Review Session MS Theory - 2018 Exam #1 & 2

$\boxed{Y|X \sim N(X, x) \qquad X \sim U(0,1)}$ → Given

#1a)

KNOW YOUR ITERATIVE/DOUBLE EXPECTATION/VARIANCE/ COVARIANCE FORMULAS

b)

$E(Y) = E[E(Y|X)] \qquad Y|X \sim N(x, x)$

$= E(X) = \frac{1+0}{2} = \frac{1}{2}$

$Var(Y) = E[Var(Y|X)] + Var[E(Y|X)]$

$= E(X) + Var(X) = \frac{1+0}{2} + \frac{(1-0)^2}{12}$

$= 7/12$

$Cov(Y, X)$

Method #1: $= \underbrace{E(XY)} - E(Y)E(X) = \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{12}$

Apply double expectation

$E(XY) = E[E(XY|X)] = E[X\,E(Y|X)] = E(X^2)$

$= Var(X) + [E(X)]^2$

$Var(X) = E(X^2) - [E(X)]^2 \qquad = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$

Method #2:

$Cov(Y, X) = E[Cov(X, Y|X)] + Cov(E(X|X), E(Y|X))$

Can condition on any rv in the same space, as long as you use the same one

$Cov(X, Y|X) = E[XY|X] - E(Y|X)E(X|X)$

$= XE(Y|X) - XE(Y|X) = 0$

$Cov(Y, X) = 0 + Cov(X, X) = 0 + Var(X) = \frac{1}{12}$

b) $Cov(Y-X, X) = E[X(Y-X)] - E(Y-X)E(X)$

$= [E(XY) - E(X^2)] - [E(Y) - E(X)]E(X)$

Applying linearity of expectation

$= 0 - 0 = 0$

In part 1a, we showed $E(XY) = E(X^2)$ and $E(Y) = E(X)$

OR

$Cov(Y-X, X) = E[\underbrace{Cov(X, Y-X|X)}] + Cov(\underbrace{E(Y-X|X)}, E(X|X))$

This will be 0 because covariance of X & something conditioned on X

$E(Y-X|X)$

$= E(Y|X) - E(X|X)$

$= X - X = 0$

$Cov(k, X) = 0$

constant

So, $Cov(Y-X, X) = 0$

Now, are $Y-X$, X independent? Let $T = Y-X$.

Showing Independence:

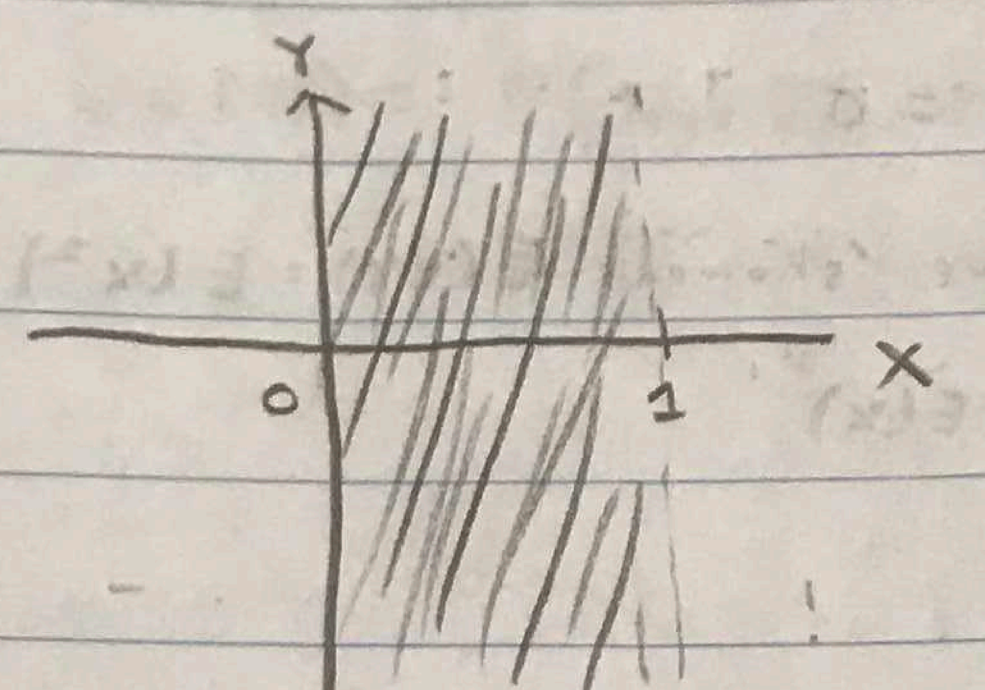
1) $f_{T,X}(t,x) = f_T(t) \cdot f_X(x)$ for all (t,x)

2) $F_{T,X}(t,x) = F_T(t) \cdot F_X(x)$ for all (t,x)

These two methods are about finding the joint pdf/cdf, then factoring into distinct / separate functions of T and X for independence. DO NOT DO THIS HERE.

First, you'd need to find $f_Y(y) = \int_{\underbrace{x \in X}} f_X(x) \cdot f_{Y|X}(y|x)\,dx$

here is 0 to 1 b/c $X \sim U(0,1)$



3) Show that the distribution $T|X$ does not depend on X for all (t,x)

$Y|X \sim N(X, X) \Rightarrow$ Given

Thus,

$Y - X|X \sim N(0, X)$ We let $T = Y - X$

$\Downarrow$

$f_{T|X}(t|x) = \frac{1}{\sqrt{2\pi x}} e^{-t^2/2x}$

Since $T|X$ does depend on X for some (t,x), T and X not independent

The key here is recognizing that b/c $Y|X$ is normal, so is $Y - X|X$.

Again, avoid integrating where you can

c)

$$\boxed{\text{Double expectation of probabilities} \quad P(A) = E[P(A|X)]}$$

☆

$$P(X > Y) = 1 - \underbrace{P(Y < X)} = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$\underbrace{P(Y < X)}_{A} = E\left[\underbrace{P(Y < X | X)}\right] = E(\tfrac{1}{2}) = \tfrac{1}{2}$$

Here, we are looking at P where $Y|X < X$.
Since $Y|X \sim N(X, X)$,
by symmetry, the P that $Y|X$ is less than its mean is $\tfrac{1}{2}$.

d) $P(Z_{n+1} < Z_n \mid \underbrace{Z_n = \max(Z_1, \dots, Z_n)}_{\text{Call } A}) = \underbrace{1 - P(Z_{n+1} > Z_n | A)}$

Easier to work like this b/c we want/now have $Z_{n+1} >$ the largest of $Z_1, \dots, Z_n$

Apply Bayes' Rules

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(Z_{n+1} > Z_n \mid A) = \frac{P(A,\ Z_{n+1} > Z_n)}{P(A)}$$

B/c $Z_1, \dots, Z_{n+1}$ is a sequence of iid rvs, this becomes a counting problem

$$P(A) = \frac{(n-1)!}{n!}$$

$\rightarrow Z_n$ is fixed at the top spot, so $(n-1)!$ free spots

$\rightarrow n!$ ways to order $Z_1, \dots, Z_n$

$$= \tfrac{1}{n}$$

Z_n, Z_{n+1} fixed at top end; $n-1$ free

$$P(A, Z_{n+1} > Z_n) = \frac{(n-1)!}{(n+1)!} \rightarrow (n+1)! \text{ orderings of } Z_1, \dots, Z_{n+1}$$

$$P(Z_{n+1} < Z_n \mid A) = 1 - \frac{\frac{(n-1)!}{(n+1)!}}{\frac{(n-1)!}{n!}} = 1 - \frac{n!}{(n+1)!} = 1 - \frac{1}{n+1} = \boxed{\frac{n}{n+1}}$$

e) Let $W = Z_1 - Z_2$

B/c Z_1, Z_2 are iid $N(0,1)$, $W \sim N(0,2)$

We could then say $T = \frac{Z_1 - Z_2}{\sqrt{2}} \sim N(0,1)$

$$E[|T|] = \int_{-\infty}^{\infty} |t| \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$ → or recognize ~~both pieces are~~ symmetrical about 0

$$= \int_{-\infty}^{0} -t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt + \int_{0}^{\infty} t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

2a)

$$= 2\int_{0}^{\infty} t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} t e^{-t^2/2} \, dt$$

Let $u = -t^2/2$

$du = -t \, dt$

$$= -\frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{u} \, du$$

$$= -\sqrt{\frac{2}{\pi}} \left[e^{-t^2/2} \Big|_{0}^{\infty} \right]$$

$$= -\sqrt{\frac{2}{\pi}} \left[0 - 1 \right] = \sqrt{\frac{2}{\pi}}$$

We want $E[|Z_1 - Z_2|] = E[|W|] = \sqrt{2}\, E[|T|] = \frac{2}{\sqrt{\pi}}$

Qaqish also did this.

$$2 \int_0^{\infty} t \frac{e^{-t^2/2}}{\sqrt{2\pi}} = 2 \int_0^{\infty} t\, \phi(t)\, dt$$

$$\phi(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}$$

$$\frac{d\, \phi(t)}{dt} = \frac{e^{-t^2/2}}{\sqrt{2\pi}} \cdot -t = -t\, \phi(t)$$

Thus, $2 \int_0^{\infty} t\, \phi(t)\, dt = 2 \left[\phi(t) \right] \Big|_{\infty}^{0}$

as you're integrating $-\frac{d\, \phi(t)}{dt}$ w/ respect to t

2a) First, it should be evident that $E(\bar{Y}) = \mu$ by WLLN

$\left[\text{or } E\left(\frac{Y_1 + Y_2}{2} \right) = \frac{1}{2}\left[E(Y_1) + E(Y_2) \right] = \frac{1}{2}\left[\mu + \mu \right] = \mu \right]$

$$f_{Y_i}(y|\mu) = \prod_{i=1}^{2} \frac{1}{\mu} e^{-Y_i/\mu} = \left(\frac{1}{\mu} \right)^2 e^{-\sum_{i=1}^{2} Y_i / \mu}$$

This can be written in exp. family form

$h(y) = 1$, $c(\theta) = \left(\frac{1}{\mu} \right)^2$, $w(\theta) = -\frac{1}{\mu}$, $t(y) = \sum_{i=1}^{2} Y_i$

I'm not sure what $\{ (w_1(\theta), \dots, w_k(\theta)) : \theta \in \Theta \}$ contains an open set in R^k means, but since $w(\theta) = -\frac{1}{\mu}$ and $\mu > 0$, I'm assuming it's okay.

So, $T(Y) = \sum_{i=1}^{2} Y_i$ is a CSS for μ.

c)

Since $E(\bar{Y}) = \mu$, so $\bar{Y}$ is an unbiased estimator for μ and $\bar{Y}$ is a function only of $\sum_{i=1}^{2} Y_i$ (a CSS for μ), by Lehmann-Scheffe Theorem, $\bar{Y}$ is the UMVUE for μ.

b) $X_1 = Y_{(1)} = \min(Y_1, Y_2)$

$$F_{Y_1}(y) = P(Y_{(1)} \le y)$$
$$= 1 - P(Y_{(1)} \ge y)$$
$$= 1 - [P(Y_1 \ge y)]^2$$
$$= 1 - [1 - P(Y_1 \le y)]^2 = 1 - [e^{-y/\mu}]^2$$

$$P(Y_1 \le y) = \int_0^y \frac{1}{\mu} e^{-y/\mu} \, dy = -\int_0^y e^t \, dt$$

$t = \frac{-y}{\mu}$
$$= -\left[e^{-y/\mu} \Big|_0^y \right]$$

$dt = -\frac{1}{\mu} dy$
$$= -\left[e^{-y/\mu} - 1 \right]$$

$$= 1 - e^{-y/\mu}$$

$$f_{Y_{(1)}}(y) = 0 - 2 \cdot [e^{-y/\mu}] \cdot [e^{-y/\mu}] \cdot \left(-\frac{1}{\mu}\right)$$
$$= \frac{2}{\mu} e^{-2y/\mu} \qquad 0 < y < \infty, \quad \mu > 0$$

Thus, $X_1 = \min(Y_1, Y_2)$ is distributed as exponential w/ mean $\mu/2$

Think of finding the expected value of a die roll

Basically, we can "break up" or partition the E(R) into the two possible distinct events multiplied by the probability of that event occurring

c) $E[R] = \underbrace{E[R \mid Y_1 > Y_2]}_{\text{Event 1}} \underbrace{P(Y_1 > Y_2)}_{P(\text{Event 1})} + \underbrace{E[R \mid Y_2 > Y_1]}_{\text{Event 2}} \underbrace{P(Y_2 > Y_1)}_{P(\text{Event 2})}$

Since R is $|Y_1 - Y_2|$, the two possibilities are $Y_1 > Y_2$ & $Y_2 > Y_1$.

$P(Y_1 > Y_2) = P(Y_2 > Y_1) = \frac{1}{2}$ b/c we have two iid continuous random variables

$E[R \mid Y_1 > Y_2] = E[Y_1 - Y_2 \mid Y_1 > Y_2]$

$E[R \mid Y_2 > Y_1] = E[Y_2 - Y_1 \mid Y_2 > Y_1]$

Written this way b/c

$R = \max(Y_1, Y_2) - \min(Y_1, Y_2)$

$E[R \mid Y_1 > Y_2] = E[\,E[Y_1 - Y_2 \mid Y_1 > Y_2, Y_2]\,]$

We use iterative expectation & add a second condition so Y_2 is treated as a constant

Apply the memoryless property & apply to expectations

$\Rightarrow$ Essentially $E[Y_1 - Y_2 \mid Y_1 > Y_2, Y_2] = E[Y_1 + k \mid Y_1 > k]$

$= E[Y_1] = \mu$

By the same logic, $E[R \mid Y_2 > Y_1] = E[E[Y_2 - Y_1 \mid Y_2 > Y_1, Y_1]]$

$= E[Y_2] = \mu$

So, $E[R] = \frac{1}{2}\mu + \frac{1}{2}\mu = \mu$

Using mgfs to find the distribution of R

$M_{Y_1 - Y_2}(t) = M_{Y_1}(t) \cdot M_{Y_2}(-t)$

As Y_1, Y_2 are iid, the difference of the two can be expressed as the product of their mgfs

$$= \frac{1}{1-\mu t} \cdot \frac{1}{1+\mu t} = \frac{1}{1^2 - \mu^2 t^2}$$

This is a double exp. / Laplace distribution

w/ $\mu = 0$ and $\sigma = \mu > 0$ (in our dist. table)

$$f_T(t) = \frac{1}{2\mu} e^{-|t|/\mu} \qquad -\infty < t < \infty$$

$R = |T|$ => monotone from $(-\infty, 0)$ & monotone over $(0, \infty)$

$(-\infty, 0)$ -> Abs. value removed; $t<0$, so two negatives cancel, then plug in $-r$ for t

$R = |T|$; $T = -R$, $t < 0$

$$f_R(r) = f_T(-r) \left| \frac{d}{dr} - r \right| = \frac{1}{2\mu} e^{-r/\mu}$$

$(0, \infty)$ -> $t > 0$, so we have $e^{-t/\mu}$; plug in r for t

$R = |T|$; $T = R$, $t > 0$

$$f_R(r) = f_T(r) \left| \frac{d}{dr} r \right| = \frac{1}{2\mu} e^{-r/\mu}$$

Summing the two pieces, you get

$$f_R(r) = \frac{2}{2\mu} e^{-r/\mu} = \frac{1}{\mu} e^{-r/\mu} \qquad \mu > 0, \quad r > 0$$

Thus $R \sim Exp(\mu)$, so $Var(R) = \mu^2$

d) Recognize that $Y_1 + Y_2 = X_1 + X_2$

$$E(Y_1) + E(Y_2) = E(X_1) + E(X_2)$$

$$E(X_2) = E(Y_1) + E(Y_2) - E(X_1)$$

$$= \mu + \mu - \mu/2 = \frac{3\mu}{2}$$

$$\underbrace{Var(Y_1 + Y_2)}_{Y_1, Y_2 \text{ iid}} = \underbrace{Var(X_1 + X_2)}_{X_1, X_2 \text{ as order stats are not iid}}$$

$$Var(Y_1) + Var(Y_2) = Var(X_1) + Var(X_2) + 2\,Cov(X_1, X_2)$$

$$Cov(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2)$$

$$= E(Y_1 Y_2) - E(X_1)E(X_2)$$

$$= \underbrace{E(Y_1)E(Y_2)}_{b/c\ Y_1, Y_2 \text{ iid}} - E(X_1)E(X_2)$$

$$= \mu^2 - \mu/2 \cdot \frac{3\mu}{2} = \mu^2 - \frac{3\mu^2}{4} = \frac{\mu^2}{4}$$

$$Var(X_2) = \mu^2 + \mu^2 - \frac{\mu^2}{4} - \frac{2\mu^2}{4} = \frac{8\mu^2}{4} - \frac{3\mu^2}{4} = \frac{5\mu^2}{4}$$

Some function of $\bar{Y}$

e) The key is first recognizing that $\underbrace{E[R \mid \bar{Y}]}_{} = g(\bar{Y})$.

Then, from part A, recognize $\bar{Y}$ is the UMVUE & CSS for μ

$$E[E[R \mid \bar{Y}]] = E[g(\bar{Y})] = E(R) = \mu$$

1) Using def of CSS:

$E[g(\bar{Y})] = \mu$ & $E(\bar{Y}) = \mu$

Let $h(\bar{Y}) = g(\bar{Y}) - \bar{Y}$

$$E[\underbrace{g(\bar{Y}) - \bar{Y}}] = 0$$

$h(\bar{Y})$ is a function of a CSS of μ

Thus, $h = 0$ is the only function that has $E(h(\bar{Y})) = 0$ for all $\mu > 0$. Since $h(\bar{Y})$ is 0 function, $g(\bar{Y}) = \bar{Y}$, so $E[R \mid \bar{Y}] = \bar{Y}$.

2) Uniqueness of UMVUE

$E[g(\bar{Y})] = \mu$ & $g(\bar{Y})$ is function of only CSS of μ $(\bar{Y})$

By Lehmann - Scheffe, $g(\bar{Y})$ is UMVUE of μ. But $\bar{Y}$ is the UMVUE of μ. By uniqueness of UMVUE, $g(\bar{Y}) = E[R \mid \bar{Y}] = \bar{Y}$