Question 1 | 2014, MS-1

X = fathers ht, Y = sons ht

$(X, Y) \sim$ Biv. Norm

$EX = EY = 68$

$\text{Var}X = \text{Var}Y = 4$

$\text{Cor}(X, Y) = 0.6$

a) P(father taller than son)

$= P(X > Y) = P(X - Y > 0)$ → need distn. of $X-Y$ ... linear combo of normals will be normal.

$W = X - Y \sim N(0,\ 4 + 4 - 2(0.6))$

$\sim N(0,\ 6.8)$

C

$P(W > 0)$

$= P\Big(\underbrace{\frac{W-0}{\sqrt{6.8}}}_{\sim N(0,1)} > \frac{0-0}{\sqrt{6.8}}\Big)$

$= P(Z > 0) = 0.5$

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b) $P(X - Y \geq 4)$ prob father (x) is at least 4 in taller than son (y)

$P(W \geq 4)$

$= P\left(Z \geq \frac{4-0}{\sqrt{6.8}}\right)$

C

$= 1 - P(Z < 4/\sqrt{6.8}) = 1 - \Phi(4/\sqrt{6.8}) = 1 - \Phi(1.53393) = 1 - 0.9370$

$= \sim 6.3\%$

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c) conditional distribution will be N as well

$E(Y \mid X = x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 68 + 0.15(74 - 68) = 68.9$

$\text{Var}(Y \mid X = x) = (1 - \rho^2)\sigma_Y^2 = (1 - 0.15^2)\,4$ C $= 3.91$

$\text{cor} = \frac{0.6}{\sqrt{4.4}} = 0.15$

d) Given the father is 74 inches tall, find the probability that the son is taller than the father.

$P(Y > 74 \mid X = 74)$

$P\left(\frac{Y - 71.6}{\sqrt{2.56}} > \frac{74 - \cancel{71.6}\ 68.6}{\cancel{\sqrt{2.56}}\ 3.91} \right) = \cancel{P(Z > 1.5) = 1 - \Phi(1.5) = 1 - 0.9332} \sim 6.7\%$

$= P(Z > 2.57) = 1 - \Phi(2.57) = < 1\%$

✓

e) 100 father-son pairs sampled.

joint distn of $\bar{X}, \bar{Y}$

still biv. normal, but

$E\bar{X} = E\bar{Y} = 68$ $\qquad = 68$

$Var\bar{X} = Var\bar{Y} = \frac{4}{100}$ $\qquad = 0.04$ $\qquad$ C

$Cov(\bar{X}, \bar{Y}) = \frac{1}{n} cov(x, y) = \frac{0.6}{100} = 0.006$

f) Prob 2 sample averages are w/in 3 in of each other

$W = \bar{X} - \bar{Y} \sim N(0,\ 0.04 + 0.04 - 2(0.006))$

$\sim N(0,\ 0.068)$

$P(|\bar{X} - \bar{Y}| < 3)$

$= P(-3 < W < 3)$

$= P(-3/\sqrt{0.68} < Z < 3/\sqrt{0.68})$ $\qquad$ C

$= \Phi(3/\sqrt{0.068}) - \Phi(-3/\sqrt{0.068}) = \Phi(11.50) - \Phi(-11.50) \sim 100\%$

Question 2

$y \sim$ # common colds (actual)

$x \sim$ expected # common colds

$y|x \sim \text{Pois}(x)$ $\quad E\,y|x = x \quad Var\,y|x = x$

$x \sim \text{Uni}(0,2) \quad f(x) = 0.5 \quad E\,x = 1 \quad Var = \frac{2^2}{12} = \frac{4}{12} = \frac{1}{3}$

a) Find Ey and $Vary$

$Ey = E_x\, E\,y|x = E_x X = 1$

$Vary = Var_x\, E\,y|x + E_x\, Var\,y|x$

$= Var\,X + EX$

$= \frac{1}{3} + 1 = \frac{4}{3}$

$y \neq$ Poisson
$Ey = 1$
$Vary = 4/3$

$Ey = Vary$ if $y \sim$ Poisson

---

b) $corr(x,y) = \frac{cov(x,y)}{\sqrt{Varx \cdot Vary}} = \frac{EXY - EXEY}{\sqrt{Varx \cdot Vary}}$

$Varx = EX^2 - (EX)^2$

$E(XY) = E\,E\,XY|X$

$= E(X\,E\,Y|X) = EX^2 = VarX + (EX)^2 = \frac{1}{3} + (1)^2 = 4/3$

$\Rightarrow \frac{4/3 - 1(1)}{\sqrt{1/3\,(4/3)}} = \frac{1/3}{\sqrt{4/9}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \boxed{\frac{1}{2} = corr(x,y)}$

c) $W = 4X - Y + 4$. Compute $\text{cov}(W, Y)$

$\text{cov}(W, Y) = E(WY) - EW\,EY$

$EY = 1$

$EW = E(4X - Y + 4) = 4EX - EY + 4 = 4 - 1 + 4 = 7$

$\text{Var}(W+Y) = \text{Var}W + \text{Var}Y + 2\text{cov}(W, Y)$

$\text{Var}(4X+4) = (4^2\text{Var}X + \text{Var}Y - 2(4)\text{cov}(X, Y)) + \text{Var}Y + 2\text{cov}(W, Y)$

$16\left(\frac{1}{3}\right) = \left(\frac{16}{3} + \frac{4}{3} - 8\left(\frac{1}{3}\right)\right) + \frac{4}{3} + 2\text{cov}(W, Y)$

$\frac{16}{3} = \left(\frac{12}{3}\right) + \frac{4}{3} + 2\text{cov}(W, Y)$

$\text{cov}(W, Y) = 0$

$\Rightarrow$ does not imply independence.

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d) i. Is Y an unbiased predictor of X?
ii. Compute the MSE?

i. $E(Y - X) = 0$ if unbiased

$EY - EX = 1 - 1 = 0$ ✓

ii. $MSE = \text{Var}(Y - X) = E\left((Y-X)^2\right) - \cancel{E(Y-X)}^{\,0\,2}$

$= \text{Var}Y + \text{Var}X - 2\text{cov}(X, Y)$

$= \frac{4}{3} + \frac{1}{3} - 2\left(\frac{1}{3}\right)$

$= \frac{4}{3} + \frac{1}{3} - \frac{2}{3} = 1$

Question 2 | 2014, MS-1

$y = \#$ actual common welds

$x =$ expected common welds

$Y \mid X=x \sim \text{Pois}(x)$ $\quad E(Y|X) = x$ $\quad \text{Var}(Y|X) = x$ $\quad f(y|x) = \frac{e^{-x}x^y}{y!}$

$X \sim \text{Uni}(0,2)$ $\quad E(X) = 1$ $\quad \text{Var}(x) = \frac{(2-0)^2}{12} = \frac{4}{12} = \frac{1}{3}$ $\quad f(x) = \frac{1}{2}$

---

a) Find EY and Vary. Does Y have a Poisson distribution.

$$EY = EEY|X = E_x(X) = 1$$

$$\text{Var}y = \text{Var}EY|X + E\text{Var}Y|X = \text{Var}_x(X) + E_x(X) = \frac{1}{3} + 1 = \frac{4}{3}$$

$y \neq$ poisson

$EY = \text{Var}y$

---

b) $\text{corr}(x,y) = \frac{\text{cov}(x,y)}{\sqrt{\text{Var}X\,\text{Var}y}} = \frac{\frac{1}{3}}{\sqrt{\frac{1}{3}\cdot\frac{4}{3}}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

$$\begin{aligned}
\text{cov}(x,y) &= EXY - EXEY \\
&= EEXY|X - EXEY \\
&= EXEY|X - EXEY \\
&= EX^2 - EXEY \\
&= (\text{Var}X + (EX)^2) - EXEY \\
&= \left(\tfrac{1}{3} + 1^2\right) - (1)(1) \\
&= \left(\tfrac{4}{3}\right) - 1 = \tfrac{1}{3}
\end{aligned}$$

c) $W = 4X - Y + 4$

$\text{cov}(W,Y) = EWY - EW\,EY$ or $\text{Var}(W+Y) = \text{Var}W + \text{Var}Y + 2\text{cov}(W,Y)$

$EW = 4EX - EY + 4$

$= 4 - 1 + 4 = 7$

$EWY = ?$

$W + Y = 4X + 4$

$\text{Var}(4X+4) = 16\,\text{Var}X = \frac{16}{3}$

$\text{Var}(4X - Y + 4)$

$16\,\text{Var}X - \text{Var}Y - 2(4)(1)\text{cov}(X,Y)$

$= 16\left(\frac{1}{3}\right) - \frac{4}{3} - 2(4)(1)\left(\frac{1}{3}\right)$

$= \frac{16}{3} - \frac{4}{3} - \frac{8}{3} = \frac{12}{3}$

$\frac{16}{3} = \frac{12}{3} + \frac{4}{3} + 2(\text{cov}(W,Y))$

$0 = \text{cov}(W,Y)$

$\Rightarrow$ Just because $\text{cov}(W,Y) = 0$

$\not\Rightarrow$ that W and Y are independent

Question 2 | 2014, MS-1

d) X is not observable.
Is Y an unbiased predictor of X?
compute the MSE = $E\{(U-V)^2\}$

$$Var\{(Y-X)\} = E\{(Y-X)^2\} - \{\overset{0}{\cancel{E(Y-X)}}\}^2$$

$$E(Y-X) = EY - EX$$
$$= 1 - 1 = 0 \Rightarrow \text{unbiased}$$

$$EXX = EX^2 = Var X + (EX)^2$$
$$= \frac{1}{3} + \frac{3}{3} = \frac{4}{3}$$

$$E\{(Y-X)^2\} = Var\{(Y-X)\} + \{E(Y-X)\}^2$$

$$EY^2 - 2EXY + EX^2 = Var Y + Var X - 2cov(X,Y) + \cancel{(0)}^2$$

$$\left(\frac{4}{3} - (1)^2\right) - 2\left(\frac{4}{3}\right) + \left(\frac{1}{3} + (1)^2\right) = \frac{4}{3} + \frac{1}{3} - 2\left(\frac{1}{3}\right)$$

$$\frac{3}{3} = \frac{3}{3}$$

$$= 1$$

$\boxed{MSE = 1}$

$$Var(Y-X) = E(Y-X)^2 - (\overset{0}{\cancel{E(Y-X)}})^2$$

e) Find constants a and b such that
$a+bY$ is an unbiased predictor and the pred. var is as small as possible.

$$= E(a+bY-X)=0 \quad \text{(unbiased)}$$

$$= a+bEY-EX = 0$$

$$a+b(1)-(1)=0$$

$$a = 1-b$$

$Var(a+bY-X)$ is minimized

$$b^2 Var\,Y + Var\,X - 2b\,Cov(X,Y) = 0$$

$$\cancel{2}b\,Var\,Y - \cancel{2}\,Cov(X,Y) = 0$$

$$b = \frac{Cov(X,Y)}{Var(Y)} = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4}$$

$$b = \frac{1}{4}$$

$$a = \frac{3}{4}$$

Question 2 | 2014, MS-1

f) What is the probability that a subject gets <u>no</u> common colds during the study period? $\Rightarrow$ numeric value.

$$P(\text{no common colds}) = P(y=0)$$
$$= f(y=0)$$

$$f(y) = \int_x f(y|x)\, f(x)\, dx \quad \text{and eval @ } y=0$$

$$f(y|x)\, f(x) = \frac{e^{-x} x^y}{y!} \cdot \frac{1}{2}$$

$$\text{when } y=0 \quad = \frac{e^{-x}}{2}$$

$$\int_0^2 \frac{e^{-x}}{2}\, dx \qquad x \sim uni(0,2)$$

$$= -\frac{e^{-x}}{2}\Big|_0^2 = -\frac{e^{-2} - -e^{0}}{2} = \frac{1-e^{-2}}{2}$$

g) compute the conditional mean of X for a subject w/ no common colds during the study period.

$$E(X|Y=0) = \int_X x \cdot f(x|y)\, dx$$

$$= \int_X x \cdot \frac{f(y|x)\cdot f(x)}{f(y)}\, dx \quad \text{at } y=0$$

$$= \int_0^2 x \cdot \frac{\cancel{\tfrac{1}{2}}\left(e^{-x}\, \cancel{x^y}^{1} / \cancel{y!}^{1}\right)}{\frac{(1-e^{-2})}{\cancel{2}}}\, dx$$

$$= \int_0^2 x\, \frac{e^{-x}}{(1-e^{-2})}\, dx$$

$\int u\,dv = uv - \int v\,du$

$u = x \qquad v = -e^{-x}$

$du = dx \qquad dv = e^{-x}$

$$= \frac{1}{(1-e^{-2})} \int_0^2 x\, e^{-x}\, dx$$

$$= \frac{1}{(1-e^{-2})} \left\{ -x\, e^{-x}\Big|_0^2 + \int_0^2 e^{-x}\, dx \right\}$$

$$= \frac{1}{(1-e^{-2})} \left\{ -2\, e^{-2} + -e^{-x}\Big|_0^2 \right\}$$

$$= \frac{1}{(1-e^{-2})} \left\{ -2\, e^{-2} + \left(-e^{-2} - -\cancel{e}^{1} \right) \right)$$

$$= \frac{1 - 3e^{-2}}{(1-e^{-2})}.$$

e) $\cdot\ a + bY$ = unbiased for X and Var minimized

$E(a + bY - X) = 0$

$a + bEY - EX = 0$

$a + b(1) - 1 = 0$

$a = 1 - b$

$Var(a + bY - X) = $ minimized

$b^2 Var Y + Var X - 2b\, cov(X,Y)$

$2b Var Y - 2 cov(X,Y) = 0$

$$b = \frac{cov(X,Y)}{Var Y} = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4}$$

$b = \frac{1}{4}$

$a = \frac{3}{4}$

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f) $P(Y=0)$ $\quad \int_x f(y|x) \cdot f(x)\, dx = f(y)$

$$= \int_0^2 \frac{e^{-x} \cancel{x^y}}{\cancel{y!}}\Big|^{y=0} \cdot \frac{1}{2}\, dx$$

$$= \frac{1}{2}\int_0^2 e^{-x} = -\frac{1}{2} e^{-x}\Big|_0^2 = -\frac{1}{2}(e^{-2} - 1) = \frac{1 - e^{-2}}{2}$$

g) conditional mean of X for a subject w/ no common cold

$E(X \mid y=0) = \int_x x\, f(x|y)\, dx$ = conditional mean

at $y=0$

$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{f(\cancel{y}^{0}|x) f(x)}{f(y) \neq 0}$

but we don't have f(y)

$\Downarrow$

it is part f)

$$= \int_0^2 x \cdot \frac{\left(\frac{1}{2} e^{-x}\right)}{\left(\frac{1-e^{-2}}{2}\right)}\, dx$$

$$= \frac{\cancel{\frac{1}{2}}}{\cancel{\frac{1}{2}}(1-e^{-2})} \int_0^2 x e^{-x}\, dx$$

$u = x \quad v = -e^{-x}$

$du = dx \quad dv = e^{-x}$

$$= (1-e^{-2})^{-1} \left\{ -x e^{-x}\Big|_0^2 + \int_0^2 e^{-x}\, dx \right\}$$

$$= (1-e^{-2})^{-1} \left\{ -2e^{-2} - (e^{-2} - 1) \right\}$$

$$= (1-e^{-2})^{-1} \left\{ 1 - 3e^{-2} \right\}$$

Question 3 | 201#, MS-1

$X_1 \ldots X_n \sim$ iid from $f(x|\lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}$ $\quad x = 0, 1, \ldots \quad \lambda > 0$

a) Find the MLE of the parameter $\theta = P(X=0)$

$$P(X=0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} = \theta$$

$$\hat{\theta}_{MLE} = e^{-\hat{\lambda}} \qquad \text{(so, need } \hat{\lambda}\text{)}$$

$$L(\lambda | \underset{\sim}{x}) = \frac{\lambda^{\Sigma x_i} e^{-n\lambda}}{\Pi x!}$$

$$\ell(\lambda | \underset{\sim}{x}) = \Sigma x_i \log \lambda - n\lambda + \Sigma \log x_i$$

$$\frac{d\ell}{d\lambda} = \frac{\Sigma x_i}{\lambda} - n \qquad \hat{\lambda}_{ME} = \frac{\Sigma x_i}{n} = \bar{x}$$

$$\frac{d^2\ell}{d\lambda^2} = \frac{-\Sigma x_i}{\lambda^2} < 0 \checkmark$$

$$\hat{\theta} = e^{-\bar{x}}$$

b) Show $\hat{\theta} = (1-\frac{1}{n})^Y$ is an unbiased estimator of θ

$Y = \Sigma X_i$

$= (1-\frac{1}{n})^{n\bar{X}}$ $\Sigma X_i = Y \sim Pois(n\lambda)$

$$E(\hat{\theta}) = \sum_{y=0}^{\infty} (1-\tfrac{1}{n})^y \frac{e^{-n\lambda}(n\lambda)^y}{y!}$$

$$= \sum_{y=0}^{\infty} \frac{(n\lambda-\lambda)^y}{y!} \frac{(e^{-n\lambda})e^{\lambda}}{e^{\lambda}} = \underbrace{\sum_{y=0}^{\infty} \frac{(n\lambda-\lambda)^y e^{-(n\lambda-\lambda)}}{y!}}_{1} \cdot \underbrace{e^{-\lambda}}_{\theta}$$

$= \theta \Rightarrow$ unbiased estimator.

$E\hat{\theta} = e^{-\lambda}$

---

c) Derive the variance of $\hat{\theta}$ $\hat{\theta} = (1-\frac{1}{n})^Y \leftarrow$ function of $\Sigma X_i \sim Pois(n\lambda)$

$\operatorname{Var}\hat{\theta} = E\hat{\theta}^2 - E(\hat{\theta})^2$ switch those

$((1-\frac{1}{n})^Y)^2 = ((1-\frac{1}{n})^2)^Y$ yes b/c n and y are positive.

$$E\hat{\theta}^2 = \sum_{y=0}^{\infty} \{(1-\tfrac{1}{n})^2\}^y \frac{e^{-n\lambda}(n\lambda)^y}{y!} = \sum_{y=0}^{\infty} (1-\tfrac{2}{n}+\tfrac{1}{n^2})^y \frac{e^{-n\lambda}(n\lambda)^y}{y!}$$

$$= \sum_{y=0}^{\infty} \frac{e^{-n\lambda}}{y!} \cdot (n\lambda - 2\lambda + \tfrac{\lambda}{n})^y = \sum_{y=0}^{\infty} \frac{(n\lambda-2\lambda+\frac{\lambda}{n})^y}{y!} \cdot \frac{e^{-(n\lambda-2\lambda+\frac{\lambda}{n})}}{e^{2\lambda-\frac{\lambda}{n}}}$$

$\to y \sim Pois(n\lambda - 2\lambda + \frac{\lambda}{n})$

$$E\hat{\theta}^2 = e^{-(2\lambda-\frac{\lambda}{n})}$$

$$\operatorname{Var}\hat{\theta} = e^{-2\lambda+\frac{\lambda}{n}} - e^{-2\lambda}$$

$$= e^{-2\lambda}(e^{\lambda/n} - 1)$$

☺ craig

More C

Compare to CRLB (?)

$$\mathrm{Var}\,\hat{\theta} = e^{-2\lambda}\left(e^{-\lambda/n} - 1\right)$$

CRLB for unbiased est of $\theta = e^{-\lambda}$

$$= \frac{\left\{\frac{d}{d\theta}\tau(\theta)\right\}^2}{E\left(-\frac{d^2}{d\theta^2}\log f(x|\theta)\right)} = \frac{e^{-2\lambda}}{\ }$$

λ

$$E\left(+\frac{\sum X_i}{\lambda^2}\right) = \frac{\sum X_i}{\lambda^2} = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

$$\mathrm{CRLB} = \frac{e^{-2\lambda}\cdot\lambda}{n}$$

$$e^{-2\lambda}\left(\frac{\lambda}{n}\right) \overset{?}{=} e^{-2\lambda}\left(e^{-\frac{\lambda}{n}} \cancel{-1}\right)$$

$$\Downarrow$$

$$\cancel{1} + \frac{\lambda}{n} + \frac{\lambda^2}{n^2 2} \ldots$$

$$\mathrm{CRLB} < \mathrm{Var}(\hat{\theta})$$

d) observe

$$Z_i = \begin{cases} 1 & X_i > 0 \\ 0 & X_i = 0 \end{cases}$$

Find MLE $\tilde{\lambda}$ as a function of $Z_1 \ldots Z_n$ and ~~derive~~ its limiting variance in explicit form.

$$p = P(X_i > 0)$$
$$= 1 - P(X_i = 0) = 1 - e^{-\lambda} \quad \Rightarrow \quad \hat{p} = 1 - e^{-\tilde{\lambda}}$$
$$\Rightarrow \quad \tilde{\lambda} = \log(1 - \hat{p})$$

① Find $\hat{p}$

$\Rightarrow \bar{Z}$ by Bernoulli

② $\tilde{\lambda} = \log(1 - \bar{Z})$

③ limiting var by delta method.

$$\sqrt{n}(\bar{Z} - p) \xrightarrow[d]{} N(0, p(1-p)) \quad \text{by CLT}$$

$$g(w) = \log(1-w)$$

$$\sqrt{n}(g(\bar{Z}) - g(p)) \xrightarrow[d]{} N(0, g'(p)^2 p(1-p))$$

$$g'(w) = \frac{-1}{(1-w)}$$

$$g'(w)^2 = (1-w)^{-2}$$

$$\sqrt{n}(\tilde{\lambda} - \lambda) \xrightarrow[d]{} N(0, \underbrace{(1-p)^{-2} p(1-p)}))$$

$$\Downarrow$$

$$\frac{p}{(1-p)} = \frac{1 - e^{-\lambda}}{\cancel{1-(1-}e^{-\lambda})} = e^{\lambda} - 1$$

e)

$$\lambda \overset{?}{=} e^{\lambda} - 1 \overset{?}{=} \cancel{1} + \lambda + \frac{\lambda^2}{2} \cdots \cancel{-1}$$

$$\operatorname*{var}_{n \to \infty} \bar{X} < \operatorname*{var}_{n \to \infty} \tilde{\lambda}$$

$X_1 \ldots X_n$ random sample from $N(0, \sigma^2)$.

Test $H_0: \sigma = \sigma_0$ vs. $H_1: \sigma \neq \sigma_0$ it is suggested one can use:

$$\delta(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum X_i^2 < c_1 \text{ or } \sum X_i^2 > c_2 \\ 0 & \text{else} \end{cases}$$

---

a) Find c_1 and c_2 such that the test is size α

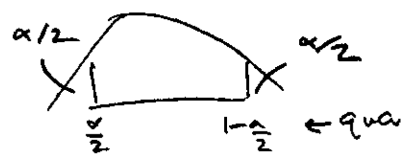
$\alpha = P(X \in R \mid H_0)$

$= P(\sum X_i^2 < c_1 \text{ or } \sum X_i^2 < c_2 \mid \sigma = \sigma_0)$ ← see part b) → hint to make it χ^2_n

$= P(\sum X_i^2 < c_1 \mid \sigma = \sigma_0) + P(\sum X_i^2 < c_2 \mid \sigma = \sigma_0)$

$= P\left(\sum \frac{X_i^2}{\sigma_0^2} < \quad \right) + P\left(\sum \left(X_i^2 / \sigma_0^2\right) < c_2 / \sigma_0^2\right)$

$\sum \left(\frac{X_i}{\sigma_0}\right)^2 \sim \chi^2_n$



$\Rightarrow \frac{c_1}{\sigma_0^2} = \chi^2_{n, \alpha/2}$

$c_1 = \chi^2_{n, \frac{\alpha}{2}} \cdot \sigma_0^2$

$\frac{c_2}{\sigma_0^2} = \chi^2_{n, 1-\frac{\alpha}{2}}$

$c_2 = \chi^2_{n, 1-\frac{\alpha}{2}} \cdot \sigma_0^2$

b) Show that, for a certain choice of c_1 and c_2, the power function is

$$\beta(\sigma) = G_n\left\{ \frac{\sigma_0^2 \chi^2_{n,\alpha/2}}{\sigma^2} \right) + \left(1 - G_n\left\{ \frac{\sigma_0^2 \chi^2_{n,1-\alpha/2}}{\sigma^2} \right\}\right.$$

G_n = cdf of χ^2_n distn

$\chi^2_{n,\alpha/2}$ is $\alpha/2$ quantile of χ^2_n distn

power = $P(X \in R \mid H_A)$

$$= P\left(\sum X_i^2 < \sigma_0^2 \cdot \chi^2_{n,\alpha/2} \mid \sigma \neq \sigma_0\right) + P\left(\sum X_i^2 > \sigma_0^2 \cdot \chi^2_{n,1-\frac{\alpha}{2}} \mid \sigma \neq \sigma_0\right)$$

* let $\sigma = \sigma$ be the value of σ under the alternative $(\sigma \neq \sigma_0)$

$$\Rightarrow P\left(\underbrace{\sum \frac{X_i^2}{\sigma^2}}_{\text{again, } \chi^2_n} < \frac{\sigma_0^2 \chi^2_{n,\alpha/2}}{\sigma^2}\right) + P\left(\sum \frac{X_i^2}{\sigma^2} > \frac{\sigma_0^2 \cdot \chi^2_{n,1-\alpha/2}}{\sigma^2}\right)$$

$$= G_n\left(\frac{\sigma_0^2 \chi^2_{n,\alpha/2}}{\sigma^2}\right) + \left(1 - G_n\left(\frac{\sigma_0^2 \cdot \chi^2_{n,1-\alpha/2}}{\sigma^2}\right)\right)$$

<

b/c >

d) $H_0: \sigma = \sigma_0$ vs. $H_1: \sigma > \sigma_0$

let $H_1: \sigma = \sigma_1$ where $\sigma_1 > \sigma_0$. Then generalize to

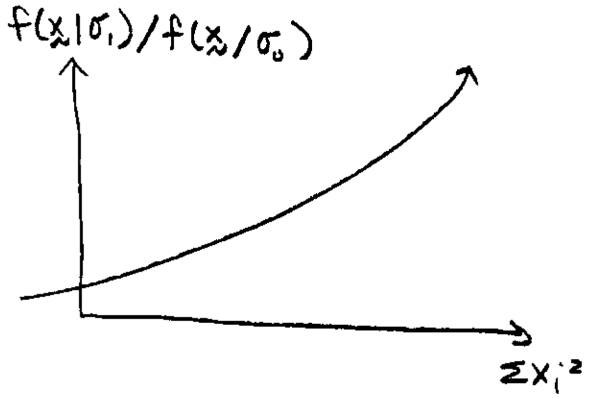
$\Rightarrow$ for s.v. simple use NP Lemma

$$\frac{f(\underset{\sim}{x} \mid \sigma_1)}{f(\underset{\sim}{x} \mid \sigma_0)} > c \Rightarrow \text{UMP test of size } \alpha$$

$$f(\underset{\sim}{x}) = (2\pi)^{-1/2} \frac{1}{\sigma} \exp(-\Sigma x_i^2 / 2\sigma^2)$$

ratio:

$$\frac{\cancel{(2\pi)^{-1/2}}\, \sigma_1^{-1} \exp(-\Sigma x_i^2 / 2\sigma_1^2)}{\cancel{(2\pi)^{-1/2}}\, \sigma_0^{-1} \exp(-\Sigma x_i^2 / 2\sigma_0^2)} = \underbrace{\left(\frac{\sigma_0}{\sigma_1}\right)}_{>1} \exp\Big(\underbrace{\Sigma x_i^2}_{\text{pos}} \underbrace{\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)}_{\text{pos}}\Big)$$



$$R = \left\{ \underset{\sim}{x} : \left(\frac{\sigma_0}{\sigma_1}\right) \exp\left(\Sigma x_i^2 \left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)\right) > c \right\}$$

↳ this is an increasing function of Σx_i^2, and so an equivalent R is

$$\{ \underset{\sim}{x} : \Sigma x_i^2 > c \}$$

and the size α test is

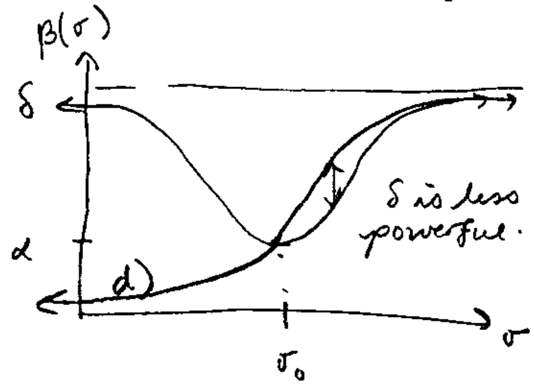
$$\alpha = P(\Sigma x_i^2 > c \mid \sigma = \sigma_0)$$

$$= P\Big(\underbrace{\Sigma \frac{x_i^2}{\sigma_0^2}}_{\to \chi^2_n} > c/\sigma_0^2\Big)$$

$$\frac{c}{\sigma_0^2} = \chi^2_{n,1-\alpha}$$

$$= P(\Sigma x_i^2 > \chi^2_{n,1-\alpha} \cdot \sigma_0^2)$$

e) $\beta(\sigma) = 1 - G_n\left(\frac{\chi^2_{n,1-\alpha} \cdot \sigma_0^2}{\sigma^2}\right)$



Question 4 cont.)

c) Prove or disprove that the $\delta(\underset{\sim}{x})$ is the UMP test of its size.

e.g. one could use something like this:

$R = \{ \Sigma X_i^2 > c \}$

size $\alpha = P(\Sigma X_i^2 > c \mid \sigma = \sigma_0)$

$$= P\left(\Sigma \frac{X_i^2}{\sigma_0^2} > c/\sigma_0^2\right)$$

$$\frac{c}{\sigma_0^2} = \chi^2_{n,1-\alpha}$$

$$c = \chi^2_{n,1-\alpha} \cdot \sigma_0^2$$

$$\beta(\sigma) = P(\Sigma X_i^2 > \sigma_0^2 \chi^2_{n,1-\alpha} \mid \sigma > \sigma_0)$$

$$= P\left(\Sigma\left(\frac{X_i^2}{\sigma^2}\right) > \frac{\sigma_0^2 \chi^2_{n,1-\alpha}}{\sigma^2}\right)$$

$$= 1 - G_n\left(\frac{\sigma_0^2 \chi^2_{n,1-\alpha}}{\sigma^2}\right)$$

so, if $\sigma <<<$, the $\delta(\underset{\sim}{x})$ is more powerful.

But if $\sigma >>>$, then this test is more powerful.

But they are the same size α.

So $\delta(\underset{\sim}{x})$ is not UMP.

(S v. complex usually do not have UMP.

#2014

C = craig answer

① X = fathers ht inches
Y = Sons height inches } distributed Biv. Normal

$E(X) = E(Y) = 68$ $Var(X) = Var(Y) = 4$ $Cov(X,Y) = 0.6$

a) Prob father taller than son?

$$P(X>Y) = P(X-Y>0)$$
$$= 1 - P(X-Y<0)$$
$$= 1 - P(W<0)$$
$$= 1 - P\left(\frac{W-0}{\sqrt{6.8}} < \frac{0-0}{\sqrt{6.8}}\right)$$
$$= 1 - P(Z<0) = 0.5 \text{ C}$$

$X \sim N(68, 4)$ $8 - 1.2 = 6.8$

$Y \sim N(68, 4$

$X-Y \sim N(0, 4+4-2(0.6))$

$W \sim N(0, 6.8)$

$\frac{W-E(W)}{\sqrt{Var W}} \sim N(0,1)$

(b/c symmetric)

b) What is the probability that the father is at least 4 in taller than son?

$$P(X-Y \geq 4) = P(W \geq 4)$$
$$= P(Z \geq 4/\sqrt{6.8})$$
$$= 1 - P(Z < 4/\sqrt{6.8})$$ ✓C then finish.

c) What is the distribution of the heights of sons whose fathers are 74 inches tall?

= Distn of $Y|X=x$ will be Normal, w/

$$E(Y|X=x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x-\mu_X) = 68 + 0.15\left(\frac{\cancel{4}}{\cancel{4}}\right)(74-68) = 68.9$$

$$Var(Y|X=x) = (1-\rho^2)\,\sigma_Y^2 = (1-0.15^2)\,4 = 3.91$$

$$\rho = \frac{cov}{\sqrt{\sigma_x \sigma_y}} = \frac{0.6}{\sqrt{4(4)}} = \frac{0.6}{4} = 0.15$$

$$Y|X=74 \sim N(68.9,\ 3.91)$$

good from notes

d) Given the father = 74 in, find the prob the son is tall than the father (greater than 74 in)

$\sqrt{3.91} = 1.977$

$$P(Y>74 \mid X=74) = P(Y>74) = P\left(\frac{Y-68.9}{\sqrt{3.91}} > \frac{74-68.9}{\sqrt{3.91}}\right)$$

$$= P(Z > 3.03)$$

$$= 1-P(Z<3.03) \approx \text{small } \#$$

C

e) 100 father - son pairs are sampled

$\bar{X}$ = avg. fathers
$\bar{Y}$ = avg. sons
> what is joint $\bar{X}, \bar{Y}$ distr.

$(X,Y) \Rightarrow$ BIV. norm. so $(\bar{X}, \bar{Y}) \Rightarrow$ Biv Norm also

$\bar{X} \sim N(68, \frac{4}{100})$

$\bar{Y} \sim N(68, 4/100)$

corr says the same?

$0.15 = \frac{cov}{\sqrt{\sigma_{\bar{X}}^2 \sigma_{\bar{Y}}^2}} = 0.006 \equiv \frac{0.6}{100}$

f) What is prob the two sample avgs. are w/in 3 inches of each other

$P(|\bar{X}-\bar{Y}| < 3)$

$\bar{X}-\bar{Y} \sim N(68-68, \frac{4}{100} + \frac{4}{100} - 2(0.006))$

$\sim N(0, 0.068)$

$= P(-3 < \bar{X}-\bar{Y} < 3)$

$= P(\frac{-3}{\sqrt{0.068}} < z < 3/\sqrt{0.068})$

$= P(-11.. < z < 11...)$

$= P(z < 11.\ ) - P(z < -11..) = \sim 1$

② y = # common colds (RANDOM)
x = expected # common colds

$$Y|X \sim Pois(x) \quad E(Y|X) = X \quad Var(Y|X) = X \quad \text{(by Pois)}$$

$$X \sim Uni(0,2) \quad f_X(x) = \begin{cases} 1/2 & 0<x<2 \\ 0 & \text{else} \end{cases} \quad EX = \frac{2+0}{2} = 1 \quad VarX = \frac{(2-0)^2}{12} = \frac{4}{12} = \frac{1}{3}$$

a) find $E(Y)$ and $Var(Y)$

$$EY = E_X E(Y|X) = E_X(X) = 1$$

$$VarY = E_X Var Y|X + Var_X E Y|X$$
$$= E_X(X) + Var_X(X) = 1 + \frac{1}{3} = 4/3$$

$\Rightarrow$ NO, Y is not Pois because $EY \neq VarY$

---

b) Find $corr(X,Y)$

$$corr(X,Y) = \frac{\overset{\text{need this}}{Cov(X,Y)}}{\underset{\text{know these}}{\sqrt{VarX \cdot VarY}}}$$

$$\Rightarrow corr = \frac{\frac{1}{3}}{\sqrt{\frac{1}{3}\cdot\frac{4}{3}}} = \frac{\frac{1}{3}}{\sqrt{4/9}} = \frac{\frac{1}{3}}{2/3} = \frac{1}{2}$$

know these

pick one to condition upon

$$Cov(X,Y) = EXY - \underbrace{EX\,EY}_{\text{know these}}$$
$$= E\,E\,\overset{\text{cons.}}{X}Y|X - 1(1)$$
$$= EX^2 - 1$$
$$= (VarX + (EX)^2) - 1$$
$$= \frac{1}{3} + 1^2 - 1 = \frac{1}{3}$$

$$= E\,E(XY|X)$$
$$= E(X \cdot EY|X)$$
$$= EX^2$$

c) $W = 4X - Y + 4$ compute $\operatorname{cov}(W,Y)$ are W and Y indep? justify.

$EW = 4EX - EY + 4$
$= 4 - X + 4 = 8 - X$

$\operatorname{Var}W = 4^2 \operatorname{Var}X - \operatorname{Var}Y + \cancel{\operatorname{Var}4}^{\,0}$ no variance of a constant

$= 4^2\left(\frac{1}{3}\right) + 1^2\left(\frac{4}{3}\right) - \overset{2\ a\ b\ \operatorname{cov}(x,y)}{2(4)(1)\left(\frac{1}{3}\right)}$

$= \frac{16}{3} + \frac{4}{3} - \frac{8}{3} = \frac{12}{3} = 4$

if cov = 0 they are not necessarily indep.

$\operatorname{cov}(W,Y) = \underbrace{E(WY)}_{\text{need this}} - \underbrace{EW\,EY}_{\text{have these}}$ $EW = 8-X$ $EY = 1$

$\operatorname{Var} W+Y = \operatorname{Var}W + \operatorname{Var}Y + 2\operatorname{cov}(W,Y)$

$\operatorname{Var}(4x - y + 4 + y)$

$\operatorname{Var}(4X+4)$

$16 \operatorname{Var}X = 4 +$

$\frac{16}{3} = 4 + \frac{4}{3} + 2\operatorname{cov}(W,Y)$

$\operatorname{cov}(W,Y) = \frac{\left(\frac{16}{3} - 4 - \frac{4}{3}\right)}{2} = 0/2 = 0$

(range of W depends on Y)

$\operatorname{cov}(W,Y) = 0$
$\Rightarrow$ does not mean they are independent.

d) X is not observable, but Y is
is Y unbiased predictor of X? $\Rightarrow$ is $E(Y-X) = 0$

$E(Y-X) = E(Y) - E(X) = 1 - 1 = 0$ $\Rightarrow$ Y unbiased predictor of X

Compute prediction $MSE = E[(Y-X)^2]$

$E(X-Y) = EX - EY$
$= 1 - 1 = 0$

$\operatorname{Var}(Y-X) = PMSE - \cancel{[E(Y-X)]^2}\ 0$

$\operatorname{Var}Y + \operatorname{Var}X - 2\operatorname{cov}(X,Y)$

$\frac{4}{3} + \frac{1}{3} - 2\left(\frac{1}{3}\right) = \frac{3}{3} = 1$

PMSE = 1

e). Find constants a and b such that $a+bY$ is an unbiased predictor of X and such that the prediction var is <<<

$E(X-a-bY)=0$

$Var(X-a-bY)$ is small / minimized

$a=0,\ b=1$?

I said d) was unbiased..(?)

APPL

$E(X)-a-bE(Y)=0$

$1-a-b(1)=0$

$a+b=1$

$a=1-b$

$Var X + \cancel{Var(a)} + b^2 Var y - 2b\, cov(x,y)$

$\left\{\frac{1}{3}+b^2\left(\frac{4}{3}\right)-2b\left(\frac{1}{3}\right)\right\}\cdot \frac{d}{db}$

$2b\left(\frac{4}{3}\right)-\frac{2}{3}$

$\frac{8}{3}b-\frac{2}{3}=0$ $\qquad \frac{d}{db^2}=\frac{8}{3}>0$

$b=\frac{2}{3}\cdot\frac{3}{8}=\frac{1}{4}$ and $a=1-1/4$

$a=3/4$

f) $P(y=0)$ use $f(y) \Rightarrow$ get $f(y)$ by $\int_0^2 f(x)\cdot f(y|x)\,dx$

$fx\cdot fy|x = \frac{1}{2}\cdot\frac{e^{-x}x^y}{y!}=\frac{e^{-x}x^y}{2y!}$

$\int_0^2 \frac{e^{-x}x^y}{2y!}\,dx =$ at $y=0$ → integrate over x to get f(y) at f(y=0), that's the prob of no colds. Just did backwards

$=\int_0^2 \frac{e^{-x}}{2}\,dx = \frac{-e^{-x}}{2}\Big|_0^2$

$=-\left(\frac{e^{-2}-1}{2}\right)=$ prob (y=0 = no common colds over all possible X colds)

$v=-e^{-x}$ $\qquad u=x^y$

$dv=e^{-x}$ $\qquad du=yx^{y-1}\,dx$

$uv-\int v\,du$

$x^y(-e^{-x})+\int e^{-x}\,y\,x^{y-1}\,dx$

⋮

neverending?

g) $E(X|y=0)$

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{f(x)\,f(y|x)}{f(y\neq 0)} = \frac{f(x,y=0)}{f(y=0)}$$

$$= \frac{\frac{1}{2}\cdot\frac{e^{-x}\cancel{x^{0}}}{\cancel{0!}}}{\frac{1}{2}-\frac{e^{-2}}{2}} =$$

number would be helpful ...

$= 0.432$

$$= \frac{\frac{1}{2}e^{-x}}{0.432} = 1.16e^{-x}$$

$$\int_0^2 x \cdot f(x|y=0)\, dx$$

$$= \int_0^2 1.16\, x e^{-x}\, dx \qquad \begin{array}{ll} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x} \end{array}$$

$$= \left\{ -xe^{-x}\Big|_0^2 + \int_0^2 e^{-x}\, dx \right\} 1.16 \qquad uv - \int v\, du$$

$$= (-2(e^{-2})) - e^{-x}\Big|_0^2$$

$$= \left\{ (-2e^{-2} - (e^{-2} - 1) \right\} 1.16$$

③ $X_1 \dots X_n$ iid

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x = 0, 1, \dots \infty \quad \lambda > 0$$

a) find MLE of $\theta = P(X=0)$

$$P(X=0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda}$$

$e^{-\hat{\lambda}}$ is MLE $\Rightarrow$ get this by invariance

$$L(\lambda|\underset{\sim}{x}) = \frac{\lambda^{\Sigma x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}$$

$$\ell(\lambda|\underset{\sim}{x}) = \Sigma x_i \log \lambda - n\lambda - \Sigma \log x_i!$$

$$\frac{d\ell}{d\lambda} = \frac{\Sigma x_i}{\lambda} - n = 0 \qquad \frac{d^2\ell}{d\lambda^2} = \frac{-\Sigma x_i}{\lambda^2}$$

$$\hat{\lambda} = \frac{\Sigma x_i}{n} = \bar{x}$$

$$E\left(\frac{\Sigma x_i}{\lambda^2}\right) = \frac{\Sigma E x_i}{\lambda^2} = \frac{n}{\lambda}$$

$$\hat{\theta}_{MLE} = P(X=0) = e^{-\bar{x}}$$

b) Show that $\hat{\theta} = (1 - \frac{1}{n})^y$ is an unbiased estimator of θ where $y = \Sigma x_i$

$Y \sim \text{Pois}(n\lambda)$

should $= \theta$ if unb.

$$E(\{1-\tfrac{1}{n}\}^y) = \sum_{y=0}^{\infty} (1 - \tfrac{1}{n})^y \left(\frac{n\lambda^y e^{-n\lambda}}{y!}\right) = \sum_{y=0}^{\infty} \frac{(n\lambda - \lambda)^y e^{-n\lambda}}{y!}$$

$$= \underbrace{\sum_{y=0}^{\infty} \frac{(n\lambda-\lambda)^y e^{-(n\lambda - \lambda)}}{y!}}_{\text{Pois }(n\lambda-\lambda)\text{ pdf}} \cdot \frac{1}{e^{\lambda}} = e^{-\lambda} = P(X=0) = \theta \Rightarrow \text{unbiased estimator}$$

c) ① Derive $Var\,\hat{\theta}$. ② Does it achieve the CRLB for unbiased est. of θ? $e^{-\lambda}$

1) $Var\,\hat{\theta} = E(\hat{\theta}^2) - (E\hat{\theta})^2$

$\quad * \quad - (e^{-\lambda})^2$

distribute 2 through

$$* = E\left(\left(\left\{1-\tfrac{1}{n}\right\}^Y\right)^2\right) = \sum_{y=0}^{\infty}\left(\left(1-\tfrac{1}{n}\right)^{2y}\right)\frac{(n\lambda)^y e^{-n\lambda}}{y!}$$

$$= \sum_{y=0}^{\infty}\left(1-\frac{2}{n}+\frac{1}{n^2}\right)^y \frac{(n\lambda)^y e^{-n\lambda}}{y!}$$

$$= \sum_{y=0}^{\infty}\frac{\left(n\lambda-2\lambda+\frac{\lambda}{n}\right)^y}{y!} e^{-n\lambda} \qquad \text{kernel}$$

$$= \underbrace{\sum_{y=0}^{\infty}\frac{\left(n\lambda-2\lambda+\frac{\lambda}{n}\right)^y e^{-\left(n\lambda-2\lambda+\frac{\lambda}{n}\right)}}{y!}}_{=1} \cdot \frac{1}{e^{2\lambda-\frac{\lambda}{n}}}$$

$$= e^{-\left(2\lambda-\frac{\lambda}{n}\right)}$$

$$Var\,\hat{\theta} = e^{-2\lambda+\frac{\lambda}{n}} - e^{-2\lambda} = e^{-2\lambda}e^{\lambda/n} - e^{-2\lambda}$$

$$= e^{-2\lambda}\left(e^{\lambda/n}-1\right)$$

② CRLB =

$$\frac{\left\{\frac{d}{d\lambda}e^{-\lambda}\right\}^2}{n/\lambda} = \frac{e^{-2\lambda}\cdot\lambda}{n}$$

$E\left(-\frac{d^2}{d\lambda^2}f(\underline{x}|\lambda)\right)$

$Var\,\hat{\theta} > CRLB \Rightarrow$ does not reach minimum possible variance d.) Can only observe

$$Z_i = \begin{cases} 1 & X_i > 0 \\ 0 & X_i = 0 \end{cases}$$

find expression for MLE $\hat{\lambda}$ of λ as a function of $z_1, \dots z_n$ and derive limiting variance as $n \to \infty$

$$P(X_i > 0) = 1 - P(X_i = 0)$$
$$p = 1 - e^{-\lambda}$$

$p = 1 - e^{-\lambda}$

$\hat{p} = 1 - e^{-\hat{\lambda}}$ by invariance

$\hat{\lambda} = -\log(1-\hat{p})$ algebra

① find $\hat{p}$ MLE

$$L(p \mid \underset{\sim}{z}) = \prod_{i=1}^{n} p^{z_i}(1-p)^{1-z_i} = p^{\Sigma z_i}(1-p)^{n-\Sigma z_i}$$

$$\ell(p \mid \underset{\sim}{z}) = \Sigma z_i \log p + (n - \Sigma z_i)\log(1-p)$$

$$\frac{d\ell}{dp} = \frac{\Sigma z_i}{p} - \frac{(n-\Sigma z_i)}{(1-p)} = 0 \quad \text{solve for } p$$

$$\frac{d^2\ell}{dp^2} = -\frac{\Sigma z_i}{p} - \frac{(n-\Sigma z_i)}{(1-p)^2} < 0 \checkmark$$

$$= \Sigma z_i(1-p) - (n - \Sigma z_i)p = 0$$

$$\hat{p} = \frac{\Sigma z_i}{n} = \bar{z}$$

② $\hat{\lambda} = -\log(1-\bar{z})$

$\sqrt{n}(\hat{\lambda} - \lambda) \xrightarrow{d} N(0, ?)$ * by Δ method *

$g(w) = -\log(1-w)$

$g(w)' = \frac{+1}{1-w}$

$(g(w)')^2 = (1-w)^{-2}$

$\sqrt{n}(\bar{z} - p) \xrightarrow{d} N(0, p(1-p))$

$\sqrt{n}(g(\bar{z}) - g(p)) \xrightarrow{d} N(0, (g'(p))^2 p(1-p))$

$$\sqrt{n}(\hat{\lambda} - \hat{\lambda}) \xrightarrow{d} N(0, (1-p)^{-2} p(1-p))$$
$$N(0, p(1-p)^{-1}) \qquad p = 1 - e^{-\lambda}$$

$$\frac{(1-e^{-\lambda})}{(1-(1-e^{-\lambda}))} = \frac{(1-e^{-\lambda})}{e^{-\lambda}} = e^{\lambda} - 1$$

e) $\bar{X} = \frac{\sum X_i}{n}$

$\sqrt{n}\,(\bar{X} - \lambda) \xrightarrow{d} N(0, \lambda)$ ~~yes~~ $e^{-\lambda} = 1 - \lambda \; \ldots$

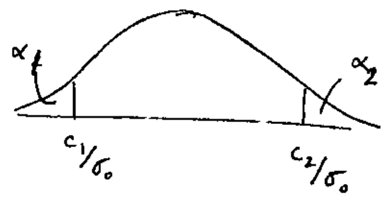
$ARE(\bar{X}, \hat{\lambda}) = \frac{\lambda}{e^{\lambda} - 1} < 1 \Rightarrow \text{var } \bar{X} \text{ as } n \to \infty < \text{var } \hat{\lambda} \text{ as } n \to \infty$

$\Rightarrow$ makes sense. lose info on $X_1 \ldots X_n$ as we dichotomize into $Z_1 \ldots Z_n$ (to get $\hat{\lambda}$), and by losing info our variance increases.

(4) $X_1 \ldots X_n \sim N(0, \sigma^2)$

$H_0: \sigma = \sigma_0$ vs. $H_1: \sigma \neq \sigma_0$

$$\delta(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } \sum X_i^2 < c_1 \text{ or } \sum X_i^2 > c_2 \\ 0 & \text{otherwise} \end{cases}$$



a) $\alpha = P(\sum X_i^2 < c_1) + P(\sum X_i^2 > c_2) \mid \sigma = \sigma_0$

$\alpha = P\left(\frac{\sum X_i^2}{\sigma_0^2} < \frac{c_1}{\sigma_0^2}\right) + P\left(\frac{\sum X_i^2}{\sigma_0^2} > c_2/\sigma_0^2\right)$ $\qquad \frac{\sum X_i^2}{\sigma_0} \sim \chi^2_n$

$= P(\chi^2_n < \chi^2_{n,\alpha_1}) + P(\chi^2_n > \chi^2_{n,1-\alpha_2})$ $\qquad \alpha_1 + \alpha_2 = \alpha$

$\Rightarrow$ $c_1 = (\chi^2_{n,\alpha_1})\,\sigma_0^2$

$c_2 = (\chi^2_{n,1-\alpha_2})\,\sigma_0^2$ C

b) show that for c_1 / c_2 choices, the power function of the test is ..

$\beta(\sigma) = P(X \in R)$

$= P\left(\frac{\sum X_i^2}{\sigma^2} < \frac{\chi^2_{n,\alpha/2} \cdot \sigma_0^2}{\sigma^2}\right) + P\left(\frac{\sum X_i^2}{\sigma^2} > \frac{\chi^2_{n,1-\alpha/2} \cdot \sigma_0^2}{\sigma^2}\right)$ χ^2_n

$\overset{CDF}{=} G_n\left(\frac{\chi^2_{n,\alpha/2} \cdot \sigma_0^2}{\sigma^2}\right) + \left(1 - G_n\left(\frac{\chi^2_{n,1-\alpha/2} \cdot \sigma_0^2}{\sigma^2}\right)\right)$

of χ^2_n

c) prove / disprove that δ is the UMP test of its size

$\Rightarrow$ complex $(\sigma \neq \sigma_0)$ tests often / do not have UMP tests.

e.g. $\delta_{2}(x) = \begin{cases} 1 & \text{if } \sum x_i^2 > c^* \\ 0 & \text{else} \end{cases}$

$R = \{ \sum x_i^2 > c^* \} \Rightarrow \left\{ \frac{\sum x_i^2}{\sigma_0^2} > c^*/\sigma_0 \right\}$

χ^2_n, $\chi^2_{n,1-\alpha}$

$$\frac{c^*}{\sigma_0^2} = \chi^2_{n,1-\alpha}$$

$$c^* = \chi^2_{n,1-\alpha} \cdot \sigma_0^2$$

$$\beta(\sigma) = 1 - G_n\left(\frac{\chi^2_{n,1-\alpha} \cdot \sigma_0^2}{\sigma^2} \right)$$

$\hookrightarrow$ This is more powerful than $\delta(x)$

when $\sigma^2 >>>$ the $\delta_2(x)$ is more powerful

just an understanding. no actual solving

d) $H_0: \sigma = \sigma_0$ vs. $H_1: \sigma > \sigma_0$ let $H_1: \sigma = \sigma_1$ $\sigma_1 > \sigma_0$ ... then extrapolate to comp.

N-P

$$\frac{f(\underline{x}|\sigma_1)}{f(\underline{x}|\sigma_0)} = \frac{(2\pi\sigma_1^2)^{-n/2}}{(2\pi\sigma_0^2)^{-n/2}} \frac{\exp(-\frac{1}{2\sigma_1^2}\sum x_i^2)}{\exp(-\frac{1}{2\sigma_0^2}\sum x_i^2)} > c \quad \text{is UMP}$$

$$= \left(\frac{\sigma_0}{\sigma_1}\right)^n \cdot \exp\left(\frac{\sum x_i^2}{2}\left(-\frac{1}{\sigma_1^2} + \frac{1}{\sigma_0^2}\right)\right)$$

$\Rightarrow \sum x_i^2 > c$ by MLR

χ^2_n distributed, $\chi^2_{n,1-\alpha}$

$R = \{ \sum x_i^2 > c \} \Rightarrow P\left\{ \frac{\sum x_i^2}{\sigma_0^2} > \frac{c}{\sigma_0^2} \right\} = \alpha$

$$\chi^2_{n,\alpha} = \frac{c}{\sigma_0^2}$$

$= P\{ \chi^2_n > \chi^2_{n,1-\alpha} \} = \alpha$

$$c = \sigma_0^2 \cdot \chi^2_{n,1-\alpha}$$

e) $P(\sum x_i^2 > c \mid \sigma = \sigma_0)$

$P\left(\frac{\sum x_i^2}{\sigma_0^2} > \chi^2_{n,\alpha} \cdot \sigma_0^2 \right)$

(yes and craig did actual math comparing the two functions.