1. For $i = 1, 2, \ldots, n$, suppose that the conditional distribution of Y_i given $X = x_i$ is normal with mean $E(Y_i|X = x_i) = \alpha + \beta x_i$ and with variance $V(Y_i|X = x_i) = \sigma^2$, **where σ^2 has a known value**. Assume that (x_i, Y_i), $i = 1, 2, \ldots, n$, constitute a set of mutually independent pairs of data (i.e., given the fixed x_i, the Y_i constitutes a set of mutually independent random variables).

   (a) With β fixed, show that the maximum likelihood estimator (MLE) of α is $\tilde{\alpha} = \bar{Y} - \beta\bar{x}$, where $\bar{Y} = n^{-1}\sum_{i=1}^{n} Y_i$ and $\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$, and that $\tilde{\alpha}$ is also a method of moment estimator.

   (b) With α fixed at $\tilde{\alpha} = \bar{Y} - \beta\bar{x}$, show that the maximum likelihood estimator (MLE) of β is
   $$\hat{\beta} = \sum_{i=1}^{n}(x_i - \bar{x})Y_i / \sum_{i=1}^{n}(x_i - \bar{x})^2.$$

   (c) Show that $\hat{\beta}$ is an unbiased estimator of β, and derive an explicit expression for $V(\hat{\beta})$.

   (d) Using the fact that $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x}$ and $\hat{\beta}$ simultaneously maximize the likelihood function, find the maximum likelihood estimator (MLE) of $\theta = \alpha + \beta x_0$, where x_0 is a known fixed constant.

   (e) Let $\hat{\theta} = \bar{Y} + \hat{\beta}(x_0 - \bar{x})$. Show that $\hat{\theta}$ is an unbiased estimator of θ, and derive an explicit expression for $V(\hat{\theta})$.

2. Let $X_1, \ldots, X_n$ be a random sample from the probability density function
   $$f_X(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0.$$

   (a) Show that the expectation of score function is 0, i.e., $E(\frac{\partial}{\partial\theta}\log f(X|\theta)) = 0$.

   (b) Derive the explicit expression for the Cramer-Rao lower bound (CRLB) for unbiased estimators of $\tau(\theta) = \theta^{-1}$.

   (c) Find the uniformly minimum variance unbiased estimator (UMVUE) for $\tau(\theta) = \theta^{-1}$, and justify whether its variance achieves the CRLB.

(d) Derive the likelihood ratio test (LRT) statistic $\lambda(x)$ for $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, and show that the rejection region $R = \{x : \lambda(x) \leq c\}$ is equivalent to $R = \{x : T \geq c_1^* \text{ or } T \leq c_2^*\}$, where $T = -\sum_{i=1}^n \log(X_i)$.

[Hint: $t^n e^{-ct}$ is a concave function of t if the constant $c > 0$.]

(e) Find c_1^* and c_2^* in (d) with test size α.