Assignment 3

Problem 1

(1) The probability that the computer will experience the next breakdown within the next 3 months is given by the cumulative distribution function (CDF) of the exponential distribution. For a rate parameter $\lambda = \frac{1}{5}$ per month:

$$P(T \leq 3) = 1 - e^{-\lambda \cdot 3} = 1 - e^{-\frac{3}{5}}$$

Final Answer: $\boxed{1 - e^{-3/5}}$

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(2) Using the **memoryless property** of the exponential distribution, the probability that the computer will not experience a breakdown in the next 4 months, given it had no breakdown in the past 4 months, is equivalent to the probability of no breakdown in a 4-month interval:

$$P(T > 8 \mid T > 4) = P(T > 4) = e^{-\lambda \cdot 4} = e^{-\frac{4}{5}}$$

Final Answer: $\boxed{e^{-4/5}}$

Problem 2

(1) To calculate the proportion of middle-class families (income between $6,000 and$12,000), we use the **standard normal distribution** with mean $\mu = 9000$ and standard deviation $\sigma = 2000$.

1. Calculate Z-scores:

$$Z_{\text{lower}} = \frac{6000 - 9000}{2000} = -1.5, \quad Z_{\text{upper}} = \frac{12000 - 9000}{2000} = 1.5$$

2. Use the Z-table to find probabilities:

$$P(Z \leq -1.5) \approx 0.0668, \quad P(Z \leq 1.5) \approx 0.9332$$

3. Subtract to find the proportion between $6,000 and$12,000:

$$P(6000 < X < 12000) = 0.9332 - 0.0668 = 0.8664$$

Final Answer:

\boxed{86.64%}

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(2) To find the income cutoff for the poorest 3%, we determine the Z-score corresponding to the 3rd percentile ($P = 0.03$).

1. From the Z-table, the Z-score for $P = 0.03$ is approximately $Z = -1.88$.
2. Convert the Z-score back to the income value:

$$X = \mu + Z \cdot \sigma = 9000 + (-1.88)(2000) = 9000 - 3760 = 5240$$

Final Answer:
$\boxed{\$5240}$ (rounded to the nearest dollar)

Problem 3

Given $X \sim \mathcal{N}(\mu, \sigma^2)$, we know the quadratic equation $y^2 + 4y + X = 0$ has no real roots when its discriminant is negative. The discriminant is:

$$\Delta = 4^2 - 4(1)(X) = 16 - 4X.$$

For no real roots, $\Delta < 0$:

$$16 - 4X < 0 \implies X > 4.$$

The problem states $P(X > 4) = 0.5$. In a normal distribution, $P(X > \mu) = 0.5$ because the distribution is symmetric about the mean. Thus, $\mu = 4$.

Final Answer:
$\boxed{4}$

Problem 4

(1) PMF of X:

The random variable X represents the number of red balls picked before obtaining a green ball. The possible values of X are $0, 1, 2$. The probabilities are calculated as follows:

- $P(X = 0)$: Probability of picking a green ball first.

$$P(X = 0) = \frac{8}{10} = \frac{4}{5}$$

- $P(X = 1)$: Probability of picking red first, then green.

$$P(X = 1) = \frac{2}{10} \cdot \frac{8}{9} = \frac{16}{90} = \frac{8}{45}$$

- $P(X = 2)$: Probability of picking both red balls first.

$$P(X = 2) = \frac{2}{10} \cdot \frac{1}{9} = \frac{2}{90} = \frac{1}{45}$$

PMF Summary:

$$P(X = x) = \{ \tfrac{4}{5} \quad \text{if } x = 0,\ \tfrac{8}{45} \quad \text{if } x = 1,\ \tfrac{1}{45} \quad \text{if } x = 2.$$

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(2) Expected Value and Variance:

- **Expected Value ($E(X)$)**:

$$E(X) = \sum x \cdot P(X = x) = 0 \cdot \frac{4}{5} + 1 \cdot \frac{8}{45} + 2 \cdot \frac{1}{45} = \frac{10}{45} = \frac{2}{9}$$

- **Variance ($Var(X)$)**:

$$E(X^2) = \sum x^2 \cdot P(X = x) = 0^2 \cdot \frac{4}{5} + 1^2 \cdot \frac{8}{45} + 2^2 \cdot \frac{1}{45} = \frac{12}{45} = \frac{4}{15}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{4}{15} - \left(\frac{2}{9}\right)^2 = \frac{4}{15} - \frac{4}{81} = \frac{88}{405}$$

Final Answers:

- $E(X)$: $\boxed{\frac{2}{9}}$
- $Var(X)$: $\boxed{\frac{88}{405}}$

Problem 5

(1) Probability of Falling Inside the Unit Circle:

The unit circle has radius 1 and area π. The square $[-1, 1] \times [-1, 1]$ has side length 2 and area 4. The probability that a random point (X, Y) falls inside the unit circle is the ratio of the circle's area to the square's area:

$$P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}.$$

Final Answer:
$\boxed{\frac{\pi}{4}}$

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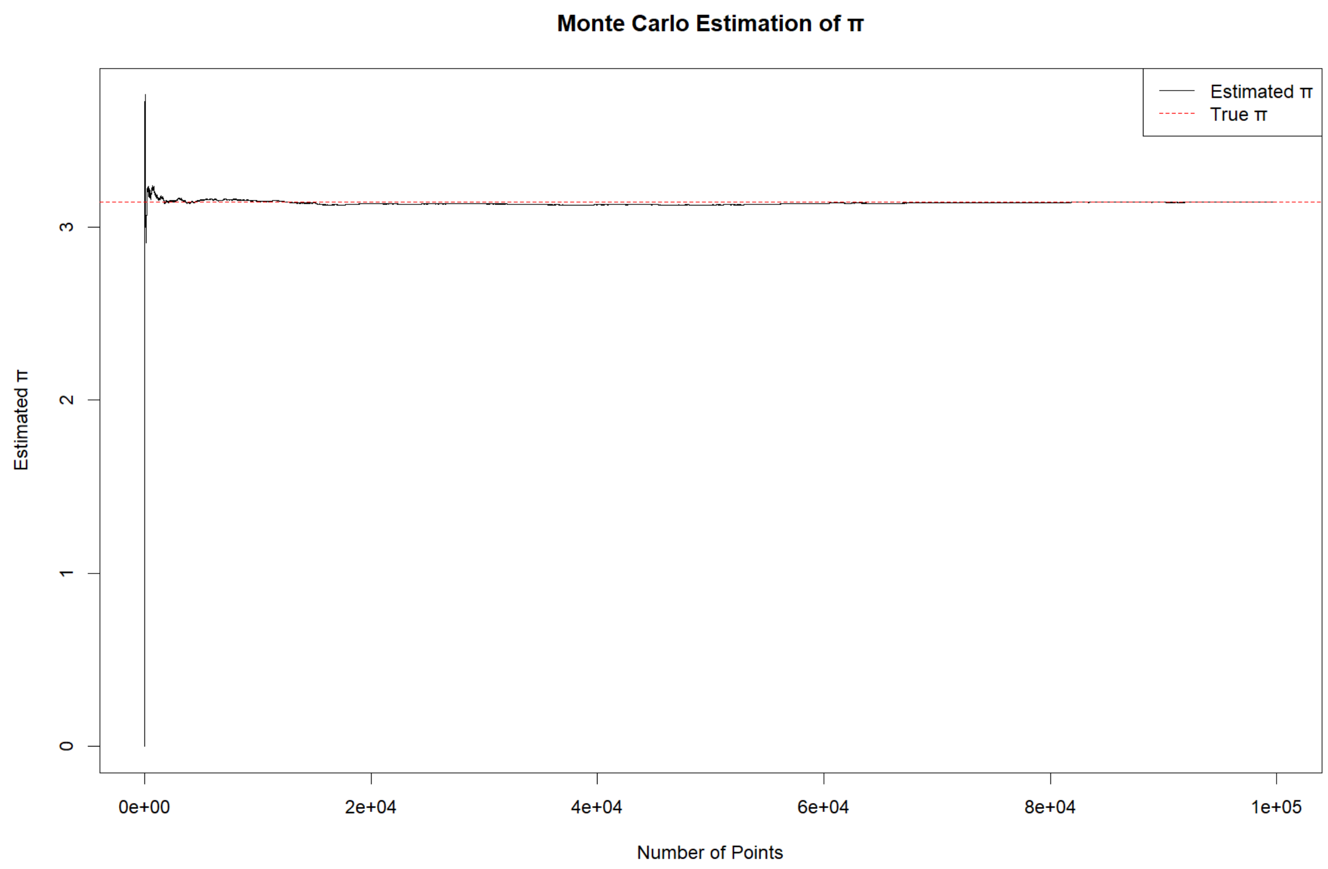
(2) R Code for Estimating π and Convergence Plot:




```r
set.seed(20250330)
n <- 100000
x <- runif(n, -1, 1)
y <- runif(n, -1, 1)
inside <- (x^2 + y^2) <= 1
cumulative_inside <- cumsum(inside)
estimate <- 4 * cumulative_inside / (1:n)

plot(1:n, estimate, type = "l",
     xlab = "Number of Points", ylab = "Estimated π",
     main = "Monte Carlo Estimation of π")
abline(h = pi, col = "red", lty = 2)
legend("topright", legend = c("Estimated π", "True π"),
       col = c("black", "red"), lty = 1:2)
```

Problem 6

See after the end of the document.

Problem 7

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