Lect5-Sequences

AMC-12

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Outline

Outline of Algebra:

- Functions:
  - Linear functions;
  - Quadratics and Polynomials;
  - Exponents and Logarithms;
  - ...
- Inequalities;
- **Sequences and Series**:
  - Arithmetic sequences;
  - Geometric sequences;
  - Fibonacci sequence;
  - ...
- Trigonometric Functions;
- Complex Number;
- ...

Fibonacci Sequence

The Fibonacci Sequence is the series of numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$$

- $1 = 0 + 1;$
- $2 = 1 + 1;$
- $3 = 1 + 2;$
- $5 = 2 + 3;$
- $8 = 3 + 5;$
- ...

- Fibonacci sequence: $\{F_n\}$;
- Rule: $F_n = F_{n-1} + F_{n-2}$;
- F_n is term number n;
- F_{n-1} is the previous term $(n-1)$;
- F_{n-2} is the term before that $(n-2)$;
- $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$

Sequence

Definition 1 (Sequence)

A sequence is a list of things (usually numbers) that are in order.

Propertis:

- Infinite or finite;
- In order;
- A rule or many rules;
- ...

Definition

In an Arithmetic Sequence the difference between one term and the next is a constant.

Example: $1, 4, 7, 10, 13, \cdots$.

Definition 2

An arithmetic sequence $\{a_n\}$ with a first term a and a common difference d, the formula for the n^{th} term is

$$a_n = a + (n-1)d.$$

Summing an Arithmetic Sequence

Define the sum of first n terms:

$$S_n = \sum_{i=1}^{n} a_n = a_1 + a_2 + \cdots + a_n,$$

For an arithmetic sequence $\{a + (n-1)d\}$, we have

$$S_n = a + (a+d) + \cdots + (a + (n-2)d) + (a + (n-1)d),$$
$$S_n = (a + (n-1)d) + (a + (n-2)d) + \cdots + (a+d) + a,$$
$$\Rightarrow$$
$$\begin{aligned} 2S_n &= (a + (n-1)d) + (a + (n-1)d) + \cdots + (a + (n-1)d) \\ &= n(a + (n-1)d) \end{aligned},$$

So

$$S_n = \frac{n}{2}(a_1 + a_n) = na + \frac{n(n-1)}{2}d.$$

Examples

Example 1: The terms of an arithmetic sequence add to 755. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and, in general, the kth term is increased by $2k-1$. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

Examples

Example 2: A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

Definition

In a Geometric Sequence each term is found by multiplying the previous term by a constant.

Example: $1, 2, 4, 8, 16, \cdots$

Definition 3

A geometric sequence $\{a_n\}$ with a first term a and a common ratio r, the formula for the n^{th} term is

$$a_n = ar^{n-1}.$$

Summing an Arithmetic Sequence

For a geometric sequence $\{ar^{n-1}\}$ with a ratio $r \neq 1$, we have

$$\begin{array}{rcccccc} S_n = & a & +ar & +\cdots & +ar^{n-2} & +ar^{n-1} & & , \\ rS_n = & & ar & +ar^2 & +\cdots & +ar^{n-1} & +ar^n & , \end{array}$$

$$\Rightarrow$$

$$(r-1)S_n = ar^n - a$$

So

$$S_n = \frac{ar^n - a}{r-1}.$$

If $|r| < 1$, we have

$$S = \sum_{i=1}^{+\infty} a_n = \frac{a}{1-r}.$$

Examples

Example 1: Zeno, a tortoise, is coming home. He is currently 40 meters from the front door. At each minute, he walks half of the remaining distance to the door. For example, for the first minute, he walks 20 meters towards the door. During the second minute, he walks 10 meters more towards the door. How many meters does he walk during the 5th minute?

Examples

Example 2:Proof:

$$0.333333\cdots = \frac{1}{3}$$

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Excercise 1

Find the sum of the first n terms of $5, 55, 555, \cdots$.

Excercise 2

Find the sum of $1^2 + 2^2 + \cdots + n^2$.

Exercise 3

Define a function of x, which denoted by

$$S(x) = 1 + x - x^2 - x^3 + x^4 + x^5 - x^6 - x^7 + \cdots$$
$$\cdots + x^{4k} + x^{4k+1} - x^{4k+2} - x^{4k+3} + \cdots$$

For how many x with $0 < x < 1$ is $S(x) = \frac{1+\sqrt{2}}{2}$?

Exercise 4

In the sequence $\{a_n\}$, $a_n = \dfrac{1}{n+1} + \dfrac{2}{n+1} + \cdots + \dfrac{n}{n+1}$. Find the sum of the first n terms of the sequence $\{b_n\}$, $b_n = \dfrac{2}{a_n a_{n+1}}$.

Excercise 5

Find the value of $S_n = \frac{2}{2} + \frac{4}{2^2} + \frac{6}{2^3} + \cdots + \frac{2n-2}{2^{n-1}} + \frac{2n}{2^n}$.

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A Short Review

- Sequence;
- Arithmetic Sequences
  - Definition;
  - Summation;
- Geometric Sequences
  - Definition;
  - Summation;
- Summation of other sequences.