α.