$\boxed{2013}$

① Collect data on

x: # respiratory infections

y: # ear infections

Suppose there is an unobserved var u: childs propensity to develop infections

$u \sim Exp(1/\mu)$ mean $\mu = 1/(1/\mu)$

$$\left.\begin{array}{l} x|u \sim Pois(u) \\ y|u \sim Pois(u) \end{array}\right\} \text{independent}$$

a) $EX = E_u E_{x|u}(X|u) = E_u(u) = \mu$

$Var X = E_u Var_{x|u}(X|u) + Var_u E_{x|u}(X|u)$

$= E_u(u) + Var(u)$

$= \mu + \mu^2 = \mu(1+\mu)$

because $EX \neq Var X$, X does not have a Poisson distribution

b) Find corr(x,y) and corr(x,u). which is larger?

① $corr(x,y) = \dfrac{cov(x,y)}{\sqrt{VarX \cdot Vary}} = \dfrac{E(XY) - \overset{\mu}{EX}\,\overset{\mu}{EY}}{\sqrt{VarX \cdot Vary}}$, $\sqrt{(\mu+\mu^2)(\mu+\mu^2)}$

$f(xy|u) = f(x|u) \cdot f(y|u)$

$= \dfrac{e^{-u}\mu^x}{x!} \cdot \dfrac{e^{-u}u^y}{y!}$

$= \dfrac{e^{-2u}u^{(x+y)}}{x!\,y!}$

* use conditional covariance

$Cov(X,Y) = E_u[Cov(X,Y|u)] + Cov(E(X|u), E(Y|u))$

$= E_u\{E(xy|u) + E(X|u)E(Y|u)\} + cov(u,u)$

$E_u\{ \quad + u^2\}$ $+ (E(u^2) + E(u)E(u))$

$+ (\{Var u + E(u)^2\} + \mu^2)$

$+ (\mu^2 + \mu^2 + \mu^2)$

$+ (3\mu^2)$

$corr(x,y) = \dfrac{?}{\sqrt{(\mu+\mu^2)(\mu+\mu^2)}}$

more on next page.

X and Y are conditionally indep

$$\operatorname{corr}(x,y) = \frac{\operatorname{cov}(x,y)}{\sqrt{\operatorname{var}x \cdot \operatorname{var}y}}$$

$$\operatorname{Var}(X+Y) = \operatorname{Var}X + \operatorname{Var}Y + 2\operatorname{cov}(x,y)$$

$$\operatorname{cov}(x,y) = \operatorname{var}(u)$$

$$\operatorname{corr}(x,y) = \frac{\operatorname{var}(u)}{\sqrt{\operatorname{var}x \cdot \operatorname{var}y}}$$

$$= \frac{\mu^2}{\sqrt{(\mu^2+\mu)(\mu^2+\mu)}} = \frac{\mu^2}{\mu^2+\mu} = \frac{\mu^2}{\cancel{\mu}(\mu+1)}$$

$$= \frac{\mu}{\mu+1}$$

$$\operatorname{cov}(x,y) = E(XY) - EX\,EY$$

$$= E_u \operatorname{cov}(x,y \mid u) + \operatorname{cov}(E(x \mid u), E(y \mid u))$$

$$= E_u\{E(xy \mid u) - E(x \mid u)E(y \mid u)\} + \operatorname{cov}(u,u)$$

$$= \cancel{E_u\{u^2 - u^2\}}^{0} + \operatorname{cov}(u,u)$$

$$= E(u^2) - E(u)E(u)$$

$$= \operatorname{Var}(u)$$

b) cont.

$$\mathrm{Corr}(X,U) = \frac{\mathrm{cov}(X,U)}{\sqrt{\mathrm{Var}X \cdot \mathrm{Var}U}} = \frac{E(XU) \overset{(-)}{+} EXEU}{\sqrt{(\mu+\mu^2)(\mu^2)}}$$

$$EXU = \iint xu \cdot f(x,u)\, du\, dx$$

constant b/c condition on u

$$\text{or } \quad E_u E(XU|U) = E_u(U \cdot E(X|U))$$

$$= E_u(U^2)$$

$$= \mathrm{Var}U + E(U)^2 = \mu^2 + \mu^2 = 2\mu^2$$

$$\mathrm{corr}(X,U) = \frac{2\mu^2 - \mu^2}{\sqrt{\mu^3+\mu^4}} = \frac{\mu^2}{\sqrt{\mu^3+\mu^4}} \quad C$$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}X \cdot \mathrm{Var}Y}} = \frac{E(XY) + EXEY}{\sqrt{\mathrm{Var}X \cdot \mathrm{Var}Y}}$$

$$E(XY) = E_u E(XY|U)$$

$$= E_u(E(X|U) \cdot E(Y|U)) \Leftarrow ???$$

$$= E_u(U \cdot U) = EU^2$$

$$= \mathrm{Var}U + (EU)^2 = \mu^2 + \mu^2 = 2\mu^2$$

$$\mathrm{corr}(X,Y) = \frac{2\mu^2 + \mu\cdot\mu}{\sqrt{(\mu+\mu^2)(\mu+\mu^2)}} = \frac{3\mu^2}{\sqrt{\mu^2 + 2\mu^3 + \mu^4}}$$

$\boxed{\text{correlation of X,U is larger.}}$ C

c) - Is X an unbiased predictor of u? ①
Compute its prediction variance. ②

① Does $E(X-u)=0$

$EX - Eu = \mu - \mu = 0 \Rightarrow$ yes, unbiased predictor

② $Var(X-u)$

$= Var X + Var u - 2cov(X,u)$

$= (\mu+\mu^2) + \mu^2 - 2(\mu^2)$ ← from messing up cov previously

$= \mu \cancel{+ 2\mu^2 - 4\mu^2} \cancel{[\mu - 4\mu^2} \cancel{[\mu(1-4\mu)}$

d) Find constants a and b such that $a + bX$ is an unbiased predictor of u and its prediction variance is minimized

$E(u - a - bX) = 0$ to be an unbiased predictor

$Eu - Ea - bEX$

$\mu - a - b\mu = 0$

$\mu(1-b) = a$

$(1-b) = \frac{a}{\mu}$

$1 - \frac{a}{\mu} = b$

$Var(u - a - bX)$

$= Var u + b^2 Var X - 2b(1) cov(X,Y)$

$= \mu^2 + b^2(\mu+\mu^2) - 2b\mu^2$

$\frac{d}{db} = 2b(\mu+\mu^2) - 2\mu^2$

$b = \frac{\cancel{2}\mu^2}{\cancel{2}(\mu+\mu^2)} = \frac{\mu}{1+\mu}$

$\frac{d^2}{db^2} = 2(\mu+\mu^2) > 0$ so min.

$\Rightarrow a = \mu\left(1 - \frac{\mu}{1+\mu}\right)$

e) Develop expressions for $\Gamma(n) = (n-1)!$

① $P(Y=0)$ and ② $P(y=0 \mid X=0)$. ③ Which one is larger?

① $P(Y=0)$ = prob no ear infections

$f(y) = \int_0^\infty f(y|u)\, f(u)\, du$

$= \int_0^\infty * \; du$

$= \dfrac{1}{(1+1/w)^y\, y}$

$f(y|u) = \text{Pois}(u) = e^{-u} u^y / y!$

$f(u) = \exp(w) = u^{-1} e^{-u/w}$

$e^{-\frac{(\mu-1)u}{\mu}}$

$e^{u\left(-\frac{(\mu-1)}{\mu}\right)}$

$f(y,u) = \dfrac{e^{-u} u^y}{y!} \cdot \dfrac{e^{-u/w}}{u}$

$= \dfrac{e^{-u-u/w}\, u^{y-1}}{y!}$ gamma kernel

$\Gamma(y+1)$

make gamma

$\alpha = y+1$

$\beta = 1/(1+1/w)$

$= \underbrace{\dfrac{e^{-u(1+1/w)}\, u^{y-1}}{\Gamma(y)\left(\frac{1}{1+1/w}\right)^y} \cdot \dfrac{\left(\frac{1}{1+1/w}\right)^y}{y}}_{*}$

So $P(y=0) = f_y(0)$

$= 0 (?)$

$$P(y=0) = \int_u f(y=0, u)\, du$$

$$= \int_u \underbrace{\frac{e^{-\mu}\mu^y}{y!}}_{y=0} \cdot \frac{1}{\mu} e^{-u/\mu}\, du$$

bounds of u $\to \infty$

$$= \frac{e^{-\mu}}{\mu} \int_0^{\infty} e^{-u/\mu}\, du \qquad x = \frac{-u}{\mu}$$

$$dx = -\frac{1}{\mu} du$$

$$= \frac{e^{-\mu}}{\mu} -\mu e^{-u/\mu} \Big|_0^{\infty} \qquad du = -\mu\, dx$$

$$= \frac{e^{-\mu}}{\mu} \left(\cancel{-\mu}(0 \cancel{-1})\right)$$

$$= e^{-\mu} = P(y=0)$$

$$P(y=0 \mid x=0)$$

$$= \frac{P(y=0 \cap x=0)}{P(x=0) = e^{-\mu}} \longrightarrow \int_u f(x=0, y=0 \mid u) \cdot f(u)\, du$$

condit. indep. $x \perp y$

$$= \int_u f(x=0 \mid u) \cdot f(y=0 \mid u) \cdot f(u)\, du$$

$$= \frac{e^{-2\mu}}{e^{-\mu}} = e^{-\mu}$$

$$= \int_w e^{-\mu} e^{-\mu} \cdot \frac{1}{\mu} e^{-w/\mu}\, dw$$

let $w = u$ b/c $u \Rightarrow \mu$:(

$$= \frac{e^{-2\mu}}{\mu} \int_0^{\infty} e^{-w/\mu}\, dw$$

$$= \frac{e^{-2\mu}}{\mu} \left(-\mu e^{-w/\mu}\right)\Big|_0^{\infty}$$

$$= \frac{e^{-2\mu}}{\cancel{\mu}} \left(\cancel{-\mu(0-1)}\right) = e^{-2\mu}$$

$$\text{ratio} = \frac{P(y=0 \mid x=0)}{P(y=0)} = 1$$

they are the same. (for this given value of y and x.

② Let $X_1, X_2 \ldots X_n$ be iid RV's from the discrete distn:

$\xrightarrow{}$ not continuous, so usual formulas don't apply

Kris barely did any tht

$f(j) = \theta_j \quad j \in (1, 2, 3) \qquad \theta_j > 0 \quad j = 1, 2, 3$

probabilities $\theta_1 + \theta_2 + \theta_3 = 1$

$T_n = X_{(1)} \quad M = \text{median} = X_{(2)}$

each X can take 3 value? ← yes

a) if $n \geq 3$, derive an expression for the probability that each of the 3 possible values will be observed ≥ 1 in the sample.

$f(1) = \theta_1$, $f(2) = \theta_2$, $f(3) = \theta_3$ } each x can take on 3 options 1, 2, 3

$$f(X_1 \ldots X_n) = \prod_{i=1}^{n} f(X_i) \quad \text{b/c iid}$$

$$= (\theta_1)^{(\#=1)} (\theta_2)^{(\#=2)} (\theta_3)^{n-(\#=1)-(\#=2)}$$

$\theta_1 + \theta_2 + \theta_3 = 1$

$\theta_1 = 1 - \theta_3 - \theta_2$

$$P(\theta_1 \geq 1, \ \theta_2 \geq 1, \ \theta_3 \geq 1) = 1 - P(\theta_1 \text{ 0 times} | \theta_2 \text{ 0 times} | \theta_3 \text{ 0 times})$$

$$= 1 - \left(P(\theta_1 \text{ 0 times}) + P(\theta_2 \text{ 0 times}) + P(\theta_3 \text{ 0 times})\right)$$

$$= 1 - \left\{P(\text{all} = \theta_2 \text{ or } \theta_3) + P(\text{all} = \theta_1 \text{ or } \theta_3) + P(\text{all} = \theta_1 \text{ or } \theta_2)\right\}$$

$$= 1 - \left\{\cancel{\theta_1^0}\, \theta_2^{(\#=2)} \theta_3^{n-(\#=2)} + \theta_1^{(\#=1)} \cancel{\theta_2^0}\, \theta_3^{n-(\#=1)} + \theta_1^{(\#=1)} \theta_2^{n-(\#=1)} \cancel{\theta_3^0}\right\}$$

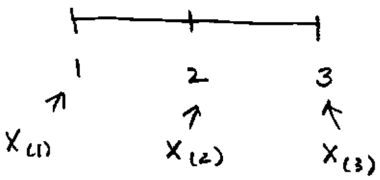
$$= 1 - \left\{\sum_{i=1}^{n} \binom{n}{i} \theta_2^i \theta_3^{n-i} + \sum_{i=1}^{n} \binom{n}{i} \theta_1^i \theta_3^{n-i} + \sum_{i=1}^{n} \binom{n}{i} \theta_1^i \theta_2^{n-i}\right\}$$

only 2 options

b) Derive the pmf of $T_n \leftarrow$ minimum

$T_n = X_{(1)}$

pmf of $X_{(1)}$ $\Rightarrow$ the min is 1



$p_j = \theta_j$ if looking for $\leq X_{j=1}$

$$= \sum_{k=1}^{n} \binom{n}{k} p_j^{\,k} (1-p_j)^{n-k} = P(X_{(1)} \leq X_j)$$

$j = 1, 2, 3$

b/c discrete : $P(X_{(1)} \leq X_j) - P(X_{(1)} \leq X_{j-1}) = P(X_{(1)} = X_j)$

$*$ if $j = 1$ then $P_{1-1} = 0$ $*$

so $\displaystyle \sum_{k=1}^{n} \binom{n}{k} P_j^{\,k} (1-P_j)^{n-k} - \sum_{k=1}^{n} \binom{n}{k} P_{j-1}^{\,k} (1-P_{j-1})^{n-k}$

$$= \sum_{k=1}^{n} \binom{n}{k} \left\{ P_j^{\,k} (1-P_j)^{n-k} - P_{j-1} (1-P_{j-1})^{n-k} \right\} = P(X_{(1)} = X_j) \quad \text{for } j = 1, 2, 3$$

where :
$P_j = \theta_j$
$P_0 = 0$

c) Prove whether or not $\underbrace{T_n \text{ converges in prob}}$ to a constant

convergence in prob:

$$\lim_{n \to \infty} P(|X_n - X| \leq \varepsilon) = 1$$

d) $n=3$

$\theta_1 = \theta_2 = \theta_3 = 1/3$

pmf of sample median

$X_{(2)}$ is sample median

cmf $P(X_{(2)} \leq j)$ $\quad j = 1, 2, 3$ (only three options

$$= \sum_{k=1}^{3} \binom{n}{k} (\theta_j)^k (1-\theta_j)^{n-k}$$

$$= \sum_{k=1}^{3} \binom{n}{k} (1/3)^k (2/3)^{n-k}$$

$\Rightarrow$ $P(X_{(2)} \leq 1) = \binom{3}{1}(1/3)(2/3)^2$

$P(X_{(2)} \leq 2) =$ " $+ \binom{3}{2}(1/3)^2(2/3)^1$

$P(X_{(2)} \leq 3) =$ " $+$ " $+ \binom{3}{3}(1/3)^3(2/3)^0$

pmf of $P(X_{(2)} = j)$

$= P(X_{(2)} \leq j) - P(X_{(2)} \leq j-1)$ $\quad$ if $j=1$ then $P(X_{(2)} \leq 0) = 0$

$$= \sum_{k=1}^{3} \binom{n}{k} \left\{ (1/3)^k (2/3)^{n-k} - (1/3)^k (2/3)^{n-k} \right) \quad (?)$$

③ $X_1 \dots X_n$ be from

$f(x|\theta) = \frac{1}{\theta^2} x e^{-x/\theta} \quad x > 0 \quad \theta > 0$

* derive explicit expressions and simplify as much as possible

a) Find minimal suff. stat for θ

gamma kernel

$= \frac{f(\underset{\sim}{x}|\theta)}{f(\underset{\sim}{y}|\theta)} \perp \theta$ iff $T(x) = T(y)$ then minimal suff.

$$= \frac{\cancel{\theta^{-2n}} (\prod x_i) e^{-\sum x_i/\theta}}{\cancel{\theta^{-2n}} (\prod y_i) e^{-\sum y_i/\theta}} = \frac{(\prod x_i)}{(\prod y_i)} \cdot \underbrace{e^{-\frac{\sum x_i}{\theta} + \frac{\sum y_i}{\theta}}}$$

$\perp \theta$ if this is 0

$\Rightarrow \sum X_i = \sum Y_i$

So $\sum X_i$ is minimal suff. stat

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b) Find MLE of θ

$L(\theta|\underset{\sim}{X}) = \theta^{-2n} (\prod X_i) e^{-\sum X_i/\theta}$

$\ell(\theta|\underset{\sim}{X}) = -2n \log\theta + \sum \log X_i - \frac{\sum X_i}{\theta}$

$\frac{d\ell}{d\theta} = \frac{-2n}{\theta} + \frac{\sum X_i}{\theta^2} = 0$

$-2n\theta + \sum X_i = 0$

$\hat{\theta} = \frac{\sum X_i}{2n}$

$$\frac{d^2\ell}{d\theta^2} = \frac{2n}{\theta^2} + \frac{-2\sum X_i}{\theta^3} = \frac{1}{\theta^3}(2n\theta - 2\sum X_i)$$

$(+) \quad (+) \quad (-)$

$$= \frac{1}{\theta^3}\left(2n\left(\frac{\sum X_i}{2n}\right) - 2\sum X_i\right) = \frac{1}{\theta^3}\left(\sum X_i(-1)\right) < 0 \checkmark$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{2n} = \frac{\cancel{n}\bar{X}}{\cancel{n}2} = \frac{\bar{X}}{2}$$

c) find E(b))

$$E\left(\frac{\Sigma X_i}{2n}\right) = \frac{\Sigma E X_1}{2n} = \frac{n(2\theta)}{2n} = \theta$$

gamma
$\beta = \theta$, $\alpha = 2$
$EX = 2\theta$

(but he integrated)

d) Is the estimator in b) the UMVUE of θ? Explain.

$\Rightarrow$ Yes. First, the estimator is unbiased. Second, it is based off of the minimum sufficient statistic. Therefore, $\hat{\theta} = \Sigma X_i / 2n$ is the UMVUE of θ.

e) CRLB for unbiased estimators of θ

$$CRLB = \frac{\left\{\frac{d}{d\theta} T(\theta)\right\}^2}{E\left(-\frac{d^2}{d\theta^2} \ell(\underset{\sim}{x}|\theta)\right)} = \frac{1}{\underbrace{E\left(-\left(\frac{2n}{\theta^2} - \frac{2\Sigma X_i}{\theta^3}\right)\right)}}$$

$$E\left(-\frac{2n}{\theta^2}\right) + E\left(+\frac{2\Sigma X_i}{\theta^3}\right) = -\frac{2n}{\theta^2} + \frac{2n(2\theta)}{\theta^3} = \frac{-2n + 4n}{\theta^2} = \frac{2n}{\theta^2}$$

$$\Rightarrow CRLB = \frac{\theta^2}{2n}$$

f) CRLB for unbiased estimators of θ^2

$$= \frac{\left\{\frac{d}{d\theta}\theta^2\right\}^2}{(2n/\theta^2)} = \frac{(2\theta)^2}{(2n/\theta^2)} = \frac{4\theta^2\,\theta^2}{2n} = \frac{2\theta^4}{n}$$

same.

3). $Y_1 \dots Y_m$ be a random sample from

$$g(y|\theta) = \frac{1}{\theta} e^{-y/\theta} \quad y > 0 \quad \theta > 0$$

steps: find $\hat{\theta}_2$

var $\hat{\theta}_1$ and var $\hat{\theta}_2$

$$L(\theta|y) = \theta^{-n} e^{-\Sigma y_i/\theta}$$

gamma $\alpha = 1$ $\beta = \theta$

$E y_1 = \theta$

$$\ell = -n \log\theta - \Sigma y_i/\theta$$

$$\frac{d\ell}{d\theta} = \frac{-n}{\theta} + \frac{\Sigma y_i}{\theta^2} = 0 \Rightarrow \theta = \frac{\Sigma y_i}{n}$$

second deriv < 0 ✓

$$E\left(\frac{\Sigma y_i}{n}\right) = \frac{n\theta}{n} = \theta$$ unbiased ✓

Both estimators are unbiased.

$$\text{Var } \hat{\theta}_1 = \text{Var}\left(\frac{\Sigma X_i}{2n}\right) = \frac{\Sigma \text{Var} X_1}{(2n)^2} = \frac{n(2\theta^2)}{4n^2} = \frac{\theta^2}{2n} = \frac{\theta^2}{64}$$

$$\text{Var } \hat{\theta}_2 = \text{Var}\left(\frac{\Sigma Y_i}{n}\right) = \frac{\Sigma \text{Var} Y_1}{n^2} = \frac{\theta^2}{n} = \frac{\theta^2}{44}$$

I would prefer $\hat{\theta}_1 \Rightarrow$ it reaches the CRLB, and based on the given sample sizes has a smaller variance.

C

④ $X_1 \dots X_n$ be random sample from pmf:

$$P(X_i = j) = p_j \quad j = 1, \dots, 4$$

The vector parameter $\theta = (p_1 \ p_2 \ p_3 \ p_4)^T$

satisfies $\sum_1^4 p_j = 1$ and $p_j > 0$ for $j = 1, \dots, 4$

Let $y_j = \#(x_i \mid x_i = j, \ i = 1 \dots n)$

<u>what</u>

a) Derive MLE of θ under H_0

(maximize likelihood function)

$L(\theta \mid \underset{\sim}{X}) =$

H_0: $\overbrace{p_1 = p_2}^{Q_1}$ AND $\overbrace{p_3 = p_4}^{Q_2}$

$p_1 - p_2 = 0 \quad p_3 - p_4 = 0$

H_a: H_0 not true

$$P(X_1 = 1) = p_1$$
$$P(X_1 = 2) = p_2$$
$$P(X_1 = 3) = p_3$$
$$P(X_i = 4) = p_4$$
$$\Sigma = 1$$

what is this

$y_1 = \#(X_i \mid X_i = 1)$ for $i = 1 \dots n$

Question 1

X ~ # respiratory infections
y ~ # ear infections
u ~ unobserved → propensity to develop more infections $EU = \gamma$, $Var\,U = \gamma^2$
~ $Exp(\mu)$ mean = μ LET $\mu = \gamma$ for clarity btwn u and μ Hint: conditioning.

$X|u \perp Y|u$ (conditional independence)

$X|u \sim Pois(u)$ $E\,X|u = u$ $Var\,X|u = u$

$Y|u \sim Pois(u)$ $E\,Y|u = u$ $Var\,Y|u = u$

a) EX and Var X. Does X have a Poisson distn?

$$EX = E_u E X|u = E_u(u) = \boxed{\gamma}$$ C

$$Var X = Var_u EX|u + E_u Var X|u = Var(u) + E(u) = \gamma^2 + \gamma = \boxed{\gamma(\gamma+1)}$$ C

⇒ No, X is not poisson distributed. $EX \neq Var X$

b) Find Corr (X,Y) and corr (X,u). Which is larger?

$$corr(x,y) = \frac{cov(x,y)}{\sqrt{varx \cdot vary}} = \frac{EXY - EXEY}{\sqrt{varx \cdot vary}} = \frac{2\gamma^2 - \gamma(\gamma)}{\sqrt{(\gamma^2+\gamma)^2}} = \frac{\gamma^2}{\gamma(\gamma+1)} = \boxed{\frac{\gamma}{\gamma+1}}$$ C

$$EXY = E_u EXY|u$$ conditional independence

$$= E_u\{EX|u \cdot EY|u\}$$

$$= E_u\{u^2\}$$

$$= Var\,u + \{Eu\}^2$$

$$= \gamma^2 + \gamma^2$$

$$= 2\gamma^2$$

cont. corr(X,u)

$$\text{corr}(X,u) = \frac{\text{cov}(X,u)}{\sqrt{\text{Var}X \cdot \text{Var}u}} = \frac{EXu - EX\,Eu}{\sqrt{\text{Var}X \cdot \text{Var}u}} = \frac{2\gamma^2 - \gamma\gamma}{\sqrt{(\gamma^2+\gamma)\cdot\gamma^2}} = \frac{\gamma^2}{\gamma\sqrt{\gamma^2+\gamma}} = \boxed{\frac{\gamma}{\sqrt{\gamma^2+\gamma}}}$$

C

$$EXu = E_u E\{Xu \mid u\}$$
$$= E_u\{u \cdot EX|u\}$$
$$= E_u\{u^2\}$$
$$= 2\gamma^2$$

$\gamma = 2$ $\quad \text{corr}(X,y) = 2/3$ $\quad \text{corr}(X,u) = 2/\sqrt{5}$ $\Rightarrow$ corr(X,u) is greater C

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c) i. Is X an unbiased predictor of u?

$E(X-u) = 0$ (?)

$\Rightarrow EX - Eu = \gamma - \gamma = 0 \Rightarrow$ Yes, X is unbiased pred. for u ¢

ii. Compute its prediction variance

$$\text{Var}(X-u) = E((X-u)^2) - \cancel{E(X-u)^2}\ 0$$
$$= \text{Var}X + \text{Var}u - 2\,\text{cov}(X,u)$$
$$= (\gamma^2+\gamma) + \gamma^2 - 2(\gamma^2) = \boxed{\gamma = \text{Var}(X-u)}$$

e) $P(y=0)$ and $P(y=0 \mid x=0)$ (develop expressions)

$\int_0^\infty f(y|u)\cdot f(u)\,du$ @ $y=0$

$= \int_0^\infty \frac{e^{-\gamma}\gamma^y}{y!}\cdot\frac{1}{\gamma}e^{-u/\gamma}\,du$

$= \int_0^\infty \frac{e^{-\gamma}}{\gamma}e^{-u/\gamma}\,du$ $\quad w = -\frac{u}{\gamma}$, $dw = -\frac{1}{\gamma}du$, $-\gamma\,dw = du$

$= \frac{e^{-\gamma}}{\gamma}\left\{-\gamma\int e^{w}\,dw\right\}$

$= -e^{-\gamma}\left\{e^{-u/\gamma}\Big|_0^\infty\right\}$

$= -e^{-\gamma}\{0-1\}$

$= \boxed{e^{-\gamma} = P(y=0)}$ C

$\int_0^\infty f(y|u)\cdot f(x|u)\cdot f(u)\,du$ @ $x=0,\ y=0$

$= \int_0^\infty \frac{e^{-\gamma}\cancel{\gamma^x}}{\cancel{y!}}\cdot\frac{e^{-\gamma}\cancel{\gamma^x}}{\cancel{x!}}\cdot\frac{1}{\gamma}e^{-u/\gamma}\,du$

$= \frac{e^{-2\gamma}}{\gamma}\int_0^\infty e^{-u/\gamma}\,du$

$= \frac{e^{-2\gamma}}{\gamma}\left\{-\gamma e^{-u/\gamma}\Big|_0^\infty\right\}$

$= -e^{-2\gamma}\{0-1\} = e^{-2\gamma} = P(x=0 \wedge y=0)$

$P(y=0 \mid x=0) = \frac{P(y=0 \wedge P(x=0)}{P(x=0)}$

$C = e^{-\gamma}$ b/c $x|u + y|u$ (?

$\Rightarrow$ So they are the same for these values of x, and y

d) Find constants a and b such that $a+bX$ is an unbiased predictor of u, and such that its pred. variance is as small as possible.

$E(a+bX-u)=0$ (a and b so that $a+bX$ is unbiased for u)

$a+bEX-Eu=0$

$a+b\gamma-\gamma=0$

$$\left.\begin{aligned} a &= \gamma(1-b) \\ \text{or} \quad b &= \frac{\gamma-a}{\gamma} = 1-\frac{a}{\gamma} \end{aligned}\right\} \text{ for an unbiased prediction}$$

$\mathrm{Var}(\overset{\text{no var}}{\cancel{a}}+bX-u) = \text{minimized}$ ($\frac{d}{da}/b$ and let $=0$)

$b^2\mathrm{Var}X + \mathrm{Var}u - 2b\,\mathrm{cov}(X,u)$

$b^2(\gamma^2+\gamma)+\gamma^2-2b\gamma^2$ $\frac{d}{db}$

$= \cancel{2}b(\gamma^2+\gamma)-\cancel{2}\gamma^2=0$

$$b=\frac{\gamma^2}{\gamma^2+\gamma}=\frac{\gamma}{\gamma+1} \quad C$$

if $b=\frac{\gamma}{\gamma+1}$

then $a=\gamma\left(1-\frac{\gamma}{\gamma+1}\right)$ C

Question 2

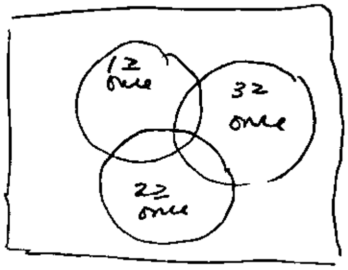
$X_1 \ldots X_n$ iid from

$f(j) = \theta_j \quad j = (1, 2, 3) \qquad \theta_1 + \theta_2 + \theta_3 = 1$

$P(X_i = j) = \theta_j \qquad T_n = X_{(1)} \quad M =$ sample median.

a) $n \geq 3$ derive an express that each of the 3 possible values (1, 2, 3) will be observed at least once

$\Rightarrow P(1 \geq \text{once} \cap 2 \geq \text{once} \cap 3 \geq \text{once})$ complement



$= 1 - P(1=0 \cup 2=0 \cup 3=0)$

$= 1 - \{P(1=0) + P(2=0) + P(3=0)\}$ not mutually exclusive

each of these is a binomial: i.e. one or not one

$= 1 - \{\binom{n}{0}(\theta_1)^0(\theta_2+\theta_3)^n + \binom{n}{0}(\theta_2)^0(\theta_1+\theta_3)^n + \binom{n}{0}(\theta_3)^0(\theta_1+\theta_2)^n\}$

$= 1 - (\theta_2+\theta_3)^n - (\theta_1+\theta_3)^n - (\theta_1+\theta_2)^n$

$= 1 - (1-\theta_1)^n - (1-\theta_2)^n - (1-\theta_3)^n$

my original work

b) Derive the pmf of $T_n = X_{(1)}$

CDF of $X_{(1)} = P(X_{(1)} \le x)$ complement

$= 1 - P(X_{(1)} > x)$ if min > then all must be >

$= 1 - P(X > x)^n$

Question 2

$X_1 \dots X_n \sim$

$f(j) = \theta_j \quad j = 1, 2, 3 \qquad \theta_1 + \theta_2 + \theta_3 = 1$

$T_n = X_{(1)}$

M = median

$\Rightarrow$ multinomial

1	2	3
θ_1	θ_2	θ_3

a) Prob (each of 3 possible values will be observed ≥ 1)

Introduce:

y_1 = # of 1's $\qquad Y_j = \sum_{i=1}^{n} I(X_i = j)$

y_2 = # of 2's $\qquad$ * focus on the event

y_3 = # of 3's $\qquad Y_j = \text{Binom}(n, \theta_j)$

$= \{ Y_1 \neq 0, \; Y_2 \neq 0, \; Y_3 \neq 0 \} = K$

$A_j = \{Y_j = 0\} \qquad A_i \cap A_j = P(Y_i = 0 \cap Y_j = 0) = P(Y_3 = n)$

$P(A_3 = n) = (\theta_3)^n$

$K = A_1^c \cap A_2^c \cap A_3^c = (A_1 \cup A_2 \cup A_3)^c$

$= 1 - \{ P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \}$

$= 1 - \{ \sum_{j=1}^{3} (1 - \theta_j)^n - \sum_{j=1}^{3} (\theta_j)^n + 0 \}$

b) Derive the PMF of $T_n = X_{(1)}$

$X_{(1)}$ will be 1 or 2 or 3

$P(X_{(1)}=1) = P(Y_1 > 0) = 1 - P(Y_1 = 0) = 1 - (1-\theta_1)^n$

↳ observe at least one ones

$P(X_{(1)}=2) = \qquad = 1 - \{1 - (1-\theta_1)^n\} - \{\theta_3^n\}$

$P(X_{(1)}=3) = P(Y_3 = n) = \qquad = \theta_3^n$

---

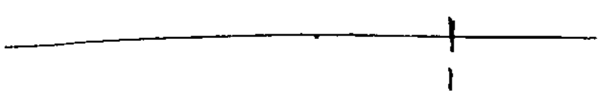
c) Prove whether or not T_n converges to a constant. If it does, find the constant.

⇒ probably $T_n \to 1$

⇒ look at the parts of the pdf

$$\left.\begin{aligned} &\lim_{n\to\infty} (1-(1-\theta_1)^n) \to 1 \\ &\lim_{n\to\infty} P(T_n = 2) \to 0 \\ &\lim_{n\to\infty} P(T_n=3) \to 0 \end{aligned}\right\} \text{converges in distribution}$$

$$\underbrace{P(|T_n - 1| < \varepsilon)}_{\text{this is what we want}}^{a_n} > \underbrace{P(|T_n - 1| = \varepsilon)}_{\hookrightarrow \text{ this goes to } 1}^{b_n}$$



↓

if this is converging to 1, then a_n must go to 1

d) consider when $n=3$ $\theta_1 = \theta_2 = \theta_3 = 1/3$ $\boxed{\text{Question 2}}$ $\boxed{\text{2013, MS-1}}$

compute the pmf of the $\underbrace{\text{sample median}}_{\text{second out of the order}}$

compute mean and variance of the sample median.

Y_1	
3	
2	
1	
0	

} 1 is median

} 1 is not med

$P(M=1) = \frac{7}{27}$ $= P\overbrace{(Y_1 = 2 \cup Y_1 = 3)}^{\text{mutually exclusive}}$

$= P(Y_1 = 2) + P(Y_1 = 3)$

$\underbrace{\binom{3}{2}\cancel{\theta_1}^{\frac{1}{3}\,2}(1-\cancel{\theta_1}^{\frac{2}{3}})^3 + \binom{3}{3}(\theta_1)^0(\theta_1)^3}_{\frac{7}{27}}$

$P(M=2) = \frac{13}{27}$

$P(M=3) = \frac{7}{27}$

same as ↗ M=1 just flipped probabilities

$E(M) = 1\left(\frac{7}{27}\right) + 2\left(\frac{13}{27}\right) + 3\left(\frac{7}{27}\right)$

see the structure of the problem.

$Var(M) = E[M^2] - E[M]^2$

$= 1\left(\frac{7}{27}\right) + 2^2\left(\frac{13}{27}\right) + 3^2\left(\frac{7}{27}\right)$

guess is $T_n \xrightarrow[p]{} 1$

$$\lim_{n\to\infty} \underbrace{P\{|T_n - 1| < \varepsilon\}} = 1$$

$P(-\varepsilon < T_n - 1 < \varepsilon)$

$P(-\varepsilon + 1 < T_n < \varepsilon + 1)$

$= P(T_n < \varepsilon + 1) - P(T_n < -\varepsilon + 1)$

$= P(T_n < 1+\varepsilon) - P(T_n < (1-\varepsilon)$

$(1 - P(T_n > 1) - (1 - P(T_n > 1)$

$P(T_n = 1 + \varepsilon)$

Question 3 |

2013, MS-1

$X_1 \ldots X_n$ iid from

$$f(x|\theta) = \frac{1}{\theta^2} x e^{-x/\theta} \quad x > 0 \quad \theta > 0$$ $\Rightarrow$ gamma $\alpha = 2$, $\beta = \theta$ by C&B

a) Find a minimal suff. stat for θ

$$f(\underset{\sim}{x}|\theta) = \theta^{-2n} (\Pi x_i) e^{-\Sigma x_i/\theta}$$

$$\frac{f(\underset{\sim}{x}|\theta)}{f(\underset{\sim}{y}|\theta)} = \frac{\cancel{\theta^{-2n}} (\Pi x_i) e^{-\Sigma x_i/\theta}}{\cancel{\theta^{-2n}} (\Pi y_i) e^{-\Sigma y_i/\theta}}$$

if $\Sigma x_i = \Sigma y_i$ then the ratio is independent of θ.

$\Rightarrow T(x) = \sum_{i=1}^{n} x_i$ is a minimal. suff. stat

C

---

b) $L(\theta|\underset{\sim}{x}) = \theta^{-2n} (\Pi x_i) e^{-\Sigma x_i/\theta}$

to find MLE, maximize the Likel/log like function

$$\log(\theta|\underset{\sim}{x}) = -2n \log\theta + \Sigma \log x_i - \frac{\Sigma x_i}{\theta}$$

$$\frac{d \log}{d\theta} = \frac{-2n}{\theta} + \frac{\Sigma x_i}{\theta^2} \Rightarrow \hat{\theta}_{MLE} = \frac{\Sigma x_i}{2n}$$ C

$$\frac{d^2 l}{d\theta^2} = \frac{2n}{\theta^2} - \frac{2\Sigma x_i}{\theta^3} = \frac{1}{\theta^3}\left(2n\theta - 2\Sigma x_i\right) = \underbrace{\frac{1}{\theta^3}}_{pos} \left(\underbrace{\Sigma x_i - 2\Sigma x_i}_{neg}\right) < 0 \quad \checkmark \Rightarrow \max.$$

$\Sigma x_i/2n$

---

c) Find $E(b)$

$EX = \alpha\beta = 2\theta$

$$E\left(\frac{\Sigma x_i}{2n}\right) = \frac{\Sigma E X_1}{2n} = \frac{\Sigma(2\theta)}{2n} = \theta$$ C

---

d) yes.

UMVUE should be unbiased and based off m.s.stat

b) = unbiased

based off Σx_i = m.s. stat

C

e) CRLB for unbiased estimators of θ.

$$\text{CRLB} = \frac{\left\{\frac{d}{d\theta} T(\theta)\right\}^2}{I_n(\theta)}$$

$I_n(\theta) = E\left(-\frac{d^2}{d\theta^2} \ell(\theta|\underset{\sim}{X})\right)$ ← $\log f(\underset{\sim}{x}|\theta)$

$$E\left(-\left(\frac{2n}{\theta^2} - \frac{2\Sigma X_i}{\theta^3}\right)\right)$$

$$= \frac{-2n}{\theta^2} + \frac{2\Sigma EX_i}{\theta^3}$$

$$= \frac{-2n}{\theta^2} + \frac{2n2\cancel{\theta}}{\theta^{\cancel{3}}} = \frac{-2n+4n}{\theta^2} = \frac{2n}{\theta^2}$$

CRLB for $T(\theta) = \theta \Rightarrow \frac{1}{(2n/\theta^2)} = \boxed{\frac{\theta^2}{2n}}$ C

---

f) CRLB for unbiased estimators of θ^2

$$\text{CRLB} = \frac{\left\{\frac{d}{d\theta}\theta^2\right\}^2}{(2n/\theta^2)} = \frac{4\theta^2}{(2n/\theta^2)} = \boxed{\frac{2\theta^4}{n}}$$ C

$\frac{d}{d\theta} = 2\theta$

$(\frac{d}{d\theta})^2 = 4\theta^2$

---

g) $\hat\theta_1 = \frac{\Sigma X_i}{2n} = \frac{\Sigma X_i}{2(32)}$ $\quad \hat\theta_2 = \frac{\Sigma Y_i}{n} = \frac{\Sigma Y_i}{44}$ $\quad E(\hat\theta_2) = \frac{\Sigma EY_i}{n} = \theta$

$\hat\theta_2 \Rightarrow L(\theta|\underset{\sim}{y}) = \theta^{-n} e^{-\Sigma y/\theta}$

$\log(\theta|\underset{\sim}{y}) = -n\log\theta - \frac{\Sigma y_i}{\theta}$

$\frac{d\ell}{d\theta} = \frac{-n}{\theta} + \frac{\Sigma y_i}{\theta^2}$ $\quad \hat\theta_2 = \frac{\Sigma y_i}{n}$

$\frac{d^2\ell}{d\theta^2} = \frac{n}{\theta^2} - \frac{2\Sigma y_i}{\theta^3} < 0 \checkmark$ max.

$\text{Var}\,\hat\theta_1 = \frac{\Sigma \text{Var} X_i}{4n^2} = \frac{n(2\theta^2)}{4n^2} = \frac{\theta^2}{2n} \Rightarrow \frac{\theta^2}{64}$

$\text{Var}\,\hat\theta_2 = \frac{\Sigma \text{Var} Y_i}{n^2} = \frac{n\theta^2}{n^2} = \frac{\theta^2}{n} \Rightarrow \frac{\theta^2}{44}$

Both are unbiased for θ.
However, the variance of $\hat\theta_1$ reaches the CRLB, and with the sample sizes specified has a smaller variance (even though the n is smaller).

C

Question 4

$X_1 \ldots X_n$ $i=1 \ldots n$ is a random sample

from $P(X_i = j)$ $j = 1, 2, 3, 4$ (four options for each X_i to be)

$= P_j$ $\quad \Sigma P_j = 1$ each $P_j > 0$

$Y_j =$