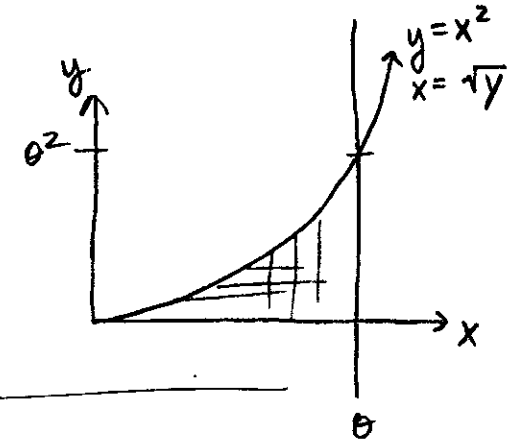
Question 2

2010, MS-1

(X,Y) distributed according to

$$f(x,y) = \frac{12xy}{\theta^6} \qquad \begin{array}{l} 0 < x < \theta \\ 0 < y < x^2 \end{array}$$



a) Find the marginal pdf of Y

$$= \int_x f(x,y)\,dx$$

$$= \int_{\sqrt{y}}^{\theta} \frac{12xy}{\theta^6}\,dx \qquad 0 < y < \theta^2$$

$$= \frac{12y}{\theta^6} \cdot \frac{x^2}{2} \Big|_{\sqrt{y}}^{\theta} = \frac{6y}{\theta^6}\left(\theta^2 - (\sqrt{y})^2\right) = \frac{6\theta^2 y - 6y^2}{\theta^6}$$

b) Find EY and Var Y

$$EY = \int_0^{\theta^2} y \cdot \left\{ \frac{6y}{\theta^4} - \frac{6y^2}{\theta^6} \right\} dy = \frac{6y^3}{3\cdot\theta^4} - \frac{6y^4}{4\theta^6} \Big|_0^{\theta^2} = \frac{6\theta^6}{3\theta^4} - \frac{6\theta^8}{4\theta^6} = 2\theta^2 - \frac{3}{2}\theta^2$$

$$= \frac{1}{2}\theta^2$$

$$\text{Var}\,Y = EY^2 - (EY)^2$$

$$\int_0^{\theta^2} \frac{6y^3}{\theta^4} - \frac{6y^4}{\theta^6}\,dy = \frac{6y^4}{4\theta^4} - \frac{6y^5}{5\theta^6}\Big|_0^{\theta^2} = \frac{6\theta^8}{4\theta^4} - \frac{6\theta^{10}}{5\theta^6} = \frac{6}{4}\theta^4 - \frac{6}{5}\theta^4$$

$$= \frac{6}{20}\theta^4$$

$$= \frac{6}{20}\theta^4 - \frac{1}{4}\theta^4 = \frac{1}{20}\theta^4 = \text{Var}\,Y$$

c) Find the conditional pdf of Y given X

$$f(y|x) = \frac{f(x,y)}{f(x)} \quad (= f(y) \text{ if } y \perp x)$$

$$f(y|x)\cdot f(x) = \frac{2y}{x^4}\cdot\frac{6x^5}{\theta^6} = \frac{12xy}{\theta^6}$$

$$= \frac{\left\{\frac{12xy}{\theta^6}\right\}}{\left\{\frac{6x^5}{\theta^6}\right\}} = \frac{12xy}{6x^5} = \boxed{\frac{2y}{x^4} \quad \begin{matrix} 0<y<x^2 \\ 0<x<\theta \end{matrix}}$$

$$f(x) = \int_y f(x,y)\,dy = \int_0^{x^2} \frac{12xy}{\theta^6}\,dy = \frac{12xy^2}{2\theta^6}\Big|_0^{x^2} = \frac{6x^5}{\theta^6} \qquad 0<x<\theta$$

d) Find $E(y|x)$

$$= \int_0^{x^2} y\cdot f(y|x)\,dy = \int_0^{x^2} \frac{2y^2}{x^4}\,dy = \frac{2}{3}\frac{y^3}{x^4}\Big|_0^{x^2} = \frac{2}{3}\frac{x^6}{x^4} = \boxed{\frac{2}{3}x^2 = EY|X}$$

check → $\int_0^\theta \frac{2}{3}x^2\cdot\frac{6x^5}{\theta^6}\,dx = \int_0^\theta \frac{12}{3}\frac{x^7}{\theta^6} = \frac{12}{3\cdot 8}\frac{x^8}{\theta^6}\Big|_0^\theta = \boxed{\frac{1}{2}\theta^2 = EY}$ ⇒ correct

e) Find $\text{Corr}(x,y)$

$$= \frac{\text{cov}(x,y)}{\sqrt{\text{var}x\cdot\text{var}y}} = \frac{Exy - EX\,EY}{\sqrt{\text{var}x\cdot\text{var}y}}$$

$$Exy = \int_0^\theta\int_0^{x^2} xy\cdot\frac{12xy}{\theta^6}\,dy\,dx = \int_0^\theta x\cdot\int_0^{x^2}\frac{12xy^2}{\theta^6}\,dy\,dx = \int_0^\theta x\cdot\frac{12xy^3}{3\theta^6}\Big|_0^{x^2}\,dx$$

$$= \int_0^\theta x\cdot\frac{12x\cdot x^6}{3\theta^6}\,dx = \int_0^\theta 4\frac{x^8}{\theta^6}\,dx = \frac{4}{9}\frac{x^9}{\theta^6}\Big|_0^\theta = \frac{4}{9}\theta^3$$

(or, can use pseudo conditional expectation):

$$E(xy) = E\,E(xy|x) = E(x\cdot \underbrace{E(y|x)}_{\text{already have}})$$

e) Question 2 cont. 2010, MS-I

so $\text{cov}(x,y) = \frac{4}{9}\theta^3 - EX\,EY$

$EX = \int_0^\theta x \cdot \frac{6x^5}{\theta^6}\,dx = \frac{6}{7}\frac{x^7}{\theta^6}\Big|_0^\theta = \frac{6}{7}\theta$

$\Rightarrow \frac{4}{9}\theta^3 - \left(\frac{6}{7}\theta\right)\left(\frac{1}{2}\theta^2\right) = \left(\frac{4}{9} - \frac{6}{14}\right)\theta^3 = \frac{1}{63}\theta^3$ craig got

$\Rightarrow \text{corr}(x,y) = \dfrac{\frac{1}{63}\theta^3}{\sqrt{\frac{1}{20}\theta^4 \cdot \frac{3}{196}\theta^2}}$ = a number,

prob good, craig's EXY is wrong.

$EX^2 = \int_0^\theta x^2 \cdot \frac{6x^5}{\theta^6} = \frac{6}{8}\frac{x^8}{\theta^6}\Big|_0^\theta = \frac{3}{4}\theta^2$

$\text{Var}X = \frac{3}{4}\theta^2 - \left(\frac{6}{7}\theta\right)^2 =$

$= \left(\frac{3}{4} - \frac{36}{49}\right)\theta^2 = \frac{3}{196}\theta^2$

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f) find $\text{cov}(x, y - \frac{2}{3}x^2)$

$= \text{cov}(x,y) - \text{cov}(x, \frac{2}{3}x^2)$

$= \frac{1}{63}\theta^2 - \left\{E\left(x\cdot\frac{2}{3}x^2\right) - EX\,E\left(\frac{2}{3}x^2\right)\right\}$

$= \frac{1}{63}\theta^3 - \left\{\frac{4}{9}\theta^3 - \left(\frac{6}{7}\theta\right)\left(\frac{1}{2}\theta^2\right)\right\}$

$= \frac{1}{63}\theta^3 - \left\{\cancel{\frac{4}{9}\theta^3} - \frac{6}{14}\theta^3\right\}$

$= 0$

$E(x\cdot\frac{2}{3}x^2) = E\left(\frac{2}{3}x^3\right)$

$= \frac{2}{3}\cdot\int_0^\theta x^3\cdot\frac{6x^5}{\theta^6}dx = \left(\frac{6}{9}\frac{x^9}{\theta^6}\Big|_0^\theta\right)\frac{2}{3}$

$= \left(\frac{2}{3}\theta^3\right)\frac{2}{3} =$

$= \frac{4}{9}\theta^3$

$E\,\frac{2}{3}x^2 = \frac{2}{3}\left(\frac{3}{4}\theta^2\right)$

$= \frac{1}{2}\theta^2$

$\frac{2}{3}\text{cov}(x, x^2)$

$\frac{2}{3}\{EX^3 - (EX\,EX^2)\}$ (but it should be same answer. way easier this way)

g) find. $P(y<x)$

$$\equiv P(y-x<0)$$

$$= \int_0^1\int_0^{x^2} \frac{12xy}{\theta^6}\,dy\,dx + \int_1^\theta\int_0^x \frac{12xy}{\theta^6}\,dy\,dx$$

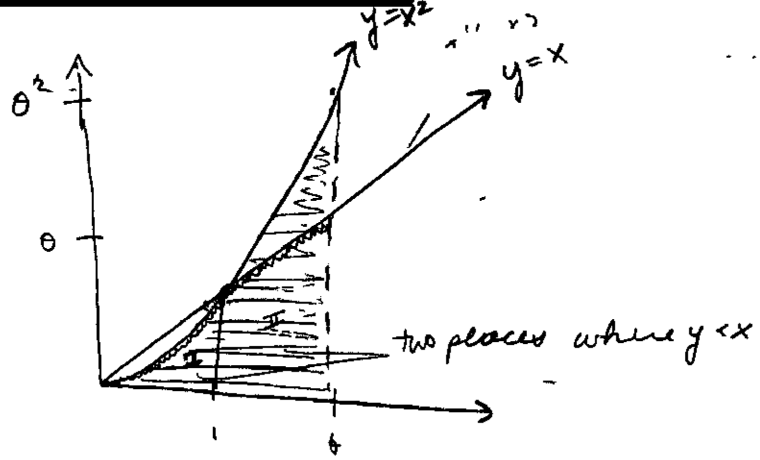
$$= \int_0^1 \frac{12xy^2}{2\theta^6}\Big|_0^{x^2} dx + \int_1^\theta \frac{12xy^2}{2\theta^6}\Big|_0^{x} dx$$

$$= \int_0^1 \frac{6x^5}{\theta^6}\,dx + \int_1^\theta \frac{6x^3}{\theta^6}\,dx$$

$$= \frac{x^6}{\theta^6}\Big|_0^1 + \frac{6}{4}\frac{x^4}{\theta^6}\Big|_1^\theta$$

$$= \frac{1}{\theta^6} + \frac{3}{2}\cancel{\frac{4}{4}}\left(\frac{\theta^4}{\theta^6} - \frac{1}{\theta^6}\right)$$

$$= \frac{3}{2}\cdot\frac{1}{\theta^2} - \frac{1}{2}\frac{1}{\theta^6} \quad (?)$$



confident

Question 2 | 2010, MS-1

c) Find the conditional pdf of y given x

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$f(x) = \int_0^{x^2} \frac{12xy}{\theta^6}\,dy = \frac{12xy^2}{2\theta^6}\Big|_0^{x^2} = \frac{6x^5}{\theta^6}$$

$$f(y|x) = \frac{\frac{12xy}{\theta^6}}{\frac{6x^5}{\theta^6}} = \frac{2y}{x^4} \qquad \begin{array}{l} 0 < x < \theta \\ 0 < y < x^2 \end{array}$$

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d) $E(y|x)$

$$\int_0^{x^2} y \cdot \frac{6y}{x^4}\,dy = \frac{6y^3}{3x^4}\Big|_0^{x^2} = \frac{6x^6}{3x^4} = 2x^2 = E\,y|x$$

$$\int_0^{\theta} 2x^2 \cdot \frac{2x^5}{\theta^6}\Big|_0^{\theta} = \frac{4x^8}{8\theta^6}\Big|_0^{\theta} = \frac{1}{2}\theta^2 = Ey \;☺$$

---

e) Find $\mathrm{corr}(x,y)$

$= \underbrace{\mathrm{cov}(x,y)} / \sqrt{\mathrm{var}x \cdot \mathrm{var}y}$

$\{EXY - EX\,EY\} / \sqrt{\mathrm{var}x \cdot \mathrm{var}y}$

① ② ③

$$= \frac{\frac{4}{9}\theta^3 - \frac{2}{7}\theta\{\frac{1}{2}\theta^2\}}{\sqrt{}}$$

plug #s in

① $$\int_0^{\theta}\int_0^{x^2} xy \cdot \frac{12xy}{\theta^6}\,dy\,dx = \int_0^{\theta} \frac{12x^2}{\theta^6} \cdot \frac{y^3}{3}\Big|_0^{x^2} dx$$

$$= \int_0^{\theta} \frac{4x^8}{\theta^6}\,dx = \frac{4x^9}{9\theta^6}\Big|_0^{\theta} = \frac{4}{9}\theta^3$$

② $$\int_0^{\theta} \frac{6x^6}{\theta^6}\,dx = \frac{6}{7}\theta$$

③ $$\int_0^{\theta} \frac{6x^7}{\theta^6}\,dx = \frac{6}{8}\theta^2$$

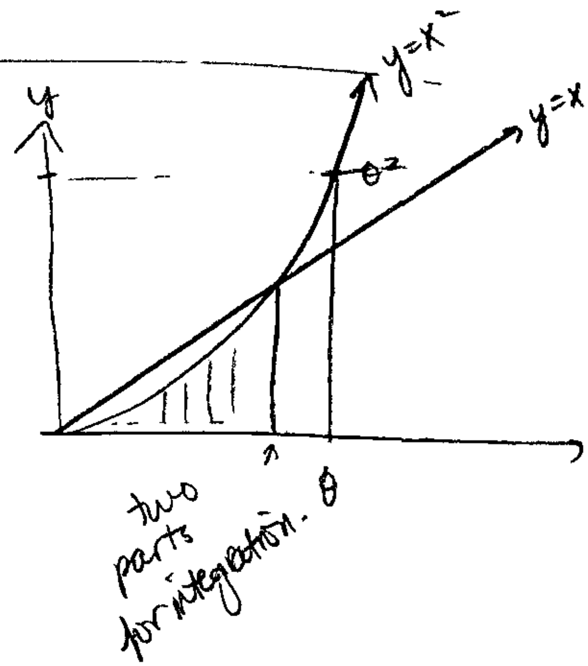
$$\frac{2}{8}\theta^2 - \left(\frac{6}{7}\theta\right)^2 =$$

f) Find $\operatorname{cov}(x, y - \frac{2}{3}x^2)$

$$= \operatorname{cov}(x, y) - \operatorname{cov}(x, \tfrac{2}{3}x^2)$$

$$= \frac{1}{63}\theta^3 - \left\{ E(\tfrac{2}{3}x^3) - E(x)E(\tfrac{2}{3}x^2) \right\}$$

g) $P(y<x) = P(y-x<0)$



Question 3 | 2010, MS1

③ $(x_1\ y_1) \ldots (X_n\ y_n)$ random sample from..

$$f(x,y) = \frac{12xy}{\theta^6} \quad 0 < x < \theta \quad 0 < y < x^2$$

a) Find a minimal suff. stat for θ

$$\frac{f(\underset{\sim}{x},\underset{\sim}{y} \mid \theta)}{f(\underset{\sim}{w},\underset{\sim}{z} \mid \theta)} = \frac{\prod_{i=1}^{n}\{(12x_iy_i)/\cancel{\theta^6} \cdot I(X_i < \theta)\}}{\prod_{i=1}^{n}\{(12w_iz_i)/\cancel{\theta^6} \cdot I(w_i < \theta)\}} = \frac{\prod_{i=1}^{n} 12x_iy_i}{\prod_{i=1}^{n} 12w_iz_i} \cdot \frac{I(x_{(1)} < \theta)}{I(w_{(1)} < \theta)}$$

if these are = then the ratio is $\perp \theta$... so $X_{(1)}$ is M.S.S. for θ

---

b) Show that the marginal pdf of x is

$$f(x) = \frac{6x^5}{\theta^6} \quad 0 < x < \theta$$

$$\int_y f(x,y)\, dy = f(x)$$

$$\int_0^{x^2} \frac{12xy}{\theta^6}\, dy = \frac{12xy^2}{2\theta^6}\Big|_0^{x^2} = \frac{6x^5}{\theta^6} \quad 0 < x < \theta$$

if one performs the transformation $W = x/\theta^6$ $\quad x = w\theta$, $\; dx/dw = \theta$

$$f(w) = f_X(w(\theta)) \cdot \theta$$

$$= \frac{6(w\theta^1)^5 \cdot \theta^1}{\theta^6} = 6w \quad \text{and this distribution is free of } \theta.$$

Therefore, this density belongs to a scale family.

c) $\bar{X} = \frac{\Sigma X_i}{n}$ Find k_1 such that $\hat{\theta}_1 = k_1 \bar{X}$ is unbiased for θ

$$E\bar{X} = E\left(\frac{\Sigma X_i}{n}\right) = \frac{\Sigma (EX_1)}{n}$$

$$EX_1 = \int_0^\theta x \cdot \frac{6x^5}{\theta^6}\,dx = \int_0^\theta \frac{6x^6}{\theta^6}\,dx = \frac{6x^7}{7}\cdot\frac{1}{\theta^6}\Big|_0^\theta = \frac{6}{7}\cdot\theta$$

$$\frac{n\left(\frac{6}{7}\theta\right)}{n} = \frac{6}{7}\theta = E\bar{X}$$

therefore, $\frac{7}{6}\bar{X} \Rightarrow$ unbiased for θ

$$E\left(\frac{7}{6}\frac{\Sigma X_i}{n}\right) = \frac{\frac{7}{6}n\left(\frac{6}{7}\theta\right)}{n} = \theta \checkmark$$

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d) $X_{(n)}$ max of X^{th} sample. Find k_2 such that $\hat{\theta}_2 = k_2 X_{(n)} \Rightarrow$ unbiased for θ

$$f_{X_{(n)}}(x) = n f(x)\, F(x)^{n-1}$$

$$F(x) = \int_0^x \frac{6t^5}{\theta^6}\,dt = \frac{t^6}{\theta^6}\Big|_0^x = \left(\frac{x}{\theta}\right)^6$$

$$EX_{(n)} = \int_0^\theta x\, f_{X_{(n)}}(x)\,dx$$

$$= \int_0^\theta x \cdot n \cdot \frac{6x^5}{\theta^6}\cdot\frac{x^{6(n-1)}}{\theta^{6(n-1)}}\,dx$$

$$= n\int_0^\theta 6\,\frac{x^{6n}}{\theta^{6n}}\,dx$$

$$= n6\,\frac{x^{6n+1}}{(6n+1)}\cdot\frac{1}{\theta^{6n}}\Big|_0^\theta = \frac{n6}{(6n+1)}\cdot\frac{\theta^{6n+1}}{\theta^{6n}} = \frac{6n}{6n+1}\cdot\theta$$

$$\Rightarrow k_2 = \frac{6n+1}{6n}$$

so $Ek_2X_{(n)}$ unbiased for θ

Question 3 | 2010, MS-1

e). Of two unbiased estimators, which one is preferable? Explain rigorously?

both unbiased...

CRLB for unbiased est. of θ =

$$\frac{\{\frac{d}{d\theta}\theta\}^2}{E\left(-\frac{d^2}{d\theta^2} \ell f(\underset{\sim}{x}|\theta)\right)} = \frac{1}{\ }$$

$\ell(\underset{\sim}{x}|\theta) = 30 \sum \log x_i - 6n\log\theta$

$\frac{d\ell}{d\theta} = \frac{-6n}{\theta}$ $\quad \frac{d^2\ell}{d\theta} = \frac{6n}{\theta^2}$ (?)

CRLB = $-\frac{\theta^2}{6n}$

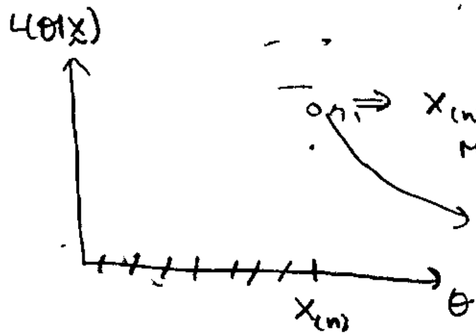
NO NEG

possibly not exchangeable?

CRLB

is one UMVUE? → unbiased and based off S.S.

$L(\theta|\underset{\sim}{x})$ $\quad \frac{6^n \cdot \prod_{i=1}^n x_i^5}{\theta^{6n}} \cdot I(x_{(n)} < \theta)$



⇒ $X_{(n)}$ is the MLE

Because $\hat{\theta}_2$ is ~~the~~ an unbiased estimator based on the MLE of θ $(X_{(n)})$ $\hat{\theta}_2$ is the most efficient estimator of θ.

(UMVUE?)

f) Show that $X_{(n)}/\theta$ is a pivotal quantity. ①
Use it to construct the shortest $1-\alpha$ CI ②
Justify rigorously why this interval is the shortest. ③.

① Distn. of $X_{(n)}/\theta$ should be free of θ

$$f_{X_{(n)}}(x) = n f(x)\, F(x)^{n-1}$$
$$= n\left(\frac{6x^5}{\theta^6}\right)\left(\frac{x}{\theta}\right)^{6(n-1)}$$

let $y = \frac{X_{(n)}}{\theta}$ $\quad X_{(n)} = y\theta$ $\quad \frac{dX_{(n)}}{dy} = \theta$

$$f_Y(y) = f_{X_{(n)}}(y\theta)\cdot\left|\frac{d}{dy} g^{-1}(y)\right|$$
$$= n\left(\frac{6(y\theta)^5}{\theta^6}\right)\left(\frac{y\theta}{\theta}\right)^{6(n-1)}\cdot\theta$$
$$= n\, 6y^5 y^{6(n-1)}$$

Weibull? $6n = \gamma$ $\beta = ?$

$\frac{6n}{1} x^{6n-1} \cancel{e^{-x^{6n}/1}}$

$f_Y(y) = n6y^{6n-1}$ $\Rightarrow$ this distribution is free of θ, and so $X_{(n)}/\theta$ is a pivotal quantity

$\rightsquigarrow$ but what distn is it? (need that for bounds)

② $1-\alpha = (a < X_{(n)}/\theta < b)$ where $\underbrace{P(X_{(n)}/\theta < b)}_{1-\alpha/2} - \underbrace{P(X_{(n)}/\theta < a)}_{\alpha/2} = 1-\alpha$

$$1-\alpha = \left(\left(\alpha/2\right)^{1/6n} \le X_{(n)}/\theta \le (1-\alpha/2)^{1/6n}\right)$$
$$= \left\{ \underbrace{\left(\tfrac{\alpha}{2}\right)^{1/6n}\cdot X_{(n)}}_{\text{upper bound}} \ge \theta \ge \underbrace{\left(1-\tfrac{\alpha}{2}\right)^{1/6n}\cdot X_{(n)}}_{\text{lower bound}} \right\}$$

$1-\frac{\alpha}{2} = \int_0^b 6ny^{6n-1}\,dy$ $\qquad \frac{\alpha}{2} = \int_0^a 6ny^{6n-1}\,dy$

$1-\frac{\alpha}{2} = y^{6n}\Big|_0^b$ $\qquad \frac{\alpha}{2} = y^{6n}\Big|_0^a$

$1-\frac{\alpha}{2} = b^{6n}$ $\qquad \frac{\alpha}{2} = a^{6n}$

$(1-\alpha/2)^{1/6n} = b$ $\qquad (\frac{\alpha}{2})^{1/6n} = a$

$a < \frac{X_{(n)}}{\theta} < b$ $\qquad 1-\alpha = \int_a^b f(y)\,dy$

$\frac{X_{(n)}}{b} < \theta < \frac{X_{(n)}}{a}$ $\qquad 1-\alpha = y^{6n}\Big|_a^b$

$1-\alpha = b^{6n} - a^{6n}$

$\frac{1}{a} - \frac{1}{(a^{6n}-(1-\alpha))^{1/6n}}$

length: $X_{(n)}\underbrace{\left(\frac{1}{a} - \frac{1}{b}\right)}_{\text{minimize}}$

Question 4

2010, MS-1

$Y_1 \dots Y_n$ sample from

$f(y|\theta) = \theta^y(1-\theta) \qquad y = 0,1,2,3\dots \qquad \theta \in (0,1)$

$\hookrightarrow$ y failures before first success

---

a) $f(y|\theta)$

$\exp(\log(\theta^y(1-\theta)))$

$= \exp(\log(1-\theta) + y\log\theta)$

$= (1-\theta)\exp^{y\log\theta}$

$\Rightarrow$ by the exponential family rules, $t(y) = y$ and so $T(y) = \sum_{i=1}^n y_i$ is a complete sufficient statistic

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b) Compute the CRLB for the variance of unbiased estimators of θ

$$CRLB = \frac{\{\frac{d}{d\theta}\tau(\theta)\}^2}{E\left(-\frac{d^2}{d\theta^2}\log f(y|\theta)\right)} \qquad \tau(\theta) = \theta \qquad \cancel{\tfrac{d}{d\theta}} = 1 \qquad * \; EY_1 = \frac{\theta}{(1-\theta)}$$

$$f(y|\theta) = \prod_{i=1}^n \theta^{y_i}(1-\theta) = \theta^{\Sigma y_i}(1-\theta)^n$$

$\to$ so $CRLB = \dfrac{1}{\frac{n}{\theta(1-\theta)^2}} = \dfrac{\theta(1-\theta)^2}{n}$

$$l(f(y|\theta)) = \Sigma y_i \log\theta + n\log(1-\theta)$$

$$\frac{dl}{d\theta} = \frac{\Sigma y_i}{\theta} + \frac{-n}{(1-\theta)}$$

$$\frac{d^2l}{d\theta^2} = \frac{-\Sigma y_i}{\theta^2} - \frac{n}{(1-\theta)^2}$$

$$E\left(\frac{\Sigma y_i}{\theta^2} + \frac{n}{(1-\theta)}\right) = \frac{\Sigma EY_1}{\theta^2} + \frac{n}{(1-\theta)^2}$$

$$= \frac{\frac{n\theta}{(1-\theta)}}{\theta^2} + \frac{n}{(1-\theta)^2} = \frac{n(1-\theta)}{(1-\theta)^2\theta} + \frac{n\theta}{\theta(1-\theta)^2} = \frac{n}{\theta(1-\theta)^2}$$

Question 3 2010, MS1

$f(x)$ is unimodal pdf. interval $[a, b]$

i. $\int_a^b f(x)\,dx = 1-\alpha$

ii $f(a) = f(b) > 0$, and

iii. $a \le x^* \le b$ where x^* is mode of $f(x)$

the $[a, b]$ is shortest among intervals

???

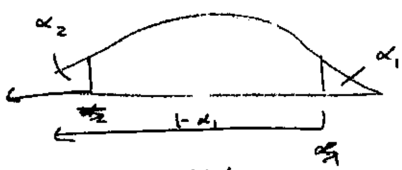
$$1-\alpha = P\left(c_1 < \frac{X_{(n)}}{\theta} < c_2\right)$$

$$1-\alpha = P\left(\frac{X_{(n)}}{\theta} < c_2\right) - P\left(\frac{X_{(n)}}{\theta} < c_1\right)$$

$$= P\left(\frac{X_{(n)}}{\theta} < c_2\right) = 1-\alpha_1$$

$$P\left(\frac{X_{(n)}}{\theta} < c_1\right) = \alpha_2$$

$\alpha_1 + \alpha_2 = \alpha$



36/3

g) $(x_1 = 5, y_1 = 20)$
$(x_2 = 13, y_2 = 151)$
$(x_3 = 18, y_3 = 222)$

$$\hat{\theta}_1 = k_1 \bar{X} = \frac{7}{6}\left(\frac{5+13+18}{3}\right)$$

$$= \frac{7}{6}(12) = 14$$

$$\hat{\theta}_2 = k_2 X_{(n)} = \left(\frac{6n+1}{6n}\right) X_{(n)}$$

$$= \frac{18+1}{18} X_{(n)}$$

$$= \frac{19}{18}(18) = 19$$

$\hat{\theta}_1$ weakness?

↳ depends on all obs?

MLE's are unstable.

④ Question 4 $y_1 \cdots y_n$ $n > 1$ from

let θ = prob failure $1-\theta$ = prob success
y = # failures before first success

$$f(y|\theta) = \theta^y (1-\theta) \quad y = 0, 1, 2 \ldots \quad \theta \in [0,1] \text{ unknown}$$

DISCRETE alternative param. geometric

$\theta = (1-p)$

$\Rightarrow f(y|p) = (1-p)^y (p) \quad E(Y) = \frac{(1-p)}{p}$

$E(y|\theta) = \frac{\theta}{(1-\theta)}$

a) Find a complete suff. stat for θ.

$$S.S = f(y|\theta) = \prod_{i=1}^{n} \theta^{y_i}(1-\theta) = \underbrace{\theta^{\Sigma y_i}(1-\theta)^n}_{g(T(Y)|\theta)} \underbrace{(1)}_{h(x)}$$

$\Rightarrow$ by Factorization Thm, Σy_i is sufficient for θ

or $\exp(\log(\theta^y(1-\theta)))$

$= \exp(y_i \log\theta + \log(1-\theta))$

$= \exp(y_i \log\theta) \exp(1-\theta)$ $\Rightarrow$ by the exponential family rules, Σy_i is a complete ($w(\theta) = \log\theta$ is bounded by the open set) and sufficient stat

$$\Rightarrow \underbrace{\exp(y_i \log\theta)}_{e^{t(x)w(\theta)}} \underbrace{\exp(\log(1-\theta))}_{c(\theta)} \underbrace{(1)}_{h(x)},$$

b) Compute CRLB for unbiased estimators of θ

$$= \frac{\{\frac{d}{d\theta}\theta\}^2}{E\left(-\frac{d^2}{d\theta^2}\log(f(y|\theta))\right)} = \frac{1}{n/(1-\theta)^2\theta} = \frac{\theta(1-\theta)^2}{n}$$

$$EY_1 = \sum_{y=0}^{n} y\,\theta^y(1-\theta) = \frac{\theta}{(1-\theta)} \text{ from above}$$

$f(y|\theta) = \theta^{\Sigma y_i}(1-\theta)^n$

$l = \Sigma y_i \log\theta + n\log(1-\theta)$

$$\frac{dl}{d\theta} = \frac{\Sigma y_i}{\theta} + \frac{-n}{(1-\theta)}$$

$$\frac{d^2}{d\theta^2} = -\frac{\Sigma y_i}{\theta^2} - \frac{n}{(1-\theta)^2}$$

$$E\left(\frac{\Sigma y_i}{\theta^2} + \frac{n}{(1-\theta)^2}\right) = \frac{\Sigma EY_1}{\theta^2} + \frac{n}{(1-\theta)^2} = \frac{n\left(\frac{\theta}{1-\theta}\right)}{\theta^2} + \frac{n}{(1-\theta)^2} = \frac{n}{(1-\theta)\theta} + \frac{n}{(1-\theta)^2} = \frac{n(1-\theta)+n\theta}{(1-\theta)^2\theta} = \frac{n}{(1-\theta)^2\theta}$$

c) Find the UMVUE of θ.

Hint : $f(0) = ?$

Does it achieve the CRLB?

UMVUE $\Rightarrow$ take unbiased estimator & condition on C.SS.

$f(0) = \theta^0(1-\theta) = 1-\theta$

$1-f(0) = 1 - \cancel{\theta^0}(1-\theta) = \theta$

$\Rightarrow$ an unbiased estimator of θ is $1 - P(y=0)$

$E(W \mid \Sigma y_i = t)$ where $W = 1 - I(y_1 = 0)$

$= E\{1 - P(y_1 = 0) \mid \Sigma y_i = t\} = P(y_1 > 0 \mid \Sigma y_i = t)$

2010, MS-1

a) Question 4) ~~Derive~~ an explicit expression for $\hat{\theta}_n$ the MLE of θ

is $\hat{\theta}_n$ unbiased for θ? If biased, in what direction

$$L(\theta|y) = \theta^{\Sigma y_i}(1-\theta)^n$$

$$l = \Sigma y_i \log\theta + n\log(1-\theta)$$

$$\frac{dl}{d\theta} = \frac{\Sigma y_i}{\theta} + \frac{-n}{(1-\theta)} = 0$$

$$= \Sigma y_i(1-\theta) - n\theta = 0$$

$$= \Sigma y_i - \Sigma y_i\theta - n\theta = 0$$

$$\hat{\theta}_n = \frac{\Sigma y_i}{\Sigma y_i + n} = \frac{n\bar{y}}{n\bar{y}+n} = \frac{n\bar{y}}{n(1+\bar{y})} = \frac{\bar{y}}{(1+\bar{y})} = \hat{\theta}_n$$

$$E\{\hat{\theta}_n\} = E\left(\frac{\bar{y}}{1-\bar{y}}\right)$$ omg.

e) $\hat{\theta}_n$ satisfies

$$\sqrt{n}\,(\hat{\theta}_n - \theta) \xrightarrow[d]{} N(0, \gamma) \quad \text{what is } \gamma?$$

$$\sqrt{n}\left(\frac{\Sigma y_i}{n} - EY_1\right) \xrightarrow[d]{} N(0, \operatorname{Var} Y_1) \rightarrow = \frac{(1-p)}{p^2} \cdot \frac{\theta}{(1-\theta)^2}$$

by Delta method

$$g(w) = \frac{w}{1-w}$$

$$\sqrt{n}\,(g(\bar{Y}) - g(EY_1)) \xrightarrow[d]{} N(0, \{g'(EY_1)\}^2 \operatorname{Var} Y_1) \qquad g'(w) = \frac{1}{(1-w)^2}$$

$$(g'(w))^2 = (1-w)^{-4}$$

$$\sqrt{n}\,(\hat{\theta}_n - \theta) \xrightarrow[d]{} N\left(0, (1-\theta)^{-4} \frac{\theta}{(1-\theta)^2}\right)$$

f) approx interval using N distn.

$$0.95 = \left(-1.96 < \frac{\sqrt{n}\,(\hat{\theta}_n - \theta)}{\sqrt{\theta/(1-\theta)^6}} < 1.96\right)$$

sub $\hat{\theta}_n$ here, then solve for regular θ by mult / sub / ÷ by -1

$$= \left(-1.96 < \frac{\sqrt{25}\,(0.6-\theta)}{\sqrt{0.6/(1-0.6)^6}} < 1.96\right)$$

$$= \left(-1.96 < \frac{5(0.6-\theta)}{12.1031} < 1.96\right)$$

$$= \left(\frac{-1.96(12.1031)}{5} - 0.6 < -\theta < \frac{1.96(12.1031)}{5} - 0.6\right)$$

$$= \left(0.6 \pm \frac{1.96(12.1031)}{5}\right) = (-4.144, 5.344) \Rightarrow 95\% \text{ CI for } \theta$$

d) Derive an explicit expression for $\hat{\theta}_n$, the MLE of θ.
Is it unbiased? What is, if any, the direction of the bias?

$$L(\theta|y) = \prod_{i=1}^{n} \theta^{y_i}(1-\theta)$$

$$= \theta^{\Sigma y_i}(1-\theta)^n$$

$$\log L(\theta|y_i) = \Sigma y_i \log\theta + n\log(1-\theta)$$

$$\frac{d\log}{d\theta} = \frac{\Sigma y_i}{\theta} + \frac{-n}{(1-\theta)} = 0$$

$$\Sigma y_i(1-\theta) - n\theta = 0$$

$$\hat{\theta} = \frac{\Sigma y_i}{\Sigma y_i + n} = \frac{\bar{y}}{\bar{y}+1}$$

or, let $\theta = (1-p)$

$$\theta + (1-\theta) = 1$$

$$(1-p) + p = 1$$

$$\Rightarrow \Sigma y_i \log(1-p) + n\log p$$

$$\frac{-\Sigma y_i}{(1-p)} + \frac{n}{p} = 0$$

$$-\Sigma y_i(p) + (n - np) = 0$$

$$p = \frac{n}{n+\Sigma y_i}$$

Question 4 |

2010, MS-I

c) Find the UMVUE for θ.

Hint : $f(0) = ?$

Does it achieve the CRLB?

$$f(0) = \theta^0 (1-\theta) = 1-\theta$$

$$1 - f(0) = 1 - (1-\theta) = \theta$$

$$1 - P(Y=0) = \theta$$

Question 4 — 2010, MS-1

e) $\sqrt{n}\,(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, V)$. Find an explicit expression for V

$V = I_1(\theta)^{-1}$

$I_1(\theta) = E\left(-\frac{d}{d\theta} \log f(x_1|\theta)\right)$

$= \frac{1}{\theta(1-\theta)^2}$

$I_1(\theta)^{-1} = (1-\theta)^2(\theta)$

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f) $n = 25$, observed $\hat{\theta}_n = 0.6$ compute an approx 95% CI

$$0.95 = P\left(-1.96 < \frac{5(0.6-\theta)}{\sqrt{(1-0.6)^2(0.6)}} < 1.96\right)$$

$$= P\left(\frac{-1.96\sqrt{(1-0.6)^2(0.6)}}{5} - 0.6 < -\theta < \frac{1.96\sqrt{0.4^2(0.6}}{5} - .6\right)$$

$\leadsto$ finish.