4.4. Revisit the XOR problem described in Table 3.1. Here, we solve it using an RBF network, whose structure is given in Figure 4.3.

Define a pair of Gaussian hidden functions as

$$\phi_j(x) = \exp\left(-\|x - c_j\|^2\right), \quad j = 1, 2$$

where the centers c_1 and c_2 are

$$c_1 = \begin{bmatrix}1\\1\end{bmatrix}, \quad c_2 = \begin{bmatrix}0\\0\end{bmatrix}$$

1. Calculate the weight vector of the RBF network and explain why this kind of network can be utilized to deal with the XOR problem successfully.
2. Study the necessity of adding the bias b. Take off the bias and the fixed input, and then point out if the corresponding network can still solve the problem or not.

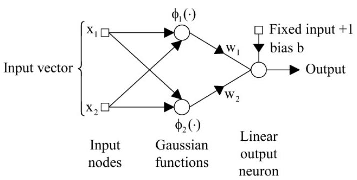

FIGURE 4.3 Structure of the RBF network in Exercise 4.4.

TABLE 3.1 Pattern Classification

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

1.

$\begin{bmatrix}x_1\\x_2\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$
$\phi_1(x)$	e^{-2}	e^{-1}	e^{-1}	1
$\phi_2(x)$	1	e^{-1}	e^{-1}	e^{-2}
$z = w_1\phi_1(x) + w_2\phi_2(x) + b$	0.007	0.994	0.994	0.007
XOR	0	1	1	0

Calculate:

$$\begin{cases} e^{-2}w_1 + w_2 + b = 0 \\ e^{-1}w_1 + e^{-1}w_2 + b = 1 \\ w_1 + e^{-2}w_2 + b = 0 \end{cases}$$

$$\Rightarrow \begin{cases} w_1 = -2.503 \\ w_2 = -2.503 \\ b = 2.841 \end{cases}$$

OR use matrix

$$W^T = [w_1 \ w_2 \ b]^T = \begin{bmatrix} e^{-2} & 1 & 1 \\ e^{-1} & e^{-1} & 1 \\ e^{-1} & e^{-1} & 1 \\ 1 & e^{-2} & 1 \end{bmatrix}^{-1} \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

This kind of network implement non-linear layer by using RBF. so it can deal with the XOR problem.

2. Take off the bias:

$$\begin{cases} e^{-2}w_1 + w_2 = 0 \\ e^{-1}w_1 + e^{-1}w_2 = 1 \\ w_1 + e^{-2}w_2 = 0 \end{cases}$$

No result for w_1, w_2.

So bias b is necessary.

or use matrix:

$$\begin{bmatrix} e^{-2} & 1 \\ e^{-1} & e^{-1} \\ e^{-1} & e^{-1} \\ 1 & e^{-2} \end{bmatrix}$$ has no inverse.