Sweeps using Linear DG on Tetrahedra

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Want to solve

$$\Omega \cdot \nabla_x f + \sigma_t f = Q \tag{1}$$

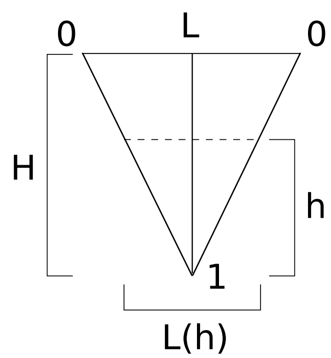
via linear DG on tetrahedra. Each DG basis is linear on a given cell C and 0 outside cell C. There are four basis elements in a given cell C which will be denoted b_0, b_1, b_2, b_3. The interpolatory basis defined by $b_i(x_j) = \delta_{ij}$ where x_j are the four vertices of the tetrahedron C will be used. Some preliminary facts will be needed.

Fact 1 On a triangle T, the integral of a linear function ℓ which evaluates to 1 at one vertex and 0 at the other two vertices has integral

$$\int_T \ell dx = \frac{1}{3} A \tag{2}$$

where A is the area of the triangle.

Proof. Orient the triangle so that the vertex with evaluation 1 is at $(0, 0)$ and the opposite edge is parallel to the x-axis. Below is a picture of such an orientation.



Then, since ℓ is constant over $L(h)$ for any h

$$\int_T \ell \, dx = \int_0^H \int_{L(h)} \ell \, dL \, dh = \int_0^H \ell(0, h) L(h) \, dh \tag{3}$$

with $\ell(0,h) = 1 - \frac{h}{H}$ and $L(h) = L\frac{h}{H}$. Evaluating the integral leads to

$$\begin{aligned}
\int_T \ell \, dx &= \int_0^H \ell(0,h)L(h)\, dh \\
&= \int_0^H \left(1 - \frac{h}{H}\right) L\frac{h}{H}\, dh \\
&= LH \int_0^1 (1-u)u\, du \\
&= LH\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}LH = \frac{1}{3}A.
\end{aligned}$$

$\square$

Fact 2 On a triangle T, the integral of $\ell_i\ell_j$ where ℓ_k is a linear function which evaluates to 1 at vertex k and 0 at the other two vertices has integral

$$\int_T \ell_i\ell_j dx = \begin{cases} \frac{1}{12}A, & \text{if } i \neq j \\ \frac{1}{6}A, & \text{if } i = j. \end{cases} \tag{4}$$

Proof. Suppose $i = j$. Then the proof is similar to the one in Fact 1, but $\ell(0,h)$ is replaced with its square

$$\begin{aligned}
\int_T \ell_i^2 \, dx &= \int_0^H \ell(0,h)^2 L(h)\, dh \\
&= \int_0^H \left(1 - \frac{h}{H}\right)^2 L\frac{h}{H}\, dh \\
&= LH \int_0^1 (1-u)^2 u\, du \\
&= LH\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{12}LH = \frac{1}{6}A.
\end{aligned}$$

Now suppose $i \neq j$. Use the same idea for the proof as in Fact 1 with ℓ_i evaluating to 1 at the bottom vertex. This means ℓ_j evaluates to 1 at one of the top vertices. The integral is given by

$$\int_T \ell_i\ell_j \, dx = \int_0^H \int_{L(h)} \ell_i\ell_j \, dL\, dh = \int_0^H \ell_i(0,h) \int_{L(h)} \ell_j \, dL\, dh \tag{5}$$

since ℓ_i is constant on $L(h)$. On $L(h)$, ℓ_j evaluates to zero at one end and $\frac{h}{H}$ at the other end. This makes the integral

$$\int_{L(h)} \ell_j \, dL = \frac{1}{2}\frac{h}{H}L(h) = \frac{L}{2}\left(\frac{h}{H}\right)^2. \tag{6}$$

Therefore

$$\int_T \ell_i \ell_j \, dx = \int_0^H \ell_i(0,h) \frac{L}{2} \left(\frac{h}{H}\right)^2 dh \tag{7}$$

$$= \int_0^H \left(1 - \frac{h}{H}\right) \frac{L}{2} \left(\frac{h}{H}\right)^2 dh \tag{8}$$

$$= \frac{1}{2} LH \int_0^1 (1-u)\, u^2 \, du \tag{9}$$

$$= \frac{1}{2} LH \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12} A. \tag{10}$$

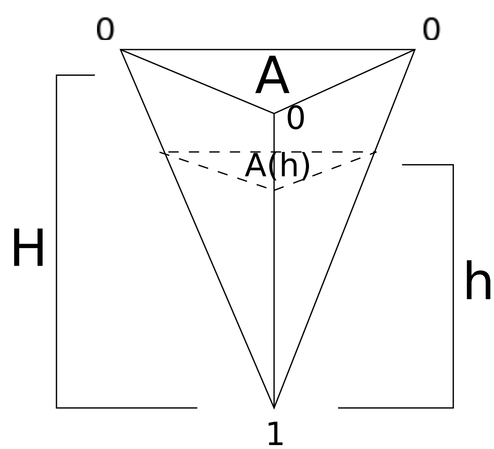
□

Fact 3 On a tetrahedron cell C, the integral of a linear function ℓ which evaluates to 1 at one vertex and 0 at the other three vertices has integral

$$\int_T \ell dx = \frac{1}{4} V \tag{11}$$

where V is the volume of C.

Proof. Orient the cell so that the vertex with evaluation 1 is at the origin and the opposite face is parallel and above the x,y-axes. Below is a picture of such an orientation.



Then using similar logic as before

$$\int_C \ell \, dx = \int_0^H \int_{A(h)} \ell \, dA \, dh = \int_0^H \ell(0,0,h) A(h) \, dh \tag{12}$$

with $\ell(0,0,h) = 1 - \frac{h}{H}$ and $A(h) = A\frac{h^2}{H^2}$. Evaluating the integral leads to

$$\begin{aligned}
\int_C \ell\, dx &= \int_0^H \ell(0,0,h)A(h)\, dh \\
&= \int_0^H \left(1 - \frac{h}{H}\right) A\frac{h^2}{H^2}\, dh \\
&= AH \int_0^1 (1-u)u^2\, du \\
&= AH\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}AH = \frac{1}{4}V.
\end{aligned}$$

□

Fact 4 On a tetrahedron cell C, the integral of the multiplication of two linear functions ℓ_i, ℓ_j which evaluate to 1 at one vertex and 0 at the other three vertices has integral

$$\int_C \ell_i \ell_j dx = \begin{cases} \frac{1}{20}V, & \text{if } i \neq j \\ \frac{1}{10}V, & \text{if } i = j \end{cases} \tag{13}$$

where V is the volume of C.

Proof. Suppose $i = j$. Then the integral is calculated as in Fact 3

$$\begin{aligned}
\int_C \ell_i^2 dx &= \int_0^H \ell_i(0,0,h)^2 A(h)\, dh \\
&= \int_0^H \left(1 - \frac{h}{H}\right)^2 A\frac{h^2}{H^2}\, dh \\
&= AH \int_0^1 (1-u)^2 u^2\, du \\
&= AH\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{30}AH = \frac{1}{10}V.
\end{aligned}$$

Suppose $i \neq j$. Orient the cell so that ℓ_i evaluate to 1 at $(0,0,0)$ and the opposite face is parallel to the x,y-axes. Then ℓ_j will evaluate to 1 at one of the top vertices. This leads to

$$\begin{aligned}
\int_C \ell_1 \ell_2\, dx &= \int_0^H \ell_1(0,0,h) \int_{T(h)} \ell_2\, dA\, dh \\
&= \int_0^H \left(1 - \frac{h}{H}\right) \frac{1}{3}\frac{h}{H} A(h)\, dh \\
&= \int_0^H \left(1 - \frac{h}{H}\right) \frac{1}{3}\frac{h}{H} A\frac{h^2}{H^2}\, dh \\
&= \frac{1}{3}AH \int_0^1 (1-u)u^3\, du \\
&= \frac{1}{3}AH\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{20}V.
\end{aligned}$$

Line 1 to 2 above follows from Fact 1 and the fact that ℓ_j at height h has vertices evaluate to 0, 0 and $\frac{h}{H}$. □

DG Equations

Now to solve the sweep equations with linear DG. The governing equation is

$$\Omega \cdot \nabla_x f + \sigma_t f = Q. \tag{14}$$

Apply test function b_j with $j = 0, 1, 2, 3$ on C

$$\int_C \Omega \cdot \nabla_x f b_j dx + \int_C \sigma_t f b_j dx = \int_C Q b_j dx. \tag{15}$$

Integrate by parts

$$-\int_C \Omega \cdot \nabla_x b_j f dx + \sum_k \int_{F_k} (\Omega \cdot \nu_k) b_j \hat{f} dA + \int_C \sigma_t f b_j dx = \int_C Q b_j dx \tag{16}$$

where F_k for $k = 0, 1, 2, 3$ is the faces of cell C and ν_k is the outward normal to F_k. The hat on the f represents the fact that f on the boundary is not well-defined (since f can be discontinuous at the boundary) so the approximation scheme must define it. This paper will assume upwind fluxes defined as

$$\hat{f} = \begin{cases} \text{f from C}, & \text{if } \Omega \cdot \nu_k > 0 \\ \text{f from cell adjacent to C}, & \text{if } \Omega \cdot \nu_k < 0. \end{cases} \tag{17}$$

Write

$$f = f_0 b_0 + f_1 b_1 + f_2 b_2 + f_3 b_3 \qquad \text{and} \qquad Q = Q_0 b_0 + Q_1 b_1 + Q_2 b_2 + Q_3 b_3.$$

Then write the flux

$$\hat{f}|_{F_k} = \hat{f}_0^{(k)} b_0 + \hat{f}_1^{(k)} b_1 + \hat{f}_2^{(k)} b_2 + \hat{f}_3^{(k)} b_3.$$

Denote A_j the area of face j, V the volume of cell C, and $\bar{A}_j = (\Omega \cdot \nu_j) A_j$.

Equation

$$\int_C f b_j dx = \sum_i \int_C f_i b_i b_j dx = \frac{1}{20} V (f_0 + f_1 + f_2 + f_3 + f_j) \tag{18}$$

Equation Similarly

$$\int_C Q b_j dx = \frac{1}{20} V (Q_0 + Q_1 + Q_2 + Q_3 + Q_j) \tag{19}$$

Equation Note that $\Omega \cdot \nabla_x$ is a directional derivative in direction Ω. Split the directional derivative in two with one direction parallel to ν_j and the other perpendicular to ν_j

$$\Omega \cdot \nabla_x = (\Omega \cdot \nu_j) \nu_j \cdot \nabla_x + \Omega^{\perp} \cdot \nabla_x. \tag{20}$$

The point of this is $\Omega^{\perp} \cdot \nabla_x b_j = 0$ and $\nu_j \cdot \nabla_x b_j = \frac{1}{L_j}$ where L_j is the height of the tetrahedron perpendicular to face j. Then using $\frac{1}{3} L_j A_j = V$

$$\int_C \Omega \cdot \nabla_x b_j f dx = \sum_i \int_C (\Omega \cdot \nu_j) \frac{1}{L_j} f_i b_i dx \tag{21}$$

$$= \sum_i \frac{1}{4} V f_i (\Omega \cdot \nu_j) \frac{1}{L_j} \tag{22}$$

$$= \sum_i \frac{1}{12} f_i (\Omega \cdot \nu_j) A_j = \frac{1}{12} \bar{A}_j (f_0 + f_1 + f_2 + f_3). \tag{23}$$

Equation

$$\sum_k \int_{F_k} (\Omega \cdot \nu_k) b_j \hat{f} dA = \sum_{i,k} \hat{f}_i^{(k)} (\Omega \cdot \nu_k) \int_{\partial F_k} b_j b_i dA \tag{24}$$

If $j = k$ or $i = k$ then the integral is zero since either b_i or b_j is zero on the triangle. In the case $j \neq k$ and $i \neq k$, one can use Fact 2. This yields

$$\sum_k \int_{F_k} (\Omega \cdot \nu_k) b_j \hat{f} dA = \sum_{i \neq k, j \neq k} \begin{cases} \frac{1}{12} \hat{f}_i^{(k)} \bar{A}_k, & \text{if } i \neq j \\ \frac{1}{6} \hat{f}_i^{(k)} \bar{A}_k, & \text{if } i = j \end{cases} \tag{25}$$

Putting it all together Putting all the equations together leads to a system of 4 equations, one per test function b_j. The equations are:

$$\begin{aligned} &\frac{1}{12}\bar{A}_1 \left(2\hat{f}_0^{(1)} + \hat{f}_2^{(1)} + \hat{f}_3^{(1)}\right) + \frac{1}{12}\bar{A}_2 \left(2\hat{f}_0^{(2)} + \hat{f}_1^{(2)} + \hat{f}_3^{(2)}\right) + \frac{1}{12}\bar{A}_3 \left(2\hat{f}_0^{(3)} + \hat{f}_1^{(3)} + \hat{f}_2^{(3)}\right) \\ &\quad + \frac{1}{12}\bar{A}_0 \left(f_0 + f_1 + f_2 + f_3\right) + \frac{\sigma_t V}{20} \left(2f_0 + f_1 + f_2 + f_3\right) + \frac{V}{20} \left(2Q_0 + Q_1 + Q_2 + Q_3\right) \end{aligned}$$

$$\begin{aligned} &\frac{1}{12}\bar{A}_0 \left(2\hat{f}_1^{(1)} + \hat{f}_2^{(1)} + \hat{f}_3^{(1)}\right) + \frac{1}{12}\bar{A}_2 \left(\hat{f}_0^{(2)} + 2\hat{f}_1^{(2)} + \hat{f}_3^{(2)}\right) + \frac{1}{12}\bar{A}_3 \left(\hat{f}_0^{(3)} + 2\hat{f}_1^{(3)} + \hat{f}_2^{(3)}\right) \\ &\quad + \frac{1}{12}\bar{A}_1 \left(f_0 + f_1 + f_2 + f_3\right) + \frac{\sigma_t V}{20} \left(f_0 + 2f_1 + f_2 + f_3\right) + \frac{V}{20} \left(Q_0 + 2Q_1 + Q_2 + Q_3\right) \end{aligned}$$

$$\begin{aligned} &\frac{1}{12}\bar{A}_0 \left(\hat{f}_1^{(1)} + 2\hat{f}_2^{(1)} + \hat{f}_3^{(1)}\right) + \frac{1}{12}\bar{A}_1 \left(\hat{f}_0^{(2)} + 2\hat{f}_2^{(2)} + \hat{f}_3^{(2)}\right) + \frac{1}{12}\bar{A}_3 \left(\hat{f}_0^{(3)} + \hat{f}_1^{(3)} + 2\hat{f}_2^{(3)}\right) \\ &\quad + \frac{1}{12}\bar{A}_2 \left(f_0 + f_1 + f_2 + f_3\right) + \frac{\sigma_t V}{20} \left(f_0 + f_1 + 2f_2 + f_3\right) + \frac{V}{20} \left(Q_0 + Q_1 + 2Q_2 + Q_3\right) \end{aligned}$$

$$\begin{aligned} &\frac{1}{12}\bar{A}_0 \left(\hat{f}_1^{(1)} + \hat{f}_2^{(1)} + 2\hat{f}_3^{(1)}\right) + \frac{1}{12}\bar{A}_1 \left(\hat{f}_0^{(2)} + \hat{f}_2^{(2)} + 2\hat{f}_3^{(2)}\right) + \frac{1}{12}\bar{A}_2 \left(\hat{f}_0^{(3)} + \hat{f}_1^{(3)} + 2\hat{f}_3^{(3)}\right) \\ &\quad + \frac{1}{12}\bar{A}_3 \left(f_0 + f_1 + f_2 + f_3\right) + \frac{\sigma_t V}{20} \left(f_0 + f_1 + f_2 + 2f_3\right) + \frac{V}{20} \left(Q_0 + Q_1 + Q_2 + 2Q_3\right) \end{aligned}$$