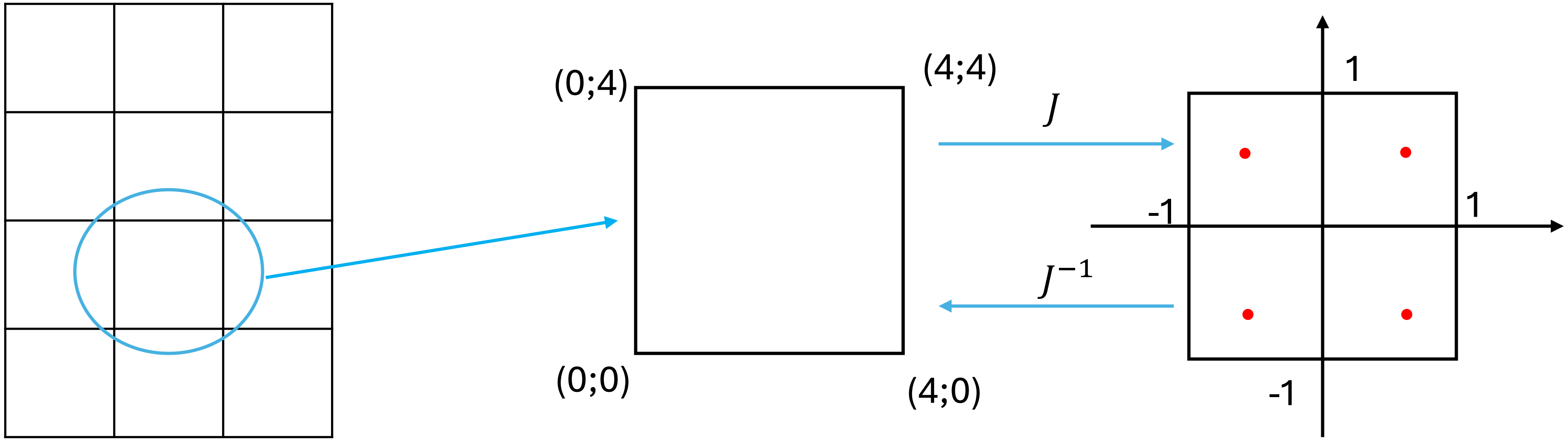
MES – Jakobian

ID	1	2	3	4
x	0	4	4	0
y	0	0	4	4



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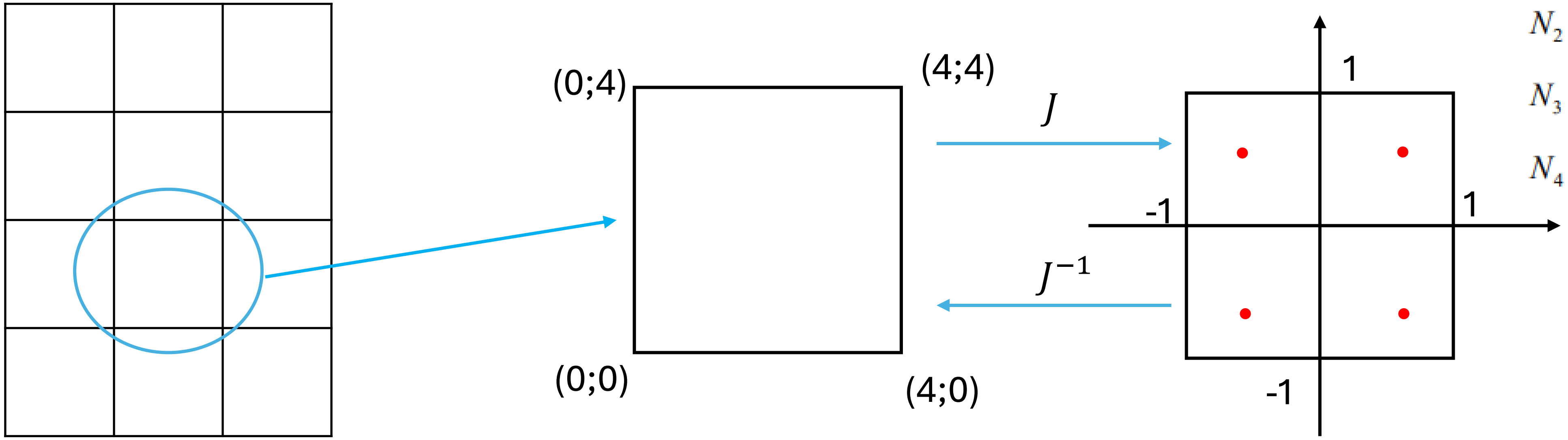
$$[H]=\int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T+\left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T+\left\{\frac{\partial\{N\}}{\partial z}\right\}\left\{\frac{\partial\{N\}}{\partial z}\right\}^T\right)dV$$

$N_1=0.25(1-\xi)(1-\eta)$

$N_2=0.25(1+\xi)(1-\eta)$

$N_3=0.25(1+\xi)(1+\eta)$

$N_4=0.25(1-\xi)(1+\eta)$



Interpolacja współrzędnej podano w następujący sposób:

$$x = \sum_{i=1}^{np} (N_i x_i) = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 = \{N\}^T \{x\}$$

$$y = \sum_{i=1}^{np} (N_i y_i) = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 = \{N\}^T \{y\}$$

Rozwiązanie krok po kroku dla 1D

$$N_1 = 0.5(1-\xi)$$

$$N_2 = 0.5(1+\xi)$$

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} \text{ czyli } \frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi}$$

$$\frac{\partial N_1}{\partial \xi} = -0.5$$

$$\frac{\partial N_2}{\partial \xi} = 0.5$$

$$x = \sum_{i=1}^{np} (N_i x_i) = N_1 x_1 + N_2 x_2$$

$$\det J\left[\frac{\partial x}{\partial \xi}\right] = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2$$

$$\det J\left[\frac{\partial x}{\partial \xi}\right] = -0.5 x_1 + 0.5 x_2 = 0.5 * (x_2 - x_1)$$

Dlatego do obliczenia pochodnych względem ksi oraz eta korzysta się n następujących zależności:

$$\frac{\partial N(x,y)}{\partial \xi} = \frac{\partial N(x,y)}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N(x,y)}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial N(x,y)}{\partial \eta} = \frac{\partial N(x,y)}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N(x,y)}{\partial y}\frac{\partial y}{\partial \eta}$$

Powyższe wzory można przedstawić w sposób macierzowy:

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}$$

Aby wyznaczyć wektor:

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}$$

Korzystamy z zasady rozwiązania układu równań:

$$[A]\{b\} = \{c\} \Rightarrow \{b\} = [A]^{-1}\{C\}$$

Jeżeli $[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to macierz odwrotna $[A]^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Stąd:

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

Jeżeli $[A]=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to macierz odwrotna $[A]^{-1}=\frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Stąd:

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det[J]}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

Powyższe sformułowanie nazywa się jakobianem przekształcenia. Jakobian obliczany jest dla każdego punktu całkowania osobno.

Obliczamy pochodne funkcji kształtu względem ksi oraz eta:

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta) \qquad \frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta) \qquad \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta) \qquad \frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta)_4$$

$$\frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi) \qquad \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi) \qquad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi) \qquad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)_4$$

Wyznaczamy jakobian przekształcenia:

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 + \frac{\partial N_4}{\partial \xi}x_4 = \frac{1}{4}(\eta-1)x_1 + \frac{1}{4}(1-\eta)x_2 + \frac{1}{4}(1+\eta)x_3 - \frac{1}{4}(1+\eta)x_4$$

$$\frac{\partial x}{\partial \xi} = \frac{\eta}{4}x_1 - \frac{1}{4}x_1 + \frac{1}{4}x_2 - \frac{\eta}{4}x_2 + \frac{1}{4}x_3 + \frac{\eta}{4}x_3 - \frac{1}{4}x_4 - \frac{\eta}{4}x_4 = \frac{\eta}{4}(x_1 - x_2 + x_3 - x_4) + \frac{1}{4}(-x_1 + x_2 + x_3 - x_4)$$

Obliczyć pochodne przy dwupunktowym schemacie całkowania w pierwszym punkcie:

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta) \qquad \frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta) \qquad \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta) \qquad \frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta)_4$$

$$\frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi) \qquad \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi) \qquad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi) \qquad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)_4$$

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta) = -0.39434 \qquad \frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi) = -0.39434$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta) = 0.39434 \qquad \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi) = -0.106$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta) = 0.106 \qquad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi) = 0.106$$

$$\frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta) = -0.106 \qquad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi) = 0.39434$$

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Obliczyć składowe jakobianu przy założeniu że

$$x = \sum_{i=1}^{np}(N_i x_i) = N_1 x_1 + N_2 x_2$$

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Przykład:

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 + \frac{\partial N_4}{\partial \xi}x_4 = \frac{1}{4}(\eta-1)x_1 + \frac{1}{4}(1-\eta)x_2 + \frac{1}{4}(1+\eta)x_3 - \frac{1}{4}(1+\eta)x_4$$

$$\frac{\partial x}{\partial \xi} = \frac{\eta}{4}x_1 - \frac{1}{4}x_1 + \frac{1}{4}x_2 - \frac{\eta}{4}x_2 + \frac{1}{4}x_3 + \frac{\eta}{4}x_3 - \frac{1}{4}x_4 - \frac{\eta}{4}x_4 = \frac{\eta}{4}(x_1 - x_2 + x_3 - x_4) + \frac{1}{4}(-x_1 + x_2 + x_3 - x_4)$$

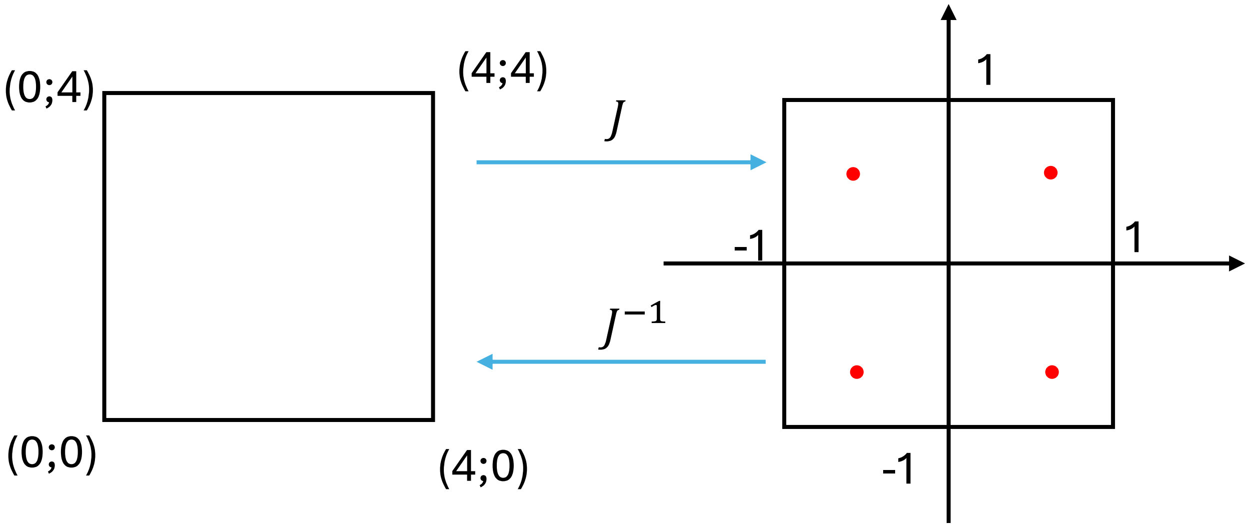
$$\frac{\partial x}{\partial \xi} = 2, \frac{\partial y}{\partial \eta} = 2, \frac{\partial x}{\partial \eta} = 0, \frac{\partial y}{\partial \xi} = 0$$

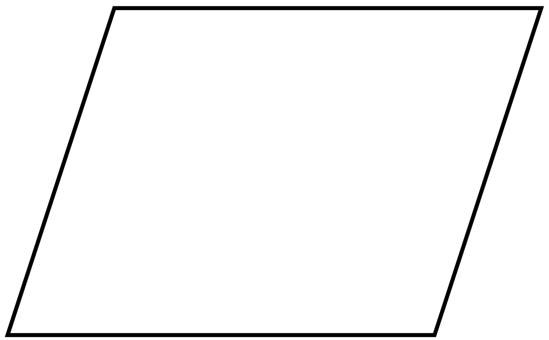
$$J = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{4}\left(2\frac{\partial N_1}{\partial \xi} + 0\frac{\partial N_1}{\partial \eta}\right)$$

$$\frac{\partial N_1}{\partial y} = \frac{1}{4}\left(0\frac{\partial N_1}{\partial \xi} + 2\frac{\partial N_1}{\partial \eta}\right)$$





- Obliczyć $\frac{\partial x}{\partial \xi}$ i $\frac{\partial y}{\partial \eta}$ w 1 pc dla

ID	1	2	3	4
x	0	4	4	0
y	0	0	4	5

- Obliczyć $\frac{\partial x}{\partial \xi}$ i $\frac{\partial y}{\partial \eta}$ w 1 pc dla

ID	1	2	3	4
x	0	4	4	0
y	0	0	4	5

Pkt całk	1
J_1_1	2
J_1_2	-0.10566
J_2_1	0
J_2_2	2.394338

Praca domowa

Struct Elem4{

Ksi [1x4(9)]

Eta [1x4(9)]

Pochodne(ξ i η)

}

Oblicz [J]

Oblicz det[J]

Oblicz [J]^-1

To we wszystkich pkt całkowania

Dodać możliwość wczytywania współrzędnych punktów z pliku

	$\frac{\partial N_1}{\partial \xi}$	$\frac{\partial N_2}{\partial \xi}$	$\frac{\partial N_3}{\partial \xi}$	$\frac{\partial N_4}{\partial \xi}$
pc_1				
pc_2				
pc_3				
pc_4				

Wynikiem pracy jest funkcja licząca Jakobian, det J oraz odwrotności Jakobianu w każdym z punktów całkowania. Obliczony Jakobian, det J oraz odwrotność Jakobianu należy wypisać na ekran

Praca ma zwracać poprawny wynik dla przykładu omawianego na zajęciach.