Cheyenne Crowley
BIOS 667 MT2 Redo
HW 6

1.

$\nu_{ij} = E[Y_{ij} | b_i] = (\beta_1 + b_{i1}) + (\beta_2 + b_{i2}) x_{ij}$

with $x_{i1} = 0$, $x_{i2} = 1$, $\beta_1 = 1$ $\beta_2 = 1$ $G = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E[Y_{ij} | b_i] = 1 + b_{i1} + (1 + b_{i2}) I(j = 2)$

1A

$\beta_2 \to$ conditional: contrasts in ν

$\to$ marginal: contrasts in μ.

1B

$$\text{corr}(Y_{i1}, Y_{i2}) = \frac{\text{cov}(Y_{i1}, Y_{i2})}{\sqrt{\text{var}(Y_{i1})\,\text{var}(Y_{i2})}}$$

$\text{cov}(Y_{i1}, Y_{i2}) = E[Y_{i1} Y_{i2}] - E[Y_{i1}] E[Y_{i2}]$

$E[Y_{i1}] = E[E[Y_{i1} | b_i]] = E[1 + b_{i1}] = 1 + E[b_{i1}] = 1 + 0 = 1$

$E[Y_{i2}] = E[E[Y_{i2} | b_i]] = E[1 + b_{i1} + (1 + b_{i2})] = 1 + E[b_{i1}] + 1 + E[b_{i2}]$
$= 1 + 0 + 1 + 0 = 2$

$\text{var}(Y_{i1}) = E(\text{var}(Y_{i1} | b_i)) + \text{var}(E[Y_{i1} | b_i])$

$G = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} = \text{cov}(b_i) \Rightarrow$ between subjects

$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{cov}(Y_i | b_i) \Rightarrow$ within subjects

$\Rightarrow \text{var}(Y_{i1}) = E[1] + \text{var}(1 + b_{i1}) = 1 + \text{var}(b_{i1}) = 1 + 1 = 2$

$\text{var}(Y_{i2}) = E[\text{var}(Y_{i2} | b_i)] + \text{var}(E[Y_{i2} | b_i])$

$= E[1] + \text{var}(1 + b_{i1} + 1 + b_{i2})$

$= 1 + \text{var}(b_{i1} + b_{i2}) = 1 + \text{var}(b_{i1}) + \text{var}(b_{i2}) + 2\text{cov}(b_{i1}, b_{i2}$

$= 1 + 1 + 4 + 2(1) = 8$

$E[Y_{i1}Y_{i2}] = E[E[Y_{i1}Y_{i2}|b_i]]$ $\quad \begin{pmatrix} Y_{i1} \\ Y_{i2} \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{bmatrix} 2 & x \\ x & 8 \end{bmatrix}\right) \ldots$

Law of total covariance

$$\text{cov}(X,Y) = E(\text{cov}(X,Y|Z)) + \text{cov}(E[X|Z], E[Y|Z])$$

$$\text{cov}(Y_{i1}, Y_{i2}) = E[\underbrace{\text{cov}(Y_{i1}, Y_{i2}|b_i)}_{R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}] + \text{cov}(E[Y_{i1}|b_i], E[Y_{i2}|b_i])$$

$$= E[0] + \text{cov}(1 + b_{i1}, 2 + b_{i1} + b_{i2})$$

$$= \text{cov}(b_{i1}, b_{i1} + b_{i2}) = \underbrace{\text{var}(b_{i1}) + \text{cov}(b_{i1}, b_{i2})}_{G = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}}$$

$$= 1 + 1 = 2$$

$$\Rightarrow \text{corr}(Y_{i1}, Y_{i2}) = \frac{\text{cov}(Y_{i1}, Y_{i2})}{\sqrt{\text{var}(Y_{i1})\text{var}(Y_{i2})}} = \frac{2}{\sqrt{2(8)}} = \frac{2}{4} = 1/2$$

$\boxed{1C}$ $\quad Y_i = \begin{bmatrix} Y_{i1} \\ Y_{i2} \end{bmatrix}$

From 1B, we found $Y_{i1} \sim N(1, 2)$ $\quad Y_{i2} \sim N(2, 8)$
and $\text{cov}(Y_{i1}, Y_{i2}) = 2$

$$\Rightarrow Y_i \sim N\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}\right)$$

$\boxed{1D}$ BLUP $\quad b_{i2} | Y_{i2}$ $\quad \begin{pmatrix} Y_i \\ b_i \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_i \\ 0 \end{pmatrix}, \begin{bmatrix} \Sigma_i & GZ_i^T \\ GZ_i^T & G \end{bmatrix}\right]$

$$\hat{b}_i = G Z_i^T \Sigma_i^{-1} (Y_i - \hat{\mu}_i)$$

$Z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \Sigma_i = \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}$

$$b_{i2} = [1 \; 4] \begin{bmatrix} 1 \\ 1 \end{bmatrix} [8]^{-1} (Y_{i2} - 2)$$

$G = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ $\quad \mu_i = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= 5(1/8)(Y_{i2} - 2)$$

$$\boxed{\hat{b}_{i2} = \frac{5}{8}(Y_{i2} - 2)}$$

$\boxed{1.E.}$ BLUP predicted MSE = variance

$$var(\hat{b_i} - b_i) = var(\hat{b_{i2}}) + var(b_{i2}) - 2\,cov(\hat{b_{i2}}, b_{i2})$$

$$= var\left(\tfrac{5}{8}[Y_{i2} - 2]\right) + var(b_{i2}) - 2\,cov\left(\tfrac{5}{8}(Y_{i2}-2), b_{i2}\right)$$

$$= \frac{25}{64} var(Y_{i2}) + 4 - 2\left(\tfrac{5}{8}\right) \underbrace{cov(Y_{i2}, b_{2})}_{GZ_i^T}$$

$$= \frac{25}{64}(8) + 4 - 2\left(\tfrac{5}{8}\right)(5)$$

$$= \frac{25}{8} + \frac{32}{8} - \frac{50}{8}$$

$$\Rightarrow var(\hat{b_{i2}} - b_{i2}) = 7/8$$

$\boxed{1F}$ $\hat{b_{i2}} = \frac{5}{8}(Y_{i2} - 2)$

if a subject has 0 observations

$\Rightarrow \hat{b_{i2}} = E[\hat{b_{i2}}] = 0$

$var(\hat{b_{i2}}) = var(b_{i2}) = 4$

$\boxed{1G}$ $D = Y_{i2} - Y_{i1}$

$$E[Y_{i2} - Y_{i1}] = E[E[Y_{i2} - Y_{i1} | b_i]]$$

$$= E[E[Y_{i2}|b_i] - E[Y_{i1}|b_i]]$$

$$= E[-\beta_1 - b_{i1} + (\beta_1 + b_{i1} + \beta_2 + b_{i2})]$$

$$= E[\beta_2 + b_{i2}] = E[\beta_2] + E[b_{i2}]$$

$$= E[\beta_2] = \beta_2$$

$$var(Y_{i2} - Y_{i1}) = var(Y_{i2}) + var(Y_{i1}) - 2\,cov(Y_{i1}, Y_{i2})$$

we know $\begin{pmatrix} Y_{i1} \\ Y_{i2} \end{pmatrix} \sim N\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}\right)$ $\Rightarrow$ $= 2 + 8 - 2(2) = 6$
and var <u>isnt</u> calculated based on β

$\Rightarrow D_i \sim N(\beta_2, 6)$

$\Rightarrow B_2$ can be estimated by D_i b/c $E[D_i] = \beta_2$

Given $var(\beta_2 | D_i) = \delta / K$ Let $K = \#$ of subjects

BLUE of $\beta \Rightarrow \hat{\beta} = (X^T \Sigma^{-1} X)^{-1} X^T \Sigma^{-1} Y$

$\hat{\beta}_2 = (X^T \Sigma^{-1} X)^{-1} X^T \Sigma^{-1} (D_i)$

$var(\hat{\beta}_2 | D_i = Y_{i2} - Y_{i1}) = \frac{1}{K} var(D_i) = \frac{6}{K} = \frac{\delta}{K} \Rightarrow \delta = 6$

* the BLUE (IG) of β is the usual WLS estimator

$\boxed{1H}$ $\hat{\beta} = (X^T \Sigma^{-1} X)^{-1} X^T Y$ $\Rightarrow \gamma = 6 = \delta$

$var(\hat{\beta}_2) = var((X^T \Sigma X)^{-1} X^T Y)$

$= \frac{1}{K} var(Y_{i2} - Y_{i1}) = \frac{6}{K} = \frac{\gamma}{K} \Rightarrow \gamma = 6.$

$\boxed{2}$ $\gamma_{ij} = E[Y_{ij} | b_i] = (\beta_1 + b_{i1}) + (\beta_2 + b_{i2}) x_{ij}$

FULL MODEL : $-2\log L = 200$

Reduced : $-2\log L = 205$

$q = 1$ Let $\alpha = 0.05$

$H_0: g_{22} = 0$ $H_1: g_{22} > 0$

$\chi^2_1 = 3.841$

$\chi^2_2 = 5.991$

$205 - 200 = 5$

$\Rightarrow$ p-value $= \dfrac{P\{\chi^2_1 > 5\} + P\{\chi^2_2 > 5\}}{2}$

$= \dfrac{0.025 + 0.082}{2} = 0.0535$

If we use an $\alpha = 0.05$ and the correct p-value is 0.0535 we fail to reject Ho.