Assignment 3

Problem 1

(1) The probability that the computer will experience the next breakdown within the next 3 months is given by the cumulative distribution function (CDF) of the exponential distribution. For a rate parameter $\lambda = \frac{1}{5}$ per month:

$$P(T \leq 3) = 1 - e^{-\lambda \cdot 3} = 1 - e^{-\frac{3}{5}}$$

Final Answer: $\boxed{1 - e^{-3/5}}$

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(2) Using the **memoryless property** of the exponential distribution, the probability that the computer will not experience a breakdown in the next 4 months, given it had no breakdown in the past 4 months, is equivalent to the probability of no breakdown in a 4-month interval:

$$P(T > 8 \mid T > 4) = P(T > 4) = e^{-\lambda \cdot 4} = e^{-\frac{4}{5}}$$

Final Answer: $\boxed{e^{-4/5}}$

Problem 2

(1) To calculate the proportion of middle-class families (income between $6,000 and 12,000), we use the **standard normal distribution** with mean $\mu = 9000$ and standard deviation $\sigma = 2000$.

1. Calculate Z-scores:

$$Z_{\text{lower}} = \frac{6000 - 9000}{2000} = -1.5, \quad Z_{\text{upper}} = \frac{12000 - 9000}{2000} = 1.5$$

2. Use the Z-table to find probabilities:

$$P(Z \leq -1.5) \approx 0.0668, \quad P(Z \leq 1.5) \approx 0.9332$$

3. Subtract to find the proportion between $6,000 and 12,000:

$$P(6000 < X < 12000) = 0.9332 - 0.0668 = 0.8664$$

Final Answer:

\boxed{86.64%}

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(2) To find the income cutoff for the poorest 3%, we determine the Z-score corresponding to the 3rd percentile ($P = 0.03$).

1. From the Z-table, the Z-score for $P = 0.03$ is approximately $Z = -1.88$.
2. Convert the Z-score back to the income value:

$$X = \mu + Z \cdot \sigma = 9000 + (-1.88)(2000) = 9000 - 3760 = 5240$$

Final Answer:
$\boxed{\$5240}$ (rounded to the nearest dollar)

Problem 3

Given $X \sim \mathcal{N}(\mu, \sigma^2)$, we know the quadratic equation $y^2 + 4y + X = 0$ has no real roots when its discriminant is negative. The discriminant is:

$$\Delta = 4^2 - 4(1)(X) = 16 - 4X.$$

For no real roots, $\Delta < 0$:

$$16 - 4X < 0 \implies X > 4.$$

The problem states $P(X > 4) = 0.5$. In a normal distribution, $P(X > \mu) = 0.5$ because the distribution is symmetric about the mean. Thus, $\mu = 4$.

Final Answer:
$\boxed{4}$

Problem 4

(1) PMF of X:

The random variable X represents the number of red balls picked before obtaining a green ball. The possible values of X are $0, 1, 2$. The probabilities are calculated as follows:

- $P(X = 0)$: Probability of picking a green ball first.

$$P(X = 0) = \frac{8}{10} = \frac{4}{5}$$

- $P(X = 1)$: Probability of picking red first, then green.

$$P(X = 1) = \frac{2}{10} \cdot \frac{8}{9} = \frac{16}{90} = \frac{8}{45}$$

- $P(X = 2)$: Probability of picking both red balls first.

$$P(X = 2) = \frac{2}{10} \cdot \frac{1}{9} = \frac{2}{90} = \frac{1}{45}$$

PMF Summary:

$$P(X = x) = \{ \tfrac{4}{5} \quad \text{if } x = 0,\ \tfrac{8}{45} \quad \text{if } x = 1,\ \tfrac{1}{45} \quad \text{if } x = 2.$$

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(2) Expected Value and Variance:

- **Expected Value ($E(X)$)**:

$$E(X) = \sum x \cdot P(X = x) = 0 \cdot \frac{4}{5} + 1 \cdot \frac{8}{45} + 2 \cdot \frac{1}{45} = \frac{10}{45} = \frac{2}{9}$$

- **Variance ($Var(X)$)**:

$$E(X^2) = \sum x^2 \cdot P(X = x) = 0^2 \cdot \frac{4}{5} + 1^2 \cdot \frac{8}{45} + 2^2 \cdot \frac{1}{45} = \frac{12}{45} = \frac{4}{15}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{4}{15} - \left(\frac{2}{9}\right)^2 = \frac{4}{15} - \frac{4}{81} = \frac{88}{405}$$

Final Answers:

- $E(X)$: $\boxed{\frac{2}{9}}$
- $Var(X)$: $\boxed{\frac{88}{405}}$

Problem 5

(1) Probability of Falling Inside the Unit Circle:

The unit circle has radius 1 and area π. The square $[-1, 1] \times [-1, 1]$ has side length 2 and area 4. The probability that a random point (X, Y) falls inside the unit circle is the ratio of the circle's area to the square's area:

$$P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}.$$

Final Answer:
$\boxed{\frac{\pi}{4}}$

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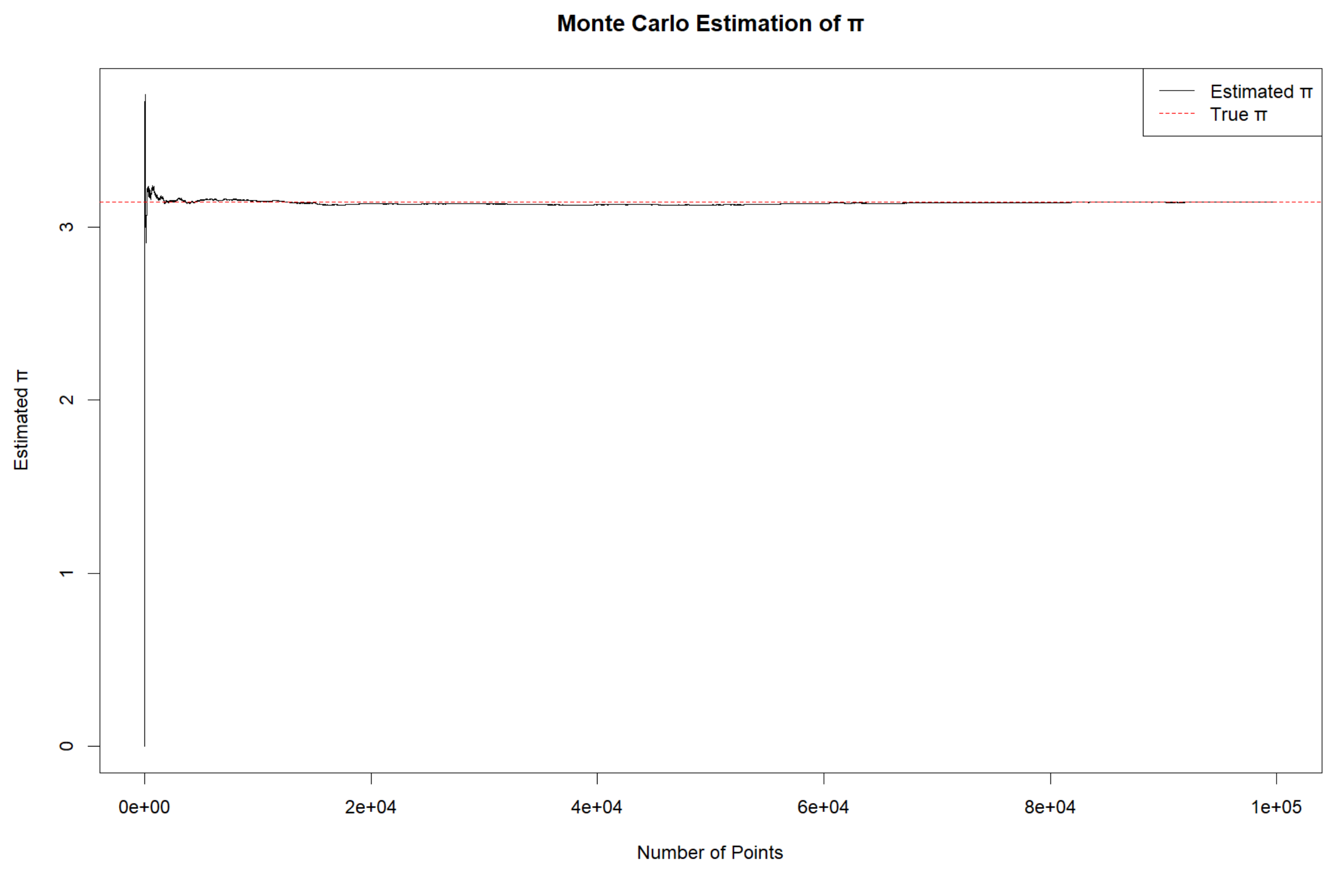
(2) R Code for Estimating π and Convergence Plot:




```r
set.seed(20250330)
n <- 100000
x <- runif(n, -1, 1)
y <- runif(n, -1, 1)
inside <- (x^2 + y^2) <= 1
cumulative_inside <- cumsum(inside)
estimate <- 4 * cumulative_inside / (1:n)

plot(1:n, estimate, type = "l",
     xlab = "Number of Points", ylab = "Estimated π",
     main = "Monte Carlo Estimation of π")
abline(h = pi, col = "red", lty = 2)
legend("topright", legend = c("Estimated π", "True π"),
       col = c("black", "red"), lty = 1:2)
```

Problem 6

See after the end of the document.

Problem 7

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Galton Board Experiment

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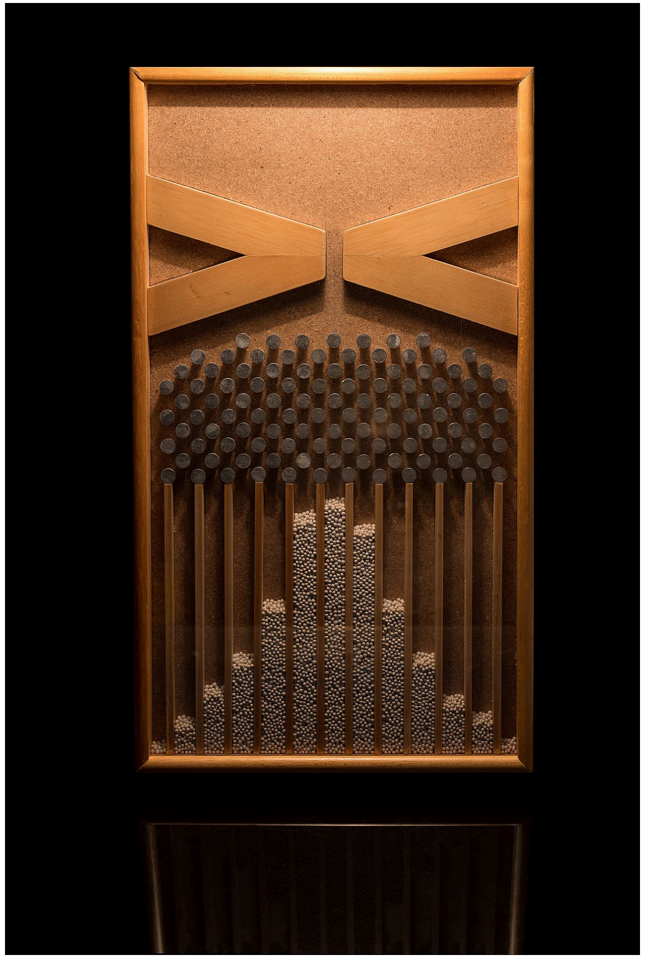
What is Galton board?

- The Galton board, also known as the Galton box or quincunx or bean machine (or incorrectly Dalton board), is a device invented by Francis Galton to demonstrate the central limit theorem, in particular that with sufficient sample size the binomial distribution approximates a normal distribution.



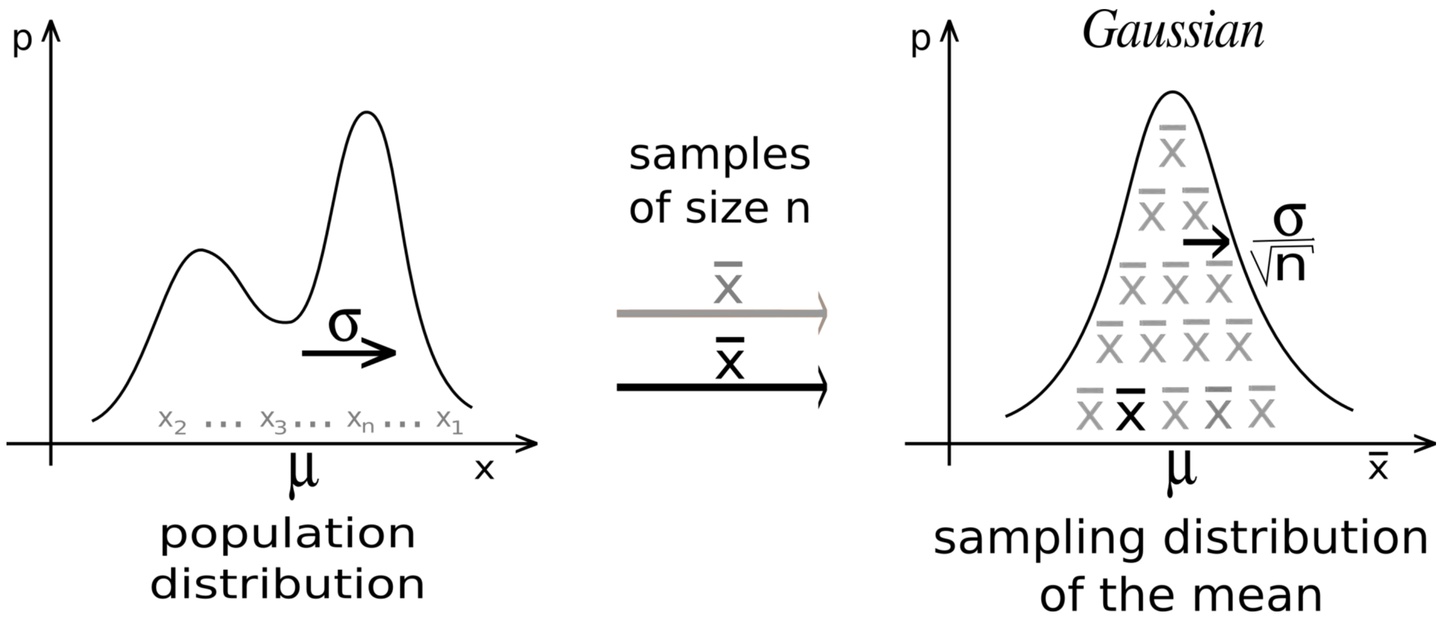
Description

- The Galton board consists of a vertical board with interleaved rows of pegs. Beads are dropped from the top and, when the device is level, bounce either left or right as they hit the pegs. Eventually they are collected into bins at the bottom, where the height of bead columns accumulated in the bins approximate a bell curve. Overlaying Pascal's triangle onto the pins shows the number of different paths that can be taken to get to each bin.

Before and after the spin

Central limit theorem

- In probability theory, the central limit theorem (CLT) states that, under appropriate conditions, the distribution of a normalized version of the sample mean converges to a standard normal distribution. This holds even if the original variables themselves are not normally distributed. There are several versions of the CLT, each applying in the context of different conditions.



Reference

- Wikipedia. (n.d.). Galton board. Retrieved April 1, 2025, from https://en.wikipedia.org/wiki/Galton_board
- Wikipedia. (n.d.). Central limit theorem. Retrieved April 1, 2025, from https://en.wikipedia.org/wiki/Central_limit_theorem