Converge in Probability

$X_n \xrightarrow{p} X$ as $n \to \infty$
$\lim_{n\to\infty} P(|X_n - X| < \epsilon) = 1$

WLLN

a) Xs iid **b)** $E(X_i) = \mu$ **c)** $Var(X_i) = \sigma^2 < \infty$
Let $\bar{X}_n = \sum_{i=1}^n X_i$ Then for any $\epsilon > 0$:
$\lim_{n\to\infty} P(|\bar{X}_n - \mu| < \epsilon) = 1$
$\bar{X}_n \xrightarrow{p} \mu$ (consistency of $\bar{X}_n$)
The condition $E(X_i)$ exists and is finite is sufficient in WLLN

Converge in Distribution

Let F_{X_n} be cdf of X_n
Then $X_n \xrightarrow{d} X$ as $n \to \infty$
If $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at all points where $F_X(x)$ is continuous
For large n, the cdf of X_n becomes close to the cdf of X
Convergence in distribution does not imply X_n and X approximate each other

CLT

a) Xs iid **b)** $E(X_i) = \mu$ **c)** $Var(X_i) = \sigma^2 < \infty$
Let $\bar{X}_n = \sum_{i=1}^n X_i$ (so far same setup as WLLN)
Let $Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ and let G_n denote cdf of Z_n
Then for any $-\infty < z < \infty$: $\lim_{n\to\infty} G_n(z) = \Phi(z)$
Z_n has a limiting standard normal distribution: $Z_n \xrightarrow{d} N(0,1)$ as $n \to \infty$

Relationships between Modes of Convergence

$X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$

Slutsky's Theorem

If **a)** $X_n \xrightarrow{d} X$ and **b)** $Y_n \xrightarrow{p} a$ (a is finite constant)
Then: **1)** $Y_n X_n \xrightarrow{d} aX$ **2)** $Y_n + X_n \xrightarrow{d} a + X$ **3)** $X_n/Y_n \xrightarrow{d} X/a$ (if $a \neq 0$)
X_n and Y_n don't have to be independent
Slutsky's allows substituting consistent estimators when proving convergence in distribution

Convergence of Transformed Sequences

If $X_n \overset{p}{\to} X$ then $h(X_n) \overset{p}{\to} h(X)$
If $X_n \overset{d}{\to} X$ then $h(X_n) \overset{d}{\to} h(X)$
h only has to be continuous on range of X
$S_n^2 \overset{a.s}{\to} \sigma^2$ as $n \to \infty$ $\quad \bar{X_n}^2 \overset{a.s}{\to} \mu^2$ as $n \to \infty$

Delta Method - Univariate

Suppose $\{T_n\}$ is a random sequence with
$\sqrt{n}(T_n - \theta) \overset{d}{\to} N(0, \sigma^2)$ and
g is a function with $g'(\theta)$ exists and $\neq 0$ Then:
$\sqrt{n}\{g(T_n) - g(\theta)\} \overset{d}{\to} N(0, \{g'(\theta)\}^2\sigma^2)$
θ is the asymptotic mean of T_n
θ may or may not be mean of T_n or mean may not even exist