1. Let X_1 and X_2 be independent and identical exponential random variables with pdf

$$f_X(x|\beta) = \frac{1}{\beta}e^{-x/\beta}, \quad 0 < x < \infty, \quad 0 < \beta < \infty,$$

and let $X_{(1)}$ and $X_{(2)}$ are order statistics.

   (a) Let

$$U_1 = \frac{2X_{(1)}}{X_1 + X_2} \quad \text{and} \quad U_2 = X_1 + X_2.$$

   Find the joint density function $f_{U_1,U_2}(u_1, u_2)$ using the Jacobian method.

   (b) Show that U_1 and U_2 are independent.

   (c) Find the marginal pdf of U_1 and U_2, and find $E(U_1)$.

   (d) Show that U_2 is a complete sufficient statistic and U_1 is an ancillary statistic for β.

   (e) Use Basu's Theorem to find $E(U_1)$. Is it identical to the answer in (c)?

2. Let $X_1, \ldots, X_n$ constitute a random sample from $N(0, \sigma^2)$ with probability density function

$$f_X(x|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}}, \quad -\infty < x < \infty, \quad 0 < \sigma^2 < \infty.$$

   (a) Show that the distribution belongs to an exponential family by identifying $h(x)$, $c(\sigma^2)$, $w(\sigma^2)$, and $t(x)$.

   (b) Show that $T = \sum_{i=1}^n X_i^2$ is a sufficient statistic for unknown $\theta = \sigma^r$, where r is a known positive integer.

   (c) What is the exact distribution of T/σ^2? Justify your answer.

   (d) If a random variable Y follows Gamma$(\alpha = 2, \beta = n/2)$, then

$$E(Y) = \alpha\beta = n \quad \text{and} \quad E(Y^{r/2}) = \frac{\Gamma(n/2 + r/2)}{\Gamma(n/2)}2^{r/2}.$$

   Develop an explicit expression for an unbiased estimator $\hat{\theta}$ that is a function of T. You need to show that $E\{\hat{\theta}(T)\} = \sigma^r$.

3. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, 1)$.

   (a) Find the limiting distribution of $U_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i - \mu)$ by the central limit theorem.

   (b) Show that $V_n = \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 \to 1$ in probability by the weak law of large numbers.

   (c) Find the limiting distribution of $W_n = U_n/V_n$.

(d) Find the limiting distribution of $\sqrt{n}(\bar{X}^2 - \mu^2)$, where $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$.

(e) Construct a 95% confidence interval for μ^2, either under a finite n (exact) or $n \to \infty$ (limiting).

4. It has been known that Pareto distribution can be used to model the distribution of family incomes in certain population, where Pareto probability density function is defined by
$$f_Y(y) = \theta\gamma^\theta y^{-(\theta+1)}, \quad 0 < \gamma < y < \infty, \quad 0 < \theta < \infty.$$
with cumulative density function
$$F_Y(y) = 1 - \left(\frac{y}{\gamma}\right)^{-\theta}.$$
Let $Y_1, \ldots, Y_n$ constitute a random sample from this density function where γ is a **known** positive constant (minimal wage, for example) and where θ is an **unknown** parameter (known as Pareto index).

(a) Consider a random variable $T_n = \theta n(Y_{(1)} - \gamma)/\gamma$. Show that the cumulative density function of T_n is
$$P(T_n > t) = \left\{\left(1 + \frac{t}{\theta n}\right)^n\right\}^{-\theta}.$$

(b) Given that $\lim_{n\to\infty}(1 + x/n)^n = e^x$ for some constant x, show that the limiting distribution of T_n follows exponential distribution with mean 1, i.e., $T_n \to_d T$ where $f_T(t) = e^{-t}$, $0 < t < \infty$.

(c) Using the limiting result in (b), find a 95% confidence interval for θ given that $P(T < 0.025) = 0.025$ and $P(T < 3.689) = 0.975$.

(d) Comment on if the limiting property brings any advantage for the statistical inferences for θ. Can we use the result in (a) to construct a 95% confidence interval for θ? Explain.