Lect4-Inequalities

AMC-12

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Outline

Outline of Algebra:

- Functions:
  - Linear functions;
  - Quadratics and Polynomials;
  - Exponents and Logarithms;
  - ...
- **Inequalities**:
  - Mean Inequality;
  - Cauchy Inequality;
- Sequences and Series;
- Trigonometric Functions;
- Complex Number;
- ...

Basic Properties

Here are some basic properties for inequalites:

- $x > y \Leftrightarrow y < x$;
- $x > y, y > z \Rightarrow x > z$;
- $x > y \Leftrightarrow x + z > y + z$;
- $x > y, z > 0 \Rightarrow xz > yz$; $x > y, z < 0 \Rightarrow xz < yz$;
- $x > y, m > n \Rightarrow x + m > y + n$;
- $x > y > 0, m > n > 0 \Rightarrow xm > yn$;
- $x > y > 0, n > 0 \Rightarrow x^n > y^n$; $x > y > 0, n < 0 \Rightarrow x^n < y^n$;
- ...

Basic Ineuqlity I

Theorem 1 (Basic Ineuqlity (AM-GM))

For any $a, b \geq 0$,

$$\frac{a+b}{2} \geq \sqrt{ab},$$

with equality iff $a = b$.

- Arithmetic mean: $\dfrac{a+b}{2}$,
- Geometric mean: $\sqrt{ab}$,
- Harmonic mean: $\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$,
- Quadratic mean: $\sqrt{\dfrac{a^2+b^2}{2}}$.

Basic Ineuqlity II

HM-GM-AM-QM Inequality: for any $a, b > 0$

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}},$$

The equality holds iff $a = b$.

Definition of Means

- Quadratic mean:

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}},$$

- Arithmetic mean:

$$\frac{x_1 + x_2 + \cdots + x_n}{n},$$

- Geometric mean:

$$\sqrt[n]{x_1 x_2 \cdots x_n},$$

- Harmonic mean:

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

Mean Inequality

Theorem 2 (Mean Inequality (QM-AM-GM-HM))

For $x_i > 0, i = 1, 2, \cdots, n,$

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 x_2 \cdots x_n}$$

$$\leq \frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}},$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

Examples

Example 1: Caculate AM, GM, HM, QM of $4, 36, 45, 50, 75$.

Examples

Example 2: Find the greatest possible value of xyz, where x, y and z are positive real numbers with $x + y + z = 12$.

Examples

Example 3: Determine the minimum value of the function

$$f(x,y,z) = \frac{x^2+y^2+z^2}{xy+yz},\ x>0, y>0, z>0.$$

Cauchy-Schwarz Inequality

Theorem 3

For two sequences of real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$, we have

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)$$

$$\geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2.$$

Equality holds iff $b_1 = b_2 = \cdots = b_n = 0$ or there exist a k that $a_i = k b_i$ for all $i = 1, 2, \cdots, n$.

Example:

$$(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2.$$

Examples

Example 1: Given $a_1, a_2, \cdots, a_n \in \mathbb{R}$ such that $a_1 + a_2 + \cdots + a_n = 1$, prove that

$$a_1^2 + a_2^2 + \cdots a_n^2 \geq \frac{1}{n}.$$

Examples

Example 2: If $x^2 + y^2 + z^2 = 1$, what is the maximum value of $x + 2y + 3z$?

Examples

Example 3: For positive reals $a, b, c > 0$, show that

$$abc(a+b+c) \le a^3b + b^3c + c^3a.$$

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Excercise 1

Let a, b, c, d, e, f be real numbers

$$a + b + c + d + e + f = 10,$$

$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 + (e-1)^2 + (f-1)^2 = 6,$$

find the maximum value of f.

Excercise 2

Determine the minimum value of the function

$$f(x,y,z) = \frac{x^2+y^2+z^2}{xy+2yz},\ x>0, y>0, z>0.$$

Exercise 3

Determine the maximum possible value of the function

$$f(x) = 4x + 3\sqrt{1 - x^2},$$

where $0 < x < 1$.

Excercise 4

Let a, b, c, d be positive real numbers such that

$$a + b + c + d = 1,$$

what is the minimum value of

$$\frac{1}{a} + \frac{4}{b} + \frac{9}{c} + \frac{16}{d}.$$

Exercise 5

For positive reals $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$, prove that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

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A Short Review

- Properties of inequalities;
- Mean inequality:
    - Definitions of AM, GM, HM, QM;
    - QM-AM-GM-HM inequality;
- Cauchy-Schwarz inequality:
    - C-S inequality;
    - Proof of C-S inequality;