Lect2-Polynomials

AMC-12

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Outline

Outline of Algebra:

- Functions:
  - Linear functions;
  - Quadratics and **Polynomials**;
  - Exponents and Logarithms;
  - ...
- Inequalities;
- Sequences and Series;
- Trigonometric Functions;
- Complex Number;
- ...

Definition of Polynomial

Definition 1 (Polynomial)

A polynomial is of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$, n is the degree of this polynomial.

Remark 1.1

*When $n = 1$, it is **linear**; $n = 2$, it is **quadratic**; $n = 3$, it is **cubic**; $n = 4$, it is **quartic**.*

Factoring

Definition 2 (Factor)

If polynomial $A(x), B(x)$ and $Q(x)$ satisfy

$$A(x) = B(x)Q(x),$$

$B(x)$ is called the factor of $A(x)$, i.e., polynomial $A(x)$ is divisible by polynomial $B(x)$.

Theorem 3 (Factor theorem)

c is a root of a polynomial $P(x)$, that is $P(c) = 0$, if and only if $x - c$ is a factor of $P(x)$.

Useful Factoring Method

- $x^2 + (b+c)x + bc = (x+b)(x+c)$;
- $x^2 \pm 2ax + a^2 = (x \pm a)^2$;
- $x^3 \pm 3ax^2 + 3a^2x \pm a^3 = (x \pm a)^3$;
- $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + a^{n-1})$;
- $x^n - a^n = (x+a)(x^{n-1} - x^{n-2}a + \cdots + xa^{n-2} - a^{n-1})$, n is even;
- $x^n + a^n = (x+a)(x^{n-1} - x^{n-2}a + \cdots - xa^{n-2} + a^{n-1})$, n is odd.

Examples

- Example 1: Factor each of the following polynomials

$$x^2 + 15x + 36;$$

$$24x^3 + 81.$$

- Example 2: Factor each of the following polynomials

$$x^4 + y^4 - 7x^2y^2;$$

$$(x - y)^3 + (y - z)^3 + (z - x)^3.$$

Roots and inequalities

Theorem 4 (Fundamental theorem of algebra)

Any polynomial of degree n has n roots in $\mathbb{C}$.

Theorem 5 (Intermediate value theorem)

If function $f(x)$ is continuous on the interval $[a, b]$, then for all w between $f(a)$ and $f(b)$, there must be at least one value $a \leq c \leq b$ such that $f(c) = w$.

So how to solve a polynomial inequality?
Example: Solve $(x-1)^3(x-2)^2(x-3)^5 \geq 0$.

Vieta's Theorem

Theorem 6 (Vieta's theorem for polynomial)

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ *be a polynomial with complex coefficients and degree* n*, having complex roots* $r_1, r_2, \cdots, r_n$*. Then for any integer* $0 \le k \le n$*,*

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} r_{i_1} r_{i_2} \cdots r_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}.$$

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Excercise 1

Polynomial $P(x) = 2x^4 - 3x^3 + ax^2 + 5x + b$ is divisible by $(x+1)(x-2)$. What is a and b?

Exercise 2

Let $f(x) = x^4 - 7x^3 - 3x + 10$. How many negative numbers a are there such that $f(a) = 0$?

Exercise 3

Let $f(x) = x^3 - 4x^2 + 2x + 2$. How many real a satisfy $f(a) = 0$?

Excercise 4

Let $f(x)$ be a polynomial such that

$$x^2 f(x-1) = (x-1)^2 f(x).$$

What is $f(x)$?

Exercise 5

Let $f(x) = x^3 - 3x^2 + 5x - 15$, and let a, b, and c be the roots of this polynomial. Compute $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

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A Short Review

- Definition of polynomial;
- Factor;
- Roots and inequalities;
- Vieta's theorem.