Assignment 1

Problem 1

Solution:

Given:

- Probability of motherboard (MB) problems, $P(MB) = 0.4$
- Probability of hard drive (HD) problems, $P(HD) = 0.3$
- Probability of both MB and HD problems, $P(MB \cap HD) = 0.15$

To find the probability that a computer has **no MB or HD problems**, we first calculate the probability of having **at least one problem** using the principle of inclusion-exclusion:

$$P(MB \cup HD) = P(MB) + P(HD) - P(MB \cap HD)$$

$$P(MB \cup HD) = 0.4 + 0.3 - 0.15 = 0.55$$

The probability of **no problems** is the complement of the union:

$$P(\text{No MB/HD problems}) = 1 - P(MB \cup HD)$$

$$P(\text{No MB/HD problems}) = 1 - 0.55 = 0.45$$

Answer: The probability of a fully functioning MB and HD is $\boxed{0.45}$.

Problem 2

Solution:

Given:

- Probability a programmer knows Java, $P(J) = 0.7$
- Probability a programmer knows Python, $P(P) = 0.6$
- Probability a programmer knows both, $P(J \cap P) = 0.5$

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1. Probability of not knowing Python and not knowing Java:

First, calculate the probability of knowing *at least one language* using inclusion-exclusion:

$$P(J \cup P) = P(J) + P(P) - P(J \cap P)$$

$$P(J \cup P) = 0.7 + 0.6 - 0.5 = 0.8$$

The probability of knowing **neither** language is the complement:

$$P(\text{Neither}) = 1 - P(J \cup P) = 1 - 0.8 = 0.2$$

Answer: $\boxed{0.20}$

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2. Probability of knowing Java but not Python:

This is the probability of knowing Java *minus* the overlap with Python:

$$P(J \setminus P) = P(J) - P(J \cap P)$$

$$P(J \setminus P) = 0.7 - 0.5 = 0.2$$

Answer: $\boxed{0.20}$

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3. Probability of knowing Java given they know Python:

Use the definition of conditional probability:

$$P(J \mid P) = \frac{P(J \cap P)}{P(P)}$$

$$P(J \mid P) = \frac{0.5}{0.6} \approx 0.8333$$

Answer: $\boxed{\frac{5}{6}} \quad \text{(or approximately 0.8333)}$

Problem 3

Solution:

When selecting k elements from n distinct elements **with replacement**, the number of permutations and combinations are derived as follows:

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1. Permutations (Order Matters):

Each selection is ordered, and elements can be repeated. For each of the k positions, there are n choices. Thus:

$$\text{Number of permutations} = n \times n \times \cdots \times n = n^k$$

Example: For $n = 3$, $k = 2$: Permutations = $3^2 = 9$ (e.g., AA, AB, AC, BA, BB, BC, CA, CB, CC).

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2. Combinations (Order Does Not Matter):

Elements can repeat, but order is irrelevant. This is equivalent to distributing k identical items into n distinct bins (stars and bars method). The formula is:

$$\text{Number of combinations} = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}$$

Example: For $n = 3$, $k = 2$: Combinations = $\binom{3+2-1}{2} = \binom{4}{2} = 6$ (e.g., {AA, AB, AC, BB, BC, CC}).

Final Answer:

- Permutations with replacement: $\boxed{n^k}$
- Combinations with replacement: $\boxed{\binom{n+k-1}{k}}$

Problem 4

Solution:

We need to compute the probability that **at least one couple** is paired together when 4 male-female couples are randomly paired for a dance.

Total Pairings: There are $4! = 24$ ways to pair 4 males with 4 females.

Using the Principle of Inclusion-Exclusion:

Let A_i be the event that the i-th couple is paired together. The probability of at least one A_i is:

$$P\left(\bigcup_{i=1}^{4} A_i\right) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

1. **Single Couple Pairing:**
   - $\binom{4}{1}$ ways to choose 1 couple.
   - Probability: $\frac{3!}{4!} = \frac{1}{4}$.
   - Total: $4 \cdot \frac{1}{4} = 1$.
2. **Two Couples Paired:**
   - $\binom{4}{2} = 6$ ways to choose 2 couples.
   - Probability: $\frac{2!}{4!} = \frac{1}{12}$.
   - Total: $6 \cdot \frac{1}{12} = 0.5$.
3. **Three Couples Paired:**
   - $\binom{4}{3} = 4$ ways to choose 3 couples.
   - Probability: $\frac{1!}{4!} = \frac{1}{24}$.
   - Total: $4 \cdot \frac{1}{24} = \frac{1}{6}$.
4. **All Four Couples Paired:**
   - Probability: $\frac{1}{4!} = \frac{1}{24}$.

Combining these:

$$P(\text{At least one couple}) = 1 - 0.5 + \frac{1}{6} - \frac{1}{24} = \frac{15}{24} = \frac{5}{8}.$$

Alternative Derangement Approach: The number of derangements (no couple paired) for $n = 4$ is $!4 = 9$. Thus:

$$P(\text{No couples}) = \frac{9}{24} = \frac{3}{8} \implies P(\text{At least one couple}) = 1 - \frac{3}{8} = \frac{5}{8}.$$

Final Answer: $\boxed{\frac{5}{8}}$

Problem 5

Solution:

We need to find the probability that the sum of two randomly chosen numbers X and Y from the interval $[0, 1]$ is less than $\frac{7}{5} = 1.4$.

Geometric Interpretation: The problem is equivalent to finding the area of the region $X + Y < \frac{7}{5}$ within the unit square $[0, 1] \times [0, 1]$.

1. **Breakdown of the Region:**
   - For $0 \leq X \leq 0.4$, Y can range from 0 to 1 (since $\frac{7}{5} - X \geq 1$).
   - For $0.4 < X \leq 1$, Y can range from 0 to $\frac{7}{5} - X$.

2. **Area Calculation:**
   - **First Region (Rectangle):**

     $$\text{Area}_1 = 0.4 \times 1 = 0.4$$

   - **Second Region (Integral):** $$ \text{Area}*2* = *int*{0.4}^1 \left(\frac{7}{5} - X\right) dX $$

     $$= \left[\frac{7}{5}X - \frac{1}{2}X^2\right]_{0.4}^{1}$$

     $$= \left(\frac{7}{5}(1) - \frac{1}{2}(1)^2\right) - \left(\frac{7}{5}(0.4) - \frac{1}{2}(0.4)^2\right)$$

     $$= (1.4 - 0.5) - (0.56 - 0.08) = 0.9 - 0.48 = 0.42$$

3. **Total Area:**

   $$\text{Total Area} = 0.4 + 0.42 = 0.82$$

Probability: The probability is the total area divided by the unit square area (1):

$$P\left(X + Y < \frac{7}{5}\right) = \frac{41}{50} = 0.82$$

Final Answer: $\boxed{\frac{41}{50}}$

Problem 6

See screenshot after the `Problem 7` solution.

Problem 7

1. Satisfy the requirements of the `Introductory Course`

2. Review the theroy of `Probability` and `Combinatorics`

3. Our teacher is beautiful and kind !!!

Bertrand’s Paradox

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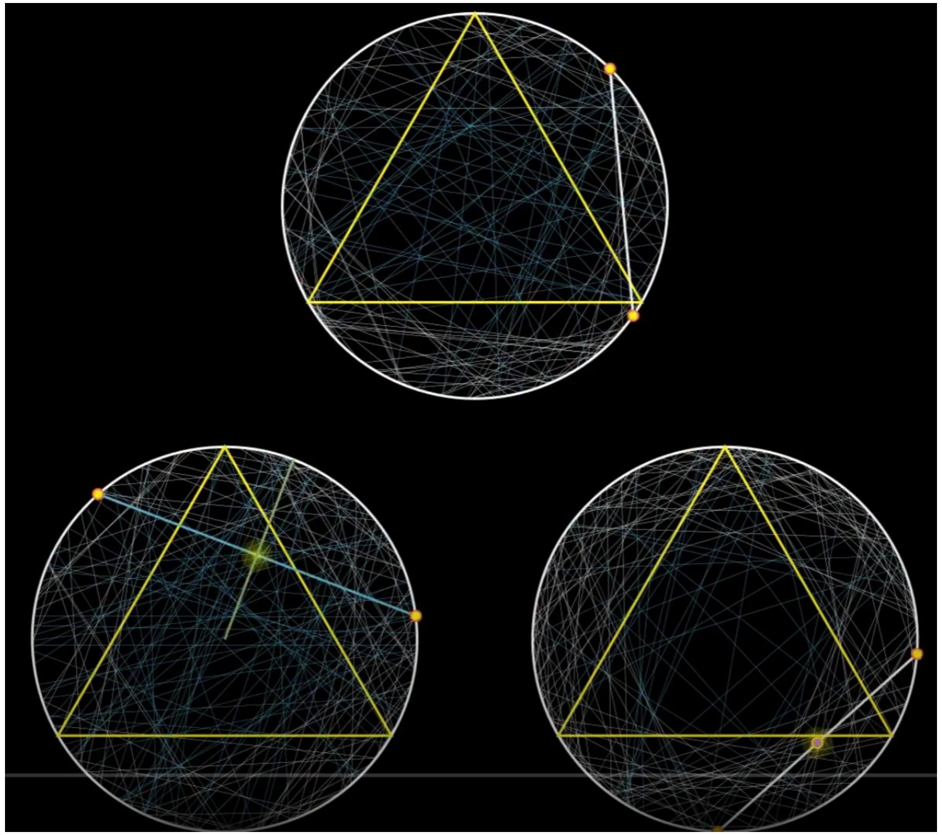
What is Bertrand's Paradox?

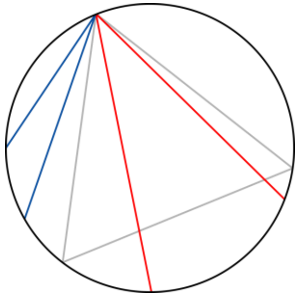
- Consider an equilateral triangle that is inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?

$\frac{1}{2}$? $\frac{1}{3}$? $\frac{1}{4}$?



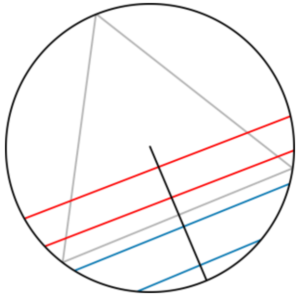
Three different way

- **Bertrand gave three arguments (each using the principle of indifference), all apparently valid yet yielding different results:**
  - The **"random endpoints"** method:
    - Choose two random points on the circumference of the circle and draw the chord joining them.



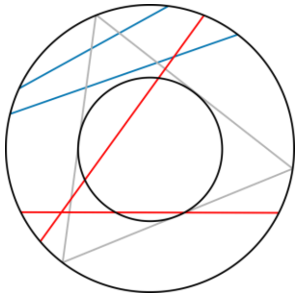
$\frac{1}{2}$

  - The **"random radial point"** method:
    - Choose a radius of the circle, choose a point on the radius and construct the chord through this point and perpendicular to the radius.



$\frac{1}{3}$

  - The **"random midpoint"** method:
    - Choose a point anywhere within the circle and construct a chord with the chosen point as its midpoint.

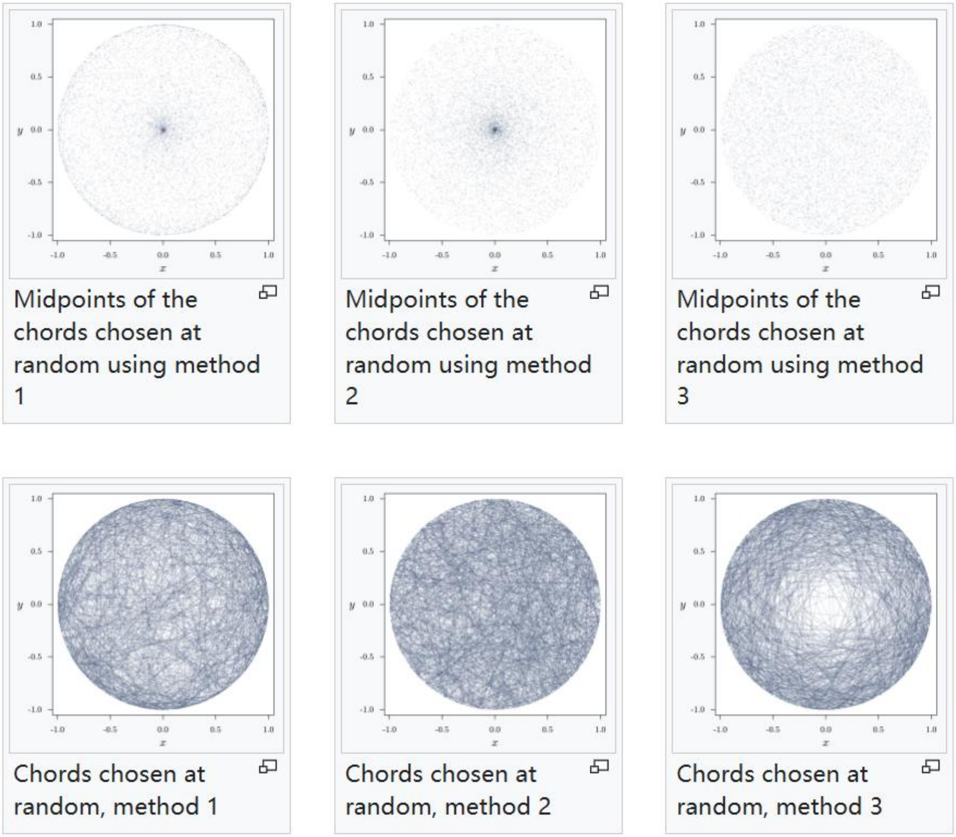


$\frac{1}{4}$

Why & Conclusion

- These three selection methods differ as to the weight they give to chords which are diameters. This issue can be avoided by "regularizing" the problem so as to exclude diameters, without affecting the resulting probabilities. But as presented above, in method 1, each chord can be chosen in exactly one way, regardless of whether or not it is a diameter; in method 2, each diameter can be chosen in two ways, whereas each other chord can be chosen in only one way; and in method 3, each choice of midpoint corresponds to a single chord, except the center of the circle, which is the midpoint of all the diameters.

Scatterplots showing simulated Bertrand distributions, midpoints/chords chosen at random using the above methods.



Midpoints of the chords chosen at random using method 1

Midpoints of the chords chosen at random using method 2

Midpoints of the chords chosen at random using method 3

Chords chosen at random, method 1

Chords chosen at random, method 2

Chords chosen at random, method 3

Reference

- Wikipedia contributors. (n.d.). Bertrand paradox (probability). Wikipedia, The Free Encyclopedia. Retrieved March 5, 2025, from https://en.wikipedia.org/wiki/Bertrand_paradox_(probability)

- 3Blue1Brown. (2020, May 18). Bertrand's paradox || Expected values and probabilities [Video]. YouTube. https://www.youtube.com/watch?v=mOwTedemFzQ