a red, b blue $\quad P(\text{red}) = \frac{a}{a+b} \quad P(\text{blue}) = \frac{b}{a+b}$

$\Rightarrow$ pick red $\Rightarrow P(\text{red}) = \frac{a}{a+b}, \quad P(\text{blue}) = \frac{b}{a+b}$

pick blue $\Rightarrow P(\text{red}) = \frac{a+1}{a+b}$

$P(\text{blue}) = \frac{b-1}{a+b}$

$\Rightarrow$ pick red $\Rightarrow P(\text{red}) = \frac{a}{a+b}$

$P(\text{blue}) = \frac{b}{a+b}$

pick blue $\Rightarrow P(\text{red}) = \frac{a+2}{a+b}$

$P(\text{blue}) = \frac{b-2}{a+b}$

$\Rightarrow$ pick red $= \quad P(\text{red}) = \frac{a}{a+b} \quad P(\text{blue}) = \frac{b}{a+b}$

pick blue $= \quad P(\text{red}) =$

draw #	red	blue	
1	a	b	$\Rightarrow \frac{a}{a+1}$ red
2	a or $a+1$	b or $b-1$	

4) Find the probability that the $(n+1)$st ball drawn is red, and show this prob converges to 1 as $n \to \infty$

$\Rightarrow$ dichotomous outcome

$$R_n = \begin{cases} 1 & \text{if } n^{th} \text{ draw is red} = 1 \\ 0 & \text{if } n^{th} \text{ draw is blue} \end{cases}$$

$P((n+1)\text{st draw} = \text{red}) = P(R_{n+1} = 1)$

$= E(R_{n+1}) = E\,E(R_{n+1} \mid X_n)$

$= E\left(\frac{X_n}{a+b}\right)$ $\rightarrow$ prob of drawing red after n draws $\Rightarrow$ we expect X_n red balls / $a+b$ total

$= \frac{EX_n}{a+b}$

$$\underbrace{\frac{1}{a+b}\left(a+b-b\left(1-\frac{1}{a+b}\right)^n\right)}_{\text{simplify, then take lim } n \to \infty}$$

c) Show $E\{X_{n+1}\} = \left(1 - \frac{1}{a+b}\right) E(X_n) + 1$

$$\underbrace{EX_{n+1} = E(EX_{n+1} \mid X_n)}_{\text{have to see this}}$$

$$E(X_{n+1} \mid X_n) = \underbrace{\left(\frac{X_n}{a+b}\right)}_{P(\text{red})} \cdot \underbrace{X_n}_{(\#\text{ red stays the same})} + \underbrace{\left(\frac{\overbrace{a+b}^{\text{total}} - \overbrace{X_n}^{\#\text{ reds}}}{a+b}\right)}_{P(\text{blue})} \underbrace{(X_n + 1)}_{\text{blue increases by 1}}$$

simplify

$$= \left(1 - \frac{1}{a+b}\right) X_n + 1$$

$$E\,E(X_{n+1} \mid X_n) = E\left(\left(1 - \frac{1}{a+b}\right) X_n + 1\right)$$

$$= \left(1 - \frac{1}{a+b}\right) EX_n + 1 = E(X_{n+1})$$

---

d) Use c) to show that $\longrightarrow$ implies induction. Show true $n=1$
assume true $n=n$
show true $n=n+1$

$$E(X_n) = a + b - b\left(1 - \frac{1}{a+b}\right)^n$$

$$E(X_1) = a + b - b\left(1 - \frac{1}{a+b}\right)^1$$

$$= a + b - b + \frac{b}{a+b} = a + \frac{b}{a+b} \Rightarrow \text{TRUE}$$

Assume true for $n=n$

$$EX_{n+1} = a + b - b\left(1 - \frac{1}{a+b}\right)^{n+1} = \left(1 - \frac{1}{a+b}\right) E(X_n) + 1$$

$$= \left(1 - \frac{1}{a+b}\right)\left(a + b - b\left(1 - \frac{1}{a+b}\right)^n\right) + 1 \Rightarrow \text{simplify to}$$

Question 2

2015, MS-1

X = time to recurrence $\sim \text{Uni}(0,1)$ $EX = \frac{1}{2}$ $\text{Var } X = \frac{1}{12}$

$Y|X \sim \text{Uni}(X, X+1)$ $\quad EY|X = \frac{(X+1)+X}{2} = \frac{1+2X}{2}$ $\quad \text{Var}Y|X = \frac{(X+1-X)^2}{12} = \frac{1}{12}$

Y = time until death

a) $EY = E_X EY|X = E(\frac{1}{2} + X) = \frac{1}{2} + EX = \frac{1}{2} + \frac{1}{2} = 1$

$f(x) = I(0<x<1)$
$f(y|x) = I(x<y<x+1)$

$$\text{Var}y = E\,\text{Var}y|X + \text{Var}\,Ey|X$$
$$= E(\tfrac{1}{12}) + \text{Var}(\cancel{\tfrac{1}{2}} + X)$$
$$= \tfrac{1}{12} + \tfrac{1}{12} = \tfrac{1}{6}$$

---

b) $\text{corr}(x, y)$

$$= \frac{\text{cov}(x,y)}{\sqrt{\text{Var}X \cdot \text{Var}Y}} = \frac{EXY - EXEY}{\sqrt{\text{Var}X \cdot \text{Var}Y}}$$

$\text{Var}X + (EX)^2 = \frac{1}{12} + \cancel{\frac{1}{4}}^{3/12} = \frac{4}{12} = \frac{1}{3}$

$$EXY = E(X \cdot EY|X)$$
$$= E(X \cdot (\tfrac{1}{2} + X)) = \tfrac{1}{2}(\tfrac{1}{2}) + EX^2 = \tfrac{1}{4} + \tfrac{1}{3} = \tfrac{7}{12}$$

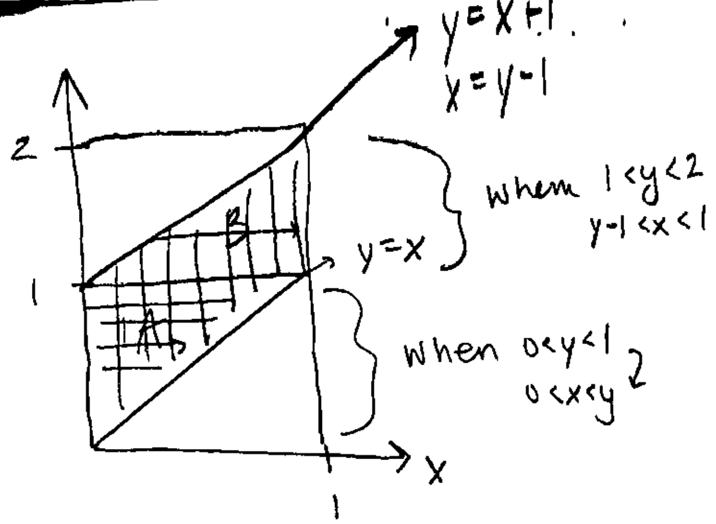
$$\text{corr}(x,y) = \frac{\frac{7}{12} - \frac{1}{2}(1)}{\sqrt{(\frac{1}{12})(\frac{1}{6})}} = \frac{1/12}{\sqrt{1/72}} = \frac{\frac{1}{2}}{\frac{1}{6\sqrt{2}}} = \frac{3}{\sqrt{2}}$$

c) Find $f_Y(y)$

if $y < 1$ then $\int_0^y (1)\, dx = y$

$y > 1$ then $\int_{y-1}^1 (1)\, dx = 1 - (y-1)$

$2 - y$



$$f(y) = \begin{cases} y & \text{if } y < 1 \\ 2-y & \text{if } y > 1 \end{cases}$$

Qagish's 'easier way'

let $W = Y - X \sim Uni(0,1)$

$W \perp X$

$\Rightarrow Y = W + X$

$\underbrace{E(Y|Y)}_{Y} = \underbrace{E(X|Y) + E(W|Y)}$

$= 2E(X|Y)$ b/c X and W are symm. and iid

$\Rightarrow E(X|Y) = Y/2$

d) $f(x|y) = \dfrac{f(x,y)}{f(y)}$

$$f(x|y) = \begin{cases} 1/y & 0 < x < y, \quad 0 < y < 1 \\ 1/2-y & y-1 < x < 1, \quad 1 < y < 2 \end{cases}$$

bounds of x, from the above joint range

$X = Y - W$

$E\left(\frac{X}{Y}\right) = E\left(\frac{Y-W}{Y}\right) = E\left(\frac{Y}{Y}\right) - E\left(\frac{W}{Y}\right)$

$E\left(\frac{X}{Y}\right) = 1 - E\left(\frac{W}{Y}\right)$

$2E\left(\frac{X}{Y}\right) = 1$

$E\left(\frac{X}{Y}\right) = \frac{1}{2}$

Question 2 cont

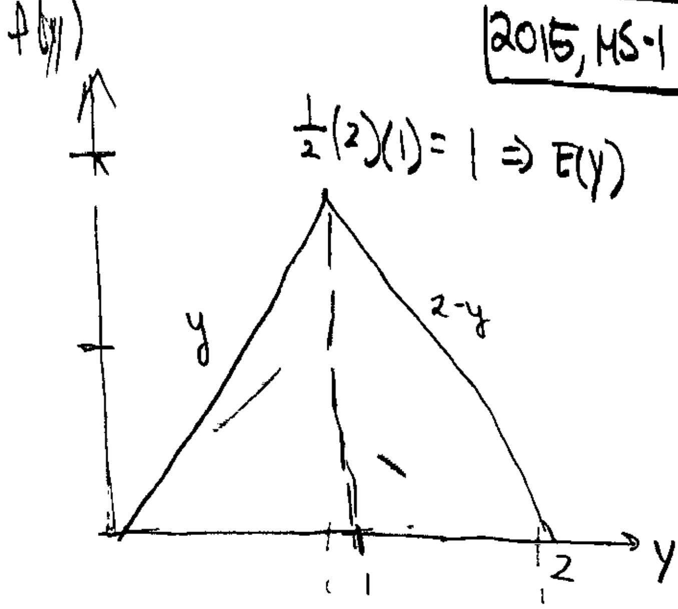
$\boxed{2015, MS\text{-}1}$

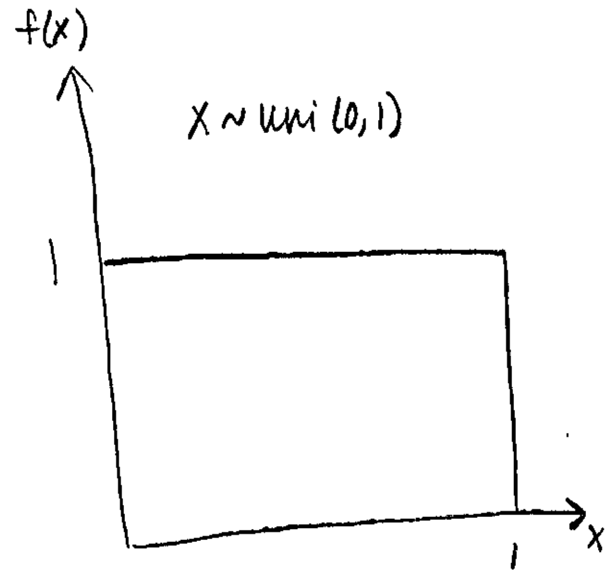
e) $E(X|y) = \int_0^y x \cdot \frac{1}{y}\, dx \qquad \int_{y-1}^{1} x \cdot \frac{1}{(2-y)}\, dx$

$= \frac{x^2}{2}\Big|_0^y \cdot \frac{1}{y} \qquad \frac{x^2}{2}\Big|_{y-1}^{1} \cdot \frac{1}{(2-y)}$

$= \frac{y^2}{2y} \qquad \frac{1-(y-1)^2}{2} \cdot \frac{1}{(2-y)} \qquad y^2-2y+1$

$= \frac{y}{2} \qquad \frac{\overset{y(2-y)}{-y^2+2y}}{2(2-y)} = \frac{y}{2}$

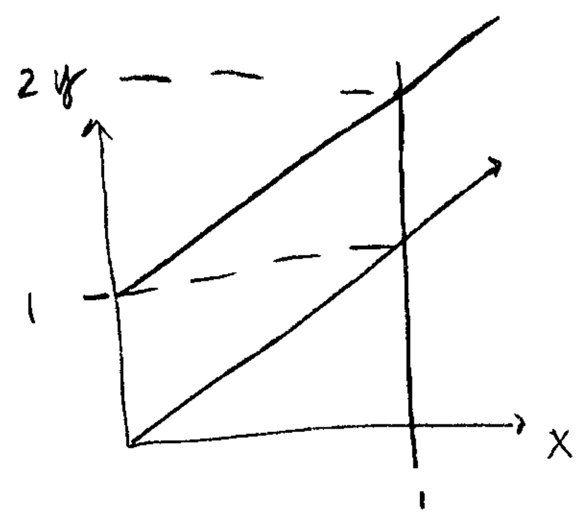




f) $E\left(\frac{X}{Y}\right) = E\left\{E\left(\frac{X}{Y} \,\middle|\, y\right)\right\}$

$= E\left(\frac{1}{y} E(X|y)\right)$

$= E\left(\frac{1}{y}\left(\frac{y}{2}\right)\right) = E\left(\frac{1}{2}\right) = \frac{1}{2}$



$f(x|y) = \frac{f(x) \cdot f(y|x)}{f(y)} = \frac{f(x,y)}{f(x)}$

f) $E\left(\frac{X}{Y}\right) = \iint \frac{x}{y} f(x,y)\, dx\, dy$

$$= \int_0^1 \int_0^y \frac{x}{y}\, dx\, dy + \int_1^2 \int_{y-1}^1 \frac{x}{y}\, dx\, dy$$

$$= \int_0^1 \frac{1}{y}\left\{ \frac{x^2}{2}\Big|_0^y \right\} dy + \int_1^2 \frac{1}{y}\left\{ \frac{x^2}{2}\Big|_{(y-1)}^1 \right\} dy$$

$$= \int_1^1 \frac{y^2}{2y}\, dy + \int_1^2 \frac{1-(y-1)^2}{2y}\, dy$$

$$= \int_0^1 \frac{y}{2}\, dy + \int_1^2 \frac{-y}{2} + 1\, dy$$

$$= \frac{y^2}{2}\Big|_0^1 + \frac{-y^2}{4} + y\Big|_1^2$$

$$= \frac{1}{4} + \left\{(-1+2) - \left(-\frac{1}{4}+1\right)\right.$$

$$= \frac{1}{4} - 1 + 2 + \frac{1}{4} - 1 = \frac{1}{2}$$

$(y-1)^2 =$

$y^2 - 2y + 1$

$1 - y^2 + 2y - 1$

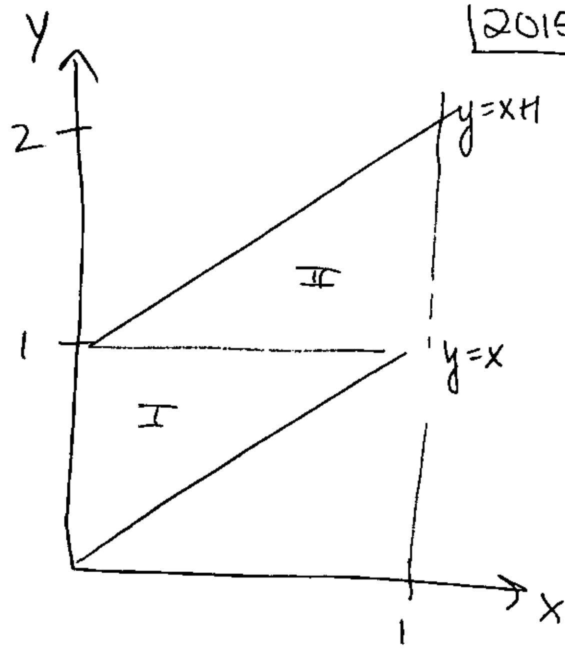
$\frac{-\cancel{y^2}}{2\cancel{y}} + \frac{2y}{2y}$

Question 2

2015, MS-1

$X \sim Uni(0,1)$

$Y|X \sim Uni(X, X+1)$ $0 < x < y$



a) $EY = EE\,Y|X$
$= E_X\left(\frac{(x+1)+x}{2}\right)$
$= E_X\left(\frac{2x+1}{2}\right) = E(x) + \frac{1}{2}$
$= \frac{1}{2} + \frac{1}{2} = 1$

$Var\,Y|X = \frac{(x+1-x)^2}{12}$

$Var\,Y = Var\,EY|X + E\,Var\,Y|X$
$= Var_X(X + \frac{1}{2}) + E_X(\frac{1}{12})$
$= Var\,X + \frac{1}{12} = \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$

$EY = 1 \quad Var\,Y = \frac{1}{6}$

b) $corr(x,y) = \frac{cov(x,y)}{\sqrt{varX \cdot varY}} = \frac{E(XY) - EXEY}{\sqrt{varX \cdot varY}}$

$E(xy) = EE(xy|x)$
$= E(XEY|X) = E(x \cdot (x + \frac{1}{2})) = E(x^2 + \frac{1}{2}x)$
$= EX^2 + \frac{1}{2}EX = (Var\,X + (EX)^2) + \frac{1}{2}EX = \frac{1}{12} + (\frac{1}{2})^2 + \frac{1}{2}(\frac{1}{2})$
$= \frac{1}{12} + \frac{1}{4} + \frac{1}{4} = \frac{1}{12} + \frac{3}{12} + \frac{3}{12} = \frac{7}{12}$

$corr = \frac{\frac{7}{12} - \frac{6}{12}(\frac{12}{12})}{\sqrt{\frac{1}{12}(\frac{1}{6})}} = \frac{\frac{7}{12} - \frac{6}{12}}{\sqrt{1/72}} = \frac{\frac{1}{12}}{6\sqrt{2}} = \frac{1}{2\sqrt{2}}$

c) find $f_Y(y) \Rightarrow \int_X f(x,y)\,dx$ $\qquad f(x,y) = \left\{\frac{1}{(1-0)}\right\}\left\{\frac{1}{(x+1)-x}\right\} = 1$

I: $\int_0^y (1)\,dx$ $\qquad$ II: $\int_{y-1}^{1} 1\,dx$

$x\Big|_0^y = y$ $\qquad$ $x\Big|_{y-1}^{1} = y-2$

$$f(y) = \begin{cases} y & \text{if } y<1, \cancel{\;y\;} \\ \cancel{y/2}\ 2-y & \text{if } y>1, \cancel{\;y\;} \end{cases}$$

bounds only contain y
⇒ marginal does not depend
on x at all

---

d) find $f(x|y) = \frac{1}{f(y)}$

$$f(x|y) = \begin{cases} 1/y & \text{if } y \cancel{\;\;}<1 & 0<x<y \\ 1/\cancel{(y/2)}\ (2-y) & \text{if } y>1 & y-1<x<1 \end{cases}$$

e) Find $E X|y$ $\quad \int_X x \cdot f(x|y)\, dx$

$$= \int_0^y x \cdot \frac{1}{y}\, dx = \frac{x^2}{2} \cdot \frac{1}{y} \Big|_0^y = \frac{y^2}{2} \cdot \frac{1}{y} = \frac{y}{2} \quad \text{if } y < 1$$

$$= \int_{y-1}^1 x \cdot \frac{1}{(2-y)}\, dx = \frac{x^2}{2} \cdot \frac{1}{(2-y)} \Big|_{y-1}^1 = \left\{\frac{1}{2} \cdot \frac{1}{(2-y)}\right\} - \left\{\frac{(y-1)^2}{2} \cdot \frac{(1)}{(2-y)}\right\}$$

$$\frac{1}{2(2-y)} - \frac{y^2-2y+1}{2(2-y)} = \cancel{y(y-2)} = \frac{y\,(2-y)}{2(2-y)} = \frac{y}{2} \quad \text{if } y > 1$$

$$E X|y = \begin{cases} \frac{y}{2} & \text{if } y < 1 \\ \frac{y}{2} & \text{if } y > 1 \end{cases} \Rightarrow \frac{y}{2} \text{ for } 0 < y < 2$$

---

f) Find $E\left(\frac{X}{Y}\right)$

$$E\left(\frac{X}{Y}\right) = E\,E\left(\frac{X}{Y} \Big| Y\right) = E\left(\frac{1}{Y} E X|Y\right) = E\left(\frac{1}{Y} \cdot \frac{Y}{2}\right) = E\left(\frac{1}{2}\right) = \frac{1}{2}$$

Question 3 | CO | 2015, MS-1 | [illegible]

$X_1 \dots X_n \sim f(x|\theta) = \theta e^{-\theta x}$

a) Derive the MLE, $\hat{\theta}$. Show that $1/\hat{\theta}$ is the UMVUE of all unbiased estimators of $1/\theta$

$L(\theta|\underset{\sim}{X}) = \theta^n e^{-\theta \Sigma X_i} \qquad = \prod_{i=1}^{n} \theta e^{-\theta X_i}$

$l(\theta|\underset{\sim}{X}) = n\log\theta - \theta \Sigma X_i \qquad = \Sigma \log\theta - \theta X_i$

$u(\theta|\underset{\sim}{X}) = n/\theta - \Sigma X_i \qquad = \Sigma(1/\theta - X_i)$

$J(\theta|\underset{\sim}{X}) = -(-n/\theta^2) \qquad = -\Sigma(-1/\theta^2)$

$I_1(\theta) = E(1/\theta^2) = 1/\theta^2$

$I_n(\theta) = E(n/\theta^2) = n/\theta^2$

$\Rightarrow$ MLE = max likelihood or log function (derive & set = 0)

$\frac{n}{\theta} - \Sigma X_i = 0 \qquad \hat{\theta}_{MLE} = \frac{n}{\Sigma X_i} = \bar{X}^{-1}$

$\Rightarrow 1/\hat{\theta} = 1/\bar{X}^{-1} = \bar{X}$ (by invariance of MLE's.)

$\Rightarrow$ Show UMVUE.

1) It is an unbiased estimator based solely on the comp + m.s.s.

$f(x|\theta) = \theta e^{-\theta x} \Rightarrow h(x)=1 \quad c(\theta)=\theta \quad t(x)=x \quad w(\theta)=\theta$

$\Rightarrow \Sigma X_i$ is m.s.s + complete

2) $Var\,\bar{X} = \frac{\Sigma Var X_i}{n^2} = \frac{n(\frac{1}{\theta^2})}{n^2} = \frac{1}{\theta^2 n}$

$CRLB = \frac{\{\frac{d}{d\theta}\tau(\theta)\}^2}{n/\theta^2} = \frac{(\frac{1}{\theta^2})^2}{\frac{n}{\theta^2}} = \frac{1}{\theta^2 n}$

reaches the CRLB.

b) Derive the limiting variance of $\underbrace{\sqrt{n}(\hat{\theta}-\theta) \xrightarrow[d]{} N(0, I_1(\theta)^{-1})}_{\text{by properties of the MLE}}$

$I_1(\theta)^{-1} = \theta^2$

CRLB for $\theta = \dfrac{1}{n/\theta^2} = \dfrac{\theta^2}{n}$

CRLB* $n = \theta^2$

$\Rightarrow$ the limiting variance does reach the CRLB * n

---

c) Suppose $Z_1 \dots Z_n$ are not observed. τ constant.

$Z_i = \begin{cases} 1 & \text{if } X_i > \tau \\ 0 & \text{if } X_i \le \tau \end{cases}$

Find MLE of θ $(\tilde{\theta})$ based on $Z_1 \dots Z_n$, and derive the large sample distn. in explicit form.

$p = P(X_i > \tau)$

$= 1 - \int_0^\tau \theta e^{-\theta x} dx = 1 + e^{-\theta x}\Big|_0^\tau = 1 + (e^{-\theta\tau} - \cancel{e^{-\theta(0)}}) = e^{-\theta\tau}$

$p = e^{-\theta\tau} \Rightarrow \tilde{\theta} = -\dfrac{\log(\tilde{p})}{\tau}$ $\Leftarrow$ the new MLE of θ by invariance

$\tilde{p} = \bar{Z}$

$L(p|\underset{\sim}{z}) = \prod p^{z_i}(1-p)^{1-z_i} = p^{\Sigma z_i}(1-p)^{n-\Sigma z_i}$

$\ell(p|\underset{\sim}{z}) = \Sigma z_i \log p + (n - \Sigma z_i)\log(1-p)$

$\dfrac{d\ell}{dp} = \dfrac{\Sigma z_i}{p} + \dfrac{-(n-\Sigma z_i)}{(1-p)} = 0$

$\Sigma z_i(1-\cancel{p}) - np + \cancel{\Sigma z_i p}$

$\tilde{p} = \dfrac{\Sigma z_i}{n} = \bar{z}$

$\Rightarrow$

Therefore, $\boxed{\tilde{\theta} = -\frac{\log \bar{Z}}{\tau}}$

$\Rightarrow$ large sample distn. $\rightarrow$ use δ method.

$\sqrt{n}\,(\bar{Z} - p) \xrightarrow{d} N(0,\ p(1-p))$ by CLT $\quad \sqrt{n}\left(\frac{\Sigma X_i}{n} - EX_1\right) \xrightarrow{d} N(0, Var X_1)$

$$\sqrt{n}\left(\underbrace{\frac{\log \bar{Z}}{-\tau}}_{g(p)} - \theta\right) \xrightarrow{d} N\left(0,\ \underbrace{\overbrace{p(1-p)}^{Var X_1}\ \overbrace{\frac{1}{p^2\tau^2}}^{g'(p)^2}}\right)$$

$g(w) = \frac{\log w}{-\tau}$

$g' = \frac{1}{-w\tau}$

$(g')^2 = \frac{1}{(w\tau)^2}$

$$\frac{p(1-p)}{p^2\tau^2} = \frac{(1-p)}{p\,\tau^2} = \frac{1-e^{-\theta\tau}}{e^{-\theta\tau}\tau^2} = \frac{(e^{\theta\tau}-1)}{\tau^2}$$

d) Take $\tau = EX_1 = \frac{1}{\theta}$, compare asymptotic variances

$$ARE(\tilde{\theta}, \hat{\theta}) = \frac{\frac{e^{\theta\tau}-1}{\tau^2}}{\theta^2} = \frac{(e^{1}-1)\,\theta^2}{\theta^2} > 1$$

$\Rightarrow \tilde{\theta}$ has the greater asymptotic variance.

$\Rightarrow$ Expected because we dichotomized, which means we are losing information and therefore variability

~~Question #4~~ 2013

$X_1 \dots X_n \sim f(x|\theta) = \theta e^{-\theta x}$

$Y_1 \cdots Y_n \sim f(y|\beta) = \beta e^{-\beta x}$

Test $H_0: \phi = 1$ vs. $H_1: \phi \neq 1$ $\qquad \phi = \dfrac{\theta}{\beta}$

$\Rightarrow \quad \theta - \beta = 0$

① Likelihood Ratio

$2(\ell(\hat{\theta}) - \ell(\theta_0)) \xrightarrow[d]{} \chi^2_1$ ← is this true for 2 samples? I don't think so.

② Score Test (?)

③ Wald

$$\frac{(\hat{\theta} - \hat{\beta})}{\sqrt{I_n(\hat{\theta}) + I_n(\hat{\beta})}} \quad (?)$$

Question 3 | 2015, MS-I

$X_1 \dots X_n \sim f(x|\theta) = \theta e^{-\theta x} \quad x > 0 \quad \theta > 0$

a) Derive the MLE $\hat{\theta}$. Show that $1/\hat{\theta}$ has the uniformly minimum variance among all unbiased estimators of $1/\theta$

$L(\theta | \underset{\sim}{X}) = \theta^n e^{-\theta \Sigma X_i}$

$l(\theta | \underset{\sim}{X}) = n \log \theta - \theta \Sigma X_i$

$\frac{d \log}{d\theta} = \frac{n}{\theta} - \Sigma X_i \qquad \hat{\theta} = \frac{n}{\Sigma X_i} = \bar{X}^{-1}$

$\frac{d \log}{d\theta} = \frac{-n}{\theta^2} < 0$ ✓

· To be the UMVUE, it is required to be unbiased and be based soley upon the minimum, sufficient, and complete stat

$\frac{\cancel{\theta^n} e^{-\theta \Sigma X_i}}{\cancel{\theta^n} e^{-\theta \Sigma Y_i}} \rightsquigarrow$ to be $\perp$ θ, $\Sigma X_i = \Sigma Y_i \quad T(X) = \Sigma X_i$ is minimally sufficient

ΣX_i is also complete by the exponential families

$E(\hat{\theta}) = E(\bar{X}^{-1})$

$E(1/\hat{\theta}) = E(\bar{X}) = \frac{n(1/\theta)}{n} = \frac{1}{\theta} \Rightarrow$ unbiased

$\Rightarrow$ based on ΣX_i, the m.s.s. + complete

$\Rightarrow$ UMVUE.

b) $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \sigma^2)$

MLE's are consistent

CRLB for unbiased estimators of θ

$= \dfrac{1}{E(-\frac{d^2}{d\theta^2}\log f(x|\theta))}$

$\sigma^2 = I_1(\theta)^{-1}$ by properties of the MLE

$I_1(\theta) = \frac{1}{\theta^2}$

$I_1(\theta)^{-1} = \theta^2$

$= E(-{}^{n}/_{\theta^2}) = \frac{n}{\theta^2}$

$\Rightarrow {}^{1}/_{(n/\theta^2)} = {}^{\theta^2}/_{n}$ = CRLB for unbiased est. of θ

$\Rightarrow \sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \theta^2)$

$CRLB * n = \frac{\theta^2}{n}\cdot n = \theta^2$ ✓ yes, the asymptotic variance reaches the CRLB.

---

c) $X_1 \dots X_n$ not observed $Z_1 \dots Z_n$ are observed

$Z_i = \begin{cases} 1 & w/\ P(X_i > \tau) \\ 0 & w/\ P(X_i \le \tau) \end{cases}$

Find alternative MLE for θ ($\tilde{\theta}$) based off of this. Derive the asymptotic distn.

$P = P(X_i > \tau) = 1 - P(X_i < \tau)$

$= 1 - \int_0^{\tau} \theta e^{-\theta x}\,dx = 1 - (-e^{-\theta x}\big|_0^{\tau}) = 1 + e^{-\theta\tau} - 1 = e^{-\theta\tau}$

$p = e^{-\theta\tau} \Rightarrow \theta = \dfrac{\log p}{-\tau}$ find MLE $\tilde{p}$, then use it to find $\tilde{\theta}$

$\tilde{p} = \bar{Z}$ (by Bernoulli)

$\tilde{\theta} = \dfrac{\log \bar{Z}}{-\tau}$

$\sqrt{n}(\bar{Z} - p) \xrightarrow{d} N(0, \overset{p(1-p)}{\operatorname{Var} Z_1})$

$g(w) = \dfrac{\log w}{-\tau}$

$g'(w) = \dfrac{1}{-\tau w}$

$g'(w)^2 = (\tau w)^{-2}$

$\sqrt{n}\left(\dfrac{\log \bar{Z}}{-\tau} - \theta\right) \xrightarrow{d} N(0, \underbrace{p(1-\overset{e^{-\theta\tau}}{p})\cdot(\tau\theta)^{-2}}_{\frac{e^{-\theta\tau}(1-e^{-\theta\tau})}{(\tau\theta)^2}})$

d) if $\tau = EX_1 = \frac{1}{\theta}$

compare Var of $\hat{\theta}$ and $\tilde{\theta}$

$\hat{\theta} = \theta^2$

$\tilde{\theta} = \frac{e^{-1}(1-e^{-1})}{(1)^2} = e^{-1} - e^{-2}$

the var of $\tilde{\theta}$ showed be lager ⇒ lose info by dichotomizing and so increase the uncertaintly / variance

---

e) $Y_1 \dots Y_n \sim f(y|\beta) = \beta e^{-\beta y}$ $y > 0$, $\beta > 0$

$H_0: \phi = 1$ vs. $H_1: \phi \neq 1$ $\phi = \frac{\theta}{\beta}$ $H_1: \theta = \beta$

large sample wald

$$\{(\hat{\theta}-\hat{\beta}) - (\theta-\beta)\} \xrightarrow{d} N(0, I_n(\hat{\theta})^{-1} + I_m(\hat{\beta})^{-1})$$

likelihood ratio: $\gamma = \theta = \beta$ in joint likelihood then maximize.

$-2\{ \ell(\hat{\theta}, \hat{\beta}) - \ell(\gamma_0)$

$X_1 \ldots X_n$ iid $\mathcal{E}xp(\frac{1}{\theta})$

$\sim f(x|\theta) = \theta e^{-\theta x}$

a) Derive the MLE of $\theta = \hat{\theta}$

$L(\theta|\underset{\sim}{x}) = \theta^n e^{-\theta \Sigma x_i}$

$\ell(\theta|\underset{\sim}{x}) = n \log \theta - \theta \Sigma x_i$

$\frac{d\ell}{d\theta} = \frac{n}{\theta} - \Sigma x_i = 0 \qquad \hat{\theta} = \frac{n}{\Sigma x_i} = \bar{x}^{-1}$

$\frac{d^2\ell}{d\theta^2} = \frac{-n}{\theta^2} < 0$ ✓ $\Rightarrow$ maximum

$\frac{1}{\theta} = \frac{1}{\bar{x}^{-1}} = \bar{x}$

$E(\bar{x}) = \frac{\Sigma(EX_i)}{n} = \frac{\frac{1}{\theta}(n)}{n} = \frac{1}{\theta} \Rightarrow$ unbiased

$f(\underset{\sim}{x}|\theta) = \theta^n e^{-\theta x}$ $\qquad \Rightarrow \Sigma x_i$ is suff-stat (minimal, and complete

$\Rightarrow \frac{1}{\theta}$ is the UMVUE

CRLB = unbiased ests of $1/\theta$

$\frac{\cancel{E}\{\frac{d}{d\theta}\frac{1}{\theta}\}^2}{E(- - \frac{n}{\theta^2})} = \frac{\frac{1}{\theta^4}}{\frac{n}{\theta^2}} = \frac{1}{\theta^2 n}$

also, achieves the CRLB

$Var(\bar{x}) = \frac{\Sigma Var X}{n^2} = \frac{n \frac{1}{\theta^2}}{n^2} = \frac{1}{\theta^2 n}$

b) $\sqrt{n}(\hat{\theta} - \theta)$ as $n \to \infty$

$\hat{\theta} = \bar{X}^{-1} \Rightarrow$ need Δ method

$$\sqrt{n}\left(\frac{\Sigma X_i}{n} - \frac{1}{\theta}\right) \xrightarrow[d]{} N\left(0, \frac{1}{\theta^2}\right) \text{ as } n \to \infty$$

$$\sqrt{n}\left(\frac{1}{\bar{X}} - \underbrace{\theta}_{g\left(\frac{1}{\theta}\right)}\right) \xrightarrow[d]{} N\left(0, \frac{\theta^4}{\theta^2} = \theta^2\right)$$

$g(w) = \frac{1}{w}$

$g'(w) = \frac{-1}{w^2}$

$g'w^2 = w^{-4}$

$\left(\frac{1}{\theta}\right)^{-4} = \theta^4$

CRLB for unbias of θ

$$\frac{1}{\frac{n}{\theta^2}} = \frac{\theta^2}{n}$$

CRLB $* n = \theta^2$

$\Rightarrow$ limiting var reaches the CRLB

c) $Z_i = \begin{cases} 1 & \text{if } (X_i > \tau) \\ 0 & \text{if } X_i \le \tau \end{cases}$

$e^{-\theta x}$ $\quad u = \theta x$

$du = -\theta dx$

$-\frac{du}{\theta} = dx$

$$p = P(X_i > \tau)$$

$$= 1 - \int_0^\tau \theta e^{-\theta x} dx = 1 + \int_0^\tau e^u du$$

$$= 1 + e^{-\theta x}\Big|_0^\tau = 1 + \left(e^{-\theta\tau} - \cancel{e^{-\theta(0)}}\right)$$

$$= e^{-\theta\tau}$$

$p = e^{-\theta\tau}$

$\Rightarrow \theta = \frac{\log p}{-\tau}$

$\tilde{\theta} = \frac{\log(\hat{p})}{-\tau}$ by invariance

$\hat{p} = \bar{Z}$ by Bernoulli

$\sqrt{n}(\bar{Z} - p) \xrightarrow{d} N(0, p(1-p))$ as $n \to \infty$

$g(w) = \frac{\log(w)}{-\tau}$

$g'(w) = \frac{1}{-\tau w}$

$g'(w)^2 = (\tau w)^{-2}$

$\sqrt{n}\left(\frac{\log \bar{Z}}{-\tau} - \frac{\log p}{-\tau}\right) \xrightarrow{d} N(0,$

$\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N(0, (\tau p)^{-2} e^{-\theta\tau}(1 - e^{-\theta\tau}))$

d) $\tau = EX_1 = \frac{1}{\theta}$

ARE of $\hat{\theta}$ and $\tilde{\theta}$

$\hat{\theta} = \theta^2$

$$\tilde{\theta} = \frac{e^{-\theta\tau}(1-e^{-\theta\tau})}{\cancel{(\tau\theta)^2}\ \tau^2 e^{-2\theta\tau}} = \frac{e^{-1}(1-e^{-1})}{\left(\frac{1}{\theta}\right)^2 \cancel{(1)^2}\ e^{-2}} = \theta^2\left(\frac{e^{-1}-e^{-2}}{e^{-2}}\right)$$

$$= \theta^2(e-1)$$

$$\frac{\theta^2(e-1)}{\theta^2}$$

Var $\tilde{\theta}$ is larger

$\Rightarrow$ makes sense, based off dichotomous info $\rightarrow$ losing variability/information

e) $Y_1 \dots Y_n \sim \beta e^{-\beta y}$

$H_0: \phi = 1$ vs $H_1: \phi \neq 1$ $\quad \phi = \frac{\theta}{\beta}$

$\Rightarrow$ use Wald test. need $\operatorname{var} \phi$

$\Rightarrow$ take log to get sums

$\log \phi = \log \theta - \log \beta$ ← indep so can just sub once have indiv. asymptotic distns.

$\cancel{\log \theta}$ $\quad \sqrt{n}\,(\bar{X}^{-1} - \theta) \xrightarrow[d]{} N(0, \theta^2)$ $\qquad \log w$

$\frac{1}{w}$

$\sqrt{n}\,(\log(\bar{x}^{-1}) - \log\theta) \xrightarrow[d]{} N(0, 1)$ $\qquad \frac{1}{w^2}$

$\sqrt{n}\,(\log \bar{y}^{-1} - \log\beta) \xrightarrow[d]{} N(0, 1)$

$\log(\theta/\beta) = \log(1)$

$\log\theta - \log\beta = 0$

$\Rightarrow \sqrt{n}\left\{(\log \bar{x}^{-1} - \log\bar{y}^{-1}) - (\log\theta - \log\beta)\right\} \xrightarrow[d]{} N(0, 2)$

as $n \to \infty$

$$\frac{\sqrt{n}(\log(\bar{x}^{-1}) - \log(\bar{y}^{-1}))}{\sqrt{2}} \sim N(0, 1)$$

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$\text{Uni}(a, \theta) \sim X \quad 0 \le a < \theta < \infty$ $\quad$ $a = \text{KNOWN}$, $\theta = \text{UNKNOWN}$ $\quad$ $F(x) = \int_a^x \frac{1}{(\theta - a)}\, dx = \frac{x-a}{\theta-a}$

concerned w/ $X_{(n)} = \theta$

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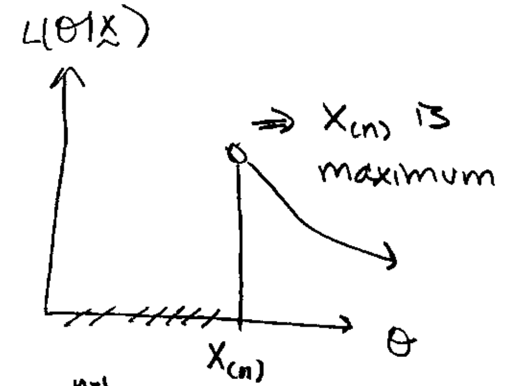
a) Show MLE $= X_{(n)}$

Biased at finite, but consistent as $n \to \infty$



$$L(\theta \mid \underset{\sim}{X}) = \frac{1}{(\theta - a)^n} I(a < X_{(1)})\, I(X_{(n)} < \theta)$$

$$E(X_{(n)}) = \int_a^\theta x \cdot f_{X_{(n)}}(x)\, dx$$

$$= \cancel{\int_a^\theta x \cdot \frac{n x^{n-1}}{(\theta-a)^n}\, dx}$$

$$= \cancel{\frac{n}{n+1} \frac{x^{n+1}}{(\theta-a)^n} \Big|_a^\theta} = \cancel{\frac{n}{n+1} \frac{(\theta-a)^{n+1}}{(\theta-a)^n}}$$

$$= \frac{n}{n+1} (\theta - a)$$

$$f_{X_{(n)}}(x) = n F(x)^{n-1} f(x) = n \frac{1}{(\theta-a)} \frac{(x-a)^{n-1}}{(\theta-a)^{n-1}} = \frac{n x^{n-1}}{(\theta-a)^n}$$

$E(X_{(n)}) \ne \theta \Rightarrow$ BIASED ESTIMATOR.

$$\cancel{\operatorname{Var} X_{(n)} = E X_{(n)}^2 - (E X_{(n)})^2}$$

$$\cancel{\int_a^\theta x^2 \cdot \frac{n x^{n-1}}{(\theta-a)^n}\, dx - \left(\frac{n}{n+1}\right)^2 (\theta - a)^2}$$

$$\cancel{= \frac{n}{(n+2)} \frac{x^{n+2}}{(\theta-a)^n} \Big|_a^\theta}$$

$$\cancel{= \frac{n}{n+2}(\theta-a)^2 - \frac{n^2}{(n+1)^2}(\theta-a)^2}$$

$$\cancel{= (\theta-a)^2 \left(\frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)}\right)}$$

$$\cancel{\frac{n(n^2+2n+1) - n^3 - 2n^2}{(n+1)^2(n+2)}} = \cancel{\frac{n}{(n+1)^2(n+2)}}$$

$$\lim_{n\to\infty} E(X_{(n)}) = \lim_{n\to\infty} \frac{n}{n+1}(\theta - a) = (\theta - a)$$

$$\lim_{n\to\infty} \operatorname{Var}(X_{(n)}) = \lim_{n\to\infty} (\theta-a)^2 \left(\frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)}\right)$$

$$= 0$$

b) Unbiased estimator of θ as a function of $X_{(n)}$

$\Rightarrow \quad \frac{n+1}{n} X_{(n)} + a$

$$E\left(\frac{n+1}{n} X_{(n)} + a\right) = \frac{n+1}{n} EX_{(n)} + a = \frac{n+1}{n} \cdot \frac{n}{n+1}(\theta - a) + a = \theta - a + a = \theta$$

Comment on why the CRLB fails in this situation.

$\Rightarrow$ the integral and derivative are not exchangeable

$$E\left(-\frac{d}{d\theta} \log f(\underset{\sim}{x} \mid \theta)\right)$$

$$= \int_a^\theta x \cdot -\frac{d}{d\theta} \log(\theta \mid \underset{\sim}{x})\, dx \neq -\frac{d}{d\theta} \int_a^\theta x \cdot \log f(\underset{\sim}{x} \mid \theta)\, dx$$ in this case

(show (?))

redo A)

$u = x \quad v = \frac{(x-a)^{n}}{n}$

$du = dx \quad dv = (x-a)^{n-1}$

$$EX_{(n)} = \int_a^\theta x \cdot \frac{n(x-a)^{n-1}}{(\theta-a)^n}\,dx = \frac{n}{(\theta-a)^n}\int_a^\theta x(x-a)^{n-1}\,dx$$

$\int u\,dv = uv - \int v\,du$

$$= \frac{n}{(\theta-a)^n}\left\{ x\frac{(x-a)^n}{n}\Big|_a^\theta - \int_a^\theta \frac{(x-a)^n}{n}\,dx \right\}$$

$$= \frac{n}{(\theta-a)^n}\left(\left\{\theta\frac{(\theta-a)^n}{n} - \cancel{\frac{a(0)^n}{n}}\right\} - \left\{\frac{(x-a)^{n+1}}{n(n+1)}\Big|_a^\theta\right\}\right)$$

$$= \frac{n}{(\theta-a)^n}\left\{\frac{\theta(\theta-a)^n}{n} - \frac{(\theta-a)^{n+1}}{n(n+1)}\right\}$$

$$= \theta - \frac{(\theta-a)}{n+1} = \frac{\theta n + \cancel{\theta} - \cancel{\theta} + a}{(n+1)} = \frac{n\theta + a}{(n+1)}$$

$E(X_{(n)}) \neq \theta \Rightarrow$ BIASED let $y = \dfrac{x-a}{\theta-a}$ $\Rightarrow$ linear transform, so ordered stats are still ordered stats

$\Rightarrow y \sim \text{Uni}(0,1)$

$$X_{(i)} \sim \text{Beta}(i,\ n-i+1)$$

$$E X_{(i)} = \frac{\alpha}{\alpha+\beta} = \frac{i}{n+1}$$

$E(X_{(n)}) \Rightarrow E$

$$E(Y_{(n)}) = \frac{E X_{(n)} - a}{\theta - a}$$

$$E(X_{(n)}) = E(Y_{(n)})(\theta-a) + a$$
$$= \frac{n}{n+1}(\theta-a) + a$$

$$F(y) = P(Y \le y)$$
$$= P\left(\frac{X-a}{\theta-a} < y\right) = P(X < y(\theta-a) + a)$$
$$= F_X(y(\theta-a)+a)$$

$$\frac{(\overbrace{y(\theta-a)+a}\ -a)}{(\theta-a)} = y \qquad \text{if } F_Y(y) = y \text{ then } y \sim \text{Uni}(0,1)$$

MAKE YOUR OWN TRANSFORMS
DO TRICKY MATH THINGS.

$$\mathrm{Var}X_{(n)} = \underset{\substack{\uparrow\\ \text{need}}}{EX_{(n)}^2} - \underset{\substack{\uparrow\\ \text{have}}}{(EX_{(n)})^2}$$

$$EX_{(n)}^2 = \int_a^\theta x^2 \cdot \frac{n}{(\theta-a)^n} \cdot (x-a)^{n-1}\,dx$$

* by parts is hard / convoluted use:

$$(x-a)^{n+1} = (x-a)^2(x-a)^{n-1}$$

$$= x^2(x-a)^{n-1} - 2ax(x-a)^{n-1} + a^2(x-a)^{n-1}$$

$$= \frac{n}{(\theta-a)^n}\left\{\int_a^\theta (x-a)^{n+1} + \underbrace{2ax(x-a)^{n-1}}_{EX_{(n)}\text{ part}} + a^2(x-a)^{n-1}\,dx\right\}$$

$$= \frac{n}{(\theta-a)^n}\left\{\frac{(x-a)^{n+2}}{n+2}\Bigg|_a^\theta + 2a\left(\frac{\theta(\theta+a)^n}{n} - \frac{(\theta-a)^{n+1}}{n(n+1)}\right) + \frac{a^2(x-a)^n}{n}\Bigg|_a^\theta\right\}$$

$$= \frac{n}{(\theta-a)^n}\left\{\frac{(\theta-a)^{n+2}}{n+2} + \frac{2a\theta(\theta+a)^n}{n} - \frac{2a(\theta-a)^{n+1}}{n(n+1)} - \frac{a^2(\theta-a)^n}{n}\right\}$$

c) Find the UMP size α test for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$ w/ clear specification for the cutoff in the R.

$\Rightarrow$ use $\theta = \theta_1$ in alternative, $\theta_1 > \theta_0$, then generalize

N-P lemma S-S

$$\frac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > c \quad \Rightarrow \text{this is the UMP test of size } \alpha$$

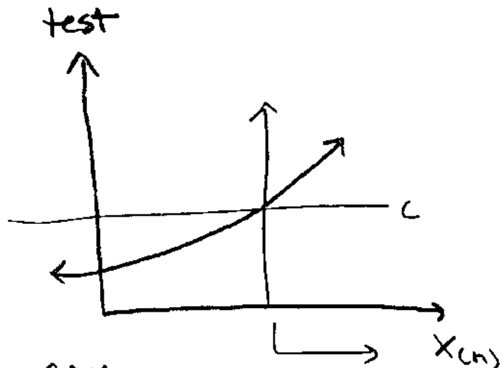
$$f(\underset{\sim}{x}) = \frac{1}{(\theta-a)^n}$$

$$\Rightarrow \frac{\frac{1}{(\theta_1-a)^n}}{\frac{1}{(\theta_0-a)^n}} \frac{I(\theta_1 > X_{(n)})}{I(\theta_0 > X_{(n)})} = \left(\frac{\theta_0-a}{\theta_1-a}\right)^n \frac{I(\theta_1 > X_{(n)})}{I(\theta_0 > X_{(n)})}$$

if $X_{(n)} < \theta_0$ then $X_{(n)} < \theta_1$

if $X_{(n)} < \theta_1$ its not necessarily $> \theta_0$

$$\text{UMP test}: \begin{cases} \left(\frac{\theta_0-a}{\theta_1-a}\right)^n & \text{if } X_{(n)} < \theta_0 \\ \text{undef.} & \text{if } \theta_0 < X_{(n)} < \theta_1 \\ \infty & \text{if } X_{(n)} > \theta_1 \end{cases}$$



$$\alpha = P\left(\left(\frac{\theta_0-a}{\theta_1-a}\right)^n > c \;\middle|\; \theta = \theta_0\right)$$

because MLR, can base the test on $X_{(n)}$

$$= P(X_{(n)} > c \mid \theta = \theta_0)$$

$$= 1 - P(X_{(n)} < c)$$

$$= 1 - P(X < c)^n$$

$$\alpha = 1 - \left(\frac{c-a}{\theta_0 - a}\right)^n \quad \leftarrow \text{sub } \theta_0$$

$\Rightarrow$

$$c = (1-\alpha)^{1/n}(\theta_0 - a) + a$$

d) Show as $n \to \infty$, the R region is indep of α.

$R = \{ x : x_{(n)} > (1-\alpha)^{1/n} (\theta_0 - a) \}$

*JUSTIFY $\lim_{n \to \infty} (1-\alpha)^{1/n}$

$\lim_{n \to \infty} (1-\alpha)^{1/n} (\theta_0 - a) = \theta_0 - a$

$(1-?)^{\infty} = 1$

$\Rightarrow \{ X_{(n)} > \theta_0 - a \}$ as $n \to \infty$

This makes sense because as we take a larger sample,

we are more and more sure

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$Uni(a, \theta)$ a constant, θ unknown parameter.

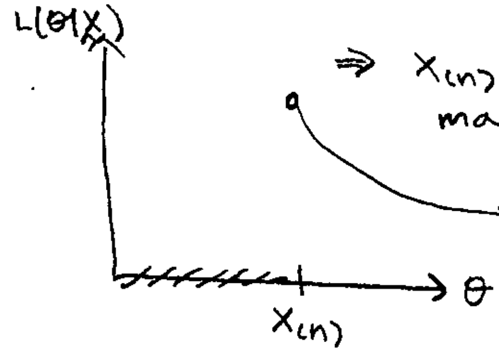
a) Show MLE is max order stat $\{X_{(n)}\}$ $\quad 0 \leq a < \theta < \infty$

Prove it is biased but consistent as $n \to \infty$ $\quad a < X < \theta$

Find MLE by maximizing the likelihood

$$L(\theta | \underset{\sim}{X}) = \prod_{i=1}^{n} \frac{1}{(\theta - a)} I(X_i < \theta)$$

$$= (\theta - a)^{-n} I(X_{(n)} < \theta)$$



$\Rightarrow X_{(n)}$ is the maximum

$\Rightarrow$ show biased : $EX_{(n)} \neq \theta$

$$EX_{(n)} = \int_a^\theta x \cdot f_{X_{(n)}}(x)\, dx$$

$$f_{X_{(n)}}(x) = n f(x)(F(x))^{n-1} = \frac{n}{(\theta - a)} \cdot \left\{\frac{(x-a)}{(\theta - a)}\right\}^{n-1}$$

$$= \int_a^\theta x \cdot \frac{n(x-a)^{n-1}}{(\theta - a)^n}\, dx$$

$$= \frac{n}{(\theta-a)^n} \int_a^\theta x (x-a)^{n-1}\, dx \qquad u = x \quad v = \frac{(x-a)^n}{n} \quad du = dx \quad dv = (x-a)^{n-1}$$

$$= \frac{n}{(\theta-a)^n} \left\{ x \frac{(x-a)^n}{n} \Big|_a^\theta - \int_a^\theta \frac{(x-a)^n}{n} dx \right\}$$

$$= \frac{n}{(\theta-a)^n} \left\{ \frac{\theta(\theta-a)^n}{n} - \frac{(x-a)^{n+1}}{n(n+1)} \Big|_a^\theta \right\}$$

$$= \frac{n\theta(\theta-a)^n}{n(\theta-a)^n} - \frac{n(\theta-a)^{n+1}}{n(n+1)(\theta-a)^n} = \theta - \frac{(\theta-a)}{(n+1)} \Rightarrow \text{BIASED}$$

$$\frac{n\theta + \theta - \theta + a}{n+1} = \frac{n\theta + a}{n+1} = \frac{n}{n+1}\theta + \frac{a}{n+1}$$

$\Rightarrow$ show consistent

$$\lim_{n \to \infty} \theta - \frac{(\theta - a)}{(n+1)} \to \theta$$

$$\operatorname{Var} X_{(n)} = EX_{(n)}^2 - EX_{(n)}^2 \qquad (EX_{(n)})^2 = \left\{ \frac{n}{n+1}\theta + \frac{a}{n+1} \right\}^2$$

$$= \frac{n^2\theta^2}{(n+1)^2} + 2\frac{(n\theta a)}{(n+1)^2} + \frac{a^2}{(n+1)^2}$$

$$EX_{(n)}^2 = \int_a^\theta x^2 \cdot \frac{n}{(\theta-a)^n} \cdot (x-a)^{n-1}\, dx$$

$$= \frac{n}{(\theta-a)^n}\int_a^\theta x^2(x-a)^{n-1} \qquad \begin{array}{ll} u = x^2 & v = \frac{(x-a)^n}{n} \\ du = 2x & dv = (x-a)^{n-1} \end{array}$$

$$= \frac{n}{(\theta-a)^n}\left\{ x^2\frac{(x-a)^n}{n}\Big|_a^\theta - \int_a^\theta \frac{(x-a)^n}{n}\cdot 2x\, dx \right\} \qquad \begin{array}{ll} u = x & v = \frac{(x-a)^{n+1}}{n+1} \\ du = dx & dv = \frac{(x-a)^n}{n} \end{array}$$

$$= \frac{n}{(\theta-a)^n}\left\{ \frac{\theta^2(\theta-a)^n}{n} - 2\left\{ \frac{x(x-a)^{n+1}}{n+1}\Big|_a^\theta - \int_a^\theta \frac{(x-a)^{n+1}}{n+1}dx \right\}\right\}$$

$$= \frac{n}{(\theta-a)^n}\left\{ \frac{\theta^2(\theta-a)^n}{n} - 2\left\{ \frac{\theta(\theta-a)^{n+1}}{n+1} - \frac{(x-a)^{n+2}}{(n+1)(n+2)}\Big|_a^\theta \right\}\right\}$$

$$= \frac{n}{(\theta-a)^n}\left\{ \frac{\theta^2(\theta-a)^n}{n} - 2\left\{ \frac{\theta(\theta-a)^{n+1}}{n+1} - \frac{(\theta-a)^{n+2}}{(n+1)(n+2)} \right\}\right\}$$

$$= \frac{n\theta^2(\theta-a)^n}{n(\theta-a)^n} - \frac{2n\theta(\theta-a)^{n+1}}{(n+1)(\theta-a)^n} + \frac{2n(\theta-a)^{n+2}}{(\theta-a)^n(n+1)(n+2)}$$

$$= \theta^2 - \frac{2n\theta(\theta-a)}{(n+1)} + \frac{2n(\theta-a)^2}{(n+1)(n+2)}$$

$$(n+1)(n+2)$$
$$n^2+3n+2$$

$$= \theta^2 - \frac{(2n\theta^2 - 2n\theta a)}{(n+1)} + \frac{2n\theta^2 - 4n\theta a + 2na^2}{(n+1)(n+2)}$$

$$= \frac{\theta^2(n^2+3n+2) - (2n\theta^2 - 2n\theta a)(n+2) + 2n\theta^2 - 4n\theta a + 2na^2}{(n+1)(n+2)}$$

$$= \frac{\cancel{\theta^2 n^2} + \cancel{3n\theta^2} + 2\theta^2 - (\cancel{2n^2\theta^2} + \cancel{4n\theta^2} - 2n^2\theta a - \cancel{4n\theta a}) + \cancel{2n}\theta^2 - \cancel{4n\theta}a + 2na^2}{(n+1)(n+2)}$$

$$= \frac{-n^2\theta^2 + n\theta^2 + 2\theta^2 + 2n^2\theta a + 2na^2}{(n+1)(n+2)}$$

$(n+1)(n-2)$

$$= \frac{-\theta^2(-n^2 - n - 2) + 2na(n\theta + a)}{(n+1)(n+2)}$$

$$Var X_{(n)} = \left\{ \frac{\theta^2(-n^2+n+2) + 2na(n\theta+a)}{(n+1)(n+2)} \right\} - \left\{ \frac{n^2\theta^2}{(n+1)^2} + \frac{2n\theta a}{(n+1)^2} + \frac{a^2}{(n+1)^2} \right\}$$

take $\lim_{n\to\infty}$ $= \left\{ \theta^2 + \theta - \theta^2 + \theta + a^2 - a^2 \right\} = 0$

$\Rightarrow$ although $X_{(n)}$ is biased, because

$\lim_{n\to\infty} E X_{(n)} = \theta$

$\lim_{n\to\infty} Var X_{(n)} = 0$

we can say $X_{(n)}$ is a consistent estimator.

b) Find an unbiased estimator as a function of $X_{(n)}$

$$E(X_{(n)}) = \frac{n}{n+1}\theta + \frac{a}{n+1}$$

$$\frac{n+1}{n} X_{(n)} - \frac{a}{n} \Rightarrow \text{unbiased}$$

$$E\left(\frac{n+1}{n} X_{(n)} - \frac{a}{n}\right) = \frac{n+1}{n}\left(\frac{n}{n+1}\theta + \frac{a}{n+1}\right) - \frac{a}{n}$$

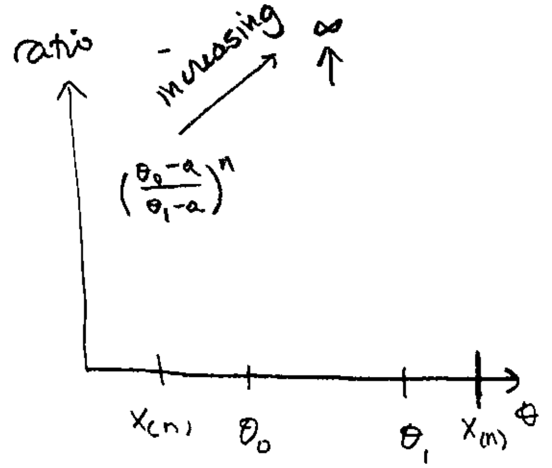
$$= \theta + \frac{a}{n} - \frac{a}{n} = \theta \quad \checkmark$$

The CRLB fails in this situation because the derivative and integral are not exchangeable in the denom.

c) UMP test of size α for $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$

w/ clear specification for the cutoff

$\Rightarrow$ define $H_1^*: \theta = \theta_1 \quad \theta_1 > \theta_0 > a$. Then generalize the test to H_1



$\Rightarrow$ use N-P Lemma

$$\frac{f(\underset{\sim}{x} \mid \theta_1)}{f(\underset{\sim}{x} \mid \theta_0)} = \frac{(\theta_1 - a)^{-n} I(X_{(n)} < \theta_1)}{(\theta_0 - a)^{-n} I(X_{(n)} < \theta_0)}$$

$$= \left(\frac{\theta_0 - a}{\theta_1 - a}\right)^n \frac{I(X_{(n)} < \theta_1)}{I(X_{(n)} < \theta_0)}$$

$$= \begin{cases} \left(\frac{\theta_0 - a}{\theta_1 - a}\right)^n & \text{if } X_{(n)} < \theta_0 < \theta_1 \quad \left(\frac{1}{1}\right) \\ \infty & \text{if } \theta_0 < X_{(n)} < \theta_1 \quad \left(\frac{1}{0}\right) \\ 0 & \text{if } \theta_0 < \theta_1 < X_{(n)} \quad \left(\frac{0}{0}\right) \end{cases}$$

$\frac{0}{1}$ not possible

$X_{(n)} > \theta_1$ but $X_{(n)} < \theta_0$ not possible

$\Rightarrow$ The test is only a function of $X_{(n)}$ $\rightarrow$ the actual test is not dependent on θ_0 or θ_1

$\Rightarrow$ Test $\Rightarrow$ $R = \{ \underset{\sim}{x} : X_{(n)} > c \}$

$$\alpha = P(X_{(n)} > c \mid \theta = \theta_0)$$
$$= 1 - P(X_{(n)} < c \mid \theta = \theta_0)$$
$$= 1 - P(X < c \mid \theta = \theta_0)^n$$
$$= 1 - \left(\frac{c - a}{\theta_0 - a}\right)^n$$

$\alpha = 1 - \left(\frac{c-a}{\theta_0 - a}\right)^n$ {solve for c}

$c = (1-\alpha)^{1/n}(\theta_0 - a) + a$

d) Show as $n \to \infty$ the R is c) is indep of α

Give an intuitive interpretation.

$$\lim_{n \to \infty} P\left(X_{(n)} > (1-\alpha)^{1/n}(\theta_0 - a) + a \right)$$

$$P\left(X_{(n)} > \theta_0 \right)$$

$\Rightarrow$ This makes sense, as

2015

① Urn contains a red and b blue balls

Ball drawn

↳ returned if red

↳ replaced w/ red if blue

X_n = # red balls after n draws

$EX = \sum_{i=1}^{n} x_i P(X=x_i)$

number of outcomes

$EX = \sum x \cdot P(X=x)$

$P(\text{red}) = \frac{a}{a+b}$ } doesn't change if red

$P(\text{blue}) = \frac{b}{a+b}$ } if blue ⟶ $P(\text{red}) = \frac{a+1}{a+b}$

$P(\text{blue}) = \frac{b-1}{a+b}$

a)

$E(X_1)$ = expected number of red balls after 1 draw

= there can be a or a+1 red balls (drew red or drew blue) on your one draw

$= (a)\left(\frac{a}{a+b}\right) + (a+1)\left(\frac{b}{a+b}\right) = (\#)(P(\text{red})) + (\#)P(\text{blue})$

$= ap + (a+1)(1-p)$

$= \sum_{i=0}^{1} \underbrace{(a+i)}_{x} \underbrace{p^{1-i}(1-p)^i}_{P(X=x)}$ where $p = \frac{a}{a+b}$

$EX_1^2 = a^2\left(\frac{a}{a+b}\right) + (a+1)^2\left(\frac{b}{a+b}\right)$

b)

$M_X(t) = E(e^{tX}) = \sum e^{ti} P(X=i)$

$= \sum_{i=0}^{1} e^{t(a+i)} p^{1-i} (1-p)^i$ ← MGF of X_1

$= e^{ta} p + e^{t(a+1)}(1-p)$

c) Show that

$$E(X_{n+1}) = \left(1 - \frac{1}{a+b}\right) E(X_n) + 1$$

EX_n = expected number of reds balls after n draws

EX_{n+1} = exp # of red after n+1 draws

$$1 - \frac{1}{a+b}$$

$$= \frac{a+b-1}{a+b} = \frac{\text{total\#} - 1}{\text{total\#}}$$

$$EX_1 = \underset{\substack{\uparrow \\ \text{original \#}}}{a} \underset{\substack{\uparrow \\ P(\text{red})}}{\left(\frac{a}{a+b}\right)} + \underset{\substack{\uparrow \\ \text{one blue drawn so increment by 1}}}{(a+1)} \underset{\substack{\uparrow \\ P(\text{blue})}}{\left(\frac{b}{a+b}\right)}$$

red blue

$\left(\frac{a}{a+b}\right)\left(\frac{b}{a+b}\right)$ or

$\underset{\text{blue}}{\left(\frac{b}{a+b}\right)}\underset{\text{red}}{\left(\frac{a+1}{a+b}\right)}$ ← red added by first blue

EX_n = 0 2 red, 0 blue

1 1 red, 1 blue drawn ← in what order?

increment to original a 2 0 red, 2 blue

$$= (a+0)\left(\frac{a}{a+b}\right) + (a+1)$$

② Patients treated → disease free

then develop recurrence → disease progresses until death

X = time to recurrence

Y = time to death

both measured from the end of treatment (time 0)

$0 < X < Y$

$X \sim Uni(0,1) \quad \rightarrow E(X) = \frac{1+0}{2} = 1/2 \qquad Var X = \frac{1}{12}$

$Y|X \sim Uni(x, x+1) \quad \rightarrow E(Y|X) = \frac{x+x+1}{2} = \frac{2x+1}{2} = x + \frac{1}{2}$

d) Find mean and variance of Y → conditioning!

$Var(Y|X) = \frac{(x+1-x)^2}{12} = \frac{1}{12}$

$E(Y) = E_x E_{y|x}(Y|X)$

$= E_x(X + \frac{1}{2}) = EX + 1/2 = 1/2 + 1/2 = 1$ ← 1 year is expected time to death. ok, makes sense

$Var(Y) = E_x Var_{y|x}(Y|X) + Var_x E_{y|x}(Y|X)$

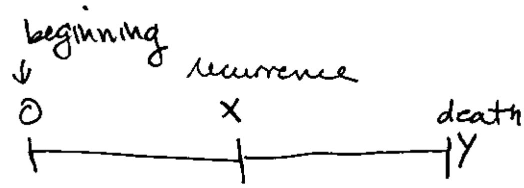
$= E_x(\frac{1}{12}) + Var_x(X + \frac{1}{2})$

$= \frac{1}{12} + Var X + \cancel{Var \frac{1}{2}} + 2\cancel{cov}(X, 1/2)$

no variance of a constant (0) ← no covariance w/ a constant (0)

$= \frac{1}{12} + \frac{1}{12}$

$= \frac{2}{12} = \frac{1}{6}$



Y is not $Uni(0,2)$

$Var\,Y = \frac{(2-0)^2}{12} = \frac{4}{12} = \frac{1}{3}$ (not true)

b) Find correlation btwn X and Y

$$\text{corr} = \frac{\text{cov}(X,Y)}{\sqrt{\text{Var}X \,\text{Var}Y}}$$ know these items

formula from HW1 661

$\text{cov}(X,Y) = E(XY) - EX\,EY$ know these

$= E\,E(XY|X)$ pick one to condition upon

$= E(X \cdot E(Y|X))$ X conditioned on X is a constant

we know this

$= E(X \cdot (X + \frac{1}{2}))$ distribute through

$= E(X^2 + \frac{1}{2}X)$ distribute expectation

$= EX^2 + \frac{1}{2}EX$

$\text{Var}X = EX^2 - (EX)^2$

$= (\text{Var}X + (EX)^2) + \frac{1}{2}E(X)$

$= \frac{1}{12} + (\frac{1}{2})^2 + \frac{1}{2}(\frac{1}{2})$

$= \frac{1}{12} + \cancel{\frac{1}{4}}\frac{3}{12} + \cancel{\frac{1}{4}}\frac{3}{12}$

$= \frac{7}{12} - \underbrace{(\frac{1}{2})}_{EX}\underbrace{(X + \frac{1}{2})}_{EY}$

$= \frac{7}{12} - \cancel{\frac{1}{2}}\frac{6}{12}X - \cancel{\frac{1}{4}}\frac{3}{12} = \frac{4-6X}{12} = \frac{2-3X}{6}$

$$\text{corr} = \frac{\left(\frac{2-3X}{6}\right)}{\sqrt{\frac{1}{2}(X+\frac{1}{2})}} = \frac{\frac{2-3X}{6}}{\sqrt{\frac{1}{2}X + \frac{1}{4}}}$$

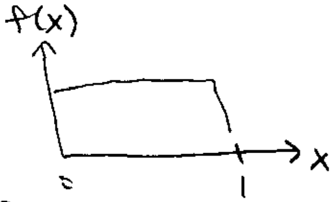
c) Find $f_Y(y)$ hint: support of x, y. It may help to split Y into two parts ?

$f(x) = \frac{1}{1-0} = 1$

$\Rightarrow$ integrate x out of the joint pdf

$f(y|x) = \frac{1}{(x+1)-x} = 1$

$f(x,y) = f(x) \cdot f(y|x)$
$= (1)(1)$
$= 1 \quad 0 < x < 1$



$I(x<1)\, I(y<x+1) \quad x < y|x < x+1$ for overall x's so $0 < y < 2$

$I(x<1)\, I(y-1 < x)\, I(y > x)$

$$f(x,y) = \begin{cases} 1 & \text{if } 0<x<1 \text{ and } x<y<x+1 \\ 0 & \text{else} \end{cases}$$

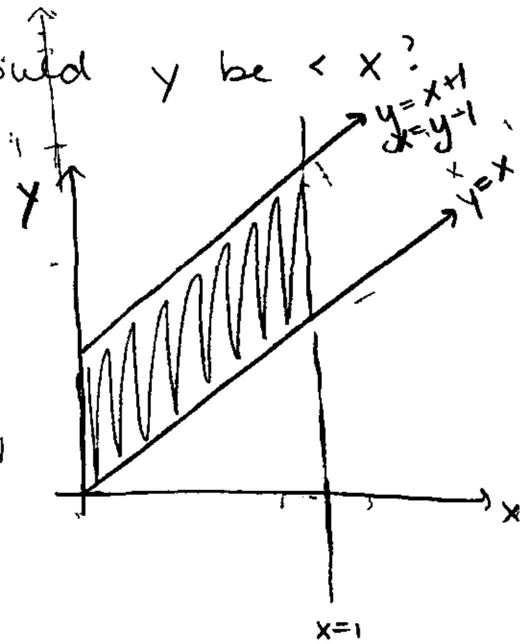
y is <u>not</u> uniform

could y be $< x$?

does the joint integrate to 1? not w/ $0<y<2$
yes w/ $x<y<x+1$

$$\int_0^1 \int_x^{x+1} 1 \, dy \, dx$$

$$= \int_0^1 (y)\Big|_x^{x+1} dx = \int_0^1 (x+1-x)\, dx = \int_0^1 1\, dx = 1$$



$$f_1(y) = \int_y^{y-1} (1)\, dx = x\Big|_y^{y-1} = (y-1) - y = -1$$ ☹

(3) $X_1 \ldots X_n$

$$f(x|\theta) = \theta e^{-\theta x} \quad x > 0 \quad \theta > 0$$

a) Derive $\hat{\theta}$ MLE. Show $1/\hat{\theta}$ has minimum variance among all unbiased estimators of $1/\theta$.

$$L(\theta|x) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \Sigma x_i}$$

$$\ell(\theta|x) = n\log\theta - \theta \Sigma x_i$$

$$\frac{d\ell}{d\theta} = \frac{n}{\theta} - \Sigma x_i = 0 \qquad \frac{d\ell^2}{d\theta^2} = \frac{-n}{\theta^2} = \frac{-n}{(n/\Sigma x_i)^2} = \frac{-\Sigma x_i^2}{n} < 0 \ \checkmark \text{ so max}$$

$$\hat{\theta} = \frac{n}{\Sigma x_i} = \bar{x}^{-1}$$

$$\hat{\theta}_{MLE} = \bar{x}^{-1}$$

$\frac{1}{\hat{\theta}} = \frac{1}{\bar{x}^{-1}} = \bar{x}$ by invariance of MLE's

by exp. family, Σx_i is sufficient stat.

Therefore, because $\bar{x}$ is unbiased for $\frac{1}{\theta}$ $\left(E\left(\frac{\Sigma x_i}{n}\right) = \frac{\Sigma E x_i}{n} = \frac{n\left(\frac{1}{\theta}\right)}{n} = \frac{1}{\theta}\right)$ and based solely on the sufficient statistic Σx_i, it is the UMVUE for $\frac{1}{\theta}$

b) Derive the limiting distn. of below, and comment on whether it reaches the CRLB times n

$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0,$

$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, I_1(\theta)^{-1})$ known, so what is this

$$I_1(\theta) = E\left(-\frac{d^2}{d\theta^2} l_1(\theta|X)\right)$$

$$= E\left(-\frac{-1}{\theta^2}\right) = \frac{1}{\theta^2}$$

$$I_1(\theta)^{-1} = \theta^2$$

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \theta^2) \quad \text{as } n \to \infty$$

CRLB for $\theta = \tau(\theta)$

$$\text{CRLB} = \frac{(1)^2}{nE(-J_1(\theta|X))} = \frac{1}{nI_1(\theta)} = \frac{1}{n/\theta^2} = \frac{\theta^2}{n}$$

$$\text{CRLB} * n = \frac{\theta^2}{n} \cdot n = \theta^2$$

$\Rightarrow$ Yes, the MLE reaches the CRLB.

This is expected → MLE's are the most efficient estimators

c) Suppose $X_1 \dots X_n$ are not observed, but dichotomized $Z_1 \dots Z_n$ are.

$$Z_i = \begin{cases} 1 & \text{if } X_i > \tau \\ 0 & \text{if } X_i \le \tau \end{cases} \qquad \tau \ge 0 \text{ known constant}$$

find MLE of θ called $\tilde{\theta}$ based on $Z_1 \dots Z_n$ and derive its large sample form

$Z_i \sim Bern(p = P(X_i > \tau))$

$$p = P(X_i > \tau) = 1 - P(X_i < \tau)$$
$$= 1 - F(\tau)$$
$$= 1 - \int_0^\tau \theta e^{-\theta x}\,dx$$
$$= 1 - \left(-e^{-\theta x}\Big|_0^\tau\right)$$
$$= 1 - \left(-e^{-\theta t} - -\cancel{e^{-\theta(0)}}^{1}\right)$$
$$p = e^{-\theta\tau}$$
$$-\frac{\log(p)}{\tau} = \theta$$

→ find $\hat{p}$ MLE
→ by invariance, find $\hat{\theta}$ MLE

$\hat{p}$ MLE $= \bar{Z}$ (known) by Bernoulli

or:

$$L(p|z) = \prod_{i=1}^n p^{z_i}(1-p)^{1-z_i}$$
$$= p^{\Sigma z_i}(1-p)^{n-\Sigma z_i}$$

$$\ell(p|z) = \Sigma z_i \log p + (n - \Sigma z_i)\log(1-p)$$

$$\frac{d\ell}{dp} = \frac{\Sigma z_i}{p} + -\frac{(n-\Sigma z_i)}{(1-p)} = 0$$
$$= \Sigma z_i(1-p) - (n - z_i)p$$
$$= \Sigma z_i - np = 0$$
$$\hat{p} = \frac{\Sigma z_i}{n} = \bar{z}$$

so $\tilde{\theta} = -\frac{\log \hat{p}}{\tau} = -\frac{\log \bar{z}}{\tau}$

$$\sqrt{n}\left(\frac{\Sigma X_i}{n} - EX_1\right) \xrightarrow{d} N(0, Var X)$$

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, (g'(\theta))^2\sigma^2)$$

$$\sqrt{n}(\bar{z} - EZ_1) \xrightarrow{d} N(0, Var Z_1) \text{ by CLT}$$

$$\sqrt{n}(\bar{z} - p) \xrightarrow{d} N(0, \underbrace{p(1-p)}_{\sigma^2})$$

$$g(w) = -\frac{\log w}{\tau}$$
$$g'w = -\frac{1}{w\tau}$$
$$(g'w)^2 = (w\tau)^{-2}$$

⇒ by Delta method:

$$\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N(0, (\theta\tau)^{-2}(p)(1-p))$$
$$\xrightarrow{d} N(0, (\tau\theta)^{-2}(e^{-\theta\tau})(1-e^{-\theta\tau}))$$

d) $\tau = EX_1 \doteq \theta$ compare asymp. var of $\hat{\theta}$ and $\tilde{\theta}$

(could use ARE to compare) $\qquad \exp\left(\frac{1}{\theta}\right)$

Var of $\hat{\theta} = \theta^2$

$$\tilde{\theta} = (\theta^2)^{-2} \frac{(1-e^{-\theta\tau})}{e^{\theta t}}$$

the var of $\hat{\theta}$ is smaller.

This makes sense, because the MLE is always the most efficient estimator.

Additionally, through dichotomization, we are losing information, thereby increasing the variance of $z_1 \dots z_n$.

e) $X_1 \dots X_n \sim f(x|\theta) = \theta e^{-\theta x}$

$Y_1 \dots Y_m \sim f(y|\beta) = \beta e^{-\beta y}$

$H_0: \phi = 1$ vs. $H_1: \phi \neq 1 \qquad \phi = \frac{\theta}{\beta}$

do all for practice.(?)

Derive a large sample size α test of H_0 vs. H_1

$\frac{\theta}{\beta} = 1 \qquad \theta = \beta$

likelihood ratio test

$$LRT = \frac{\sup \phi \in \Phi_0 \, L(\theta, \beta | x, y)}{\sup \text{ overall } L(\theta, \beta | x, y)}$$

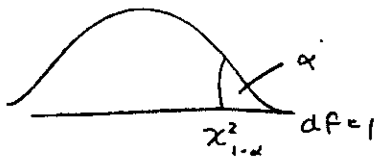
$$L(\theta, \beta | x, y) = \theta^n e^{-\theta \Sigma x_i} \beta^m e^{-\beta \Sigma y_i} = \theta_0^{n+m} e^{-\theta_0(\Sigma x_i + \Sigma y_i)}$$

$$\ell(\theta_0 | x, y) = (n+m)\log\theta_0 - \theta_0(\Sigma x_i + \Sigma y_i)$$

$$\frac{d\ell}{d\theta_0} = \frac{n+m}{\theta_0} - (\Sigma x_i + \Sigma y_i)$$

$$\hat{\theta}_0 = \frac{n+m}{\Sigma x_i + \Sigma y_i}$$

$$\frac{d^2\ell}{d\theta_0^2} = \frac{-(n+m)}{\theta_0^2} < 0 \checkmark$$



$$LRT \text{ test} = -2\log\lambda(\underset{\sim}{x}, y) = 2(\ell(\hat{\theta}) - \ell(\theta_0)) \sim \chi^2_1$$

$$= 2(\ell(\hat{\theta}, \hat{\beta}) - \ell(\hat{\theta}_0)) \xrightarrow{d} \chi^2_1$$

so $\alpha = P(-2\log\lambda(\underset{\sim}{x}, y) > \chi^2_{1,1-\alpha} \mid \phi = 1)$

Score test

$$\frac{n^{-1/2}\, u(\theta_0)}{\sqrt{I_1(\theta_0)}} \xrightarrow[d]{} N(0,1) \qquad \theta_0 = \theta = \beta$$

$$u(\theta_0) = \frac{(n+m)}{\theta_0} - (\Sigma x_i + \Sigma y_i)$$

$$I_1(\theta_0) = E\left(- - \frac{1}{\theta_0}\right) = \frac{1}{\hat{\theta}_0}$$

Wald test

$$\frac{E(\hat{\theta} - \hat{\beta}) - \overbrace{(\theta - \beta)}^{0 \text{ by expectation}}}{\sqrt{(n/\theta^2)^{-1} + (n/\beta^2)^{-1}}}$$

$$\begin{aligned} &Var(\hat{\theta} - \hat{\beta}) \\ &= Var\,\hat{\theta} + Var\,\hat{\beta} \quad + \text{no cov b/c indep.} \\ &= I_n(\hat{\theta})^{-1} + I_m(\hat{\beta})^{-1} \\ &= \left(\frac{n}{\theta^2}\right)^{-1} + \left(\frac{n}{\beta^2}\right)^{-1} \leftarrow \text{from a) derivations} \end{aligned}$$

(4) Radiation dose $\sim \mathrm{Uni}(a, \theta)$ $\quad 0 \le a < \theta$

a KNOWN (constant) $\quad a \le x \le \theta$

θ unknown

a) show MLE of θ is $X_{(n)}$ and prove it is biased but consistent $\quad \lim_{n\to\infty} E = \theta$, $\lim_{n\to\infty} Var = 0$

$$L(\theta|a,x) = \left(\frac{1}{\theta - a}\right)^n I(x_{(n)} \le \theta) \qquad 0 \text{ if } X_{(n)} > \theta$$

$\Rightarrow x_{(n)}$ (max orderstat) is the MLE of θ, from the graph

$$F(X_{(n)}) = P(X_{(n)} < x) = P(X<x)^n = \left(\frac{x}{\theta - a}\right)^n$$

$$f(X_{(n)}) = n\left(\frac{x}{\theta a}\right)^n\left(\frac{1}{\theta a}\right)$$

$$E(X_{(n)}) = \int_a^\theta x\, f_{X_{(n)}}(x)\, dx \qquad f_{X_{(n)}}(x) = n f(x)(F(x))^{n-1} = n\left(\frac{1}{\theta - a}\right)\left(\frac{x}{\theta - a}\right)^{n-1} = \frac{n}{(\theta - a)^n}\cdot x^{n-1}$$

$$= \int_a^\theta \frac{n x^n}{(\theta - a)^n}\, dx$$

$$= \frac{n}{n+1}\cdot \frac{x^{n+1}}{(\theta - a)^n}\Big|_a^\theta$$

$$= \frac{n}{n+1}\left(\frac{\theta^{n+1}}{(\theta-a)^n} - \frac{a^{n+1}}{(\theta-a)^n}\right) = \frac{n}{n+1}\left(\frac{(\theta - a)^{n+1}}{(\theta-a)^n}\right) = \frac{n}{n+1}(\theta - a)$$

$\Rightarrow E(X_{(n)}) \ne \theta \Rightarrow$ biased estimator

$$\lim_{n\to\infty} \frac{n}{n+1}\theta - \frac{n}{n+1}a \Rightarrow \theta \quad \text{if } a = 0 \ (?)$$

$$Var X_{(n)} = E X_{(n)}^2 - (E X_{(n)})^2$$

$$= \int_a^\theta x^2 \frac{n x^{n-1}}{(\theta - a)^n}\, dx$$

$$= \frac{n x^{n+2}}{(\theta-a)^{n}(n+2)}\Big|_a^\theta$$

$\Rightarrow Var X_{(n)}$ limit $\to 0$

$$= \frac{n}{n+2}\cdot\left(\frac{(\theta-a)^{n+2}}{(\theta-a)^n}\right)$$

$$= \frac{n}{n+2}(\theta-a)^2 - \left(\frac{n}{n+1}(\theta - a)\right)^2$$

$$= (\theta-a)^2\left(\frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)}\right)$$

b) Find an unbiased estimator of θ as a function of $X_{(n)}$. Comment on why the CRLB fails in this situation

↳ adjust to be $= \theta$ by

$\frac{n+1}{n}(X_{(n)} + a)$ is unbiased

The integral and derivative are not exchangeable

$$\frac{d}{d\theta}\int_a^\theta h(x) f(x|\theta)\,dx \neq \int_a^\theta h(x) \frac{d}{d\theta} f(x|\theta)\,dx$$ (the assumptions for the CRLB are violated

c) UMP test size α for $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$ w/ clear specification for the cutoff in the R.

$\Rightarrow$ let $H_1: \theta = \theta_1$, $\theta_1 > \theta_0$ (then will generalize to composite.

N-P.:

$$\frac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > c = \frac{\left(\frac{1}{\theta_1 - a}\right)^n I(X_{(n)} < \theta_1)}{\left(\frac{1}{\theta_0 - a}\right)^n I(X_{(n)} < \theta_0)} = \left(\frac{\theta_0 - a}{\theta_1 - a}\right)^n \frac{I(X_{(n)} \leq \theta_1)}{I(X_{(n)} < \theta_0)}$$

$\Rightarrow$ if $X_{(n)} < \theta_0$, then its less than θ_1

$$\text{so UMP test} = \begin{cases} 0 & \text{if } X_{(n)} > \theta_0 \\ \left(\frac{\theta_0 - a}{\theta_1 - a}\right)^n & \text{if } X_{(n)} < \theta_0 \end{cases}$$

$$\alpha = P\left(\underbrace{\left(\frac{\theta_0 - a}{\theta_1 - a}\right)^n}_{\text{can base on } X_{(n)}} > c \mid \theta = \theta_0\right)$$

$$\alpha = P(X_{(n)} > c \mid \theta = \theta_0)$$

$$= 1 - P(X_{(n)} < c \mid \theta = \theta_0)$$

$$= 1 - P(X < c \mid \theta = \theta_0)^n$$

$$\alpha = 1 - \left(\frac{c}{\theta_0 - a}\right)^n \Rightarrow c = (1-\alpha)^{1/n}(\theta_0 - a)$$