TRIANGULARITY OF A PLANE CURVE SINGULARITY

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Abstract. In this paper, we introduce *triangularity* Δ a curious numerical plane curve singularity invariant. We calculate values of Δ for ADE singularities, among others, and show that there is no rational linear dependence between Δ and other common invariants. Finally, we provide an application of Δ to a classical enumerative question regarding triangles, extending a formula of Collino & Fulton in [2]

Note to reader: This manuscript is in progress. I have placed text in red to indicate that more information needs to be included. I also highlight unfinished sections. Comments for myself are also present.

1. Introduction.

Theorem 1.1. *For ADE singularities, we have the following formulas for triangularity,*

$$\Delta(A) = \mu + 1 \quad \Delta(D) = \mu + 8 \quad \Delta(E) = 2\mu + 4,$$

where μ is the Milnor number of the singularity.

2. Preliminaries.

The Hilbert scheme $\mathrm{Hilb}^3\,\mathbb{P}^2$ is a smooth six dimension complex variety [3]. We construct an analytic chart U_0 near the fat point at origin as follows: Define

$$M = \begin{bmatrix} x-a & y-b & c \\ d & x-(a+e) & y-(b+f) \end{bmatrix}.$$

For any choice of $a, \dots, f \in \mathbb{C}$, the ideal generated by the 2×2 minors has length three in $\mathbb{C}[x, y]$ and thus, defines an element of $\mathrm{Hilb}^3\,\mathbb{A}^2$. We obtain a chart for $\mathrm{Hilb}^3\,\mathbb{P}^2$ near the fat point at origin by allowing $a, \dots, f$ to vary. In this chart, the fat locus $\mathtt{F}$—the subvariety of all fat points in $\mathrm{Hilb}^3\,\mathbb{P}^2$—is realized as a plane, cut out by the linear system

$$c = d = e = f = 0.$$

Blowing up along the plane gives a chart $\mathtt{U}_0$ for the *space of complete triangles* $\mathtt{CT}$, which is the blow of $\mathrm{Hilb}^3\,\mathbb{P}^2$ along the fat locus $\mathtt{F}$.

Triangularity is calculated locally in the chart $\mathtt{U}_0$. Suppose a curve, singular at origin, is defined by the equation $P(x, y) = 0$ near origin. The Hilbert scheme of triplets on this curve, W_P, is a threefold in $\mathrm{Hilb}^3 \mathbb{P}^2$ and a local complete intersection. The polynomial P reduces, modulo the ideal of 2×2 minors of M, to a linear expression

$$P \equiv P_x \cdot x + P_y \cdot y + P_{\mathrm{const.}}.$$

The coefficients of this linear expression are called the *resultants* of P; we compute them in a systematic way, first reducing all instances of x^2, xy, y^2 in that order, using the 2×2 minors of M. The space W_P is cut out locally by these coefficients in U_0. Set-theoretically, the chart for $\mathtt{CT}$ appears as

$$\{(a, \ldots, f) \times [C : D : E : F] : Cd = Dc, Ce = Ec, Cf = Fc\} \subset U_0 \times \mathbb{P}^3.$$

Let Z_P denote the preimage of W_P in $\mathtt{CT}$. Because the curve contains origin, Z_P contains the $\mathbb{P}^3$-fiber lying in $\mathtt{U}_0$ above origin in U_0. Triangularity is, by definition, the multiplicity of Z_P over this threefold. For computation, it is usually more convient to work in affine coordinates. Fix $C \neq 0$, set $\kappa = \mathbb{C}(D, E, F)$, and let $R = \kappa[a, b, c]_{(a,b,c)}$. Then

$$\Delta = \ell(R/(P_x, P_y, P_{\mathrm{const.}}))$$

where the resultants are now realized as polynomials in $\kappa[a, b, c]$. Because this ring is also Artinian, triangularity is entirely dependent on a local expression for the curve near its singularitiy. In other words, we conclude:

Theorem 2.1. *Triangularity Δ is a numerical invariant of plane curve singularities (up to analytic-isomorphism).*

For any polynomial P, set $R^P := R/(P_x, P_y, P_{\mathrm{const.}})$. As long as P is reduced with a singularity at origin, this ring is an Artinian, local complete intersection with maximal ideal $\mathfrak{m}_P$. The residue field is always κ; thus,

$$\Delta = \ell(R^P) = \sum_{n=0}^{\infty} \dim_\kappa \mathfrak{m}_P^n / \mathfrak{m}_P^{n+1}.$$

For brevity, we refer to the quotient $\mathfrak{m}_P^n / \mathfrak{m}_P^{n+1}$ as the *nth grade* of R^P. Its dimension over the residue field κ is the *nth grading of the singularity.* These numbers also define numerical invariants.

Insofar, the resultants of P are cruical for calculating the triangularity of the plane curve singularity defined by P. Proposition 2.2 sheds light on some of their basic characteristics.

Proposition 2.2. *Let P be a homogeneous degree d polynomial in variables x, y. The polynomials P_x, P_y are homogeneous in $\kappa[a,b,c]$ of degree $d-1$, while $P_{\text{const.}}$ is homogeneous with degree d.*

Proof. This follows by induction on d. Suppose the resultants of P are homogeneous of the expected degrees. Then consider $x \cdot P$. We have

$$x \cdot P \equiv P_x \cdot x^2 + P_y \cdot xy + P_{\text{const.}} \cdot x.$$

The quadric terms x^2 and xy reduce to

$$(2a + Ec)x + Dcy - a^2 + Eac - Dbc \text{ and } (b + Fc)x + ay - ab - Fac + Dc^2.$$

respectively. Making the proper substitutions, we see that $x \cdot P$ has resultants of the appropriate degree, all homogeneous by the inductive assumption. A similar analysis for $y \cdot P$ completes the proof. □

As an application of Proposition 2.2, we calculate the triangularity of d lines intersecting at origin. In $\mathbb{P}^2$, the union of lines is realized as the zero locus of a degree d homogeneous polynomial P in two (projective) variables x, y. Each resultant defines an algebraic subset of the affine space

$$\mathbb{A}^3_\kappa := \operatorname{Spec} \kappa[a, b, c].$$

Now, suppose $\mathbb{A}^3_\kappa$ is an affine chart of some larger projective space $\mathbb{P}^3_\kappa$, and let $V_x, V_y, V_{\text{const.}}$ denote the projective closures of the (respective) zero loci of the resultants. The scheme $V_x \cap V_y \cap V_{\text{const.}}$ is a complete intersection, consisting of a finite number of points. Bézout's theorem applies to this situation despite κ not being algebraically closed [see Corollary 3.3.3 of Toni Annala's master thesis]; thus, by the degree calculations in Proposition 2.2, Find a better reference.

$$d(d-1)^2 = \sum_{p \in V_x \cap V_y \cap V_{\text{const.}}} \operatorname{mult}_p(V_x \cap V_y, V_{\text{const.}})$$

The multiplicity at origin in $\mathbb{A}^3_\kappa$ is simply the triangularity Δ. Because the resultants are already homogeneous in a, b, c, no other intersections can occur otherwise R^P would not be of the appropriate dimension. Hence, the triangularity of d lines intersecting at origin is $d(d-1)^2$.

Theorem 2.3. *Fix any square-free homogeneous complex polynomial P of degree d in variables x, y, and another polynomial Q vanishing to order $d+1$ or greater at origin. Let $P_t = P + t \cdot Q$ be a pencil of polynomials in parameter t. Then for generic t, the triangularity of the singularity at origin is $d(d-1)^2$.*

Proof. Allow I_t to denote the ideal generated by the resultants of P_t, and set $R^t := R/I_t$ with maximal ideal $\mathfrak{m}_t$. The length function $t \mapsto \ell(R^t)$ is lower semi-continuous. We claim there is an inequality of gradings

Needs clarification because of κ v.s. $\mathbb{C}$.

$$\dim_\kappa \mathfrak{m}_0^n/\mathfrak{m}_0^{n+1} \geq \dim_\kappa \mathfrak{m}_t^n/\mathfrak{m}_t^{n+1}$$

for all n. The triangularity of the curve $P_t(x, y) = 0$ is then maximized at $t = 0$. By lower semi-continuity, for all but finitely many choices of t, the triangularity at origin of the curve $P_t(x, y) = 0$ is $d(d-1)^2$.

The inequalities appear by first noting that the resultants of P_t vary linearly in parameter t. For instance,

$$(P_t)_x = P_x + t \cdot Q_x.$$

Now, for any ideal $I \subset R$, let $\mathrm{Ld}(I, n)$ denote the collection of leading homogeneous parts of elements in I of degree n—it is a subspace of the collection of homogeneous polynomials in a, b, c of degree n. Then the dimension of $\mathfrak{m}_t^n/\mathfrak{m}_t^{n+1}$ is the codimension of $\mathrm{Ld}(I_t, n)$. Proposition 2.2 tells us that the degrees of the resultants of P are always less than the degrees of the (respective) resultants of Q, so there is an inclusion $\mathrm{Ld}(I_0, n) \subset \mathrm{Ld}(I_t, n)$ for all n and t, which completes the proof. $\square$

Remark. Theorem 2.3 tells us that we should expect the triangularity of a ordinary d-fold intersection to be $d(d-1)^2$. It also suggests that triangularity is heavily dependent on the tangent cone of the singularity—however, as our calculations for ADE singularities demonstrate, it is not entirely dependent on tangent cone. Further, any moduli space of ordinary plane curve singularities should realize the collection of points with non-generic triangularity as a closed subspace of codimension one or more. See *Section 6* for a more detailed discussion.

Write section discussing pencils of plane curve singularities. Talk about flatness and consequences that κ is not algebraically closed.

3. ADE Singularities. A plane curve is said to have an *ADE singularity* if in some chart around the singularity, the curve is the zero locus of one of the following algebraic expressions:

$$A_k : y^2 + x^{k+1} \quad k \geq 1 \qquad D_k : xy^2 + x^{k-1} \quad k \geq 4$$

$$E_6 : y^3 + x^4 \qquad E_7 : y^3 + xy^3 \qquad E_8 : y^3 + x^5$$

An *ADE* singularity is sorted first by type (A, D or E) and then by Milnor number μ which we display as a subscript [1]. For instance, there are only three analytically distinct type E singularities which are distinguished by their Milnor numbers ($6, 7, 8$ respectively). Geometers regularly come across ADE singularities—common examples include nodes and cusps, which are of types A_1 and A_2 respectively.

The goal of this section is to prove Theorem 1.1. The manner in which we compute triangularity will depend on its subtype. The infinite nature of type A and D singularities require a different approach than those of type E, which we do manually with `Macaulay2`. All calculations are preformed by first computing the gradings and then taking the sum. For every ADE singularity, a distinct pattern appears in the sequence of gradings. We outline a general pattern for AD types, where we make use of the fact that the tangent cone remains constant as the Milnor number k increases.

Type A: Denote by $R^{A_k} = R/I_{A_k}$ the ring produced in the previous section with the curve defined by $P = y^2 + x^{k+1}$. The resultants have the form

$$P_x = c + Q_x, \quad P_y = 2b + Fc + Q_y, \quad P_{\text{const.}} = ac + b^2 + Fbc + Ec^2 - Q_{\text{const.}}$$

where $Q = x^{k+1}$. If $k > 1$, the leading homogeneous parts are just the resultants of y^2. Set

$$I_A := (c, 2b + Fc, ac + b^2 + Fbc + Ec^2) = (b, c),$$

and consider the "truncated" ring $R^A = R/I_A$. Note that R^A is not Artinian; in fact, it is one-dimensional. Importantly, however, there are inequalities

$$\dim_\kappa \mathfrak{m}_{A_k}^n/\mathfrak{m}_{A_k}^{n+1} \leq \dim_\kappa \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$$

for all n and k. The calculation of Δ for type A_k singularities then follows by comparing R^{A_k} to the much simplier ring R^A.

We need the following lemma:

Lemma 3.1. *There is an equality* $\mathrm{Ld}(I_{A_k}, n) = \mathrm{Ld}(I_A, n)$ *for* $n \leq k$ *and a strict inclusion* $\mathrm{Ld}(I_A, k+1) \subset \mathrm{Ld}(I_{A_k}, k+1)$ *when* $k > 1$.

Proof. An element of I_{A_k} takes the form

$$t \cdot P_x + s \cdot P_y + r \cdot P_{\text{const.}}$$

where t, s, r are elements of R. Suppose it belongs to $\mathrm{Ld}(I_{A_k}, n)$ but not $\mathrm{Ld}(I_A, n)$. If

$$t \cdot c + s \cdot (2b + Fc) + r \cdot (ac + b^2 + Fbc + Ec^2)$$

is nonzero, then the leading homogeneous part of this polynomial also belongs to $\mathrm{Ld}(I_A, n)$. Thus, to obtain a polynomial in I_{A_k} whose leading homogeneous part *is not one* of I_A, we must have a dependence relation

$$t \cdot c + s \cdot (2b + Fc) + r \cdot (ac + b^2 + Fbc + Ec^2) = 0.$$

Clearly, the degrees of t, s, r cannot all be zero, so the leading homogeneous part of

$$t \cdot P_x + s \cdot P_y + r \cdot P_{\text{const.}} = t \cdot Q_x + s \cdot Q_y + r \cdot Q_{\text{const.}}$$

has degree at least $k+1$ by Proposition 2.2. Hence, the equality in the Lemma statement always holds.

To complete the proof, we construct a polynomial in I_{A_k} whose leading homogeneous part is of degree $k+1$ but not in $\mathrm{Ld}(I_A, k+1)$. Consider

$$(2a + 3Fb + 2Ec) \cdot P_x - b \cdot P_y - 2P_{\text{const.}}.$$

The lower degree terms cancel, leaving

$$(2a + 3Fb + 2Ec) \cdot Q_x - b \cdot Q_y - 2 \cdot Q_{\text{const.}}$$

which is homogeneous of degree $k+1$. A simple induction reveals that its expansion into a sum of monomials leaves a^{k+1} with a nonzero coefficient. Thus, the polynomial cannot belong to $I_A = (b, c)$. $\square$

Provide the explicit induction.

From Lemma 3.1, we deduce that for $k > 1$, there are equalities

$$\dim_\kappa \mathfrak{m}_{A_k}^n / \mathfrak{m}_{A_k}^{n+1} = \dim_\kappa \mathfrak{m}_A^n / \mathfrak{m}_A^{n+1}$$

for $n \leq k$ and there is a strict inequality

$$\dim_\kappa \mathfrak{m}_{A_k}^{k+1} / \mathfrak{m}_{A_k}^{k+2} < \dim_\kappa \mathfrak{m}_A^{k+1} / \mathfrak{m}_A^{k+2}.$$

The ring R^A is isomorphic to $\kappa[a]_{(a)}$, so each $\mathfrak{m}_A^n / \mathfrak{m}_A^{n+1}$ has dimension one. Thus,

$$\boxed{\Delta(A_k) = \sum_0^k 1 = k+1.}$$

The case of the node ($k = 1$) is dealt with separately but its gradings follow the same pattern.

Type D: We now proceed—in similar fashion—to calculate the triangularity of type D singularities. As before, let $R^{D_k} = R/I_{D_k}$ denote the ring formed in the previous section by considering the curve cut out by $P = xy^2 + x^{k-1}$ for $k \geq 4$. The case $k = 4$ is simply the union of three lines intersecting at origin; its triangularity is $3(3-1)^2 = 12$. Assume $k > 4$, and let I_D denote the ideal generated by the resultants of xy^2. By Proposition 2.2, I_D is the ideal generated by the leading homogeneous parts of the resultants of P. As before, there are inequalities between the gradings of the truncated ring $R^D = R/I_D$ and those of any R^{D_k}.

The ring R^D is much more complicated than R^A. The ideal I_D cannot be generated by two elements, and R^D is one-dimensional, so unlike R^A, it is *not* a local complete intersection. The following lemma proves this claim and calculates the dimensions of $\mathfrak{m}_D^n/\mathfrak{m}_D^{n+1}$ for all n.

Lemma 3.2. *The ring R^D is one-dimensional, and the table*

grade	dim
0th	1
1th	3
2nd	4
3rd	3
4th	1
$\vdots$	$\vdots$
nth	1

gives the dimension of each graded piece.

Proof. The ring I_D is generated by (homogeneous) two quadratics and a cubic, so it cannot contain any polynomials of degree one. Thus, there is a strict inclusion of ideals

$$I_D \subset (b, c) \subset (a, b, c)$$

where the latter two are prime. It then follows that the quotient R^D is one-dimensional.

The dimension of $R^D/\mathfrak{m} = \kappa$ is immediate. Because I_D is generated by homogeneous polynomials of degree greater than one, $\mathfrak{m}_D/\mathfrak{m}_D^2$ is three-dimensional with basis a, b, c. The polynomials in $\mathrm{Ld}(I_D, 2)$ are κ-linear combinations of the two homogeneous quadrics. Since $\mathfrak{m}/\mathfrak{m}^2$ is six dimensional, the dimension of $\mathfrak{m}_D/\mathfrak{m}_D^2$ is $6 - 2 = 4$. Finally, we check via explicit calculation that $\mathfrak{m}_D^2/\mathfrak{m}_D^3$ is three-dimensional and that $\mathfrak{m}_D^3/\mathfrak{m}_D^4$ is one-dimensional; see Appendix A. Since the 3rd grade is one, all proceeding grades must also be one-dimensional. Indeed, none of them can be trivial, otherwise R would be zero-dimensional. This completes our proof. $\square$

We complete our calculation with the following lemma, analogous to Lemma 3.1:

Lemma 3.3. *There is an equality* $\mathrm{Ld}(I_{D_k}, n) = \mathrm{Ld}(I_D, n)$ *for $n \leq k$ and a strict inclusion* $\mathrm{Ld}(I_A, k+1) \subset \mathrm{Ld}(I_{A_k}, k+1)$ *when $k > 4$.*

Proof. Needs to be completed. Follows almost the same argument as Lemma 3.1.

The singularity D_4 also follows the same grading rule, although we don't explicitly confirm this as its triangularity is already known to be 12. We now produce the formula

$$\boxed{\Delta(D_k) = 1 + 3 + 4 + 3 + 1 + \cdots (k-3 \text{ times}) \cdots + 1 = k + 8.}$$

Take note that any grade past the kth is identically zero because of the strict inclusion in Lemma 3.3 and the gradings for R^D shown in Lemma 3.2.

Type E: All triangularity calculations for type E singularities are preformed in the computer algebra system `Macaulay2`. The code is provided in Appendix A. Below, we summarize the calculations for E_6, E_7, E_8 with tables depicting their gradings.

E₆

grade	dim
0th	1
1th	3
2nd	4
3rd	4
4th	3
5th	1

E₇

grade	dim
0th	1
1th	3
2nd	4
3rd	4
4th	3
5th	2
6th	1

E₈

grade	dim
0th	1
1th	3
2nd	4
3rd	4
4th	4
5th	3
6th	1

The triangularity of a type E singularity therefore follows the rule

$$\boxed{\Delta(E_k) = 2k + 4.}$$

Altogether these calculations prove Theorem 1.1.

4. Triangularity and other Numerical Invariants.

5. Application to the Enumerative Question.

6. Remaining Questions.

Appendix A. Include formatted `Macaulay2` code.

References.

[1] M. Bátorová & M. Valikova & P. Chalmoviansky, *Desingularization of ADE singularities via deformation*, Proceedings - SCCG: 29th Spring Conference on Computer Graphics (2013).

[2] A. Collino & W. Fulton, *Intersection rings of spaces of triangles*, Mémoires de la Société Mathématique de France 38 (1989), pp. 75-117.

[3] A. Deopurkar & A. Patel, *Counting 3-uple Veronese surfaces,* arXiv: 2411.14232 (2024).

[4] G. Elencwajg & P. Le Barz, *Explicit computations in* $\mathrm{Hilb}^3\,\mathbb{P}^2$, In: Algebraic Geometry Sundance 1986. Lecture Notes in Mathematics, vol 1311. Springer, Berlin, Heidelberg (1988).