Question 1

2011, MS-1

n female rats each produce a number of offspring w/ pmf:

$$f(x) = \begin{cases} \frac{a}{x!} & x = 0, 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

a) for what value of a is a legitimate distribution?

$$\sum_{x=0}^{\infty} \frac{a}{x!} = 1$$

$$a \sum_{x=0}^{\infty} \frac{1^x}{x!} = 1$$

$$a(e^{-1}) = 1$$

$$a = e^{1}$$

$\Rightarrow$ $f(x) = \frac{e}{x!}$ $x = 0, 1, 2 \ldots$

$X \sim \text{Pois}(1) \Rightarrow EX = 1, \ \text{Var}X = 1$

b) By Poisson, $EX = 1$ and $\text{Var}X = 1$

Question 1 | 2011, MS-1

c) Let Y_1 = total # of offspring produced in 1 year from n parents. Find the PMF, E, var of Y_1 in terms of n

if X_i = offspring from i rats, $i=1\dots n$, and are Poisson(1)

then $Y_1 = \sum_{i=1}^{n} X_i$ after year 1

$\sum^{n}$ poissons = Pois$(n\lambda)$

$\Rightarrow Y_1 \sim \text{Pois}(n)$

$EY_1 = n$

$\text{Var}Y_1 = n$

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d) Assume no female rats ever die. So, there are $Y_1 + n$ females rats to produce offspring after year 1.

year	female	females produced
1	n	$Y_1 = \sum X_i$
2	$n + \sum_{i=1}^{n} X_i$	Y_2
3	$n + \underbrace{\sum X_i}_{=Y_1} + Y_2$	Y_3
4	$n + Y_1 + Y_2 + Y_3$	Y_4
⋮		
k	$\underbrace{n + Y_1 + \dots Y_{K-1}}_{n + \sum_{i=1}^{k-1} Y_i}$	Y_K

$Y_K \mid \sim \text{Pois}(n + \sum_{i=1}^{k-1} Y_i)$

Question 1

2011, MS-1

e) Inductively show $EY_K = 2^{K-1} n$

① Show true for $k=1$

② assume true for $n=k$

<u>when $k=1$</u>

$$EY_1 = 2^{\cancel{1-1}\,0}(n)$$

$$= n$$

$\Rightarrow$ Because $Y_1 \sim Pois(n)$, $EY_1 = n$ and so this is true for $K=1$

<u>When $k=2$</u>

$$EY_2 = 2^{2-1}(n) = 2n$$

$Y_2 \sim Pois(n + Y_1)$

$$EY_2 = E\underbrace{EY_2|Y_1}_{n+Y_1}$$

$$= E(n+Y_1) = n + EY_1 = 2n$$

$$Y_K = 2^{K-1}(n)$$

$Y_K \sim Pois(n + Y_1 + \dots + Y_{K-1})$

Question 2

$Y_1 \dots Y_n \sim Uni(0,1)$ $f(y) = 1$ $E(y) = 0.5$ $Var\, y = \frac{1}{12}$

$Y_{(k)} \sim k^{th}$ ordered stat

a) Find the MGF and characteristic function of Y_1

$Y_{(1)}$

$$MGF = E(e^{ty})$$
$$= \int_0^1 e^{ty} \cdot f_{y_1}(y) \; dy$$

$u = ty$
$du = t\,dy$
$\frac{du}{t} = dy$

$$= \int_0^1 e^{ty}\, dy$$
$$= \frac{1}{t} e^{ty} \Big|_0^1$$
$$= \frac{1}{t}\left\{e^{t(1)} - e^{t(0)}\right\} = \frac{1}{t}\left\{e^t - 1\right\}$$

$$CF = E(e^{ity})$$

$i = \sqrt{-1}$

$$= \int_0^1 e^{ity}\, dy$$

how do you integrate this?

b) Show that $Y_{(1)} \to_p 0$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} \underbrace{P(|Y_{(1)} - 0| < \varepsilon)} = 0$ definition of approaches in probability

$P(-\varepsilon < Y_{(1)} \cancel{-0} < \varepsilon)$

$= P(Y_{(1)} < \varepsilon) - P(Y_{(1)} < -\varepsilon)$

$= F_{Y_{(1)}}(\varepsilon) - F_{Y_{(1)}}(-\varepsilon)$

$= 1 - (\varepsilon)^n - (1 - (-\varepsilon)^n)$

$= (-\varepsilon)^n - \varepsilon^n$

$F_{Y_{(1)}} = P(Y_{(1)} < y)$

$= 1 - P(Y_{(1)} > y)$

$= 1 - P(Y > y)^n$

$= 1 - (y)^n$

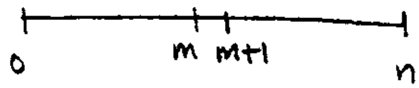
$\lim_{n\to\infty} \{(-\varepsilon)^n - (\varepsilon)^n\} = 0$

$\Rightarrow Y_{(1)} \xrightarrow[p]{} 0$

Question 2/ (2011, MS-1)

c) Compute the density function of the median of $Y_1 \dots Y_n$ when n is EVEN

$$T = \frac{Y_{(m)} + Y_{(m+1)}}{2}$$



$\Rightarrow$ even

$Y_{(k)} \sim \beta(k, n+1-k)$

$\Rightarrow$ joint density of $Y_{(m)}$ and $Y_{(m+1)}$.

$\Rightarrow$ convert via jacobian method.

let $u = Y_{(m)}$
$v = Y_{(m+1)}$

$$f_{Y_{(m)} Y_{(m+1)}}(y_i, y_j) = \frac{n!}{(m-1)!\,(\cancel{m+1-m}-1)!\,(n-m+1)!} \cdot F(y_i)^{m-1} f(y_i) f(y_j) (1 - F(y_j))^{n-(m+1)}$$

$$= \frac{n!}{(m-1)!\,(n-m+1)!} u^{m-1} (1-v)^{n-(m+1)}$$

$m-1 = n-(m+1)$

$m = \frac{n}{2}$

$$= \frac{n!}{(\frac{n}{2}-1)!\,(n-\frac{n}{2}+1)!} \cdot u^{\frac{n}{2}-1} (1-v)^{n-\frac{n}{2}+1}$$

$0 < u < v < 1$

define $x = \frac{u+v}{2}$ $\quad w = v$ $\Rightarrow$ $v = w$, $u = 2x - w$

$$J = \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = |0 - 2| = 2$$

$$f(y, x) = f(u = 2x - w, v = w) \cdot 2$$

$$= \frac{n!}{(\frac{n}{2}-1)!\,(n-\frac{n}{2}+1)!} \cdot (2x-w)^{\frac{n}{2}-1} (1-w)^{n-\frac{n}{2}+1}$$

$0 < y < 1$

$y < x < y+1$

Question 2

$$f\left(x=\frac{u+v}{2}\right)=\frac{2n!}{\left(\frac{n}{2}-1\right)!\left(n-\frac{n}{2}+1\right)!}\int_0^1 (2x-w)^{\frac{n}{2}-1}(1-w)^{n-\frac{n}{2}+1}\,dw = f\left(x=\frac{u+v}{2}\right)=$$ density of median

$w = v = Y_{(m+1)}$

$\sim B(m+1,\ n+1-(m+1))$

$\sim B(m+1,\ n-m)$

$$=\frac{\Gamma(m+1+n-m)}{\Gamma(m+1)\,\Gamma(n-m)}x^{m}(1-x)^{n-m-1}$$

$$=\frac{n!}{m!\,(n-m-1)!}x^{m}(1-x)^{n-m-1}$$

Question 2 | 2011, MS-1

d) Find MGF of $-\ln(Y_{(K)})$ and show it has the same distri of $\underbrace{\sum_{i=1}^{n-k+1} X_i}$ where the X_i's are iid $\underline{\text{exponentrae RV's}}$ w/ parameter β

$\longrightarrow$ sum of exp's

$= \Gamma(n-k+1, \beta)$

$MGF = \left(\frac{1}{1-\beta t}\right)^{n-k+1}$

$MGF = E(e^{tx})$

$= \int e^{tx} \cdot f_{-\ln(Y_{(K)})}$

so if $X_i = -\ln(Y_{(K)})$

and ΣX_i = gamma,

$X_i \sim Exp(\beta)$

define $N = -\ln(Y_{(K)})$ $\quad n = g(y)$

$$f_{Y_{(K)}}(y) = \frac{n!}{(k-1)!(n-k)!} F(y)^{k-1} f(y) (1-F(y))^{n-k}$$

$$= \frac{n!}{(k-1)!(n-k)!} (y)^{k-1} (1-y)^{n-k}$$

$-n = \ln(Y_{(K)})$

$Y_{(K)} = e^{-n}$

$Y_{(K)} = e^{-n} \quad = g^{-1}(n)$

$dy/dn = -e^{-n} \quad = \frac{d}{dy} g^{-1}(n)$

$f(n) = f_x(e^{-n}) \left|-e^{-n}\right|$

$$f_N(n) = \frac{n!}{(k-1)!(n-k)!} (-e^{-n})^{k-1} (1-e^{-n})^{n-k} \cdot -e^{-n}$$

$$= \frac{n!}{(k-1)!(n-k)!} (1-e^{-n})^{n-k} e^{-nk}$$

MGF of N

$= E($

d) Try again

$Y_{(K)} \sim \text{Beta}(k, n-k+1)$

$$f_{Y_{(k)}}(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot y^{\alpha-1}(1-y)^{\beta-1} = \frac{\Gamma(k+n-k+1)}{\Gamma(k)\Gamma(n-k+1)} y^{K-1}(1-y)^{n-k+1-1}$$

$$= \frac{n!}{(K-1)!\,(n-k)!} y^{K-1}(1-y)^{n-k} \qquad 0<y<1$$

define $W = -\ln(Y_{(K)})$

$Y_{(K)} = e^{-W}$

$$f(w) = f_y(e^{-w})\,(|-e^{-w}|)$$

$$= \frac{n!}{(k-1)!\,(n-k)!} (e^{-w})^{k-1}(1-e^{-w})^{n-k} e^{-w} \qquad W>0 \quad (?)$$

$$= \frac{n!}{(k-1)!\,(n-k)!} e^{-wk}(1-e^{-w})^{n-k}$$

MGF of $W =$

fact. $\displaystyle\int_0^\infty e^{tw} e^{-wk}(1-e^{-w})^{n-k}\,dw$

$\displaystyle\int_0^\infty e^{-w(k-t)}(1-e^{-w})^{n-k}\,dw$

Question 4 | 2001, MS-1

$f(x|\theta) = (1-\theta) + \dfrac{\theta}{2\sqrt{x}} \quad 0<x<1 \quad 0\le\theta\le 1 \quad X_1, \dots X_n$ random iid

a) Derive the UMP level α test for $H_0: \theta=0$ vs. $H_1: \theta = 1$
Specify the C.R. and justify your answer.

$$f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \left\{ (1-\theta) + \frac{\theta}{2\sqrt{x_i}} \right\}$$

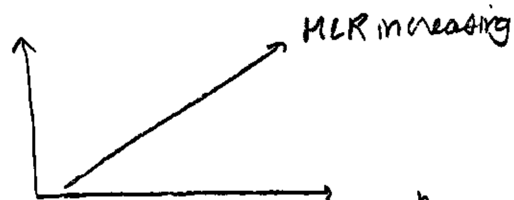
use N-P lemma b/c simple v. simple

$$\frac{f(\underset{\sim}{x}|\theta_1=1)}{f(\underset{\sim}{x}|\theta_0=0)} > c \quad \text{gives UMP test}$$

$$\Rightarrow \frac{f(\underset{\sim}{x}|\theta_1=1)}{f(\underset{\sim}{x}|\theta_0=0)} = \frac{\prod_{i=1}^{n} \left\{ \cancel{(1-1)}^{0} + \frac{1}{2\sqrt{x_i}} \right\}}{\prod_{i=1} \left\{ (1-0) + \cancel{\frac{0}{2\sqrt{x_i}}}^{0} \right\}} = \frac{\prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}}}{\cancel{1^n} = 1} = \frac{1}{2} \cdot \prod_{i=1}^{n} x_i^{-1/2}$$

take log

$$R = \left\{ \underset{\sim}{x} : \frac{1}{2}\prod_{i=1}^{n} x_i^{-1/2} > c \right\}$$



can base test off of $\prod_{i=1}^{n} \frac{1}{\sqrt{x_i}}$

$$\alpha = P\left(\frac{1}{2}\prod_{i=1}^{n} x_i^{-1/2} > c \;\middle|\; \theta_0 = 0 \right)$$

$\frac{1}{2} \Sigma \log x_i^{-1/2}$

$$\equiv P(\Sigma y_i > c \mid \theta_0 = 0)$$

$$\alpha = P(\Sigma y_i > \chi^2_{df=2n,\,1-\alpha})$$

$y = -\frac{1}{2} \log x_i \sim \text{Exp } 2$

$\Sigma \left(-\frac{1}{2}\log x_i\right) \sim \text{Gamma}(n, 2)$

$\Gamma(n,2) \equiv \chi^2_{2n}$

b) When $n=5$ and $\alpha = 0.01$ find the C.R.

Find the exact power of the test, so express it in CDF's of standard distributions.

$$0.01 = P\left(\sum_{i=1}^{5} Y_i > \chi^2_{2n,1-0.01}\right)$$

$$= 1 - P\left(\sum_{i=1}^{n} Y_i < \chi^2_{2n,0.01}\right)$$

$$0.01 = 1 - G_n\left(\sum_{i=1}^{5} -\tfrac{1}{2}\log X_i\right)$$ where G_n is the CDF of $\chi^2_{2n=10}$

$$\text{power} = P\left(\sum_{i=1}^{n} Y_i > \chi^2_{2n,0.99} \mid \theta = 1\right)$$ under alt, $\sum Y_i \sim \text{Gamma}(n,1)$ $\cancel{\sim \chi^2_{2n}}$

$$\text{power} = 1 - G_n\left(\chi^2_{\underset{10}{\cancel{2n}},0.99}\right)$$

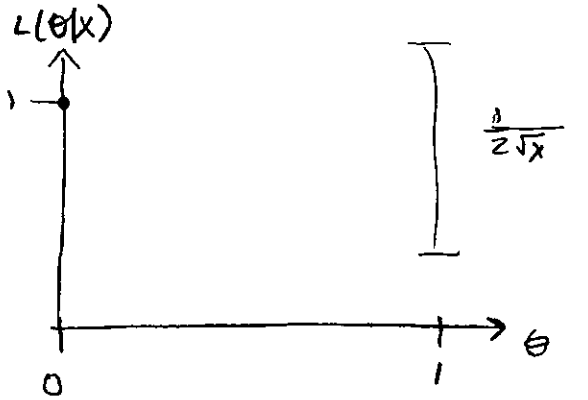
where G_n is cdf of the gamma$(n,1)$ distn

Question 4

d) Special case when $n=1$. Find $\hat{\theta}$, the MLE of θ.

Show it is biased

Find constants a and b such that $a+b\hat{\theta}$ is unbiased.



$$L(\theta|x) = (1-\theta) + \frac{\theta}{2\sqrt{x}}$$

$$\hat{\theta} \sim \text{Bern}\left(p = P\left(1 < \frac{1}{2}\frac{1}{\sqrt{x}}\right)\right)$$

$$1 > \frac{1}{2}\frac{1}{\sqrt{x}} \Rightarrow \text{MLE} = 0$$

$$1 < \frac{1}{2}\frac{1}{\sqrt{x}} \Rightarrow \text{MLE} = 1$$

$$E(\hat{\theta}) = P\left(1 < \frac{1}{2}\cdot\frac{1}{\sqrt{x}}\right)$$

$$= P\left(x < \left(\tfrac{1}{2}\right)^2\right)$$

$$= P\left(x < \tfrac{1}{4}\right) = F_x\left(\tfrac{1}{4}\right)$$

$$F_x(x) = \int_0^x (1-\theta) + \frac{\theta}{2}t^{-1/2}\,dt$$

$$= (1-\theta)x + \theta x^{1/2}$$

$$F_x\left(\tfrac{1}{4}\right) = (1-\theta)\tfrac{1}{4} + \theta\sqrt{\left(\tfrac{1}{4}\right)}$$

$$= \tfrac{1}{4} - \tfrac{1}{4}\theta + \tfrac{1}{2}\theta = \tfrac{1}{4} + \tfrac{1}{4}\theta =$$

$$\Rightarrow 4\hat{\theta} - \tfrac{1}{4}$$

$$4E\hat{\theta} - 1 = 4\left(\tfrac{1}{4} + \tfrac{1}{4}\theta\right) - 1$$

$$= 1 + \theta - 1 = \theta$$

Question 4 | 2011, MS-1

b) ~~When $n=5$, $\alpha=0.01$ Find the exact C.R.~~ and exact power.

c) T I error $= 0.01$ T II $= 0.01 \Rightarrow$ power $= 0.99$

$\rightarrow \Sigma y_i \sim \chi^2_{2n}$ under $\theta=0$

$0.01 = P(\Sigma y_i \geq \chi^2_{2n,0.99})$

$\sim$ Gamma$(n,1)$

$0.01 = 1 - P(\Sigma y_i > \chi^2_{2n,0.99} \mid \theta=1)$

$0.01 = P(\Sigma y_i < \chi^2_{2n,0.99} \mid \theta=1)$

$$0.01 = \underbrace{\Gamma_{n,1}\{\chi^2_{2n,0.99}\}}_{(?)}$$

④ $X_1, \dots X_n$

$$f(x|\theta) = (1-\theta) + \frac{\theta}{2\sqrt{x}}$$

$$= (1-\theta) + \frac{1}{2}\theta x^{-1/2} \qquad 0 < x < 1 \qquad 0 \le \theta \le 1$$

a) UMP test for $H_0: \theta = 0$, $H_1: \theta = 1$
Specify the C.R. as concisely and experitity as possible

$\Rightarrow$ use NP lemma

$$\frac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > c \qquad f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \left\{ (1-\theta) + \frac{1}{2}\theta x^{-1/2} \right\}$$

$$= \frac{\prod_{i=1}^{n} \left\{ (1-\overset{\nearrow \theta_1 = 1}{\theta_1}) + \frac{1}{2}\theta_1 x_i^{-1/2} \right\}}{\prod_{i=1}^{n} \left\{ (1-\underset{\hookrightarrow \theta_0 = 0}{\theta_0}) + \frac{1}{2}\theta_0 x_i^{-1/2} \right\}} = \frac{\prod_{i=1}^{n} \left\{ 0 + \frac{1}{2}x_i^{-1/2} \right\}}{\prod_{i=1}^{n} \{1\}} = \frac{\frac{1}{2}\prod_{i=1}^{n} x_i^{-1/2}}{1} = \frac{1}{2}\prod_{i=1}^{n} x_i^{-1/2}$$

$$R = \left\{ \underset{\sim}{x} : \frac{1}{2}\prod_{i=1}^{n} x_i^{-1/2} > c \right\} \Rightarrow \left\{ x : \prod_{i=1}^{n} x_i^{-1/2} > c^* \right\} \quad c^* = 2c$$ UMP test for given hypotheses

b) For $n = 5$ and $\alpha = 0.01$, Find the exact C.R. and exact power

$$0.01 = P\left(\prod_{i=1}^{n} x_i^{-1/2} > c^* \mid \theta = 0\right) \quad \text{find } c^*$$

$$\text{power} = P\left(\prod x_i^{-1/2} > c^* \mid \theta = 1\right) \quad \text{using } c^* \text{ exactly}$$

c) Let Y_1 denote the total # of female offspring that come from the n parents

$Y_1 = \Sigma X_i$

sum of Poissons ⇒ Pois$(n\lambda)$

$\Sigma \text{Pois}(1) = \text{Pois}(n)$

$$\text{pmf} = \frac{e^{-n} n^{y_1}}{y_1!} \qquad y_1 = 0, 1, 2, \ldots$$

(0 female offspring produced etc. from all n)

$EY_1 = n$

$\text{Var}\, Y_1 = n$

d) Assume no female rats ever die.

Then there are $n + Y_1$ female rats

let Y_K denote the # female OFFSPRING produced in k^{th} year

what is the distn. of $Y_K \mid Y_{K-1}, Y_{K-2} \ldots Y_1$

$X_i \sim \text{Pois}(1)$

$n \to Y_1 \qquad \Sigma X_i = Y_1 \sim \text{Pois}(n)$ ← total number of

$= Y_2 \sim \text{Pois}(Y_1 + n)$

$n + Y_1 \to Y_2 \qquad Y_3 \sim \text{Pois}(Y_2 + Y_1 + n)$

$n + Y_1 + Y_2 \to Y_3$

$\vdots$

$$Y_K \sim \text{Pois}\left(n + \sum_{i=1}^{K-1} Y_i\right)$$

① How quickly rat pop grows. 2011

n female rats

offspring each produces follows $f(x) = \frac{\alpha}{x!}$ $x = 0, 1, 2 \ldots$

0 else

for some fixed α. All offspring are able to breed in the next year and all rats behave independently

a) For what value of α is the above a legitimate distribution?

$\sum_{x=0}^{\infty} \frac{\alpha}{x!} = 1$ (to be a true pdf.

$\alpha \sum_{x=0}^{\infty} \frac{1}{x!} = 1 \quad = \alpha \sum_{x=0}^{\infty} \frac{(1)^x = 1}{x!} = \alpha e^{-1}$

$\alpha (e) = 1$

$\alpha = 1/e = e^{-1}$

$X \sim Pois(\lambda = 1)$

$f(x) = \frac{e^{-1}(1)^x}{x!} = \frac{e^{-1}}{x} \quad x = 0, 1, \ldots$

b) Expected value of offspring each year? The variance

$\left.\begin{array}{l} EX = 1 \\ VarX = 1 \end{array}\right\}$ by Poisson, easiest solution.

or,

$EX = \sum_{x=0}^{\infty} \frac{e^{-1}}{x!} \cdot x$

$= \sum_{x=0}^{\infty} \frac{e^{-1}}{(x-1)!} \quad \ldots ?$

f)

$$\mathrm{cov}(Y_1, Y_2 - Y_1) = E(Y_1(Y_2 - Y_1)) - E(Y_1)E(Y_2 - Y_1)$$

$$= E(Y_1 Y_2 - Y_1^2) - E(Y_1)\{E(Y_2) - E(Y_1)\}$$

$$f_{Y_1} = \mathrm{Pois}(n) = \frac{e^{-n}(n)^{y_1}}{y_1!}$$

$$f_{Y_2|Y_1} = \mathrm{Pois}(n + y_1) = \frac{e^{-(n+y_1)}(n+y_1)^{y_2}}{y_2!}$$

g) An exterminator hired to expose all female rats to a chemical.

exposure causes rat to die w/ probability $= 1-p$

Female rats that die are removed from the population / don't produce new off.

Expected # of female rats after the first yr → surviving + new off.

n original rats

p = prob. survival

$W = \text{Binom}(n, p)$ $\quad W = \#$ surviving rats (each of n rats survives w/ probability p)

⇒ W rats left to produce.

$EW = np$

np females rats to produce.

reproduction = pois

survival = binom

$Y_1 =$

Pois | Binom

$Y_1 | W \sim$

$W \sim \text{Binom}(n, p)$

Question 3 2011, MS-I

c)

$$E\,\delta(X) = a\left(2(1-\theta) - 2\sqrt{c}(1-\theta) - \frac{\theta}{2}\log c + b\right) = \theta$$

$$a = \frac{\theta}{2(1-\theta) - 2\sqrt{c}(1-\theta) - \frac{\theta}{2}\log c + b}$$

$a \to 0$ as $c \to 0$

$\log 0 \to \infty$

$\theta/\infty \to 0$

$$b = \frac{\theta}{a} - 2(1-\theta) + 2\sqrt{c}(1-\theta) + \frac{\theta}{2}\log c$$

d)

$$Var_0\,\delta(X) = a^2\,Var_0(X^{-1/2})$$

$$E_0(X^{(-1/2)2}) = E_0(X^{-1}) = \int_c^1 X^{-1}\left((1-0) + \cancel{\frac{0}{2\sqrt{X}}}\,dx\right) = \log x \Big|_c^1$$

$$= \cancel{\log 1}^{\,0} - \log c$$

$$= -\log c$$

$$\{E_0(X^{-1/2})\}^2 = \left(2(1) - 2\sqrt{c}(1) - \frac{1}{2}\log c\right)^2$$

$$= \left(2(1-\sqrt{c}) - \frac{1}{2}\log c\right)^2$$

$$\cancel{-\log c} - \left(2(1-\sqrt{c})\right)^2 - 2(1-\sqrt{c})\left(\frac{1}{2}\log c\right)2 - \cancel{\frac{1}{2}(\log c)^2}\,)$$

$$-\left((2 - 2\sqrt{c})^2 - 2(1-\sqrt{c})\log c\right.$$

Question 3

2011, MS-1

b) $\delta(x) = \begin{cases} a(x^{-1/2} + b) & c < x < 1 \\ 0 & 0 < x \le c \end{cases}$

$E_\theta\, \delta(x)$

$E(X^{-1/2})$

$$E(a(X^{-1/2} + b)) = a\left\{\left(\int_c^1 X^{-1/2}\left((1-\theta) + \frac{\theta}{2\sqrt{x}}\right) dx\right) + b\right\}$$

$$= a\left\{\left(\int_c^1 x^{-1/2}(1-\theta) + \frac{\theta}{2}x^{-1}\, dx\right) + b\right\}$$

$$= a\left\{\left(\frac{x^{1/2}}{\frac{1}{2}}(1-\theta) + \frac{\theta}{2}\log x \Big|_c^1\right) + b\right\}$$

$$= a\left\{\left(2(1-\theta) + 0 - 2\sqrt{c}(1-\theta) - \frac{\theta}{2}\log c\right) + b\right\}$$

$$= a\left((2-2\sqrt{c})(1-\theta) - \frac{\theta}{2}\log c + b\right) = E_\theta(x)$$

$$= a\left(2 - 2\theta - 2\sqrt{c} + 2\theta\sqrt{c} - \frac{\theta}{2}\log c + b\right) = 2a - 2a\theta - 2a\sqrt{c} + 2a\theta\sqrt{c} - \frac{a\theta}{2}\log c + ab$$

$$= (1-\theta)\{a(2-2\sqrt{c}) + b\} + \theta\{a(-\tfrac{1}{2}\log c + b)\}$$

$$= (1-\theta)(2a - 2a\sqrt{c} + ab) + \theta\left(-\frac{a}{2}\log c + ab\right)$$

$$= (1-\theta)2a - (1-\theta)2a\sqrt{c} + (1-\theta)ab + \frac{-\theta a}{2}\log c + \theta ab$$

$$= 2a - 2a\theta - 2a\sqrt{c} + 2a\sqrt{c}\,\theta + ab - \cancel{\theta ab} - \frac{\theta a}{2}\log c + \cancel{\theta ab}$$

$$= 2a - 2a\theta - 2a\sqrt{c} + 2a\sqrt{c}\,\theta + ab - \frac{a\theta}{2}\log c = E_\theta(\delta(x))$$

Question 3 | 2011, MS-I

③ $X \in (0,1)$

$$f(x\mid\theta) = (1-\theta) + \tfrac{1}{2}\theta (x)^{-1/2} \qquad 0<x<1 \qquad 0\le\theta\le 1 \qquad n=1$$

a) Find $E_\theta(X)$ {the expectation depends on θ}

$$EX = \int_{\mathcal{X}} x\cdot f(x\mid\theta)\,dx$$

$$= \int_0^1 x\cdot\left\{(1-\theta) + \tfrac{1}{2}\theta x^{-1/2}\right\} dx$$

$$= \int_0^1 (1-\theta)x + \tfrac{1}{2}\theta x^{1/2}\, dx$$

$$= \frac{(1-\theta)x^2}{2} + \tfrac{1}{2}\theta x^{3/2}\cdot\tfrac{2}{3}\ \Big|_0^1$$

$$= (1-\theta)\left(\tfrac{1}{2}\right) + \tfrac{1}{3}\theta$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\theta + \tfrac{1}{3}\theta = \boxed{\tfrac{1}{2} - \tfrac{1}{6}\theta \qquad 0\le\theta\le 1}$$ yes.

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b) Consider

$$\delta(x) = \begin{cases} a\left(X^{-1/2} + b\right) & \text{if } c < X < 1 \\ 0 & \text{if } 0 < X \le C \end{cases}$$

a, b, c are given constants.

Find $E_\theta(\delta(x)) =$

greater than some specified #

$$E_\theta\left(a\left(X^{-1/2}+b\right)\right) = a\left\{EX^{-1/2} + b\right\} \quad \text{when } c<X<1$$

$$E_\theta(0) = 0 \quad \text{when } 0<X\le C$$

$$EX^{-1/2} = \int_c^1 x^{-1/2}\left\{(1-\theta) + \tfrac{1}{2}\theta x^{-1/2}\right\} dx \cancel{= \int_0^1 (1-\theta)x^{-1/2} + \tfrac{1}{2}\theta x^{-1}\, dx}$$

$$\cancel{= (1-\theta)x^{1/2}\cdot 2 + \tfrac{1}{2}\theta\log x\ \Big|_0^1 = 2(1-\theta)}$$

$E_\theta(\delta(X)) = a\{2(1-\theta)+b\}$ or 0

$E_0(\delta(X)) = a(2+b)$

$E_1(\delta(X)) = ab$

no

$$\begin{aligned}
E_\theta(\delta(X)) &= 2a(1-\theta)+ab \\
&= (1-\theta)\underbrace{(2a+ab)}^{E_{\theta=0}} + \theta\overbrace{(ab)}^{E_{\theta=1}} \\
&= 2a+ab-2a\theta-\cancel{\theta ab}+\cancel{\theta ab} \\
&= 2a(1-\theta)+ab \\
&= a\{2(1-\theta)+b\} \equiv E_\theta(\delta(X))
\end{aligned}$$

c) Show that for any $c \in (0,1)$ there exist a and b such that $\delta(X)$ is an unbiased estimator of θ. Derive explicit expressions of a and b (in terms of c). Show that $a \to 0$ and $c \to 0$

$E(\delta(X)) = a\{2(1-\theta)+b\} = 0$

$E(\delta(X) - \theta) = 0$ if unbiased

c) For any c, there are a, b, such that $\delta(x)$ is unbiased. Find a and b in terms of c

$E(\delta(x)) = \theta$ if unbiased

$$2a - \underline{2a\theta} - 2a\sqrt{c} + \underline{2a\sqrt{c}\,\theta} + ab - \underline{\frac{a}{2}\theta \log c} = \theta$$

$$\cancel{\theta\left(-2a + 2a\sqrt{c} - \frac{a}{2}\log c\right) + 2a - 2a\sqrt{c} + ab = \theta}$$

$$\cancel{2a\left(1 - c + \frac{b}{2}\right) = \theta\left(1 - 2a + 2a\sqrt{c} - \frac{a}{2}\log c\right)}$$

$$a\left(2 - 2\theta - 2\sqrt{c} + 2\sqrt{c}\,\theta + b - \frac{\theta}{2}\log c\right) = \theta$$

$$a\left(\cancel{2(1-\theta) - 2\sqrt{c}(1-\theta) + b} - \frac{\theta}{2}\log c\right) = 0$$

Question 1

2011 Theory

④ Seems most reasonable to start w/...

$X_1 \ldots X_n$ iid from

$$f(x|\theta) = (1-\theta) + \frac{\theta}{2\sqrt{x}} \qquad \begin{matrix} 0 < x < 1 \\ 0 \le \theta \le 1 \end{matrix} \qquad \theta \text{ unknown}$$

a) Derive the UMP test of SIZE α for $H_0: \theta = 0$ v. $H_1: \theta = 1$

Justify your answers, no approximations.

(simple v. simple ⇒ use NP lemma)

$\frac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > c$ gives UMP test

$$f(\underset{\sim}{x}) = \prod_{i=1}^{n} \left\{ (1-\theta) + \frac{\theta}{2\sqrt{x}} \right\}$$

③ X distributed on $(0,1)$ w/ pdf: $n=1$

$$f(x|\theta) = (1-\theta) + \frac{\theta}{2\sqrt{x}} \qquad \begin{array}{l} 0<x<1 \\ 0\le\theta\le 1 \end{array}$$

a) Find $E(X)$

d) Question 3 | 2011, MS-1

$\underset{\theta=0}{Var}(\delta(x))$

$= Var_0(a(x^{-1/2}+b))$

$= a^2 Var\, x^{-1/2} + \cancel{Var\, b}$

$= a^2$

$\{E(x^{-1/2})^2\} + (EX^{-1/2})^2$

$EX^{-1} + (2(1-\theta))^2$

$\int_0^1 x^{-1}(1-\theta) + \frac{1}{2}\theta x^{-3/2}\,dx + (2(1-\theta))^2$

$= \underbrace{(1-\theta)\log x + \frac{1}{2}\theta x^{-1/2}\cdot -2 \Big|_0^1}_{(?)} + (2(1-\theta))^2$

$(2-2\theta)^2$

$= -\theta + (4 - 8\theta + 4\theta^2)$

$= 4 - 9\theta + 4\theta^2 \ldots (?)$

g) All female rats exposed to a chemical

$\Rightarrow$ survive with probability p

compute expected # of female rats after the first year

$0 \rightarrow n$ rats $\rightarrow n(p)$ live $\quad$ (number surviving) $\sim \text{Binom}(n, p)$

$1 \rightarrow np$ rats to produce, they still reproduce $\text{Pois}(1)$

$\rightarrow np + \underbrace{np \text{ offspring}}_{\sum_{i=1}^{np} X_i \Rightarrow Y_1 \sim \text{Pois}(np)}$

$\rightarrow 2np$ rats

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h) for what values of p does the # of rats decrease from one year to the next?

guess $\rightarrow p < 1/2$

$2np < np$ $\quad$ (new # females) < original # females

$\frac{np}{2np} < 1$ $\quad$ when $p = 1/2$, the total # of surving females + their offspring is less than the original population

f) Compute the covariance btwn Y_1 and $Y_2 - Y_1$

$$\begin{aligned}
\text{cov}(Y_1, Y_2 - Y_1) &= E(Y_1 * (Y_2 - Y_1)) - EY_1\, E(Y_2 - Y_1) \\
&= EE(Y_1 Y_2 - Y_1^2 \mid Y_1) - E(Y_1)\, E\, E(Y_2 - Y_1 \mid Y_1) \\
&= E(Y_1 E(Y_2 \mid Y_1) - Y_1^2) - E(Y_1)(E(EY_2 \mid Y_1) - Y_1) \\
&= E(Y_1 (n + Y_1) - Y_1^2) \\
&= E\, n Y_1 \qquad - EY_1\, E(n + Y_1 - Y_1) \\
&= n(n) \qquad - n(n) = 0
\end{aligned}$$