BIOS 660 Homework 05

Question 1, 2, 3, 4, and 5

Question 1

(a) $P(\text{double}) = \left(\frac{1}{6}\right)^2 \times 6 = \boxed{\frac{1}{6}}$

(b) There are 6 outcomes with sum of 4 or less, namely, (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)

There are 2 of them being double; namely (1, 1), (2, 2).

$P(\text{sum 4 or less}) = 6/36 = 1/6$. $P(\text{double and sum 4 or less}) = 2/36 =$

$P(\text{double} \mid \text{sum 4 or less}) = P(\text{double and sum 4 or less}) / P(\text{sum 4 or le}$

$= (1/18)/(1/6) = \boxed{1/3}$

(c) $P(\text{At least one } 6) = P(\text{first is } 6) + P(\text{second is } 6) - P(\text{Both are } 6)$

$= 1/6 + 1/6 - 1/36$

$= \boxed{11/36}$

(d) $P(\text{One die is } 6 \mid \text{different numbers})$

$= P(\text{One die is } 6 \text{ and different numbers}) / P(\text{different numbers})$

$= (11/36 - 1/36)/(1 - 1/6)$

$= (10/36)/(30/36)$

$= \boxed{1/3}$

Question 2

Proof by induction:

(1) Base case: $k=1$

the probability of getting a white ball is $\frac{m}{m+n}$

(2) Inductive step: Assuming the probability of getting a white b when $k=n$ is $\frac{m}{m+n}$, then the probability of getting a white ball when $k=n+1$ is:

$$p = P(\text{white } n+1 \mid \text{white } n) P(\text{white } n) + P(\text{white } n+1 \mid \text{black } n) P(\text{black } n$$

$$= \left(\frac{m+1}{m+n+1}\right)\frac{m}{m+n} + \left(\frac{m}{m+n+1}\right)\frac{n}{m+n}$$

$$= \frac{m^2+m+mn}{(m+n)(m+n+1)}$$

$$= \frac{m(m+1+n)}{(m+n)(m+n+1)}$$

$$= \frac{m}{m+n}.$$

(3). For all $k \in \mathbb{N}$ and $k \geq 1$, the probability that the last ball is white is the same as the probability that the first ball is white, i.e. it is $\frac{m}{m+n}$.

Question 3

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{P(A\cap B\cap C) + P(A\cap B\cap C^c)}{P(B)}$$

$$= \frac{P(A\cap B\cap C)}{P(B)} + \frac{P(A\cap B\cap C^c)}{P(B)}$$

$$= \frac{P(B\cap C)}{P(B)} \cdot \frac{P(A\cap B\cap C)}{P(B\cap C)} + \frac{P(B\cap C^c)}{P(B)} \cdot \frac{P(A\cap B\cap C^c)}{P(B\cap C^c)}$$

$$= P(C|B)\,P(A|B\cap C) + P(C^c|B)\,P(A|B\cap C^c)$$

Assuming all conditional events have positive probability.

Question 4

(1) Proof that A and B^c are independent:

$P(A\cap B^c) = P(A) - P(A\cap B)$

$= P(A) - P(A)P(B)$ because A and B are independen

$= P(A) - P(A)(1 - P(B^c))$

$= P(A) - P(A) + P(A)P(B^c)$

$= P(A)P(B^c)$.

Therefore A and B^c are independent.

(2) Proof that A^c and B^c are independent:

In part (1) we have shown that A and B^c are independent

$P(A^c\cap B^c) = P(B^c) - P(B^c\cap A)$

$= P(B^c) - P(B^c)P(A)$ because A and B^c are independ

$= P(B^c) - P(B^c)(1 - P(A^c))$

$= P(B^c) - P(B^c) + P(A^c)P(B^c)$

$= P(A^c)P(B^c)$

Therefore A^c and B^c are independent.

Question 5

Let A and B be the random variable that represents the points that UNC team gets in the first and second game, respectively.

Then for A, $f(a) = \begin{cases} 0.2 & \text{for } y = 2 \\ 0.2 & \text{for } y = 1 \\ 0.6 & \text{for } y = 0 \end{cases}$

for B, $f(b) = \begin{cases} 0.35 & \text{for } b = 2 \\ 0.35 & \text{for } b = 1 \\ 0.3 & \text{for } b = 0. \end{cases}$

Let Y be the random variable that represents the number of points that the team ears over the weekend: $Y = A + B$.

Since A and B are independent,

then $f(y) = \begin{cases} 0.18 & \text{for } y = 0 \checkmark \\ 0.27 & \text{for } y = 1 \checkmark \\ 0.34 & \text{for } y = 2 \checkmark \\ 0.14 & \text{for } y = 3 \checkmark \\ 0.07 & \text{for } y = 4 \checkmark \\ 0 & \text{otherwise} \end{cases}$

BIOS 660 HW 5 Part 2 Solution

Problem 6. a.) What is the probability that Harry wins the match?

$$
\begin{aligned}
P(\text{Harry Wins}) &= P(\text{Harry wins on game 1 or game 2 or ... game 10}) \\
&= \sum_{i=1}^{10} P(\text{Harry wins on game i}) \\
&= \sum_{i=1}^{10} P(\text{Harry wins on game i and the first i-1 games are draws}) \\
&= \sum_{i=1}^{10} 0.3 * (0.3)^{i-1} \\
&= \sum_{i=1}^{10} 0.3^{i} \\
&\approx 0.4286
\end{aligned}
$$

b.) What is the PMF of the duration of the match?

Let X be the random variable corresponding to the duration of the match.

$$
\begin{aligned}
P(X = i) &= P(\text{Match ends on game i}) \\
&= P(\text{Game is won on game i and the previous i-1 games are draws}) \\
&= 0.7 * (0.3)^{i-1} \text{ for } i = 1, ..., 9 \\
P(X = 10) &= P(\text{ The first 9 games are draws}) \\
&= 0.3^{9}
\end{aligned}
$$

$$P(X=i)=\begin{cases} 0.7*(0.3)^{i-1} & , \quad i=1,...,9 \\ 0.3^9 & , \quad i=10 \\ 0 & , \quad \text{Else} \end{cases}$$

Problem 7. CB 1.33
Let CB be the event that a person is color blind, M be the event that a person is male, and F be the event that a person is female.

Given:
$P(CB|M) = 0.05$
$P(CB|F) = 0.0025$

$$\begin{aligned} P(M|CB) &= \frac{P(M\cap CB)}{P(CB)} \\ &= \frac{P(CB|M)P(M)}{P(CB|M)P(M)+P(CB|F)P(F)} \\ &= \frac{0.05*(0.5)}{0.05*(0.5)+0.0025*(0.5)} \\ &\approx 0.9524 \end{aligned}$$

Problem 8. CB 1.36

a.) Let T be a r.v. for the number of times a target has been hit.

$$\begin{aligned} P(\text{Target is hit at least once}) &= P(T\geq 2) \\ &= 1-P(T<2) \\ &= 1-P(T=0)-P(T=1) \\ P(T=0) &= .8^{10} \\ P(T=1) &= \binom{10}{1}0.2(0.8)^9 \\ P(T\geq 2) &\approx 0.624 \end{aligned}$$

b.)

$$
\begin{aligned}
P(T \geq 2 | T \geq 1) &= \frac{P(T \geq 2, T \geq 1)}{P(T \geq 1)} \\
&= \frac{P(T \geq 2)}{1 - P(T = 0)} \\
&= \frac{1 - P(T = 0) - P(T = 1)}{1 - P(T = 0)} \\
&\approx 0.699
\end{aligned}
$$

Problem 9. CB 1.38

a.) Let $P(B) = 1$

$$
\begin{aligned}
P(B^c) &= 1 - P(B) \\
&= 0 \\
P(A \cap B^c) &\leq P(B^c) \\
P(A \cap B^c) &= 0 \\
P(A) &= P(A \cap B) + P(A \cap B^c) \\
&= P(A \cap B) \\
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{P(A)}{P(B)} \\
&= P(A)
\end{aligned}
$$

b.) Let $A \subset B$

$$
\begin{aligned}
A &\subset B \\
A \cap B &= A \\
P(B|A) &= \frac{P(A \cap B)}{P(A)} \\
&= \frac{P(A)}{P(A)} \\
&= 1 \\
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{P(A)}{P(B)}
\end{aligned}
$$

c.) Let A and B be mutually exclusive ($A \cap B = \emptyset$).

$$
\begin{aligned}
P(A|A \cup B) &= \frac{P(A \cap (A \cup B))}{P(A \cup B)} \\
&= \frac{P((A \cap A) \cup (A \cap B)}{P(A) + P(B) - P(A \cap B)} \\
&= \frac{P(A \cup \emptyset)}{P(A) + P(B)} \\
&= \frac{P(A)}{P(A) + P(B)}
\end{aligned}
$$

d.)

$$
\begin{aligned}
P(A \cap B \cap C) &= P(A \cap (B \cap C)) \\
&= P(A|B \cap C)P(B \cap C) \\
&= P(A|B \cap C)P(B|C)P(C)
\end{aligned}
$$

Problem 10. CB 1.39

1. Let A and B be mutually exclusive.

$$
\begin{aligned}
P(A)P(B) &> 0 \\
A \cap B &= \emptyset \\
P(A \cap B) &= 0 \\
P(A \cap B) &\neq P(A)P(B)
\end{aligned}
$$

Then A and B are not independent.

Therefore: If A and B are mutually exclusive, they cannot be independent.

2. Let A and B be independent.

$$
\begin{aligned}
P(A \cap B) &= P(A)P(B) \\
&> 0
\end{aligned}
$$

Then A and B are not mutually exclusive.

Therefore: If A and B are independent, they cannot be mutually exclusive.

BIOS 660 HW 5 Q11-16

02 October 2018

11.

P(at least 10 questions correct) = P(10 questions correct) + P(11 questions correct) + ...

If the student is guessing, than each of the 4 possible answers has equal probability of being chosen, and only 1 of these answers is correct.

P(question is correct) $= \frac{1}{4}$

P(i questions correct) $= \binom{20}{i}(\frac{1}{4})^i(\frac{3}{4})^{20-i}$

P(at least 10 questions correct) $= \Sigma_{i=10}^{20}\binom{20}{i}(\frac{1}{4})^i(\frac{3}{4})^{20-i}$

12.

$P_X(X = x_i) = P(\{s_j \in S : X(s_j) = x_i\})$

Sample Space $S = \{s_1, ..., s_n\}$

We are told that the range of our random variable X is $\mathcal{X} = \{x_1, ..., x_m\}$. It can be seen that our range $\mathcal{X}$ is finite, and thus $\sigma(\mathcal{X})$ is the σ-generated field of $\mathcal{X}$, which is the set that includes all subsets of $\mathcal{X}$, including $\mathcal{X}$ itself (from textbook Example 1.2.2). To satisfy Kolmogorov's axioms, it must be the case that our probability function with domain $\sigma(\mathcal{X})$ satisfies:

(1) $P(A) \geq 0 \ \forall A \in \sigma(\mathcal{X})$ (2) $P(\sigma(\mathcal{X})) = 1$ (3) If $A_1, A_2, ... \in \sigma(\mathcal{X})$ are pairwise disjoint, then $P(\cup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} P(A_i)$

(1) If $A_k \in \sigma(\mathcal{X})$, then $P_X(X = A_k) = P(\cup_{x_i \in A_k}\{s_j \in S : X(s_j) = x_i\}) \geq 0$, as we are told that P is a probability function.

(2) It can be seen that $\{\cup_{i=1}^{m}\{s_j \in S : X(s_j) = x_i\}\}$ is the set that contains all elements in the sample space, i.e. the sample space itself S. Thus $P_X(\mathcal{X}) = P(\cup_{i=1}^{m}\{s_j \in S : X(s_j) = x_i\}) = P(S) = 1$.

(3) If we assume that $A_1, A_2, ... \in \sigma(\mathcal{X})$ are pairwise disjoint, then we can see that $P(\cup_{k=1}^{\infty} A_k) = P(\cup_{k=1}^{\infty}\{\cup_{x_i \in A_k}\{s_j \in S : X(s_j) = x_i\}\})$ as P is a probability function, this tells us that this is equivalent to $\Sigma_{k=1}^{\infty} P(\cup_{x_i \in A_k}\{s_j \in S : X(s_j) = x_i\}) = \Sigma_{k=1}^{\infty} P_X(A_k)$.

Thus, all 3 of Kolmogorov's axioms are satisfied.

13. A function $F(x)$ is a CDF iff $lim_{x \to -\infty} F(x) = 0$ and $lim_{x \to \infty} F(x) = 1$, $F(x)$ is a nondecreasing function of x, and $F(x)$ is right continuous.

a.

$lim_{x \to -\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$ Since $lim_{x \to -\frac{\pi}{2}} \tan(x) = -\infty$, we can reciprocate this relationship for the inverse $\tan^{-1}(x)$, i.e. $lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$, so $lim_{x \to -\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2} + \frac{1}{\pi}(-\frac{\pi}{2}) = \frac{1}{2} - \frac{1}{2} = 0$

Similarly, as $lim_{x \to \frac{\pi}{2}} \tan(x) = \infty$ we can reciprocate this relationship for the inverse $\tan^{-1}(x)$, i.e. $lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$, so $lim_{x \to \infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2} + \frac{1}{\pi}(\frac{\pi}{2}) = \frac{1}{2} + \frac{1}{2} = 1$

$\frac{d}{dx}(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)) = \frac{1}{\pi(1+x^2)} > 0$.

$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$ is continuous, and thus right continuous. As the three conditions have been satisified, we can say that $F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$ is a cdf.

b.

$$\lim_{x \to -\infty} F_X(x) = 0 \quad \text{since} \quad \lim_{x \to -\infty} e^{-x} = \infty$$

and

$$\lim_{x \to \infty} F_X(x) = 1 \quad \text{since} \quad \lim_{x \to \infty} e^{-x} = 0.$$

Differentiating $F_X(x)$ gives

$$\frac{d}{dx} F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0, \text{ showing that } F_X(x) \text{ is increasing.}$$

c. $\lim_{x\to-\infty} e^{-e^{-x}} = 0$, $\lim_{x\to\infty} e^{-e^{-x}} = 1$, $\frac{d}{dx} e^{-e^{-x}} = e^{-x} e^{-e^{-x}} > 0$.

d. $\lim_{x\to-\infty} (1 - e^{-x}) = 0$, $\lim_{x\to\infty} (1 - e^{-x}) = 1$, $\frac{d}{dx}(1 - e^{-x}) = e^{-x} > 0$.

e. $\lim_{y\to-\infty} \frac{1-\epsilon}{1+e^{-y}} = 0$, $\lim_{y\to\infty} \epsilon + \frac{1-\epsilon}{1+e^{-y}} = 1$, $\frac{d}{dx}(\frac{1-\epsilon}{1+e^{-y}}) = \frac{(1-\epsilon)e^{-y}}{(1+e^{-y})^2} > 0$ and $\frac{d}{dx}(\epsilon + \frac{1-\epsilon}{1+e^{-y}}) > 0$, $F_Y(y)$ is continuous except on $y = 0$ where $\lim_{y\downarrow 0}(\epsilon + \frac{1-\epsilon}{1+e^{-y}}) = F(0)$. Thus is $F_Y(y)$ right continuous.

Since all the functions are continuous, then they are also right-continuous.

14. a. $\lim_{y\to-\infty} F_Y(y) = \lim_{y\to-\infty} 0 = 0$ and $\lim_{y\to\infty} F_Y(y) = \lim_{y\to\infty} 1 - \frac{1}{y^2} = 1$. For $y \le 1$, $F_Y(y) = 0$ is constant. For $y > 1$, $\frac{d}{dy}F_Y(y) = 2/y^3 > 0$, so F_Y is increasing. Thus for all y, F_Y is nondecreasing. Therefore F_Y is a cdf. Note: Since lim y↓c F(y)=F(c), then F(y) is right continuous.

b. The pdf is $f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} 2/y^3 & \text{if } y > 1 \\ 0 & \text{if } y \le 1. \end{cases}$

c. $F_Z(z) = P(Z \le z) = P(10(Y-1) \le z) = P(Y \le (z/10)+1) = F_Y((z/10)+1)$. Thus,

$$F_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ 1 - \left(\frac{1}{[(z/10)+1]^2}\right) & \text{if } z > 0. \end{cases}$$

15. a.

Choose c s.t. $c\int_0^{\pi/2} \sin(x)dx = 1$

$\int_0^{\pi/2} \sin(x)dx = -\cos(\pi/2) - (-\cos(0)) = -(0) + 1 = 1$

$1c = 1$

$c = 1$

b.

Choose c s.t. $c\int_{-\infty}^{\infty} e^{-|x|}dx = 1$

$\int_{-\infty}^{\infty} e^{-|x|}dx = \int_{-\infty}^{0} e^{-(-x)}dx + \int_0^{\infty} e^{-(x)}dx = (e^0 - e^{-\infty}) + (e^{-\infty} - e^0) = (1-0) + (1-0) = 1 + 1 = 2$

$2c = 1$

$c = \frac{1}{2}$

16.

$P(V = 5) = P(V \le 5) = P(T < 3) = \int_0^3 \frac{1}{1.5}e^{-t/1.5}dt = (-e^{-2}) - (-e^0) = 1 - e^{-2}$

For the rest of the values of V, i.e. $v \ge 6$, $V = 2T$, so $P(V \le v) = P(2T \le V) = P(T \le v/2) = \int_0^{v/2} \frac{1}{1.5}e^{-t/1.5}dt = (-e^{-v/3}) - (-e^0) = 1 - e^{-v/3}$

So, we have

$$P(V \le v) = \begin{cases} 0 & 0 < v < 5, \\ 1 - e^{-2} & 5 \le v < 6, \\ 1 - e^{-v/3} & 6 \le v \end{cases} .$$