Ann Marie Weideman, MS Part I, 201

1. $(X,Y) \sim N(\mu, \Sigma)$

where $\mu = \begin{pmatrix} 68 \\ 68 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 4 & 0.6 \\ 0.6 & 4 \end{pmatrix}$

a) Find $P(X>Y)$

$P(X>Y) = P(X-Y>0)$ *

Know that $E[X-Y] = E[X] - E[Y] = 68 - 68 = 0$

$$Var[X-Y] = Var[X] + Var[Y] - 2\,Cov[X,Y]$$
$$= 4 + 4 - 2(0.6) = 6.8$$

$\Rightarrow X-Y \sim N(0, 6.8)$ ✓ checked against chegg

Then, $P(X-Y>0) = \boxed{0.5}$ since $X-Y$ is symmetric about 0.

b) Find $P(X>Y+4)$

$P(X>Y+4) = P(X-Y>4) = 1 - \Phi\left(\frac{4-0}{\sqrt{6.8}}\right) = \boxed{1 - \Phi\left(4/\sqrt{6.8}\right)}$

✓ checked against chegg

c) Find dist of $Y | X = 74$

Know that in the bivariate case $X_1 | X_2 = x_2 \sim N\left(\mu_1 + \frac{\sigma_1}{\sigma_2}\rho(x_2 - \mu_2), (1-\rho^2)\sigma_1^2\right)$

where ρ is the correlation coefficient between X_1 and X_2

Then, $Y | X = 74 \sim N\left(68 + \frac{2}{2} 0.15 (74 - 68), (1 - 0.15^2) \cdot 4\right)$

$$\equiv \boxed{N(68.9, 3.91)}$$ ✓ checked against chegg

where $\rho = \frac{Cov(X,Y)}{\sqrt{Var(X) Var(Y)}} = \frac{0.6}{\sqrt{4 \cdot 4}} = 0.15$

d) Find $P(Y>74 \mid X=74)$

$$P(Y>74 \mid X=74) = P\left(\frac{Y-68.9}{\sqrt{3.91}} > \frac{74-68.9}{\sqrt{3.91}} \,\Big|\, X=74\right)$$

$$= 1 - \Phi\left(\frac{74-68.9}{\sqrt{3.91}}\right) = \boxed{1 - \Phi\left(\frac{5.1}{\sqrt{3.91}}\right)}$$

e) Find dist of $(\bar{X}, \bar{Y})$.

Know mean is unchanged: $\tilde{\mu} = \begin{pmatrix} 68 \\ 68 \end{pmatrix}$

Divide covariance structure by n (like we do to variance in univariate dist)

$$\Sigma = \begin{bmatrix} 4/n & 0.6/n \\ 0.6/n & 4/n \end{bmatrix}$$

✓ checked by Dr. Q.

So, $$\boxed{(\bar{X}, \bar{Y}) \sim N\left(\begin{pmatrix} 68 \\ 68 \end{pmatrix}, \begin{bmatrix} 4/n & 0.6/n \\ 0.6/n & 4/n \end{bmatrix}\right)}$$

$$\begin{bmatrix} 4/n & 0.6/n \\ 0.6/n & 4/n \end{bmatrix} \equiv \frac{1}{n}\begin{bmatrix} 4 & 0.6 \\ 0.6 & 4 \end{bmatrix}$$

f) Find $P(\bar{X} - \bar{Y} \leq 3)$

$P(|\bar{x} - \bar{y}| \leq 3) = \ ?$

Know $E[\bar{x} - \bar{y}] = E[\bar{x}] - E[\bar{y}] = 68 - 68 = 0$

$$Var[\bar{x} - \bar{y}] = Var[\bar{x}] - Var[\bar{y}] - 2\,cov(\bar{x}, \bar{y})$$

$$= 4/n + 4/n - 2(0.6/n) = 6.8/n$$

Then, $P(|\bar{x} - \bar{y}| \leq 3) = P(-3 \leq \bar{x} - \bar{y} \leq 3) = P(\bar{x} - \bar{y} \leq 3) - P(\bar{x} - \bar{y} \leq -3)$

$$= P(\bar{x} - \bar{y} \leq 3) - \underbrace{P(\bar{x} - \bar{y} \geq 3)}_{\text{since symmetric}} = P(\bar{x} - \bar{y} \leq 3) - (1 - P(\bar{x} - \bar{y} \leq 3))$$

$$= \boxed{2\,\Phi\left(\frac{3}{\sqrt{6.8/n}}\right) - 1}$$

$\boxed{\text{As } n \to \infty, \text{ then have } 2\,\Phi\left(\frac{3}{\sqrt{6.8/n}}\right) - 1 \to 1}$

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2. Given $E[Y|X] = X$ $Y|X \sim Pois(X)$

$X \sim Unif(0,2)$

$$f_X(x) = \begin{cases} 0.5 & , 0 < x < 2 \\ 0 & , \text{else} \end{cases}$$

a) Find $E[Y]$ and $Var(Y)$. Does Y have Poisson dist? Verify

$E[Y] = E[E[Y|X]] = E[X] = \frac{0+2}{2} = \boxed{1}$

$Var[Y] = E[Var[Y|X]] + Var[E[Y|X]]$

$= E[X] + Var[X] = 1 + \frac{(2-0)^2}{12} = \boxed{4/3}$

✓ checked against Anjie

Since $E[Y] \neq Var[Y]$, then Y CANNOT have a Poisson dist.

b) Find $Corr(X, Y)$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{\underbrace{Var(X)}_{1/3}\underbrace{Var(Y)}_{4/3}}}$$

Need $Cov(X,Y) = E[Cov(X,Y|X)] + Cov(E[X|X], E[Y|X])$

$= E[\cancel{0}] + Cov(X,X) = Var(X) = 1/3$

$$\Rightarrow Corr(X,Y) = \frac{1/3}{\sqrt{1/3 \cdot 4/3}} = \frac{1/3}{\sqrt{4/9}} = \frac{1/3}{2/3} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2} = \boxed{\frac{1}{2}}$$ ✓

checked against Anjie

Ann Marie W.
MS Part I, 2014

2 c) Define $W = 4X - Y + 4$. Compute $Cov(W, Y)$.
Are W & Y independent? Justify

$$Cov(4X - Y + 4, Y) = Cov(4X, Y) - Cov(Y, Y) + \cancel{Cov(4, Y)}^{0}$$

$$= 4\,Cov(X, Y) - Var(Y) = 4(1/3) - 4/3 = \boxed{0}.$$

Let $\begin{cases} W = 4X - Y + 4 \\ V = Y \end{cases} \Rightarrow \begin{cases} X = \frac{1}{4}(4 - V - W) = 1 - \frac{1}{4}V - \frac{1}{4}W \\ Y = V \end{cases}$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} -1/4 & -1/4 \\ 1 & 0 \end{vmatrix} = |+1/4| = 1/4$$

Know. $f_{X,Y}(x, y) = f_{Y|X}(y|x) \cdot f_X(x) = \frac{x^y e^{-x}}{y!} \cdot \frac{1}{2}$, $\quad 0 < x < 2$, $y \in \{0, 1, 2, \dots\}$

$$\Rightarrow f_{X,Y}\left(1 - \tfrac{1}{4}V - \tfrac{1}{4}W, V\right) = \frac{(1 - 1/4V - 1/4W)^V e^{-(1 - 1/4V - 1/4W)}}{V!} \cdot \frac{1}{2} \cdot \frac{1}{4}$$

$$= \frac{1}{8} \cdot \frac{(1 - 1/4V - 1/4W)^V e^{-1 + 1/4V + 1/4W}}{V!}, \quad V \in \{0, 1, 2, \dots\}, \ -\infty < W \leq 12$$

However, due to the piece $(1 - 1/4V - 1/4W)^V$ in the numerator, there is no way to factor $f_{X,Y}(1 - 1/4V - 1/4W, \underset{Y}{V})$ into separate functions of $Y = V$ and W $\Rightarrow$ not independent.

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d) Is Y an unbiased predictor of X? Compute the MSE 0.432

$E[Y-X] = E[Y] - E[X] = 1 - 1 = 0 \Rightarrow$ Y is an unbiased predictor of X.

Know $MSE = \text{variance} + \text{bias}^2 = Var(Y) + (E[Y-X])^2$

$\Rightarrow MSE = Var(Y) + 0^2 = \boxed{4/3}$ ✓ checked by Dr. Q.

e) Find the constants a & b ∋ a+bY is an unbiased predictor of X and that the prediction variance is as small as possible.

$E[X-a-bY] = 0 \Rightarrow E[X] - a - bE[Y] = 1 - a - b \cdot 1 = 0$

$\Rightarrow 1 - a = b$

$Var(X-a-bY) = Var(X) + b^2 Var(Y) - 2b\,Cov(X,Y) = 1/3 + b^2 \cdot 4/3 - 2b \cdot 1/3$

Then $-2/3\, b + 4/3\, b^2 + 1/3$ ✓

To find minimum, take derivative & set equal to 0.

$\Rightarrow \frac{d}{db}\left(4/3\, b^2 - 2/3\, b + 1/3\right) = 8/3\, b - 2/3 \overset{set}{=} 0 \Rightarrow 8/3\, b = +2/3 \Rightarrow \boxed{b = 1/4}$

$\Rightarrow 1 - a = 1/4 \Rightarrow 1 - 1/4 = a \Rightarrow \boxed{a = 3/4}$ ✓ checked by Dr. Q.

f) Find P(Y=0)

$P(Y=0) = \int_X f_{X,Y=0}(x,0)\,dx = \int_X f_{Y=0|X}(0|x) \cdot f_X(x)\,dx$

$= \int_0^2 \frac{1}{2} \frac{x^0 e^{-x}}{0!}\,dx = \frac{1}{2}\int_0^2 e^{-x}\,dx = -\frac{1}{2}e^{-x}\Big|_0^2 = -\frac{1}{2}e^{-2} + \frac{1}{2}$

$\approx \boxed{0.432}$ ✓ checked against Angie

2. g) Compute $E[X \mid Y=0]$.

Know $Y|X \sim Pois(X)$ and $X \sim Unif(0,2)$

First find $f_{X,Y=0}(x,0) = f_{Y=0|X}(Y=0|x) f_X(x) = \frac{x^0 e^{-x}}{0!} \cdot \frac{1}{2}$

$= \frac{1}{2} e^{-x}$

Then, $f_{X|Y=0}(x|Y=0) = \frac{f_{X,Y=0}(x,0)}{f_Y(0)} = \frac{\frac{1}{2}e^{-x}}{\int_X f_{X,Y=0}(x,0)\,dx} = \frac{\frac{1}{2}e^{-x}}{\int_0^2 \frac{1}{2}e^{-x}dx}$

$= \frac{\frac{1}{2}e^{-x}}{-\frac{1}{2}e^{-x}\Big|_0^2} = \frac{\frac{1}{2}e^{-x}}{-\frac{1}{2}e^{-2}+\frac{1}{2}} = \frac{e^{-x}}{1-e^{-2}}, \quad 0<x<2$

Then, finally, $E[X|Y=0] = \int_X x\, f_{X|Y}(x|y)\,dx = \int_0^2 \frac{xe^{-x}}{1-e^{-2}}\,dx$

$= \left(\frac{1}{1-e^{-2}}\right)\int_0^2 xe^{-x}dx$ * (need to use integration by parts)

Let $u = x$ $dv = e^{-x}$
$du = 1$ $v = -e^{-x}dx$ $\Big\} \Rightarrow uv - \int v\,du = -xe^{-x}\Big|_0^2 - \int_0^2(-e^{-x})\,dx$

$= -xe^{-x}\Big|_0^2 + \int_0^2 e^{-x}dx = (-2e^{-2}+0) + \left(-e^{-x}\Big|_0^2\right)$

$= -2e^{-2} - e^{-2} + 1 = (-2-1)e^{-2}+1 = -3e^{-2}+1$

$\Rightarrow * = \frac{-3e^{-2}+1}{1-e^{-2}} = \boxed{\frac{1-3e^{-2}}{1-e^{-2}}}$.

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3 a) Find $\hat{\theta}_{MLE}$ where $\theta = P(X=0)$

① find $\hat{\lambda}_{MLE}$

② Use invariance property to find $\hat{\theta}_{MLE}$

$$\mathcal{L}(\lambda|x) = \prod_{i=1}^{n} \frac{\lambda^{x} e^{-\lambda}}{x!} = \frac{e^{-\lambda n} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\Rightarrow \ell(\lambda|x) = -\lambda n + \sum_{i=1}^{n} x_i \log(\lambda) - \sum_{i=1}^{n} \log(x_i!)$$

$$\Rightarrow \frac{\partial \ell}{\partial \lambda} = -n + \sum_{i=1}^{n} x_i / \lambda \overset{set}{=} 0 \Rightarrow \hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{X}$$

Since $\left.\frac{\partial^2 \ell}{\partial \lambda^2}\right|_{\lambda=\hat{\lambda}} = \left. -\sum_{i=1}^{n} x_i / \lambda^2 \right|_{\lambda=\hat{\lambda}} < 0 \Rightarrow \hat{\lambda}_{MLE}$ occurs @ global max.

✓ checked against Angie

Since $\theta = P(X=0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda}$, by invariance property, $\boxed{\hat{\theta}_{MLE} = e^{-\bar{X}}}$

b) Show that $\hat{\theta} = (1 - 1/n)^{Y}$ is an unbiased estimator of θ, where $Y = \sum_{i=1}^{n} X_i$

Need to show $E[\hat{\theta}] = \theta$. Know $Y = \sum_{i=1}^{n} X_i \sim Pois(n\lambda)$

$$E[(1-1/n)^{Y}] = \sum_{i=0}^{\infty} \frac{(1-1/n)^{Y} (n\lambda)^{Y} e^{-n\lambda}}{y!} = e^{-n\lambda} \sum_{i=0}^{\infty} \frac{(n\lambda - \lambda)^{Y}}{y!}$$

$$= e^{-n\lambda} \cdot \underbrace{e^{n\lambda - \lambda}}_{\text{use Taylor series sum: } e^{x} = \sum_{i=0}^{\infty} \frac{x^i}{i!}} = e^{-\lambda} = \theta \checkmark$$

checked against Claire

$$\Rightarrow \hat{\theta} = (1 - 1/n)^{Y} \text{ is an unbiased estimator of } \theta.$$

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3 c) Will ask Dr. Q next week & update.

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3 d. Given $z_i = \begin{cases} 1, & x_i > 0 \\ 0, & x_i = 0 \end{cases}$

Find $\hat{\lambda}_{MLE}$ as a function of $z_1, \dots, z_n$

Then, derive the limiting variance of $\sqrt{n}\,\hat{\lambda}$ as $n \to \infty$

$z_i \sim Bern(P(x_i > 0))$ where $P(x_i > 0) = 1 - P(x_i \le 0) = 1 - P(x_i = 0)$

$= 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-\lambda}$

$\Rightarrow z_i \sim Bern(\overbrace{1 - e^{-\lambda}}^{p})$

Then, $\mathcal{L}(p \mid \underset{\sim}{z}) = \prod_{i=1}^{n} (p)^{z_i} (1-p)^{1-z_i} = p^{\sum z_i} (1-p)^{n - \sum z_i}$

$\Rightarrow \ell(p \mid \underset{\sim}{z}) = \sum z_i \log(p) + (n - \sum z_i) \log(1-p)$

$\Rightarrow \frac{\partial \ell}{\partial p} = \frac{\sum z_i}{p} - \frac{(n - \sum z_i)}{(1-p)} \overset{set}{=} 0$

$\Rightarrow \frac{\sum z_i}{p} = \frac{(n - \sum z_i)}{1-p} \Rightarrow \frac{1-p}{p} = \frac{n - \sum z_i}{\sum z_i}$

$\Rightarrow \frac{1}{p} \cancel{-1} = \frac{n}{\sum z_i} \cancel{-1} \Rightarrow \hat{p} = \frac{1}{n} \sum z_i = \bar{z}$

Since $\left.\frac{\partial^2 \ell}{\partial \lambda^2}\right|_{\lambda = \hat{\lambda}} = \left. -\frac{\sum z_i}{p^2} - \frac{(n - \sum z_i)}{(1-p)^2} \right|_{\lambda = \hat{\lambda}} = \frac{-n\cancel{\bar{z}}}{\bar{z}^{\cancel{2}}} - \frac{n(\cancel{1-\bar{z}})}{(1-\bar{z})^{\cancel{2}}}$

$= \frac{-n}{\bar{z}} - \frac{n}{(1-\bar{z})} = \frac{-n(1-\bar{z})}{\bar{z}(1-\bar{z})} - \frac{n\bar{z}}{\bar{z}(1-\bar{z})} = \frac{-n + \cancel{n\bar{z}} - \cancel{n\bar{z}}}{\bar{z}(1-\bar{z})} = \frac{-n}{\bar{z}(1-\bar{z})} < 0$

$\Rightarrow \hat{p}$ occurs at a global max.

$\Rightarrow \hat{p} = (1 - e^{-\hat{\lambda}}) = \bar{z} \Rightarrow -e^{-\hat{\lambda}} = \bar{z} - 1 \Rightarrow e^{-\hat{\lambda}} = 1 - \bar{z}$

$\Rightarrow \hat{\lambda} = -\log(1 - \bar{z})$ ✓

next pg →

checked against Angie

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3 d) To derive the limiting variance of $\sqrt{n}\,\hat{\lambda}$ as $n \to \infty$.

① By CLT, have $\sqrt{n}(\hat{p} - p) \xrightarrow{d} N(0, I_1^{-1}(p))$

Need to find $I_1^{-1}(p) = E(J_1(p|z)) = E\left(\frac{z}{p^2} + \frac{(1-z)}{(1-p)^2}\right)$

$$= \frac{E(z)}{p^2} + \frac{1}{(1-p)^2} - \frac{E(z)}{(1-p)^2} = \frac{p}{p^2} + \frac{1}{(1-p)^2} - \frac{p}{(1-p)^2}$$

$$= \frac{(1-p)^2}{p(1-p)^2} + \frac{p}{p(1-p)^2} - \frac{p^2}{p(1-p)^2} = \frac{1-2p+\cancel{p^2}+p-\cancel{p^2}}{p(1-p)^2}$$

$$= \frac{\cancel{1-p}}{p(1-p)^{\cancel{2}}} = \frac{1}{p(1-p)}$$

$$\Rightarrow I^{-1}(p) = p(1-p)$$

② Then, using delta method,

$$\sqrt{n}\,(g(\hat{p}) - g(p)) \xrightarrow{d} N(0, \{g'(p)\}^2\, p(1-p))$$

Have $g(p) = -\log(1-p) \Rightarrow g'(p) = \frac{1}{(1-p)} \Rightarrow \{g'(p)\}^2 = \frac{1}{(1-p)^2}$

$$\Rightarrow \sqrt{n}(g(\hat{p}) - g(p)) \xrightarrow{d} N\left(0, \frac{1}{(1-p)^{\cancel{2}}}\, p(\cancel{1-p})\right)$$

$p = 1 - e^{-\lambda}$

$\Rightarrow 1 - p = e^{-\lambda}$

$$\Rightarrow \sqrt{n}(\hat{\lambda} - \lambda) \xrightarrow{d} N\left(0, \frac{1-e^{-\lambda}}{e^{-\lambda}}\right)$$

$$\equiv N(0, e^{\lambda} - 1)$$

✓ checked against Arjie & online resource

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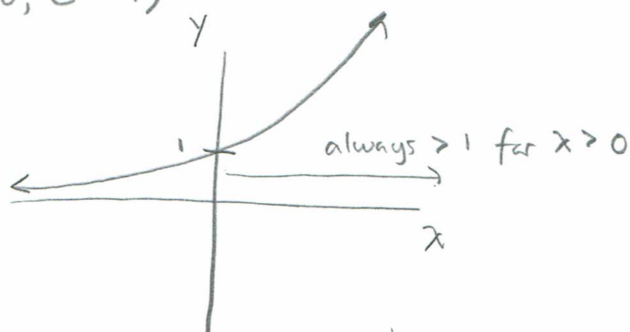
3e) Show that $\sqrt{n}\,\bar{X}$ has smaller limiting variance than $\sqrt{n}\,\hat{\lambda}$ & give an explanation as to why this makes sense.

[Given that $\sqrt{n}(\bar{X}-\lambda) \xrightarrow{d} N(0,\lambda)$

In part d), have $\sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow{d} N(0, e^{\lambda}-1)$

Then, the ratio $\frac{e^{\lambda}-1}{\lambda}$ looks like



λ	Y
1×10^{-10}	~ 1
1	$e-1$
2	$(e^2-1)/2$

Since Z_i was dichotomized (bernoulli RV), we lose information that was otherwise present in X_i. Thus, we would normally expect that this would increase the limiting variance (Keep in mind that it is sometimes better to dichotomize in the case of lots of error in raw data.]

✓ checked against Angie

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4. $X_1, \dots, X_n \overset{iid}{\sim} N(0, \sigma^2)$

Want to test $H_0: \sigma = \sigma_0$ vs. $H_1: \sigma \neq \sigma_0$

Use $\delta(X_1, \dots, X_n) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i^2 < c_1 \text{ or } \sum_{i=1}^{n} X_i^2 > c_2 \\ 0, & \text{else} \end{cases}$

a) Find c_1 and c_2 for α-level test.

Have $\alpha = P\left(\sum_{i=1}^{n} X_i^2 < c_1 \text{ or } \sum_{i=1}^{n} X_i^2 > c_2 \mid \sigma = \sigma_0\right)$

$= P\left(\sum_{i=1}^{n} X_i^2 < c_1 \mid \sigma = \sigma_0\right) + P\left(\sum_{i=1}^{n} X_i^2 > c_2 \mid \sigma = \sigma_0\right)$

Since $X_i \sim N(0, \sigma^2)$ for $i = 1, \dots, n \Rightarrow X_i/\sigma \sim N(0,1)$

$\Rightarrow \sum_{i=1}^{n} \left(\frac{X_i}{\sigma}\right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} X_i^2 \sim \chi_n^2$

Then $\alpha = P\left(\frac{1}{\sigma_0^2} \sum_{i=1}^{n} X_i^2 < \frac{1}{\sigma_0^2} c_1\right) + P\left(\frac{1}{\sigma_0^2} \sum_{i=1}^{n} X_i^2 > \frac{1}{\sigma_0^2} c_2\right)$

$\Rightarrow \frac{1}{\sigma_0^2} c_1 = \chi^2_{n, \alpha/2}$ and $\frac{1}{\sigma_0^2} c_2 = \chi^2_{n, 1-\alpha/2}$

$\Rightarrow \boxed{c_1 = \sigma_0^2 \chi^2_{n, \alpha/2}}$ and $\boxed{c_2 = \sigma_0^2 \chi^2_{n, 1-\alpha/2}}$

✓ checked against chegg

4 b) ⌈ Power function of test; $H_0: \sigma = \sigma_0$ vs. $H_1: \sigma \neq \sigma_0$ is,

$$\beta(\sigma) = P_\sigma\left(\sum_{i=1}^n X_i^2 < \sigma_0^2 \chi^2_{n,\alpha/2}\right) + P_\sigma\left(\sum_{i=1}^n X_i^2 > \sigma_0^2 \chi^2_{n,1-\alpha/2}\right)$$

$$= P_\sigma\left(\frac{\sum_{i=1}^n X_i^2}{\sigma^2} < \frac{\sigma_0^2 \chi^2_{n,\alpha/2}}{\sigma^2}\right) + P_\sigma\left(\frac{\sum_{i=1}^n X_i^2}{\sigma^2} > \frac{\sigma_0^2 \chi^2_{n,1-\alpha/2}}{\sigma^2}\right)$$

$$= G_n\left(\frac{\sigma_0^2 \chi^2_{n,\alpha/2}}{\sigma^2}\right) + 1 - G_n\left(\frac{\sigma_0^2 \chi^2_{n,1-\alpha/2}}{\sigma^2}\right)$$ ⌋ ✓ checked against chegg

4 c) Prove or disprove that the test δ is the UMP test of its size.

⌈ From Dr. Lin's class, we know that UMP tests DO NOT exist for two-sided tests (e.g., δ).

Take $\sigma = 2$, $\sigma_0 = 1$, $\alpha = 0.05$, and $n = 10$.

Then, $\beta(2) = G_{10}\left(\frac{1\,\chi^2_{10,0.025}}{4}\right) + 1 - G_{10}\left(\frac{1\,\chi^2_{10,0.975}}{4}\right)$

in R $= 6.56 \times 10^{-5} + 1 - 0.117036 = 0.883$

Let us look at a one-sided critical region w/ the same parameters (and, of course, the same α), where $R = \{x : \sum_{i=1}^n X_i^2 > \chi^2_{10,0.95}\}$

Then $\beta^*(2) = P\left(\sum_{i=1}^n X_i^2 > \chi^2_{10,0.95} \mid \sigma = 2\right)$

$$= 1 - P\left(\sum_{i=1}^n X_i^2 \le \chi^2_{10,0.95} \mid \sigma = 2\right)$$

$$= 1 - P\left(\frac{\sum_{i=1}^n X_i^2}{4} \le \frac{\chi^2_{10,0.95}}{4}\right) = 0.9176$$

Since $\beta^*(2) = 0.9176 > \beta(2) = 0.883$,

δ is NOT the UMP test of its size. ⌋ ✓ checked against chegg

4 d) Find critical region used to test $H_0: \sigma = \sigma_0$ vs. $H_1: \sigma > \sigma_0$

First, find MP (most powerful) test using Neyman Pearson.

Have, $R = \left\{ x : \dfrac{f(x \mid \sigma_1)}{f(x \mid \sigma_0)} > c \right\}$ for $H_0: \sigma = \sigma_0$ vs $H_1: \sigma = \sigma_1$ ($\sigma_1 > \sigma_0$)

Then, $R = \left\{ x : \dfrac{\left(\frac{1}{\cancel{\sqrt{2\pi}}\,\sigma_1}\right)^n e^{-\sum_{i=1}^n x_i^2 / 2\sigma_1^2}}{\left(\frac{1}{\cancel{\sqrt{2\pi}}\,\sigma_0}\right)^n e^{-\sum_{i=1}^n x_i^2 / 2\sigma_0^2}} > c \right\}$

$= \left\{ x : \underbrace{\left(\dfrac{\sigma_0}{\sigma_1}\right)^n e^{-\sum_{i=1}^n x_i^2 \left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right)}}_{} > c \right\}$

This ratio is monotone increasing for $\sigma_1^2 > \sigma_0^2$

$\Rightarrow \dfrac{1}{\sigma_1^2} < \dfrac{1}{\sigma_0^2}$

$\Rightarrow R^* = \left\{ x : \sum_{i=1}^n x_i^2 > c^* \right\}$

Then $\alpha = P\left(\sum_{i=1}^n x_i^2 > c^* \mid \sigma = \sigma_0\right) = P\left(\underbrace{\dfrac{\sum_{i=1}^n x_i^2}{\sigma_0^2}}_{\sim \chi^2_n} > \dfrac{c^*}{\sigma_0^2}\right)$

$\Rightarrow \dfrac{c^*}{\sigma_0^2} = \chi^2_{n,1-\alpha} \Rightarrow c^* = \sigma_0^2 \chi^2_{n,1-\alpha}$

✓ checked against chegg, they had small mistake

$\Rightarrow$ $\boxed{R = \left\{ x : \sum_{i=1}^n x_i^2 > \sigma_0^2 \chi^2_{n,1-\alpha} \right\}}$. Since the critical region doesn't depend on σ_1, it is also the UMP test for $H_0: \sigma = \sigma_0$ vs. $H_1: \sigma > \sigma_0$

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4 e) i) Find the power function used to test the hypothesis in d)
ii) Show that δ is less powerful than the UMP test derived in d)

T i) Power function of test, $H_0: \sigma = \sigma_0$ vs. $H_1: \sigma > \sigma_0$ is

$$\beta^*(\sigma) = P_\sigma\left(\sum_{i=1}^n X_i^2 > \sigma_0^2 \chi^2_{n,1-\alpha}\right)$$

$$= P_\sigma\left(\frac{\sum_{i=1}^n X_i^2}{\sigma^2} > \frac{\sigma_0^2 \chi^2_{n,1-\alpha}}{\sigma^2}\right)$$ ✓ checked against chegg

$$= \boxed{1 - G_n\left(\frac{\sigma_0^2 \chi^2_{n,1-\alpha}}{\sigma^2}\right)}$$

for $G_n(\cdot)$ the CDF of a chi-squared dist w/ n df and $\chi^2_{n,1-\alpha}$ the $1-\alpha$ quantile

ii) We already showed this in part 4c), but briefly again

Since $\beta^*(2) = 0.9176 > \beta(2) = 0.883$, δ is less powerful than the UMP test in d).

1. Red Blue

$$\begin{array}{ccc} & a & b & \\ n \text{ times} \left[\begin{array}{c} \vdots \end{array} \right. & & \left. \begin{array}{c} \vdots \end{array} \right] n \text{ times} \\ & X_n & a+b-X_n & \end{array}$$

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a) Find $E[X_1]$

First write PMF for X_1 so we can then use expectation formula for a discrete RV.

$$P(X_1 = x_1) = \begin{cases} \frac{a}{a+b}, & x_1 = a \\ \frac{b}{a+b}, & x_1 = a+1 \end{cases}$$

Then, $E[X_1] = a\left(\frac{a}{a+b}\right) + (a+1)\left(\frac{b}{a+b}\right) = \frac{a^2+ab+b}{a+b} = \frac{a(\cancel{a+b})}{(\cancel{a+b})} + \frac{b}{a+b}$

$$= \boxed{a + \frac{b}{a+b}}$$

b) Find the MGF of X_1

Using PMF from last step.

$$M_{X_1}(t) = E[e^{tX_1}] = \boxed{\frac{a}{a+b} e^{at} + \frac{b}{a+b} e^{(a+1)t}}$$

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1 c) <u>Show that $E[X_{n+1}] = \left(1 - \frac{1}{a+b}\right) E[X_n] + 1$</u>

⌐ First, write PMF for X_n based on picture on first page, will draw again here:

	Red	Blue	
Start:	a	b	
n times	$\vdots$	$\vdots$	n times
	X_n	$a+b-X_n$	

$$P(X_n = x) = \begin{cases} \frac{X_n}{a+b}, & x = X_n \\ \frac{a+b-X_n}{a+b}, & x = X_n + 1 \end{cases}$$

Then, $E(X_{n+1} \mid X_n) = X_n\left(\frac{X_n}{a+b}\right) + (X_n+1)\left(\frac{a+b-X_n}{a+b}\right)$

$$= \frac{\cancel{X_n^2} + aX_n + bX_n - \cancel{X_n^2} + a + b - X_n}{a+b} = \frac{(a+b-1)X_n + (a+b)}{a+b}$$

Then, $E[X_{n+1}] = E[E(X_{n+1} \mid X_n)] = \boxed{\left(1 - \frac{1}{a+b}\right) E[X_n] + 1}$ ⌟

1 d) Use part c) to show that $E[X_n] = a + b - b\left(1 - \frac{1}{a+b}\right)^n$

⌐ <u>Base Case</u>: $E[X_1] = a + \frac{b}{a+b}$ (from part a)

$$E[X_1] = a + b - b\left(1 - \frac{1}{a+b}\right)^1 = a + b - b\left(1 - \frac{1}{a+b}\right) = a + \cancel{b} - \cancel{b} + \frac{b}{a+b}$$

$$= a + \frac{b}{a+b}$$

<u>Inductive Step</u>: Assume true for $n-1$. Prove true for n.

$E[X_{n-1}] = a + b - b\left(1 - \frac{1}{a+b}\right)^{n-1}$. From c), know $E[X_n] = \left(1 - \frac{1}{a+b}\right) E[X_{n-1}] + 1$

$$= \left(1 - \frac{1}{a+b}\right)\left[a + b - b\left(1 - \frac{1}{a+b}\right)^{n-1}\right] + 1 = (a+b)\left(1 - \frac{1}{a+b}\right) - b\left(1 - \frac{1}{a+b}\right)^n + 1$$

$$a + b - b\left(1 - \frac{1}{a+b}\right)^n$$ ⌟

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1e) Find the probability that the $(n+1)$st ball drawn is red and show that this probability converges to 1 as $n \to \infty$.

Let A be the event in which the $(n+1)$st ball is red.

Then, $P(A \mid X_n) = \frac{X_n}{a+b}$

$\therefore$ $P(A) = E[P(A \mid X_n)] = E\left(\frac{X_n}{a+b}\right) = \frac{1}{a+b} E(X_n) = \frac{1}{a+b}\left(a+b-b\left(1-\frac{1}{a+b}\right)^n\right)$ ✓

$= 1 - \frac{b}{a+b}\underbrace{\left(1-\frac{1}{a+b}\right)}_{<1}^n \longrightarrow 1$ as $n \to \infty$.

1f) Show that $X_n \xrightarrow{P} a+b$

Want to show $\lim_{n\to\infty} P\{|X_n - (a+b)| > \varepsilon\} = 0$

Going to use Chebychev's ineq., which says that

$P[g(X) \geq r] \leq \frac{E[g(X)]}{r}$ for X a RV, $g(X)$ a non-negative function, and $r > 0$.

Let $g(X_n) = |X_n - (a+b)|$, then, by Chebychev's have:

$$P[|X_n - (a+b)| \geq \varepsilon] \leq \frac{E[|X_n - (a+b)|]}{\varepsilon}$$

$$= \frac{E[(a+b) - X_n]}{\varepsilon} \quad (\text{since } X_n \leq a+b)$$

$$= \frac{a+b - E(X_n)}{\varepsilon}$$

$$= a+b - \left(a+b-b\left(1-\frac{1}{a+b}\right)^n\right)/\varepsilon$$

$$= \frac{b\left(1-\frac{1}{a+b}\right)^n}{\varepsilon} \xrightarrow{0} \text{ as } n\to\infty \text{ since } \left(1-\frac{1}{a+b}\right) < 1$$

✓

Thus, $\lim_{n\to\infty} P[|X_n - (a+b)| \geq \varepsilon] = 0$, so $X_n \xrightarrow{P} a+b$.

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2. X = time to occurance
Y = time to death
$X \sim Unif(0,1)$, $Y|X \sim Unif(x, x+1)$

a) Find the mean i) & variance ii) of Y

i) $E[Y] = E[E(Y|X)] = E\left[\frac{(x+1+x)}{2}\right] = E\left[\frac{2x+1}{2}\right] = E[X + 1/2]$

$= E[X] + 1/2 = 1/2 + 1/2 = \boxed{1}$ ✓

ii) $Var[Y] = E[Var(Y|X)] + Var[E(Y|X)]$ (by Law of Total Variance)

$= E\left[\frac{(x+1-x)^2}{12}\right] + Var\left[x + 1/2\right] = E[1/12] + Var[X] = 1/12 + 1/12 = \boxed{1/6}$ ✓

b) Find the correlation between X & Y

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\overbrace{E(Cov(X,Y|X)) + Cov(E(X|X), E(Y|X))}^{\text{By Law of Total Covariance}}}{\sqrt{\underbrace{(1/12)}_{Var(X)}\underbrace{(1/6)}_{Var(Y)\text{ from a)}}}}$$

$$\begin{array}{r} 12 \\ \times\ 6 \\ \hline 72 \end{array}$$

$$= \frac{0 + Cov(X, X + 1/2)}{\sqrt{1/72}} = \frac{Cov(X,X) + \cancelto{0}{Cov(X, 1/2)}}{\sqrt{1/72}}$$

$$= \frac{Var(X)}{\sqrt{1/72}} = \frac{1/12}{\sqrt{1/72}} = \frac{1}{\cancel{12}\,2} \cdot \frac{\cancel{6}\sqrt{2}}{1} = \boxed{\frac{\sqrt{2}}{2}}$$ ✓

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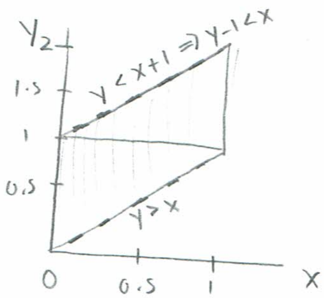
2c) Find $f_Y(y)$

Know $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} \Rightarrow f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x)$

$= \left(\frac{1}{1-0}\right)\left(\frac{1}{x+1-x}\right) = 1, \quad \{(x,y) : 0 < x < y < x+1\}$



Then, $f_Y(y) = \int_0^y f_{X,Y}(x,y)\,dx = \int_0^y 1\,dx = y$

&

$f_Y(y) = \int_{y-1}^1 1\,dx = x\Big|_{y-1}^1 = 1-y+1 = 2-y$

$\therefore \boxed{f_Y(y) = \begin{cases} y, & 0 < y \le 1 \\ 2-y, & 1 < y < 2 \end{cases}}$ ✓

2d) Find $f_{X|Y}(x|y)$

$f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \boxed{\begin{cases} \dfrac{1}{y}, & \begin{matrix} 0 < x < y \\ 0 < y \le 1 \end{matrix} \\[2ex] \dfrac{1}{2-y}, & \begin{matrix} y-1 < x < 1 \\ 1 < y < 2 \end{matrix} \end{cases}}$ ✓

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2 e) Find $E[X|Y]$

$$E[X|Y] = \int_x x \, f_{X|Y}(x|y)\,dx$$

$$= \int_0^y x \frac{1}{y}\,dx + \int_{y-1}^1 x \frac{1}{2-y}\,dx = \frac{1}{y}\int_0^y x\,dx + \frac{1}{2-y}\int_{y-1}^1 x\,dx$$

$$= \frac{1}{y}\left(\frac{1}{2}x^2\right)\Big|_0^y + \frac{1}{2-y}\left(\frac{1}{2}x^2\right)\Big|_{y-1}^1 = \frac{1}{y}\left(\frac{1}{2}y^2\right) + \frac{1}{2-y}\left(\frac{1}{2}(1)^2 - \frac{1}{2}(y-1)^2\right)$$

$$= \frac{1}{2}y + \frac{1}{2-y}\left(\frac{1}{2} - \frac{1}{2}(y-1)(y-1)\right) = \frac{1}{2}y + \frac{1}{2-y}\left(\frac{1}{2} - \frac{1}{2}(y^2 - 2y + 1)\right)$$

$$= \frac{1}{2}y + \frac{1}{2-y}\left(\cancel{\frac{1}{2}} - \frac{1}{2}y^2 + y - \cancel{\frac{1}{2}}\right) = \frac{1}{2}y + \frac{1}{2-y}\left(-\frac{1}{2}y^2 + y\right)$$

$$= \frac{\frac{1}{2}y(2-y)}{2-y} + \frac{-\frac{1}{2}y^2 + y}{2-y} = \frac{y - \frac{1}{2}y^2 - \frac{1}{2}y^2 + y}{2-y} = \frac{2y - y^2}{2-y}$$

$$= \frac{y(2-y)}{(2-y)} = \boxed{y}$$ ✓

2 f) Find $E\left[\frac{X}{Y}\right]$

$$E\left[\frac{X}{Y}\right] = E\left[E\left(\frac{X}{Y}\Big|Y\right)\right] = E\left[\frac{1}{Y}E(X|Y)\right] = E\left[\frac{1}{Y}\cdot Y\right]$$

$$= E[1] = \boxed{1}.$$ ✓

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3. Given $X_1, \ldots, X_n \overset{iid}{\sim} \theta e^{-\theta x}$, $x>0$, $\theta>0$

a) $\underbrace{\text{Derive the MLE, } \hat{\theta}}_{i)}$. $\underbrace{\text{Show that } 1/\hat{\theta} \text{ is the UMVUE}}_{ii)}$.

i) $L(\theta|X) = \theta^n e^{-\theta \sum_{i=1}^n x_i}$

$\ell(\theta|X) = n\log(\theta) - \theta \sum_{i=1}^n x_i$

$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n x_i \overset{set}{=} 0 \Rightarrow \frac{n}{\theta} = \sum_{i=1}^n x_i \Rightarrow \frac{\theta}{n} = \frac{1}{\sum_{i=1}^n x_i}$ (allowed to do since sum > 0)

$\Rightarrow \theta = \frac{n}{\sum_{i=1}^n x_i} = \frac{\cancel{n}}{\cancel{n}\bar{X}} = \frac{1}{\bar{X}} \Rightarrow \hat{\theta} = \frac{1}{\bar{X}}$

Since $\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2} < 0 \Rightarrow \hat{\theta}$ occurs at a global max. ✓

---

ii) To show that $1/\hat{\theta}$ is the UMVUE need to:

1) Find a complete & sufficient statistic for θ

2) Adjust the complete & sufficient statistic to be unbiased, which will give the UMVUE.

1) $f(x|\theta) = \underbrace{I(x>0)}_{h(x)} \underbrace{\theta}_{c(\theta)} \exp(\underbrace{-\theta}_{w(\theta)} \underbrace{x}_{t(x)})$. Then $T(x) = \sum_{i=1}^n x_i$ is a complete & sufficient statistic since $w(\theta) = -\theta \in (-\infty, 0)$ is an open set in $\mathbb{R}^1$.

2) $E[T(x)] = E[\sum_{i=1}^n x_i] = \sum_{i=1}^n E[x_i] = n\left(\frac{1}{\theta}\right) = \frac{n}{\theta}$

However, $E[T] = \frac{n}{\theta}$ is biased. To be unbiased, need $E[\,?\,] = \frac{1}{\theta}$.

Thus $E[\frac{1}{n}\sum_{i=1}^n x_i] = \frac{1}{\theta}$

✓

Thus, $\frac{1}{n}\sum_{i=1}^n x_i = \bar{X} = \frac{1}{\hat{\theta}}$ (b/c $\hat{\theta} = \frac{1}{\bar{X}}$ in part i)) is the UMVUE.

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3b) i) Derive the limiting dist. of $\sqrt{n}(\hat{\theta}-\theta)$ as $n \to \infty$.

ii) Comment on whether the limiting variance of $\sqrt{n}(\hat{\theta}-\theta)$ reaches the CRLB.

i) Two methods to derive limiting distribution of $\sqrt{n}(\hat{\theta}-\theta)$.

Method 1: By CLT, know $\sqrt{n}(\bar{x}-\frac{1}{\theta}) \xrightarrow{d} N(0, \frac{1}{\theta^2})$

Then, by Delta Method have $\sqrt{n}(\underbrace{g(\bar{x})}_{= \hat{\theta}} - \underbrace{g(\frac{1}{\theta})}_{= \theta}) \xrightarrow{d} N(0, \{g'(\frac{1}{\theta})\}^2 \cdot \frac{1}{\theta^2})$

where $g(y) = \frac{1}{y}$

$\Rightarrow g'(y) = -\frac{1}{y^2} \Rightarrow g'(\frac{1}{\theta}) = \theta^2 \Rightarrow \{g'(\frac{1}{\theta})\}^2 = \theta^4$

Thus, $\boxed{\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \theta^2)}$

Method 2: Using asymptotics, know $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \frac{1}{I_1(\theta)})$

Know $\underbrace{Var(u_1(\theta))}_{\text{variance of score function}} = I_1(\theta) \Rightarrow \mathcal{L}(\theta|x) = \theta e^{-\theta x}$

$\Rightarrow l(\theta|x) = \log(\theta) - \theta x$

$\Rightarrow \frac{\partial l}{\partial \theta} = \frac{1}{\theta} - x \Rightarrow Var(\frac{1}{\theta} - x) = -Var(x)$

Then, $u_1(\theta) = -(-Var(x)) = Var(x) = \frac{1}{\theta^2}$

Thus, $\boxed{\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \theta^2)}$

ii) Know that $CRLB = I_n^{-1}(\theta) = (n I_1(\theta))^{-1} = \frac{1}{\underbrace{n I_1(\theta)}_{\text{since scalar here}}} = \frac{1}{n(1/\theta^2)}$

$= \frac{\theta^2}{n}$

So yes, the limiting variance θ^2 reaches $n \cdot CRLB = n(\frac{\theta^2}{n}) = \theta^2$.

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c) Given $Z_1, \dots, Z_n$ observed.

$Z_i = 1$ if $X_i > \tau$ and $Z_i = 0$ if $X_i \le \tau$. Find MLE $\tilde{\theta}$ of θ

based on $Z_1, \dots, Z_n$ and derive the large sample dist. of $\tilde{\theta}$

in explicit form.

Given $I(X_i > \tau) \Rightarrow E[\underbrace{I(X_i > \tau)}_{Z_i}] = P(X_i > \tau) = \underbrace{e^{-\theta\tau}}_{\gamma}$

Let $\gamma = e^{-\theta\tau}$.

Then, $\mathcal{L}(\gamma \mid z) = \prod_{i=1}^{n} \gamma^{z}(1-\gamma)^{1-z} = \gamma^{\sum z_i}(1-\gamma)^{n-\sum z_i}$

$\Rightarrow \ell(\gamma \mid z) = \sum z_i \log(\gamma) + (n - \sum z_i)\log(1-\gamma)$

$\Rightarrow \frac{\partial \ell}{\partial \gamma} = \frac{\sum z_i}{\gamma} - \frac{(n - \sum z_i)}{1-\gamma} \overset{set}{=} 0 \Rightarrow \hat{\gamma} = \frac{1}{n}\sum z_i = \bar{z}$

Since $\left.\frac{\partial^2 \ell}{\partial \gamma^2}\right|_{\gamma=\hat{\gamma}} = \left.\frac{-\sum z_i}{\gamma^2} - \frac{(n - \sum z_i)}{(1-\gamma)^2}\right|_{\gamma=\hat{\gamma}} = \frac{-n\cancel{\bar{z}}}{\bar{z}^2} - \frac{n\cancel{(1-\bar{z})}}{(1-\bar{z})^2} < 0$

$\Rightarrow \hat{\gamma}$ occurs @ a global max.

checked w/ Dr Lin's answer

Since $\hat{\gamma} = \bar{z} \Rightarrow e^{-\tilde{\theta}\tau} = \bar{z} \Rightarrow \boxed{\tilde{\theta} = -\frac{1}{\tau}\log(\bar{z})}$ by invariance property.

Need to employ delta method to derive large sample dist. of $\tilde{\theta}$.

Know $\sqrt{n}(\bar{z} - \gamma) \xrightarrow{d} N(0, \gamma(1-\gamma))$

According to delta method, have:

$\sqrt{n}(g(\bar{z}) - g(\gamma)) \xrightarrow{d} N(0, \{g'(\gamma)\}^2 \gamma(1-\gamma))$

Let $g(\gamma) = -\frac{1}{\tau}\log(\gamma) \Rightarrow g'(\gamma) = -\frac{1}{\tau\gamma} \Rightarrow \{g'(\gamma)\}^2 = +\frac{1}{\tau^2\gamma^2}$

Thus, $\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N\left(0, \frac{1}{\tau^2\gamma^2}\gamma(1-\gamma)\right) \equiv N\left(0, \frac{(1-\gamma)}{\tau^2\gamma}\right)$

$\equiv N\left(0, \frac{1-e^{-\theta\tau}}{\tau^2 e^{-\theta\tau}}\right) \equiv \boxed{N\left(0, \frac{1}{\tau^2}(e^{\theta\tau}-1)\right)}$

checked w/ Dr. Lin's answer

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d) Let $\tau = E[X_1]$. Compare the asymptotic variances of $\hat{\theta}$ and $\tilde{\theta}$. Which one is larger? Is the anticipated result, and why?

Let $\tau = E[X_1] = \theta$

Then, $\sqrt{n}(\hat{\theta} - \theta) \to N(0, \theta^2)$

&

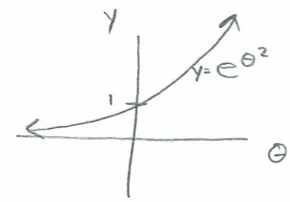
$\sqrt{n}(\tilde{\theta} - \theta) \to N\left(0, \frac{1}{\theta^2}(e^{\theta^2} - 1)\right)$

Take ratio of [limiting $Var(\tilde{\theta})$ / limiting $Var(\hat{\theta})$].

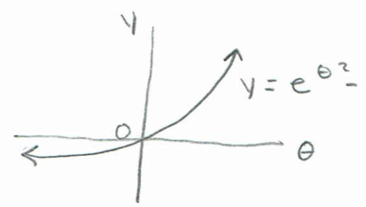
If ratio > 1, then limiting $Var(\tilde{\theta})$ is larger. Else limiting $Var(\hat{\theta})$ is larger.

$$\frac{\frac{1}{\theta^2}(e^{\theta^2} - 1)}{\theta^2} = \frac{1}{\theta^4}(e^{\theta^2} - 1) > 0 \quad \text{since } e^{\theta^2} > 1 \text{ for } \theta > 0.$$

$$\Rightarrow e^{\theta^2} - 1 > 0 \text{ for } \theta > 0.$$



$\Downarrow$



Thus, limiting $Var(\tilde{\theta})$ > limiting $Var(\hat{\theta})$.

This makes sense because you lose information when you dichotomize a continuous RV (which is why it is typically not recommended to bin continuous values if you can avoid it). However, if there was substantial error involved in measuring X_i, then dichotomizing this variable may have actually reduced the variance.

because $\hat{\theta}$ would be an inconsistent estimator of X_i

✓ Check w/ Dr Lin. soln.

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3 e) Given hazard function corresponding to f_X as

$$h_X(x|\theta) = \frac{f_X(x|\theta)}{\int_x^\infty f_X(s|\theta)\,ds}, \quad x > 0$$

In addition to $X_1, \ldots, X_n$ another independent random sample is available : $Y_1, \ldots, Y_n \overset{iid}{\sim} f_Y(y|\beta) = \beta e^{-\beta y}, \ y>0, \ \beta > 0$

Want to compare the two hazard functions $h_X(x|\theta) = \theta, \ x>0$ and $h_Y(y|\beta) = \beta, \ y>0$ by testing $H_0: \Psi = 1$ vs. $H_1: \Psi \neq 1$ where $\Psi = \theta/\beta$.

Derive a large sample (likelihood ratio, score, or Wald test) size α test of H_0 vs. H_1.

First, find joint likelihood function for x & y is:

$$\mathcal{L}(\theta, \beta | x, y) = \mathcal{L}(\theta|x) \cdot \mathcal{L}(\beta|y) = \theta^n e^{-\theta \sum_{i=1}^n x_i} \beta^n e^{-\beta \sum_{i=1}^n y_i} = \theta^n e^{-\theta n \bar{x}} \beta^n e^{-\beta n \bar{y}}$$

$$= (\theta\beta)^n e^{-n(\theta\bar{x} + \beta\bar{y})}$$

Under H_0: $\theta = \beta = \mu_0 \Rightarrow \mathcal{L}(\theta_0, \beta_0 | x, y) = (\mu_0)^{2n} e^{-n(\mu_0 \bar{x} + \mu_0 \bar{y})} = (\mu_0)^{2n} e^{-n\mu_0(\bar{x} + \bar{y})}$

which has MLE $\hat{\mu}_0 = \dfrac{2}{\bar{x} + \bar{y}}$

Under H_1: $\hat{\theta} = \dfrac{1}{\bar{x}}$ and $\hat{\beta} = \dfrac{1}{\bar{y}} \Rightarrow \mathcal{L}(\hat{\theta}, \hat{\beta} | x, y) = \left(\dfrac{1}{\bar{x}} \dfrac{1}{\bar{y}}\right)^n e^{-n\overbrace{\left(\frac{1}{\bar{x}}\bar{x} + \frac{1}{\bar{y}}\bar{y}\right)}^{2}}$

$$= \left(\frac{1}{\bar{x}\bar{y}}\right)^n e^{-2n}$$

Then, $\lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \dfrac{\sup_{H_0} \mathcal{L}(\theta, \beta | x, y)}{\sup_{H_0 \cup H_1} \mathcal{L}(\theta, \beta | x, y)} = \dfrac{\mathcal{L}(\hat{\mu}_0 | x, y)}{\mathcal{L}(\hat{\theta}, \hat{\beta} | x, y)} =$

$\alpha = P(-2\log\lambda(\underset{\sim}{x}, \underset{\sim}{y}) > c_1^*)$ where $c_1^* = \chi^2_{1,1-\alpha}$

$\Uparrow$

$$\frac{\left(\frac{2}{\bar{x}+\bar{y}}\right)^{2n} e^{-n\left(\frac{2}{\cancel{\bar{x}+\bar{y}}}\right)\cancel{(\bar{x}+\bar{y})}}}{\left(\frac{1}{\bar{x}\bar{y}}\right)^n e^{-2n}} = \frac{\left(\frac{2}{\bar{x}+\bar{y}}\right)^{2n} \cancel{e^{-2n}}}{\left(\frac{1}{\bar{x}\bar{y}}\right)^n \cancel{e^{-2n}}} = \frac{\left(\frac{2}{\bar{x}+\bar{y}}\right)^{2n}}{\left(\frac{1}{\bar{x}\bar{y}}\right)^n} = \boxed{\left\{\frac{4\bar{x}\bar{y}}{(\bar{x}+\bar{y})^2}\right\}^n}$$

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4. $X_1, \dots, X_n \overset{iid}{\sim} Unif(a, \theta)$ ← known

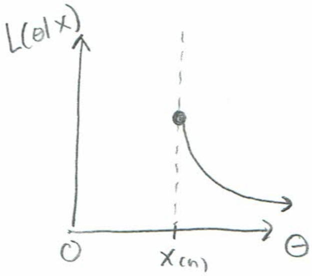
θ = max dose

a) i) Show the MLE of θ is the maximum order statistic $X_{(n)}$

ii) Prove that it is a biased estimator of θ at each finite n, but a consistent estimator as $n \to \infty$.

[i) Take $\mathcal{L}(\theta|x) = \prod_{i=1}^{n} \left(\frac{1}{\theta - a}\right) I(a \le x_i \le \theta) = \underbrace{\left(\frac{1}{\theta - a}\right)^n}_{>0} I(X_{(n)} \le \theta)$

Decreasing function of θ for $\theta \ge X_{(n)}$



$\Rightarrow \hat{\theta}_{MLE} = X_{(n)}$

ii) $F_{X_{(n)}}(y) = P(X_{(n)} \le y) = \{P(X \le y)\}^n = \left[\int_a^y \frac{1}{(\theta - a)} dx\right]^n = \left[\frac{x}{(\theta - a)}\Big|_a^y\right]^n$

$= \left(\frac{y - a}{\theta - a}\right)^n$

$\Rightarrow E[X_{(n)}] = \int_a^\theta y \cdot \frac{d}{dy}\left(\frac{y-a}{\theta - a}\right)^n dy = \int_a^\theta y \cdot \frac{n}{(\theta - a)}\left(\frac{y-a}{\theta - a}\right)^{n-1} dy \quad *$

Let $u = y - a \Rightarrow du = dy$
Lower bound: $u = a - a = 0$
Upper bound: $u = \theta - a$

$\Rightarrow * = \int_0^{\theta - a} (u + a) \cdot \frac{n}{(\theta - a)}\left(\frac{u}{\theta - a}\right)^{n-1} du$

$= n \int_0^{\theta - a} \frac{u^n + a u^{n-1}}{(\theta - a)^n} du = \frac{n}{(\theta - a)^n} \int_0^{\theta - a} u^n + a u^{n-1} du = \frac{n}{(\theta - a)^n}\left[\frac{1}{n+1} u^{n+1} + \frac{a}{n} u^n\right]\Big|_0^{\theta - a}$

$= \frac{n}{(\theta - a)^n}\left[\frac{1}{n+1}(\theta - a)^{n+1} + \frac{a}{n}(\theta - a)^n\right] = \frac{n}{n+1}\frac{(\theta - a)^{n+1}}{(\theta - a)^n} + a\frac{(\theta - a)^n}{(\theta - a)^n}$

$= \frac{n}{n+1}(\theta - a) + a \ne \theta$, so biased estimator

Cont'd on next pg. →

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4. a) ii) cont'd

$$Var[X_{(n)}] = E[X_{(n)}^2] - E[X_{(n)}]^2$$

where $E[X_{(n)}^2] = \int_a^\theta y^2 \frac{d}{dy}\left(\frac{y-a}{\theta-a}\right)^n dy = \int_a^\theta y^2 \cdot \frac{n}{(\theta-a)}\left(\frac{y-a}{\theta-a}\right)^{n-1} dy \quad *$

Let $u = y-a \Rightarrow du = dy$
Lower bound: $u = a-a = 0$
Upper bound: $u = \theta - a$

$$\Rightarrow * = \int_0^{\theta-a} (u+a)^2 \cdot \frac{n}{(\theta-a)}\left(\frac{u}{\theta-a}\right)^{n-1} du$$

$$= n\int_0^{\theta-a} \frac{(u^2+2au+a^2)(u^{n-1})}{(\theta-a)^n} du = n\int_0^{\theta-a} \frac{(u^{n+1}+2au^n+a^2u^{n-1})}{(\theta-a)^n} du$$

$$= \frac{n}{(\theta-a)^n}\left[\frac{1}{n+2}u^{n+2} + \frac{2a}{n+1}u^{n+1} + \frac{a^2}{n}u^n\right]\Bigg|_0^{\theta-a}$$

$$= \frac{n}{(\theta-a)^n}\left[\frac{1}{n+2}(\theta-a)^{n+2} + \frac{2a}{n+1}(\theta-a)^{n+1} + \frac{a^2}{n}(\theta-a)^n\right]$$

$$= \frac{n}{n+2}(\theta-a)^2 + \frac{2an}{n+1}(\theta-a) + a^2$$

Then, $Var[X_{(n)}] = \frac{n}{n+2}(\theta-a)^2 + \frac{2an}{n+1}(\theta-a) + a^2 - \left[\frac{n}{n+1}(\theta-a)+a\right]^2$

$$= \frac{n}{n+2}(\theta-a)^2 + \cancel{\frac{2an}{n+1}(\theta-a)} + \cancel{a^2} - \frac{n^2}{(n+1)^2}(\theta-a)^2 - \cancel{\frac{2an}{n+1}(\theta-a)} - \cancel{a^2}$$

$$= \frac{n}{n+2}(\theta-a)^2 - \left(\frac{n}{n+1}\right)^2(\theta-a)^2$$

Since $\lim_{n\to\infty} E[X_{(n)}] = \lim_{n\to\infty}\left(\frac{n}{n+1}(\theta-a)+a\right) = \theta - a + a = \theta$

&

$\lim_{n\to\infty} Var[X_{(n)}] = \lim_{n\to\infty}\left(\frac{n}{n+2}(\theta-a)^2 - \left(\frac{n}{n+1}\right)^2(\theta-a)^2\right) = (\theta-a)^2 - (\theta-a)^2 = 0$

$\Rightarrow \hat{\theta} = X_{(n)}$ is a consistent estimator of θ.

HOLY CRAP!

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4 b) Find an unbiased estimator of θ as a function of $X_{(n)}$.

Comment on why the CRLB fails in this situation.

$$E[X_{(n)}] = \frac{n}{n+1}(\theta - a) + a = \left(\frac{n}{n+1}\right)\theta - \frac{na}{n+1} + a = \left(\frac{n}{n+1}\right)\theta \; \frac{-\cancel{na} + \cancel{na} + a}{n+1}$$

$$= \frac{n\theta - a}{n+1}$$

$$\Rightarrow E\left[(n+1)X_{(n)}\right] = n\theta - a \quad \Rightarrow \quad E\left[(n+1)X_{(n)} + a\right] = n\theta$$

$$\Rightarrow E\left[\frac{(n+1)}{n}X_{(n)} + \frac{a}{n}\right] = \theta$$

$$\Rightarrow E\left[\frac{(n+1)X_{(n)} + a}{n}\right] = \theta \quad \Rightarrow \quad \frac{(n+1)X_{(n)} + a}{n}$$ is an unbiased estimator of θ.

The CRLB fails in this situation because the support of $f(x|\theta)$ depends on the parameter θ (i.e., $f(x|\theta) = \frac{1}{\theta - a}$, $0 \le a < X < \theta < \infty$). This is because $f(x|\theta)$ must be such that the order of integration of $f(x|\theta)$ w.r.t. x and differentiation w.r.t. θ can be interchanged.

This is a violation, specifically, of one of the three regularity conditions of the CRLB (see wiki: "Cramer Lower Bound" for info).

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4. $X_i \overset{iid}{\sim} \text{Unif}(a, \theta)$, a known, θ unknown

$i = 1, \ldots, n$ iid

c) Find the UMP size α test for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$ w/ clear specification

Let $f(x) = \dfrac{1}{\theta - a}$, $a < x < \theta$. $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$

$R = \left\{ x : \dfrac{f(x|\theta_1)}{f(x|\theta_0)} > c \right\}$ where $\theta_1 > \theta_0$. Then,

$$\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{(\theta_1 - a)^{-n} I(a < x_{(n)} < \theta_1)}{(\theta_0 - a)^{-n} I(a < x_{(n)} < \theta_0)} = \begin{cases} \left(\dfrac{\theta_0 - a}{\theta_1 - a}\right)^n, & a < x_{(n)} < \theta_0 \\ \infty, & \theta_0 < x_{(n)} < \theta_1 \end{cases}$$

which is a non-decreasing function of $x_{(n)}$.

$\Leftrightarrow R^* = \{ x : x_{(n)} > c^* \}$. Then, $\alpha = P(x_{(n)} > c^* | \theta = \theta_0)$

$= 1 - F_{x_{(n)}}(c^* | \theta = \theta_0) = 1 - \left(\dfrac{c^* - a}{\theta_0 - a}\right)^n \Rightarrow 1 - \alpha = \left(\dfrac{c^* - a}{\theta_0 - a}\right)^n$

$\Rightarrow (1-\alpha)^{1/n} = \dfrac{c^* - a}{\theta_0 - a} \Rightarrow (1-\alpha)^{1/n}(\theta_0 - a) = c^* - a$

$\Rightarrow (1-\alpha)^{1/n}(\theta_0 - a) + a = c^*$

where $R = \{ x : x_{(n)} > (1-\alpha)^{1/n}(\theta_0 - a) + a \}$

Note: This almost exact problem for $f(x|\theta) = \frac{1}{\theta}$ is in 2015 final exam from Dr. Lin's class.

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4 d) Show that asymptotically, as $n \to \infty$, the rejection region in c) is independent of α.

Give an intuitive interpretation of this phenomenon.

From c), have $R = \{x : X_{(n)} > \underbrace{(1-\alpha)^{1/n}}_{\text{will go to 1 as } 1/n \to 0} (\theta_0 - a) + a\}$

Then, $\lim_{n \to \infty} (R) = \{x : X_{(n)} > \theta_0\}$, which doesn't depend on α.

I'm not sure this makes sense to me.

I thought that frequentists tend to fix α & as $n \to \infty$, Type II error $\to 0$ and hence power $\to 1$.

In contrast, what we are seeing here is that, as $n \to \infty$, Type I error is non-existent. Now, whether it limits to 0 can't be determined due to n in the exponent.

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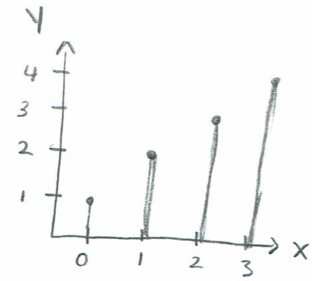
1. Given $P(X=x) = (x+1)p^2 q^x$, $x = 0,1,2,\ldots$

$0 < p < 1$, prob of success

$q = 1-p$



a) Find joint density of X and Y.

$Y \mid X \sim Unif(0, x+1) \equiv \frac{1}{(x+1)} I(0 < y < x+1)$

Then, $f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x)$

$$= \cancel{(x+1)} p^2 q^x \frac{1}{\cancel{(x+1)}} I(0<y<x+1)$$

$$= \boxed{p^2 q^x I(0<y<x+1) \text{ for } \{(x,y) : x = \{0,1,2,\ldots\} \text{ and } 0<y<x+1\}}$$

b) Find $f_Y(y)$, the density of y.

$f_Y(y) = \sum_{x \in A(y)} f_{X,Y}(x,y) = \sum_{x \in A(y)} p^2 q^x I(0<y<x+1)$

where $A(y) = \{\lfloor y \rfloor, \lfloor y \rfloor + 1, \ldots\}$

The $f_Y(y) = \sum_{x=\lfloor y \rfloor}^{\infty} p^2 q^x I(0<y<x+1)$

(Note: Remember formula for geometric series: $\sum_{n=M}^{\infty} cr^n = \frac{cr^M}{1-r}$)

$$= p^2 \sum_{x=\lfloor y \rfloor}^{\infty} q^x I(0<y<x+1) = \frac{p^2 q^{\lfloor y \rfloor}}{\underbrace{1-q}_{p}} = \boxed{p q^{\lfloor y \rfloor}, \; 0 < y < \infty}$$

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1 c) Find $E[Y]$

Take $E[E[\underbrace{Y|X}_{\sim Unif(0,x+1)}]] = E\left[\frac{X+1}{2}\right] = \frac{1}{2}\underbrace{E(X)}_{\sim Nbinom(2,p)} + \frac{1}{2} = \cancel{\frac{1}{2}}\left[\frac{\cancel{2}(1-p)}{p}\right] + \frac{1}{2}$

$= \boxed{q/p + 1/2}$

d) Find $Cov(X,Y)$

Use double expectation formula for covariance

$Cov(X,Y) = E[\underbrace{Cov(X,Y|X)}_{=0}] + Cov[E(Y|X), E(X|X)]$

$= \cancel{E[0]}^{0} + Cov\left[\frac{X+1}{2}, X\right] = \frac{1}{2}\underbrace{Var(X)}_{\sim Nbinom(2,p)} = \frac{1}{2}\left[\frac{2q}{p^2}\right] = \boxed{\frac{q}{p^2}}$

e) Define $T = 2Y - X$. Find $Cov(T,X)$

$Cov(T,X) = Cov(2Y-X, X) = 2\,Cov(Y,X) - Cov(X,X)$

$= 2\left(\frac{q}{p^2}\right) - Var(X) = \frac{2q}{p^2} - \frac{2q}{p^2} = 0.$

f) Are T and X independent?

Hint: On these types of problems, they will try to trick you into saying that $Cov(T,X) = 0 \Rightarrow T \perp\!\!\!\perp X$. However, this is NOT true unless T and X are jointly normal.

Take $P(2Y - X = 2Y \mid X = 0) = 1 \neq P(2Y - X = 2Y)$.

Thus, $T = 2Y - X$ is not independent from X.

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2. $X_1, \ldots, X_n \sim N(\mu, 1)$

Define $\theta = P(X>0)$. Use $\Phi(t)$ to denote CDF of $N(0,1)$ evaluated @ t.

a) Express $P(X>0)$ as a function of μ.

$$P(X>0) = 1 - P(X \le 0) = 1 - P\Big(\underbrace{\frac{X-\mu}{1}}_{\sim N(0,1)} \le -\mu\Big) = 1 - \Phi(-\mu) = \boxed{\Phi(\mu)}$$

b) Find an unbiased estimator of $P(X>0)$

Want an unbiased estimator of $\theta = P(X>0)$ (a.k.a. $E[\overset{\text{unbiased}}{\text{estimator}}] = \theta$).

Since the thing we want to estimate is a probability, this is a hint that the estimator is likely to be an indicator function.

Take $E[I(X>0)] = P(X>0)$. Thus, $I(X>0)$ is an unbiased estimator of $\theta = P(X>0)$.

c) Find the MLE of $P(X>0)$.

First, find MLE of μ.

$$L(\mu|x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^n (x_i-\mu)^2/2} \Rightarrow \ell(\mu|x) = \log\left[\left(\frac{1}{\sqrt{2\pi}}\right)^n\right] - \sum_{i=1}^n (x_i-\mu)^2/2$$

$$\Rightarrow \frac{\partial \ell}{\partial \mu} = -\cancel{2}\sum_{i=1}^n (x_i-\mu)/\cancel{2} \overset{\text{set}}{=} 0 \Rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{X}$$

Since $\frac{\partial^2 \ell}{\partial \mu^2} = -n < 0 \Rightarrow \hat{\mu}$ occurs @ a global max.

Then, since $P(X>0) = 1 - \Phi(-\mu)$, as in a), have $\hat{\theta} = \widehat{P(X>0)} = 1 - \Phi(-\hat{\mu})$

$= 1 - \Phi(-\bar{x}) = \Phi(\bar{x})$ by the invariance property of MLE.

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2 d) Find the Cramer-Rao Lower bound on the variance of unbiased estimators of $P(X>0)$.

To calculate CRLB, can use formula:

$$CRLB = \frac{\left\{ \frac{d\tau(\theta)}{d\theta} \right\}^2}{-E\left\{ \frac{\partial^2}{\partial\theta^2} \log f(\underset{\sim}{x}|\theta) \right\}}$$

where $\tau(\theta) = E(w)$ where w is an unbiased estimator of θ.

Here $\tau(\theta) = \theta \Rightarrow \left\{ \frac{d\tau(\theta)}{d\theta} \right\}^2 = (1)^2 = 1$

$$-E\left\{ \frac{\partial^2}{\partial\theta^2} \left(\log\left[\left(\frac{1}{\sqrt{2\pi}}\right)^n\right] - \sum_{i=1}^{n} (x_i-\theta)^2 \right) \right\} = -E(-n) = n$$

Thus, $\boxed{CRLB = 1/n}$

2e) Find the UMVUE of $P(X>0)$.

First, show that $\sum_{i=1}^{n} X_i$ is a CSS for μ.

Have $f(\underset{\sim}{x},\mu) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n}(x_i-\mu)^2/2} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n}(x_i^2 - 2\mu x_i + \mu^2)/2}$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n} x_i^2/2}}_{h(x)} \cdot e^{\sum_{i=1}^{n} \underbrace{\mu}_{w(\mu)} \overbrace{x_i}^{t(x)}} \cdot \underbrace{\underbrace{e^{-\sum_{i=1}^{n}\mu^2/2}}_{e^{-n\mu^2/2}}}_{c(\mu)}$$

where $\mu \in (-\infty, \infty)$, an open set in $\mathbb{R}^1$.

$\Rightarrow \sum_{i=1}^{n} X_i$ is a CSS for $\mu \Rightarrow \overline{X}$ is also a CSS for μ.

Since $\overline{X}$ is a CSS for μ and since $\Phi(\overline{x})$ is the MLE of $\Phi(\mu)$, then $\Phi(\overline{X})$ is the UMVUE of $E(\Phi(\overline{X})) = \Phi(\mu) = P(X>0)$.

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3. Given $X_1 \in \{a_1, a_2, a_3, a_4\}$

	a_1	a_2	a_3	a_4
θ_1	0.3	0.4	0.1	0.2
θ_2	0.4	0.1	0.2	0.3
θ_3	0.2	0.1	0.5	0.2

Note: No summation in likelihood since given single obs, X.

a) Find the MLE of θ under different values of X

Given one observation, X, need to find the value θ_i at which the global max occurs.

$L(\theta|x) = P(a_i|\theta)$ for $i \in \{1,2,3\}$

From table, can see that
$$\hat{\theta} = \begin{cases} \theta_2, & x = a_1 \\ \theta_1, & x = a_2 \\ \theta_3, & x = a_3 \\ \theta_2, & x = a_4 \end{cases} = \boxed{\begin{cases} \theta_2, & x = a_1, a_4 \\ \theta_1, & x = a_2 \\ \theta_3, & x = a_3 \end{cases}}$$

b) Derive the critical region of the LRT for $H_0: \theta = \theta_1$ vs. $H_1: \theta \neq \theta_1$ with type I error prob $\alpha = 0.1$ and $\Theta = \{\theta_1, \theta_2, \theta_3\}$

$$\lambda(x) = \frac{\sup_{\theta \in \Theta_0} L(\theta|x)}{\sup_{\theta \in \Theta} L(\theta|x)} = \frac{\sup_{\theta \in \Theta_0} P(a_i|\theta)}{\sup_{\theta \in \Theta} P(a_i|\theta)}$$

$$= \begin{cases} \frac{P(a_1|\theta_1)}{P(a_1|\theta_2)} = \frac{0.3}{0.4} = \frac{3}{4}, & x = a_1 \\ \frac{P(a_2|\theta_1)}{P(a_2|\theta_1)} = \frac{0.4}{0.4} = 1, & x = a_2 \\ \frac{P(a_3|\theta_1)}{P(a_3|\theta_3)} = \frac{0.1}{0.5} = \frac{1}{5}, & x = a_3 \\ \frac{P(a_4|\theta_1)}{P(a_4|\theta_2)} = \frac{0.2}{0.3} = \frac{2}{3}, & x = a_4 \end{cases}$$

Then, the critical region $R = \{x : \lambda(x) \le c\}$ for $c \in [0,1]$.

$\Rightarrow \underset{0.1}{\alpha} = \sup_{\theta \in \Theta_0} P(\lambda(x) \le c)$

$= P(\lambda(x) \le c \mid \theta = \theta_1)$

Then, $P(a_3|\theta_1) = 0.1 = P(\lambda(x) \le 1/5)$

$\Rightarrow \boxed{R = \{x : \lambda(x) \le 1/5\}}$

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3 c) Give the test function of the LRT in b) in explicit form. Explain explicitly how one would apply the testing procedure given a single obs. X.

Have $\lambda(x) = \dfrac{\sup_{\theta \in \Theta_0} L(\theta|X)}{\sup_{\theta \in \Theta} L(\theta|X)}$ is the likelihood ratio test for testing

$H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_0^c$. Let $\hat{\theta}_0$ denote the restricted MLE over Θ_0 and let $\hat{\theta}$ denote the unrestricted MLE over $\Theta = \Theta_0 \cup \Theta_0^c$.

Then, the LRT statistic is:

rejection region

$\lambda(x) = \dfrac{L(\theta_0|x)}{L(\hat{\theta}_{MLE}|x)}$. Then, $R = \{x : \lambda(x) \le c\}$ for $c \in [0,1]$

$\Leftrightarrow R^* = \{x : \hat{\theta}_{MLE} \ge c^* \text{ or } \hat{\theta}_{MLE} \le c^*\}$ where $\hat{\theta} \ge c^*$ or $\hat{\theta} \le c^*$ follows the direction of H_1.

No clue if this is what they want.

The question isn't very "explicit." -pun intended

3 d) Find the UMP test for testing $H_0: \theta = \theta_1$ vs. $H_1: \theta = \theta_2$ w/ $\alpha = 0.1$

$$R = \left\{ x \in \{a_1, a_2, a_3, a_4\} : \frac{f(x|\theta_2)}{f(x|\theta_1)} > k \right\}$$

The ratios of pmfs give:

$$\frac{f(a_1|\theta_2)}{f(a_1|\theta_1)} = \frac{0.4}{0.3} = \frac{4}{3}, \qquad \frac{f(a_2|\theta_2)}{f(a_2|\theta_1)} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$\frac{f(a_3|\theta_2)}{f(a_3|\theta_1)} = \frac{0.2}{0.1} = 2 \qquad \frac{f(a_4|\theta_2)}{f(a_4|\theta_1)} = \frac{0.3}{0.2} = \frac{3}{2}$$

If we choose $\frac{1}{4} < k < 2$, the Neyman-Pearson Lemma says that the test rejects H_0 if $x = a_3$ is the UMP level $\alpha = P(X = a_3 | \theta_1) = 0.1$ test.

3e) Comment on whether the UMP test for the hypothesis in d) is also the UMP test for the hypothesis in b). If you think it is, provide the rationale. If you think it is not, derive the UMP test for the hypothesis in b).

No, we cannot compare the UMP test in 3d) to the UMP test in 3b) because 3b) does not have a UMP test. For 3b), $H_0: \theta = \theta_1$ vs. $H_1: \theta \neq \theta_1$. Simply put, a UMP test does not exist for testing as given in 3b) because critical regions turn out to be different for $\theta > \theta_1$ and $\theta < \theta_1$. This means, there are only UMP tests for one-sided hypotheses in which we can use the N-P lemma.

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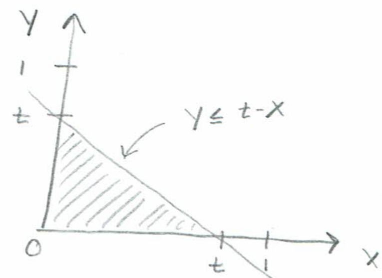
1. Given $X \sim \text{Unif}(0,1)$, $Y \sim \exp(1)$ where $X \perp\!\!\!\perp Y$.
Define $T = X + Y$.

a) Given a constant $t \in (0,1)$, derive an explicit expression for $P(T \le t)$.

$$P(T \le t) = P(X + Y \le t) = P(Y \le t - X)$$



$$= \int_{x=0}^{t} \int_{y=0}^{t-x} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{x=0}^{t} \int_{y=0}^{t-x} e^{-y}\, dy\, dx \qquad \left(\begin{array}{l}\text{since } X \perp\!\!\!\perp Y\text{, can write joint pdf} \\ \text{as the product of the pdf of } X \text{ \& the pdf of } Y\end{array}\right)$$

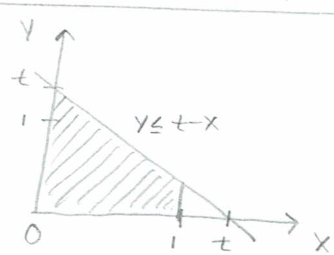
$$= \int_{x=0}^{t} -e^{-y} \Big|_0^{t-x} dx = \int_{x=0}^{t} \left(-e^{-(t-x)} + 1\right) dx = -e^{-(t-x)} + x \Big|_0^{t}$$

$$= \left(-e^{-(t-t)} + t\right) - \left(-e^{-(t-0)} + 0\right) = \boxed{-1 + t + e^{-t}, \quad t \in (0,1)}$$

✓

b) Given a constant $t \in (1,\infty)$, derive an explicit expression for $P(T \le t)$.

$$P(T \le t) = P(X + Y \le t) = P(Y \le t - X)$$



$$= \int_{x=0}^{1} \int_{y=0}^{t-x} e^{-y}\, dy\, dx$$

$$= \int_{x=0}^{1} \left(-e^{-(t-x)} + 1\right) dx$$

$$= -e^{-(t-x)} + x \Big|_0^{1} = \left(-e^{-(t-1)} + 1\right) - \left(-e^{-(t-0)} + 0\right) = \boxed{-e^{-t+1} + 1 + e^{-t}, \quad t \in (1,\infty)}$$

✓

next pg. →

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1. c) Find $E[T]$, $Var[T]$, and $Corr(X,T)$

Find $E[T] = E[X+Y] = E[X] + E[Y] = 1/2 + 1 = \boxed{1.5} = \boxed{3/2}$ ✓

Find $Var[T] = Var[X+Y] = Var[X] + Var[Y] + \underbrace{2(0)}_{X \perp\!\!\!\perp Y} = \frac{1}{12} + 1 = \boxed{\frac{13}{12}}$ ✓

Find $Corr(X,T)$ (Hard method) $= \dfrac{Cov(X,T)}{\sqrt{Var(X)}\sqrt{Var(T)}} = \dfrac{E[X \cdot T] - E[X]E[T]}{\sqrt{Var(X)}\sqrt{Var(T)}}$ $*$

Where $E[X \cdot T] = E[X(X+Y)] = E[X^2 + XY] = E[X^2] + E[XY]$

$= \underbrace{E[X^2] + E[X]E[Y]}_{\text{Since } X \perp\!\!\!\perp Y} = \underbrace{Var[X] + E[X]^2}_{E[X^2]} + E[X]E[Y] = \frac{1}{12} + \frac{1}{4} + \frac{1}{2}(1)$

$= \frac{1}{12} + \frac{3}{12} + \frac{6}{12} = \frac{10}{12} \div \frac{2}{2} = \frac{5}{6}$

Then, $* = \dfrac{\frac{5}{6} - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{\sqrt{1/12}\sqrt{13/12}} = \dfrac{\frac{10}{12} - \frac{9}{12}}{\sqrt{13/12}} = \dfrac{\frac{1}{12}}{\frac{\sqrt{13}}{12}} = \frac{1}{\cancel{12}} \cdot \frac{\cancel{12}}{\sqrt{13}} = \boxed{\frac{1}{\sqrt{13}}}$

Note: I'm an idiot; would have been much easier to find $Corr(X,T)$ using

Find $Corr(X,T)$ (Easy Method) $= \dfrac{Cov(X,T)}{\sqrt{Var(X)}\sqrt{Var(T)}} = \dfrac{Cov(X,X+Y)}{\sqrt{Var(X)}\sqrt{Var(T)}} = \dfrac{\overbrace{Cov(X,X)}^{Var(X)} + \overbrace{Cov(X,Y)}^{=0 \text{ by independence}}}{\sqrt{Var(X)}\sqrt{Var(T)}}$

$= \dfrac{Var(X)}{\sqrt{Var(X)}\sqrt{Var(T)}} = \dfrac{\frac{1}{12}}{\sqrt{1/12}\sqrt{13/12}} = \dfrac{\frac{1}{12}}{\sqrt{13}/12} = \boxed{\frac{1}{\sqrt{13}}}$ ✓

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d) Define $W = 13X - T$. Find $Cov(T, W)$. Are T and W indep? Justify.

$$Cov(T,W) = Cov(X+Y, 12X - Y) = 12\,Cov(X,X) - \cancel{Cov(X,Y)}^{0} + 12\,\cancel{Cov(X,Y)}^{0} - Cov(Y,Y)$$

$$= 12 Var(X) - Var(Y) = 12\left(\tfrac{1}{12}\right) - 1 = 0$$

Note that $Cov(T,W) = 0 \not\Rightarrow T \perp\!\!\!\perp W$ unless T & W are jointly bivariate normal.

Given $\begin{cases} W = 13X - T \\ T = X + Y \end{cases} \Rightarrow \begin{cases} X = \dfrac{W+T}{13} \\ Y = T - X = \dfrac{13T}{13} - \left(\dfrac{W+T}{13}\right) = \dfrac{12T - W}{13} \end{cases}$

$$\Rightarrow |J| = \begin{vmatrix} \frac{\partial x}{\partial W} & \frac{\partial x}{\partial T} \\ \frac{\partial y}{\partial W} & \frac{\partial y}{\partial T} \end{vmatrix} = \begin{vmatrix} 1/13 & 1/13 \\ -1/13 & 12/13 \end{vmatrix} = \left| \frac{12}{13^2} + \frac{1}{13^2} \right| = \frac{1}{13}$$

Then, since $X \perp\!\!\!\perp Y$ can write $f_{X,Y}(x,y) = 1 \cdot e^{-y} = e^{-y}$, $0 \le y < \infty$

$$\Rightarrow f_{X,Y}\left(\frac{W+T}{13}, \frac{12T-W}{13}\right) = \frac{1}{13} e^{\frac{-12T+W}{13}}, \quad \begin{matrix} 0 \le T < \infty \\ -\infty < W \le 12 \end{matrix}$$

with $f_T(t) = \int_W f_{X,Y}\left(\frac{W+T}{13}, \frac{12T-W}{13}\right) dw = \int_{-\infty}^{12} \frac{1}{13} e^{\frac{-12t+W}{13}} dw = \frac{1}{\cancel{13}}\left(\frac{\cancel{13}}{1}\right) e^{\frac{-12t+W}{13}} \Big|_{-\infty}^{12}$

$$= e^{\frac{-12T+12}{13}}, \quad 0 \le T < \infty,$$

$$f_W(w) = \int_0^\infty \frac{1}{13} e^{\frac{-12t+W}{13}} dt = \frac{1}{\cancel{13}} \cdot \left(-\frac{\cancel{13}}{12}\right) e^{\frac{-12t+W}{13}} \Big|_0^\infty$$

$$= -\frac{1}{12} e^{W/13}, \quad -\infty < w \le 12$$

However, since $f_{T,W}(T,W) = \frac{1}{13} e^{\frac{-12T+W}{13}} \ne \left(-\frac{1}{12} e^{12/13}\right) e^{\frac{-12T+W}{13}} = f_T(T) f_W(w)$,

(a.k.a. the joint is not equal to the product of the marginals), then T is not independent of W.

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e) Find constants a and b $\ni E[a+bT-X]=0$ and $Var(a+bT-X)$ is minimized.

$$E[a+bT-X]=0 \Rightarrow E[a]+bE[T]-E[X]=0$$

$$\Rightarrow a+b(3/2)-1/2=0 \Rightarrow a=-\tfrac{3}{2}b+\tfrac{1}{2}$$

$$Var(a+bT-X) = b^2 \underbrace{Var(T)}_{13/12} + \underbrace{Var(X)}_{1/12} - 2b\,\underbrace{Cov(T,X)}$$

$$= Cov(X+Y, X)$$
$$= Cov(X,X) + \cancel{Cov(X,Y)}^{\,0}$$
$$= Var(X) = 1/12$$

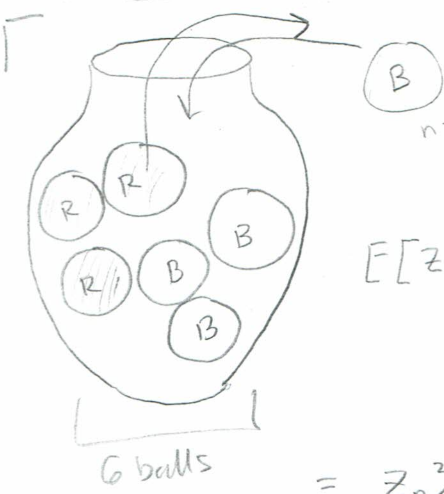
$$= \frac{13}{12}b^2 - \frac{2}{12}b + \frac{1}{12} \overset{set}{=} 0$$

$\Rightarrow 13b^2 - 2b + 1 = 0$. Take 1st derivative to find minimum.

$$26b - 2 = 0 \Rightarrow \boxed{b = 1/13} \checkmark$$

$$\text{Then } a = -\frac{3}{12}\left(\frac{1}{13}\right) + \frac{1}{2} = \frac{-3}{26} + \frac{13}{26} = \frac{10}{26} \div \frac{2}{2} = \boxed{\frac{5}{13}} = a \checkmark$$

f) Not writing this one out; too long.



$$\begin{array}{ccc} & R & B & \\ n\text{ trials} & \left[\begin{matrix} 3 \\ \vdots \\ Z_n \end{matrix}\right. & \left.\begin{matrix} 3 \\ \vdots \\ 6-Z_n \end{matrix}\right] & n\text{ trials} \end{array}$$

$$E[Z_{n+1}|Z_n] = \underbrace{(Z_n-1)}_{\substack{\#\text{ red remaining} \\ \text{if } Z_{n+1} \text{ is red}}}\underbrace{(Z_n/6)}_{P(red)} + \underbrace{(Z_n+1)}_{\substack{\#\text{ red} \\ \text{remaining if} \\ Z_{n+1} \text{ is blue}}}\underbrace{\left(1-\frac{Z_n}{6}\right)}_{P(blue)}$$

$$= \cancel{Z_n^2/6} - Z_n/6 + Z_n - \cancel{Z_n^2/6} + 1 - Z_n/6 \checkmark$$

$$= \frac{-2Z_n}{6} + Z_n + 1 = \frac{4Z_n}{6} + 1 = \frac{2}{3}Z_n + 1$$

$$\text{Then, } E[Z_{n+1}] = E\left[E[Z_{n+1}|Z_n]\right] = E\left[\tfrac{2}{3}Z_n + 1\right] = \tfrac{2}{3}E[Z_n] + 1$$

Then, since $Z_1 = 3 \Rightarrow E[Z_2] = \frac{2}{3}(3)+1 = 3$, $E[Z_3] = \frac{2}{3}(3)+1 = 3$,

$\ldots\ E[Z_n] = \frac{2}{3}(3)+1 = 3.$

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2. Given $f(y|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1}\exp(-y/\beta)$, $y>0, \alpha>0, \beta>0$

where $Y_i \overset{iid}{\sim} \text{gamma}(\alpha,\beta)$

a) Assume α Known. Derive the MLE of $\hat{\beta}$ and show that $\hat{\beta}$ is an unbiased estimator of β.

$$L(\beta|y) = \prod_{i=1}^n \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1}\exp(-y/\beta) = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^n y_i^{\alpha-1}\right)\exp\left(-\sum_{i=1}^n y_i/\beta\right)$$

$$\Rightarrow \ell(\beta|y) = -n\log(\Gamma(\alpha)\beta^\alpha) + \sum_{i=1}^n(\alpha-1)\log(y_i) - \sum_{i=1}^n y_i/\beta$$

$$= -n\log(\Gamma(\alpha)) - n\log(\beta^\alpha) + \sum_{i=1}^n(\alpha-1)\log(y_i) - \sum_{i=1}^n y_i/\beta$$

$$\Rightarrow \frac{\partial \ell}{\partial \beta} = \underbrace{\frac{-n\alpha\beta^{\alpha-1}}{\beta^\alpha}}_{=\frac{-n\alpha}{\beta}} + \sum_{i=1}^n y_i/\beta^2 \overset{set}{=} 0$$

$$\Rightarrow \frac{n\alpha}{\beta} = \sum_{i=1}^n y_i/\beta^2 \Rightarrow \boxed{\hat{\beta} = \frac{1}{n\alpha}\sum_{i=1}^n y_i} = \boxed{\frac{1}{\alpha}\bar{Y}}$$

Know that $\hat{\beta}$ occurs @ a global max b/c:

$$\left.\frac{\partial^2 \ell}{\partial \beta^2}\right|_{\beta=\hat{\beta}} = \left.\frac{n\alpha}{\beta^2} - \frac{2\sum_{i=1}^n y_i}{\beta^3}\right|_{\beta=\hat{\beta}} = \frac{n\alpha}{\left(\frac{1}{n\alpha}\sum_{i=1}^n y_i\right)^2} - \frac{2\sum_{i=1}^n y_i}{\left(\frac{1}{n\alpha}\sum_{i=1}^n y_i\right)^3} = \frac{n^3\alpha^3}{\left(\sum_{i=1}^n y_i\right)^2} - \frac{2n^3\alpha^3}{\left(\sum_{i=1}^n y_i\right)^2}$$

$$= \frac{-n^3\alpha^3}{\left(\sum_{i=1}^n y_i\right)^2} < 0.$$

$$E[\hat{\beta}] = E\left[\frac{1}{n\alpha}\sum_{i=1}^n y_i\right] = \frac{1}{n\alpha}\sum_{i=1}^n E[y_i] = \frac{1}{\cancel{n\alpha}}(\cancel{n\alpha}\beta) = \beta$$

due to independence

Since $E[\hat{\beta}] = \beta \Rightarrow \hat{\beta}$ is an unbiased estimator of β.

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2. b) Derive the MLE of $S(t) = P(Y > t)$, given that $\alpha = 1$.

For $\alpha = 1$, $f(y|\beta) = \frac{1}{\beta} \exp(-y/\beta)$

$$\Rightarrow F(y|\beta) = \int_0^y \frac{1}{\beta} \exp\left(-\frac{t}{\beta}\right) dt = \frac{1}{\beta} \cdot \frac{-\beta}{1} \exp\left(\frac{-t}{\beta}\right)\Big|_0^y = -\exp\left(-\frac{t}{\beta}\right)\Big|_0^y$$

$$= 1 - \exp(-y/\beta)$$

$$\Rightarrow S(t) = P(Y > t) = 1 - P(Y \le t) = 1 - \left(1 - \exp\left(-\frac{t}{\beta}\right)\right) = \exp\left(-\frac{t}{\beta}\right)$$

Then, by the invariance property of the MLE,

have $\hat{S}(t) = \exp\left(-\frac{t}{\hat{\beta}}\right) = \exp\left(\frac{-\alpha t}{\bar{Y}}\right)$

$\parallel \leftarrow \alpha = 1$

$$\boxed{\exp\left(-\frac{t}{\bar{Y}}\right)}, \quad t > 0,\ y > 0$$

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2\. c) Let $V_1 = \begin{cases} 1, & Y_1 > t \\ 0, & \text{else} \end{cases}$. Show that V_1 is an unbiased estimator of $S(t)$.

$E[V_1] = E[I(Y_1 > t)] = P(Y_1 > t) = S(t)$ ✓

Since $E[V_1] = S(t)$, then V_1 is an unbiased estimator of $S(t)$.

2 d) Fix $\alpha = 1$. Show that the conditional pdf of Y_1 given $U = \sum_{i=1}^{n} Y_i$ is

$$f_{Y_1|U}(y_1|u) = \begin{cases} \frac{n-1}{u^{n-1}}(u-y_1)^{n-2}, & 0 < y_1 < u \\ 0, & \text{else} \end{cases}$$

Note: $Y_1, \ldots, Y_n$ $\perp\!\!\!\perp$ and $\sim$ gamma$(1, \beta)$

$$f_{Y_1|U}(y_1|u) = \frac{f_{Y_1,U}(y_1,u)}{f_U(u)} = \frac{f_{Y_1, U-Y_1}(y_1, u-y_1)}{f_U(u)}$$

$$= \frac{\left(\frac{1}{\Gamma(1)\beta^1}\cancel{\exp(-y_1/\beta)}\right)\left(\frac{1}{\Gamma(n-1)\beta^{n-1}}(u-y_1)^{n-2}\cancel{\exp(-(u-y_1)/\beta)}\right)}{\left(\frac{1}{\Gamma(n)\beta^n}u^{n-1}\cancel{\exp(-u/\beta)}\right)}$$

$$= \frac{1}{\Gamma(n-1)\cancel{\beta^n}} \cdot \frac{\Gamma(n)\cancel{\beta^n}}{1} \cdot \frac{1}{u^{n-1}} (u-y_1)^{n-2}$$

✓

$$= \frac{(n-1)!}{(n-2)!} \frac{1}{u^{n-1}} (u-y_1)^{n-2} = \begin{cases} \frac{n-1}{u^{n-1}}(u-y_1)^{n-2}, & 0 < y_1 < u \\ 0, & \text{else} \end{cases}$$

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2e)

i) Show that $E[V_1|u] = \left(1-\frac{t}{u}\right)^{n-1} I(u>t)$

⌈ Know $E[V_1|u] = E[I(Y_1>t)|u] = P(Y_1|u)$

$$= \int_0^u f_{Y_1|u}(y_1|u)\,dy_1 = \left[\int_t^u f_{Y_1|u}(y_1|u)\,dy_1\right] I(Y_1>t)$$

$$= \left[\int_t^u \frac{n-1}{u^{n-1}} (u-y_1)^{n-2}\,dy_1\right] I(Y_1>t)$$

$$= \left[\frac{(\cancel{n-1})}{u^{n-1}} \cdot \frac{-1}{(\cancel{n-1})} (u-y_1)^{n-1} \Big|_t^u\right] I(Y_1>t)$$

$$= \left[\frac{-1}{u^{n-1}} \cancelto{0}{(u-u)^{n-1}} + \frac{1}{u^{n-1}} (u-t)^{n-1}\right] I(Y_1>t)$$

$$= \left(\frac{u-t}{u}\right)^{n-1} I(u>t) = \boxed{\left(1-\frac{t}{u}\right)^{n-1} I(u>t)}$$ ⌋

ii) Now, want to show that $E[V_1|U]$ is the UMVUE

⌈ ① Show $U = \sum_{i=1}^n Y_i$ is a complete & sufficient statistic

② Show $E(V_1|U)$ is unbiased estimator of $S(t)$

③ Then conclude by Lehmann Scheffe that $E(V_1|U)$ is the UMVUE.

① $f(\underset{\sim}{y}|\alpha=1,\beta) = \underbrace{I(y_i>0)}_{h(y)} \underbrace{\frac{1}{\beta}}_{c(\beta)} e^{\overbrace{-\frac{1}{\beta}}^{w(\beta)} \overbrace{\sum_{i=1}^n y_i}^{t(Y)=U}}$ ✓ Done by Dr. Q in class

$\Rightarrow$ U is a sufficient statistic that is also complete since $w(\beta) = -\frac{1}{\beta} \in (-\infty, 0) \in \mathbb{R}$.

Also, $E(E(V_1|U)) = E(V_1) = P(Y_1>t) = S(t) \Rightarrow E(V_1|U)$ is an unbiased estimator of $S(t)$.

Thus, by Lehmann Scheffe, $E(V_1|U)$ is the UMVUE. ⌋

3. Given $X_1, \ldots, X_n \overset{iid}{\sim} F$ with EDF defined as $F_n(x)$.

a) Let $Y_i = I(X_i \le x)$. EDF can be written as $\frac{1}{n}\sum_{i=1}^{n} Y_i$

Show $F_n(x)$ is a consistent estimator of $F(x)$.

$$\lim_{n\to\infty} E\{F_n(x)\} = \lim_{n\to\infty}\left[E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} E(Y_i)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} E(I(X_i \le x))\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} P(X_i \le x)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} F(x)\right] = \lim_{n\to\infty}\left[\frac{1}{\cancel{n}}(\cancel{n}F(x))\right] = F(x).$$

$$\lim_{n\to\infty} Var\{F_n(x)\} = \lim_{n\to\infty}\left[Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)\right] = \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} Var(I(X_i \le x))\right] = \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} P(X_i \le x)(1 - P(X_i \le x))\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^{\cancel{2}}}(\cancel{n}F(x)(1-F(x)))\right] = 0$$

blows up w/ limit

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3.b) Given a specific $x \in A := \{t : 0 < F(t) < 1\}$, describe the asymptotic distribution of $F_n(x)$ when $n \to \infty$, and derive an approximate 95% CI for $F(x)$ when n is large.

From part a) Know that $E\{F_n(x)\} = F(x)$ and $Var\{F_n(x)\} = \frac{F(x)(1-F(x))}{n}$

Then, by CLT, the asymptotic dist. of $F_n(x)$ when $n \to \infty$ is:

$$\boxed{\sqrt{n}\,(F_n(x) - F(x)) \xrightarrow{d} N(0, F(x)(1-F(x)))}$$

Also, want 95% CI $= \left(-z_{\alpha/2} \le \frac{F_n(x) - F(x)}{\sqrt{\frac{F(x)(1-F(x))}{n}}} \le z_{\alpha/2}\right)$

$$\approx \left(-z_{\alpha/2} \le \underbrace{\frac{F_n(x) - F(x)}{\sqrt{\frac{F_n(x)(1-F_n(x))}{n}}}}_{\text{holds by Slutsky's}} \le z_{\alpha/2}\right) = \left(-z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}} \le F_n(x) - F(x) \le z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}}\right)$$

$$= \left(-z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}} - F_n(x) \le -F(x) \le z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}} - F_n(x)\right)$$

$$= \left(F_n(x) - z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}} \le F(x) \le F_n(x) + z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}}\right)$$

$$\Rightarrow \boxed{95\%\ CI(F(x)) = \left(F_n(x) - z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}},\ F_n(x) + z_{\alpha/2}\sqrt{\frac{F_n(x)(1-F_n(x))}{n}}\right)}$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$-quantile of the std normal dist.

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3c) Test $H_0: F(y)=0.5$ vs. $H_1: F(y) \neq 0.5$

Find the LRT statistic and its distribution under H_0.

$H_0: F(y) = 0.5$ vs. $H_1: F(y) \neq 0.5$.

Let $p = F(y)$.

Model each $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Bern}(p)$

$$\Rightarrow L(p|\underset{\sim}{y}) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i} = p^{\sum_{i=1}^{n} y_i}(1-p)^{n-\sum_{i=1}^{n} y_i}$$

$$\Rightarrow \ell(p|\underset{\sim}{y}) = \sum_{i=1}^{n} y_i \log(p) + \left(n - \sum_{i=1}^{n} y_i\right)\log(1-p)$$

$$\Rightarrow \frac{\partial \ell}{\partial p} = \frac{\sum_{i=1}^{n} y_i}{p} - \frac{\left(n - \sum_{i=1}^{n} y_i\right)}{1-p} \overset{set}{=} 0$$

$$\Rightarrow \hat{p} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$$

Not that $\hat{p}$ occurs at a global max since:

$$\left.\frac{\partial^2 \ell}{\partial p^2}\right|_{p=\hat{p}} = \frac{-n\cancel{\bar{y}}}{\bar{y}^{\cancel{2}}} - \frac{n\cancel{(1-\bar{y})}}{(1-\bar{y})^{\cancel{2}}} = \frac{-n}{\bar{y}} - \frac{n}{(1-\bar{y})} < 0$$

$\Rightarrow \hat{p}$ occurs at a global max.

The LRT test statistic is: $\lambda(\underset{\sim}{y}) = \dfrac{\sup_{p=0.5} L(p|y)}{\sup_{0 \le p \le 1} L(p|y)} = \dfrac{L(0.5)}{L(\hat{p})}$

$$= \frac{\prod_{i=1}^{n}(0.5)^{y_i}(0.5)^{1-y_i}}{\prod_{i=1}^{n}(\bar{y})^{y_i}(1-\bar{y})^{1-y_i}} = \frac{(0.5)^n}{\bar{y}^{\sum_{i=1}^{n} y_i}(1-\bar{y})^{n-\sum_{i=1}^{n} y_i}} = \frac{(0.5)^n}{\bar{y}^{n\bar{y}}(1-\bar{y})^{n-n\bar{y}}}$$

$$= \left(\frac{0.5}{\bar{y}^{\bar{y}}(1-\bar{y})^{1-\bar{y}}}\right)^n \Bigg\} \text{ LRT statistic}$$

✓ Checked by Dr. Qaqish

As $n \to \infty$, $-2\log(\lambda(\underset{\sim}{y})) = -2\log\left(\dfrac{0.5}{\bar{y}^{\bar{y}}(1-\bar{y})^{1-\bar{y}}}\right)^n$

$$= -2n\left[\log(0.5) - \bar{y}\log(\bar{y}) - (1-\bar{y})\log(1-\bar{y})\right] \xrightarrow{d} \chi_1^2.$$

✓ Checked by Dr. Q

Ann Marie W., Bios Theory, 2015

3 d) Derive the critical region of the UMP test, given a single observation X_1 and Type I error α.

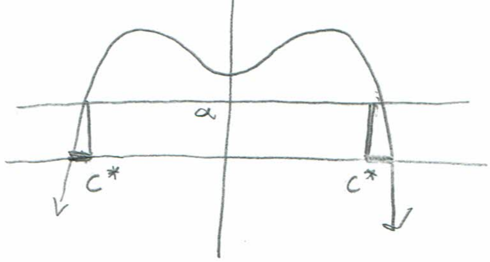
UMP test by N-P: Testing $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$, where the pdf corresponding to θ_0 is $g(x)$ and the pdf corresponding to θ_1 is $h(x)$.

Rejection region: $R = \left\{ x_1 : \dfrac{g(x_1)}{h(x_1)} < c \right\} = \left\{ x_1 : \dfrac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right)}{\frac{1}{\pi(1+x_1)^2}} < c \right\}$

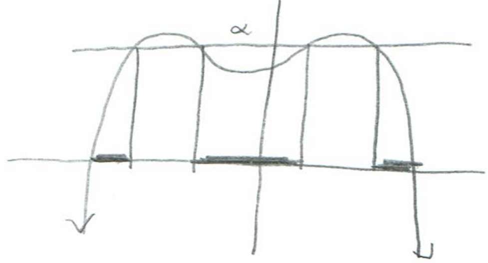
$= \left\{ x_1 : \sqrt{\frac{\pi}{2}} (1+x_1)^2 \exp\left(\frac{-x_1^2}{2}\right) < c \right\}$ for $c \geq 0$, $-\infty < x_1 < \infty$

Two possiblities for critical region:

Case I: Small α → two critical regions



Case II: large α → three critical regions



Since case II only happens for super large values of α (e.g., 0.6), I am going to assume α is reasonably small and write the rejection region as in Case I.

$\Rightarrow R^* = \{ x_1 : |x_1| > c^* \} \Rightarrow \alpha = P(|x_1| > c^* \mid x_1 \sim N(0,1))$

$\Rightarrow c^* = \Phi^{-1}(1 - \alpha/2)$ ✓ Done in class by Dr. Q

$\Rightarrow R = \{ x_1 : |x_1| > \Phi^{-1}(1 - \alpha/2) \}$

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MS Bios Theory, 2017

3 e) Derive the statistical power of the decision rule of the UMP test in d) under H_1.

Note: Under H_1: $X_1 \sim t_1$ and $X_1^2 \sim F_{1,1}$

For power, always "given the alternative."

Have $\beta(\theta) = P\left(|X_1| > \Phi^{-1}(1-\alpha/2) \mid X_1 \sim t_1\right)$

By squaring, we have the benefit of dropping the abs. value on the X_1.

$= P\left(X_1^2 > \{\Phi^{-1}(1-\alpha/2)\}^2 \mid X_1^2 \sim F_{1,1}\right)$

$= 1 - \underbrace{P\left(X_1^2 \le \{\Phi^{-1}(1-\alpha/2)\}^2\right)}_{\text{CDF of } F_{1,1} \text{ dist RV}}$

$= \boxed{1 - F_{1,1}\left(\{\Phi^{-1}(1-\alpha/2)\}^2\right)}$

✓

Done in class by Dr. Q

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6/6/19

MS Exam 2018

1. Given $Y|X \sim N(X,X)$ and $X \sim Unif(0,1)$

a) Find $E[Y]$

$$E[Y] = E[\underbrace{E[Y|X]}_{X}] = E[X] = \frac{1+0}{2} = \boxed{\frac{1}{2}}$$ ✓

Find $Var[Y]$

$$Var[Y] = Var[E[Y|X]] + E[Var[Y|X]] = Var[X] + E[X] = \frac{1}{12} + \frac{1}{2} = \boxed{\frac{7}{12}}$$ ✓

Find $Cov[Y,X]$ **(Method 1)**

now treat as constant

$$Cov[Y,X] = E[XY] - \underbrace{E[X]}_{1/2}\underbrace{E[Y]}_{1/2} = E[E[XY|X]] - 1/4$$

$$= E[X \underbrace{E[Y|X]}_{X}] - 1/4 = \underbrace{E[X^2]}_{} - 1/4 = Var[X] + E[X]^2 - 1/4 = \frac{1}{12} + \cancel{\frac{1}{4}} - \cancel{1/4}$$

$Var(X) = E(X^2) - E(X)^2$
$\Rightarrow E(X^2) = Var(X) + E(X)^2$

$$= \boxed{\frac{1}{12}}$$

Find $Cov[Y,X]$ **(Method 2)**

$$\underbrace{Cov[Y,X]}_{} = E[Cov[X,Y|X]] + Cov[E[X|X], E[Y|X]]$$

Double expectation formula for covariance (a.k.a., law of total covariance)

$$= E[0] + Cov[X,X] = Var[X] = \boxed{\frac{1}{12}}$$ ✓

covariance btwn a constant & a variable = 0

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1. Given $Y|X \sim N(X,X)$ and $X \sim Unif(0,1)$

b) Find $Cov(Y-X, X)$. Is $Y-X \perp\!\!\!\perp X$ ($\perp\!\!\!\perp$ = independent)? Justify your answer.

$$Cov(Y-X,X) = \underbrace{Cov(Y,X)}_{1/12 \text{ in a)}} - Cov(X,X) = 1/2 - Var(X) = \frac{1}{12} - \frac{1}{12} = 0.$$

(Note: You could have also used law of total covariance)

Method 2

Note: $Cov(Y-X,X) = 0 \not\Rightarrow Y-X \perp\!\!\!\perp X$ unless the joint distribution of $Y-X$ and X is bivariate normal.

Know that $Y-X|X \sim N(0,X)$ ✓

Dr. Qaqish said I could have stopped here & that would have been sufficient.

Also know that $f_{Y-X|X}(y-x|x) = \dfrac{f_{X,Y-X}(x,y-x)}{f_X(x)}$

$$\Rightarrow f_{X,Y-X}(x,y-x) = \underbrace{f_{Y-X|X}(y-x|x)}_{\frac{1}{\sqrt{2\pi x}} e^{-y^2/2x}} \cdot \underbrace{f_X(x)}_{1}$$

Since $f_{X,Y-X}(x,y-x) = \dfrac{1}{\sqrt{2\pi x}} e^{-y^2/2x}$ cannot be factored into functions of $y-x$ and x, then $Y-X \not\perp\!\!\!\perp X$.

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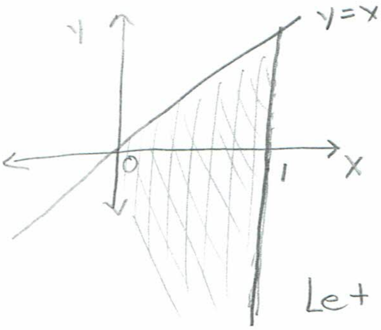
1. c) Find the numerical value of $P(X>Y)$

Method 1 (Easy Way)

$$P(X>Y) = P(Y<X) = E[P(Y<X \mid X)] = E[P(Y-X<0 \mid X)] \quad *$$

Know $Y-X \mid X \sim \underbrace{N(0,X)}_{\text{Centered around } 0}$, so $* = E[1/2] = \boxed{1/2}$ ✓

Method 2 (Super tough Way)



$$\int_0^1 \int_{-\infty}^{x} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^1 \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi x}} e^{\frac{-(y-x)^2}{2x}}\,dy\,dx \quad *$$

NOTE: $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

$\Rightarrow f_{X,Y}(x,y) = \underbrace{f_X(x)}_{=1}\ \underbrace{f_{Y|X}(y|x)}_{=\text{pdf of } N(x,x)}$

Let $u = \dfrac{y-x}{\sqrt{2x}} \Rightarrow du = \dfrac{1}{\sqrt{2x}}\,dy \Rightarrow \sqrt{2x}\,du = dy$

Upper Lim: $u = \dfrac{x-x}{\sqrt{2x}} = 0$

Lower Lim: $u = \lim_{y\to-\infty}\left(\dfrac{y-x}{\sqrt{2x}}\right) = -\infty$

By substitution: $* = \displaystyle\int_0^1 \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi x}} e^{-u^2} (\sqrt{2x}\,du)\,dx$

$$= \int_0^1 \int_{-\infty}^{0} \frac{1}{\sqrt{\pi}} e^{-u^2}\,du\,dx = \int_0^1 \left(\frac{1}{2}\right) \int_{-\infty}^{0} \frac{2}{\sqrt{\pi}} e^{-u^2}\,du\,dx = \int_0^1 \left(-\frac{1}{2}\right) \underbrace{\int_0^{-\infty} \frac{2}{\sqrt{\pi}} e^{-u^2}\,du}_{\operatorname{erf}(-\infty)=-1}\,dx$$

mult. by $\frac{2}{2}$

flipped integral limits

$$= \int_0^1 \left(-\frac{1}{2}\right)(-1)\,dx = \frac{1}{2}x\Big|_0^1 = \frac{1}{2} - 0 = \boxed{1/2}$$ ✓

Holy guacamole!

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d) Suppose that $Z_1, Z_2, \ldots$ is a sequence of iid RV each $\sim N(0,1)$.
For $n=1,2,\cdots$, derive a general expression for:

$$P(Z_{n+1} < Z_n \mid Z_n = \max(Z_1, \ldots, Z_n))$$

Call $Z_n = \max(Z_1, \ldots, Z_n)$ as A.

Then, want $P(Z_{n+1} < Z_n \mid A) = 1 - P(Z_{n+1} > Z_n \mid A)$

$= 1 - \frac{P(A \cap Z_{n+1} > Z_n)}{P(A)}$ (by conditional probability)

Since have $\underbrace{(\underbrace{\text{Any ordering } Z_1, \ldots, Z_{n-1}}_{(n-1)!\text{ possibilities}}, Z_n, Z_{n+1})}_{\text{out of } (n+1)! \text{ total}}$ in $P(A \cap Z_{n+1} > Z_n)$

$\Rightarrow P(A \cap Z_{n+1} > Z_n) = \frac{(n-1)!}{(n+1)!} = \frac{\cancel{(n-1)(n-2)\cdots(1)}}{(n+1)(n)\cancel{(n-1)(n-2)\cdots(1)}} = \frac{1}{(n+1)(n)}$

Also $P(A) = P(Z_n = \max(Z_1, \ldots, Z_n)) = 1/n$

Thus, $1 - \frac{P(A \cap Z_{n+1} > Z_n)}{P(A)} = 1 - \frac{\frac{1}{(n+1)(n)}}{\frac{1}{n}} = 1 - \frac{1}{(n+1)\cancel{(n)}} \cdot \frac{\cancel{n}}{1}$

$= 1 - \frac{1}{n+1} = \frac{n+\cancel{1}-\cancel{1}}{n+1}$

$= \boxed{\frac{n}{n+1}}$ ✓

e) Find the numerical value of $E[|Z_1 - Z_2|]$

Let $W = Z_1 - Z_2 \sim N(0,2)$

Then, $E[|Z_1 - Z_2|] = E[|W|] = \int_{-\infty}^{\infty} |w| \frac{1}{\sqrt{4\pi}} e^{-w^2/4}\, dw$

$= 2\int_0^{\infty} w \frac{1}{\sqrt{4\pi}} e^{-w^2/4}\, dw$ (due to symmetry) $= \frac{1}{\sqrt{\pi}} \int_0^{\infty} w e^{-w^2/4}\, dw$ $*$

Let $u = w^2/4 \Rightarrow du = \frac{w}{2}\, dw \Rightarrow \frac{2}{w}\, du = dw$

By substitution, $* = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \cancel{w} e^{-u} \left(\frac{2}{\cancel{w}}\, du\right) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u}\, du = \frac{2}{\sqrt{\pi}} (-e^{-u}) \Big|_0^{\infty}$

$= \frac{2}{\sqrt{\pi}} (e^{-u}) \Big|_{\infty}^{0} = \frac{2}{\sqrt{\pi}} \left[e^0 - \lim_{u \to \infty} (e^{-u})\right] = \frac{2}{\sqrt{\pi}} [1 - 0] = \boxed{\frac{2}{\sqrt{\pi}}}$ ✓

(I flipped limits w/ negative)

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2. a) Prove that $\overline{Y}$ is the UMVUE of μ.

Know $Y_i \overset{iid}{\sim} Exp(\mu)$ for $i=1,2$

minimal order statistic

Then, $f(\underset{\sim}{y}|\mu) = \prod_{i=1}^{2} \frac{1}{\mu} e^{-y_i/\mu} I(y_i \geq 0) = \underbrace{I(Y_{(1)} \geq 0)}_{h(y)} \underbrace{\left(\frac{1}{\mu}\right)^2}_{c(\mu)} e^{\underbrace{-\frac{1}{\mu}}_{w(\mu)} \underbrace{\sum_{i=1}^{2} y_i}_{t(\mu)}}$

Then, $T(y) = \sum_{i=1}^{2} y_i$ is a minimal sufficient statistic.

$\Rightarrow \frac{1}{2}\sum_{i=1}^{2} y_i = \overline{y}$ is also a minimal sufficient statistic.

Note that $\overline{Y}$ is also complete since $w(\mu) = -\frac{1}{\mu} \in (-\infty, \infty)$ in an open set in $\mathbb{R}$ (in fact, it's all of $\mathbb{R}$!!!)

Thus, $\overline{Y}$ is a CSS.

Also have $E[\overline{Y}] = \frac{1}{2}E[Y_1 + Y_2] = \frac{1}{2}[2\mu] = \mu \Rightarrow \overline{Y}$ is an unbiased estimator of μ.

Thus, by Lehmann Scheffe, $\overline{Y}$ is the unique UMVUE of μ.

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2b) Prove w/out using the formula for the pdf of order statistics that $X_1 \sim Exp(\mu/2)$.

Take $F_{X_1}(x) = P(X_1 \leq x) = 1 - P(X_1 > x) = 1 - P(Y_1 > x, Y_2 > x)$

$= 1 - P(Y_1 > x) \cdot P(Y_2 > x)$ by indep of Y_1, Y_2.

$= 1 - \left[\int_x^\infty \frac{1}{\mu} e^{-t/\mu} du \right]^2 = 1 - \left[-e^{-t/\mu} \Big|_x^\infty \right]^2 = 1 - \left[e^{-x/\mu} \right]^2$

$= 1 - e^{-2x/\mu}, \quad x \geq 0.$

✓

Then, $f_{X_1}(x) = \frac{d}{dx}\left[1 - e^{-2x/\mu} \right] = \underbrace{\frac{2}{\mu} e^{-2x/\mu}}_{\text{pdf of } exp(\mu/2)}, \quad x \geq 0$

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2c) Show that $E[R]=\mu$ and find $Var(R)$.

(Method 1) (Harder Method)

Recall the memoryless property of the exponential.

Namely, $E[Y_1 \mid Y_1 > c] = c + E[Y_1] = c+\mu$

$$E[Y_1 - c \mid Y_1 > c] = E[Y_1] = \mu$$

$$E(R) = E[R \mid Y_1 > Y_2]\underbrace{P(Y_1>Y_2)}_{=1/2} + E[R \mid Y_2 > Y_1]\cdot \underbrace{P(Y_2>Y_1)}_{=1/2}$$

$$= E[Y_1 - Y_2 \mid Y_1 > Y_2](1/2) + E[Y_2 - Y_1 \mid Y_2 > Y_1](1/2)$$

$$= E[E[(Y_1 - Y_2 \mid Y_1 > Y_2) \mid Y_2]](1/2) + E[E[(Y_2 - Y_1 \mid Y_2 > Y_1) \mid Y_1]](1/2)$$

$$= E[E[Y_1]](1/2) + E[E[Y_2]](1/2) \quad \text{(by memoryless property)}$$

$$= E(\mu)(1/2) + E(\mu)(1/2) = \mu/2 + \mu/2 = \boxed{\mu}.$$

To find $Var(R)$

Know $Var(R) = E(R^2) - E(R)^2$

where $E(R)=\mu$ and $E(R^2) = E[(X_2 - X_1)^2] = E[(Y_1-Y_2)^2] = Var(Y_1 - Y_2) + [E(Y_1-Y_2)]^2$

$$= \underbrace{Var(Y_1) + Var(Y_2)}_{\perp\!\!\!\perp \text{ so } 0 \text{ covariance}} + [E(Y_1) - E(Y_2)]^2 = \mu^2 + \mu^2 + [\cancel{\mu - \mu}^{\,0}]^2 = 2\mu^2$$

Then, $Var(R) = 2\mu^2 - \mu^2 = \boxed{\mu^2}$.

(Method 2) (Easier Method)

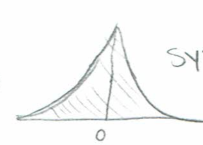
NOTE: The moment generating fcn (MGF) of a RV X is:
$M_X(t) = E[e^{tX}]$, $t \in \mathbb{R}$
whenever the expectation exists.

Know $Y_1, Y_2 \overset{iid}{\sim} Exp(\mu)$

$$M_{Y_1 - Y_2}(t) = E[e^{t(Y_1 - Y_2)}]$$

$$= E[e^{tY_1}]\cdot E[e^{-tY_2}] \quad (\text{since } Y_1 \perp\!\!\!\perp Y_2)$$

MGF of double exponential (aka Laplace)



$$= \underbrace{M_{Y_1}(t)}_{\text{mgf exp}(\mu)} \cdot \underbrace{M_{Y_2}(t)}_{\text{mgf exp}(\mu)} = \left(\frac{1}{1-\mu t}\right)\left(\frac{1}{1+\mu t}\right) = \frac{1}{1-\mu^2 t^2} \Rightarrow Y_1 - Y_2 \sim \text{Double Exp}(0, \mu)$$

$$\Rightarrow R = |Y_1 - Y_2| \sim Exp(\mu)$$

$$\Rightarrow E[R] = \mu \text{ and } Var[R] = \mu^2$$

Thanks to Jean catching my mistake, corrected post-upload.

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2 d) Find $E(X_2)$, $Var(X_2)$, and $Corr(X_1, X_2)$

i) Find $E[X_2] = E[\,|Y_1 - Y_2| + X_1\,] = E\underbrace{|Y_1 - Y_2|}_{\sim Exp(\mu)} + E\underbrace{(X_1)}_{\sim Exp(\mu/2)}$

$= \mu + \mu/2 = \boxed{3\mu/2}$ ✓

ii) Find $Var[X_2] = Var[\,|Y_1 - Y_2| + X_1\,] = Var\underbrace{|Y_1 - Y_2|}_{\sim Exp(\mu)} + Var\underbrace{(X_1)}_{\sim Exp(\mu/2)}$

$= \mu^2 + \left(\frac{\mu}{2}\right)^2 = \boxed{5\mu^2/4}$ ✓

iii) Find $Corr(X_1, X_2) = \dfrac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}}$

First, $Cov(X_1, X_2) = E(X_1 \cdot X_2) - E(X_1)E(X_2)$

$= E(Y_1 Y_2) - E(X_1)E(X_2)$

$= \underbrace{E(Y_1)}_{\mu} \cdot \underbrace{E(Y_2)}_{\mu} - \underbrace{E(X_1)}_{\mu/2} \cdot \underbrace{E(X_2)}_{3\mu/2} = \mu^2/4$

Then, $Corr(X_1, X_2) = \dfrac{\mu^2/4}{\sqrt{\mu^2/4 \cdot 5\mu^2/4}} = \dfrac{\mu^2/4}{\sqrt{5\mu^4/16}} = \dfrac{1}{\sqrt{5}} = \boxed{\frac{\sqrt{5}}{5}}$ ✓

2e) Find $E[R|\bar{Y}]$

Since $\bar{Y}$ is a sufficient statistic (part a), know that $E[R|\bar{Y}] = g(\bar{Y})$ for some function g. Then, since $E[\underline{E[R|\bar{Y}]}] = E[R] = \mu \Rightarrow E[g(\bar{Y})] = \mu$.

By uniqueness of the UMVUE, $g(\bar{Y}) = \bar{Y}$. $g(\bar{Y})$

Thus, $E[R|\bar{Y}] = \boxed{\bar{Y}}$ ✓

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3. a) Derive the MLE of p and its large sample distribution.

tilda denotes a vector

Given $y_i \sim Bern(p)$

$$L(p \mid \underset{\sim}{y}) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i} = p^{\sum_{i=1}^{n} y_i}(1-p)^{n-\sum_{i=1}^{n} y_i}$$

$$\ell(p \mid \underset{\sim}{y}) = \sum_{i=1}^{n} y_i \log(p) + \left(n - \sum_{i=1}^{n} y_i\right)\log(1-p)$$

$$\frac{\partial \ell}{\partial p} = \frac{\sum_{i=1}^{n} y_i}{p} - \frac{\left(n-\sum_{i=1}^{n} y_i\right)}{1-p} = 0 \Rightarrow \frac{1-p}{p} = \frac{\left(n - \sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n} y_i}$$

$$\Rightarrow \frac{1}{p} \cancel{-1} = \frac{n}{\sum_{i=1}^{n} y_i} \cancel{-1} \Rightarrow \hat{p} = \sum_{i=1}^{n} y_i / n = \bar{y}$$

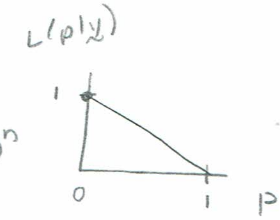
Check that $\hat{p}$ occurs at a global max.

$$\left.\frac{\partial^2 \ell}{\partial p^2}\right|_{p=\hat{p}} = \left. -\frac{\sum_{i=1}^{n} y_i}{p^2} - \frac{\left(n - \sum_{i=1}^{n} y_i\right)}{(1-p)^2}\right|_{p=\hat{p}} = \frac{-n\bar{y}}{\bar{y}^2} - \frac{(n-n\bar{y})}{(1-\bar{y})^2}$$

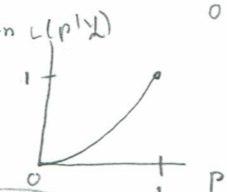
$$= \frac{-n}{\bar{y}} - \frac{n\cancel{(1-\bar{y})}}{(1-\bar{y})^{\cancel{2}}} = \frac{-n(1-\bar{y})}{\bar{y}(1-\bar{y})} - \frac{n\bar{y}}{\bar{y}(1-\bar{y})} = \frac{-n+\cancel{n\bar{y}}-\cancel{n\bar{y}}}{\bar{y}(1-\bar{y})}$$

$$= \frac{-n}{\bar{y}(1-\bar{y})} < 0 \quad \text{as long as } \bar{y} \neq 0, 1$$

If $\bar{y} = 0 \Rightarrow \sum_{i=1}^{n} y_i = 0 \Rightarrow L(p \mid \underset{\sim}{y}) = p^0 (1-p)^{n-0} = (1-p)^n$



If $\bar{y} = 1 \Rightarrow \sum_{i=1}^{n} y_i = n \Rightarrow L(p \mid \underset{\sim}{y}) = p^n \cancel{(1-p)}^{n-n}$
$= p^n$



$$\Rightarrow \hat{p}_{MLE} = \begin{cases} 0, & \text{if } \bar{y} = 0 \\ \bar{y}, & \text{if } \bar{y} \neq 0,1 \\ 1, & \text{if } \bar{y} = 1 \end{cases} = \boxed{\bar{y} \text{ for all } y \in \{0,1\}}$$

✓

The large sample dist of $\hat{p}_{MLE} = \bar{y}$ is normal w/ variance $p(1-p)$ because, by CLT,

$$\sqrt{n}(\bar{y}-p) \xrightarrow{d} \boxed{N(0, p(1-p))}$$

← not checked w/ Dr. Qaqish

AMW

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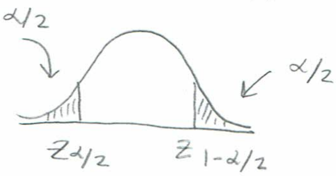
3 b) For a given $\alpha \in (0,1)$, find either an exact or approximate $(1-\alpha)$ CI for p.

Γ Want $(1-\alpha)$ CI $\ni P[T_1 < p < T_2] = 1-\alpha$.

Know $E[\bar{y}] = E\left[\sum Y_i / n\right] = \frac{1}{n} E\left[\sum Y_i\right] = \frac{1}{n}(np) = p$

$Var[\bar{y}] = Var\left[\sum Y_i/n\right] = \frac{1}{n^2} Var\left(\sum Y_i\right) = \frac{1}{n^2} p(1-p) \cdot n = \frac{p(1-p)}{n}$

Then, $1-\alpha = P\left[Z_{\alpha/2} \le \frac{\bar{y} - p}{\sqrt{\frac{p(1-p)}{n}}} \le Z_{1-\alpha/2} \right]$



$= P\left[Z_{\alpha/2} \le \frac{\bar{y} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \le Z_{1-\alpha/2} \right]$ (by Slutsky's)

$= P\left[Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le \bar{y} - p \le Z_{1-\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right]$

$= P\left[Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} - \bar{y} \le -p \le Z_{1-\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} - \bar{y} \right]$

$= P\left[\bar{y} - Z_{1-\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \bar{y} - Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right]$

Not checked w/ Dr. Qaqish

$$\boxed{\Rightarrow (1-\alpha) \text{ CI for } p: \left[\bar{y} - Z_{1-\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \ \bar{y} - Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right]}$$

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3 c) Too long too summarize question here.

Given $p_i = \dfrac{1}{1+\exp(\beta_0-\beta_1 x_i)}$

Know $L(\beta_1|\underset{\sim}{y}) = \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i}$

$$\Rightarrow \ell(\beta_1|\underset{\sim}{y}) = \sum_i y_i\log(p_i) + (1-y_i)\log(1-p_i) = y_i\log\left(\frac{1}{1+\exp(\beta_0-\beta_1 x_i)}\right) + (1-y_i)\log\left(1-\frac{1}{1+\exp(\beta_0-\beta_1 x_i)}\right)$$

$$= \sum_i \cancel{y_i\log(1)}^{\,0} - y_i\log(1+\exp(\beta_0-\beta_1 x_i)) + (1-y_i)\log\left(\frac{\exp(\beta_0-\beta_1 x_i)}{1+\exp(\beta_0-\beta_1 x_i)}\right)$$

$$= \sum_i -y_i\log(1+\exp(\beta_0-\beta_1 x_i)) + (1-y_i)\log(\exp(\beta_0-\beta_1 x_i)) - (1-y_i)\log(1+\exp(\beta_0-\beta_1 x_i))$$

$$= \sum_i \cancel{-y_i\log(1+\exp(\beta_0-\beta_1 x_i))} + (1-y_i)\log(\exp(\beta_0-\beta_1 x_i)) - \log(1+\exp(\beta_0-\beta_1 x_i)) + \cancel{y_i\log(1+\exp(\beta_0-\beta_1 x_i))}$$

$$= \sum_i (1-y_i)(\beta_0-\beta_1 x_i) - \log(1+\exp(\beta_0-\beta_1 x_i))$$

Then, $\dfrac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{n} (1-y_i)(-x_i) - \dfrac{(-x_i)\exp(\beta_0-\beta_1 x_i)}{(1+\exp(\beta_0-\beta_1 x_i))} \overset{set}{=} 0$

$$\Rightarrow \sum_{i=1}^{n} \frac{(1-y_i)(-x_i)(1+\exp(\beta_0-\beta_1 x_i)) + x_i\exp(\beta_0-\beta_1 x_i)}{(1+\exp(\beta_0-\beta_1 x_i))} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{(1-y_i)(-x_i - x_i\exp(\beta_0-\beta_1 x_i)) + x_i\exp(\beta_0-\beta_1 x_i)}{(1+\exp(\beta_0-\beta_1 x_i))} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{-x_i + x_i y_i + x_i y_i\exp(\beta_0-\beta_1 x_i)}{(1+\exp(\beta_0-\beta_1 x_i))} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{x_i(-1+y_i+y_i\exp(\beta_0-\beta_1 x_i))}{(1+\exp(\beta_0-\beta_1 x_i))} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{x_i(-1+y_i(1+\exp(\beta_0-\beta_1 x_i)))}{(1+\exp(\beta_0-\beta_1 x_i))} = 0$$

$$\sum_{i=1}^{n} x_i\left(\frac{-1}{(1+\exp(\beta_0-\beta_1 x_i))} + y_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i(y_i - p_i) = 0 \quad \checkmark$$

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3d) For $H_0: \beta_1 = 0$ vs. $H_1: \beta_1 \neq 0$, find an asymptotic test with Type I error probability α.

Three types of asymptotic tests:

Choose Wald: Under H_0, have $I(\hat{\beta}_1) = Var(U(\hat{\beta}_1)) = Var\left(\sum_{i=1}^{n} x_i(y_i - p_i)\right)$

$$= \sum_{i=1}^{n} x_i^2 \underbrace{Var(y_i)}_{\text{Note: } Var(p_i)=0} = \sum_{i=1}^{n} x_i^2 p_i(1-p_i) \text{ since } y_i \sim Bern(p_i).$$

$$\Rightarrow \underbrace{Z}_{\text{Wald statistic}} = \frac{\hat{\beta}_1 - 0}{\sqrt{\frac{1}{I(\hat{\beta}_1)}}} \sim N(0,1). \text{ Thus, reject if } |z| > z_{1-\alpha/2} = \underbrace{\Phi^{-1}(1-\alpha/2)}_{\substack{\text{quantile function of} \\ \text{standard normal dist.}}}$$

(interestingly, also called the "probit" function.)

3e) Find the constant $K \ni \hat{\theta}$ is the MLE of θ where $\hat{\theta} = K/\hat{\beta}_1$. We were told that trasmission probability $(p) = 0.95$ and that the gametocyte density threshold (x_i) is θ.
Then, we are asked to find an exact or approximate 95% CI for θ.

$$0.95 = \frac{1}{1+\exp(\beta_0 - \beta_1\theta)} \Rightarrow 1+\exp(\beta_0-\beta_1\theta) = \frac{1}{0.95} \Rightarrow \exp(\beta_0-\beta_1\theta) = \frac{1}{0.95} - 1$$

$$\Rightarrow \beta_0 - \beta_1\theta = \log\left(\frac{1}{0.95}-1\right) \Rightarrow -\beta_1\theta = \log\left(\frac{1}{0.95}-1\right) \Rightarrow \theta = -\frac{1}{\beta_1}\log\left(\frac{1}{0.95}-1\right)$$

Since $\hat{\beta}_1$ is the MLE for β_1, by the invariance property,

$$\hat{\theta} = -\frac{1}{\hat{\beta}_1}\log\left(\frac{1}{0.95}-1\right) \Rightarrow \boxed{K = -\log\left(\frac{1}{0.95}-1\right)} \approx \boxed{2.94}$$

3 e) continued

Can use Delta method according to Dr. Q, but there is an easier way.
Consider a $1-\alpha$ CI for β_1 (l, u). Consider a monotonic increasing transformation t. Then for $l < \beta_1 < u \Rightarrow t(l) < t(\beta_1) < t(u)$.

From part d), have that $Var(\hat{\beta}_1) = \frac{1}{I(\hat{\beta}_1)} = \frac{1}{\sum_{i=1}^{n} x_i^2 p_i(1-p_i)}$

Then, $(1-\alpha)$ CI for β_1 is: $\hat{\beta}_1 \pm z_{1-\alpha/2}\sqrt{\frac{1}{I(\hat{\beta}_1)}}$

$$= \left(\hat{\beta}_1 - z_{1-\alpha/2}\sqrt{\frac{1}{I(\hat{\beta}_1)}},\ \hat{\beta}_1 + z_{1-\alpha/2}\sqrt{\frac{1}{I(\hat{\beta}_1)}}\right)$$

Then, for $\theta = \frac{K}{\beta_1}$, have $(1-\alpha)$ CI for θ as: $\boxed{\left(K\left(\hat{\beta}_1 - z_{1-\alpha/2}\sqrt{\frac{1}{I(\hat{\beta}_1)}}\right)^{-1},\ K\left(\hat{\beta}_1 + z_{1-\alpha/2}\sqrt{\frac{1}{I(\hat{\beta}_1)}}\right)^{-1}\right)}$

inverse

inverse