Ann Marie Weideman, MS Exam
Part I, 2016

1. Given $X \sim (x+1)p^2 q^x = (x+1)p^2(1-p)^x$, $x \in \{0,1,2,\ldots\}$
& $Y|X \sim \text{Unif}(0, x+1)$, $0 < y < x+1$

a) What is the joint density (pdf/pmf) of X & Y? For densities, don't forget to specify the domain (support).

Want $f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x)$

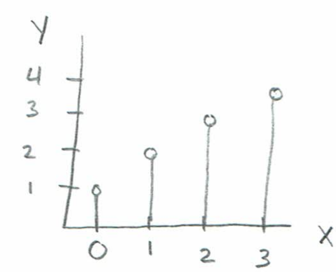
$$= \frac{1}{\cancel{(x+1)}} \cdot \cancel{(x+1)}\, p^2(1-p)^x \, I(0<y<x+1)$$

$$= p^2(1-p)^x I(0<y<x+1)$$

$$\Rightarrow \boxed{f_{X,Y} = \begin{cases} p^2(1-p)^x, & \{(x,y): x=\{0,1,2,\ldots\} \text{ and } 0<y<x+1\} \\ 0, & \text{else} \end{cases}}$$

b) Find $f_Y(y)$, the density of Y.

Note in this picture,



So, $X \in A(y)$ where $A(y) = \{\lfloor y \rfloor, \lfloor y \rfloor + 1, \lfloor y \rfloor + 2, \ldots\}$ where $\lfloor \cdot \rfloor$ = floor function.

$$\Rightarrow f_Y(y) = \sum_{x \in A(y)} f_{X,Y}(x,y) = \sum_{x=\lfloor y \rfloor}^{\infty} p^2(1-p)^x$$ * There is a generic form of the geometric series (write it on your formula sheet!) that says $\sum_{n=M}^{\infty} cr^n = \frac{cr^M}{1-r}$. So, $* = p^2 \sum_{x=\lfloor y \rfloor}^{\infty} (1-p)^x = p^2\left(\frac{1 \cdot (1-p)^{\lfloor y \rfloor}}{1-(1-p)}\right)$

$$= \frac{p^{\cancel{2}}(1-p)^{\lfloor y \rfloor}}{\cancel{p}} = \boxed{p(1-p)^{\lfloor y \rfloor}, \quad 0 < y < \infty}$$

(Note: The support of the marginal must always depend on only that variable itself (i.e., not x in this case).)

1 c) Find $E[Y]$

Note: Any time they ask you to find an expected value, first look to see if they have provided you w/ a conditional distribution. Well, in fact, here they have! It's our "lucky" day.

$$E[Y] = E[E[Y|X]] = E\left[\frac{X+1}{2}\right] = \frac{1}{2}E[X] + 1/2 = ?$$

Hmmm, let's look carefully back at the dist. of X, which denotes the # of failures before the 2nd success. This looks like a negbin$(2,p)$

Thus, $E[Y] = \frac{1}{2}E[X] + 1/2 = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}(1-p)}{p}\right) + \frac{1}{2} = \frac{1}{p} - 1 + \frac{1}{2} = \boxed{\frac{1}{p} - \frac{1}{2}}$

d) Find $Cov(X,Y)$

$$Cov(X,Y) = E[\underbrace{\cancel{Cov(X,Y|X)}}_{0}] + Cov(E[X|X], E[Y|X])$$

(if you don't have this formula on your sheet, write it Now!)

$$= Cov(X, X/2 + 1/2) = \frac{1}{2}\underbrace{Cov(X,X)}_{= Var(X)} + \underbrace{\cancel{Cov(X, 1/2)}}_{0}$$

$$= \frac{1}{2}Var(X) = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}(1-p)}{p^2}\right) = \boxed{\frac{q}{p^2}}$$

e) Define $T = 2Y - X$. Find $Cov(T,X)$.

P.S. On like every prior exam, this covariance has been 0, so I am expecting this to happen again.

$$Cov(T,X) = Cov(2Y-X, X) = 2Cov(Y,X) - Cov(X,X) = 2Cov(Y,X) - Var(X)$$

$$= 2\underbrace{\left(\frac{q}{p^2}\right)}_{\text{part d)}} - \frac{2q}{p^2} = \boxed{0}$$

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1 f) Are T & X independent? Justify.

Two RV's T & X are independent if

$$P(T=t \mid X=x) = P(T=t) \quad \forall\, t, x. \qquad (1)$$

So, in order to show they are not independent, we simply need to find one point for which (1) doesn't hold.

Let $X=0$ and $t = 2Y - X = 2Y$.

Then, $P(T=2y \mid X=0) = 1$, but $P(T=2y) \neq 1$.

Thus, $P(T=t \mid X=x) \neq P(T=t) \ \forall\, t, x \Rightarrow T$ and X not independent.

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2. Given $X_1, \dots, X_n \sim N(\mu, 1)$

Define $\theta = P(X > 0)$. Use $\Phi(t)$ to denote the CDF of the std normal dist. evaluated @ t.

a) Express $P(X>0)$ as a fcn of μ.

$$P(X>0) = 1 - P(X \le 0) = 1 - P\left(\frac{X-\mu}{1} \le \frac{-\mu}{1}\right)$$

$$= 1 - P(\underbrace{X-\mu}_{\sim N(0,1)} \le -\mu) = 1 - \Phi(-\mu) = \Phi(\mu).$$

b) Find an unbiased estimator of $P(X>0)$.

Since $E[I(X>0)] = P(X>0) \Rightarrow I(X>0)$ is an unbiased estimator of $P(X>0)$.

c) Find the MLE of $P(X>0)$.

① First, find MLE of μ.

② Then, since $P(X>0) = \Phi(\mu)$, can employ invariance property to find $\widehat{P(X>0)} = \hat{\theta}_{MLE}$.

① Have, $\mathcal{L}(\mu|x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^n (x_i-\mu)^2/2}$

$$\Rightarrow \ell(\mu|x) = -n\log(\sqrt{2\pi}) - \frac{1}{2}\sum_{i=1}^n (x_i-\mu)^2$$

$$\Rightarrow \frac{\partial \ell}{\partial \mu} = +\sum_{i=1}^n (x_i-\mu) \overset{set}{=} 0 \Rightarrow +\sum_{i=1}^n x_i - n\mu = 0$$

$$\Rightarrow \hat{\mu} = \bar{X}$$

Note that $\hat{\mu}$ occurs @ a global max since $\frac{\partial^2 \ell}{\partial \mu^2} = -n < 0$.

② Since $P(X>0) = \theta = \Phi(\mu)$ (from a)), then

$\hat{\theta}_{MLE} = \Phi(\bar{X})$.

2 d) Find the Cramer-Rao lower bound on the variance of unbiased estimators of $P(X > 0)$.

Know $CRLB = \dfrac{\left\{ \dfrac{\partial \tau(\theta)}{\partial \theta} \right\}^2}{-E\left\{ \dfrac{\partial^2}{\partial \theta^2} \log f(\underset{\sim}{x} \mid \theta) \right\}}$

However, we don't have $f(\underset{\sim}{x} \mid \theta)$. Instead, we have $f(\underset{\sim}{x} \mid \mu)$. So, the numerator also needs to be in terms of μ.

Numerator

Since $\tau(\theta) = \theta = P(X > 0) = \Phi(\mu)$ (from part a)), then

$$\frac{\partial \tau(\theta)}{\partial \theta} = \Phi'(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2} \Rightarrow \left\{ \frac{\partial \tau(\theta)}{d\theta} \right\}^2 = \frac{1}{2\pi} e^{-\mu^2}$$

Denominator

From part c) Know that $\dfrac{\partial^2}{\partial \mu^2} \log f(\underset{\sim}{x} \mid \mu) = -n \Rightarrow$

$$-E\left\{ \frac{\partial^2}{\partial \mu^2} \log f(\underset{\sim}{x} \mid \mu) \right\} = -E(-n) = n$$

Thus, $CRLB = \dfrac{\frac{1}{2\pi} e^{-\mu^2}}{n} = \boxed{\dfrac{1}{2\pi n} e^{-\mu^2}}$

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2e) Find the UMVUE of $P(X>0)$.

Using method 1 from notes for finding UMVUE:

<1> Find an unbiased estimator W for $\tau(\theta)$.
<2> Look for a complete sufficient statistic for θ.
<3> Derive $\phi(T) = E(W|T=t)$
<4> Then $\phi(T)$ is the UMVUE of $\tau(\theta)$

Step <1>: We did this in b), know $I(x>0) = W$ is an unbiased estimator for $\tau(\theta) = \theta = P(x>0)$.

Step <2>: We know a CSS for θ is $\underbrace{\sum_{i=1}^{n} x_i}_{\text{"} T}$ since $X \sim N(\mu, 1)$.

Step <3>: Derive $\phi\left(\sum_{i=1}^{n} x_i\right) = E\left(I(x>0) \mid \sum_{i=1}^{n} x_i\right)$

$= P\left(X>0 \mid \sum_{i=1}^{n} x_i\right)$ *

Need $f_{X|\Sigma X_i}$. Know $\begin{pmatrix} X \\ \sum X_i \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu \\ n\mu \end{pmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & n \end{bmatrix}\right)$

and thus, $X \mid \sum X_i \sim N\left(\mu + \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}}\left(\sum x_i - n\mu\right),\ 1 \cdot \left(1 - \frac{1}{n}\right)\right)$

$\equiv N\left(\bar{X}, \left(1-\frac{1}{n}\right)\right) \equiv N\left(\bar{x}, \frac{n-1}{n}\right)$

(using this formula:
$Y|X \sim N\left[\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_X),\ \sigma_Y^2(1-\rho^2)\right]$)

$\Rightarrow$ * $= P\left(\underbrace{\frac{X-\bar{X}}{\sqrt{\frac{n-1}{n}}}}_{\sim N(0,1)} > \frac{0-\bar{x}}{\sqrt{\frac{n-1}{n}}} \;\middle|\; \sum_{i=1}^{n} x_i\right)$

$= 1 - \Phi\left(\frac{-\bar{x}}{\sqrt{\frac{n-1}{n}}}\right) = \Phi\left(\frac{\bar{x}}{\sqrt{\frac{n-1}{n}}}\right)$.

Step <4> Then, $\Phi\left(\frac{\bar{X}}{\sqrt{\frac{n-1}{n}}}\right)$ is the UMVUE of $\tau(\theta) = P(x>0)$.

3. Given

	a_1	a_2	a_3	a_4
θ_1	0.3	0.4	0.1	0.2
θ_2	0.4	0.1	0.2	0.3
θ_3	0.2	0.1	0.5	0.2

} $P(x|\theta)$ for $x \in \{a_1, a_2, a_3, a_4\}$

Suppose that one obs. X is available (i.e., $n=1$)

a) Find $\hat{\theta}_{MLE}$ under different values of X (that is, as a fcn of X).

For $X = a_1$, largest probability occurs @ θ_2

For $X = a_2$, largest probability occurs @ θ_1

For $X = a_3$, largest probability occurs @ θ_3

For $X = a_4$, largest probability occurs @ θ_2

Thus, $\boxed{\hat{\theta}_{MLE} = \begin{cases} \theta_2, & X = a_1, a_4 \\ \theta_1, & X = a_2 \\ \theta_3, & X = a_3 \end{cases}}$

b) Derive the critical region of the LRT for $H_0: \theta = \theta_1$ vs. $H_1: \theta \neq \theta_1$ w/ type I error probability $\alpha = 0.1$.

This part is super important here.

not a vector here since a single obs

$$\lambda(X) = \frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta|x)}{\sup_{\theta \in \Theta} \mathcal{L}(\theta|x)} = \frac{\sup_{\theta \in \Theta_0} P(a_i|\theta)}{\sup_{\theta \in \Theta} P(a_i|\theta)}$$

$$\Rightarrow \lambda(x) = \begin{cases} \dfrac{P(a_1|\theta_1)}{P(a_1|\theta_2)} = \dfrac{3}{4}, & x = a_1 \\ \dfrac{P(a_2|\theta_1)}{P(a_2|\theta_1)} = 1, & x = a_2 \\ \dfrac{P(a_3|\theta_1)}{P(a_3|\theta_3)} = \dfrac{1}{5}, & x = a_3 \\ \dfrac{P(a_4|\theta_1)}{P(a_4|\theta_2)} = \dfrac{2}{3}, & x = a_4 \end{cases}$$

$\Rightarrow R = \{x : \lambda(x) \leq c\} \Rightarrow \alpha = \sup_{\theta \in \Theta_0} P(\lambda(x) \leq c)$

$\Rightarrow 0.1 = P(\lambda(x) \leq c \,|\, \theta = \theta_1)$

$\Rightarrow \boxed{R = \{x : \lambda(x) \leq 1/5\}}$

3 c) Give the test function of the LRT in b) in explicit form. Explain explicitly how one would apply the testing procedure using a single obs. X.

[As given in b),

$$\lambda(x) = \frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta | X)}{\sup_{\theta \in \Theta} \mathcal{L}(\theta | X)} = \frac{\sup_{\theta \in \Theta_0} P(a_i | x)}{\sup_{\theta \in \Theta} P(a_i | x)} \quad \text{LRT}$$

Given some rejection region $R = \{x : \lambda(x) \leq c\}$ for $0 \leq c \leq 1$, then reject X if $\lambda(x) \leq c$ and fail to reject X if $\lambda(x) > c$.]

d) Find the UMP test for testing the null $H_0 : \theta = \theta_1$ vs. $H_1 : \theta = \theta_2$ w/ type I error prob. $\alpha = 0.1$.

[$$R = \left\{ x : \frac{f(x|\theta_2)}{f(x|\theta_1)} > c \right\} \quad \text{where } \theta_2 > \theta_1.$$

The ratios of PMFs give:

$$\frac{f(a_1|\theta_2)}{f(a_1|\theta_1)} = \frac{0.4}{0.3} = \frac{4}{3}, \quad \frac{f(a_2|\theta_2)}{f(a_2|\theta_1)} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$\frac{f(a_3|\theta_2)}{f(a_3|\theta_1)} = \frac{0.2}{0.1} = 2, \quad \frac{f(a_4|\theta_2)}{f(a_4|\theta_1)} = \frac{0.3}{0.2} = \frac{3}{2}$$

Try some

If $\frac{1}{4} < c < \frac{4}{3}$, then the test that rejects H_0 if $X = a_1$ or a_3 or a_4 is

$\alpha = P(X = a_1, a_3, a_4 | \theta = \theta_1) = 0.3 + 0.1 + 0.2 = 0.6$

If $\frac{3}{2} < c < 2$, then the test that rejects H_0 if $X = a_3$ is

$\alpha = P(X = a_3 | \theta = \theta_1) = 0.1$

$\Rightarrow R = \{X = a_3\}$.]

e) Comment on whether the UMP test for the hypothesis in d) is also the UMP test for the hypothesis in b).

If you think it is, provide the rationale. If you think it is not, derive the UMP test for the UMP test for the hypothesis in b)

In d), we tested θ_1 against θ_2. Only remaining point is to test θ_1 against θ_3.

$$R = \left\{ x : \frac{f(x|\theta_3)}{f(x|\theta_1)} > c \right\}$$ where $\theta_3 > \theta_1$.

The ratios of PMFs give:

$$\frac{f(a_1|\theta_3)}{f(a_1|\theta_1)} = \frac{0.2}{0.3} = \frac{2}{3}, \quad \frac{f(a_2|\theta_3)}{f(a_2|\theta_1)} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$\frac{f(a_3|\theta_3)}{f(a_3|\theta_1)} = \frac{0.5}{0.1} = 5, \quad \frac{f(a_4|\theta_3)}{f(a_4|\theta_1)} = \frac{0.2}{0.2} = 1$$

If $1 < c < 5$, then the test that rejects H_0 if $x = a_3$

is $\alpha = P(x = a_3 | \theta = \theta_1) = 0.1 \Rightarrow R = \{x = a_3\}$

So, it looks like this test is the UMP test against θ_2 and θ_3.

It is also the UMP test for b).