2012, MS-1

Question 1

i) $f(x \mid y=0) = \dfrac{f(x, y=0)}{f(y=0)} = \dfrac{f(x)f(y=0 \mid x)}{f(y=0)}$

$f(x) = \dfrac{e^{-\mu}\mu^x}{x!}$

$f(y \mid x=x) = \dfrac{e^{-x}x^y}{y!}$

$f(x,y) = \dfrac{e^{-\mu}\mu^x}{x!} \cdot \dfrac{e^{-x}x^y}{y!}$ at $y=0$

$= \dfrac{e^{-\mu}\mu^x e^{-x}}{x!} = \dfrac{e^{-(\mu+x)}\mu^x}{x!} = f(x, y=0)$

$\sum_{x=0}^{\infty} f(x,y) \Rightarrow f(y)$

$f(y=0) = e^{-\mu}\sum_{x=0}^{\infty} \dfrac{e^{-x}\mu^x}{x!} = e^{-\mu}\sum_{x=0}^{\infty}\dfrac{(\mu/e)^x}{x!}$

$= e^{-\mu}\left(e^{\mu/e}\right)$

$f(x \mid y=0) = \dfrac{\dfrac{\cancel{e^{-\mu}}e^{-x}\mu^x}{x!}}{\cancel{e^{-\mu}}e^{\mu/e}} = \dfrac{e^{-x}\mu^x}{e^{\mu/e}x!} = \dfrac{e^{-(\mu/e)}(\mu/e)^x}{x!}$ $\quad X \sim \text{Pois}(\mu/e)$

Question 1 | 2012, MS-4

d) Find $E(X \mid y \geq 1)$

$X = 0, 1, 2, \ldots \infty$

$E(X \mid y=0) = \frac{\mu}{e}$

$E(X) = \mu$

Total expectation:

$E(X) = \sum_{i=1}^{n} E(X \mid A_i) \cdot P(A_i)$

$E(X) = E(X \mid y=0) P(y=0) + E(X \mid y \geq 1) P(y \geq 1)$

$$\mu = \underset{\uparrow \text{ c)}}{\frac{\mu}{e}} \underset{\uparrow \, f(y=0)}{(e^{-\mu} e^{\mu/e})} + \underbrace{E(X \mid y \geq 1)}_{\text{unknown}} \underbrace{(1 - e^{-\mu} e^{\mu/e})}_{1 - f(y=0)}$$

$$\frac{\mu - \frac{\mu}{e}(e^{-\mu} e^{\mu/e})}{1 - e^{-\mu} e^{\mu/e}} = E(X \mid y \geq 1)$$

$\boxed{\text{Question 1}}$ $\boxed{\text{2012, MS1}}$

e) $E\left(\frac{Y}{X+1}\right)$ Verify whether this expectation is <1 for all μ

$$= \sum_{y=0}^{\infty}\sum_{x=0}^{\infty} \frac{y}{x+1} \frac{e^{-\mu}\mu^x}{x!} \cdot \frac{e^{-x}x^y}{y!}$$

$$= \sum_y \sum_x \frac{y\, e^{-(\mu+x)}\mu^x x^y}{(x+1)(x!)(y!)}$$

$$= \sum \sum \frac{e^{-(\mu+x)}\mu^x x^y}{\Gamma(x+2)} \cdot \frac{y}{y!} \quad \frac{1}{(y-1)!}$$

$$= \sum \sum \frac{e^{-(\mu+x)}\mu^x x^y}{\Gamma(x+2)(y!)}$$

$$(x+1)(x!) = (x+1)! = \Gamma(x+2)$$

$$E\left(\frac{Y}{X+1}\right) = E\,E\left(\frac{Y}{X+1}\,\middle|\, X\right)$$

$$= E\left(\frac{1}{X+1} E(Y|X)\right)$$

$$= E\left(\frac{X}{X+1}\right)$$

$$= \sum_{x=0}^{\infty} \frac{x}{x+1}\frac{e^{-\mu}\mu^x}{x!} = \sum_{x=0}^{\infty}$$

$$= \frac{x+1}{x+1} - \sum_{x=0}^{\infty}\frac{1}{x+1}\cdot\frac{1}{x!}e^{-\mu}\mu^x = 1 - \sum_{x=0}^{\infty}\frac{e^{-\mu}\mu^x}{(x+1)!}\cdot\frac{\mu^1}{\mu^1}$$

$$= 1 - \frac{1}{\mu}\boxed{\sum_{x=0}^{\infty}\frac{e^{-\mu}\mu^{(x+1)}}{(x+1)!}}$$

$$= 1 - \frac{1}{\mu}e^{\mu}(e^{\mu}-1)$$

$$\frac{x}{x+1}\ (x)(x-1)(x-2)\ldots(1)$$

$$= \frac{x+1-1}{x+1}$$

* Taylor Expansion (?)

ee

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \sum_{n=0}^{\infty} \frac{f'(a)}{n!}(x-a)^n$$

$$\sum_{x+1=0}^{\infty}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$$

$$\frac{1}{\mu}\sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^{x+1}}{(x+1)!} = \frac{e^{-\mu}}{\mu} \sum_{x=0}^{\infty} \frac{\mu^{x+1}}{(x+1)!}$$

$$\sum_{x=0}^{\infty} \frac{\mu^x}{x!} \cdot \frac{\mu^1}{(x+1)}$$

$$\sum_{x=0}^{\infty} \frac{\mu^{x+1}}{(x+1)!} = \underbrace{\frac{\mu}{1} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} \cdots}_{} = e^{\mu} - 1$$

$$\sum_{x=0}^{\infty} \frac{\mu^x}{x!} = 1 + \frac{\mu}{1} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} \cdots = e^{\mu}$$

$$\sum_{x=0}^{\infty} \frac{\mu^{x+1}}{(x+1)!} = \frac{\mu^{(0+1)}}{(0+1)!} + \frac{\mu^{(1+1)}}{(1+1)!} + \frac{\mu^{(2+1)}}{(2+1)!} = \mu + \frac{\mu^2}{2} + \frac{\mu^3}{3}$$

$$\sum_{x=0}^{\infty} \frac{\mu^x}{x!} = e^{\mu} \text{ (known / easy)}$$

Question 2

f) $T = wX + (1-w)Y \qquad w \in (0,1)$

what is the best choice of W to be used as an estimator of μ?

$\Rightarrow$ unbiased. minimize variance

$E(WX + (1-W)Y)$-

$W EX + (1-W) EY = W\mu + (1-W)\mu$

$Var(WX + (1-W)Y) = W^2 Var X + (1-W)^2 Var Y + 2W(1-W) cov(X,Y)$

$\frac{d}{dW} = 2W Var X - 2(1-W) Var Y + 2 cov(X,Y) - 2(2W) cov(X,Y)$

$= \cancel{2}W Var X - \cancel{2} Var Y + \cancel{2}W Var Y + \cancel{2} cov(X,Y) - 2\cancel{4}W cov(X,Y) = 0$

$w(Var X + Var Y - cov(X,Y)) = Var Y - cov(X,Y)$

$$W = \frac{Var Y - cov(X,Y)}{Var X + Var Y - cov(X,Y)}$$

$$= \frac{2\mu - \mu}{\mu + 2\mu - \mu} = \frac{\mu}{2\mu} = \frac{1}{2}$$

$\frac{d^2}{dW^2} = 2 Var X + 2 Var Y - 4 cov(X,Y)$

$2(\mu) + 2(2\mu) - 4(\mu) = 2\mu > 0 \Rightarrow$ so $W = 1/2$ is a minimum.

~~c) Find $E\left(\frac{Y}{X+1}\right) = E(Y)E\left(\frac{1}{X+1}\right)$~~

b) Derive the LRT of

$H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$ w/ β unknown

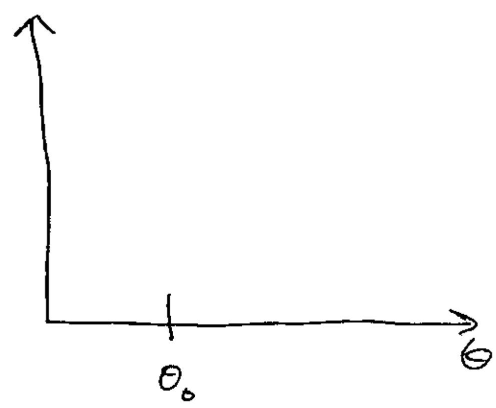
$\Rightarrow$ fix $\beta = X_{(1)} = \hat{\beta}_{MLE}$

$$\lambda(x) = \frac{\sup(\text{null}) \to \theta_0}{\text{Supp overall} \to \hat{\theta} \text{ if } \hat{\theta} > \theta_0}$$

$$= \frac{\theta_0^{-n} e^{-\Sigma(x_i - x_{(1)})/\theta_0}}{\hat{\theta}^{-n} e^{-\Sigma(x_i - x_{(1)})/\hat{\theta}}} \qquad \hat{\theta} = \frac{\Sigma(x_i - x_{(1)})}{n}$$

$$= \left(\frac{\hat{\theta}}{\theta_0}\right)^n e^{-\frac{\Sigma(x_i - x_{(1)})}{\theta_0} + \frac{\Sigma(x_i - x_{(1)})}{\hat{\theta}}}$$

$$= \underbrace{\left(\frac{\hat{\theta}}{\theta_0}\right)^n e^{-\frac{\Sigma(x_i - x_{(1)})}{\theta_0} + n}}_{k}$$



$$\lambda(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \hat{\theta} < \theta_0 \\ k & \text{if } \hat{\theta} > \theta_0 \end{cases}$$

and use $-2\log\lambda(\underset{\sim}{x})$ to approx χ^2_1

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① $X \sim \text{Pois}(\mu)$

$Y|X \sim \text{Pois}(x)$

a) Find $E(y)$ and $\text{var}(y)$

$EY = EEY|X = EX = \mu$ C

$\text{Var}Y = E\,\text{Var}Y|X + \text{Var}\,EY|X$

$= EX + \text{Var}X$

$= \mu + \mu = 2\mu$ C

b) Find $\text{corr}(x,y)$

$\text{corr}(x,y) = \dfrac{\text{cov}(x,y)}{\sqrt{\text{var}x \cdot \text{var}y}}$ ← need ← know

$\text{cov}(x,y) = EXY - EXEY$

$= EE(XY|X) - EXEY$

$= E(X\,EY|X) - EXEY$

$= E(X^2) - EXEY$

$= \{\text{Var}X + (EX)^2\} - EXEY$

$= \{\mu + \mu^2\} - \mu(\mu)$

$= \mu$

$\Rightarrow \text{corr}(x,y) = \dfrac{\mu}{\sqrt{2\mu \cdot \mu}} = \dfrac{\mu}{\mu\sqrt{2}} = 1/\sqrt{2}$ C

c) Find $f(x|y=0)$

$$f(x|y=y) = \frac{f(x,y)}{f(y)} = \frac{f(x)\cdot f(y|x)}{f(y)}$$

ⓐ marginal · conditional = joint

ⓑ ~~integrate~~ / sum x out of joint

b/c discrete

ⓐ $X \sim \text{Pois}(\mu)$ $\quad f(x) = \frac{e^{-\mu}\mu^x}{x!}$

$Y|X \sim \text{Pois}(x)$ $\quad f(y|x) = \frac{e^{-x}x^y}{y!}$

$$\Rightarrow f(x,y) = f(x)\cdot f(y|x)$$

$$= \frac{e^{-\mu}\mu^x}{x!}\cdot\frac{e^{-x}x^y}{y!} = \frac{e^{-(\mu+x)}\mu^x x^y}{x!\,y!} \quad \begin{array}{l} x=0,1,2\ldots \\ y=0,1,2\ldots \end{array}$$

sum x out to get marginal of y

and when $y=0$, $f(x,y=0) = \frac{e^{-(\mu+x)}\mu^x}{x!} = \frac{e^{-\mu}e^{-x}\mu^x}{x!}$

ⓑ

$$f(y) = \sum_{x=0}^{\infty} f(x,y) = \sum_{x=0}^{\infty}\frac{e^{-(\mu+x)}\mu^x x^y}{x!\,y!}$$

finished on other page

c) cont...

$x! = \Gamma(x+1)$

or $\Gamma(w) = (w-1)!$

joint pmf: $p(x)\,p(y|x) = p(x,y)$

$P(X=x) \sim \text{Pois}(\mu) = \dfrac{e^{-\mu}\mu^x}{x!}$

$P(Y=y|X=x) \sim \text{Pois}(x) = \dfrac{e^{-x}x^y}{y!}$

$$P(X=x, Y=y) = \frac{e^{-\mu}\mu^x}{x!} \cdot \frac{e^{-x}x^y}{y!}$$

$\mu > 0$
$x = 0, 1, 2 \ldots$
$y = 0, 1, 2 \ldots$

$$= \frac{e^{-(\mu+x)}\mu^x x^y}{x!\,y!}$$

$$P(Y=y) = \sum_{x=0}^{\infty} \frac{e^{-(\mu+x)}\mu^x x^y}{x!\,y!}$$

$$= \frac{e^{-\mu}}{y!} \sum_{x=0}^{\infty} \frac{e^{-x}\mu^x x^y}{x!}$$

$$= \frac{e^{-\mu}}{y!} \sum_{x=0}^{\infty} \frac{(\mu/e)^x x^y}{x!} \quad \text{if } y=0$$

$x^0 = 1 \quad 0! = 1$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu/e)^x}{x!}$$

$$= e^{-\mu} e^{(\mu/e)} = e^{-\mu + \mu/e}$$

And therefore:

$$P(X=x|Y=y) = \frac{\left\{\dfrac{e^{-(\mu+x)}\mu^x x^y}{x!\,y!}\right\}}{\left\{\dfrac{e^{-\mu}}{y!}\displaystyle\sum_{x=0}^{\infty}\frac{(\mu/e)^x x^y}{x!}\right\}} = \frac{\dfrac{e^{-(\mu+x)}\mu^x}{x!}}{e^{-\mu}e^{\mu/e}} = \frac{\cancel{e^{-\mu}}e^{-x}\mu^x}{x!\,\cancel{e^{-\mu}}e^{\mu/e}} = \frac{e^{-x-\mu/e}\mu^x}{x!} \Bigg\}\ \text{not obviously recognizable}$$

$x = 0, 1, 2 \ldots$

d) Find $E(X \mid Y \geq 1)$ Hint: use c) and what is known about X

strictly posi.

c) $\Rightarrow$ gives us the $P(X \mid y=0)$

$P(X \mid y \geq 1) = 1 - P(X \mid y=0)$ ok. useful

$E(X \mid y \geq 1)$

$= \sum_{X=0}^{\infty} X \cdot P(X \mid y \geq 1) = \sum_{X=0}^{\infty} X \cdot (1 - P(X \mid y=0))$

all possible values of X

$= \sum_{X=0}^{\infty} X \cdot \left(1 - \frac{e^{-X - M/e} \mu^{X}}{X!}\right)$

glucose tolerance $\sim N(6, 9)$ w/ Diabetes

" " $\sim N(4,4)$ w/o Diabetes

when testing those w/ $gt \geq 5 \Rightarrow$ diabletes

$gt < 5 \Rightarrow$ no diabetes

p = prob diabetes in real life (not through the test)

a) Prob person who has diabetes is correctly classified?

diabetes $\sim$ Bern (p) p = prob diabetes

$P((+) \text{ test} \mid \text{have diabetes}) \rightarrow N(6,9)$

$P(\text{glucose} \geq 5 \mid N(6,9))$

$P\left(\frac{X-6}{\sqrt{9}} \geq \frac{5-6}{\sqrt{9}}\right) = P(Z \geq -1/3)$

$= 1 - P(Z < -1/3) = > 50\%$ $(1 - 0.37 = 0.63)$

b) Prob person who does not have diabetes is correctly classifies

$P(- \text{ test} \mid \text{no diabetes}) \rightarrow N(4,4)$ $\quad Y \sim N(4,4)$

$P(Y < 5) = P\left(\frac{Y-4}{\sqrt{4}} < \frac{5-4}{\sqrt{4}}\right) = P(Z < 1/2) = < 50\%$ (0.69)

c) Probability of a positive test result for a randomly chosen person? (Answer is function of p)

$P(+) = \sum_{Diab} P(+ \cap Diab)$

$P(y \geq 5)$

$1 - P(y < 5) = 1 - 0.69$

$= P(+ | Diab)\, P(Diab) + P(+ / non)\, P(non)$

$= (0.63)(p) + 0.29\,(1-p)$

$= 0.63p + 0.29 - 0.29p$

$= 0.29 + 0.34p$ C

d) $P(d | +) = \dfrac{P(d \cap +)}{P(+)} = \dfrac{P(+ | d)\, P(d)}{P(+)} = \dfrac{(0.63)(p)}{0.29 + 0.34p}$ C

e) $P(\text{diabetes} \mid \text{glucose tol} = 6) = \dfrac{P(\text{diab} \cap gt=6)}{P(gt=6)} = \dfrac{P(gt=6 | d)\, P(d)}{P(gt=6)}$ ⓐ ⓑ

ⓐ $P(a) = p$

$P(gt=6 | d)$ $\dfrac{1}{\sqrt{2\pi}\,(3)} e^{-(6-6)^2/2(9)} = 0.13$

$gt \sim N(6, 9)$

$\Rightarrow 0.13p$

ⓑ $P(gt=6) = P(gt=6 \cap \text{diab/no diab})$

$= P(gt=6 | d) P(d) + P(gt=6 | nd)\, P(nd)$

$= 0.13\,(p) + 0.12\,(1-p)$

$= 0.13p + 0.12 - 0.12p$

$= 0.12 + 0.01p$

$\Rightarrow P(d | gt=6) = \dfrac{0.13p}{0.12 + 0.01p}$

seems like same approach.

7) Two gt measurements on a random diabetic individual are distributed Biv. Normal.

$r = 0.5 \qquad gt1 \sim N(6, 9) \qquad gt2 \sim N(6, 9)$

A randomly chosen subject tested today has a glucose tolerance value of 6. If the same subject is tested a month from now, what is the prob of a positive test?

gt1, gt2 are biv. Norm w/ $\rho = 0.5$

$P(gt2 \geq 5 \mid gt1 = 6)$

$$f(y \mid X = x) \sim N\left\{\mu = \mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x),\ \sigma^2 = (1-\rho^2)\sigma_y^2\right\}$$

$$\sim N\left(6 + \cancel{0.5\left(\tfrac{3}{3}\right)(6-6)}^{0},\ (1-\cancel{0.5^2})\,9\right)$$

$$0.75(9) = 6.75$$

$$\sim N(6,\ 6.75)$$

100 - 68 = 32
32/2 = 16

so, $P(gt2 \geq 5 \mid gt1 = 6)$

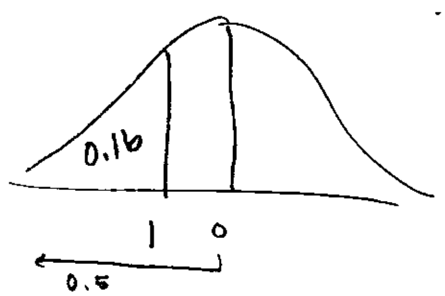
$$= 1 - P(\overset{y}{gt2} \leq 5 \mid \overset{x}{gt1} = 6)$$

$$= 1 - P\left(\frac{y-6}{\sqrt{6.75}} \leq \frac{5-6}{\sqrt{6.75}}\right)$$

$$= 1 - P(Z \leq -1/\sqrt{6.75})$$

$$= 1 - P(Z \leq -0.38)$$

$$= 1 - 0.35 \qquad = 0.65$$



If an individuals first test = 6, there is a probability of 0.65 that their second test will be positive (≥ 5)

③ $X_1 \ldots X_n$

$$\frac{3}{\theta} x^2 e^{-x^3/\theta} \qquad \theta > 0 \qquad x > 0$$

a) Find a suff. stat for θ (use Factorization Thm)

$$f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \frac{3}{\theta} x_i^2 e^{-x_i^3/\theta}$$

$$= \underbrace{\left(\frac{3}{\theta}\right)^n}_{\text{and}} \underbrace{(\Pi x_i^2)}_{h(x)} \underbrace{e^{-\Sigma x_i^3/\theta}}_{g(T(x)|\theta)}$$

by the Factorization Theorem, Σx_i^3 is sufficient for θ

b) min suff stat = $\frac{f(\underset{\sim}{x}|\theta)}{f(\underset{\sim}{y}|\theta)} \perp \theta$ iff $T(x) = T(y)$ ← m.s.s.

$$\frac{\cancel{(3/\theta)^n} (\Pi x_i^2) e^{-\Sigma x_i^3/\theta}}{\cancel{(3/\theta)^n} (\Pi y_i^2) e^{-\Sigma y_i^3/\theta}} = \left(\frac{\Pi x_i^2}{\Pi y_i^2}\right) e^{-\frac{\Sigma x_i^3}{\theta} + \frac{\Sigma y_i^3}{\theta}}$$

if $\Sigma x_i^3 = \Sigma y_i^3$ then the ratio $\perp \theta$

$\Rightarrow \Sigma x_i^3$ is a minimal suff. stat.

c) Find MLE of θ

$$L(\theta|\underset{\sim}{x}) = \prod_{i=1}^{n} \frac{3}{\theta} x_i^2 e^{-x_i^3/\theta}$$

$$= (3/\theta)^n (\Pi x_i^2) e^{-\Sigma x_i^3/\theta}$$

$$\ell(\theta|\underset{\sim}{x}) = n(\log 3 - \log \theta) + \Sigma \log x_i^2 - \frac{\Sigma x_i^3}{\theta}$$

$$\frac{d\ell}{d\theta} = -\frac{n}{\theta} + \frac{\Sigma x_i^3}{\theta^2} = 0$$

$$-n\theta + \Sigma x_i^3 = 0$$

$$\boxed{\hat{\theta}_{MLE} = \frac{\Sigma x_i^3}{n}}$$

$$\frac{d\ell}{d\theta^2} = \frac{n}{\theta^2} - \frac{2\Sigma x_i^3}{\theta^3}$$

$$= \frac{1}{\theta^3}(n\theta - 2\Sigma x_i^3) < 0 \checkmark \text{ max.}$$

$$= \frac{1}{\theta^3}(\Sigma x_i^3 - 2\Sigma x_i^3)$$

pos

d) $\hat{\theta}_{MLE} = \frac{\Sigma X_i^3}{n}$

$$E\left(\frac{\Sigma X_i^3}{n}\right) = \frac{\Sigma E X_1^3}{n}$$

$$= \frac{n\theta}{n} = \theta \quad \text{(unbiased)}$$

$EX^3 = \cancel{\int_0^\infty x^3 \cdot \frac{3}{\theta} x^2 e^{-x^3/\theta} dx}$

$\cancel{= \frac{3}{\theta}\int_0^\infty x^5 e^{-x^3/\theta} dx}$ by parts?

$u = x^5$ $v =$

$\cancel{du = 5x^4}$ $dv = e^{-x^3/\theta}$

$\Rightarrow$ ~~W.A says $= \theta$~~

$\cancel{\int e^{-x^3/\theta}}$ $u = x^3$ $du = 2x^2$

(? ...)

$$EX^n = \beta^{n/\gamma}\,\Gamma(1 + n/\gamma)$$

$$\Rightarrow E(X^3) = \theta^1\,\Gamma(1+1) = \theta$$

I'm more confident on my answer

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e) Yes. It is an unbiased estimator and based soley on the sufficient statistic

I'm wy.

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f) CRLB for unbiased estimators of θ

I'm wy.

$$CRLB = \frac{\{\frac{d}{d\theta}\tau(\theta)\}^2}{E(-\frac{d^2}{d\theta^2}f(\underset{\sim}{x}|\theta))} = \frac{(1)}{E\left(-\frac{n}{\theta^2} + \frac{2\Sigma X_i^3}{\theta^3}\right)} = \frac{1}{-\frac{n}{\theta^2} + \frac{2n\theta}{\theta^3}} = \frac{1}{n/\theta^2} = \frac{\theta^2}{n}$$

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g) Derive R for UMP $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$ $\theta_1 > \theta_0$ ~~> 0~~

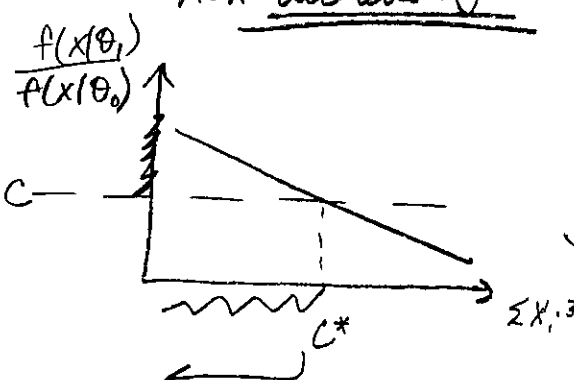
S vs. S $\Rightarrow$ use N-P lemma

$$R = \left\{x : \frac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > c\right\}$$

$$= \frac{(3/\theta_1)^n \cancel{(\Pi X_i^2)} e^{-\Sigma X_i^3/\theta_1}}{(3/\theta_0)^n \cancel{(\Pi X_i^2)} e^{-\Sigma X_i^3/\theta_0}} = \left(\frac{\theta_0}{\theta_1}\right)^n e^{-\frac{\Sigma X_i^3}{\theta_1} + \frac{\Sigma X_i^3}{\theta_0}} = \underbrace{\left(\frac{\theta_0}{\theta_1}\right)^n e^{-\Sigma X_i^3\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)}}_{\text{MLR decreasing}}$$

(+) (−) (+)

I'm wy.



could also use

$R = \{\Sigma X_i^3 < C^*\}$ as UMP test

$\Rightarrow$ yes generalitable to composite hyp b/c no dependence on θ_1

(4) $X_1 \dots X_n$

$$f(x \mid \theta, \beta) = \frac{1}{\theta} e^{-(x-\beta)/\theta} \quad x > \beta > 0 \quad \theta > 0 \qquad 0 < \beta < x$$

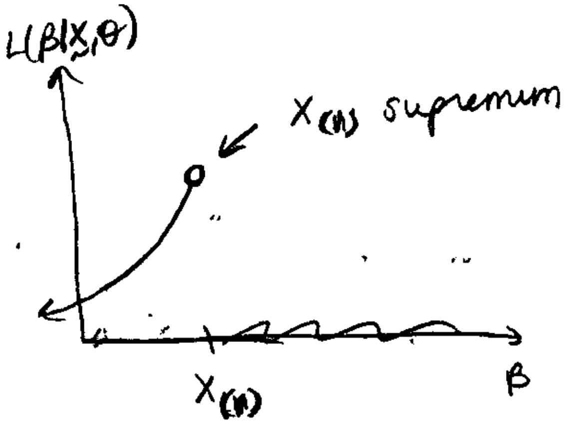
a) Find MLE's for θ and β

$$L(\theta, \beta \mid \underset{\sim}{X}) = \prod_{i=1}^{n} \left(\frac{1}{\theta}\right) e^{-(x_i-\beta)/\theta} \, I(\beta > x_i)$$

$$= \left(\frac{1}{\theta}\right)^n e^{-\sum_{i=1}^{n}(x_i-\beta)/\theta} \, I(X_{(n)} < \beta)$$

↓ what Natalie

* hold θ constant (Σx_i etc. are cons.)



$\Rightarrow X_{(n)}$ is MLE for β
now need MLE for θ

$x > \beta$
$\Rightarrow X_{(1)}$ must be $> \beta$ ⌢̈

$$L(\theta \mid \underset{\sim}{X}, x_{(n)}) = \left(\frac{1}{\theta}\right)^n e^{-\Sigma(x_i - x_{(n)})/\theta}$$

$$\ell(\theta) = n(\log 1 - \log\theta) - \frac{\Sigma(x_i - x_{(n)})}{\theta}$$

$$\frac{d\ell}{d\theta} = \frac{-n}{\theta} + \frac{\Sigma(x_i - x_{(n)})}{\theta^2} = 0$$

$$-n\theta + \Sigma(x_i - x_{(n)}) = 0$$

$$\boxed{\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n}(x_i - x_{(n)})}{n}}$$

$$\frac{d\ell^2}{d\theta^2} = \frac{n}{\theta^2} + \frac{-2\,\Sigma(x_i - x_{(n)})}{\theta^3}$$

$$\frac{1}{\theta^3}\left(n\theta - 2\Sigma(x_i - x_{(n)})\right) < 0 \quad \checkmark \text{ maximum}$$

b) Derive the LRT for testing
$H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$ with β unknown (use MLE $= X_{(n)}$)
ID a distn. approx or exact whichever is easier that may be used in constructing the rejection region

Likelihood ratio $= \dfrac{L(\theta_0 \mid \underset{\sim}{X}, X_{(n)})}{L(\hat{\theta} \mid \underset{\sim}{X}, X_{(n)})}$ (reject if < c) or $\underbrace{-2\log(\lambda(\underset{\sim}{X}))}_{\text{approx}}$ reject Ho for large values of

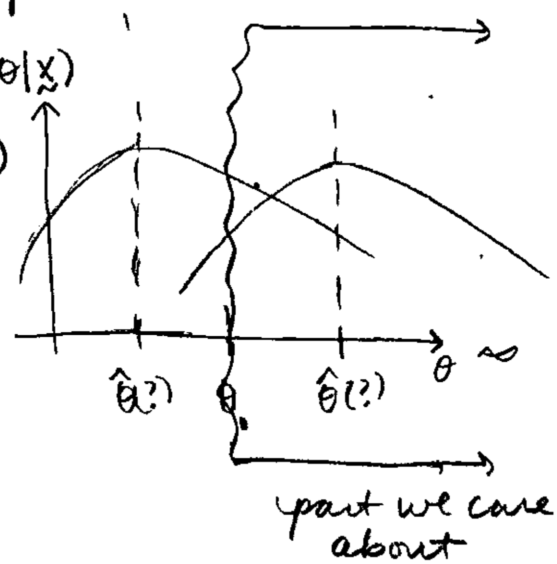
$\theta_0 = \theta_0$ in null

$\hat{\theta}$ in overall space: $[\theta_0, \infty)$

max of likelihood function is $\hat{\theta} = \dfrac{\Sigma(X_i - X_{(n)})}{n}$

$\Rightarrow$ if $\hat{\theta} < \theta_0$ then max is θ_0

$\hat{\theta} > \theta_0$ then max is $\hat{\theta}$



so LRT $= \lambda(\underset{\sim}{X})$ $= \dfrac{\left(\frac{1}{\theta}\right)^n e^{-\Sigma(X_i - X_{(n)})/\theta_0}}{\left(\frac{1}{w}\right)^n e^{-\Sigma(X_i - X_{(n)})/w}}$

$$\lambda(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } \hat{\theta} = \dfrac{\Sigma(X_i - X_{(n)})}{n} < \theta_0 \\ \left(\dfrac{\overset{\bar{x}-x_{(n)}}{\hat{\theta}}}{\theta_0}\right)^n e^{-\Sigma(X_i - X_{(n)})/\theta_0 + n} & \text{if } \hat{\theta} = \dfrac{\Sigma(X_i - X_{(n)})}{n} > \theta_0 \end{cases}$$

C

w/ $X_{(n)}$

and reject if $\lambda(\underset{\sim}{X}) < c$ (no obvious distn to find c)

OR approx:

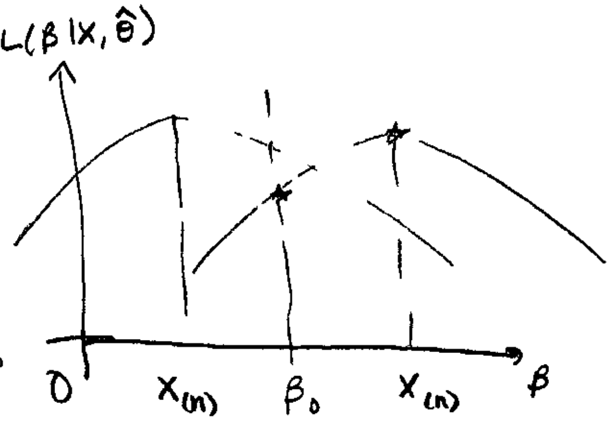
$$-2\log\lambda(\underset{\sim}{x}) = \begin{cases} 0 \\ 2\left\{\left(-n\log\hat{\theta} - \dfrac{\Sigma(X_i - X_{(n)})}{\hat{\theta}}\right) - \left(-n\log\theta_0 - \dfrac{\Sigma(X_i - X_{(n)})}{\theta_0}\right)\right\} \end{cases}$$

and reject if > C

find c via $-2\log\lambda(\underset{\sim}{x}) \sim \chi^2_1$

c) Derive the LRT for testing $H_0: \beta = \beta_0$ vs. $H_1: \beta > \beta_0$ w/ θ unknown (use MLE) and β_0 a given positive constant

ID a distribution to use in constructing the R



$$LRT = \frac{L(\beta_0 | \underset{\sim}{X}, \hat\theta)}{L(\beta | \underset{\sim}{X}, \hat\theta) \text{ sup in } [\beta_0, \infty)}$$

$$LRT = \begin{cases} 1 & \text{if } X_{(n)} < \beta_0 \\ \dfrac{\cancel{(1/\hat\theta)^n} \exp(-\Sigma(X_i - \beta_0)/\hat\theta\}}{\cancel{(1/\hat\theta)^n} \exp(-\Sigma(X_i - X_{(n)})/\hat\theta\}} & \text{if } X_{(n)} > \beta_0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } X_{(n)} < \beta_0 \\ \exp((\cancel{-\Sigma X_i} + \beta_0 \cancel{+\Sigma X_i} - X_{(n)})/\hat\theta) & \text{if } X_{(n)} > \beta_0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } X_{(n)} < \beta_0 \\ \exp\{(\beta_0 - X_{(n)})/\hat\theta\} & \text{if } X_{(n)} > \beta_0 \end{cases}$$

Reject if $LRT < c$, a specified constant.

Could also do $R = \{\underset{\sim}{x} : X_{(n)} > c\}$ as the LRT is monotone decreasing in $X_{(n)}$

or use $-2\log\lambda(\underset{\sim}{X})$

relatively w/ craig didn't do.