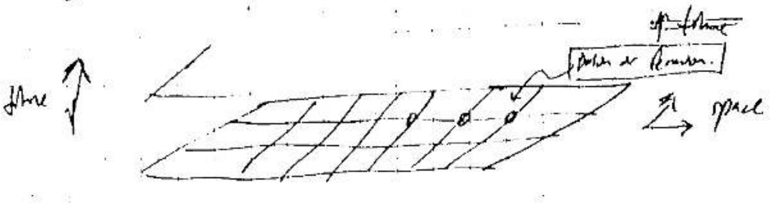
$\boxed{\text{Lecture 1}}$: Quantum Mechanics of Bosons and Fermions.

In this course, we're interested in the thermodynamics of bosonic or fermionic quantum fields. Essentially, these can be thought of as physical fields (functions) defined on all of spacetime with either a bosonic or fermionic degree of freedom living at each point:

Laine & Vuorinen 1701.01554



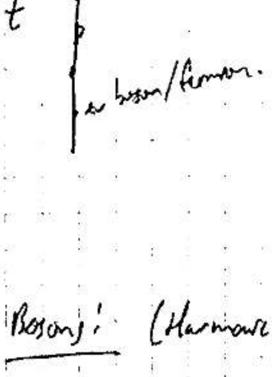
We will ~~ultimately~~ be constructing a path-integral representation of the thermodynamic partition function, which governs the thermodynamics of the system:

$$Z \equiv \mathrm{Tr}\left[e^{-\beta \hat{H}}\right], \quad \text{(tr over full set of states)}$$

This will follow similar derivations to QFT where one computes a vacuum transition rate:

$$\langle 0| e^{-i\hat{H}t} |0\rangle.$$

(~~After Noting~~ Note that $-\beta \Leftrightarrow -it$ similarity in the structure.) But first let's review some basics of thermodynamics of bosonic and fermionic systems in 0 spatial dimensions: (QM)



Thermodynamics:

$$Z(T) \equiv \mathrm{Tr}\left[e^{-\beta \hat{H}}\right] = e^{-\frac{F}{T}}, \quad \beta \equiv \frac{1}{T}$$

$$F = -T \ln Z, \quad E = \frac{1}{Z}\mathrm{Tr}\left[\hat{H} e^{-\beta \hat{H}}\right]$$

$$S = -\frac{\partial F}{\partial T} = \ln Z + \frac{1}{TZ}\mathrm{Tr}\left[\hat{H} e^{-\beta \hat{H}}\right] = -\frac{F}{T} + \frac{E}{T},$$

$$\Rightarrow (E = F + TS)$$

Bosons: (Harmonic oscillator) (creation & annihilation operators)

$$[\hat{a}, \hat{a}] = 0, \quad [\hat{a}^+, \hat{a}^+] = 0, \quad [\hat{a}, \hat{a}^+] = 1$$

$$\hat{H} = \hbar\omega\left(\hat{a}^+ a + \tfrac{1}{2}\right) = \frac{\hbar\omega}{2}\left(\hat{a}^+ a + \hat{a}\hat{a}^+\right)$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle\,,\quad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad n=0,1,2,\dots$$

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

Fermionic oscillator:

$$\{\hat{a},\hat{a}\}=0\,,\quad \{\hat{a}^+,\hat{a}^+\}=0\,,\quad \{\hat{a},\hat{a}^+\}=1$$

vacuum state: $\hat{a}|0\rangle \equiv 0.$

one-particle state: $|1\rangle \equiv \hat{a}^+|0\rangle$

No other states: $\hat{a}|1\rangle = \hat{a}\hat{a}^+|0\rangle = (1-\hat{a}^+\hat{a})|0\rangle = |0\rangle.$

$\hat{a}^+|1\rangle = \hat{a}^+\hat{a}^+|0\rangle = 0.$

The Hamiltonian is:

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^+\hat{a} - \hat{a}\hat{a}^+\right) = \hbar\omega\left(\hat{a}^+\hat{a} - \tfrac{1}{2}\right)$$

Thermodynamics:

Bosons:

$$\boxed{Z} = \sum_{n=0}^{\infty}\langle n|e^{-\beta H}|n\rangle = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(\frac{1}{2}+n)}$$

$$= \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} = \boxed{\frac{1}{2}\csc\left(\frac{\hbar\omega}{2T}\right).}$$

$$\boxed{F} = T\ln\left(e^{\frac{\hbar\omega}{2T}} - e^{-\frac{\hbar\omega}{2T}}\right) = \boxed{\frac{\hbar\omega}{2} + T\ln\left(1-e^{-\beta\hbar\omega}\right)}$$

$$\approx \begin{cases} \frac{\hbar\omega}{2} & T \ll \hbar\omega \\ -T\ln\left(\frac{T}{\hbar\omega}\right) & T \gg \hbar\omega \end{cases}$$

$$S = -\frac{\partial F}{\partial T} = \ln\left(1 \mp e^{-\beta\hbar\omega}\right) + \frac{\hbar\omega}{T}\,\frac{1}{e^{\beta\hbar\omega}-1}$$

$$\boxed{E = F + TS = \hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right)}$$

Fermions:
$$\boxed{Z} = \langle 0 | e^{-\beta H} | 0 \rangle + \langle 1 | e^{-\beta H} | 1 \rangle$$
$$= e^{+\frac{1}{2}\beta\hbar\omega} + e^{-\beta(\hbar\omega - \frac{1}{2})}$$
$$= \left[1 + e^{-\beta\hbar\omega}\right] e^{\frac{\beta\hbar\omega}{2}} = \boxed{2 \cosh\left(\frac{\hbar\omega}{2}\right)}$$
$$\boxed{F} = -T \ln\left(e^{\frac{\hbar\omega}{2T}} + e^{-\frac{\hbar\omega}{2T}}\right) = \boxed{-\frac{\hbar\omega}{2} - T\left(1 + e^{-\beta\hbar\omega}\right)}$$

Note the minus signs from the bosonic case.
$$\Rightarrow S = -\ln\left(1 + e^{-\beta\hbar\omega}\right) + \frac{\hbar\omega}{T} \frac{1}{e^{\beta\hbar\omega} + 1}$$
$$\boxed{E} = F + TS = \boxed{\hbar\omega\left(-\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} + 1}\right)}.$$

You've seen this all before, but now we want to

(i) promote these to field calculations (with dependence on spacetime coordinates)

(ii) Carry out a path-integral derivation.

Going to do this with coherent states, which is where bosonic & fermionic derivations are the most similar

Coherent states

Coherent states are ~~eigenvalues~~ ~~displaced vacuum~~ defined to be eigenvalues of the lowering operator:
$$\cancel{\hat{a}|\varphi\rangle = \varphi|\varphi\rangle} \quad \hat{a}|\varphi\rangle = \varphi|\varphi\rangle$$
and for ~~fermions~~ bosons / fermions, these are given by:
$$|\varphi\rangle \equiv e^{\pm\varphi a^{\dagger}}|0\rangle$$

Need a quick review of how to deal with fermionic (Grassmann) variables.

Let ψ_i be anticommuting variables. We define it to behave in the following ways:

Anticommutativity: $\psi_1\psi_2 = \psi_2\psi_1$ $(\Rightarrow \psi^2 = 0)$

Functions: defined by Taylor series

$$f(\psi) \equiv f(0) + \psi f'(0) \quad \text{truncates!}$$

Differentiation: $\frac{d}{d\psi}(c) = 0$, $\frac{d}{d\psi}\psi = 1$; linear

(Definite) Integration: $\int d\psi = 0$, $\int d\psi\, \psi = 1$; linear

Essentially, all operators defined algebraically.

~~Coh~~ # Coherent States (Altland & Simons) bosons/fermions $\boxed{\zeta = +1/-1}$

These states are defined as: $|\phi\rangle \equiv e^{\pm\phi a^{+}}|0\rangle$

These are eigenstates of the annihilation operator: $\boxed{\hat{a}|\phi\rangle = \phi|\phi\rangle}$ in both cases

bosons: $\hat{a}|\phi\rangle = [a, e^{\phi\hat{a}^{+}}]|0\rangle$

$$= \sum_{n=0}^{\infty} \frac{[a,(a^{+})^{n}]}{n!}\phi^{n}|0\rangle$$

$[a,(a^{+})^{n}] = n(a^{+})^{n-1}$ can easily prove this by induction:

assume: $a(a^{+})^{n} = n(a^{+})^{n-1} + (a^{+})^{n}a$ (& true for $n=1$)

$$\Rightarrow \boxed{a(a^{+})^{n+1}} = n(a^{+})^{n} + (a^{+})^{n}\underbrace{a\,a^{+}}_{1+a^{+}a}$$

$$\boxed{= (n+1)(a^{+})^{n} + (a^{+})^{n+1}a \quad \checkmark}$$

$$\Rightarrow \boxed{\hat{a}|\phi\rangle} = \sum_{n=1}^{\infty}\frac{\phi^{n}}{n!}n(a^{+})^{n-1}|0\rangle$$

$$= \phi\, e^{\phi a^{+}}|0\rangle = \boxed{\phi|\phi\rangle}$$

Fermions: $\boxed{\hat{a}|\eta\rangle} = \hat{a}\, e^{-\eta \hat{a}^\dagger}|0\rangle = \hat{a}(1-\eta a^\dagger)|0\rangle \quad 0 + \eta \hat{a}\hat{a}^\dagger|0\rangle$

$= 0 + \cancel{\quad}\, \eta|0\rangle$ } raise & lower (anticommute $\hat{a}$ with η).

$= \eta(1-\eta\hat{a}^\dagger)|0\rangle$ } $\eta^2=0$, so can add this.

$= \boxed{\eta|\eta\rangle}$

Need the following relations:

Completeness: $Id = \int d(\varphi^\dagger,\varphi)\, e^{-\varphi^\dagger\varphi}|\varphi\rangle\langle\varphi|$

overlap: $\langle\varphi'|\varphi\rangle = \exp(\varphi'^\dagger\varphi)$

Measure: $d(\varphi^\dagger,\varphi) \equiv \dfrac{d\varphi^\dagger\, d\varphi}{\pi^{(1+\zeta)/2}}$

Quick proofs:

Bosons: $\langle\varphi'|\varphi\rangle = \langle 0| e^{\varphi'^\dagger a}\, e^{\varphi\hat{a}^\dagger}|0\rangle$

Baker-Campbell-Hausdorff says $\boxed{e^A e^B} = e^{A+B+\frac{1}{2}[A,B]} = \boxed{e^B e^A e^{[A,B]}}$ (number)

$e^B e^A = e^{A+B-\frac{1}{2}[A,B]} = \cancel{\quad}$

$\Rightarrow \langle\varphi'|\varphi\rangle = \langle 0| \underbrace{e^{\varphi\hat{a}^\dagger}}_{1+(\,)a^\dagger}\, \underbrace{e^{\varphi'^\dagger a}}_{1+(\,)a}\, e^{[a,a^\dagger]\varphi'^\dagger\varphi}|0\rangle$

overlap

$= \langle 0|0\rangle \exp(\varphi'^\dagger\varphi)$ ✓

Fermions: $\langle\eta'|\eta\rangle = \langle 0|(1-\hat{a}\eta')(1-\eta\hat{a}^\dagger)|0\rangle$

$= \langle 0|0\rangle + \eta'\eta\langle 1|1\rangle = 1+\eta'\eta = \exp(\eta'\eta)$ ✓

Res. of identity:

Fermions: $\int d\eta^\dagger d\eta\, e^{-\eta^\dagger\eta}\, \overbrace{(1-\eta\hat{a}^\dagger)|0\rangle\langle 0|(1-\eta^\dagger\hat{a})}^{(|0\rangle-\eta|1\rangle)(\langle 0|-\eta^\dagger\langle 1|)}$ other way!

$= \int d\eta^\dagger d\eta\,(1-\eta^\dagger\eta)\,(|0\rangle\langle 0| + (\text{only one of } \eta \text{ or } \eta^\dagger) + \eta\eta^\dagger|1\rangle\langle 1|)$

$= \cancel{\int d\eta^\dagger d\eta\, (|0\rangle\langle 0|)} = \int d\eta^\dagger d\eta\,(-\eta^\dagger\eta|0\rangle\langle 0| + \eta\eta^\dagger|1\rangle\langle 1|) = \boxed{|0\rangle\langle 0| + |1\rangle\langle 1|}$ ✓

Bosons are actually harder:

$$\int \frac{d\phi^+ d\phi}{\pi} e^{-\phi^+\phi} \sum_{\substack{n=0\\m=0}}^{\infty} \frac{\phi^n}{\sqrt{n!}} |n\rangle\langle m| \frac{\phi^{+m}}{\sqrt{m!}}$$

ϕ^+ is the complex conjugate of ϕ. So, go to ~~radial, angular~~ polar coordinates in the complex plane: $d\phi^+ d\phi = |\phi| d|\phi| d\theta$ (θ is the angle)

$$\int \frac{|\phi| d|\phi|}{\pi} \int_0^{2\pi} d\theta \, e^{-|\phi|^2} \sum_{n,m=0}^{\infty} \frac{|\phi|^{n+m}}{\sqrt{n!m!}} \underbrace{e^{i(n-m)\theta}}_{2\pi \delta_{mn}} |n\rangle\langle m|$$

$$= \sum_{n=0}^{\infty} \int \frac{2|\phi| d|\phi|}{n!} |\phi|^{2n} e^{-|\phi|^2} |n\rangle\langle n|$$

$$= \sum_{n=0}^{\infty} \underbrace{\int \frac{d|\phi|^2}{n!} e^{-|\phi|^2} |\phi|^{2n}}_{=1 \text{ (Gamma function)}} |n\rangle\langle n| = \boxed{\sum_{n=0}^{\infty} |n\rangle\langle n|} \checkmark$$

Path Integral: ~~(Going to path~~

$$Z = \mathrm{Tr}\left[e^{-\beta \hat{H}}\right] = \sum_{n\geq 0} \langle n| e^{-\beta \hat{H}} |n\rangle$$

Insert completeness:

$$= \int d(\varphi^+, \varphi) e^{-\varphi^+\varphi} \sum_{n\geq 0} \underbrace{\langle n|\varphi\rangle}_{\phi \leftrightarrow \phi^+} \langle \varphi| e^{-\beta \hat{H}} |n\rangle$$

bosonic operator

exchange (grassmann) variables:

$$= \int d(\varphi^+, \varphi) e^{-\varphi^+\varphi} \sum_{n\geq 0} \langle \overset{\pm}{\zeta} \varphi| e^{-\beta \hat{H}} |n\rangle\langle n|\varphi\rangle$$

$$\Rightarrow \boxed{Z = \int d(\varphi^+, \varphi) e^{-\varphi^+\varphi} \langle \pm\varphi| e^{-\beta \hat{H}} |\varphi\rangle}$$

* Bosons return to same state; fermions to negative the state!

Now break up $[0,\beta]$ into N small intervals $\Delta\tau \equiv \frac{\beta}{N}$

$$Z = \int d(\varphi^+,\varphi)\, e^{-\varphi^+\varphi}\langle \pm\varphi|\, e^{-\Delta\tau_N \hat{H}}\, \underbrace{\mathbb{1}}_{\text{identity again.}}\, e^{-\Delta\tau_{N-1}\hat{H}} \dots e^{-\Delta\tau_1 \hat{H}}|\varphi\rangle$$

$$= \int d(\varphi^+,\varphi)\, d(\varphi_N^+,\varphi_N)\, e^{-\varphi^+\varphi} e^{-\varphi_N^+\varphi_N} \langle \pm\varphi| e^{-\Delta\tau_N \hat{H}}|\varphi_N\rangle \langle\varphi_N| e^{-\Delta\tau_{N-1}\hat{H}} \dots e^{-\Delta\tau_1 \hat{H}}|\varphi\rangle$$

Now deal with each term, using property of coherent state:

$$e^{-\varphi_i^+\varphi_i}\langle\varphi_i| e^{-\Delta\tau \hat{H}[\hat{a}^+,\hat{a}]}|\varphi_{i-1}\rangle$$
$$\simeq e^{-\varphi_i^+\varphi_i}\langle\varphi_i|\varphi_{i-1}\rangle e^{-\Delta\tau H[\varphi_i^+,\varphi_{i-1}]}$$
$$= \exp\{-\varphi_i^+\varphi_i + \varphi_i^+\varphi_{i-1} - \Delta\tau H[\varphi_i^+,\varphi_{i-1}]\}$$
$$= \exp\left\{-\Delta\tau\left[\varphi_i^+\left(\frac{\varphi_i-\varphi_{i-1}}{\Delta t}\right) + H[\varphi_i^+,\varphi_{i-1}]\right]\right\}$$

Taking the limit of large N then gives: (and absorbing the original $\varphi^+\varphi$ variable introduced during the trace operation):

$$\Rightarrow Z = \int \mathcal{D}\varphi^+(\tau)\,\mathcal{D}\varphi(\tau)\, \exp\left\{-\int_0^\beta d\tau \left[\varphi^+(\tau)\frac{d\varphi(\tau)}{d\tau} + H[\varphi^+(\tau),\varphi(\tau)]\right]\right\}$$

$\varphi^+(\beta) = \pm\varphi(0)$ $\qquad$ * Legendre transform

$\varphi(\beta) = \pm\varphi(0)$

For usual Hamiltonians, Legendre transform gives Euclidean Lagrangian

Going to fields now:

$$\boxed{Z = \int \mathcal{D}\varphi^+\mathcal{D}\varphi\, \exp\left\{-\int_0^\beta d\tau \int d^3x\, [\mathcal{L}_E]\right\}}$$

$\varphi^+(\beta) = \pm\varphi(\beta)$

$\varphi^+(\beta) = \pm\varphi(\beta)$

Path Integral; Summary:

"Matsubara frequencies"

- Compact "time" interval $(\beta \leftrightarrow it) \to$ discrete energies.
- Bosons periodic in imaginary time (energies $\omega_n = 2\pi T n$)
- Fermions antiperiodic in imaginary time [energies $\omega_n = 2\pi T(n+\frac{1}{2})$]
- Path integral with Euclidean Lagrangian $(t \to -i\tau)$
  (Euclidean Lagrangian b/c the $\phi^+ \frac{d\phi}{d\tau}$ term doesn't have a $+i$ like in the real-time path integral.)

* Note that "size" of compact τ direction is inversely proportional to temperature T.
  - high T, small τ dimension
  - low T, large τ dimension again ($T \to 0$ gives integral again, w/ continuous energies)

$$\longleftarrow\!\!+\!\!-\!\!+\!\!-\!\!+\!\!-\!\!+\!\!-\!\!+\!\!-\!\!+\!\!-\!\!+\!\!\longrightarrow \qquad \text{— 1 hr mark}$$
$$\underbrace{\quad}_{\sim 1/T}$$

Free Bosonic Field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \overset{\frac{1}{2}m\phi^2}{V(\phi)} = \overbrace{\underbrace{-\frac{1}{2}(\partial_t\phi)^2}}^{+\frac{1}{2}(\partial_\tau\phi)^2 \text{ Euclidean.}} + \frac{1}{2}(\nabla\phi)^2 - V(\phi)^{\,\frac{1}{2}m\phi^2}$$

~~$S(\mathcal{L}_E$ Fourier ... $T \sum_{\omega_n}$~~

Fourier of field: $\phi(\tau, \bar{x}) = T\sum_{\omega_n} \frac{1}{V}\sum_{\bar{k}} \tilde{\phi}(\omega_n, \bar{k})\, e^{i\omega_n\tau - i\bar{k}\cdot\bar{x}}$
(finite box)

~~$\Rightarrow \mathcal{L}_E \ldots \frac{T}{2V}\sum_{n,\bar{k}}$~~

now use Gaussian integral: $\int \mathcal{D}\phi_i \mathcal{D}\phi_i \exp(-\frac{1}{2}\phi_i A_{ij}\phi_j) \propto [\det(A)^{-\frac{1}{2}}]$

$$\exp(-\int \mathcal{L}_E) = \prod_{\bar{k}} \left\{ \exp\left(-\frac{T}{2V}\sum_{\omega_n}(\omega_n^2 + \bar{k}^2 + m^2)\,|\phi(\omega_n, \bar{k})|^2\right) \right\}$$

Product over $\bar{k}$ space. The path integral then gives gaussian integrals, which can be computed as: (first for the $d=0$ harmonic oscillator)

(8) $$Z^{HO}_{\omega} = \int \mathcal{D}\phi\, e^{-S_E} = \cancel{\frac{T}{\omega}\prod_{n=1}^{\infty}\frac{\omega_n^2}{\omega^2+\omega_n^2}} = \# T \prod_{n=-\infty}^{\infty} \frac{1}{(\omega^2+\omega_n^2)^{1/2}} \quad (*)$$

Using $\frac{\sinh \pi x}{\pi x} = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right)$, can show this is: $(\omega^2 = k^2+m^2)$

$$Z^{HO}_{\omega} = \exp\left\{-\frac{1}{T}\left[\frac{\omega}{2} + T\ln\left(1-e^{-\beta\omega}\right)\right]\right\} = \exp\left(-\frac{F_{HO}}{T}\right).$$

~~$\Rightarrow Z^{SFT} = \prod_k Z^{HO}$, can be written in this form:~~

~~Thus, the free energy can be written in the form (for~~

Thus, the free energy (density) of the SFT can be written: (in d dimensions)

$$\boxed{\lim_{V\to\infty}\frac{F^{SFT}}{V} = \left[\frac{e^{\gamma_E}\bar{\Lambda}^2}{4\pi}\right]^{\varepsilon}\int\frac{d^dk}{(2\pi)^d}\left[\frac{E_k}{2} + T\ln\left(1-e^{-\beta E_k}\right)\right]}$$

with $E_k = \sqrt{\omega^2+m^2}$. Note that the expression $(*)$ on p. 8 also tells us

$$\boxed{\lim_{V\to\infty}\frac{F}{V}} = \lim_{V\to\infty}\left(-\frac{T}{V}\ln Z\right) = +\lim_{V\to\infty}\frac{T}{V}\frac{1}{2}\sum_{k,\omega_n}\ln\left(\omega_n^2+\bar{k}^2+m^2\right) + \text{const.}$$

$$= \boxed{T\sum_{\omega_n}\int\frac{d^dk}{(2\pi)^d}\frac{1}{2}\ln\left(\omega_n^2+\bar{k}^2+m^2\right) + \text{const.}}$$

We usually write this as $\sum\!\!\!\!\int_K \equiv T\sum_{\omega_n}\int\frac{d^dk}{(2\pi)^d}\left(\frac{e^{\gamma_E}\bar{\Lambda}^2}{4\pi}\right)^{\varepsilon}$ (to save space). ← factor to simplify dim-reg expressions

We will frequently encounter sum-integrals... (will come back to this momentarily)

Free Dirac field

Minkowski Lagrangian (density): $\mathcal{L}_M = \bar{\Psi}\left(\gamma_M^{\mu}\partial_{\mu} + m\right)\Psi$

$\bar{\Psi} \equiv \Psi^{\dagger}\gamma_M^0$, $m \equiv m\cdot\mathbb{1}_{4\times4}$, $\{\gamma_M^{\mu}, \gamma_M^{\nu}\} = 2\eta^{\mu\nu}$, $(\gamma_M^{\mu})^{\dagger} = \gamma_M^0\gamma_M^{\mu}\gamma_M^0$

$\eta = (-1,+1,+1,+1)$. $(\gamma_M^{\mu} = -i\gamma^{\mu}_{M,\text{other}})$

Carrying out the Legendre transformation, we will find:

$$L_E = L_M(\tau = it) = \bar{\Psi}\left(\gamma_E^{\mu}\partial_{\mu} + m\right)\Psi$$

$$\gamma_E^0 = -i\gamma_M^0,\ \gamma_E^i = \gamma_M^i,\ \{\gamma_E^{\mu}, \gamma_E^{\nu}\} = 2\delta^{\mu\nu}$$ ← Can raise & lower indices w/o (-1) nuisance!

Fourier space for fermions is similar, use $\{P\}$ to denote fermionic Matsubara component of $P^0 = \hbar\omega_n^f = 2\pi T(n+\frac{1}{2})$ $n \in \mathbb{Z}$.

$$\psi(x) = \underset{\{P\}}{\SumInt} e^{iP\cdot X}\tilde{\psi}(P) \; ; \; \bar{\psi} \text{ as expected } (-iP\cdot X)$$

This gives: $S_E \equiv \underset{\{P\}}{\SumInt} \bar{\tilde{\psi}}(P)\left[i\not{P}+m\right]\tilde{\psi}(P)$.

Using the Fermionic Gaußian Integral:

$$\int \mathcal{D}\bar{\psi}_i \mathcal{D}\psi_i \, e^{-\bar{\psi}_i A^{ij}\psi_j} = \det(A_{ij})$$

Hence, We arrive at:

$$Z = \mathrm{Tr}\left[e^{-S_E}\right] = \# \prod_{\{P\}} \det\left[i\not{P}+m\right] \quad \text{product sym. in } P\to -P$$

$$= \#\left(\prod_{\{P\}} \left\{\det\left[i\not{P}+m\right]\det\left[-i\not{P}+m\right]\right\}\right)^{1/2}$$

$$= \#\left(\prod_{\{P\}} \det\left[\left(\underbrace{\not{P}\not{P}}_{P^2}+m^2\right)\underbrace{\mathbb{1}_{4\times4}}_{4}\right]\right)^{1/2}$$

$$= \# \prod_{\{P\}} \left(P^2+m^2\right)^2$$

Therefore

$$\boxed{\lim_{V\to\infty}\frac{F}{V}} = \lim_{V\to\infty}\left(-\frac{T}{V}\ln(Z)\right) = -\lim_{V\to\infty}\frac{T}{V}\times 2\sum_{\{P\}}\ln\left(P^2+m^2\right) + \text{const.}$$

$$= \boxed{-4\underset{\{P\}}{\SumInt}\frac{1}{2}\ln\left(P^2+m^2\right)+\text{const.}} \quad = -4 \times (\text{bosonic result}) \; \left(\text{but for } \underset{\{P\}}{\SumInt}\right).$$

~~* Pressure is~~ * Free energy is -1 of bosonic result (as expected)

— note this came from grassman ~~integral~~ Gaussian

* $4\times$ b/c 4 dof of Dirac field!

Manipulating bosonic + fermionic sum-integrals. [Lecture 1/2]

Consider the following bosonic thermal sum, with $f(p)$ a meromorphic function in the complex plane:

$$\sigma_b \equiv T \sum_{\omega_n} f(\omega_n). \qquad (*)$$

We seek a way to evaluate this expression. Introduce an auxiliary function

$$i n_\beta(ip) \equiv \frac{i}{\exp(i\beta p)-1}, \quad n_\beta \text{ bose distribution.}$$

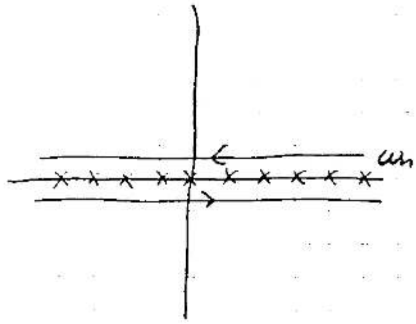
This function has poles where $\exp(i\beta p)=1$; that is at $\beta p = 2\pi n \quad n\in\mathbb{Z}$

Matsubara frequencies! Moreover, for points near one of these poles $\omega_n + z$:
($p=\omega_n$)

$$i n_\beta[i(\omega_n+z)] = \frac{i}{\exp(i\beta[\omega_n+z])-1} = \frac{i}{\exp(i\beta z)-1}$$
$$\simeq \frac{T}{z} + \mathcal{O}(1).$$

Hence, the residue at each pole is T. This means we can replace the sum in $(*)$ by the (contour) complex integral:

$$\sigma_b = \oint \frac{dp}{2\pi i} f(p) i n_\beta(ip) \equiv$$
$$= \int_{-\infty-i0^+}^{+\infty-i0^+} \frac{dp}{2\pi} f(p) n_\beta(ip)$$
$$+ \int_{+\infty+i0^+}^{-\infty+i0^+} \frac{dp}{2\pi} f(p) n_\beta(ip).$$



here sub $p\to -p$, and use

$$\boxed{n_\beta(-ip)} = \frac{1}{\exp(-i\beta p)-1} = \frac{\exp(i\beta p)-1+1}{1-\exp(i\beta p)}$$
$$= \boxed{-1-n_\beta(ip)}$$

$$\Rightarrow \boxed{\sigma_b} = \int_{-\infty-i0^+}^{+\infty-i0^+} \frac{dp}{2\pi} \left\{ f(-p) + [f(p)+f(-p)] n_\beta(ip) \right\}$$
$$\boxed{= \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p) + \int_{-\infty-i0^+}^{+\infty-i0^+} \frac{dp}{2\pi} [f(p)+f(-p)] n_\beta(ip)}$$

Bosonic thermal sum

↑ vacuum   ↑ thermal contribution

~~The then~~ Inspecting the thermal contribution, in the LHP:

$$|n_B(ip)|^{p=x-iy} = \left|\frac{1}{e^{i\beta x}e^{\beta y}-1}\right| \overset{y\gg T}{\approx} e^{-\beta y} \overset{y\gg x}{\approx} e^{-\beta|p|}$$

Hence, if $f(p)$ grows slower than exponentially (slower than $e^{\beta|p|}$) at large $|p|$ — such as polynomially — the thermal contribution can be closed in the lower-half plane, picking up the poles and residues of $f(p)+f(-p)$.

$\Rightarrow$ $\boxed{\text{Physically, thermal contribution from "on-shell" particles}}$. 2 hr mark.

Fermionic thermal sums can actually be related back to the bosonic ones above, by being clever:

$$\sigma_f \equiv T\sum_{\{\omega_n\}} f(\omega_n)$$

$$\sigma_f(T) = T\left[\cdots + f(-3\pi T) + f(-\pi T) + f(\pi T) + \cdots\right]$$

$$\cancel{= T\left[\cdots + f(-3\pi T) + f(-2\pi T) + f(\pi T) + f(2\pi T) + \cdots\right]}$$

$$\cancel{- T\left[\quad + f(2\pi T) \quad + f(2\pi T) + \cdots\right.}$$

$$= T\left[\cdots + f(-3\pi T) + f(-2\pi T) + f(-\pi T) + f(0) + f(\pi T) + f(2\pi T) + \cdots\right]$$
$$- T\left[\cdots \quad + f(2\pi T) \quad + f(0) \quad + f(2\pi T) + \cdots\right]$$

$$= 2\cdot\frac{T}{2}\left[\cdots + f\left(-6\pi\frac{T}{2}\right) + f\left(-4\pi\frac{T}{2}\right) + f\left(-2\pi\frac{T}{2}\right) + f(0) + \cdots\right]$$
$$- T\left[\quad + \; + f(2\pi T) \quad + f(0) + \cdots\right]$$

$$\Rightarrow \quad \boxed{\sigma_f(T) = 2\sigma_b\left(\frac{T}{2}\right) - \sigma_b(T)}$$

Thus, the general fermionic sum will be derived as above w/ the auxiliary function:

$$2n_B^{(\frac{T}{2})}(ip) - n_B^{(T)}(ip).$$

(12)

this can be reduced as:

$$2n_B^{(\frac{T}{2})}(ip) - n_B^{(T)}(ip) = \frac{2}{\exp(2ip\beta)-1} - \frac{1}{\exp(ip\beta)-1}$$

$$= \frac{1}{\exp(ip\beta)-1}\left[\frac{2}{\exp(ip\beta)+1} - 1\right]$$

$$= \frac{1}{\cancel{\exp(ip\beta)-1}}\left[\frac{(-1)\,\cancel{1-\exp(ip\beta)}}{\exp(ip\beta)+1}\right]$$

$$= \boxed{-\,n_F^{(T)}(ip).} \qquad n_F(p) \text{ fermi-dirac distrib.}$$

Therefore, fermionic thermal sums are ~~given by~~: evaluated by:

$$\boxed{\sigma_f(T) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} f(p) - \int_{-\infty-i0^+}^{+\infty-i0^+} \frac{dp}{2\pi}\left[f(p)+f(-p)\right] n_F(ip)}$$

Fermionic thermal sum. (is indeed $-1 \times$ Boson, w/ $n_B \to n_F$).

Example of a thermal sum: Pressure of a scalar field.

$$\frac{F}{V} = \oint_P \tfrac{1}{2}\ln(P^2+m^2) = \oint_P \tfrac{1}{2}\ln(\omega_n^2 + \underbrace{E_p}_{\equiv\sqrt{\vec{p}^2+m^2}})^2 \equiv \int_p j(E_p)$$

Let's evaluate the generalized integral

$$i(E;c) \equiv T\sum_{\omega_n} \frac{1}{(\omega_n+c)^2+E^2}, \quad c \in \mathbb{C}$$

Note that, defining $j(E,c) \equiv \frac{T}{2}\sum_{\omega_n} \ln\left[(\omega_n+c)^2+E^2\right]$

$$\cancel{\frac{1}{E}\frac{dj(E)}{dE} = \frac{1}{E}\frac{d}{dE}\,T\sum_{\omega_n}\tfrac{1}{2}\ln(\omega_n^2+E^2) = T\sum_{\omega_n}}$$

$$\frac{1}{E}\frac{dj(E,c)}{dE} = T\sum_{\omega_n}\frac{1}{(\omega_n+c)^2+E^2},$$

so if we evaluate this sum to get the answer for $j(E,0) \equiv j(E)$.

Taking: (in rac part)

$$f(p) = \frac{1}{(p+c)^2+E^2} = \frac{1}{2E}\left[\frac{1}{p+c+iE} - \frac{1}{p+c-iE}\right]$$

(Im $T>0$ part)

$$f(p)+f(-p) = \frac{i}{2E}\left[\frac{1}{p+c+iE} + \frac{1}{p-c+iE} - \frac{1}{p+c-iE} - \frac{1}{p-c-iE}\right]$$

The poles in the LHP are at $p = \pm c - iE$, with residue $\frac{i}{2E}$.

~~the~~

Vacuum term: $$\int_{-\infty}^{\infty}\frac{dp}{2\pi}f(p) = \frac{1}{2\pi}(-2\pi i)\left(\frac{i}{2E}\right) = \frac{1}{2E}$$

Thermal term: $$\int_{-\infty-i0^+}^{\infty-i0^+}\frac{dp}{2\pi}\left[f(p)+f(-p)\right] = \frac{1}{2\pi}(-2\pi i)\left[\frac{i}{2E}\right]\Big[n_B(i(c-iE)) + n_B(i(-c-iE))\Big]$$

$$\Rightarrow \boxed{i(E;c) = \frac{1}{2E}\left[1+n_B(E-ic)+n_B(E+ic)\right].}$$

Now observe that

$$n_B(x) = \frac{1}{e^{x}-1} = \frac{e^{-x}}{1-e^{-x}} = \frac{d}{dx}\left[\ln(1-e^{-x})\right]$$

Hence, we can derive

$$\boxed{j(E,c) = \text{const.} + \frac{E}{2} + \frac{T}{2}\left\{\ln\left[1-e^{-\beta(E-ic)}\right] + \ln\left[1-e^{-\beta(E+ic)}\right]\right\}}$$

Setting $c\to 0$ reproduces the bosonic pressure of a single mode!

Some notes:

* For regularization purposes, sometimes easier to derive $\sum$ of derivatives — ~~this is actually complicated b/c~~ Otherwise, Dimensional regularization is a great tool.

* Note that $ic \to -\mu$ here looks like the result with a chemical potential. In other words, chemical potential enters as $\boxed{\omega_n \mapsto \omega_n - i\mu}$ chemical potential. * The in general

~~... $H \mapsto H - \mu N$ ... $\omega_n a^\dagger a \mapsto (\omega_n - \mu) a^\dagger a$~~

Interacting field theory (perturbation theory) $\boxed{\text{Lecture 2}}$

Our time with free fields has ended. Now we want to consider a theory with a nonzero interaction term in the action:

$$S = S_0 + S_I$$

We want to expand around the free action (schematically)

$$e^{-\frac{F/V}{T}} = \int \mathcal{D}\varphi\, e^{-S_0[\varphi] - S_I[\varphi]} \simeq \int \mathcal{D}\varphi\, e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2} S_I^2 + \dots\right)$$

$$\Rightarrow \frac{F}{V} = -\frac{T}{V} \ln Z = -T \ln\left[\int \mathcal{D}\varphi\, e^{-S_0}\left(1 - S_I + \tfrac{1}{2} S_I^2\right)\right]$$

$$= \frac{F_0}{V} - \frac{T}{V} \ln\left[\langle 1 - S_I + \tfrac{1}{2} S_I^2 + \dots\rangle_0\right]$$

$$\boxed{\langle \mathcal{O}\rangle_0 \equiv \frac{\int \mathcal{D}\varphi\, \mathcal{O}\, e^{-S_0}}{\int \mathcal{D}\varphi\, e^{-S_0}}}$$

$$\simeq \frac{F_0}{V} + \Big(\langle S_I\rangle_0 - \underbrace{\tfrac{1}{2}\left[\langle S_I^2\rangle_0 - \langle S_I\rangle_0^2\right]}_{\text{connected}} + \dots\Big)\frac{T}{V}$$

$$\Rightarrow \boxed{\frac{F}{V} = \frac{F_0}{V} + \langle S_I - \tfrac{1}{2} S_I^2 + \dots\rangle_{0,c} \ \cancel{\int_x}^{\text{drop overall}}}$$

Sum over connected components only (drop overall $\int_x$ to remove $\frac{T}{V}$ prefactor).

To compute these connected components, we use Wick's theorem (derived by adding a source and differentiating)

Write $S_0[\varphi] = \int_{x,y} \frac{1}{2} \varphi_x A_{xy} \varphi_y$

~~Define~~ ~~Introduce a source~~ Define

$$Z[J] \equiv \int \mathcal{D}\varphi\, \exp\left[\int_{x,y}\left(-\tfrac{1}{2}\varphi_x A_{xy}\varphi_y + J_x \varphi_x\right)\right]$$

$$= \exp\left[\int_{xy} \tfrac{1}{2} J_x A^{-1}_{xy} J_y\right] Z[0]$$

Complete square & ~~shift~~ shift φ by constant.

Then $\boxed{\langle \varphi_1 \varphi_2 \cdots \varphi_N \rangle_{0,c}} = \frac{1}{Z[0]} \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \cdots \frac{\delta}{\delta J_N} Z[J] \Big|_{J=0}$

$$= \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \cdots \frac{\delta}{\delta J_N} \Big(1 + \int_{a,b} \frac{1}{2} J_a A^{-1}_{ab} J_b + \frac{1}{2} \int_{a,b} \frac{1}{2} J_a A^{-1}_{ab} J_b \int_{c,d} \frac{1}{2} J_c A^{-1}_{cd} J_d + \cdots \Big)$$

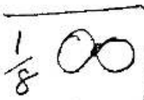
$$= \boxed{\sum_{\text{all combs.},c} A^{-1}_{1,i} A^{-1}_{i,j} \cdots A^{-1}_{k,l} A^{-1}_{l,N}}$$

<u>Example: A scalar field theory: (ϕ^4)</u>

$f \equiv \frac{F}{V}$; $f_\infty = f_0 + f_1 + f_2 + \cdots$; $\boxed{\mathcal{L} = \mathcal{L}_0 + \lambda \int_0^\beta d\tau \int_x \frac{1}{4!} \varphi^4}$

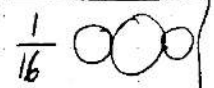
$$\boxed{f_1} = \frac{T}{V} \langle S_{\mathcal{I},0} \rangle = \frac{T}{V} \underbrace{\int_0^\beta d\tau \int_x}_{V/T} \frac{\lambda}{4!} \underbrace{\langle \phi(X)\phi(X)\phi(X)\phi(X) \rangle_0}_{\text{Transl. invariant.}}$$

$$= \frac{\lambda}{4!} \langle \varphi_0 \varphi_0 \varphi_0 \varphi_0 \rangle_0 = \frac{3}{4!} \lambda \langle \varphi_0 \varphi_0 \rangle_0 \langle \varphi_0 \varphi_0 \rangle_0$$

$= \frac{1}{8}$ 

f_2 proceeds similarly:

$$\boxed{f_2} = \left(\frac{\lambda}{4!}\right)^2 \Big[12 \int_x \langle \varphi_x \varphi_0 \rangle_0^4 + 36 \langle \varphi_0 \varphi_0 \rangle_0^2 \int_x \langle \phi_x \varphi_0 \rangle_0^2 \Big]$$

$= \frac{1}{48}$  $+ \frac{1}{16}$ 

The only contribution appearing in these expressions are $\langle \varphi_X \varphi_Y \rangle_0$ — the propagator. Let's look a little more at this function.

The (scalar) propagator:

* remember $\tau \in [0, \beta]$.

$$\langle \phi_{X}, \varphi_Y \rangle_0 \equiv = \oint_P \frac{e^{iP\cdot(X-Y)}}{P^2 + m^2}$$

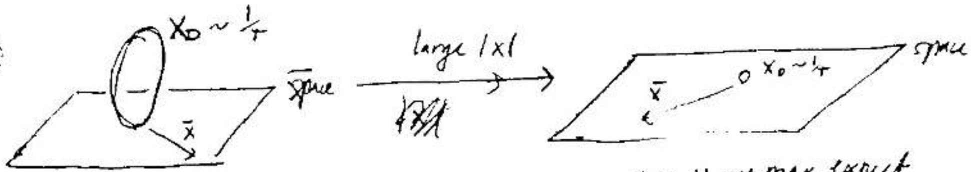
How do the limits of this compare to ~~FEO~~ vacuum theory? Consider $\langle \phi_X, \phi_0 \rangle$ WLOG:

UV limit: $|\bar{x}|, |x_0| \ll \frac{1}{T}, \frac{1}{m}$. Expect dominant regime from certain representation from $|\bar{p}||\bar{x}| \sim 1$, $|p^0||x_0| \sim 1 \Rightarrow |p|, |p^0| \gg T, m$. Hence, from the sum-integral perspective T is small $\Rightarrow \SumInt_P \to \int_P$ $\left(T\sum_{\omega_n} \to \int_{p^0}\right)$

$$\Rightarrow \langle \phi_x, \phi_y \rangle_0 \to \int_P \frac{e^{iP\cdot(X-Y)}}{P^2+m^2} \quad \text{Vacuum result}$$

$$\frac{d^4 P}{P^2} \sim (\text{mom})^2 \sim \frac{1}{|X|^2}$$

IR limit: $|x| \gg \frac{1}{T}$. In this case, the periodic ~~temporal~~ component is always small (at most $\frac{1}{T}$). * Let's take T large also, to make things clearer.



Since $\omega_n \gg 1_n$ ~~expect~~ we may expect only the zero mode to contribute to the long-distance physics. (Can show this by doing the zero-mode computation)

$$\Rightarrow \langle \phi_x \phi_0 \rangle_0 \simeq T \int_{\bar{p}} \frac{e^{i\bar{p}\cdot\bar{x}}}{p^2+m^2} \sim T \cdot (\text{mom})^1 = T|\bar{x}|^{-1}$$

can do this integral: $z \equiv \bar{p}\cdot\bar{x}/(px)$

$$= \frac{T}{(2\pi)^2} \int_{-1}^{+1} dz \int_0^\infty dp\, p^2 \frac{e^{+ipxz}}{p^2+m^2}$$

$$= \frac{T}{(2\pi)^2} \int_0^\infty \frac{dp\, p^2}{p^2+m^2} \frac{e^{+ipx} - e^{-ipx}}{ipx}$$

$$= \frac{T}{(2\pi)^2 ix} \int_{-\infty}^{\infty} \frac{dp\, p^2\, e^{ipx}}{p^2+m^2}$$

$$\boxed{\langle \phi_x \phi_0 \rangle_0 = \frac{T e^{-mx}}{4\pi x}, \quad x \gg \frac{1}{T}}$$

($d=3$ we would expect $\sim x^{-2}$)

* IR physics modifies the propagator in an essential way.

* prop $\sim |\text{mass}|^{D-2} = |\text{mass}|^{d-1}$

$\boxed{\begin{array}{l} \text{UV: all dimensions vacuum: } \sim |X|^{-2} \\ \text{IR: one dim } (x_0) \sim \frac{1}{T} \Rightarrow \sim T \cdot |x|^{-1} \end{array}}$

— Ilm.

~~IR Sensitivity & Resummation~~ # UV & IR divergences

This qualitative modification of the propagator in the IR will lead to some very interesting physics. Let us first, however, look at a few types of divergences appearing in the calculations. We have:

$$\boxed{f = \cancel{\ } J(m, T) + \frac{1}{8}\lambda\left[I(m,T)\right]^2 + \mathcal{O}(\lambda^2)}$$

(2) $J(m,T) \equiv \int_{\bar{k}} \left[\frac{E_k}{2} + T\ln\left(1 - e^{-\beta E_k}\right)\right] \quad \left(= \sum\!\!\!\!\!\!\int_K \frac{1}{2}\ln(K^2+m^2) + \text{const.}\right)$

(1) $I(m,T) \equiv \frac{1}{m}\frac{d}{dm} J(m,T) = \cancel{\ } \int_{\bar{k}} \frac{1}{2E_k}\left[1 + 2n_B(E_k)\right]$

$\left(= \sum\!\!\!\!\!\!\int_K \frac{1}{K^2+m^2}\right)$ Can't be done for general m.

Let's work at high-T to simplify the expressions. We'll do this in a slightly funny way, because the terms are not well-behaved for $m \to 0$ (and in particular $m/T \to 0$). We will ~~first split~~: $\boxed{\text{Evaluate } I(m,T) \text{ by first doing } \int_{\bar{k}} \text{ and then } \sum_{\omega_n}}$, then use:

$$J(m,T) = \int_0^m dm'\, m' I(m',T) + J(0,T)$$

$$J(0,T) = \frac{T^4}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left[1 - \exp(-\sqrt{x^2})\right] = -\frac{\pi^2 T^4}{90} \quad \text{(Mathematica)}$$

to get $J(m,T)$. Evaluate $I(m,T)$:

1.) $\sum\!\!\!\!\!\!\int_K = \sum\!\!\!\!\!\!\int_K{}' + T\int_{\bar{k}}$

2.) $I^{(n=0)} = T\int_{\bar{k}} \frac{1}{k^2+m^2} = T\overbrace{\int_{\bar{k}} \frac{1}{(k^2+m^2)^A}}^{\equiv \Phi(m,d,A)}$ for $A = 1$

3.) Dim reg results:

a.) $d^d k = \frac{\pi^{d/2}}{\Gamma(d/2)}(k^2)^{\frac{d-2}{2}} d(k^2)$

b.) $k^2 \mapsto m^2 t$ (scale out m):

$$\Phi(m,d,A) = \frac{m^{d-2A}}{\Gamma(d/2)}\frac{\pi^{d/2}}{(2\pi)^d}\overbrace{\int_0^\infty dt\, t^{\frac{d-2}{2}}(1+t)^{-A}}^{B(\frac{d}{2}, A-\frac{d}{2})}\left(\frac{e^{\gamma_E}\Lambda^2}{4\pi}\right)^\varepsilon$$

~~This is now in the form of an Euler Beta function!~~ Changing $t \to 1/s - 1$, $dt \to -ds/s^2$ gives an Euler Beta function

$$\boxed{\Phi(m,d,A)} = \frac{m^{d-2A}}{(4\pi)^{d/2}\Gamma(d/2)} \underbrace{\int_0^1 ds\, s^{A-\frac{d}{2}-1}(1-s)^{\frac{d}{2}-1}}_{B\left(A-\frac{d}{2},\,\frac{d}{2}\right)} \left(\frac{e^{\gamma_E}\bar{\Lambda}^2}{4\pi}\right)^{\varepsilon}$$

$$= \boxed{\boxed{\frac{(m^2)^{\frac{d}{2}-A}}{(4\pi)^{d/2}} \frac{\Gamma\left(A-\frac{d}{2}\right)}{\Gamma(A)}} \left(\frac{e^{\gamma_E}\bar{\Lambda}^2}{4\pi}\right)^{\varepsilon}}$$

~~Using this result leads us to ($d = 3-2\varepsilon$, ex~~

(4) $d = 3-2\varepsilon$ and expanding in ε.

Result: $\boxed{I^{(n=0)}} = T\,\Phi(m, 3-2\varepsilon, 1) = \boxed{-\frac{Tm}{4\pi} + \mathcal{O}(\varepsilon)}$

$$\left(\Rightarrow J^{(n=0)} = -\frac{Tm^3}{12\pi}\right)$$

odd powers of m from $n=0$ Matsubara modes

Remaining term:

$$I'(m,T) = T \sum_{\omega_n}{}' \int_k \frac{1}{\omega_n^2 + k^2 + m^2}$$

$k^2 \to (2\pi T)^2 k^2$

$$= T(2\pi T)^{d-2} \sum_{n=-\infty}^{\infty}{}' \int_k \frac{1}{[n^2 + k^2 + [m/(2\pi T)]^2]}$$

$$I'(m,T) = 2T^{d-1}(2\pi)^{d-2}\underbrace{\left[\sum_{n=1}^{\infty} n^{d-2}\right]}_{\zeta(2-d)}\frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)}\frac{1}{(4\pi)^{d/2}}$$

$$= \frac{T^2}{12} + \mathcal{O}(m^2) + \mathcal{O}(\varepsilon).$$

$$\boxed{I'(m,T)} = 2T(2\pi T)^{d-2}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\frac{[m/(2\pi T)]^{2l}}{[n^2+k^2]^{l+1}}(-1)^l$$

$$\underbrace{\sum_{l=0}^{\infty}\int_k\left[\quad\right]}_{=\Phi(n,d,1)\,\sim\,\frac{1}{n^{2l+2-d}}}$$

$$= \boxed{\frac{2T\left(\frac{e^{\gamma_E}\Lambda^2}{4\pi}\right)^{\varepsilon}}{(4\pi)^{d/2}(2\pi T)^{2-d}}\sum_{l=0}^{\infty}\left[\frac{-m^2}{(2\pi T)^2}\right]^{l}\frac{\Gamma(l+1-\frac{d}{2})}{\Gamma(l+1)}\zeta(2l+2-d)}$$

$$\Rightarrow I'(m,T) = \frac{T^2}{12} - \frac{2m^2}{(4\pi)}\left[\frac{1}{2\varepsilon} + \ln\left(\frac{\Lambda e^{\gamma_E}}{4\pi T}\right)\right] + \frac{2m^4\zeta(3)}{(4\pi)^4 T^2} + \mathcal{O}\left(\frac{m^6}{T^4}\right) + \mathcal{O}(\varepsilon)$$

+ zero mode

$$\Rightarrow \boxed{I(m,T) = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{2m^2}{(4\pi)^2}\left[\frac{1}{2\varepsilon} + \ln\left(\frac{\Lambda e^{\gamma_E}}{4\pi T}\right)\right] + \cdots}$$

Reordering the $T=0$ expression $I_0(m) \equiv \int_k \frac{1}{2E_k}$, we would find:

$$I(m,T) = \underbrace{-\frac{m^2}{16\pi^2}\left[\frac{1}{\varepsilon} + \ln\frac{\Lambda^2}{m^2} + 1\right]}_{\text{vacuum}} + \underbrace{I_T(m)}_{\text{thermal}}$$

$$I_T(m) \simeq \frac{T^2}{12} - \frac{mT}{4\pi} + \frac{2m^2}{(4\pi)^2}\left[\ln\left(\frac{m e^{\gamma_E}}{4\pi T}\right) - \frac{1}{2}\right] + \mathcal{O}\left(\frac{m^4}{T^2}\right) + \mathcal{O}(\varepsilon)$$

And we find:

(20.) $$\boxed{J(m,T) = -\frac{\pi^2 T^4}{90} + \frac{m^2T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{2(4\pi)^2}\left[\frac{1}{2\varepsilon} + \ln\left(\frac{\Lambda e^{\gamma_E}}{4\pi T}\right)\right] \cdots}$$

~~But then~~ But then we see something funny happening with the $\mathcal{O}(\lambda)$ term of f:

$$f_{(1)} = \frac{1}{8}\lambda\left[I(m,T)\right]^2 \ni (T\text{-dependence})\left(\tfrac{1}{\varepsilon}\text{ divergence}\right)$$

this doesn't look like vacuum renormalization, so we'll have to look into this. $\boxed{\text{* UV divergence}}$

Now consider part of the $f_{(2)}$ contribution:

0 mode: $T\int\frac{1}{(\vec{p}^2+m^2)^2} \sim \frac{T}{m}$

$$f_{(2)} \ni \frac{1}{16}\ \text{OOO} = \frac{1}{16}\lambda^2\underbrace{\left[\oint_P\frac{1}{[P^2+m^2]^2}\right]}\left[I(m,T)\right]^2$$

one prop.

as $m\to 0$, $\to \oint\frac{1}{[P^2]^2} \sim \int\frac{d\vec{p}^{\,3}}{\vec{p}^{\,4}}$ for the zero mode

~~IR divergent.~~

Can also see this here

$$\oint_P\frac{1}{[P^2+m^2]^2} = -\frac{d}{dm^2}I(m,T) = -\frac{1}{2m}\frac{d}{dm}\left(-\frac{mT}{4\pi}+\dots\right)$$

$$= \frac{T}{8\pi m} \to \infty \quad \text{for } m\to 0.$$

$\boxed{T \text{ comes from } n=0 \text{ Matsubara mode}}$

But plasma of weakly-interacting/scalar massless particles should have well-defined thermodynamics. What's going on here? $\boxed{\text{* IR divergence}}$

— End lecture 2

~~Side start~~ Renormalization: (UV)

Renormalization actually does work the same as at $T=0$, we need to consider coupling corrections to the bare mass parameter in $\mathcal{L}$:

$$m_B^2 = m_R^2 + \lambda_R\,\delta m_R^2$$

Then

This term will remove the funny behavior.

$$f(T) = \phi^{(0)}(m_B,T) + \lambda_B\,\phi^{(1)}(m_B,T)$$

$$= \phi^{(0)}(m_R,T) + \lambda_R\left[\phi^{(1)}(m_R,T) + \frac{\partial\phi^{(0)}}{\partial m_R^2}\delta m_R^2\right] + \dots$$

One can perform the renormalization by using the on-shell scheme, by performing the usual 1-PI resummation of the propagator

$$(1.)\quad \boxed{\langle\phi\phi\rangle_{full} = \left[\langle\phi\phi\rangle_0^{-1} + \Pi\right]^{-1} = \text{———} - \text{—O—} + \ldots = \frac{1}{p^2+m^2+\Pi}} \to \boxed{\langle\phi\phi\rangle_{full}^{-1} - \langle\phi\phi\rangle_0^{-1} = \Pi}$$

$$\langle\phi\phi\rangle_{full} = \left[\langle\phi\phi\rangle_0^{-1} + \cancel{\langle\phi\phi\rangle_0}\,\Pi\,\cancel{\langle\phi\phi\rangle_0}\right] = \langle\phi\phi\rangle_0\left[1+\langle\phi\phi\rangle_0\Pi\right]$$

$$\to \langle\phi\phi\rangle_c - \langle\phi\phi\rangle_0 = \cancel{\ldots}$$

$$\simeq \langle\phi\phi\rangle_0 - \langle\phi\phi\rangle_0\Pi\langle\phi\phi\rangle_0 \quad \boxed{\frac{\langle\phi\phi e^{-S_I}\rangle}{\langle e^{-S_I}\rangle} = \langle\phi\phi\rangle_{full}} \quad \langle\phi\phi\left(e^{-S}-1\right)\rangle_0 = -\langle\phi\phi\rangle_0\Pi\langle\phi\phi\rangle_0 \quad \text{Expanding exp}$$

$$(3) \Rightarrow \boxed{\langle\phi\phi\rangle_0\Pi\langle\phi\phi\rangle_0 = \langle\phi S_I\phi\rangle_c + \text{HOTs}}$$

Perform $T=0$ renorm.

$$\langle\phi\left(\frac{\lambda}{4!}\phi_x\phi_x\phi_x\phi_x\right)\phi\rangle_{0,c} = \frac{4\cdot 3}{4!}\lambda\langle\phi_y\phi_x\rangle_0\langle\phi_x\phi_x\rangle_0\langle\phi_x\phi_z\rangle_0 \qquad \propto m^2$$

$$\Rightarrow \boxed{\Pi} = \frac{\lambda}{2}\oint_p \frac{1}{p^2+m^2} = \boxed{\frac{\lambda}{2}I_0(m)} \qquad \underline{\text{Vac integral}}$$

Hence, $\langle\phi\phi\rangle_c = \oint_p \dfrac{1}{p^2+m_B^2+\frac{\lambda}{2}I_0(m_B)} + \mathcal{O}(\lambda^2)$. $\underline{\text{Vac integral}}$

$$\Rightarrow m_{pole}^2 = m_B^2 + \frac{\lambda}{2}I_0(m_B) \Rightarrow \boxed{m_B^2 = m_{pole}^2 - \frac{\lambda_R}{2}I_0(m_{pole})}$$

We can now plug this into the free energy: $\quad \frac{d}{dm^2}f(m) = \frac{1}{2m}\frac{d}{dm}f(m)$

$$f(T) = J(m_B,T) + \frac{1}{8}\lambda_B\left[I(m_B,T)\right]^2 \qquad = \frac{1}{2}I_0(m_p)$$

$$= J(m_p,T) + \left(m_B^2-m_p^2\right)\frac{\partial J(m_p,T)}{\partial m_B^2}$$

$$+ \frac{1}{8}\lambda_R\left[I(m_p,T)\right]^2 \qquad \boxed{\text{(Vac)(Mat) cancels!}}$$

$$= J(m_p,T) - \frac{\lambda_R}{4}I_0(m_p)\underbrace{I(m_p,T)}_{I_0+I_T} + \frac{1}{8}\lambda_R\underbrace{\left[I(m_p,T)\right]^2}_{[I_0+I_T]^2}$$

$J = J_0 + J_T$
$I = I_0 + I_T$

$$= \underbrace{\left\{J_0(m_p,T) - \frac{1}{8}\lambda_R I_0^2(m_p)\right\}}_{T=0,\ \text{div (finite)}} + \underbrace{\left\{J_T(m_p) + \frac{1}{8}\lambda_R I_T^2(m_p)\right\}}_{T>0\ \text{finite}}.$$

Indeed splits into vacuum & (finite) matter parts;

(22)

+ Usual $T=0$ renormalization works, even if interm. results show (vac div)×(finite T) contributions)

$$f_{(0)} \ni J^{(n=0)} \sim \boxed{m_B^3 T}$$

ratio: $\boxed{\frac{\lambda_B T^2}{m_B^2}}$

$$f_{(1)} \ni \frac{3}{4!} \lambda_B \, I'(0,T) \; I^{(n=0)}$$

$$\sim \lambda_B \left(\frac{T^2}{12}\right)\left(-\frac{m_B T}{4\pi}\right) \sim \boxed{\lambda_B m_B T^3}$$

$\cancel{f_{(2)} \ni \frac{1}{16} \lambda_B^2 \ldots}$

ratio $\boxed{\frac{\lambda_B T^2}{m_B^2}}$

$$f_{(2)} \ni \frac{1}{16} \lambda_B^2 \left(I'(0,T)\right)^2 \left(T \int_p \frac{1}{(p^2+m_B^2)^2}\right)$$

$$\sim \lambda_B^2 \left(T^2\right)^2 \quad \left(\frac{T}{m_B}\right) \quad * \sim \boxed{\frac{\lambda_B^2 T^5}{m_B}}$$

So the expansion parameter is $\lambda_B T^2 / m_B^2$; ~~suggests~~

* Clearly something goes wrong if $m_B \ll \lambda_B T^2$

* Suggests $\exists$ some new physics @ scale $\lambda_B T^2$.

(22.5)

~~EFT diagnosis~~ # Resummation (IR)

Now we turn to the IR divergences, which hint at new & interesting physics from ~~the~~ $T > 0$.

~~Note that~~ $\times$ these correspond to the limit $m_{phys}, m_B \to 0$, and seem to be connected to the zero Matsubara mode, which has no "mass term" (unlike $\omega_n \propto T$, $n \neq 0$).

Looking back at what we have above, we indeed see that the $n=0$ mode is the issue; and it contributes to odd powers of m:

$I'(0,T) = \frac{T^2}{12}$

$I^{(n=0)} = -\frac{mT}{4\pi}$

$$f_{(0)} \ni J^{(n=0)} = \cancel{\tfrac{m^3 T}{12\pi}} \sim m_B^3 T \qquad T\int \frac{1}{\vec{p}^2+m^2}$$

$$f_{(1)} \ni \frac{3}{4!}\lambda_B \cdot I'(0,T) \cdot I^{(n=0)} \sim +\lambda_B m_B T^3$$

$$f_{(2)} \ni \frac{9}{4!}\lambda_B^2 \left[I'(0,T)\right]^2 \cdot \frac{d}{dm^2} I^{(n=0)} \sim \frac{+\lambda_B^2 T^5}{m_B} \qquad T\int \frac{1}{(\vec{p}^2+m^2)^2}$$

ratio $\sim \frac{\lambda_B T^2}{m_B^2}$

$\boxed{\text{ratio} \sim \frac{\lambda_B T^2}{m_B^2}}$

(clearer on p. 22.5)

This ratio is not small if $m_B^2 \ll \lambda_B T^2$ — suggests that there is some new physics at the scale $\lambda_B T^2$.

We can deal with these difficulties in two ways:

1.) Diagrammatic Resummation

2.) EFT for the $n=0$ matsubara mode.

Let's do these in turn.

Resummation: — — — = $n=0$ mode, —— = $n\neq 0$ mode

had $f_{(0)}$ =

had $f_{(1)}$ =

had $f_{(2)}$ =

} require:

"ring" or "daisy" diagrams

Complications:

$$f(T) = \langle S_I - \tfrac{1}{2}S_I^2 + \cdots + \frac{(-1)^{N+1}}{N!} S_I^N \rangle_c, \text{ drop } \oint_x$$

$$\Rightarrow \frac{(-1)^{N+1}}{N!}\left(\frac{\lambda_B}{4!}\right)^N \langle \phi\phi\phi\phi \;\; \phi\phi\phi\phi \;\; \phi\phi\phi\phi \cdots \phi\phi\phi\phi \rangle_0$$

$$= \boxed{\frac{(-1)^{N+1}}{N!}\left(\frac{\lambda_B}{4!}\right)^N \underbrace{\binom{4}{2}^N}_{6} \underbrace{[2(N-1)][2(N-2)]\cdots[2]}_{2^{N-1}(N-1)!} \left[\frac{T^2}{12}\right]^N} \text{ — } I'(0,T) \text{ (non-zero modes)}$$

$$\boxed{\cdot\, T \int_{\bar{p}} \left(\frac{1}{p^2+m_B^2}\right)^N} \quad \uparrow \text{ zero mode}$$

Look at the 0-mode part for small N:

$$N=1: \int_{\bar p} \frac{1}{p^2+m_B^2} = -\frac{m_B}{4\pi} = \frac{-d}{dm_B^2}\left(+\frac{m_B^3}{6\pi}\right)$$

$$N=2: \int_{\bar p} \frac{1}{(p^2+m_B^2)^2} = -\frac{d}{dm_B^2}\left(-\frac{m_B}{4\pi}\right) = \left(-\frac{d}{dm_B^2}\right)\left(-\frac{d}{dm_B^2}\right)\left(+\frac{m_B^3}{6\pi}\right)$$

$$\Rightarrow \text{generally: } \boxed{\int_{\bar p} \frac{1}{(p^2+m_B^2)^N}} = -\frac{1}{N-1}\frac{d}{dm_B^2}\int_{\bar p}\frac{1}{(p^2+m_B^2)^{N-1}}$$

$$= \left(\frac{-1}{N-1}\right)\left(\frac{-1}{N-2}\right)\cdots\left(\frac{-1}{1}\right)\left(\frac{d}{dm_B^2}\right)^{N-1}\int_{\bar p}\frac{1}{p^2+m_B^2}$$

$$= \boxed{\frac{(-1)^N}{(N-1)!}\left(\frac{d}{dm_B^2}\right)^{N}\left(\frac{m_B^3}{6\pi}\right)}$$

Plugging it all in:

$$f_{(N)} \ni -\frac{T}{2}\frac{1}{N!}\left(\frac{\lambda_B T^2}{\cancel{2}\cdot 4!}\cancel{2}\right)^N\left(\frac{d}{dm_B^2}\right)^N\left(\frac{m_B^3}{6\pi}\right)$$

Taylor series

$$\Rightarrow \ni \sum_{N=0}^{\infty} \frac{-1}{N!}\left(\frac{\lambda_B T^2}{4!}\right)^N\left(\frac{d}{dm_B^2}\right)^N\left(\frac{-m_B^3 T}{12\pi}\right)$$

$$= \boxed{-\frac{T}{12\pi}\left(m_B^2 + \frac{\lambda_B T^2}{4!}\right)^{3/2}} \quad \text{— Thr}$$

(24) $\ast$ $m_B \to 0$ can be taken. $\ast$ $\lambda_B^{3/2}$ contribution in this case!

* Infrared divergences modify the order in the weak-coupling expansion!

$$\boxed{f(T)} = -\frac{\pi^2 T^4}{90} + \frac{\lambda_B T^4}{4! \times 48} - \frac{T}{12\pi}\left[\frac{\lambda_B T^2}{4!}\right]^{3/2} + \mathcal{O}(\lambda_B^2 T^4)$$

$$= \boxed{-\frac{\pi^2 T^4}{90\pi}\left[1 - \frac{5}{64}\frac{\lambda_R}{\pi^2} + \frac{5}{32}\left(\frac{\lambda_R}{\pi^2}\right)^{3/2} + \mathcal{O}(\lambda_R^2)\right]}$$

massless limit.

EFT method (Better for computing/understanding physics)

(1.) Determine $\boxed{\text{T-dependent}}$ $\boxed{\text{pole}}$ mass in the $m_B \to 0$ limit.
<u>Effective thermal mass m_{eff}^2</u>

(2.) Argue in $\lambda_R \ll 1$ limit, m_{eff}^2 only needed for $n=0$ mode.

(3.) Rewrite Lagrangian (for $m_B^2 = 0$) in the form:

$$\mathcal{L}_E = \underbrace{\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2} m_{eff}^2 \phi_{n=0}^2}_{\mathcal{L}_0} + \underbrace{\tfrac{1}{4!}\lambda_B \phi^4 - \tfrac{1}{2} m_{eff}^2 \phi_{n=0}^2}_{\mathcal{L}_I}$$

This leads to same answer as above: Contd

$$m_{eff}^2 = \lim_{m_B^2 \to 0}\left[m_B^2 + 3\lambda_B I(m_B, T)\right] = \frac{\lambda_R T^2}{4!} + \mathcal{O}(\lambda_R^2)$$

$\uparrow$ $f_p \to \oint_P$ version of above

* For $n \neq 0$, $m_{eff}^2 \ll \omega_n^2$, ~~doesn't modify mode significantly~~ Only need for zero modes!
(So indeed only need it for the $n=0$ mode)

$$\Rightarrow \boxed{f_{(0)}(T)} = \oint_P \ln(P^2)\,\tfrac{1}{2} + T\int_{\vec{p}} \tfrac{1}{2}\ln(\vec{p}^2 + m_{eff}^2) - \text{const.}$$

$$= J'(0,T) + J^{(n=0)}(m_{eff}, T) = \boxed{-\frac{\pi^2 T^4}{90} - \frac{m_{eff}^3 T}{12\pi}}$$

$$f_{(1)}(T) = \frac{3}{4!}\lambda_B \langle \phi_0 \phi_0\rangle_0 \langle \phi_0 \phi_0\rangle_0 - \tfrac{1}{2} m_{eff}^2 \langle \phi_0^{n=0} \phi_0^{n=0}\rangle_0$$

$$= \frac{3}{4!}\lambda_B\left[I'(0,T) + I^{(n=0)}(m_{eff}, T)\right]^2 - \tfrac{1}{2} m_{eff}^2 I^{(n=0)}(m_{eff}, T)$$

$$\Rightarrow f_{(1)}(T) = \frac{3}{4!}\lambda_B\left[\frac{T^4}{144} - \cancel{\frac{m_{eff}T^3}{24\pi}} + \frac{m_{eff}^2T^2}{16\pi^2}\right] + \frac{1}{2}m_{eff}^2\left(\frac{m_{eff}T}{4\pi}\right)$$

$$\underbrace{\frac{\lambda_R T^2}{4!}}_{3\frac{\lambda_R m_{eff} T^2}{8\pi \cdot 4!}}$$

$$\Rightarrow \boxed{f(T) = -\frac{\pi^2 T^4}{90} + \frac{3}{4!}\lambda_R\frac{T^4}{144} - \frac{m_{eff}^3 T}{12\pi}}$$

agrees w/ previous expression.

removes the unneeded term! (one of the progressively more IR worrying terms)

* This method works even at higher orders.
* IR divergences (/) get canceled by the reorganization (note that, in EFT styles need to compute m_{eff}^2 to higher orders as well.)

Lecture 3

QCD

Transition to more advanced part of the course now, in real QCD.

* Won't derive Path-integral w/ gauge fields. There are subtleties, related to fixing a gauge in the derivation (and A_0), but in the end, the path integral ultimately becomes the same as expected $\mathcal{L}_E = \mathcal{L}_M(t \to i\tau)$

$$Z_{QCD} = C\int_{per.} \mathcal{D}A_0^a \mathcal{D}A_u^a \int_{per.} \mathcal{D}\bar{c}^a \mathcal{D}c^a \int_{anti} \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{-S_{QCD}}$$

$$S_{QCD} = \int_0^\beta d\tau \int_{\vec{x}} \Big[\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} \quad \text{(glue)}$$

$$+ \frac{1}{2\xi}\partial_\mu A_\mu^a \partial_\nu A_\nu^a + \partial_\mu \bar{c}^a \partial_\mu c^a + g f^{abc}\partial_\mu \bar{c}^a A_\mu^b c^c \quad \text{(gauge)}$$

$$+ \bar{\Psi}_a (\not{D}_{ab} + m\mathbb{1}_{ab})\Psi_b\Big] \quad \text{(fermions)}$$

These lead to the propagator:

$\leftarrow Q -$ $- P \rightarrow$

$$\langle \tilde{A}_\mu^a(P)\tilde{A}_\nu^b(Q)\rangle_0 = \delta^{ab}\delta(P+Q)\Big[\overbrace{\frac{\delta_{\mu\nu} - P_\mu P_\nu/P^2}{P^2}}^{\mathbb{P}^{T}_{\mu\nu}} + \overbrace{\xi\frac{P_\mu P_\nu/P^2}{P^2}}^{\mathbb{P}^{L}_{\mu\nu}}\Big]$$

$$\int_x e^{i(P+Q)\cdot x} = \beta\delta_{p_n+q_n,0}(2\pi)^d\delta^{(d)}(\vec{p}+\vec{q})$$

Gluon contribution:

$$1/\text{prop} = P^2\, \mathbb{P}^{\perp}_{\mu\nu} + \frac{P^2}{\xi}\, \mathbb{P}^{\parallel}_{\mu\nu}$$

$$\mathbb{P}^{\perp}_{\mu\mu} = \delta_{\mu\mu} - 1 = d \qquad P^{\parallel}_{\mu\mu} = 1$$

$\delta^{aa} = N_c^2 - 1 \equiv d_A$ copies of this:

$$\Rightarrow f(T)\big|_{\text{gluons}} = d_A \left\{ d \cdot \frac{1}{2} \oint_P \left[\ln(P^2) + \text{const}\right] + \frac{1}{2} \oint_P \left[\ln\left(\frac{1}{\xi}P^2\right) - \text{const}\right] \right\}$$

gluons:

$$= d_A \left\{ (d+1)\, J(0,T) - \cancel{\frac{1}{2} \oint_P \ln(\xi)} \right\}$$

$\to 0$ in dim-reg (no scale!)

~~Ghosts have Each ghost has path in spacetime has:~~

The ghosts have a grassman integral (no ½ coef, but −1), but are

bosonic $\Rightarrow \oint_P$:

$$\text{ghosts: } f(T)\big|_{gh} = d_A \left\{ -2 \cdot \frac{1}{2} \oint_P \left[\ln(P^2) + \text{const}\right] \right\}$$

$$= -2 d_A\, J(0,T)$$

So that

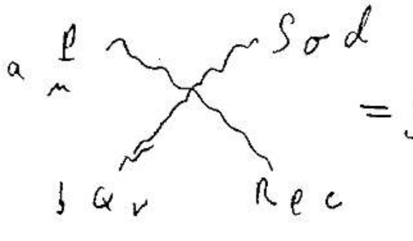
$$f\big|_{\text{gauge}} = f\big|_{gl} + f\big|_{gh} = d_A \underbrace{(d-1)}_{\#\text{ physical modes}}\, J(0,T)$$

" 2. "

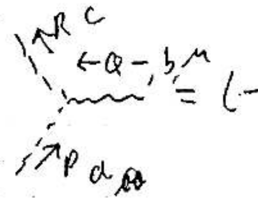
$$\langle \tilde{c}^{a}(\ell)\,\tilde{\bar{c}}^{b}(Q)\rangle_0 = \delta^{ab}\,\delta(\ell-Q)\,\frac{1}{\ell^2}\,, \quad = \quad \xrightarrow[]{Q\;\; - - \to - -\;\; \ell}$$

$$\langle \tilde{\psi}_A(\ell)\,\tilde{\bar{\psi}}_B(Q)\rangle_0 = \delta_{AB}\,\delta(\ell-Q)\,\frac{-i\not{\ell}+m}{\ell^2+m^2}\,. \quad = \quad \xrightarrow[]{Q \qquad \ell}$$

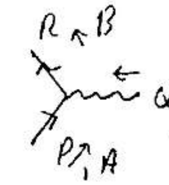
$$= ig f^{abc}\left\{\delta_{\mu\varrho}(\ell_\nu - R_\nu) + \text{cyclic}\;[(P,Q,R),\,(\mu,\nu,\varrho)]\right\}$$



$$= g^2 f^{eab} f^{ecd}\left(\delta_{\mu\varrho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\varrho}\right) + \dots \text{ cyclic } \begin{matrix}(P,Q,R,S)\\(\mu,\nu,\varrho,\sigma)\\(a,b,c,d).\end{matrix}$$



$$= (-ig f^{abc} P_\mu)\,,$$



$$= -ig\,T^a_{AB}\,\gamma^\mu$$

Noninteracting blackbody radiation:

$$\boxed{J(0,T)} = \frac{1}{2}\,\sum\!\!\!\!\!\!\int_{\ell}\left[\ln(\ell^2)^{1/\text{prop.}} - \text{const.}\right] = \boxed{-\frac{\pi^2 T^4}{90}}$$

$$\boxed{\tilde{J}(0,T)} = \frac{1}{2}\,\sum\!\!\!\!\!\!\int_{\{\ell\}}\left[\ln(\ell^2) - \text{const.}\right] = 2\cdot J\!\left(0,\tfrac{T}{2}\right) - J(0,T)$$

$$= \left(\underset{\uparrow\,\frac{1}{8}}{\frac{2}{2^4}} - 1\right)\left(\frac{\pi^2 T^4}{90}\right) = \boxed{\frac{7}{8}\,\frac{\pi^2 T^4}{90}}$$

~~The 4d (maybe) a gluon propagator can be written as:~~

$$\cancel{M_{\mu\nu} = \ell^2\delta_{\mu\nu} - \left(1-\tfrac{1}{\xi}\right)\ell_\mu\ell_\nu = \dots}$$

$$\cancel{= \ell^2\left(\delta_{\mu\nu} - \frac{\ell_\mu\ell_\nu}{\dots}\right)}$$

__Gluon:__ $$f(T) = \underbrace{(N_c^2-1)}_{\#\text{ adjoint color}}\left\{\overbrace{d \times \cancel{\ }\times J(0,T)}^{\perp} + \overbrace{1\times\cancel{\ }\times J(0,T)}^{\parallel}\right\}$$

__Ghosts:__ $$f(T) = \cancel{(N_c^2-1)\{\dots \times J(0,T)\}} - \frac{T}{V}\ln Z_{gh}$$ (clearer on p. 26.5) ← grassmann integrals

$$= -\sum\!\!\!\!\!\!\int_Q \ln\left(\Pi_a P^2\right) = -2(N_c^2-1)\,J(0,T)$$ (27)

Fermions: $f(T) = -4 \cdot \underbrace{N_c N_f}_{\#\text{ ferms}} \tilde{J}(0,T)$

$$\Rightarrow f(T) = -\frac{\pi^2 T^4}{90}\left[2(N_c^2-1) + \frac{7}{2} N_f N_c\right]$$

* Note that ghosts removed unphysical DOF of the gluons.

End lecture 3

Thermal gluon mass

Won't go through the calculation in detail. As in the scalar theory, we want: (Feynman gauge $\xi = 1$)

$$\frac{\langle A_\mu^a(K) A_\nu^b(Q) e^{-S_I}\rangle_0}{\langle e^{-S_I}\rangle_0} = \delta^{ab}\delta(K+Q)\left[\frac{\delta_{\mu\nu}}{K^2} - \frac{\Pi_{\mu\nu}(K)}{K^4} + \mathcal{O}(g^4)\right]$$

Doing lots of contractions & manipulations, shown in detail in Laine & Vuorinen, one finds: (Using the fact that $K^2 \approx 0$ from the bare gluon mass $= 0$) $\Rightarrow$ Ignore K mode of internal prop.

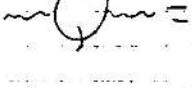
Fermions will come back to this point.

$$\langle \tilde{A}_\mu^a(K)\tilde{A}_\nu^b(Q)(-S_I)\rangle_{0,c} =$$  $$= -g^2 N_c\, d\, I_T(0) \left[\frac{\delta_{\mu\nu}}{(K^2)^2}\right]\left[\frac{\delta^{ab}\delta(K+Q)}{(K^2)^2}\right]$$

 $$\stackrel{\mathrm{II}}{=} \frac{g^2 N_c}{2}\left[\frac{\delta^{ab}\delta(K+Q)}{(K^2)^2}\right]\left\{-3\delta_{\mu 0}\delta_{\nu 0} + 7\delta_{\mu i}\delta_{\nu i}\right\}\frac{T^2}{12}$$

 $$= -\frac{g^2 N_c}{2}\left[- \right](-\delta_{\mu 0}\delta_{\nu 0} + \delta_{\mu i}\delta_{\nu i})\frac{T^2}{12}$$

$I_T(0)$ & $\tilde{I}_T(0)$ appear.

 $$= -g^2 N_f\left[\right]\left\{\delta_{\mu 0}\delta_{\nu 0} + 0 \times \delta_{\mu i}\delta_{\nu i}\right\}\frac{T^2}{6}$$

Total: $$-\frac{\delta^{ab}\delta_{\mu 0}\delta_{\nu 0}\delta(K+Q)}{(K^2)^2}\boxed{\underbrace{g^2 T^2\left(\frac{N_c}{3} + \frac{N_f}{6}\right)}_{m_E^2!}}$$

Actually gauge invariant to this order!

~~In rest frame, must be T~~

~~For zero mode $K = (0, \vec{k})$; WT says T taken computed~~

$$\Rightarrow \boxed{\langle \tilde{A}_\mu^a(K)\tilde{A}_\nu^b(Q)\rangle \stackrel{K\approx 0}{\approx} \frac{\delta^{ab}\delta_{\mu\nu}\delta(K+Q)}{K^2 + \delta_{\mu 0}\delta_{\nu 0} m_E^2}}$$

Debye mass parameter

(1) ~~Now~~ A_0 field exponentially screened in static, low mom limit.

(3.) * Actually, not taking the $k \to 0$ limit will show even more interesting structure later.

(28) (2) * "Color-magnetic" fields A_i do not get screened at this order

~~Real Thermal Loops:~~

~~Let's revisit this self energy again, but let's not neglect to the bosonic Matsubara mode — the reason is that as T gets smaller: more ...~~ QCD

I'll now show a sketch of how to compute the $\mathcal{O}(g^2)$ corrections to the $f(T)$ of pQCD at high T. The main steps are:

1.) Do the Color, Lorentz contractions to scalarize the exp. (this I won't show).

2.) Use $P\cdot K = \frac{1}{2}\left[(P-K)^2 - P^2 - K^2\right]$

To introduce inverse propagators and cancel some propagators.

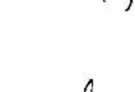
3.) Use momentum shifts to reduce to small set of master integrals, and

4.) Compute (or look up) these master integrals. As Pablo pointed out, this is always possible @ 2 loops

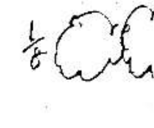
2-loop free energy of QCD is leads to fully factorizable integrals — great!

$\frac{1}{8}$  $= +\frac{g^2}{4} N_c\, d\, I_T(0) \sum\!\!\!\!\!\!\int_K \frac{\delta^{aa}\delta_{\mu\mu}}{K^2}$

$$= \frac{g^2}{4} N_c (N_c^2-1)\, d(d+1) \left[I_T(0)\right]^2$$

$\frac{1}{12}$  $= -\frac{g^2}{4} N_c\, d\, \delta^{aa} \sum\!\!\!\!\!\!\int_{K,P} \frac{K^2+(K-P)^2+P^2}{K^2P^2(K-P)^2}$

$$= -\frac{g^2}{4} N_c (N_c^2-1)\, d \times 3 \left[I_T(0)\right]^2$$

$\frac{1}{2}$  $= \frac{g^2}{4} N_c\, \delta^{aa} \sum\!\!\!\!\!\!\int_{P,K} \frac{P^2+(K-P)^2-K^2}{K^2P^2(K-P)^2}$

$$= \frac{g^2}{4} N_c (N_c^2-1) \left[I_T(0)\right]^2$$

$-\frac{1}{2}$  $= -g^2 N_f\, \delta^{aa} \frac{d-1}{2} \sum\!\!\!\!\!\!\int_{K,\{P\}} \frac{P^2+(K-P)^2-K^2}{K^2P^2(K-P)^2} = -\frac{g^2}{2} N_f\, \delta^{aa}_{F} (d-1)\left\{2 I_T(0)\, \tilde{I}_T(0) - \left[\tilde{I}_T(0)\right]^2\right\}$

Using $\boxed{d=3}$, $\boxed{I_T(0) = \frac{T^2}{12}}$, $\boxed{\tilde{I}_T(0)} = 2\, I_{\frac{T}{2}}(0) - I_T(0) = \left[\frac{2}{2^2} - 1\right]\frac{T^2}{12} = \boxed{\frac{-T^2}{24}}$

one finds:

$$f_{(1)}(T)\Big|_{QCD} = -\frac{\pi^2 T^4}{90}(N_c^2-1)\left(-\frac{5}{2}\,\frac{g^2}{4\pi^2}\right)\left(N_c + \frac{5}{4}N_f\right).$$

With the thermal mass from above, this ~~means~~ means the free energy density is:

$$\boxed{f(T)\Big|_{QCD} = -\frac{\pi^2 T^4}{45}(N_c^2-1)\left\{1 + \frac{7}{4}\,\frac{N_f N_c}{N_c^2-1} - \frac{5}{4}\left(N_c + \frac{5}{4}N_f\right)\frac{\alpha_s}{\pi} + 30\left(\frac{N_c}{3} + \frac{N_f}{6}\right)^{3/2}\left(\frac{\alpha_s}{\pi}\right)^{3/2} + \mathcal{O}(\alpha_s^2)\right\}}$$

* First relative correction is negative — Interactions decrease the pressure.
* Shows the power of (1)–(4) above. Very little work for $\mathcal{O}(\alpha_s^{3/2})$ result! (Also knowing this ~~mag~~ thermal mass idea.)
* We'll return to the general EFT description next time.

Next today, however, we'll turn to the quark sector, and the complementary limit of high μ_B, $T=0$.

EFTs of high-temperature QCD: $\boxed{\text{Lecture 4}}$

- Saw above that there's interesting IR physics in general QFTs. What happens in QCD specifically, and what is the low-energy EFT? (Looked like the $n=0$ from Matsubara mode at least ~~appr~~ becomes massive.)

$$\underline{\text{bosons}}: \quad \sum\!\!\!\!\!\!\int_{\bar{P}} f(\omega_n, \bar{p}) = \int_{\bar{p}} \left\{ \frac{1}{2} \int_{-\infty - i0^+}^{+\infty - i0^+} \frac{d\omega}{2\pi} \left[f(\omega, \bar{p}) + f(-\omega, \bar{p}) \right] \left[1 + 2n_B(i\omega) \right] \right\}$$

$$\underline{\text{fermions}}: \quad \sum\!\!\!\!\!\!\int_{\{\bar{P}\}} f(\omega_n, \bar{p}) = \int_{\bar{p}} \left\{ \frac{1}{2} \int_{-\infty - i0^+}^{+\infty - i0^+} \frac{d\omega}{2\pi} \left[f(\omega, \bar{p}) + f(-\omega, \bar{p}) \right] \left[1 - 2n_F(i\omega) \right] \right\}$$

In general, f is products of propagators and additional structures from vertices.

$$f(\omega, \bar{p}) \sim \frac{1}{\omega^2 + E_p^2} \quad \text{simplest case}$$

What dominates in these expressions? (Keeping in mind $\omega_n \sim T$ for $n \neq 0$ for bosons)

$\underline{\text{bosons}}$: LHS, zero mode: $\boxed{\dfrac{T}{E_p^2}}$

RHS, $\underline{\text{Bose enhancement}}$ near $\omega = -iE_p$:

$$\frac{1}{2}\left(\frac{-2\pi i}{2\pi}\right)\frac{2}{-2iE_p}\left[1 + 2n_B(E_p)\right]$$

$$= \frac{1}{E_p}\left(\frac{1}{2} + \frac{1}{e^{E_p/T} - 1}\right) \quad \text{high-}T$$

$$\simeq \frac{1}{E_p}\left(\frac{1}{2} + \frac{1}{E_p/T + E_p^2/(2T^2)} + \dots\right)$$

$$\simeq \boxed{\frac{T}{E_p^2}} \quad \checkmark \text{agree}.$$

$\underline{\text{fermions}}$, LHS, no zero mode: $\dfrac{T}{(\pi T)^2} \sim \boxed{\dfrac{1}{\pi^2 T}}$

RHS, similarly: $\dfrac{1}{E_p}\left(\dfrac{1}{2} - \dfrac{1}{e^{E_p/T} + 1}\right) \simeq \dfrac{1}{E_p}\left(\dfrac{1}{2} - \dfrac{1}{2 + E_p/T} + \dots\right)$

$$= \boxed{\mathcal{O}\left(\frac{1}{T}\right)} \checkmark$$

* What's the dimensionless factor associated with these loop integrals?

$$\left.\begin{array}{l} \text{– Adding loop adds factor } g^2 \text{ (vertices)} \\ \text{– Additional prop gives } T \text{ in Matsubara sum} \end{array}\right\} g^2 T$$

- Where there is bosons can give $E_p^{-\#} \to m^{-\#}$ after $\bar{p}$ integral.
- $\underline{\text{Fermions}}$ can only give $T^{-\#}$.

Hence, we expect: bosons: $\boxed{\lambda_B \sim \frac{g^2 T}{m}}$ (found this before for ring diagrams)
fermions: $\boxed{\lambda_B} \sim \frac{g^2 T}{T} \sim \boxed{g^2}$

<u>Conclusions</u>

1.) Fermions are always perturbative at high T.

2.) Bosonic zero modes have poor convergence, w/ $m \to 0$.
↳ In fact, $m^2_{eff} \sim g^2 T^2$ indicates
$$\lambda_{\substack{B,0\\A_0}} \sim \frac{g^2 T}{\sqrt{g^2 T^2}} \sim g.$$
⟹ odd powers of g in weak-coupling expansion.

3.) <u>Spatial gluonic modes</u> have $m_{eff} \lesssim g^2 T$
$$\Rightarrow \lambda_{\substack{B,\text{ spatial}\\A_i\ \text{glue}}} \sim \frac{g^2 T}{g^2 T} \sim \mathcal{O}(1) \quad \underline{\text{nonperturbative}}$$

- This agrees w/ diagrammatic analysis of Linde PLB 96, 289 (1980) ("Linde problem").

So we have scale hierarchy at high-T $\boxed{g^2 T \ll g T \ll T}$

* <u>Idea of EFT.</u> - Given full theory w/ 1PI Green Functs Γ_n
- <u>Identify</u> ~~[illegible]~~ IR DOF. Compute GFs in EFT $\bar{\Gamma}_n$
↑ use symmetries of full theory to construct operators
- For momenta & energy small $\left(p, \omega \lesssim g M \right)$ ← heavy scale; g ← small coupling

~~[illegible]~~ $\frac{\delta \bar{\Gamma}_n}{\bar{\Gamma}_n} \equiv \frac{|\bar{\Gamma}_n - \Gamma_n|}{\bar{\Gamma}_n} \leq \mathcal{O}(g^k)$ for some $k > 0$.

If masses & couplings in low-energy theory are suitably tuned. (<u>matching</u>)

<u>In hot QCD</u>: (Kajantie, Laine, Rummukainen, Schröder 0211321v2)

1.) DOF: matsubara 0-modes (of gluons A_0, $\bar{A}_i$) These are <u>T-indep.</u>
So <u>Dimensionally Reduced</u> EFT of $d = 3 - 2\varepsilon$ dimensions.

(35) 2.) Symmetries a) <u>Spatial</u> translations & rotations (medium breaks Lorentz.)
~~[illegible]~~

Symmetries b.) $\bar{A}_0 \to -\bar{A}_0$ (if no quark chem. pot.)

c.) Gauge transformation split: b/c fields are τ independent:

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1}$$

$$\hookrightarrow \boxed{\begin{aligned} \bar{A}_i' &= U\bar{A}_i U^{-1} + \frac{i}{g} U\partial_i U^{-1} \\ \bar{A}_0' &= U\bar{A}_0 U^{-1} \end{aligned}}$$

$\leftarrow$ gauge fields ($\Rightarrow$ So can't have thermal mass!)

$\leftarrow$ adjoint rep scalar.

Lagrangian (Full) for τ independent fields gives:

$$F_{0i}^a = \cancel{\partial_\tau A_i^a} - D_i^{ab} A_0^b$$

$$\to L_E = \frac{1}{4} F_{ij}^a F_{ij}^b + \frac{1}{2}(D_i^{ab} A_0)(D_i^{ac} A_0^c)$$

can rewrite as $\boxed{L_{eff}^{(0)} = \frac{1}{4}\bar{F}_{ij}^a \bar{F}_{ij}^b + \mathrm{Tr}\{[\bar{D}_i, \bar{A}_0][\bar{D}_i, \bar{A}_0]\}}$

~~Higher dim~~ Remaining terms of the form:

$$\begin{aligned} &\dim = 2: \ \mathrm{Tr}[A_0^2] \\ &\dim = 4: \ \mathrm{Tr}[A_0^4], \ (\mathrm{Tr}[A_0^2])^2 \quad \text{(actually connected for } N_c = 2,3) \\ &\dim = 6: \ \mathrm{Tr}\{[\bar{D}_i, \bar{F}_{ij}][\bar{D}_k, \bar{F}_{kj}]\} + \dots \end{aligned}$$

$\Rightarrow$ Electrostatic QCD:

$$\boxed{\begin{aligned} S_{eff} = \frac{1}{T}\int_x \Big\{ &\frac{1}{4}\bar{F}_{ij}^a \bar{F}_{ij}^a + \mathrm{Tr}([\bar{D}_i, \bar{A}_0][\bar{D}_i, \bar{A}_0]) \\ &+ \bar{m}^2 \mathrm{Tr}[\bar{A}_0^2] + \bar{\lambda}_1^{(1)}(\mathrm{Tr}[\bar{A}_0^2])^2 + \bar{\lambda}^{(2)}\mathrm{Tr}[\bar{A}_0^4] \\ &+ \dots \Big\} \quad (+ i\bar{\gamma}\, \mathrm{Tr}[\bar{A}_0^3] \text{ if } \mu_q > 0) \end{aligned}}$$

$\uparrow \int_0^\beta d\tau$

Matching gives:

$$\bar{m}^2 = g^2 T^2\left(\frac{N_c}{3} + \frac{N_f}{6}\right) + \mathcal{O}(g^4 T^2)$$

$$\bar{\lambda}^{(1)} = \frac{g^4}{4\pi^2} + \dots, \quad \bar{\lambda}^{(2)} = \frac{g^4}{12\pi^2}(N_c - N_f) + \dots$$

$$\bar{g}^2 = g^2 + \mathcal{O}(g^4), \quad \bar{\gamma} = \sum_{i=1}^{N_f} \frac{\mu_i g^3}{3\pi^2} + \mathcal{O}(g^5)$$

Starts @ $\mathcal{O}(g^3)$ in pQCD pressure

(Analyzing neglected operators suggests $\frac{\delta F}{F} \sim \left(\frac{m}{T}\right)^3 \sim g^3$). (c.f. p. 123 L&V).

(up to the order shown)

Can even go further and integrate out the massive A_0 field. (If we are interested in long-wavelength physics $|x| > 1/\bar{m}$.

$$\Rightarrow \text{Magnetostatic QCD.} \quad S''_{eff} = \frac{1}{T}\int_x \left\{ \frac{1}{4} \bar{\bar{F}}^a_{ij} \bar{\bar{F}}^a_{ij}/g + \text{HOOps} \right\}$$

* 1 scale parameter → the gauge coupling.

Starts @ $\mathcal{O}(g^6)$ of QCD pressure. Rescaling the fields to remove the $\frac{1}{T}$ gives $\boxed{\bar{\bar{g}}^2 T}$ as the coupling. $A \mapsto A T^{1/2}$ $\quad (\partial A)^2 + gAA\partial A + g^2 AAAA$

$\Rightarrow$ All dimensionful quantities $\propto [\bar{\bar{g}}^2 T]^k \times (\text{non-perturbative } \#)$

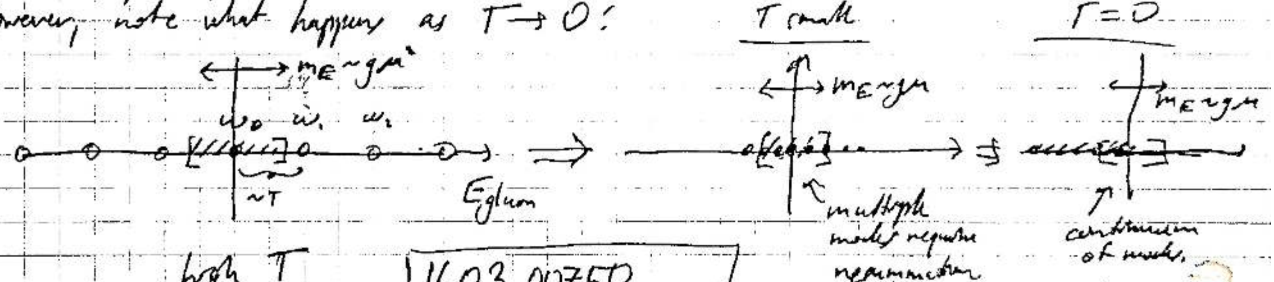
$\rightarrow [(\partial A)^2 + gT^{1/2} AA\partial A + g^2 T AAAA]\, T$

~~Note that if $\mu_q \geq 0$, then this essentially all still holds @ high T.~~ ← NEXT PAGES (MOVED HERE) ON $\mu_B \geq 0$ & low or 0 T. *

~~# High-μ_q EFTs?~~ *

@ high μ_B, high T, this essentially all still holds. (Just a few more operators in EQCD, since $A_0 \to -A_0$ no longer a symmetry.)

However, note what happens as $T \to 0$!



high T

1603.00750
Kurkela + Vuorinen.

@ high μ and small T (or T=0), Can't resum only one Matsubara mode!

$\Rightarrow$ Thermal mass is not enough; require $\Pi(K, \frac{|k^0|}{|k|}) \simeq \underbrace{\Pi_{HTL}(\frac{k^0}{k})}_{\text{Hard-thermal loop}} + k^2(\ldots)$

Require more general self-energy than just the $\Pi^{\mu\nu}|_{k^0=0}$, with nontrivial $\frac{k^0}{|k|}$ dependence.

"Hard thermal loop" self energy, since the loop integral is dominated by the ~~most~~ highest p, ω "hardest" momenta.

$$\Rightarrow \Pi^{\mu\nu}(k^0,k) = [\text{illegible}]\, g^2 N_f \delta^{ab} \oint_Q \frac{Tr(\not{Q}\gamma^\mu(\not{K}+\not{Q})\gamma^\nu)}{Q^2(P+Q)^2}$$

$\sim Q^3 \cdot \frac{1}{Q}\frac{1}{Q} \to Q$ in UV at $T=0$.

* Might show more details here.

High μ_B, $T=0$ approx. (Cold Quark matter) (** inserted on p. 37)

In this limit, $\oint_{P/\{Q\}} \longrightarrow \int_{P,\{Q\}Q}$ (4d integrals again, which is much simpler!)

Note that chemical potential enters as:

$$Z = \exp\left[-\beta(\hat{H} - \mu \hat{N})\right], \text{ but } \hat{N} = \int_x \hat{J}^0, \text{ conserved current, 0 } \cancel{\text{component, which}} \text{ component.}$$

~~Such a component normally couples to gauge field A^0's~~

~~Mink: $\partial_0 + gA_0 \xleftrightarrow{\text{Euclid.}}$ Euclid~~

~~$\downarrow$ chem. pot.~~

~~$\partial_0 + \mu \longrightarrow -i\partial_\tau + \mu = -i(\partial_\tau + i\mu)$~~

~~Add an imaginary gauge field to the Euclidean Lagrangian:~~

$$\mathcal{L}_E \ni \cancel{\ldots} \leftrightarrow (\bar{\Psi}\gamma_\mu \Psi) - (\bar{\Psi}\gamma_E^0 \Psi)\mu$$

$$\boxed{\cancel{\ldots} \to \partial_0 - \mu} \leftarrow$$

$$\boxed{i\omega_n \mapsto i\omega_n - \mu = i(\omega_n + i\mu) \text{ ; imaginary shift to frequency!}}$$

~~This means that~~ * NO more sum-integral

* ... but contour integrals still necessary b/c of $(Q^0 + i\mu)$.

E.g.: $$\tilde{I}_k \equiv \int_{\tilde{P}} \frac{1}{(\tilde{P})^{2n}} = \int_P \frac{1}{[(p^0+i\mu)^2 + \bar{p}^2]^k} = \lim_{T\to 0} \oint_P \frac{1}{[(p^0+i\mu)^2 + \bar{p}^2]^k}$$

~~Normal approach here is to do the $\bar{p}$ integral first as we did in the~~

Two possible approaches here:

1.) Proceed as we did before for the $\oint_P$ case & do the $\bar{p}$ integral first; and also keep $T>0$ first:

$$T\sum_{p^0}\int_{\bar p}\frac{1}{[\bar p^2+(p^0+i\mu)^2]^k} \xrightarrow{\text{scale out } T} = T\sum_{\omega_n}(2\pi T)^{d-2k}\int_{\bar p}\frac{1}{\left\{\left[(n+\tfrac12)+\tfrac{i\mu}{2\pi T}\right]^2+(\bar p)^2\right\}^k}$$

$$\cancel{\Phi(p^0+i\mu, d, k)}$$

$$\left(-\tfrac12+\tfrac{i\mu}{2\pi T}\right)^2 = \left(\tfrac12-\tfrac{i\mu}{2\pi T}\right)^2$$

$$= T\sum_{n=1}^{\infty}(2\pi T)^{d-2k}\left[\overbrace{\int_{\bar p}\frac{1}{[\bar p^2+(n+\frac12+\frac{i\mu}{2\pi T})^2]^k}}^{\Phi(n+\frac12+\frac{i\mu}{2\pi T},\, d,\, k)}+\overbrace{\int_{\bar p}\frac{1}{[\bar p^2+(n+\frac12-\frac{i\mu}{2\pi T})^2]^k}}^{"\,-\frac{i\mu}{2\pi T}\,"}\right]$$

$$= T\,\frac{(2\pi T)^{d-2k}}{(4\pi)^{d/2}}\,\frac{\Gamma(k-\frac d2)}{\Gamma(k)}\left(\frac{e^{\gamma_E}\Lambda^2}{4\pi}\right)^{\varepsilon}$$

$\boxed{0305183}$

Vuorinen

$$\times\left\{\underbrace{\sum_{n=1}^{\infty}\frac{1}{(n+\frac12+\frac{i\mu}{2\pi T})^{2k-d}}+\sum_{n=1}^{\infty}\frac{1}{(n+\frac12-\frac{i\mu}{2\pi T})^{2k-d}}}\right\}$$

"Hurwitz zeta function" $\rightarrow\ \zeta\left(2k-d,\ \tfrac12+\tfrac{i\mu}{2\pi T}\right)+\zeta\left(2k-d,\ \tfrac12-\tfrac{i\mu}{2\pi T}\right)$

- zero-T limit on next +2 pages

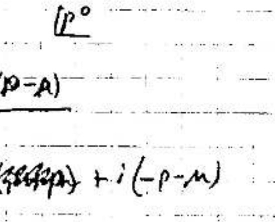
(2.) Other option is to just directly perform the integrals at $T=0$

— no $\sum_n$ necessary!

— no tricky $T\to 0$ limit at the end (note the $\frac{\mu}{2\pi T}\to\infty$).

$$\int_{\bar p}\int_{p}\frac{dp_0}{2\pi}\frac{1}{[(p^0+i\mu)^2+\bar p^2]^n} \longleftarrow \frac{1}{\{[p^0+i\mu+ip][p^0+i\mu-ip]\}^n}$$

$$\text{Gives} \propto \int_{\bar p}\Theta(p-\mu)\left(\frac{d}{dp_0}\right)^{n-1}\frac{1}{(p^0+i\mu+ip)^n}$$



* Inspecting these two results carefully, one finds that the results don't agree for $n>2$

* $\boxed{2208.14479}$ examines why this is

↳ related to non-integrable divergence @ $p_0=0$, $p=\mu$ for $n>2$

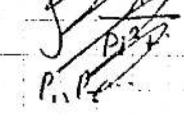
* ~~The~~ Missing ~~(derivative terms)~~ derivatives of delta functions (boundary terms), arising from $n_F\to\Theta$ ⟵ this is missing in the residue theorem!

$$\left(\frac{d}{dp^0}\right)^{n-1}\left[\frac{n_F[i\beta(p_0-i\mu)]}{(p^0+ip)^n}\right]$$

(32)

Cutting rules! At $T=0$, these residue calculations are aided by the following observation from 0912.1856, proven in 1609.04339

~~Given an N-loop integral $F(\{P_u\},\mu)$~~



Given an arbitrary 1PI N-loop n-point Feynman graph $F(\{P_u\},\mu)$, where P_u, $u=1,2\cdots n$, are the external momenta. Then

$$F(\{P_u\},\mu) = \underbrace{F_{0\text{-cut}}(\{P_u\})}_{\text{Vacuum}} + \underbrace{F_{1\text{-cut}}(\{P_u\},\mu)}_{\text{1 "cut" internal ferm. line}} + \cdots + \underbrace{F_{n\text{-cut}}(\{P_u\},\mu)}_{\text{n "cut" internal fermion lines.}}$$

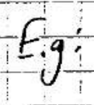
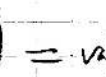
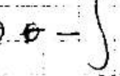
A "cut" means:

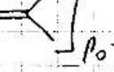
1.) Remove the cut prop. from the original graph.

2.) Evaluate the resulting $N-j$ loop, $n+2j$-point amplitude ~~at μ~~ in vacuum.

3.) Set the cut momenta Q_i on-shell; i.e. writing $q_0^i = iE_i$

4.) Integrate the resulting expression over the cut 3d momenta w/ weights $-\theta(\mu-E_i)/(2E_i)$

E.g.:  $= \text{vac}\Big[$  $\Big] - \int_p \frac{\theta(\mu-E_p)}{2E_p}\Big[$  $\Big]_{p_0\to iE_p}$

$+ \int_p \frac{\theta(\mu-E_p)}{2E_p}\int_q \frac{\theta(\mu-E_q)}{2E_q}\Big[$  $\Big]_{p_0\to iE_p,\ q_0\to iE_q}$

* Subtleties w/ higher-order propagators & intermediate divergences.

~~## Thermal Loops~~, ~~## EFTs at high T and μ~~ ~~(Remaining topics)~~

* ~~Tips~~ for calculating perturbative pressure corrections

  * ~~Scalarize~~
  * ~~Shift momenta~~
  * ~~Try your best to factorize~~

$$\lim_{T\to 0} \tilde{I}_k(\mu, T) = -\left(\frac{e^{\gamma_E}\bar{\Lambda}^2}{4\pi}\right)^{\varepsilon} \frac{i\mu}{2\pi} \frac{\Gamma(k-\frac{d}{2})}{(4\pi)^{d/2}\,\Gamma(k)\,(1+d-2k)}$$

$$\left[(i\mu)^{d-2k} - (-i\mu)^{d-2k}\right]$$

Other way ($T=0$ & regulate):

$$I_k(\mu) \xrightarrow[p_0 \text{ first}]{} -\left(\frac{e^{\gamma_E}\bar{\Lambda}^2}{4\pi}\right) \frac{\mu}{\Gamma(\frac{1}{2})} \frac{\Gamma(k-\frac{d}{2})}{(4\pi)^{d/2}\,\cancel{\Gamma(k)}\,(1+d-2k)} \mu^{d-2k}$$

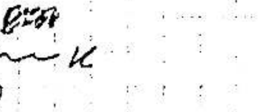
$$\Gamma(k)\Gamma(\tfrac{d}{2})$$

These only agree for $k=1$.

$$\boxed{\text{Zero-}T\text{ limit of } G_1(n,k) \text{ expression on p. (32)}}$$

(33.5)

More details of HTL derivation:



$$\Pi^{\mu\nu}(k^0,\vec{k}) = g^2 \frac{N_f}{2} \delta^{ab} \oint_{\{P\}} \frac{Tr[\not P \gamma^\mu (\not K - \not P) \gamma^\nu]}{P^2 (K-P)^2}$$

(may be off by sign)

$$= -g^2 N_f \oint_{\{P\}} \frac{\delta^{\mu\nu}[-K^2 + 2P^2 + 2K^\mu K^\nu - 4P^\mu P^\nu]}{P^2 (K-P)^2}$$

Focus on spatial components & shift $Q \to K-Q$ in one term:

$$\Pi^{ij}(K) = -g^2 N_f \oint_{\{P\}} \left[\frac{2\delta^{ij}}{P^2} + \frac{-K^2\delta^{ij} + 2k^i k^j - 4p^i p^j}{P^2 (K-P)^2} \right]$$

$T\sum_{\{\tilde p_n\}} \leftarrow \omega_n + i\mu$

Denoting $E_1 = |\vec p|$, $E_2 = |\vec p - \vec k|$, (Generalization from bose one discussed earlier)

$$T\sum_{\{\omega_n\}} \frac{1}{(\omega_n + i\mu)^2 + E_1^2} = \frac{1}{2E_1}\left[1 - n_F(E_1-\mu) - n_F(E_2+\mu)\right]$$

other, need δ functions

$$\mathcal{G} = T\sum_{\{p_n\}} \frac{1}{[\tilde p_n^2 + E_1^2][(k_n - \tilde p_n)^2 + E_2^2]}$$

$$= T\sum_{\{p_n\}} \Bigg[\underbrace{T\sum_{\{r_n\}} \beta\,\delta(\tilde r_n + k_n - \tilde p_n)}_{\int_0^\beta d\tau\, e^{i\tau(\tilde r_n + k_n - \tilde p_n)}} \Bigg] \frac{1}{[\tilde p_n^2 + E_1^2][\tilde r_n^2 + E_2^2]}$$

$$= \int_0^\beta d\tau\, e^{ik_n\tau} \underbrace{\left\{ T\sum_{\{p_n\}} \frac{e^{-i\tilde p_n\tau}}{\tilde p_n^2 + E_1^2} \right\}}_{(1)} \underbrace{\left\{ T\sum_{\{r_n\}} \frac{e^{i\tilde r_n\tau}}{\tilde r_n^2 + E_2^2} \right\}}_{(2)}$$

$$(2) = \frac{1}{2E_2}\left[n_F(E_2-\mu)\, e^{(\beta-\tau)E_2 - \beta\mu} - n_F(E_2+\mu)\, e^{\tau E_2} \right]$$

$$(1) = \frac{1}{2E_1}\left[n_F(E_1+\mu)\, e^{(\beta-\tau)E_1 + \beta\mu} - n_F(E_1-\mu)\, e^{\tau E_1} \right]$$

good exercise

Then $\mathcal{G} = \int_0^\beta d\tau \underbrace{e^{ik_n\tau}}_{\text{T}} \{ \cdots n_F(E_1 \pm \mu) n_F(E_2 \pm \mu) \underbrace{e^{\tau(\Delta E_1) + \beta(\sim)}}_{\text{T}} , \cdots \}$

$\uparrow \frac{1}{4E_1 E_2}$ $\quad \rightarrow \frac{1}{ik_n + \Delta E_1}$

$$\Rightarrow \mathcal{G} = \frac{1}{4E_1 E_2} \left\{ \frac{1}{ik_n - E_1 - E_2} \left[n_F(E_1 + \mu) + n_F(E_2 - \mu) - 1 \right] + \cdots \text{similar} \right\}$$

Now use the fact that $E_1 = p$, $E_2 = |p - k| \approx p - k_i \frac{\partial}{\partial p_i} |p|$
$= p - k_i v_i$ $\quad v \equiv \frac{\vec{p}}{p}$

Vacuum part drops out $\propto k_i^0 \bar{k}$
(vac $\delta E \propto K^2 |P^\perp(\vec{k})|$)

; manipulate integral a lot, integrate by parts on $\int dp\, n_F'$

$$\int_P dp\, n_F'(p \pm \mu) = -(d-1) \int_P \frac{1}{p} n(p \pm \mu)$$

$$\int d\Omega_v v_i v_j = \frac{\delta_{ij}}{d}$$

$$\boxed{\Pi_{ij}^{(f)}(K) = g^2 N_f (d-1) \int_P \frac{1}{p} \left[n_F(p+\mu) + n_F(p-\mu) \right] \int d\Omega_v \frac{v_i v_j\, ik_n}{ik_n - \bar{k} \cdot \bar{v}}}$$

Once you compute the other diagrams (glue)

$$\Rightarrow \boxed{\begin{aligned} \Pi_{ij}(K) &= m_E^2 \int d\Omega_v \frac{v_i v_j\, ik_n}{ik_n - k \cdot v} + \text{subleading} \\ m_E^2 &\equiv g^2 (d-1) \int \frac{1}{p} \left\{ N_f \left[n_F(p+\mu) + n_F(p-\mu) \right] + (d-1) N_c n_B(p) \right\} \\ &\overset{d=3}{=} g^2 \left[N_f \left(\frac{T^2}{6} + \frac{\mu^2}{2\pi^2} \right) + \frac{N_c T^2}{3} \right]. \end{aligned}}$$

(37-3)

The full expression turns out to be:

$$\Pi^{\mu\nu}(K) = m_E^2 \int_v \left(\delta^{\mu 0}\delta^{\nu 0} - \frac{ik^0}{K\cdot V} V^\mu V^\nu\right)\delta^{\alpha\tau}$$

$$\underset{\uparrow}{K\cdot V} \quad V = (-i, \vec{v})$$

The full result must be written as a sum of two projectors:

$$\mathbb{P}_T^{\mu\nu}(\hat{K}) \equiv \delta^{\mu i}\delta^{\nu j}\left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)$$

$$\mathbb{P}_L^{\mu\nu}(\hat{K}) = \underbrace{\mathbb{P}_L^{\mu\nu}(\hat{K})}_{D\text{-}\dim \text{ transverse}} - \mathbb{P}_T^{\mu\nu}(\hat{K})$$

These have the properties:

$$\mathbb{P}_{T/L}^{\mu\nu}(\hat{K})\, K_\mu = 0 \qquad (D\text{-dim transverse})$$

$$\mathbb{P}_T^{\mu\nu}(\hat{K})\, k_i = 0 \qquad (d\text{-dim \underline{transverse}})$$

& they are orthogonal $P_T^{\mu\alpha} P_L^{\alpha\nu} = 0$.

Then defining: $\boxed{\Pi_{ij}(K) = \mathbb{P}_{ij}^T(\hat{K})\,\Pi_T(K) + \mathbb{P}_{ij}^L(K)\,\Pi_L(K)}$

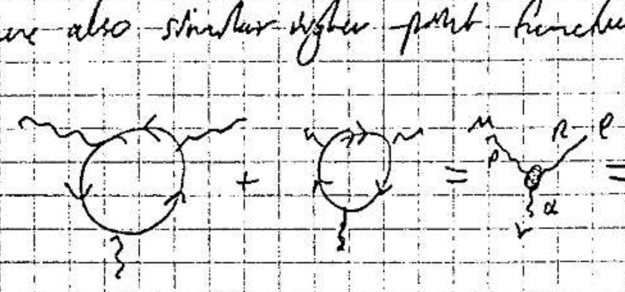
Linear algebra then gives:

$$\boxed{\begin{aligned} \Pi_T(K) &= \frac{m_E^2}{d-1}\left[-\frac{k_n^2}{k^2} + \frac{K^2}{k^2} L\right] \\ \Pi_L(K) &= \frac{m_E^2 K^2}{k^2}\left(1 - L\right) \\ L &\equiv \int_v \frac{ik_n^0}{ik_n - \vec{k}\cdot\vec{v}} \overset{d=3}{=} \frac{ik_n^0}{2k}\ln\left(\frac{ik_n + k}{ik_n - k}\right) \end{aligned}}$$

gauge invariant!

$$\Rightarrow \Pi^{\mu\nu}(K) = m_E^2 \int_v \left(\delta^{\mu 0}\delta^{\nu 0} - \frac{ik^0}{K\cdot V} V^\mu V^\nu\right) \qquad V = (-i, \vec{v}).$$

There are also similar higher point functions, e.g.:



$$= m_E^2 \int V^\mu V^\nu V^\rho \left[\frac{iQ_0}{P\cdot V\; Q\cdot V} - \frac{iR^0}{P\cdot V\; R\cdot V} \right]$$

+ glue loop gauge invariant

$\mathcal{O}(m_E^1)$ when momenta soft — same as tree vertex.

Very importantly, obeys generalized Ward Identity:

$$P^\mu \, \delta\Gamma^{\mu\nu\rho}(P,Q,R) = \Pi^{\nu\rho}(R) - \Pi^{\nu\rho}(Q)$$

HTL structure: * Can I derive this better?

$$\Pi^{\mu\nu}_{ab}(k^0, k) = m_E^2 \int_{\hat{v}} \left(\delta^{\mu 0}\delta^{\nu 0} - \frac{ik^0}{K\cdot V} V^\mu V^\nu \right) \delta_{ab}$$

$V \equiv (-i, \hat{v})$, $\hat{v} \in S^2$

normalized to 1.

@ $T=0$,

$m_E^2 = \sum_f \frac{g^2 \mu_f^2}{2\pi^2}$

Also Debye mass: $\left(\Pi^{\mu\nu}_{ab}(0,k) = m_E^2 \delta^{\mu 0}\delta^{\nu 0}\delta_{ab} \right)$.

$\Rightarrow$ Nontrivial functional dependence: $-\frac{ik^0}{2k} \ln\left(\frac{ik^0+k}{ik^0-k}\right)$.

$\Rightarrow$ Also similar HTL contributions for N-point gluon function

* At high T, also gluon contributions, which are needed for dynamics in the ~~soft~~ long-wavelength limit.

Can also write the HTL self-energy as the sum of two 4-d transverse components:

$$\Pi^{\mu\nu}_{HTL} = \overbrace{\Pi_L \mathbb{P}^{\mu\nu}_L(\hat{K})}^{1\text{ component}} + \overbrace{\Pi_T \mathbb{P}^{\mu\nu}_T(\hat{K})}^{d-1\text{ components}}$$

$$\begin{cases} \mathbb{P}^{\mu\nu}_T(\hat{k}) \equiv \delta^{\mu i}\delta^{\nu j}(\delta^{ij} - \hat{k}^i\hat{k}^j) & \text{3d transverse to } \hat{k} \\ \mathbb{P}^{\mu\nu}_L(\hat{K}) = \underset{\uparrow \text{4d transverse}}{\mathbb{P}^{\mu\nu}_{\perp}}(\hat{K}) - \mathbb{P}^{\mu\nu}_T(K) & \text{3d long. to } \hat{k} \ (\parallel \hat{k}) \end{cases}$$

Minkowski: $\boxed{\Pi_T = \frac{m_E^2}{2}\left[\frac{k_0^2}{k^2} + \left(1 - \frac{k_0^2}{k^2}\right)L(K)\right], \ \Pi_L = m_E^2\left(1-\frac{k_0^2}{k^2}\right)L(K);\ L(K) = \frac{k^0}{2k}\ln\left(\frac{k^0+k}{k^0-k}\right)}$

HTL Lagrangian does exist, but is nonlocal: (Minkowski here)

$$\delta\mathcal{L}^{HTL}_M = \cancel{\quad} + \frac{m_E^2}{2}\int_{\hat{v}} \mathrm{Tr}\left[\left(\frac{1}{V\cdot D} V^\alpha F_{\alpha\mu}\right)\left(\frac{1}{V\cdot D} V^\beta F_{\beta}{}^{\mu}\right)\right]$$

$V = (1, \hat{v})$, D covariant derivative in adjoint representation

$\exists$ local description, but additional DOF are classical fields, rather than Quantum fields, so extending relative accuracy is tricky.

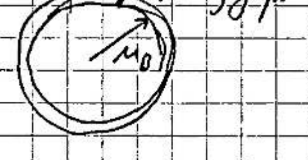
At strictly $T=0$, however, the theory is still perturbative.

$$\sim \alpha_s \int_{|P|\sim m_E} \frac{\overset{\mathcal{O}(m_E)}{\Gamma(Q,P,R)}{}^2\, d^4P}{[P^2+\Pi(P)][R^2+\Pi(R)]} \sim \mathcal{O}(\alpha_s m_E^2)$$

$\Rightarrow$ Perturbative correction to SE $\Pi \sim \mathcal{O}(m_E^2)$!

* Very importantly, @ high μ_q + $T=0$, no ~~zero~~ gluons w/ Bose enhancement, ~~so no $g^2\mu$ scale~~ which means no ultrasoft $g^2\mu$ scale with nonperturbative physics! * No zero-mode enhancement remaining g.c.
* There IS quark-pairing, leading to new boson DOF, but this is a negligible pressure contribution at high ~~T~~ μ_B

quark fermi sphere



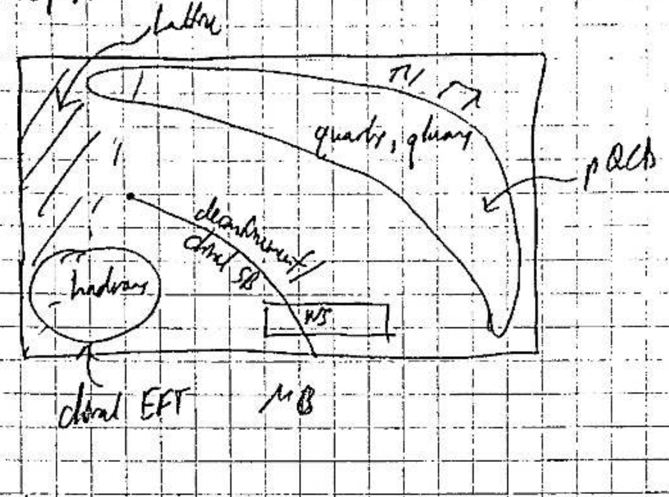
$$\text{pairing } p \sim (4\pi\mu_B^2 \cdot \Delta)\cdot\Delta \leftarrow \text{phase space}$$
$$\sim \mu_B^2 \Delta^2 \ll \mu_B^4 \text{ ~~degenerate~~ unpaired pressure.}$$

See review by Alford+ Rev Mod Phys 80 1455 (2008).

* ~~Also~~ Next time we'll focus on the pressure of Cold Quark Matter to N3LO, and discuss the EFT structure etc.

N3LO pressure of Cold Quark Matter

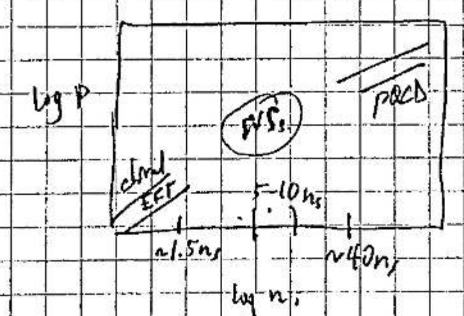
Why do we care? Pressure of dense QCD matter @ high μ_B unknown from first principles:



Q: What is the behavior of dense QCD matter?

Q: Do Neutron stars probe high enough densities to contain deconfined matter in their cores?

Approach of Kurkela + APJ 789 127 (2014); Annala, TG PRL 120, 17, (2018); Nat. Phys. 16, 9, (2020); PRX 12, 1, (2022) ...
is to combine chiral EFT, NS observations, and pQCD to constrain the NS-matter EOS + look for evidence of deconfined behavior in dense matter.



$n_s \approx 0.16\,\mathrm{fm}^{-3}$ (ave. density of ordinary nucleus).

Recent work by Komoltsev & Kurkela PRL 128, 20 (2022) and Gorda et al (APJ) 2204.11877 show that the presence of the pQCD constraint has impact on NS-EOS inference. The reason is that there are very general conditions of thermodynamic stability, causality, and consistency in the NS-matter EOS.

Observe:

Causality means

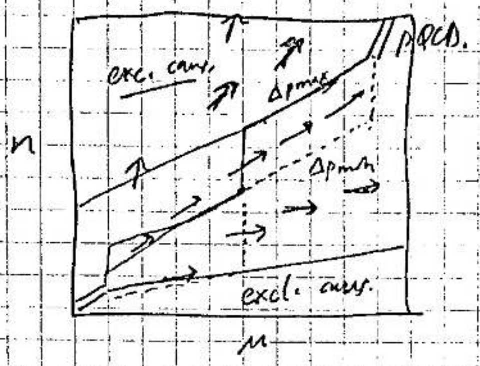
$c_s^{-2} = \frac{\mu}{n}\frac{dn}{d\mu} > 1$

$\Rightarrow n'(\mu) \geq \frac{n}{\mu}$

stability means:

$\frac{\partial^2 \Omega}{\partial \mu^2} > 0 \Rightarrow n'(\mu) > 0$

already satisfied by causality.



consistency ⇒ area under the curve is fixed

$$\int_{\mu_{CEFT}}^{\mu_{pQCD}} n(\mu)\, d\mu = \underbrace{P_{pQCD} - P_{CEFT}}_{\text{fixed} \equiv \Delta p}$$

$\boxed{\text{If } \Delta p \notin [\Delta p_{min}, \Delta p_{max}] \text{ inconsistent with CEFT + pQCD}}$

---

* Of course, fundamental pQCD calculations are also just lots of fun to do, and interesting in principle. :)

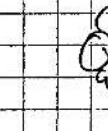
Pressure of unpaired cold Quark matter receives contributions from ~~...~~ two scales.

$$p = p_0^h + \alpha_s p_1^h \Big| + \alpha_s^2 p_2^h + \alpha_s^3 p_3^h \qquad p \sim \mu_B$$

$$\alpha_s^2 p_2^s + \alpha_s^3 p_3^s \qquad p \sim m_E \sim g\mu_B$$

$$+ \alpha_s^3 p_3^m \qquad \underline{\text{interactions between scales.}}$$

Starting at $O(\alpha_s^2)$, there are IR sensitivities, leading to the need for resummation:



$$\sim \cancel{\oint \frac{d^4P}{P^4 + \Pi_h}} \sim \int_P \frac{\Pi_{HTL}^2(k_0, u)}{(P^2)^2} \sim m_E^4 \int \frac{d^4P}{P^4} \sim \underline{\text{log divergent.}}$$

with $\Pi_{HTL} \sim m_E^2 f(k_0, u)$

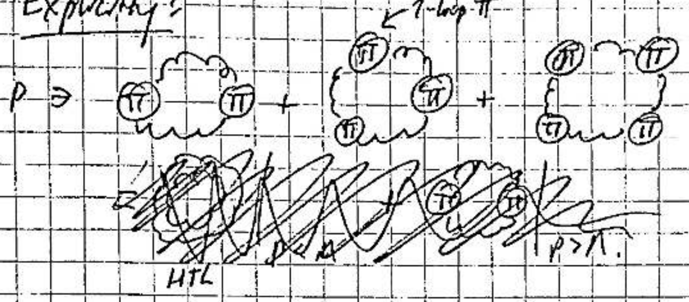
We thus expect some sort of $m_E^4 \log(\;)$ terms in the pressure, with some ratio of scales appearing in the logarithm. What scales? Expect $\frac{\mu_B}{m_E}$?

$$\cancel{\frac{d_A}{2}\int_P \log(P^2 + \Pi_E) - \frac{d_A}{2}\int_P \ln(P^2)}$$

So expect there to be a $m_E^4 \log\left(\frac{\mu_B}{m_E}\right)$ term. This is different than high-T because more than just the $n=0$ mode is contributing here.

The 0-mode at high T was giving $m_E^{3/2}$ contribution, $g^{1/2}$ enhanced. This is only $\log(g)$ enhanced. [Because of this log structure, actually easy to derive the coefficient of this logarithm. This is because it arises from between ~~the HTL~~ $\boxed{\text{UV of soft theory (HTL)} = \text{IR of hard theory.}}$

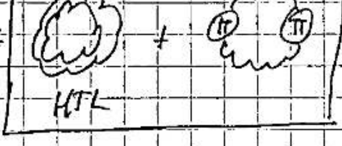
Explicitly:

$P \ni$ 

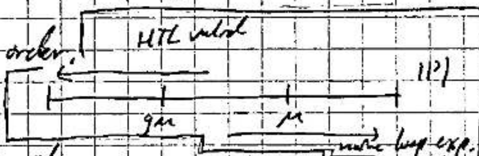
$+ \cdots = P_{res}(\Pi_1)$

$= \underbrace{\left[P_{res}(\Pi_1) - P_{res}(\Pi_{HTL})\right]}_{\text{IR safe! can loop expand!}} + P_{res}(\Pi_{HTL})$

$\boxed{P_{res}(\Pi_1)} = P_{res}(\Pi_{HTL}) + \underbrace{\left[P_{loop}(\Pi_1) - P_{loop}(\Pi_{HTL})\right]}_{= 0 \text{ in Dim Reg at } T=0}$

$=$ 

at this order.



But now consider $g\mu \ll |P| \ll \mu$. — In this range, HTL is valid enough, b/c $|P| \ll \mu$; but a loop expansion works b/c $|P| \gg g\mu$.

vacuum subtracted

$$\Rightarrow P \ni -\frac{d_A}{2}\int_{g\mu}^{\mu}\frac{d^4P}{(2\pi)^4}\left[2\ln\left(1+\frac{\Pi_T}{P^2}\right)+\ln\left(1+\frac{\Pi_L}{P^2}\right)\right]$$

$$\simeq -d_A\int_{g\mu}^{\mu}\frac{d^4P}{(2\pi)^4}\left[\frac{\Pi_T+\frac{\Pi_L}{2}}{P^2}-\frac{\Pi_T^2+\frac{\Pi_L^2}{2}}{2P^4}\right]$$

$$= -\frac{d_A}{(4\pi)^2}\left[m_\infty^2\left(\mu^2-g^2\mu^2\right)-m_\infty^4\ln\left(\frac{\mu}{g\mu}\right)\right] \ni \boxed{+\frac{d_A}{(4\pi)^2}m_\infty^4\ln\left(\frac{1}{g}\right)}$$

Here, $m_\infty^2 = \frac{m_E^2}{2}$ is the asymptotic in-medium mass: Transverse pole given by $+k_0^2 = k^2 + m_\infty^2$ approx for $k \gg g\mu$

* this agrees with the full diagrammatic calculation from Freedman & McLerran (1977) PRD 16 (multiple papers) !

(42)

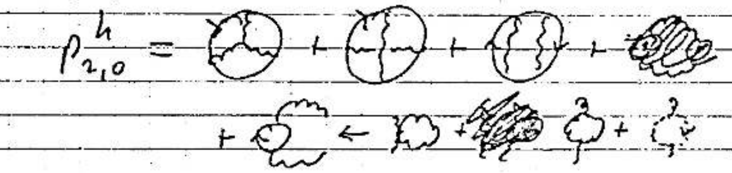
This approach was demonstrated in Gorda + PRL 121 ~~080~~ 20 (2018).

Note that we've identified soft and hard diagrams that are related to each other (the soft "expands into" the hard ones): we can group these diagrams as:

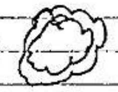
naive	1-loop HTL
$\boxed{P^h_{2,0}}$	$\boxed{P^h_{2,1}}$
	$\boxed{P^s_{2,0}}$

$\downarrow$ HTL resummation

$P_{i,j}$ — $i \leftarrow$ "hard", "soft", or "mixed". j ↗ original # of loops; ↖ resummation number of IR sensitive gluons.

$P^h_{2,0} =$ 

(not matching sum factors anymore)

$P^h_{2,1} =$  $\longrightarrow P^s_{2,0} =$ 

The "easiest" way to compute the full $O(\alpha_s^2)$ pressure of cold QM is to compute all of these terms in dim-reg and add them together.

* Since $P_{1\text{loop}}(\Pi_{HTL})$ is scale-independent in dim-reg, there is no double-counting!

Schematically, the renormalized results appear as:

$$P^s_{2,0} = \frac{d_A m_E^4}{(8\pi)^2}\left(\frac{m_E}{\Lambda_h}\right)^{-2\varepsilon}\left(\frac{P_{-1}^{N2LO}}{2\varepsilon} + P_0^{N2LO,s}\right)$$

Λ_h "factorization scale" (but still dim-reg scale)

$$P^h_{2,1} = \frac{d_A \mu^4}{(8\pi)^2}\left(\frac{\mu}{\Lambda_h}\right)^{-2\varepsilon}\left(\frac{-P_{-1}^{N2LO}}{2\varepsilon} + P_0^{N2LO,h}\right)$$

↑ divergences cancel, but leave behind $P_{-1}^{N2LO}\log\left(\frac{m_E}{\mu}\right)$.

$P^h_{2,0}$ finite after renormalization

p, p^0: polar angle, $\phi = \tan^{-1}\left(\frac{p}{p^0}\right)$.

~~At N3LO the idea is the same~~

(43) $$P^s_{2,0} \sim \int_P (d-1)\log\left[P^2+\Pi_T(\phi)\right] + \log\left(P^2+\Pi_L(\phi)\right) = \int_P \mathrm{Tr}\left[\log\left(P^2+\Pi(\phi)\right)\right]$$

The result is of the form: $\left\langle -\frac{1}{2}\mathrm{Tr}\{\Pi^2\log(\Pi)\} - \frac{1}{2}\mathrm{Tr}[\Pi^2]\log\left(\frac{m_E}{\Lambda_h}\right)\} + \dots\right\rangle_\phi$

$$\left\langle \frac{1}{\varepsilon}\left[\frac{1}{4}\mathrm{Tr}[\Pi_i^2(\phi)]\right] + \left[\cancel{\dots} \log\left(\frac{m_E}{\Lambda_h}\right) + \dots\right]\right\rangle_\phi$$

← angular average over 4d polar angle

$$\langle f\rangle_\phi \equiv \frac{\int_0^\pi d\phi \sin^2\phi\, f}{\int_0^\pi d\phi \sin^2\phi}$$

$$\to \frac{d_A m_E^4}{(8\pi)^2}\left[+\frac{1}{2\varepsilon} - \log\left(\frac{m_E}{\Lambda_h}\right) - \frac{3}{4} + \frac{\pi^2}{6} - \frac{\delta}{4} - \frac{5\log(2)}{6} + \frac{4\ln^2 2}{3}\right]$$

$\delta = -0.85638\dots$, arises from $\langle \mathrm{Tr}[\Pi^2\log(\Pi)]\rangle_\phi$ integral.

The N3LO pressure has a similar structure, but the fully IR is now the two-loop HTL diagrams:

$P^s_{3,0} =$  (not really sym factors).

And now there's a more complicated "periodic table":

hard	$P^h_{3,0}$	$P^h_{3,1}$	$P^h_{3,2}$
mixed		$P^{sm}_{3,0}$	$P^{sm}_{3,1}$
soft			$P^s_{3,0}$
	Naive	1-loop HTL	2-loop HTL

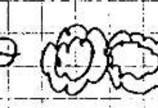
HTL resummation ↓

For example:

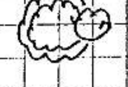
$P^3_{S,0} \ni$ 

$P^h_{3,2} \ni$ 

$P^{sm}_{3,0} \ni$  ~~2-loop HTL~~ self $\Pi^{(2)}_{HTL}$ "loop corrections"

$P^s_{3,0} \ni$  "2-loop HTL diagrams"

$P^h_{3,1} \ni$ 

$P^h_{3,0} \ni$  "$\Pi^{(1\,\mathrm{pow})}_{HTL} = K^2\tilde{\Pi}(\phi)$" power corrections

* So go beyond (LO) HTL self-energy – need "loop corrections" and power corrections

[Gorda, ~~…~~, Kurkela, Paatelainen, Säppi, Vuorinen PRL 127 16 (2021)
PRD 104 7 (2021)
computed soft 2loop HTL pressure.]

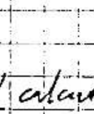
(44)

$$P^{h}_{3,0} = \frac{8 N_c d_a g^6 m_E^4}{(2\pi)^6}\left(\frac{m_E}{\Lambda_h}\right)^{-4\varepsilon}\left[\frac{p_{-2}}{(2\varepsilon)^2} + \frac{p_{-1}}{2\varepsilon} + p_0\right]$$

$$p_{-2} = \frac{11}{6}\int_\Omega \mathrm{Tr}\left[\Pi_0(\hat{u})^2\right] = \frac{11}{6}\left(\frac{\pi^2}{4}\right)$$

$$p_{-1} = \int_\Omega \left\{ \frac{19+11\pi^2}{72} - \frac{11}{6}\mathrm{Tr}\left[\Pi_0(\hat{u})^2 \ln\left[\Pi_0(\hat{u})\right]\right] \right.$$

$$\left. \int_0^{\pi/2} \frac{dx \sin(2x)\, \mathrm{Tr}\left[\delta\Gamma^2_{3g}(\hat{u}\sin x, \hat{P}\cos x)\right]}{24\left[1+\sin(2x)\,\hat{u}\cdot\hat{\ell}\right]} \right\}$$

$= 11.6840(15)$ ← from  topology, w/ bare propagators, HTL vertices.

$p_0 = 17.150(7)$

(See 2111.11948 for a readable summary of the full calculation)

The self-energy corrections $\Pi^{(2)}_{HTL,\circledcirc}$ and $\Pi^{(pow)}_{HTL}$ were first computed in QED by Manuel, Soto, Stetina PRD 94 025017 (2016); Carignano, Manuel, Soto PLB 780, 308 (2018); Carignano, Lamtyakov, Soto PLB 801, 135193 (2020) Gorda + PRD, 036012 & L031501 (2023), ~~Ekstedt~~

Gluon self energy recently computed by Ekstedt 2302.04894 @ high T and now Gorda, Paatelainen, Säppi, Seppänen 2304.09187

Ekstedt 2302.04894, 2304.09255

Interestingly, new angular structures appear, as does a new soft scale that contains a different combination of T and μ than m_E does:

$$\Pi^{(2)}_{HTL}(K) = \# \int \frac{d\Omega_v}{4\pi}\left\{ v^\mu v^\nu \left[\frac{(k^0)^2}{(v\cdot K)^2} - \frac{2k^0}{v\cdot K}\right] + \left[v^\mu n^\nu + n^\mu v^\nu\right]\frac{k^0}{v\cdot K} - n^\mu n^\nu \right\}$$

(see Ekstedt 2302.04894 for a derivation using kinetic theory).

(45) $\quad m_E^2 \sim \left(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\right), \quad \Pi^{(2)}_{HTL} \sim \left(T^2 + \frac{\mu^2}{\pi^2}\right)$ in QED (see Gorda + 23.04.09187).

* The $T=0$ pressure contribution for p^m is ongoing!