Lect3-Exponents and Logarithms

AMC-12

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Outline

Outline of Algebra:

- Functions:
  - Linear functions;
  - Quadratics and Polynomials;
  - **Exponents and Logarithms**;
  - ...
- Inequalities;
- Sequences and Series;
- Trigonometric Functions;
- Complex Number;
- ...

Exponents

Example:

$$2^2 \times 2^3 \times 2^4 = 4 \times 8 \times 16 = 512 = 2^9.$$

Rules of Exponents:

- $a^m \times a^n = a^{m+n}$, $a^n \times b^n = (a \times b)^n$;
- $a^{-n} = 1/a^n$, $a^n/a^m = a^{n-m}$, $a^n/b^n = (a/b)^n$;
- $(a^m)^n = a^{mn}$;
- $a^{1/n} = \sqrt[n]{a}$, $\sqrt[m]{a^n} = a^{n/m}$ $(a > 0)$;
- $a^{n^m} = a^{(n^m)}$;
- $a^0 = 1$ $(a \neq 0)$, $0^a = 0$ $(a > 0)$;

Examples

Example 1: Simplify

$$\left(\frac{x^5y^{-3}}{x^3y^8}\right)^2, \quad \frac{(27x^4y^{-5})^3}{(81x^2y^{-3})^2}.$$

Examples

Example 2: If $3^{x-y} = 81$, $3^{x+y} = 729$, what is x?
Example 3: If $9^{3-x} = 81^{4-2x}$, then what is x?

Examples

Example 4: Suppose that $4^{x_1} = 5, 5^{x_2} = 6, 6^{x_3} = 7, \cdots, 127^{x_{124}} = 128$. What is $x_1 x_2 \cdots x_{124}$?

Logarithms

Example: $\log_2 64 = 6,\ \log_3 81 = 4.$

Definition 1 (Logarithm)

A logarithm is the inverse of the exponent. Specifically, a logarithm is the power to which a number (the base) must be raised to produce a given number.
z is called the base-x $(x > 0, x \neq 1)$ logarithm of argument-y $(y > 0)$ if and only if $x^z = y$. In typical notation

$$\log_x y = z \Leftrightarrow x^z = y.$$

Natural logarithm: $\ln x = \log_e x,$

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818284\cdots.$$

Base 10 logarithm: $\lg x = \log_{10} x.$

Rules of Logarithm

Rules of Logarithms:

- $\log_a b + \log_a c = \log_a bc$;
- $\log_a b = -\log_a \frac{1}{b}$, $\log_a b - \log_a c = \log_a \frac{b}{c}$;
- $\log_a b^c = c \cdot \log_a b$;
- $\log_a b = \frac{1}{\log_b a}$;
- $\log_a b = \frac{\log_c b}{\log_c a} = \frac{\log_a c}{\log_b c}$;
- $a^{\log_a b} = b$;
- $\log_a 1 = 0$;

Examples

Example 1: Simplify

$$\log_2 \left(\frac{32}{9}\right)^2, \quad 2\log_4 \sqrt{5} + \frac{1}{2}\log_2 625 - \log_2 \frac{1}{5}.$$

Examples

Example 2: What are the solutions of the equation

$$\lg 2x + \lg(x-1) = \lg(x^2+3).$$

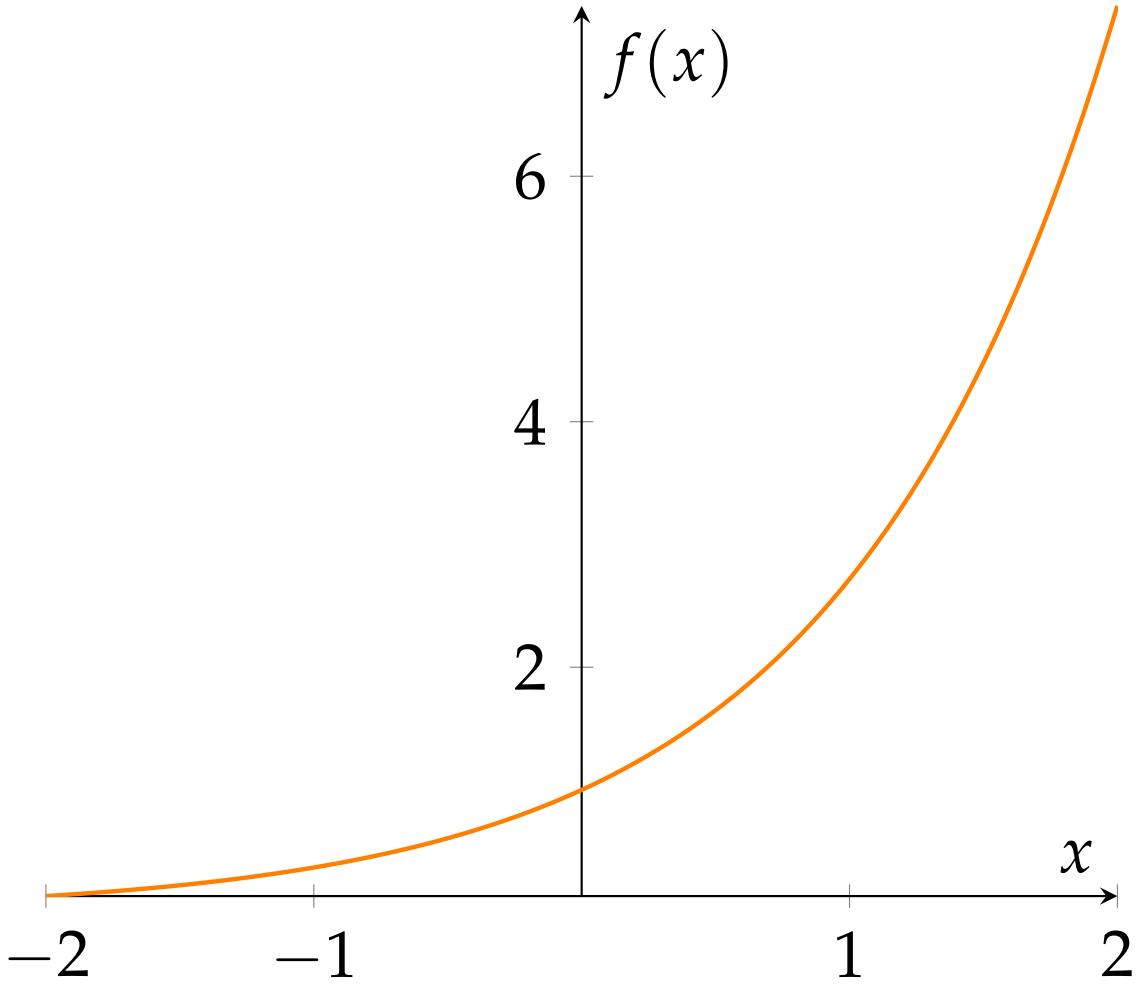
Example 3: What are the solutions of the equation

$$2(\lg x)^2 = 7\lg x - 3.$$

Examples

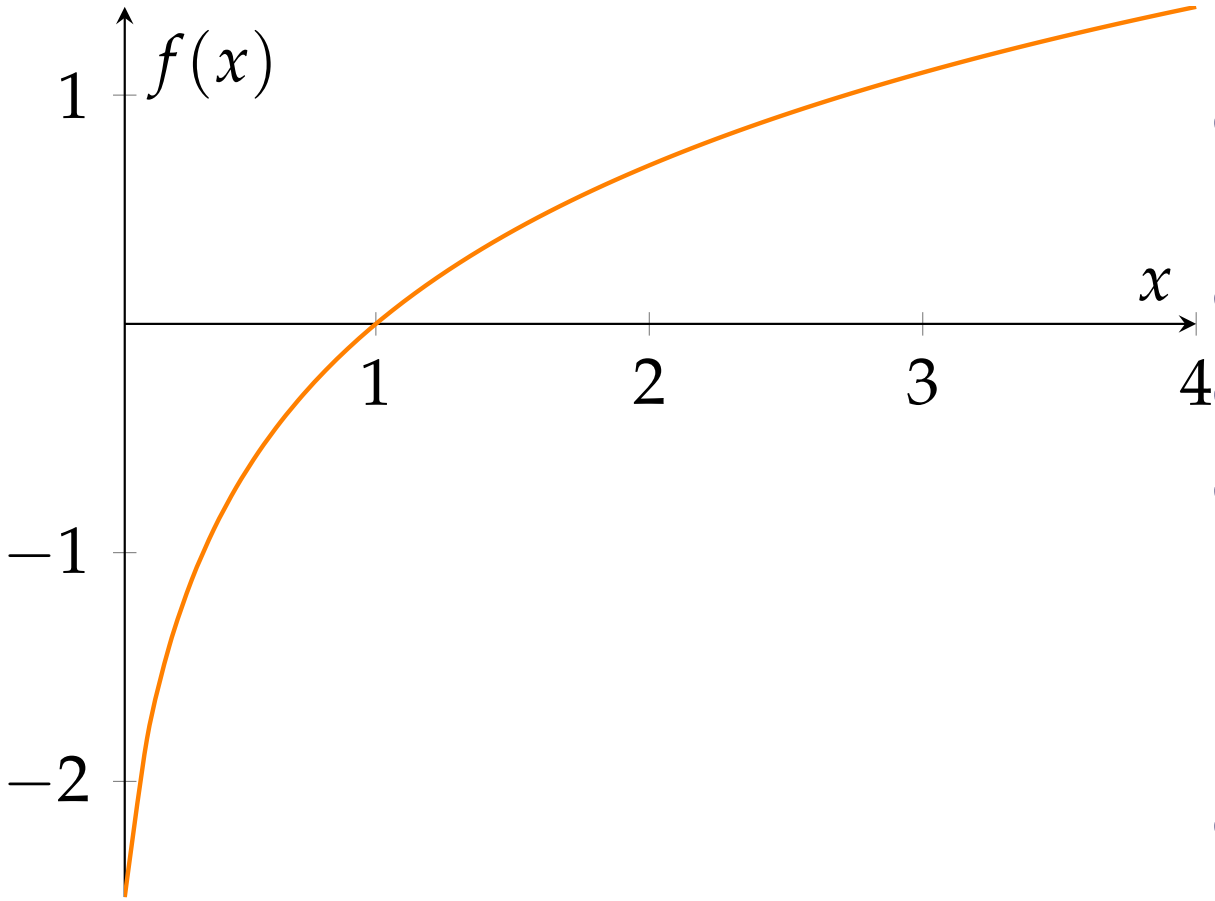
Example 4: If the solutions of the equation $x^{\log_3 x - 2} = 27$ are a and b, what is $\log_a b + \log_b a$?

Exponential Function



- $f(x) = a^x$ $(a > 0, a \neq 1)$;
- Domain:$(-\infty, +\infty)$;
- Range:$(0, +\infty)$;
- Monotonicity:$a > 1$: monotone increasing; $a < 1$: monotone decreasing.
- $f(0) = 1$.

Logarithm Function



- $f(x) = \log_a x; (a > 0, a \neq 1);$
- Domain:$(0, +\infty);$
- Range:$(-\infty, +\infty);$
- Monotonicity:$a > 1$: monotone increasing; $a < 1$: monotone decreasing.
- $f(1) = 0.$

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Excercise 1

Suppose $P = 2^m$ and $Q = 3^n$. What is 12^{mn}?
(A)P^2Q (B)P^nQ^m (C)P^nQ^{2m} (D)$P^{2m}Q^n$ (E)$P^{2n}Q^m$.

Excercise 2

How many positive integers n satisfy the following condition:

$$(130n)^{50} > n^{100} > 2^{200}?$$

Exercise 3

What is the domain of the function

$$f(x) = \log_{27}\left(\log_9\left(\log_3\left(\log_{\frac{1}{3}} x\right)\right)\right)?$$

Exercise 4

Solve the following equation for a:

$$\frac{1}{\log_2 a} + \frac{1}{\log_9 a} + \frac{1}{\log_{12} a} = 3.$$

Excercise 5

What is the solutions of the equation

$$\log_x xy \times \log_y xy + \log_x(x-y) \times \log_y(x-y) = 0?$$

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A Short Review

- Exponents
  - Definition;
  - Rules;
  - Exponential function;
- Logarithms
  - Definition;
  - Rules;
  - Logarithm function.