1. Suppose $X_1, \dots, X_n$ are iid Poisson(θ) with a probability density function
$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad \theta > 0, \quad x = 0, 1, 2, \dots.$$
   (a) Show that $I(X_1 = 0)$ is an unbiased estimator of $e^{-\theta}$, where
$$I(X_1 = 0) = \begin{cases} 1 & \text{if } X_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$
   (b) Show that $\sum_{i=1}^n X_i$ is a complete sufficient statistic.
   (c) Using Lehmann-Scheffe Theorem, show that $\phi(\sum_{i=1}^n X_i)$ is the best unbiased estimator (UMVUE) of $e^{-\theta}$, where
$$\phi\left(\sum_{i=1}^n X_i\right) = \left(1 - \frac{1}{n}\right)^{\sum_{i=1}^n X_i}$$
   (d) Compute the Cramér-Rao Lower Bound for unbiased estimators of $e^{-\theta}$.
   (e) Find the MLE of $e^{-\theta}$.

2. Suppose that the random variables $Y_1, \dots, Y_n$, $n > 2$ are independent and normally distributed with $EY_i = \theta x_i$, where $x_1, \dots, x_n$ are known constants and none of which is zero. Let $\text{Var}Y_i = \sigma^2 > 0$ and $\theta \in (-\infty, \infty)$. Assume that σ^2 is a known constant and θ is an unknown parameter.
   (a) Find the method of moments estimator $\tilde{\theta}$ of θ, matching $M_1 = n^{-1} \sum_{i=1}^n Y_i$ and $E(M_1)$.
   (b) Find the MLE $\hat{\theta}$ of θ and show that it is an unbiased estimator of θ.
   (c) Find the distribution of $\hat{\theta}$.
   (d) Let $T_1 = \sum_{i=1}^n Y_i / \sum_{i=1}^n x_i$ and $T_2 = \sum_{i=1}^n (Y_i/x_i)/n$. Show that both T_1 and T_2 are unbiased estimators of θ.
   (e) Show that the variance of $\hat{\theta}$ is smaller than the variance of both T_1 and T_2, i.e., $\text{Var}(\hat{\theta}) \le \text{Var}(T_1)$ and $\text{Var}(\hat{\theta}) \le \text{Var}(T_2)$.

   You may need the fact that $n^{-1} \sum_{i=1}^n x_i^2 \ge (n^{-1} \sum_{i=1}^n x_i^{-2})^{-1}$, i.e., arithmetic mean $\ge$ harmonic mean.

3. Let $X_1, \dots, X_n$ be a sample from the distribution with probability density function
$$f(x|\theta) = e^{-(x-\theta)}, \quad \theta \le x < \infty, \quad -\infty < \theta < \infty.$$
If one tries to test $H_0 : \theta = 0$ versus $H_1 : \theta \ne 0$.

(a) Find the probability density function of $X_{(1)} = \min\{X_1, \ldots, X_n\}$.

(b) Find the likelihood ratio test statistic $\lambda(x)$, as a function of $X_{(1)}$. If your test statistic depends on the range of $X_{(1)}$, please indicate it.

(c) Draw a figure of your test statistic $\lambda(x)$ as a function of $x_{(1)}$.

(d) By the likelihood ratio test, one rejects H_0 if $\delta(x) = 1$, where

$$\delta(x) = \begin{cases} 1 & \text{if } \lambda(x) < c, \\ 0 & \text{if } \lambda(x) > c. \end{cases}$$

Show that, equivalently, one can use the following rejection region:

$$\delta(x) = \begin{cases} 1 & \text{if } X_{(1)} > c^*, \\ 0 & \text{if } X_{(1)} < c^*. \end{cases}$$

(e) Following (d), find c^* such that the type I error probability of the test equals 0.05.