Method of Images (Electrostatics)

Project Overview

The aim of this project is to visualize the **Method of Images** in electrostatics, a clever but bizzare way of finding potential, electric field, and charge distribution in a limited number of configurations without solving Poisson's equation. We hope that the visualizations in this handouts can substitute Chapter 3.2 of *Introduction to Electrodynamics* by David J. Griffith and help reader better understand the method of images in electrostatics.

Below is a list of configurations we visualized. Each example starts with a short decription of the scenario at hand followed by key steps and equations in the derivations. We then sequentially present visualizations of the physical scenario (i.e. charge and conductor), the location of image charges (i.e. the easier-to-solve equivalence), the equipotential and electric field, and charge distribution on the conductor for some setups.

Section 1 - point charge and grounded conducting plate
Example 1: conducting sphere/point charge with a grounded conducting plate (page 124-125 Griffith 4th edition)
Example 2: point charge in between two perpendicular grounded conducting plates (Problem 3.11 Griffith 4th edition)
Example 3: point charge in between two grounded conducting plates of 60 degree angle (Problem 3.11 Griffith 4th edition)

Section 2 - point charge and conducting sphere
Example 4: point charge outside a grounded conducting sphere
Example 5: point charge inside a grounded conducting sphere

Readers who would like to challenge themselves after reading the handout can give it a try by solving the below problem:
Exercise: neutral conducting sphere in a uniform E field (Examplle 3.8 Griffith 4th edition)
The textbook solves this problem by solving the Laplace equation in spherical coordinates. But it can also be solved using the method of images.

Overview: Method of Images

The main challenge in tackling the above problems without solving Poisson's equation lies in the fact that part of the charge dirstributon of what generates the potential is unknown. Sometimes, we can make use of the symmetry of the setup to look for an equivalent problem that satisfies all the boundary conditions with a different setup and completely known charge distribution, this is the basic idea of method of image. The uniqueness theorem states that if we specify the charge distribution and the potential on the boundaries, the potential and electric field in the region we are interested in will be uniquely defined. The uniquess theorem guarantees that if we find a solution that meets all the criteria in the original problem then we have found the solution we want.

Section 1 - point charge and grounded conducting plane

Example 1: conducting sphere/point charge and a grounded conducting plate

Suppose we have a point charge q at $(0, 0, 0)$ held a distance d above an infinite, grounded, conducting plate at $(0, 0, -d)$. What is the potential above the plate caused by the given point charge and the charge induced on the conducting plate? How is the induced charge distributed on the plate (Griffith page 124-125)? The physical scenario is depicted below, where the given charge is denoted by the red dot and the conducting plates is the gray surface:

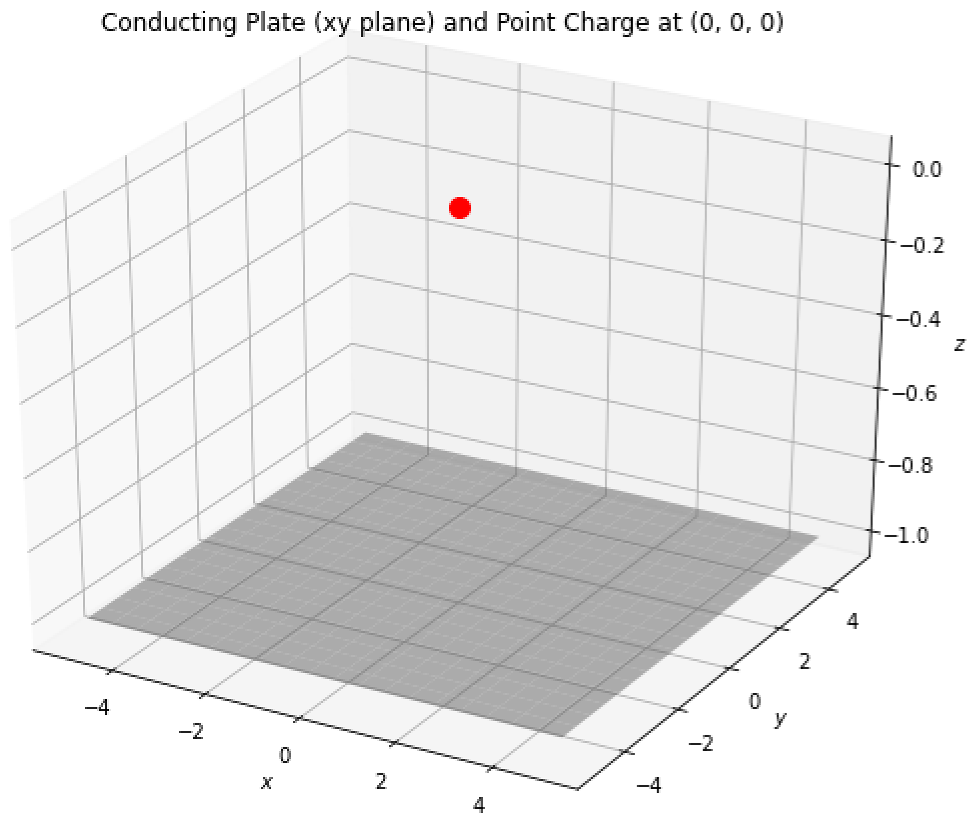


Figure 1.1 Find the potential in the region $z > -d$. Here, we set d=1 for visualization purposes.

The known charge distribution is the existing charge that we are given. The grounded conducting plate is the boundary. We can solve this problem by considering an equivalent problem that satisfies the Poisson equation for a single charge q at $(0, 0, 0)$

$$\nabla^2 V = -\frac{1}{\epsilon_0} q\delta(x)\delta(y)\delta(z)$$

boundary conditions

$$V(z = -d) = 0$$

and

$$V(r \to \infty) = 0$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

By the uniqueness theorem, if we can find an equivalent but fictitious charge distribution in $z < -d$ that, together with the existing point charge at $(0, 0, 0)$, gives $V(z = -d) = 0$, it will give us the correct potential everywhere in the space above the plate that the original problem is interested in. Note that the equivalent dictribution cannot be in the region above the conducting plane because otherwise the known charge distribution will not be satisfied.

We recall that the plane at the middle of two charge of the same magnitude and opposite sign is an equipotential. Since $V(r \to \infty) = 0$ on this plane, the potential equals 0 everywhere on this plane. Hence, by the uniqueness theorem, if we replace the conducting plate at $z = -d$ with a point change $-q$ at $(0, 0, -2d)$, the potential everywhere should stay the same. We call the equivalent charge the "image charge". If all of the induced charge on the grounded plate were gathered at one point to create the same effect as when they are dispersed naturally on the plate, that point charge would have the same charge and position as the image charge. The

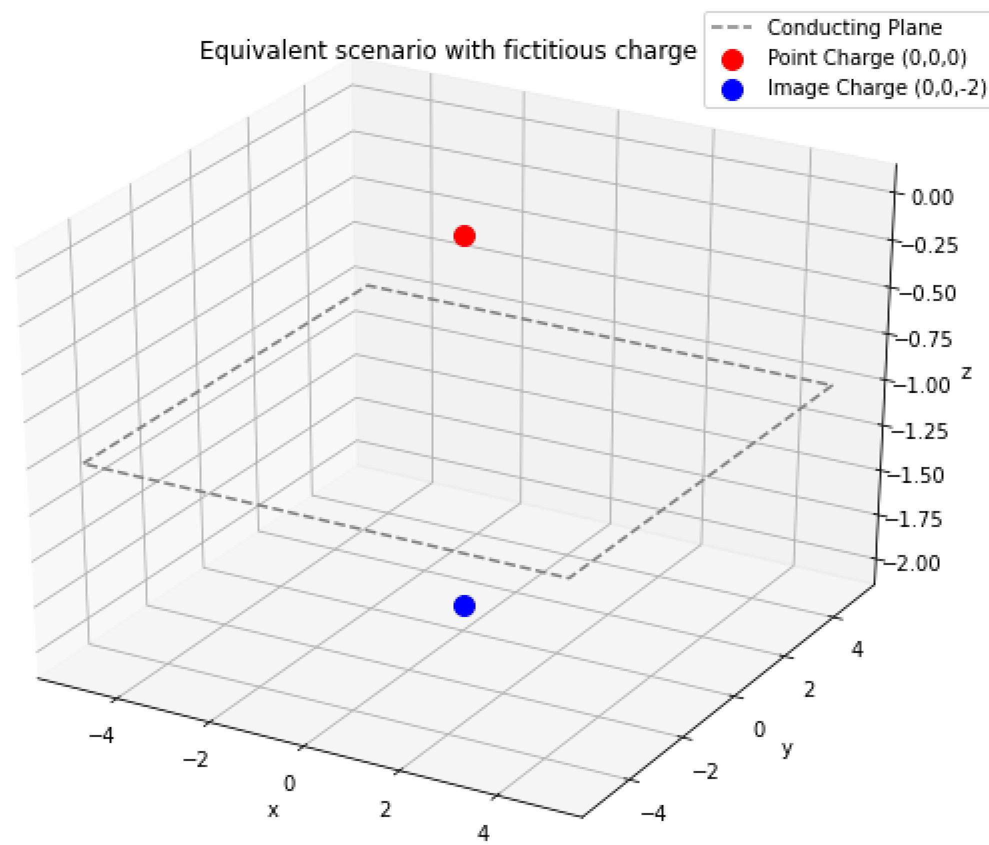


Figure 1.2 The equivalent scenario to the original problem. The red dot denotes the existing charge that is given in the original problem and the blue dot denotes the image or fictitious charge. The dashed rectangle is the conducting plane. It is dashed because in the equivalent scenario, the image charge is in place of the conducting plate to generate the same amount of potential as that would have been generated by the induced charge on the conducting plate if the conducting plate was present.

We can calculate the potential in the region $z > -d$ using the alternative solution given by the image charge:

$$V = \frac{q}{4\pi\epsilon_0}(\frac{1}{r_1} - \frac{1}{r_2}),$$

where $r_1 = \sqrt{x^2 + y^2 + z^2}$ and $r_2 = \sqrt{x^2 + y^2 + (z + 2d)^2}$ are the distance from a point in space to the existing and image charge respectively. We are now ready to calculate the electric field in the region:

$$E = -\nabla V = (-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}).$$

In particular,

$$E_z = -\frac{\partial V}{\partial z} = \frac{q}{4\pi\epsilon_0}(\frac{z}{(x^2+y^2+z^2)^{3/2}} - \frac{zd}{(x^2+y^2+(z+2d)^2)^{3/2}}),$$

which will be useful to when we calculate the charge distribution on the conducting plate.

The potential and direction of the electric field at each grid point in the xz plane $(y = 0)$ is shown in the graph below. The solid black lines indicates equipotentials. For visualization

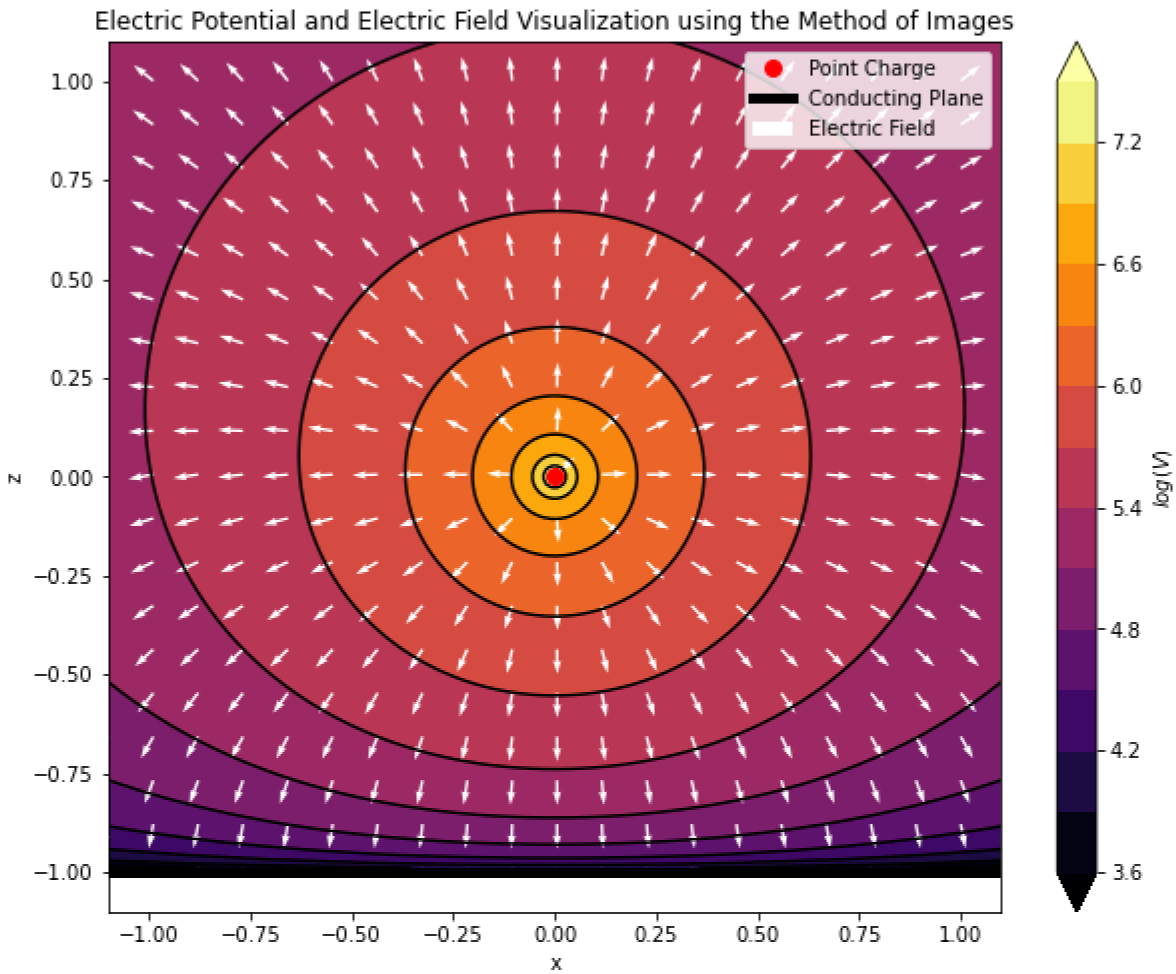


Figure 1.3 Potential (in logrithm scale) and direction of electric field in the xz plane in the vicinity of the existing charge.

We can then solve the induced charge distribution σ on the plate using the potential we calculated in the last paragraph. The charge density at a point on the conducting plane is given by the local value of electric field component that is perpendicular to the plane. In fact, at the surface of the conducting plane, the electric field must be in the normal direction to achieve electrostatics, because if there is a component parallel to the surface, the parallel component in the conductor will move the charge around and thus will not be electrostatics. In this case:

$$-\nabla V = \frac{\sigma}{\epsilon_0} = E_{\perp}(z = -d) = E_z(z = -d).$$

Hence,

$$\sigma = \epsilon_0 E_z(z = -d) = -\frac{q}{2\pi} \frac{d}{(x^2 + y^2 + d^2)^{3/2}},$$

which gives the induced charge distribution for a grounded conducting plate in the xy plane.

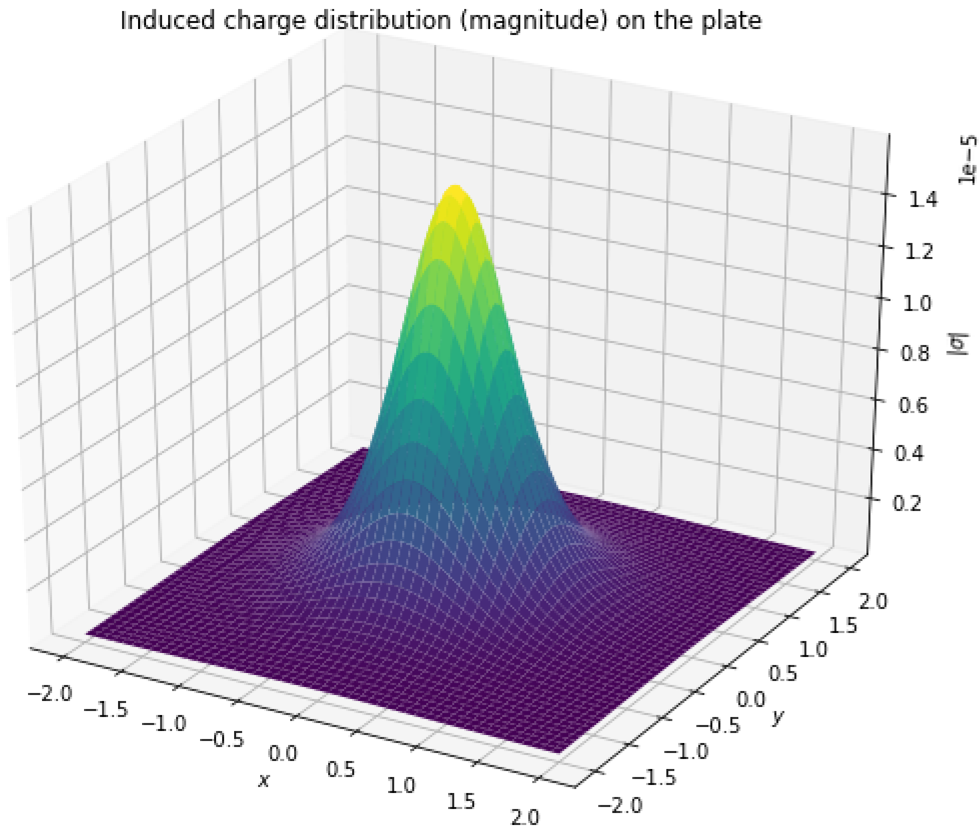
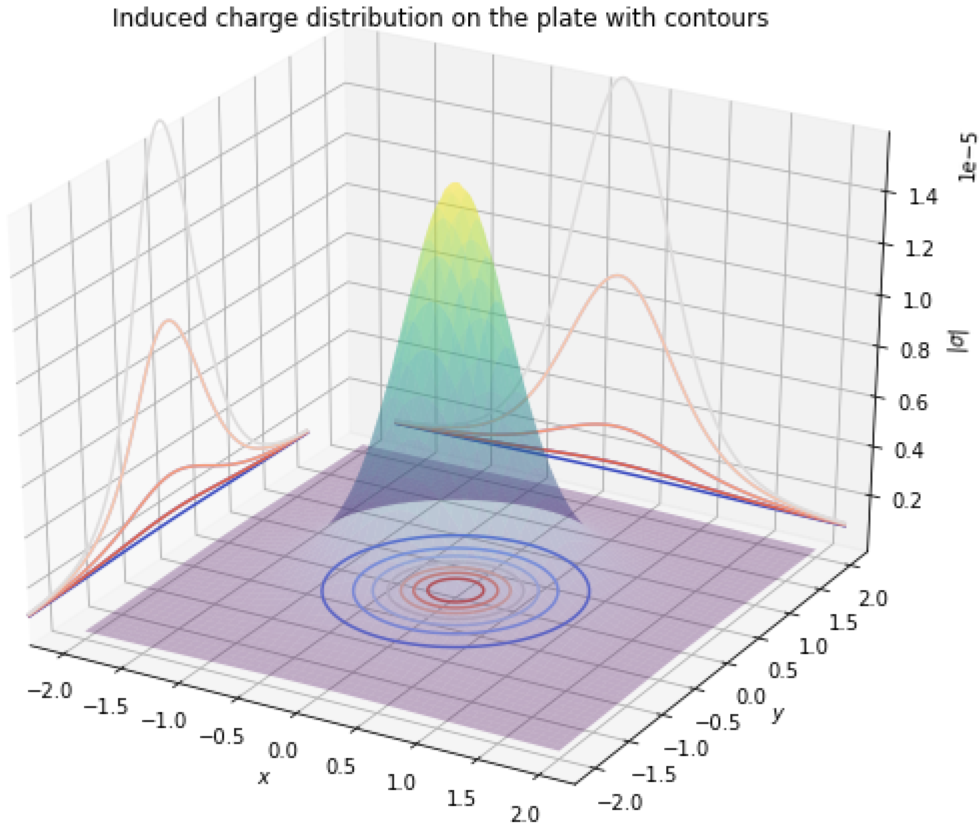


Figure 1.4 Induced charge distribution (magnitude) on the conducting plate in the xy plane with $z = -1$. The charge density reaches a peak at $(0, 0, -1)$, the point on the plate that is the closest to the point charge at $(0, 0, 0)$, as expected.

We added contours to the graph for better inspection.



Example 2: a point charge in between two perpendicular grounded conducting plates

The example analyzed above only involves one conducting plate. What happens when there are two plates involved? Consider the same point charge q situated in between two plates at the right angle, with the second plate at $x = -a$ both grounded and satisfy $V = 0$. What is the potential in the region $z > -d$ and $x > -a$. The setup is shown in the graph below.

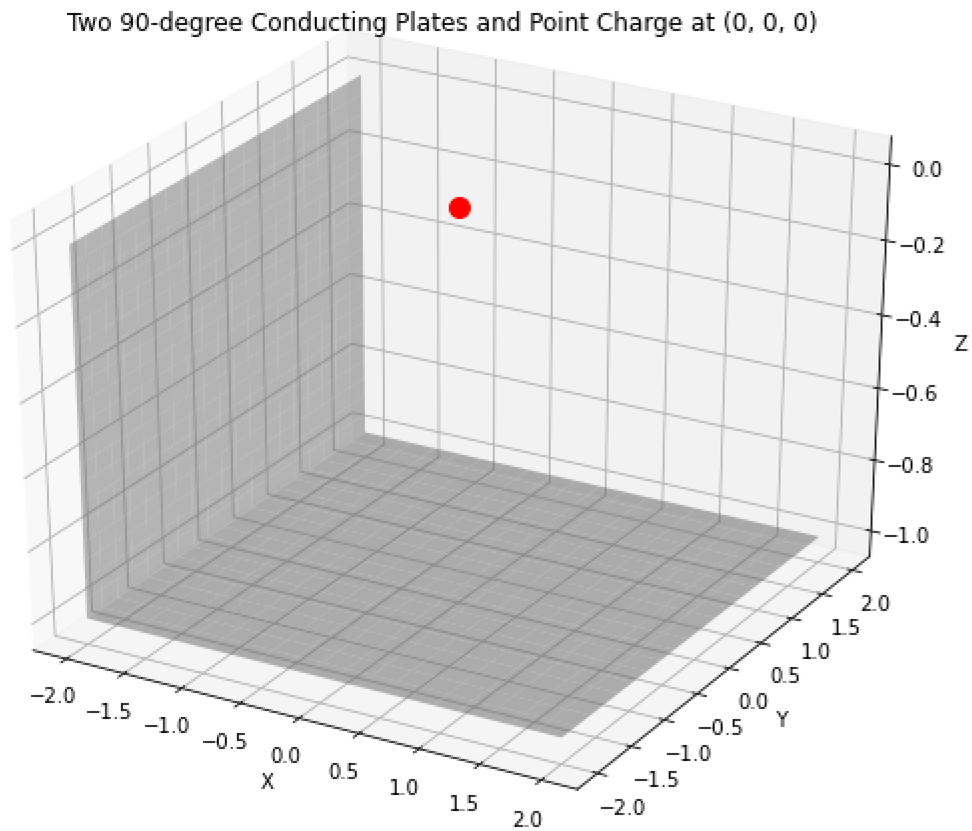


Figure 2.1 Find the potential in the region $z > -d$ and $x > -a$. Here, we set $d = 1$ and $a = 2$ for visualization purposes.

Similar to the first problem, we can solve this problem by considering an equivalent problem that satisfies the Poisson equation for a single charge q at $(0, 0, 0)$

$$\nabla^2 V = -\frac{1}{\epsilon_0} q\delta(x)\delta(y)\delta(z)$$

with an additional boundary condition

$$V(x = -a) = 0, V(z = -d) = 0$$

and

$$V(r \to \infty) = 0$$

where $r = \sqrt{x^2 + y^2 + z^2}$ the same as defined in Ex. 1.

The perfect symmetry of the system indicates the possibility of solving it with method of images. The image below shows the image configuration ($a = 2$ and $d = 1$).

We start from the same image charge as the one in Ex.1. With one image charge at $(0, 0, -2d)$ (bottom right), the surface at $z = -d$ is at zero equipotential, but the suface at $x = -a$ is not. we can follow the same logic and put an equal and opposite point charge $-q$ at $(-2a, 0, 0)$ (top left). However, this additional charge disturbs the potential at $z = -d$, so we need to put a charge q at $(-2a, 0, -2d)$ (bottome left) to counter the disturbance without affecting the

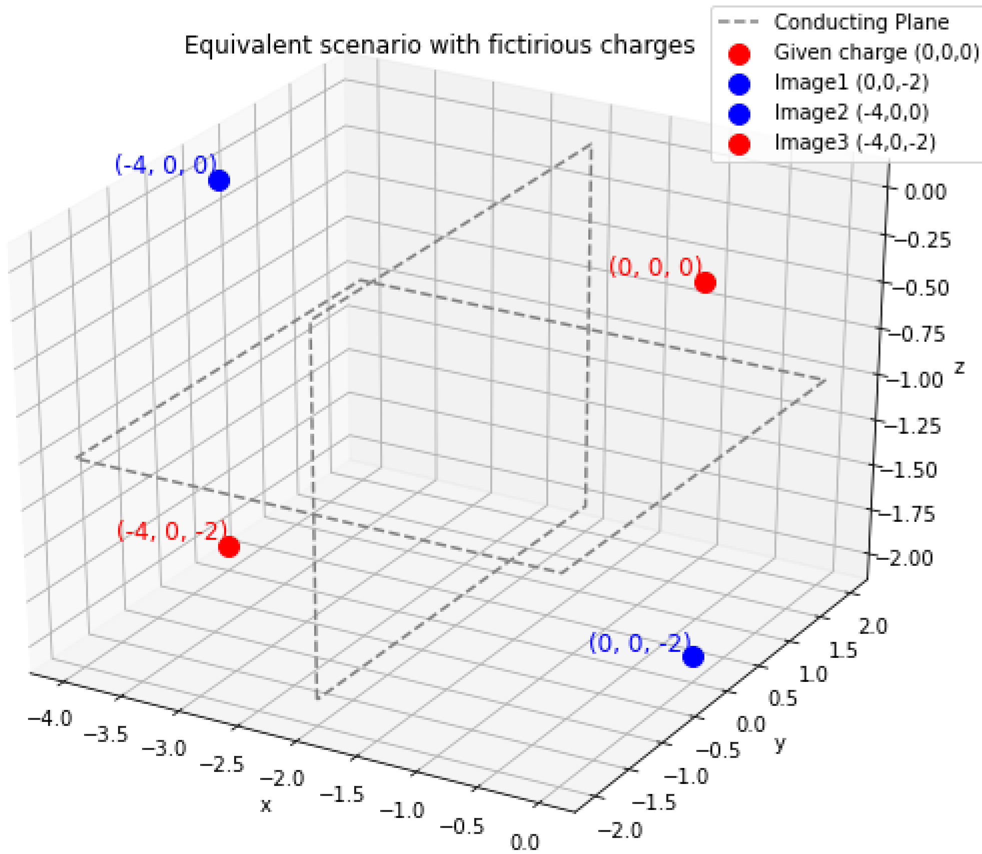


Figure 2.2 Equivalent scenario to the original problem. To have potential equals zero on both planes $z = -1$ and $x = -2$, we need three image charges.

Using the equivalence, the potential in the region $z > -d$ and $x > -a$ is easy to write down:

$$V = \frac{q}{4\pi\epsilon_0}(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}),$$

where

$r_1 = \sqrt{x^2 + y^2 + z^2}, r_2 = \sqrt{x^2 + y^2 + (z + 2d)^2}, r_3 = \sqrt{(x + 2a)^2 + y^2 + z^2}, r_4 = \sqrt{(x +$ are distances from a point in space to the existing and image charge respectively. Taking the gradient of V, we get the x and z components of the electric field in the region:

$$E_x = -\frac{\partial V}{\partial y} = \frac{q}{4\pi\epsilon_0}(\frac{x}{r_1^3} - \frac{x}{r_2^3} + \frac{x + 2a}{r_3^3} - \frac{x + 2a}{r_4^3}),$$

$$E_z = -\frac{\partial V}{\partial z} = \frac{q}{4\pi\epsilon_0}(\frac{z}{r_1^3} - \frac{z + 2d}{r_2^3} + \frac{z}{r_3^3} - \frac{z + 2d}{r_4^3}).$$

The potential and direction of the electric field at each grid point in the xz plane $(y = 0)$ is shown in the graph below. The solid black lines indicates equipotentials. For visualization purposes, we set $q = 50\mu C$.

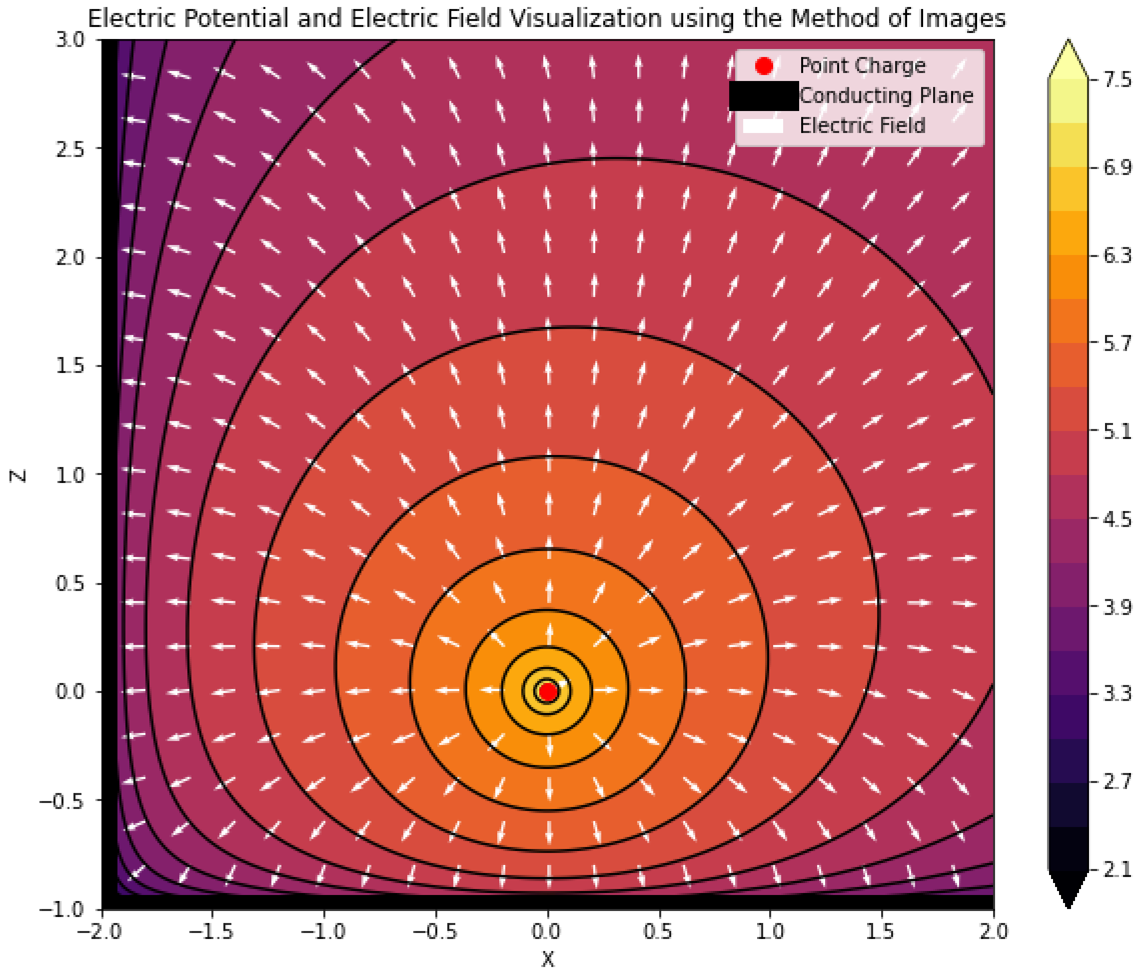
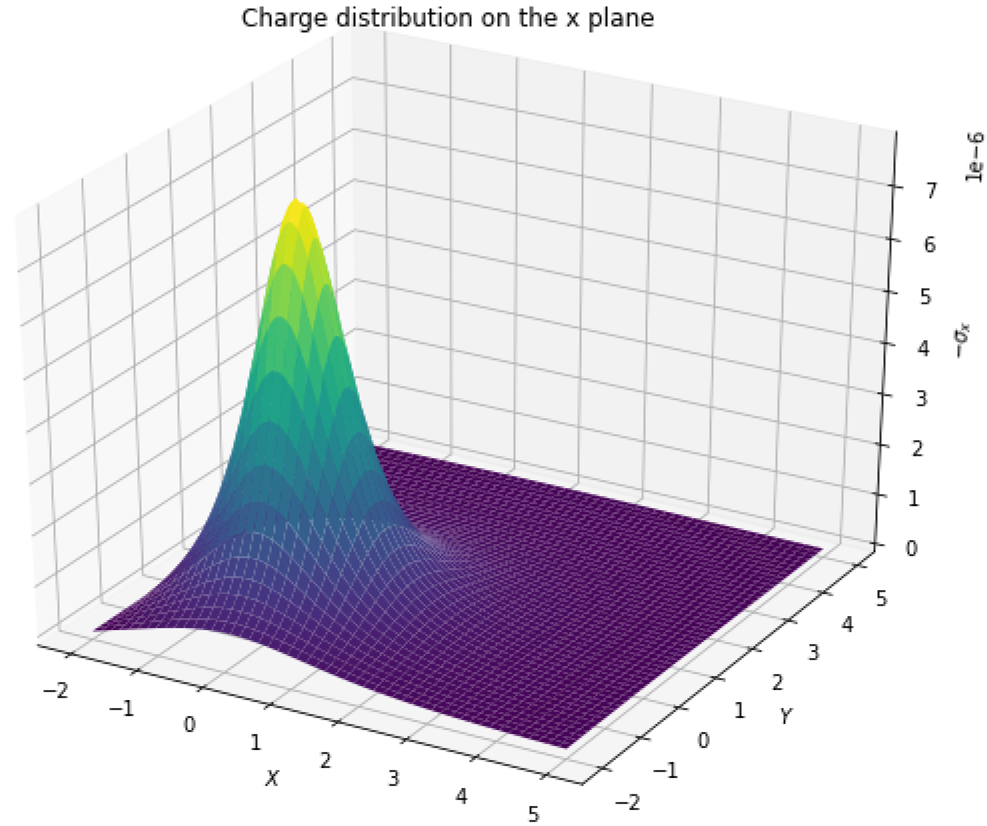


Figure 2.3 Potential (in logrithm scale) and direction of electric field in the xz plane in the vicinity of the existing charge.

The induced charge distribution σ_x and σ_z on the plates can be calculated from the x and z components of the electric field:

$$\sigma_x = \epsilon_0 E_x(x = -a)$$
$$\sigma_z = \epsilon_0 E_z(z = -d)$$

which gives the induced charge distribution for a grounded conducting plate in the xy plane. The peak is at $(0, 0)$ as expectedThe distribution in the second plane should look very similar and thus omitted. The absolute value of the charge density is visualized in the graph below:



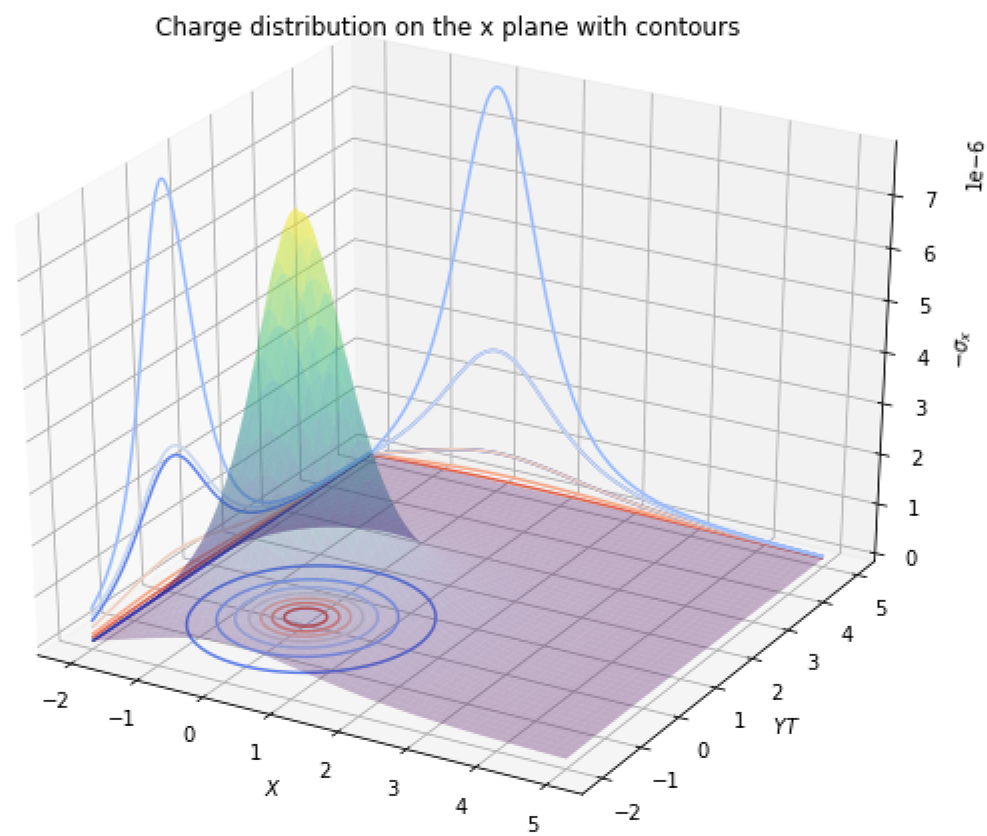


Figure 2.4 Induced charge distribution in the horizontal plane

Example 3: point charge in between two grounded conducting plates of 60 degree angle

The last example with comducting planes forming a 90-degree angle seems like a very special case. Can we still use the method of images to find the potential between two plates at a 60-degree angle with a given charge q in between? The physical setup is shown in the image below, where the gray rectangles indicate the boundary plates and the red dot signifies the given charge.

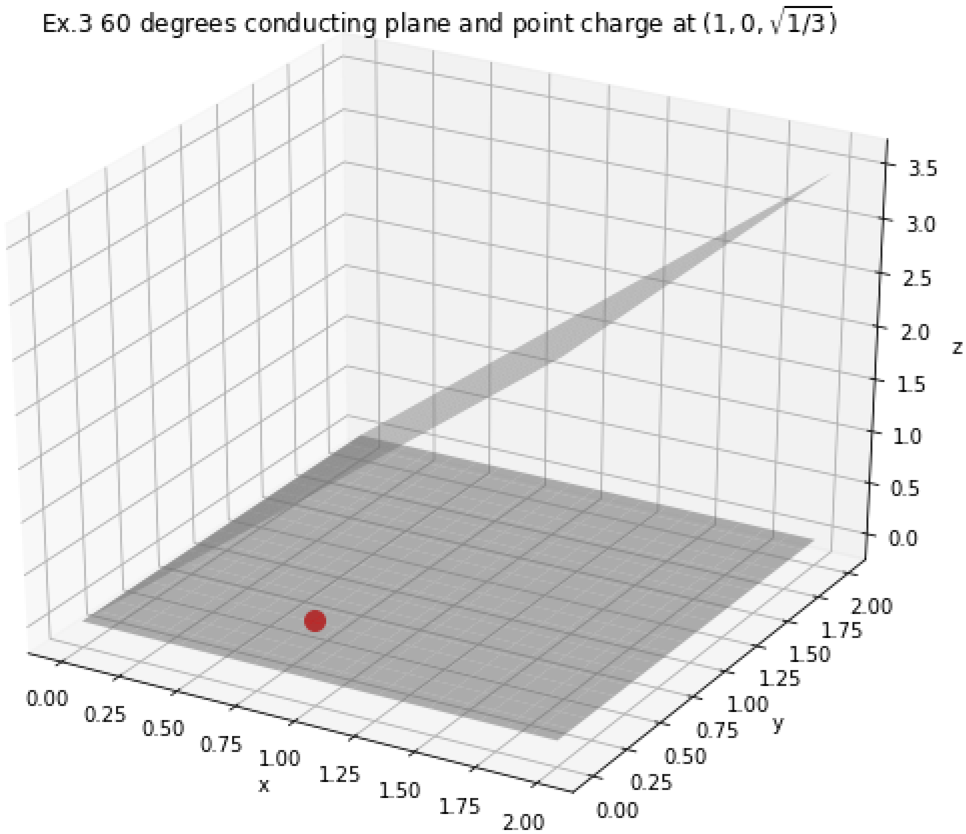


Figure 3.1

The Poisson's equation for this problem is:

$$\nabla^2 V = -\frac{1}{\epsilon_0} q\delta(x-1)\delta(y)\delta(z-\sqrt{1/3})$$

with the boundary conditions

$$V(z=\sqrt{3}x) = 0, V(z=-d) = 0$$

and

$$V(r \to \infty) = 0$$

where $r = \sqrt{x^2 + y^2 + z^2}$ the same as defined in Ex. 1

Indeed, we can still avoid solving the Poisson's equation and use the method of images to solve this problem. For easy reference, we call the horizontal surface at $z = 0$ S_1 and the other surface at $z = \sqrt{3}x$ S_2.

Following the same logic as Example 1 and 2, we can start by placing an image charge #1 $= -q$ at $(1, 0, -\sqrt{1/3})$ to make the potential of S_1 zero. Then we add image charge #2 $= -q$ at $(0, 0, 1)$,the mirroring position of the given charge with respect to S_2, so that S_2 can have zero potential. To keep the potential of S_1 at zero, we add an image charge #3 $= q$ at $(0, 0, -1)$, the mirroring position of image charge #2 with repsect to S_1. But the potential of S_2 is not zero now due to this new addition, so we must add an image charge #4 $= -q$ at $(-1, 0, -\sqrt{1/3})$, the mirroring position of image charge #3 with respect to S_2. Last but not least, we add image charge #5 $= q$ to bring S_1 back to zero potential.

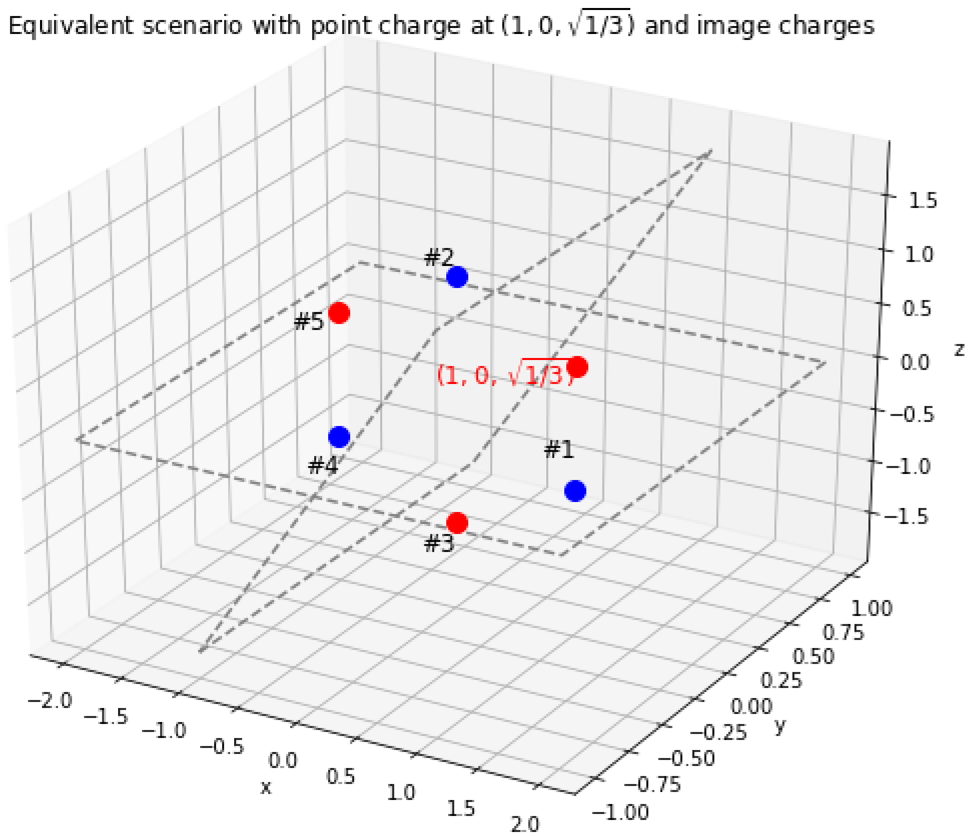


Figure 3.2

Using the equivalence, the potential in bwteen S_1 and S_2 is:

$$V = \frac{q}{4\pi\epsilon_0}(\frac{1}{r_0} - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} + \frac{1}{r_5}),$$

where r_0 ia the distance from a point in space to the given charge, and r_1, r_2, r_3, r_4, r_5 are distances from a point in space to the image charges. Taking the gradient of V, we get:

$$E = -\nabla V = (E_x, E_y, E_z),$$

which can be used to calculate the surface charge density at S_1 and S_2.

As before, the potential and direction of the electric field at each grid point in the xz plane $(y = 0)$ is shown in the graph below, followed by a graph of the surface charge density at the horizontal plane S_1. The solid black lines indicates equipotentials. For visualization purposes, we set $q = 50\mu C$.

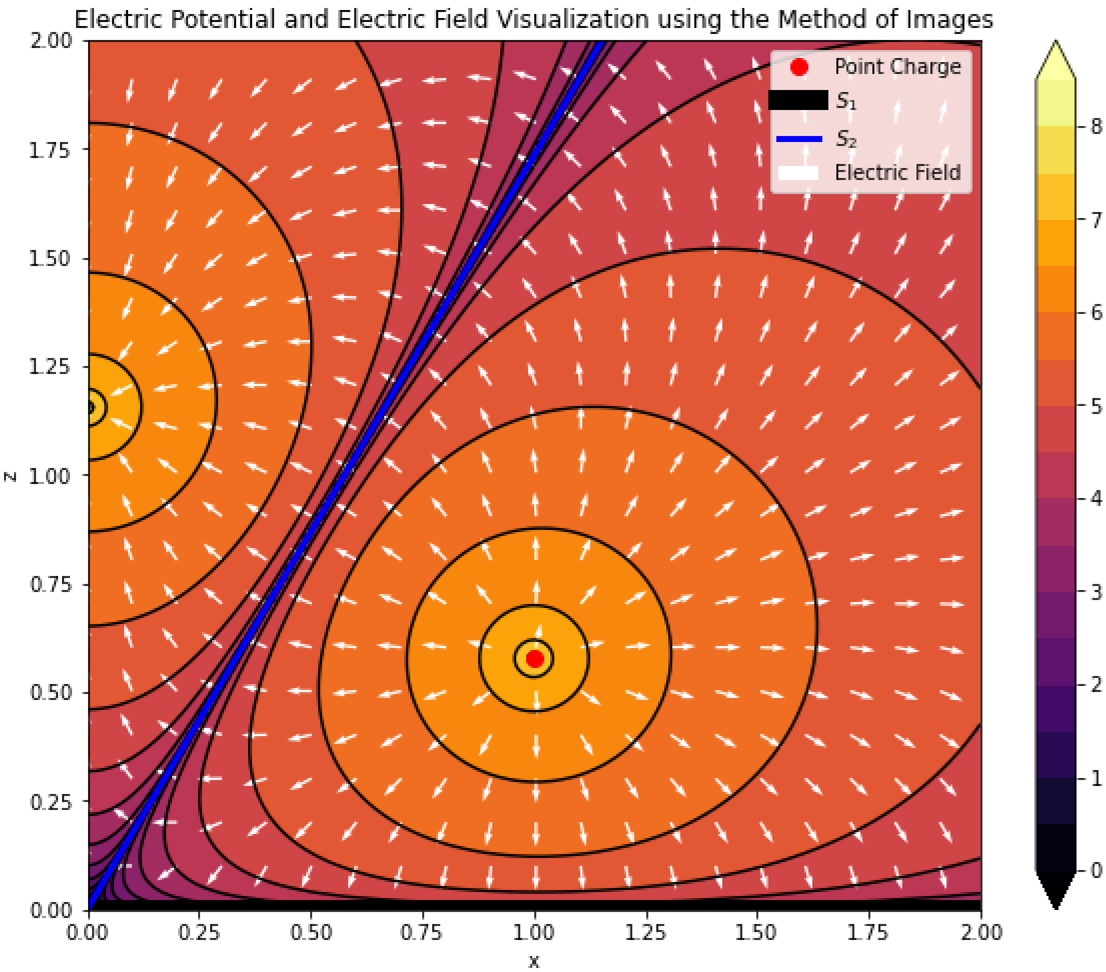


Figure 3.4

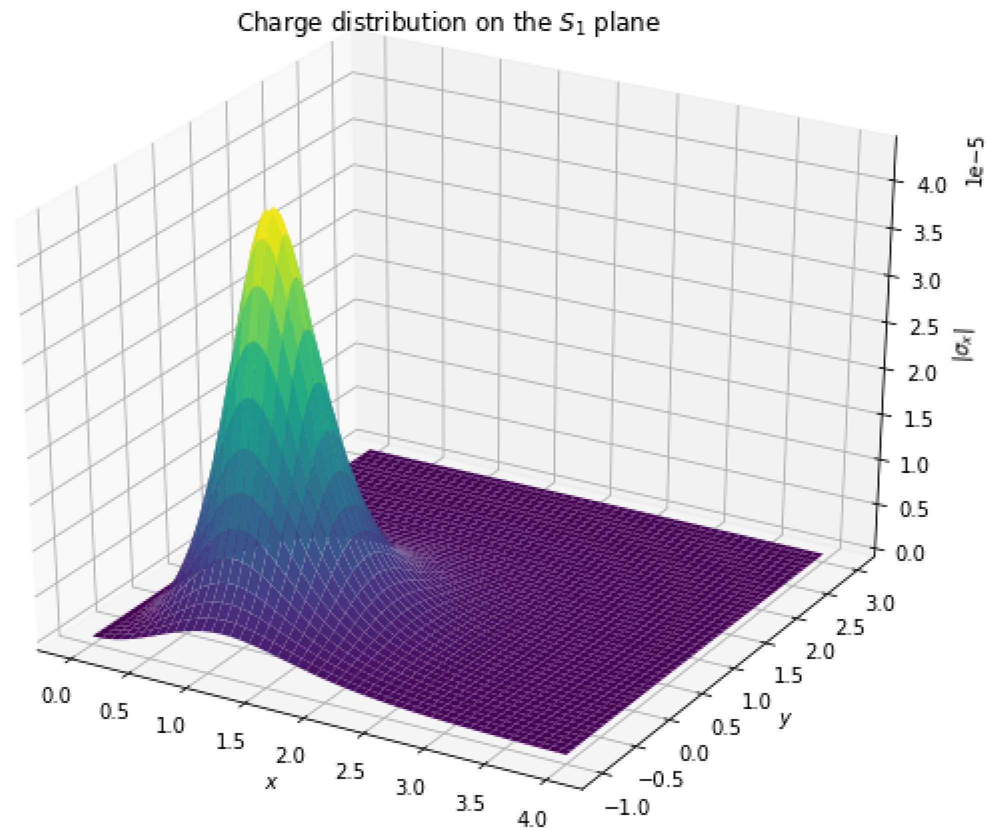


Figure 3.5

In fact, the method of image works as long as the angle between the two plates are divisible by 2π, since we need to create a symmetry so that the potentials at both surface are zero.

Section 2 - point charge and conducting sphere

Example 4: point charge outside a grounded conducting sphere

We can think of the problems in section one as a special case of a point charge near a conducting sphere where the point charge is being place very close to a giant sphere of radius close to infinity (like how we see Earth as flat when it is actually a sphere). Now let's zoom out and consider the case of putting a charge $q_1 = q$ at $(r, 0, 0)$ next to a grounded conducting sphere of radius a centered at $(0, 0, 0)$ as shown below. What is the potential generated by the point charge and the induced charge on the sphere?

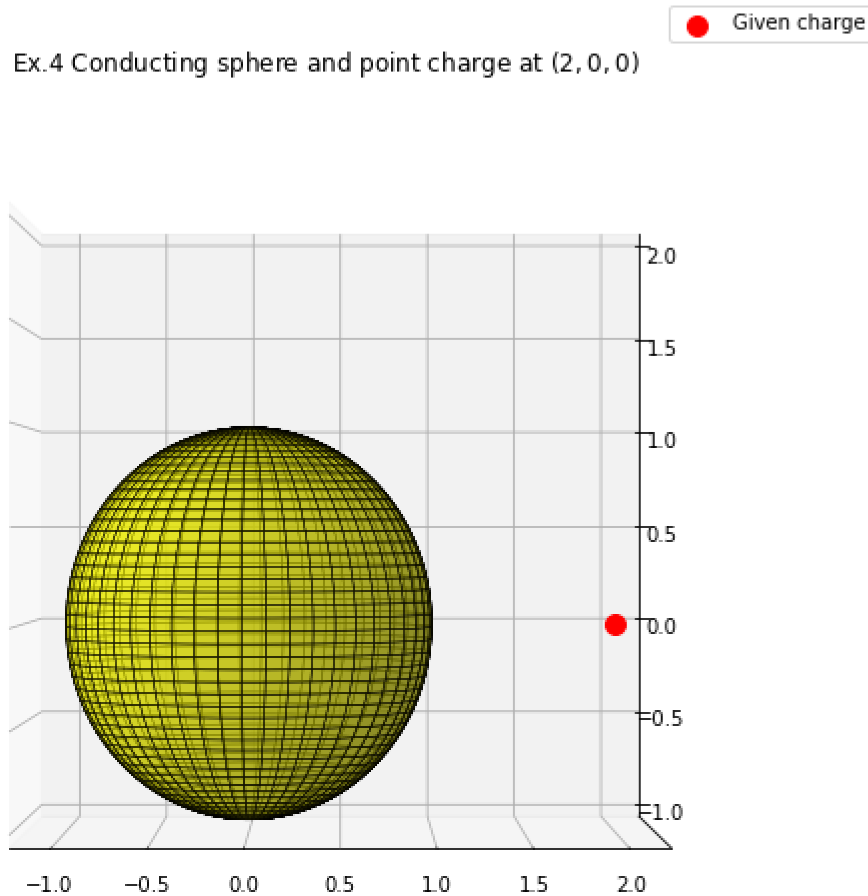


Figure 4.1

Again, we start by stating the Poisson's equation of the problem:

$$\nabla^2 V = -\frac{1}{\epsilon_0} q\delta(x-r)\delta(y)\delta(z)$$

with the boundary conditions

$$V(r_0 = a) = 0$$

and

$$V(r_0 \to \infty) = 0$$

where $r_0 = \sqrt{x^2 + y^2 + z^2}$ the same as defined in Ex. 1.

We want to find an equivalent point charge inside the sphere (so that the given charge distribution is not violated) whose potential at anywhere on the surface of the sphere cancels that of the give charge q outside. Because of the symmetry of the sphere, we can just look at a

2D cross section of this 3D problem as the image below shows (source: [2]).

The potential at a point P on the surface can be written as:

$$V(P) = \frac{1}{4\pi\epsilon_0}(\frac{q_1}{AP} + \frac{q_2}{BP}) = 0.$$

This is to say we want a point B inside the sphere such that

$$\frac{q_1}{q_2} = -\frac{AP}{BP} = constant$$

anywhere on the surface. This geometry problem was solved by Apollonius, a renowned greek geometer. Apollonius showed that a circle can be defined as the set of points in a plane that have a specified ratio of distances to two fixed points (source: [3]). This means that such point that can keep the above ratio constant along a circle/sphere indeed exists. In the context of our problem, if the original charge outside of the sphere at A is a distance d from the center of the sphere of radius a, then the image charge is a distance a^2/r from the center of the sphere, on the line to the original charge, i.e. $B(a^2/r, 0, 0)$ (source: [2]).

Therefore, $\triangle ACP \sim \triangle PCB$ and we have

$$\frac{q_1}{q_2} = -\frac{AP}{BP} = -\frac{CP}{CB} = -\frac{r}{a}.$$

Thus, the image charge $q_2 = -\frac{r}{a}q$.

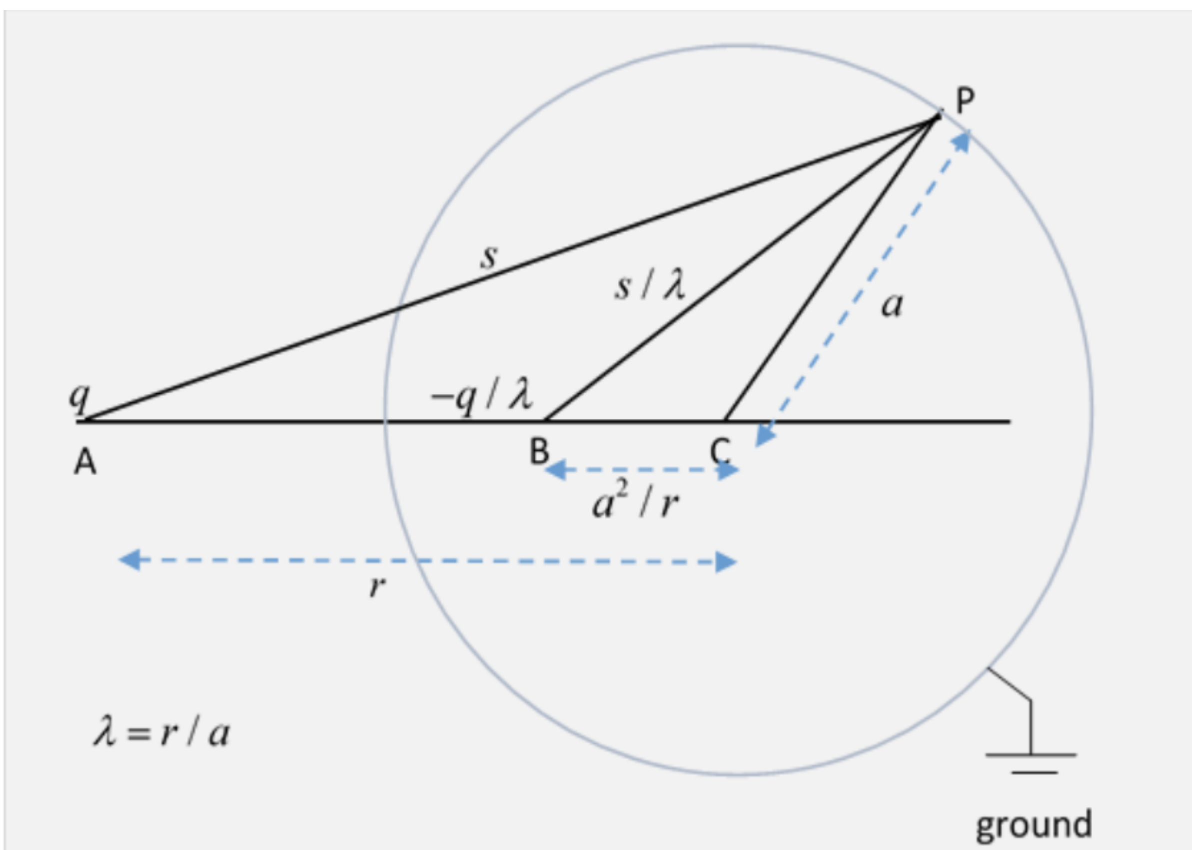


Figure 4.2

Below is a plot of the equivalent setup, with the image charge in blue located at $(a^2/r, 0, 0)$ in place of the induced charge on the surface of the sphere. For visualization purposes, we set the radius of the sphere $a = 1$ and the given charge a distance $r = 2$ from the center.

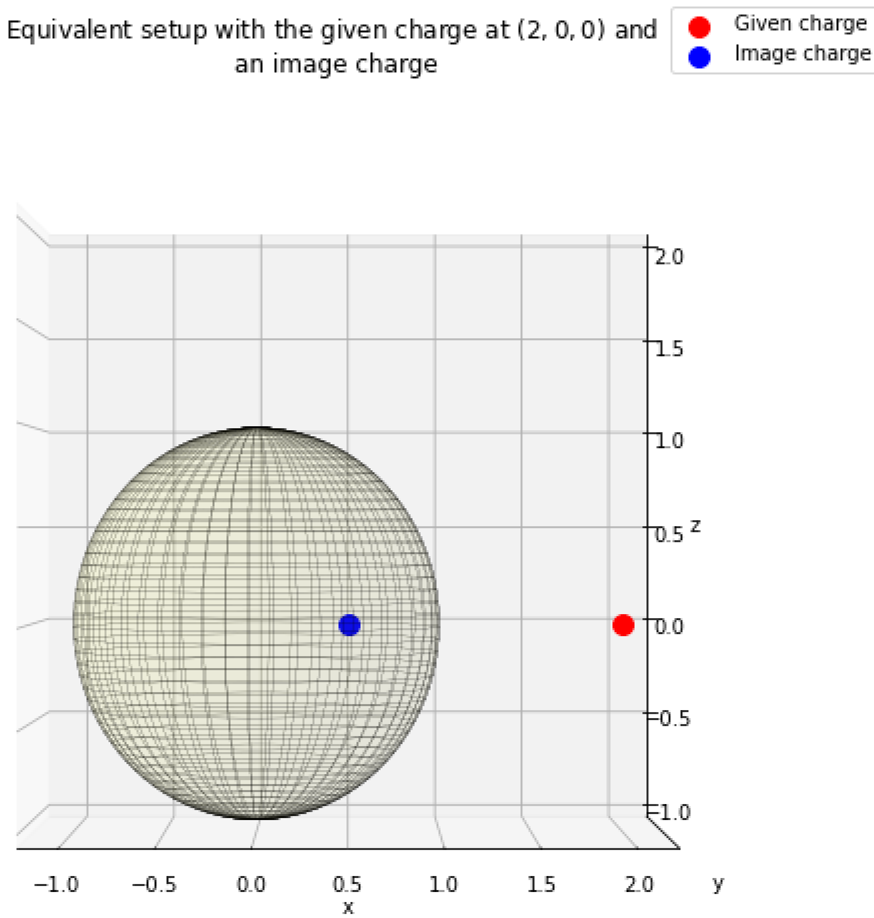


Figure 4.3

We are now ready to write the potential at any point outside the spherical conductor:

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{(x-r)^2 + y^2 + z^2}} - \frac{\frac{r}{a}q}{\sqrt{(x-(r^2/r))^2 + y^2 + z^2}}\right).$$

Once again, we take the gradient of V and get the (negative of) electric field. For simplicity, the graph below visualizes the potential and electric field outside of the sphere in the xz plane ($y = 0$). The inside of the sphere is black because it is an equipotential and should have zero potential as the grounded surface ($E = 0$ inside the conductor).

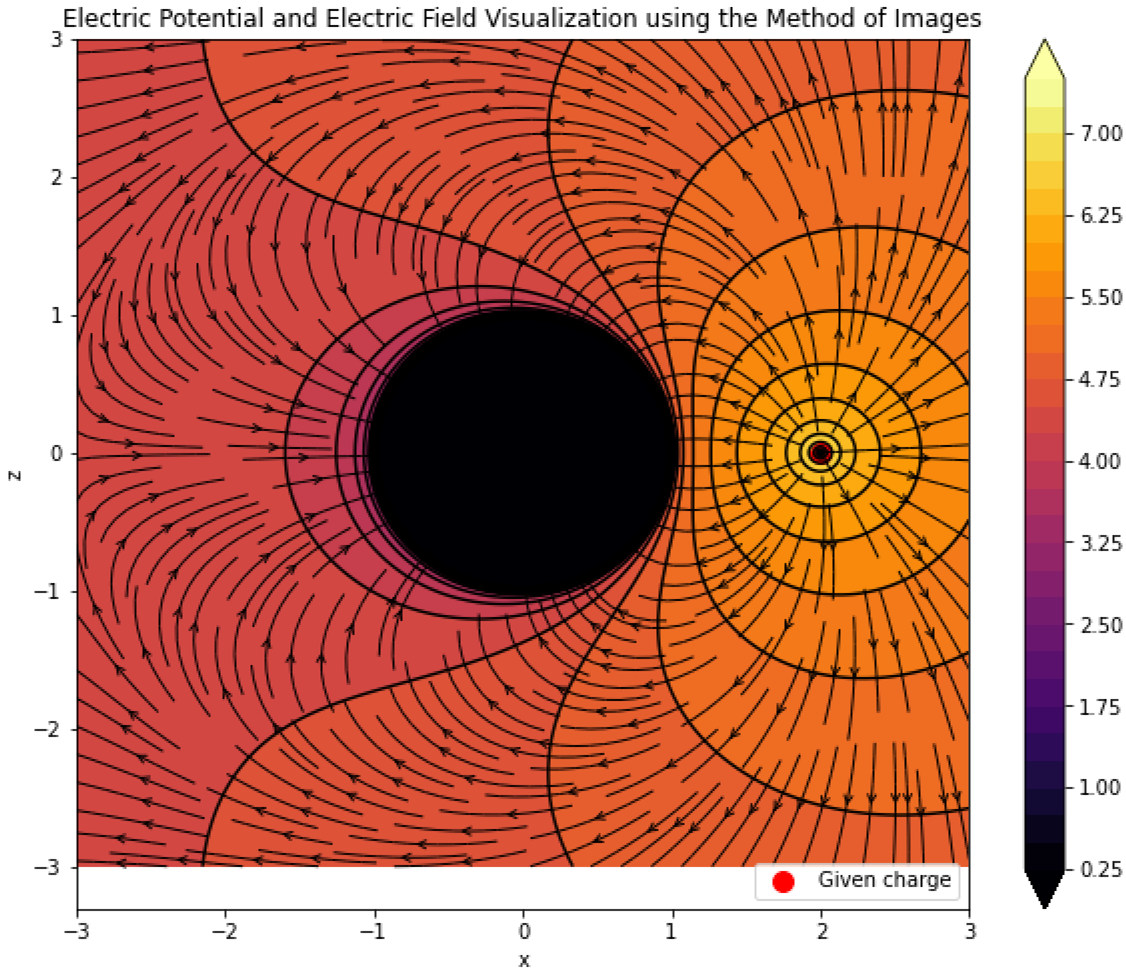


Figure 4.4

To calculate the density distribution of the induced charged at the surface, we are only interested in the radial component of the electric field at the surface because of electrostatics (the reasoning is explained in details in Example 1). Using the triangles in Figure 4.2, we can see that:

$$\overrightarrow{E_{P,q}} = \overrightarrow{E_{AC}} + \overrightarrow{E_{CP}},$$
$$\overrightarrow{E_{P,image}} = \overrightarrow{E_{BC}} + \overrightarrow{E_{CP}}.$$

Hence,

$$\overrightarrow{E_{\perp}} = \overrightarrow{E_{q,CP}} + \overrightarrow{E_{image,CP}} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{s^2}\frac{a}{s} - \frac{aq/r}{(as/r)^2}\frac{a}{as/r}\right),$$

where s and as/r are the distances from a point on the surface of the spherical conductor to the given charge and the image charge respectively. The dencity induced charge distribution is roughly visualized below, with yellow being high density and purple low density.

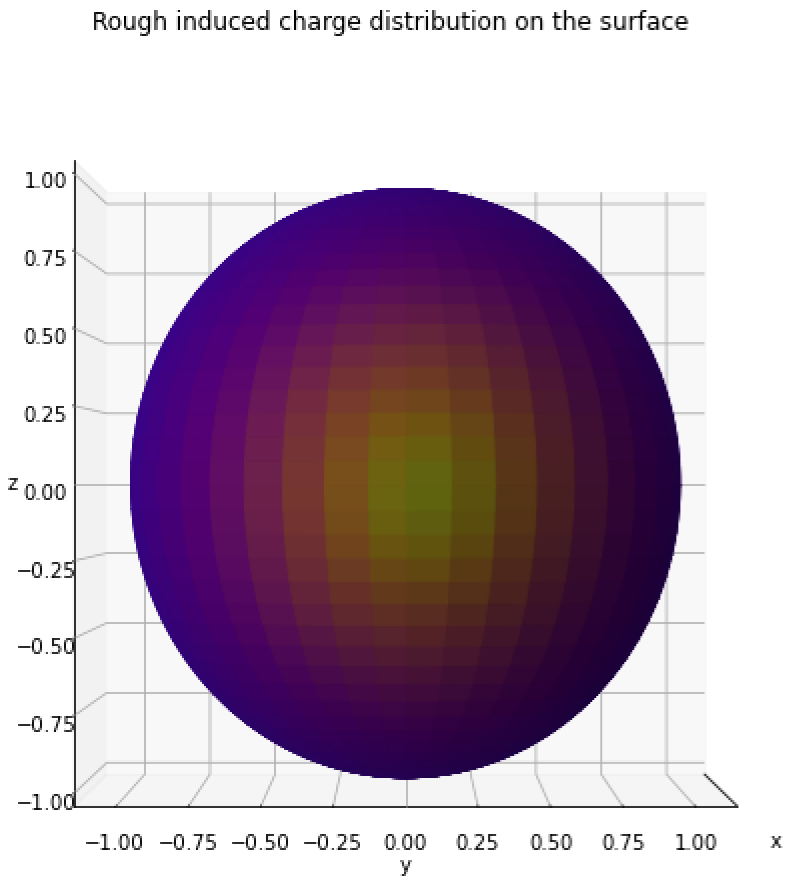


Figure 4.5

Example 5: point charge inside a grounded conducting sphere

In the last example, we consider the scenario where the given charge $q1 = q'$ is inside the conducting sphere at $(d', 0, 0)$. Hence, the area we're interested in is the region enclosed by the conducting sphere. The Poisson's equation and the boundary conditions are exactly the same as Example 4.

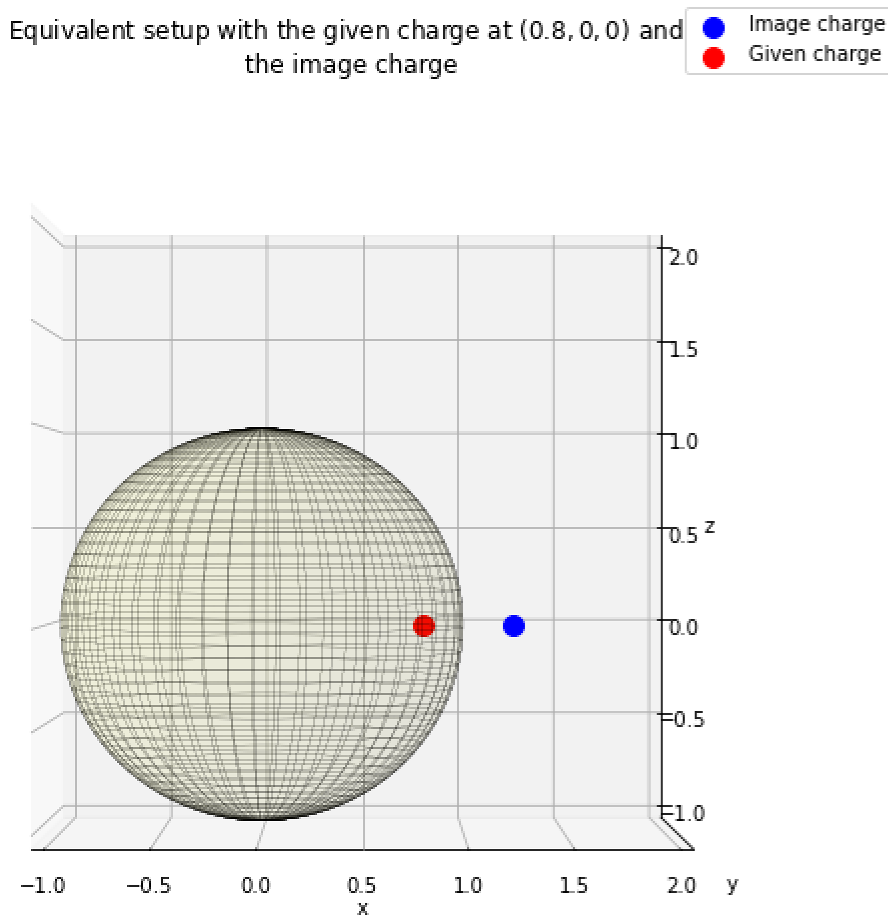
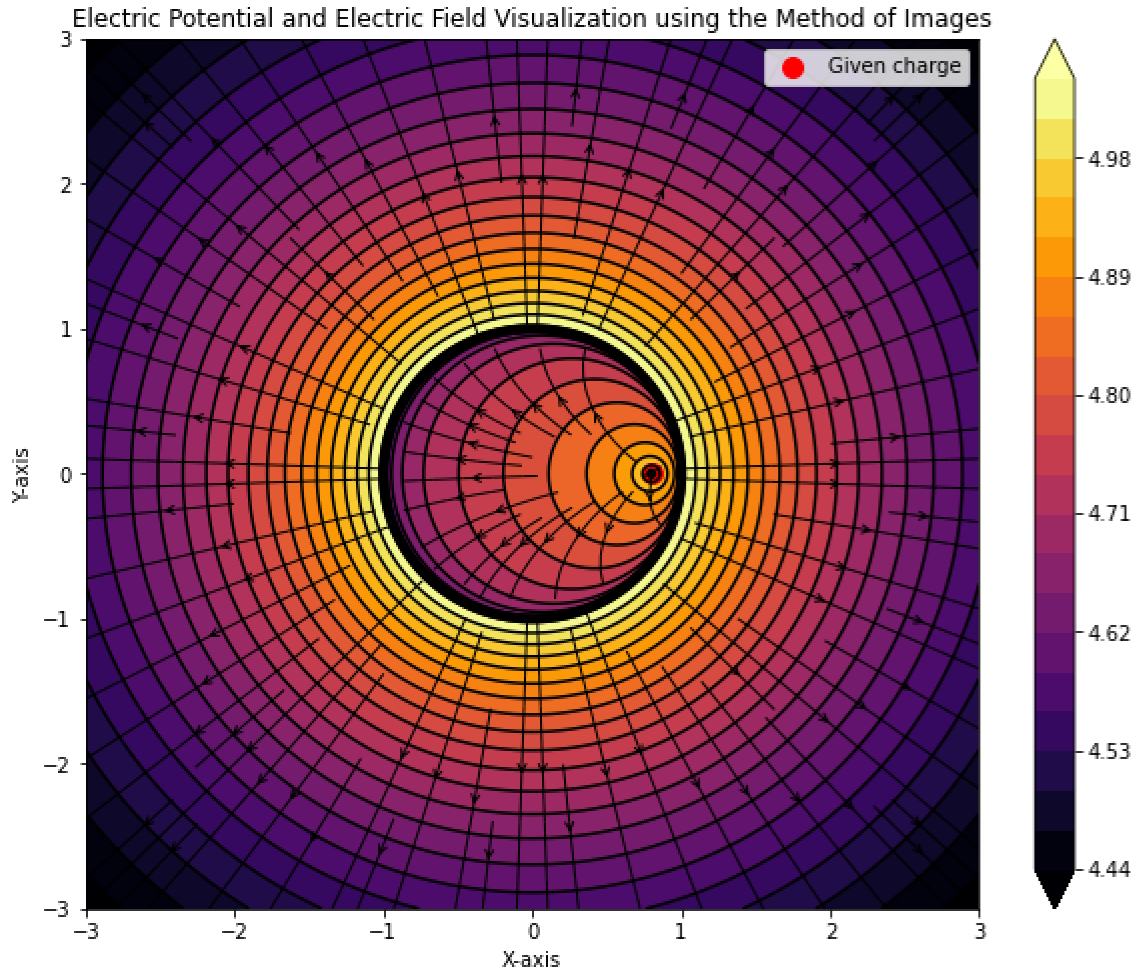


Figure 5.1

Since this is the exact same problem as Example 4, the image charge should take the place of the given charge in the last problem, located outside of the sphere on the line to the original charge and have charge $q_2 = \frac{a}{d'}q$. We can do a change of variable and quickly write down the potential inside the sphere:

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q'}{\sqrt{(x-r')^2+y^2+z^2}} - \frac{\frac{r'}{a}q'}{\sqrt{(x-(a^2/r'))^2+y^2+z^2}}\right).$$

We can get the electric field inside the sphere by taking the gradient of the potential. However, the above potential does not give any information about the field outside of the grounded conductor. Since the sphere is grounded, it has neutral charge all the time. By constructing a Gaussian surface in the region outside of the conductor, we find that The electric field outside the conducting sphere is that generated by a electric field due to the given charge at the center. The potential and electric field lines are shown int eh graph below.



Conclusion

As we have shown, the method of images can be used to solve the potential of a handful of systems with high symmetry, although its application is limited. When equivalent image charges can be found, we can usually easily write down the potential in terms of the discrete given and image charges and use it to find the electric field and induced charge distribution in the original problems. When the method works, it simplifies our work by giving the solution without us having to solve the complicated Poisson's equation.

Reference

[1] Griffith, David J. *Introduction to Electrodynamics* 4th edition.

[2] Fowler, Micheal. "8. The Image Method in Electrostatics". https://galileoandeinstein.phys.virginia.edu/Elec_Mag/2022_Lectures/EM_08_Images.html (https://galileoandeinstein.phys.virginia.edu/Elec_Mag/2022_Lectures/EM_08_Images.html)

[3] Wikipedia. "Circle of Apollonius". https://en.wikipedia.org/wiki/Circles_of_Apollonius (https://en.wikipedia.org/wiki/Circles_of_Apollonius)