Probability and Mathematical Statistics

概率与数理统计

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Course Contents

- Probability
  - Fundamentals
  - Discrete random variables
  - Continuous random variables
  - Further topics
  - The law of large numbers and the central limit theorem

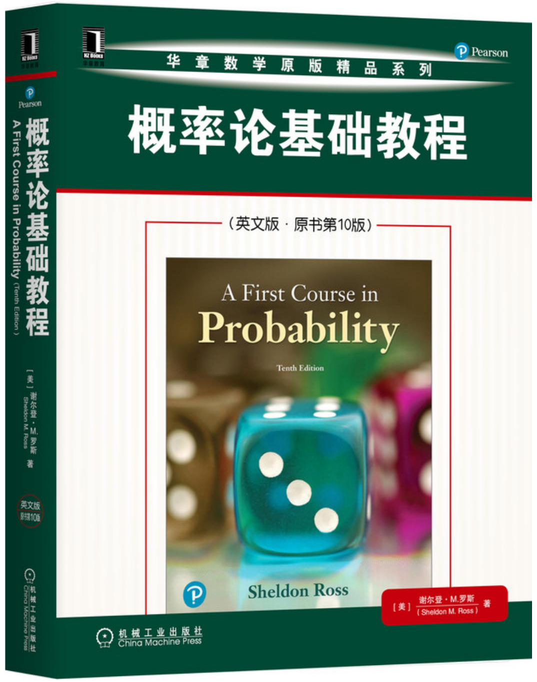
- Mathematical Statistics
  - Basic concepts
  - Parameter estimation
  - Hypothesis testing
  - Linear regression analysis

Prerequisites

- Univariate and multivariate calculus
  - Basic set theory
  - Limits, continuous functions, derivatives, Taylor expansion, Riemann integrals, Newton–Leibniz theorem, improper integrals
  - Partial derivatives, multiple and iterated integrals
  - Series, absolute convergence, rearrangements, power series
- Linear algebra
  - Vectors, matrices, matrix inverse, determinants
  - Eigenvalue decomposition, real symmetric matrices, positive (semi)definite matrices

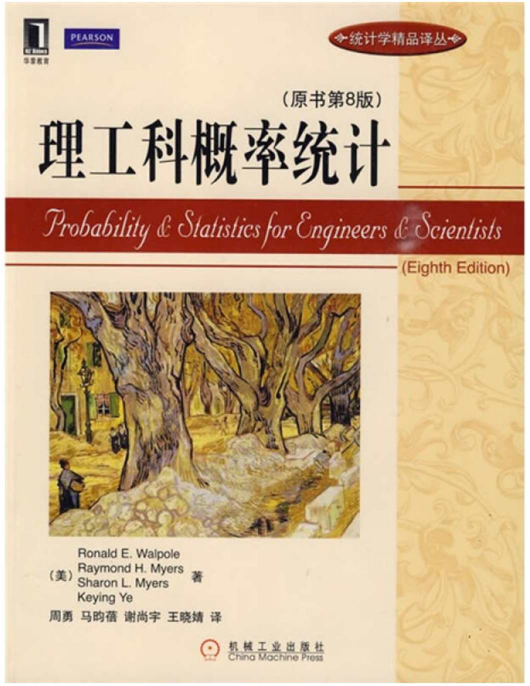
Course Contents

- No official textbook. Some recommented books:



Sheldon Ross

Dimitri P. Bertsekas,
John N. Tsitsiklis

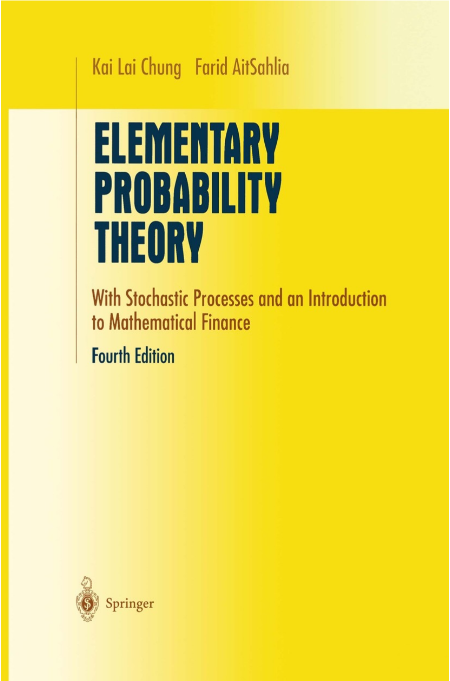


R. E. Walpole, R. H. Myers,
S. L. Myers, K. Ye

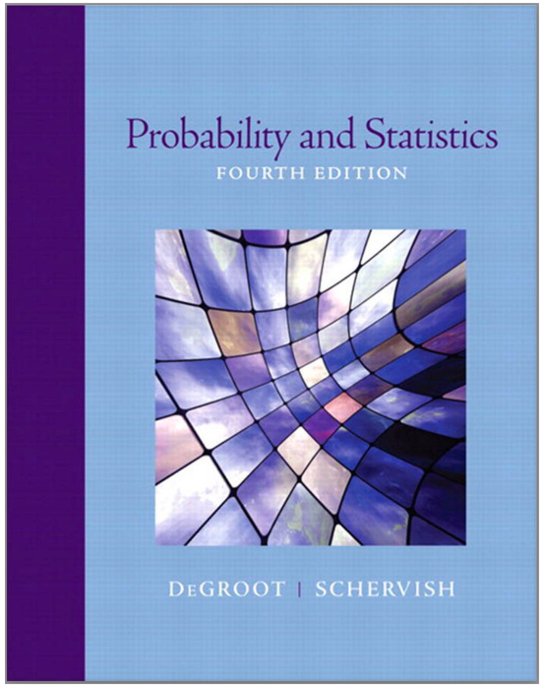
浙江大学
盛骤 谢式千 潘承毅

Course Contents

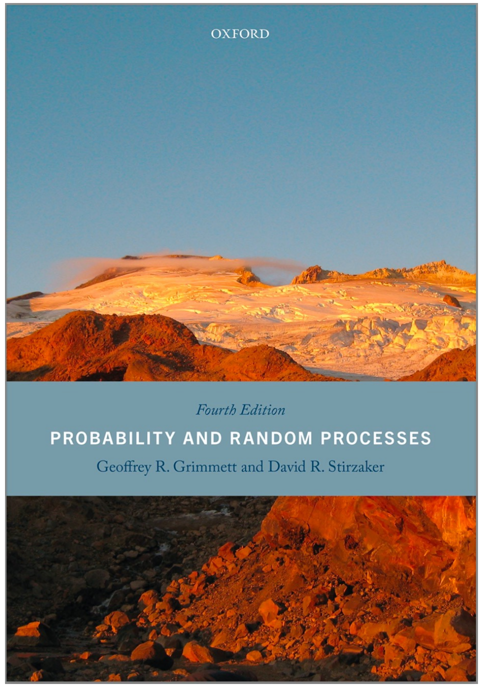
- Some other excellent books that are not in Chinese:



Kai Lai Chung, Farid AitSahlia



Morris H. DeGroot, Mark J. Schervish



Geoffrey Grimmett, David R. Stirzaker

- ❑ You don't need to read a whole book.
- ❑ Read them when you have confusions and want to see how others explain those concepts.
- ❑ Do the exercises.

Course Contents

- The slides will be mostly <u>in English</u>.
- The slides **DO NOT** include everything taught in the lectures.
- You are encouraged to <u>take your own notes</u>.
- “Official” lecture notes (in Chinese) will be provided on <u>course.pku.edu.cn</u> **AFTER** each chapter is completed.
  - My lectures notes are for you to <u>review the course materials</u>.
  - We strongly recommend that you <u>finish all the exercises in my lecture notes</u> (some of them are not easy), although you don’t need to submit your solutions.

Grading Policy

- Assignments 25%
  - Submitted by a studying group of up to **2 students**. The group will receive the same grade on each submission.
  - About 5 assignments in total.
  - Late submission will be penalized by a 50% reduction in the grade.
- Course project 15%
  - Carried out in groups. Each group consists of up to **4 students**.
  - Students in the same group will receive the same grade.
  - Will be assigned after the midterm exam.
  - Late submission will be penalized by a 50% reduction in the grade.

Grading Policy

- Midterm exam 20%
  - In class. Date to be determined (probably early November).
- Final exam 40%
  - Dec. 31^{st} afternoon
  - Problems will be similar to or even a little bit harder than assignments and examples in the handouts
- We will **NOT** 划重点 for the exams.
- Grading will be changed only when an error has been made;
  no negotiation is allowed.

- Task 1:
  - Form your homework team first.
  - Send your names and student ID numbers of homework team to our TA 王旭浩 by Sep. 20th.

Chapter 1. Fundamentals of Probability Theory

Randomness and Probability

- Phenomena with uncertainties
  - Their outcomes cannot be predicted accurately beforehand.
- But some of them seem to exhibit certain statistical regularity (统计规律) when they are repeated under roughly the same conditions for a large number of times.

Randomness and Probability

- Example: Brownian motion of a pollen particle immersed in water

Randomness and Probability

- Random experiments 随机试验: An idealized procedure
  - that can be repeated under (approximately) the same conditions;
  - whose outcomes exhibit statistical regularity

- Probability theory: How to model a random experiment mathematically?

Randomness and Probability

- **Sample space 样本空间**: The set of all possible outcomes
- **Events 事件**: An event is a **subset** of the sample space
- **Probability 概率**: Every event is assigned a number (probability) to quantify how likely this event is to occur.

Ideally: $$\boxed{\mathbb{P}(E)} = \lim_{n\to\infty} \boxed{\mathrm{Fr}_n(E)}$$

Probability of the event E

Frequency of the occurrence of E in n repeated experiments

1.1. Review of Set Theory (集合论)

Review of Set Theory (集合论)

- Given a set A and an object x, we can always talk about whether x is a member of A or not

$x \in A$: x is a member/element of A

- Two sets A and B are equal if and only if any member of A is also a member of B and vice versa.

$$A = B \quad \text{iff} \quad \forall x (x \in A \Leftrightarrow x \in B)$$

Review of Set Theory (集合论)

- Some commonly seen sets:
  - The empty set $\varnothing$
  - The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
  - The set of positive integers $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$
  - The set of integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$
  - The set of rational numbers $\mathbb{Q}$
  - The set of real numbers $\mathbb{R}$

Review of Set Theory (集合论)

- $A \subseteq B$: Any member of A is also a member of B.
- Power set 幂集 2^A : The set/family/collection of all subsets of A.
- $\{x \mid \varphi(x)\}$: The set containing all objects x satisfying the property $\varphi(x)$
  - $\{x \in A \mid \varphi(x)\}$: The subset of A containing all objects x satisfying $\varphi(x)$
  - Intervals on the real line:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

In my lecture notes, an open interval is denoted by $]a, b[$ so that you won't confuse it with the order pair

Countable and Uncountable Sets

- Countably infinite set 可数无限集: A set whose members can be <u>exhaustively enumerated</u> by a sequence.
  - Examples: $\mathbb{N},\ \mathbb{Z},\ \mathbb{Q}$
- Countable set 可数集: Finite or countably infinite
- Uncountable set 不可数集: A set that is not countable
  - Examples: $\mathbb{R}$ and its open intervals

Countable and Uncountable Sets

- Subsets of a countable set are countable.
- Given a countable set S and a mapping f defined on S, the **image** (像集/值域)

$$\operatorname{Im} f = \{f(s) \mid s \in S\}$$

is a countable set.

Set Operations

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- Commutative and associative laws 交换律与结合律:

$$A \cup B = B \cup A, \qquad (A \cup B) \cup C = A \cup (B \cup C)$$

$$A \cap B = B \cap A, \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

- Venn diagram

Set Operations

- A more complicated situation: $\mathcal{I}$ is any nonempty set, and for each $i \in \mathcal{I}$ there is one corresponding set A_i assigned to i.
  - An indexed family of sets; i is the index (指标) of A_i
- Union of an indexed family of sets:

$$\bigcup_{i \in \mathcal{I}} A_i = \{x \mid \text{there exists some } i \in \mathcal{I} \text{ such that } x \in A_i\}$$

$$= \{x \mid x \text{ belongs to at least one } A_i\}$$

Set Operations

- A more complicated situation: $\mathcal{I}$ is any nonempty set, and for each $i \in \mathcal{I}$ there is one corresponding set A_i assigned to i.
  - An indexed family of sets; i is the index (指标) of A_i

- Intersection of an indexed family of sets:

$$\bigcap_{i \in \mathcal{I}} A_i = \{x \mid \text{for all } i \in \mathcal{I} \text{ we have } x \in A_i\}$$
$$= \{x \mid x \text{ belongs to all } A_i\}$$

Set Operations

- Some examples:

$$\bigcup_{i=1}^{\infty}\left[0,\frac{i}{i+1}\right]=\bigcup_{i\in\mathbb{N}_+}\left[0,\frac{i}{i+1}\right]=\ ?$$

$$\bigcap_{i=1}^{\infty}\left(0,\frac{1}{i}\right)=\bigcap_{i\in\mathbb{N}_+}\left(0,\frac{1}{i}\right)=\ ?$$

Set Operations

- When $\mathcal{I}$ and each A_i , $i \in \mathcal{I}$ are all countable, the union and the intersection

$$\bigcup_{i \in \mathcal{I}} A_i \qquad \bigcap_{i \in \mathcal{I}} A_i$$

are countable.

Set Operations

- Distributive laws 分配律:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cap \left(\bigcup_{i \in \mathcal{I}} B_i \right) = \bigcup_{i \in \mathcal{I}} (A \cap B_i)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad A \cup \left(\bigcap_{i \in \mathcal{I}} B_i \right) = \bigcap_{i \in \mathcal{I}} (A \cup B_i)$$

Set Operations

- Difference of sets 差集: $A \backslash B = \{x \in A \mid x \notin B\}$
  - Complement 补集 B^c : When $B \subseteq A$ and A is clear from the context.

- De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

$$\left(\bigcup_{i \in \mathcal{I}} A_i \right)^c = \bigcap_{i \in \mathcal{I}} A_i^c \qquad \left(\bigcap_{i \in \mathcal{I}} A_i \right)^c = \bigcup_{i \in \mathcal{I}} A_i^c$$

Set Operations

- Cartesian product 笛卡尔积: $A \times B = \{\underbrace{(a, b)}_{\text{Ordered pair}} \mid a \in A, b \in B\}$

- The Cartesian product of two countable sets is countable.

1.2. Basics Notions of Probability Theory

Sample Space, Events and Probability

- Suppose we want to model a random experiment.
- **Sample space 样本空间**: The set containing <u>all possible outcomes</u> of this random experiment.
- In our lectures, we usually use Ω to denote the sample space, and use ω to denote a possible outcome.
- Some quick examples:
  - Throwing a dice: $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - Flipping two coins: $\Omega = \{\underbrace{(\mathrm{H}, \mathrm{H})}_{\text{Ordered pair}}, (\mathrm{H}, \mathrm{T}), (\mathrm{T}, \mathrm{H}), (\mathrm{T}, \mathrm{T})\}$

Sample Space, Events and Probability

- **Event 事件**: An event E is a subset of the sample space.
- We say that the event E occurred in one trial of the random experiment, if the outcome ω of this trial is a member of E.

- **Probability measure 概率测度**: A mapping/function $\mathbb{P}$ that assigns each event E with a real number $\mathbb{P}(E)$.
  - Probability measures have to satisfy certain properties.
    We'll talk about it later.

Quick Examples of Sample Spaces and Events

- Throwing a dice: $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - The outcome is four: $\{4\}$
  - The outcome is an even number: $\{2, 4, 6\}$
  - The outcome is a prime number: $\{2, 3, 5\}$
  - The outcome is less than seven: $\{1, 2, 3, 4, 5, 6\} = \Omega$ Certain event 必然事件
  - The outcome is not an integer: $\varnothing$ Impossible event 不可能事件

Quick Examples of Sample Spaces and Events

- Flipping three coins: $\Omega = \{(\omega_1, \omega_2, \omega_3) \mid \omega_i \in \{\mathrm{H}, \mathrm{T}\},\ i = 1, 2, 3\}$
  - The first coin lands on heads:
  - The first and the second coins turn out the same:
  - There is at most one tail:
  - At least two heads appear consecutively:
  - Four heads appear:

How many elements are there in the samples space?

How many events are there in total?

More Examples

- Generating a random number in $[0,1]$: Just set $\Omega = [0,1]$
  - The random number falls in the interval I: $I \cap [0,1]$

- Flipping a coin for (countably) infinitely many times:

$$\Omega = \{(\omega_i)_{i=1}^{\infty} \mid \omega_i \in \{\mathrm{H}, \mathrm{T}\}, \forall i \in \mathbb{N}_+\}$$

  - The frequency of heads converges to $1/2$:

$$\left\{(\omega_i)_{i=1}^{\infty} \in \Omega \,\middle|\, \lim_{n \to \infty} \frac{q_n(\omega_1, \ldots, \omega_n)}{n} = \frac{1}{2}\right\}$$

$q_n(\omega_1, \ldots, \omega_n)$ counts the number of heads in $\omega_1, \ldots, \omega_n$

Relations and Operations of Events

Events are sets, and so set relations and operations apply to them.

- $E \subseteq F$ holds if and only if E occurs implies F occurs.

$$\begin{aligned} E \subseteq F \quad &\iff \quad \text{for all } \omega \in E, \text{ we have } \omega \in F \\ &\iff \quad \text{whenever the outcome } \omega \text{ is in } E, \text{ we have } \omega \in F \\ &\iff \quad \text{whenever } E \text{ occurs, we have that } F \text{ occurs} \end{aligned}$$

- Complement of an event: $E^c = \Omega \backslash E$ occurs if and only if E does not occur.
  - E and E^c are called 对立事件.

Relations and Operations of Events

Events are sets, and so set relations and operations apply to them.

- Union of events: $E \cup F$ occurs if and only if E or F (or both) occurs.

- Intersection of events: $E \cap F$ occurs if and only if both E and F occur.
  - E and F are called disjoint or mutually exclusive 互斥 if $E \cap F = \varnothing$.
  - Disjoint events do not occur simultaneously in one trial.

Relations and Operations of Events

Events are sets, and so set relations and operations apply to them.

- More complicated situations: Given a sequence of events $E_1,\ E_2,\ E_3,\ldots$

$$\bigcup_{i=1}^{\infty} E_i = \{\omega \in \Omega \mid \omega \text{ belongs to at least one } E_i\}$$ At least one E_i occurs

$$\bigcap_{i=1}^{\infty} E_i = \{\omega \in \Omega \mid \omega \text{ belongs to all } E_i\}$$ $E_1,\ E_2,\ E_3,\ldots$ all occur

Some Mathematical Intricacy

- Unfortunately, due to some mathematical intricacy (which is well beyond the scope of this course), sometimes we **CANNOT** assign probabilities to **ALL** subsets of the sample space.
- We allow the collection of events to be a **smaller** subset of the power set 2^{Ω}, but still requires it to satisfy certain properties for theoretical development.

Event Space: Rigorous Definition

Definition. A collection $\mathcal{F}$ of subsets of Ω is called an **event space** 事件域, if

- The sample space Ω is in $\mathcal{F}$.
- For each $E \in \mathcal{F}$, the complement E^c is in $\mathcal{F}$.
- For any sequence $E_1, E_2, E_3, \ldots$ in $\mathcal{F}$, the union $\bigcup_{i=1}^{\infty} E_i$ is in $\mathcal{F}$.

* Uncountable unions of members in $\mathcal{F}$ may go out of $\mathcal{F}$.

Probability Measure: Rigorous Definition

Definition. Given a sample space Ω and an event space $\mathcal{F}$ of Ω, a mapping $\mathbb{P}: \mathcal{F} \to \mathbb{R}$ is said to be a **probability measure 概率测度** on $\mathcal{F}$, if

- $0 \leq \mathbb{P}(E) \leq 1$ for any E in the event space $\mathcal{F}$.
- $\mathbb{P}(\Omega) = 1$.
- Countable additivity 可数可加性: For any sequence of mutually exclusive sets $E_1, E_2, E_3, \ldots$ in the event space $\mathcal{F}$, we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Probability Space

- The ordered triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space 概率空间** if
  - $\mathcal{F}$ is an event space of Ω.
  - $\mathbb{P}$ is a probability measure on $\mathcal{F}$.
- Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we call members of $\mathcal{F}$ **events**, and $\mathbb{P}(E)$ the **probability** of E for any event E.

- In probability theory, we model a random experiment by a probability space.

Problems Studied by Probability Theory

- Given a random experiment, suppose we know **i)** its sample space, and **ii)** the probabilities of some simple events (or their relations).
  - How to calculate <u>the probabilities of other events</u> we are interested in?
  - How to <u>update/correct the probabilities</u> if we <u>obtain some partial information</u> on the outcome of the random experiment?

- Problems not entirely within the scope of probability theory:
  - How to find those probabilities of simple events (or their relations) from the <u>underlying physics</u> or from <u>data</u>?
  - We need tools from **mathematical statistics**.

Some Remarks on Event Spaces

- When the sample space Ω is countable, we always choose the event space to be the power set 2^{Ω}.
- When the sample space is uncountable, things are complicated…
    - Out of the scope of this course.
    - Only theorists focus on this issue. You don’t need to worry about it.
- In this course, we’ll always assume an appropriate event space $\mathcal{F}$ is given after the sample space is specified.

Properties of Probability

- The sample space (必然事件) has probability one (by definition).
- The empty set (不可能事件) has probability zero.

- An event with probability one is not necessarily certain to occur.
  - Such events are called **almost sure** (几乎必然) events.
- An event with probability zero is not necessarily impossible.

Properties of Probability

- Countable additivity implies **finite additivity**:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i) \quad \text{if } E_1, \ldots, E_n \text{ are mutually exclusive}$$

- $\mathbb{P}(E^c \cap F) = \mathbb{P}(F) - \mathbb{P}(E \cap F)$. Particularly, $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$
- $\mathbb{P}(E) \leq \mathbb{P}(F)$ whenever $E \subseteq F$.
- Use Venn diagram to develop their intuitions.

Properties of Probability

- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$
  - E and F are arbitrary.
  - Quite intuitive if we look at the Venn diagram.
  - Generalization to finitely many events:

$$\begin{aligned}\mathbb{P}(E_1 \cup \cdots \cup E_n) \\ = \sum_{i=1}^{n} \mathbb{P}(E_i) - \sum_{i_1 < i_2} \mathbb{P}(E_{i_1} \cap E_{i_2}) + \cdots \\ + (-1)^{k-1} \sum_{i_1 < \cdots < i_k} \mathbb{P}(E_{i_1} \cap \cdots \cap E_{i_k}) + \cdots \\ + (-1)^{n-1} \mathbb{P}(E_1 \cap \cdots \cap E_n)\end{aligned}$$

We won’t prove it now. You’ll learn an easier way to prove it after you learn expectation (期望).

Properties of Probability

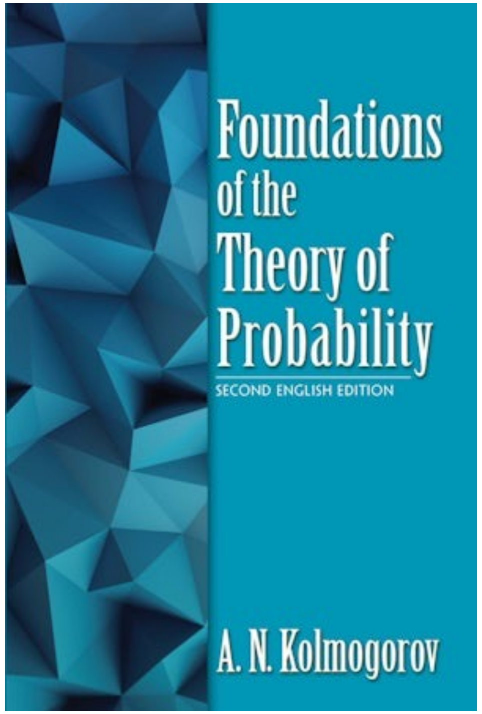
- Boole's inequality:

$$\mathbb{P}(E_1 \cup \cdots \cup E_n) \leq \mathbb{P}(E_1) + \cdots + \mathbb{P}(E_n)$$

  - The proof using mathematical induction in this case is quite clean.

Remarks on the Definition of Probability

- The definition of probability seems a bit indirect.
  - They do not tell you what probabilities are. They only tell you how probabilities should behave.
  - Some people call it the axiomatic definition of probability (概率的公理化定义).
  - First summarized and proposed by Kolmogorov (柯尔莫哥洛夫) in his 1933 book.
  - The foundation of modern probability theory.



Remarks on the Definition of Probability

- Why not define probability directly as the limit of frequency?
  - It’s going to be complicated…
  - We’ll face infintely repeated trials of random experiments from the start.
  - We’ll see that the repeated trials of random experiments need to be **independent**, which is a concept we haven’t introduced yet.
- In modern probability theory, it becomes a **theorem** that frequency converges to probability (almost surely).
- Modern probability theory can even apply to problems not related to random experiments with statistical regularity.

1.3. Classical Probability (古典概型)

Classical Probability (古典概型)

- You have learned it in high school.
  - There is really not much more to say in this course.
  - Some problems/quizzes of classical probability are very tricky, though.
- How does classical probability fit in modern probability theory?

Classical Probability (古典概型)

- For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be a classical probability model:
  - The sample space Ω should be a finite set.
  - The event space should be the power set 2^{Ω}.
  - The probability measure should satisfy

$$\mathbb{P}(\{\omega\}) = \frac{1}{|\Omega|}, \quad \forall \omega \in \Omega \qquad \Longrightarrow \qquad \mathbb{P}(E) = \frac{|E|}{|\Omega|}, \quad \forall E \subseteq \Omega.$$

All outcomes are equally probable.

Just count!

Overview of Basic Principles of Counting

- 乘法原理
- 加法原理

- 排列数 $\dfrac{n!}{(n-k)!}$
- 组合数 binomial coefficient $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$
- 二项式定理 binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Overview of Basic Principles of Counting

- 多项式系数 multinomial coefficient: Suppose $n_1 + n_2 + \cdots + n_r = n$

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

  - The number of ways to partition a set A with n elements into r subsets $A_1, A_2, \ldots, A_r$, where A_i has n_i elements.
  - The number of permutations of r groups of objects, where the ith group consists of n_i indistinguishable objects.

Overview of Basic Principles of Counting

- 多项式定理 multinomial theorem

$$(x_1 + \cdots + x_r)^n = \sum_{\substack{n_1 + \cdots + n_r = n \\ n_1, \ldots, n_r \in \mathbb{N}}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

- We recommend that you do some exercises in Chapter 1 of *A First Course in Probability* by Sheldon Ross, to get acquainted with basic techniques of counting.

The Banach Matchbox Problem

- A mathematician carries 2 matchboxes, 1 in left pocket and 1 in right pocket. Each time he needs a match, he is equally likely to pick a box from either pocket. Consider the moment when the mathematician first discovers that the box picked is already empty. Assuming that both matchboxes initially contained N matches, what is the probability that there are exactly k matches ($0 \leq k \leq N$) in the other box?

1.4. Conditional Probability (条件概率)

Conditional Probability

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
- Assume we observe that some event F has occurred. How should we update the probability measure?

Conditional Probability

Definition. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let F be an event satisfying $\mathbb{P}(F) > 0$. For any event E, The conditional probability of E given F is defined as

$$\mathbb{P}(E \,|\, F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

- If $\mathbb{P}(F) = 0$, the conditional probability $\mathbb{P}(E \,|\, F)$ is **UNDEFINED**.

Conditional Probability

- Example: Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a classical probability model. Let F be an event that is not empty.
- Then for any event E, we have

$$\mathbb{P}(E \,|\, F) = \frac{|E \cap F|}{|F|} = \sum_{\omega \in E \cap F} \frac{1}{|F|}$$

- Outcomes in F are equally probable.
- Outcomes not in F have probability zero.

Conditional Probability as a Probability Measure

Proposition. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let F be an event satisfying $\mathbb{P}(F) > 0$. Then the mapping

$$E \in \mathcal{F} \mapsto \mathbb{P}(E \,|\, F) \in \mathbb{R}$$

is a probability measure on $\mathcal{F}$ satisfying

- $\mathbb{P}(F \,|\, F) = 1$.
- $\dfrac{\mathbb{P}(E_1 \,|\, F)}{\mathbb{P}(E_2 \,|\, F)} = \dfrac{\mathbb{P}(E_1)}{\mathbb{P}(E_2)}$ whenver E_1, $E_2 \subseteq F$ and $\mathbb{P}(E_2) > 0$.

We can adopt $\mathbb{P}(\,\cdot \,|\, F)$ as the updated probability measure after we observe F has occurred.

Multiplication Rule 乘法公式

- We have

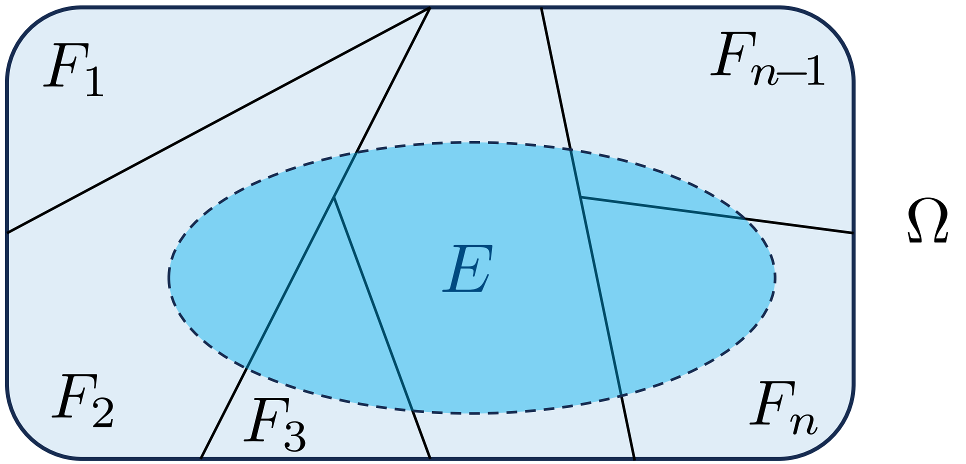
$$\mathbb{P}(E_1 \cap \cdots \cap E_n) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2 \,|\, E_1) \cdot \mathbb{P}(E_3 \,|\, E_1 \cap E_2) \cdots \cdot \mathbb{P}(E_n \,|\, E_1 \cap \cdots \cap E_{n-1})$$

provided that $\mathbb{P}(E_1 \cap \cdots \cap E_{n-1}) > 0$.

Law of Total Probability 全概率公式

- Suppose the events $F_1, \ldots, F_n$ satisfy
  1. $F_1 \cup \cdots \cup F_n = \Omega$
  2. $F_i \cap F_j = \varnothing, \ \forall i \neq j$
  3. $\mathbb{P}(F_i) > 0, \ \forall i$
- Then for any event E,

$$\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \,|\, F_i)\, \mathbb{P}(F_i)$$

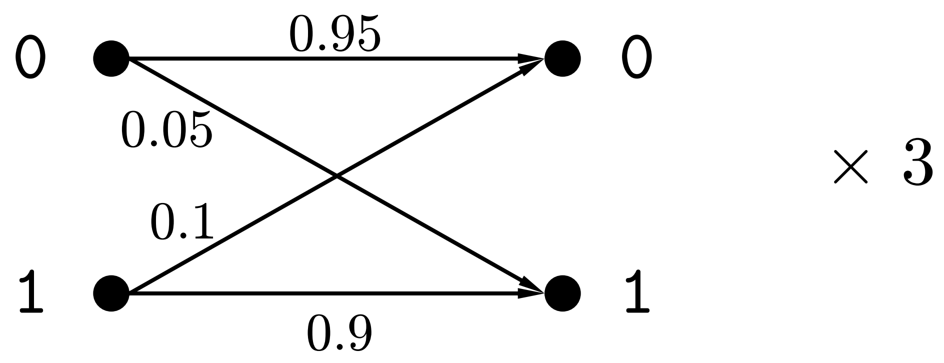


- A special case:

$$\mathbb{P}(E) = \mathbb{P}(E \,|\, F)\, \mathbb{P}(F) + \mathbb{P}(E \,|\, F^c)\, \mathbb{P}(F^c)$$

Exercise: Communication Through Noisy Channels

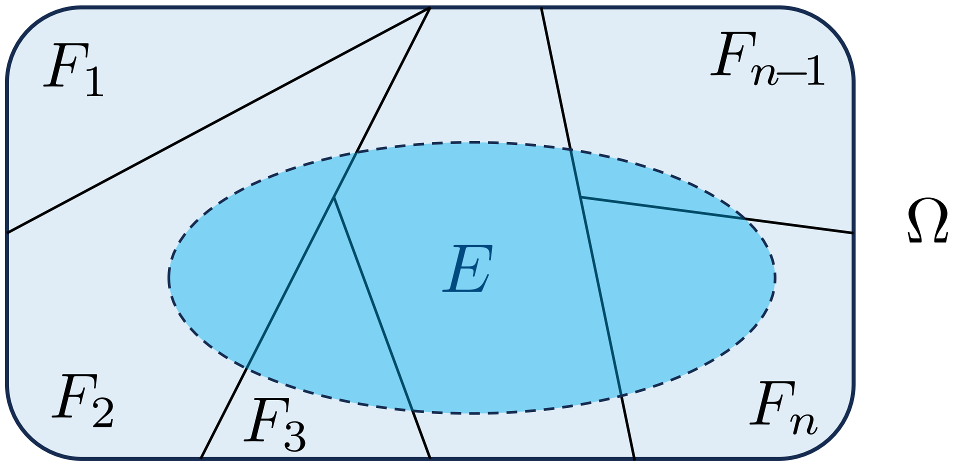
- A sensor sends data (a string of 0/1 symbols) to the server, and each symbol needs to be transmitted through three noisy channels sequentially.
- When 0 is transmitted through each channel, the error probability is 0.05. When 1 is transmitted through each channel, the error probability is 0.1.



- Suppose a symbol 0 is sent from the sensor. What is the probability that the server receives the symbol correctly?

Bayes' Theorem 贝叶斯定理

- Suppose the events $F_1, \ldots, F_n$ satisfy
  1. $F_1 \cup \cdots \cup F_n = \Omega$
  2. $F_i \cap F_j = \varnothing, \ \forall i \neq j$
  3. $\mathbb{P}(F_i) > 0, \ \forall i$
- Then for any i and any event E with a positive probability,



Posterior probability 后验概率

$$\boxed{\mathbb{P}(F_i \mid E)} = \frac{\mathbb{P}(E \mid F_i)\boxed{\mathbb{P}(F_i)}}{\sum_{j=1}^{n} \mathbb{P}(E \mid F_j)\boxed{\mathbb{P}(F_j)}}$$

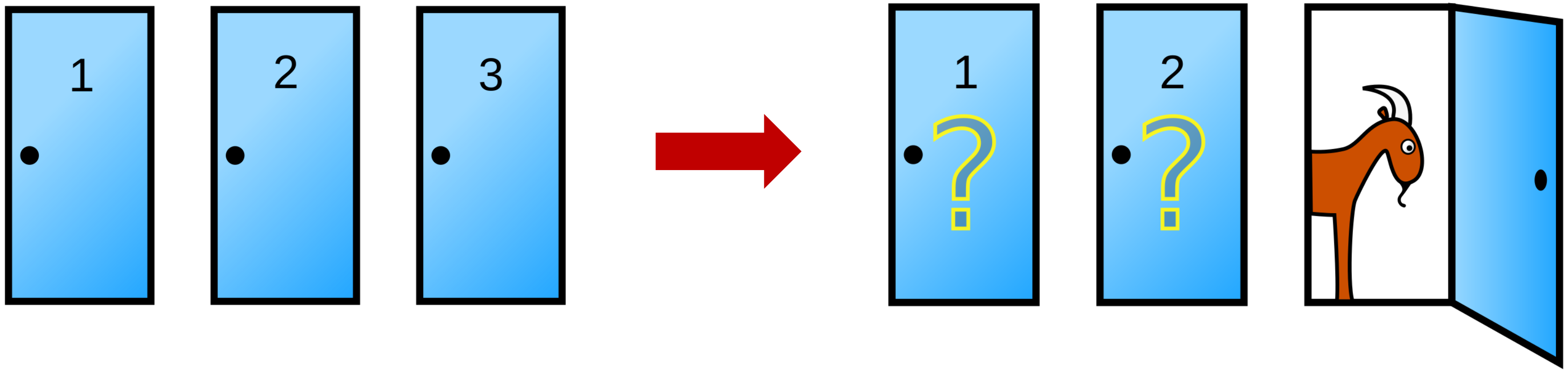
Prior probability 先验概率

Exercise: Diagnosis of A Rare Disease

- A rare disease has an incidence of 1 in 10^5 in the population at large. There is a diagnostic test, but it is imperfect. If a person has the disease, the test is positive with probability 99%; otherwise, the test is positive with probability 1%.
- Now suppose a person is randomly picked from the population, and the test result for this person turns out to be positive. What is the probability that this person has the disease?

Exercise: Monty Hall Problem

- Suppose you're on a television show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats.
- You pick a door, say No. 1.
- The host, who knows what's behind the doors, then opens another door that has a goat (say No. 3), and asks you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?



1.5. Independence of Events (事件的独立性)

Independence of Events

Definition. We say that two events E and F are **independent**, if

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F).$$

- If F has a positive probability, then

  E and F are independent $\quad \Longleftrightarrow \quad \mathbb{P}(E \,|\, F) = \mathbb{P}(E)$

- If E has a positive probability, then

  E and F are independent $\quad \Longleftrightarrow \quad \mathbb{P}(F \,|\, E) = \mathbb{P}(F)$

Knowing the occurrence of one does not affect the probability of the other.

Independence of Events

- An easy exercise: Let E and F be any events. The following statements are equivalent:

1) E and F are independent.
2) E and F^c are indepenent.
3) E^c and F are independent.
4) E^c and F^c are independent.

Independence of Multiple Events

- **Definition**. Let $\mathcal{I}$ be any nonempty set, and suppose for each $i \in \mathcal{I}$ there is one corresponding set E_i assigned to i. We say that this family of events $\{E_i,\ i \in \mathcal{I}\}$ are **mutually independent**, if for any <u>nonempty finite</u> subset $\mathcal{J} \subseteq \mathcal{I}$, we have

$$\mathbb{P}\left(\bigcap_{j \in \mathcal{J}} E_j\right) = \prod_{j \in \mathcal{J}} \mathbb{P}(E_j).$$

Independence of Multiple Events

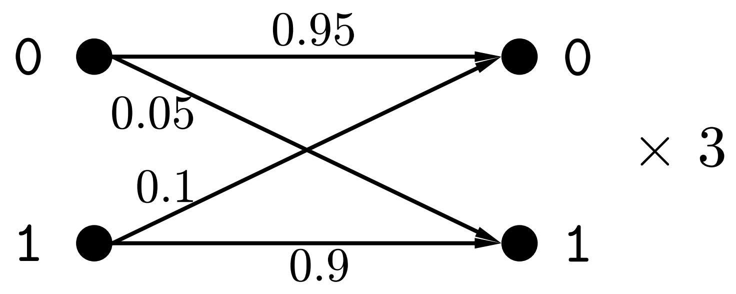
- Example: For three events $E,\ F$ and G, they are mutually indepenenendent if and only if all the following identities hold:

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F) \qquad \mathbb{P}(E \cap G) = \mathbb{P}(E)\,\mathbb{P}(G)$$

$$\mathbb{P}(F \cap G) = \mathbb{P}(F)\,\mathbb{P}(G) \qquad \mathbb{P}(E \cap F \cap G) = \mathbb{P}(E)\,\mathbb{P}(F)\,\mathbb{P}(G)$$

Exercise: Communication Through Noisy Channels

- Still consider the case where a sensor transmits a string of 0/1 symbols to the server through three noisy channels sequentially.



- Further assume errors in different symbol transmissions are independent.
- To improve reliability, each symbol is transmitted three times and the server decodes the symbol by majority rule.
- What is the probability that 0 can be decoded correctly?

Exercise: Communication Through Noisy Channels

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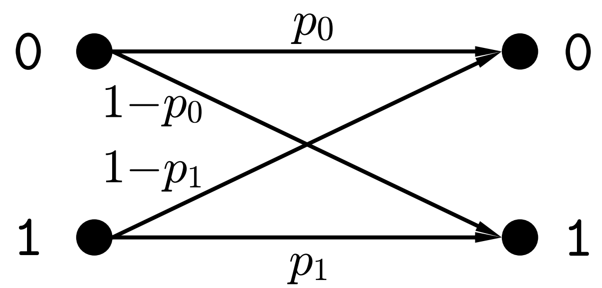

- Further assume errors in different symbol transmissions are independent.
- To improve reliability, each symbol is transmitted three times and the server decodes the symbol by majority rule.
- What is the probability that 0 can be decoded correctly?

Summary of Chapter 1

- **Sample space, events and probability measure**
  - Axiomatic definition & basic properties
- **Classical probability**
  - All outcomes are equally probable, so just count!
- **Conditional probability**
  - When you know something has occurred but still don’t know the exact outcome
- **Independence of events**
  - More to come when you study random variables