1. Suppose that iid random variables $X_1, \ldots, X_n$ follow a uniform distribution on the interval $(0, 1)$ with pdf
$$f_X(x) = 1, \quad 0 < x < 1.$$
Let random variables $U = X_{(1)}$ and $V = 1 - X_{(n)}$, where $X_{(1)} = \min_i X_i$ and $X_{(n)} = \max_i X_i$ are minimum and maximum order statistics, respectively.
   (a) Find an explicit expression for the joint distribution of the random variables U and V.
   (b) Let $R = nU$ and $S = nV$. Show that
   $$P(R > r, S > s) = \left(1 - \frac{r}{n} - \frac{s}{n}\right)^n.$$
   (c) Following the result in (b), show that R and S are asymptotically independent. You may need the fact that
   $$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$
   (d) What is the asymptotic distribution of R and S?

2. A certain simple biological system involves exactly two independently functioning components. If one of these two components fails, then entire systems fails. For $i = 1, 2$, let Y_i be the random variable representing the time to failure of the ith component, with the pdf of Y_i being
$$f_{Y_i}(y_i) = \theta_i e^{-\theta_i y_i}, \quad 0 < y_i < \infty, \quad \theta_i > 0.$$
Clearly, if this biological system fails, then only two random variables are observable, namely U and W, where $U = \min(Y_1, Y_2)$ and
$$W = \begin{cases} 1, & \text{if } Y_1 < Y_2, \\ 0, & \text{if } Y_2 < Y_1. \end{cases}$$
   (a) Show that the joint distribution $f_{U,W}(u, w)$ of random variables U and W is
   $$f_{U,W}(u, w) = \theta_1^{(1-w)}\theta_2^{w} e^{-(\theta_1+\theta_2)u}, \quad 0 < u < \infty, \quad w = 0, 1,$$
   by working on the derivation of $P(U \le u, W = 0)$ and $P(U \le u, W = 1)$.
   (b) Find the marginal distribution $f_W(w)$ of the random variable W.
   (c) Find the marginal distribution $f_U(u)$ of the random variable U.
   (d) Are U and W independent random variables?

3. Suppose that $X_1, X_2, \dots, X_n$ are iid random variables distributed as Poisson with mean $\mu > 0$. Denote $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$. We are interested in constructing a confidence interval for μ.
   (a) State the central limit theorem for $\bar{X}_n$.
   (b) What is the asymptotic variance of $T_n = \sqrt{n}(\bar{X}_n - \mu)$?
   (c) What is the appropriate function $h(\bar{X}_n)$ so that $h(\bar{X}_n)T_n \to_d N(0,1)$? What theorem(s) are needed to justify such claim?
   (d) Use the last part to construct an approximate 95% confidence interval for μ. Give the upper and lower limits in explicit form.
   (e) Another approach to eliminate μ from the asymptotic variance is to find a function g such that $\sqrt{n}(g(\bar{X}_n) - g(\mu)) \to_d N(0,1)$. Find an explicit expression for $g(\mu)$.
   (f) Use the last part to construct an approximate 95% confidence interval for μ. Give the upper and lower limits in explicit form.

4. Let $Y_1, \dots, Y_n$ constitute a random sample from $N(0, \sigma^2)$.
   (a) Show that $T = \sum_{i=1}^{n} Y_i^2$ is a sufficient statistic for unknown $\theta = \sigma^r$, where r is a known positive integer.
   (b) What is the distribution of T/σ^2? Justify your answer.
   (c) If a random variable X follows Gamma($\alpha = n/2, \beta = 2$), then

$$E(X) = \alpha\beta = n \quad \text{and} \quad E(X^{r/2}) = \frac{\Gamma(n/2 + r/2)}{\Gamma(n/2)} 2^{r/2}.$$

   Develop an explicit expression for an unbiased estimator $\hat{\theta}$ that is a function of T. You need to show that $E\{\hat{\theta}(T)\} = \sigma^r$.

5. (a) Let X and Y be random variable such that $E(X^k)$ and $E(Y^k) \neq 0$ exist for a positive integer k. If the ratio of X/Y and its denominator Y are independent, prove that $E\{(X/Y)^k\} = E(X^k)/E(Y^k)$.
   (b) Let $X_{(1)}, \dots, X_{(n)}$ be the order statistics of a random sample $X_1, \dots, X_n$ from a distribution that has pdf

$$f_X(x|\beta) = \frac{1}{\beta} e^{-x/\beta}, \quad 0 < x < \infty, \quad 0 < \beta < \infty.$$

   Show that $T = \sum_{i=1}^{n} X_{(i)}$ is complete sufficient and $nX_{(1)}/T$ is ancillary for β.
   (c) Use the result in (a) and Basu's Theorem to determine $E(R)$, where $R = nX_{(1)}/T$.