Lect1-Quadratic Fuctions

AMC-12

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Outline

Outline of Algebra:

- Functions:
  - Linear functions;
  - **Quadratics** and Polynomials;
  - Exponents and Logarithms;
  - ...
- Inequalities;
- Sequences and Series;
- Trigonometric Functions;
- Complex Number;
- ...

Three Forms of Quadratic Function

Expanded form

$$f(x) = ax^2 + bx + c \ (a \neq 0)$$

Factored form

$$f(x) = a(x - x_1)(x - x_2) \ (a \neq 0)$$

Vertex form

$$f(x) = a(x - h)^2 + k \ (a \neq 0)$$

Properties of Quadratic Functions

Symmetric Axis and Vertex

The symmetric axis of a quadratic function $f(x) = ax^2 + bx + c$ is

$$x = -\frac{b}{2a},$$

and its vertex is

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).$$

Properties of Quadratic Functions

Discriminant and Roots

The discriminant of a quadratic function $f(x) = ax^2 + bx + c$ is $\Delta = b^2 - 4ac$, if $\Delta \geq 0$, the two real solutions of $f(x) = 0$ are

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}.$$

Quadratic Inequalities

- - Parameters a, b, c and function's graph;
  - Discriminant Δ and roots;
  - Equations and inequalities.
- Solution of $ax^2+bx+c>0(a>0)$, discriminant $\Delta=b^2-4ac$,
  - $\Delta>0$, $x<\dfrac{-b-\sqrt{\Delta}}{2a}$ or $x>\dfrac{-b+\sqrt{\Delta}}{2a}$;
  - $\Delta=0$, $x\neq\dfrac{-b}{2a}$;
  - $\Delta<0$, $x\in\mathbb{R}$.
- Solution of $ax^2+bx+c<0(a>0)$, discriminant $\Delta=b^2-4ac$,
  - $\Delta>0$, $\dfrac{-b-\sqrt{\Delta}}{2a}<x<\dfrac{-b+\sqrt{\Delta}}{2a}$;
  - $\Delta\leq 0$, $x\in\emptyset$.

Examples of Quadratic Inequalities

- Example 1: $x^2 + 4x - 21 \geq 0$
- Example 2: $x^2 - 3x - 1 < 0$
- Example 3: $x^2 - 5|x| + 6 > 0$

Vieta's Theorem

Vieta's Theorem for Quadratic

If x_1 and x_2 are two roots of the quadratic equation $f(x) = ax^2 + bx + c = 0$, then

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}.$$

Examples of Vieta's Theorem

- Example 1: Let r and s be the roots of the polynomial $x^2 + x + 1 = 0$. What is $r^2 + s^2 + r + s + rs$?
- Example 2: If the sum of two squares of the real roots for the quadratic equation $2x^2 + ax - 2a + 1$ is $\frac{29}{4}$. What is the value of a?

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Excercise 1

$x, y \in \mathbb{R}$ and $x^2 + xy + y^2 = 3$, what is the possible maximum of $x - y$?

Exercise 2

Assume a, b are two roots of the equation $2x^2 + x - 7$. Find

1. $a^2 + b^2$;
2. $\frac{1}{a} + \frac{1}{b}$;
3. $a^3 + b^3$.

Exercise 3

For what values of k do the quadratics $x^2 + kx + 1$ and $kx^2 + x + 1$ share exactly one root?

Exercise 4

Suppose that the equation $x^2 - bx + a$ has roots $a - b$ and $a + b$, for non-zero real numbers a, b. Find a.

Exercise 5

Find the unique positive integer n for which $n^2 + 7n$ is a perfect square.

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A Short Review

- Three forms of quadratic function;
- Geometric properties of quadratic functions;
- Roots of quadratic functions;
- Quadratic inequalities;
- Vieta's theorem.