An ε–δ Discriminant for Projection-Induced Irregularity in Recurrent Navier–Stokes Observations

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Abstract

We propose a representation-level validity criterion for separating intrinsic dynamics from artifacts introduced by reduced observations used in turbulence analysis (e.g., POD/autoencoder latent projections and Fourier truncations). Viewing time discretization as a recurrence and observations as $z_n = \pi(u_n)$, we test whether the induced observation recurrence remains locally single-valued/continuous via an ε–δ correspondence discriminant; instability signals projection folds and closure failure. In a CPU-first, Docker-reproducible viscous Burgers demo with Fourier top-k projection, the test yields PASS/FAIL trust regions that can serve as a validity gate before fitting low-dimensional (AI/ROM) dynamics such as Neural ODEs.

Introduction

1. Classical formulation and the regularity problem

The incompressible Navier–Stokes equations on $\mathbb{R}^3$ (or a smooth domain $\Omega \subset \mathbb{R}^3$) describe the evolution of a velocity field $u(x,t)$ and pressure $p(x,t)$:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \tag{1}$$
$$\nabla \cdot u = 0, \tag{2}$$

where $\nu > 0$ is the kinematic viscosity. The global existence of smooth solutions for smooth initial data in 3D remains one of the central open questions in mathematical fluid dynamics. While the classical regularity problem is often stated in terms of whether solutions remain smooth for all time, practical turbulence phenomena are usually encountered through *observations* and *numerical representations* rather than direct access to an exact solution. This motivates a complementary question: *when irregularity is observed, is it necessarily an intrinsic analytic singularity of the flow, or can it arise from the way the flow is represented and projected into an observation space?*

Motivation from reduced representations (AI/ROM). In modern data-driven turbulence workflows, one often compresses a high-dimensional flow field into a low-dimensional representation (e.g., POD coefficients or an autoencoder latent state) and then fits a low-dimensional time-evolution model (e.g., Neural ODE/RNN) to forecast dynamics. Such pipelines implicitly assume that the chosen projection π yields an observation space in which the induced recurrence is sufficiently well-posed and stable; when this assumption fails, apparent irregularity and poor generalization may be artifacts of projection/closure rather than intrinsic flow behavior. This work proposes a minimal, reproducible *validity gate* for reduced observations before committing to model fitting.

Related context (projection effects). Map/recurrence viewpoints for Navier–Stokes and turbulence have long been studied in dynamical-systems settings [4, 5], and projection/filtering operations are

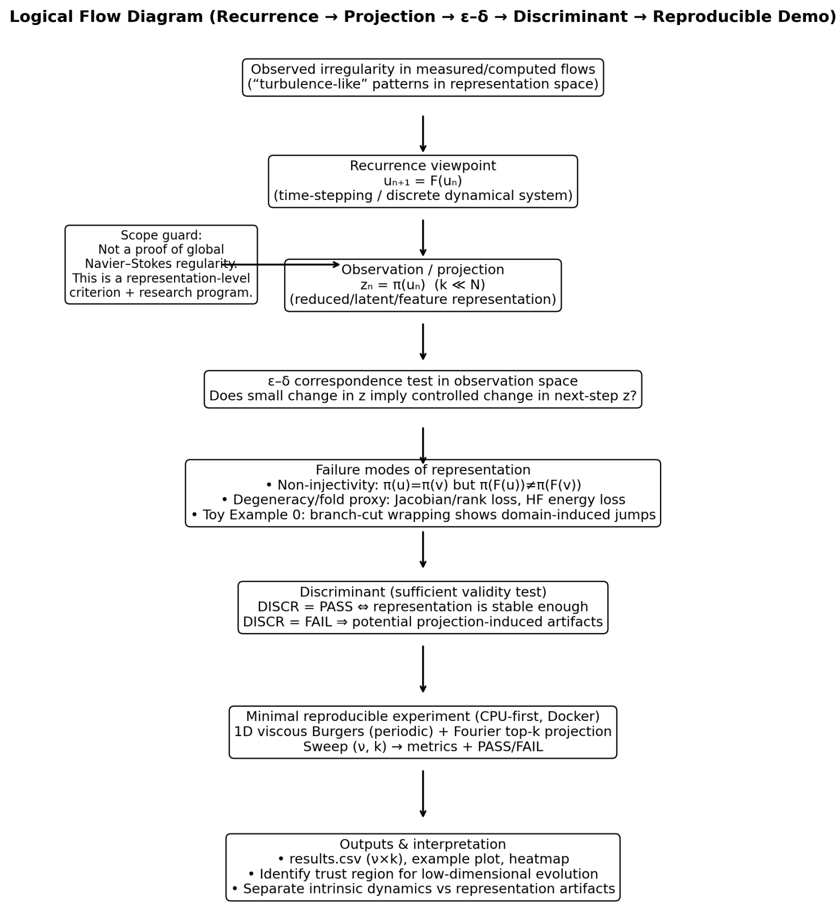

Figure 1: Logical flow: recurrence $\to$ projection $\to$ ε–δ test $\to$ discriminant $\to$ reproducible demo.

known to introduce representation-dependent effects such as commutator errors and closure/stability challenges [6, 3, 2, 1].

Contribution. We treat observed irregularity as a representation-level phenomenon and evaluate reduced observations (Fourier truncations and, conceptually, POD/autoencoder latents) via an ε–δ correspondence test and a sufficient PASS/FAIL discriminant, validated by a reproducible Burgers+Fourier top-k demo.

1 Framework

We adopt a recurrence-first view of time stepping and treat projection/representation as explicit operators on the dynamics.

Why time discretization yields a recurrence. A time-dependent PDE can be viewed as an (infinite-dimensional) ODE of the form $\partial_t u = \mathcal{G}(u)$ on a state space $\mathcal{X}$. On a time grid $t_n = n\Delta t$, the forward difference approximation $\partial_t u(t_n) \approx (u^{n+1} - u^n)/\Delta t$ gives

$$\frac{u^{n+1} - u^n}{\Delta t} = \mathcal{G}(u^n), \qquad \Longrightarrow \qquad u^{n+1} = u^n + \Delta t\, \mathcal{G}(u^n). \tag{3}$$

Defining the one-step map $F_{\Delta t}(u) := u + \Delta t\, \mathcal{G}(u)$ yields the recurrence $u^{n+1} = F_{\Delta t}(u^n)$, and iteration gives $u^{n+m} = F_{\Delta t}^m(u^n)$. In incompressible Navier–Stokes, one may write $\mathcal{G}(u) = P\big(-(u \cdot \nabla)u + \nu\Delta u + f\big)$ using the Leray projection P to enforce $\nabla \cdot u = 0$.

1.1 Navier–Stokes as a recurrence map

We emphasize that the "recurrence" viewpoint is not an ad-hoc reinterpretation: it is the natural consequence of time discretization. In practice, both computation and many conceptual models operate through one-step evolution operators, yielding a discrete-time map

$$u^{n+1} = F_{\Delta t}(u^n). \tag{4}$$

This section records a standard derivation (at a high level) that makes the map structure explicit.

Starting point (incompressible Navier–Stokes). Let $u(x, t)$ and $p(x, t)$ solve

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u + f, \qquad \nabla \cdot u = 0, \tag{5}$$

on a periodic box (or a domain with compatible boundary conditions), where $\nu > 0$ is viscosity and f is an optional forcing term.

Leray projection and pressure elimination. Let P denote the Leray projection onto divergence-free vector fields. Formally, applying P removes the pressure gradient and yields

$$\partial_t u = P\big(-(u \cdot \nabla)u + \nu\Delta u + f\big). \tag{6}$$

This representation highlights that incompressibility is enforced via a projection operator (a theme aligned with the representation-level viewpoint of this paper).

Time discretization (one-step map). For a time step $\Delta t > 0$, a forward-Euler discretization gives the recurrence

$$u^{n+1} = u^n + \Delta t\, P\big(-(u^n \cdot \nabla)u^n + \nu\Delta u^n + f^n\big) =: F_{\Delta t}(u^n). \tag{7}$$

More stable schemes treat diffusion semi-implicitly; for example,

$$(I - \nu\Delta t\, \Delta)\, u^{n+1} = u^n - \Delta t\, P\big((u^n \cdot \nabla)u^n\big) + \Delta t\, P(f^n), \tag{8}$$

which again defines a (generally nonlinear) map $u^{n+1} = F_{\Delta t,\nu}(u^n)$.

Why viscosity matters. Viscosity does not "create" recurrence; rather, it stabilizes the one-step map by damping high-frequency components. On periodic domains, Fourier modes diagonalize the Laplacian:

$$\widehat{\nu\Delta u}_k = -\nu|k|^2\, \hat{u}_k, \tag{9}$$

making the smoothing action explicit and motivating reduced observations such as Fourier top-k truncation.

Iterates and self-consistency. Once the one-step map is defined, substitution yields the iterates

$$u^{n+m} = F_{\Delta t}^m(u^n), \qquad F_{\Delta t}^m := \underbrace{F_{\Delta t} \circ \cdots \circ F_{\Delta t}}_{m \text{ times}}, \tag{10}$$

which is the precise sense in which recurrence "closes" under iteration.

Connection to projection-based discriminants. Our discriminant DISCR is not a test of continuity of $F_{\Delta t}$ on the intrinsic state space. Instead, it evaluates whether the induced evolution in an observation space (after projection/truncation) remains stable and effectively single-valued over the regime of interest.

1.2 Weak/Energy form and a Fourier viewpoint (motivation for projection)

A convenient way to connect the Navier–Stokes dynamics to representation-level questions is to move from the strong PDE form to an energy/weak form and then to a modal (Fourier) viewpoint. This does not replace classical analysis; rather, it provides a principled setting in which projection and truncation can be discussed as explicit operators on the dynamics.

Weak/Energy form (integration by parts). Let Ω be a periodic box (or a domain with suitable boundary conditions) and let u be a divergence-free velocity field. Testing the momentum equation against a smooth divergence-free test function φ and integrating by parts yields a weak formulation of the form

$$\langle \partial_t u, \varphi \rangle + \langle (u \cdot \nabla) u, \varphi \rangle + \nu \langle \nabla u, \nabla \varphi \rangle = \langle f, \varphi \rangle, \tag{11}$$

where $\langle \cdot, \cdot \rangle$ denotes an L^2 pairing and f is a forcing term (optional). In particular, choosing $\varphi = u$ (formally) gives the classical energy balance/inequality, with the viscous term producing dissipation through $\|\nabla u\|_2^2$. This highlights why boundary conditions matter (boundary terms) and motivates our use of periodic settings as minimal reproducible configurations for isolating core mechanisms.

Fourier/modal representation. On periodic domains, write

$$u(x,t) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k(t)\, e^{ik \cdot x}, \tag{12}$$

so that phase information appears naturally through complex coefficients $\hat{u}_k$. The viscous term becomes diagonal in Fourier space:

$$\widehat{\nu \Delta u}_k(t) = -\nu |k|^2 \hat{u}_k(t), \tag{13}$$

which makes explicit the damping of high-frequency modes and clarifies how viscosity stabilizes discrete-time evolution.

Projection as an operator (top-k truncation). A reduced observation or model can be written as an explicit projection π acting on modes, e.g. a top-k truncation

$$(\pi u)(x,t) = \sum_{|k| \le k_0} \hat{u}_k(t)\, e^{ik \cdot x}. \tag{14}$$

This provides a concrete and mathematically transparent instance of "representation": the dynamics may be smooth in the full state space, while projection (or chart selection) can introduce degeneracy, loss of single-valuedness, and ε–δ mismatch in the observed evolution. Our discriminants are designed to detect precisely when such projection-induced effects become significant.

2 Discriminant for representation validity

We consider discrete-time evolution $u_{n+1} = F(u_n)$ on an intrinsic state space and an observation/projection $z_n = \pi(u_n)$. The central question is whether the induced observed evolution can be treated as stable and effectively single-valued. We formalize this via an ε–δ correspondence test in observation space and a sufficient PASS/FAIL discriminant.

Interpretation of PASS. In this paper, `PASS` is interpreted as a *sufficient* validity condition for the chosen representation/projection: it indicates the observed evolution is stable enough to be treated as effectively single-valued over the tested regime. `FAIL` indicates potential projection-induced degeneracy, but is not a necessary condition for physical singularity.

2.1 Recurrence vs. Discriminant: why they are linked

Setup. Let $\mathcal{X}$ be a state space (e.g., a function space discretized to $\mathbb{R}^N$). Let $F : \mathcal{X} \to \mathcal{X}$ denote the one-step evolution operator (time integrator), and let $\pi : \mathcal{X} \to \mathcal{Z}$ be an observation/projection map into a reduced space $\mathcal{Z}$ (e.g., $\mathbb{R}^k$ with $k \ll N$). We write the discrete-time (recurrence) dynamics as

$$u_{n+1} = F(u_n), \qquad u_n \in \mathcal{X}, \tag{15}$$

and the observed sequence as $z_n = \pi(u_n) \in \mathcal{Z}$.

Definition 1 (Recurrence (discrete-time dynamics))**.** A *recurrence* is the iteration of a map F on a state space $\mathcal{X}$:

$$u_{n+1} = F(u_n). \tag{16}$$

When F is continuous, it defines a continuous discrete-time dynamical system on $\mathcal{X}$.

Proposition 1 (Continuity of iterates)**.** If $F : \mathcal{X} \to \mathcal{X}$ is continuous, then for every $n \in \mathbb{N}$ the iterate

$$F^n := \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}} \tag{17}$$

is continuous on $\mathcal{X}$.

Proof. Continuity is preserved under composition: if f and g are continuous, then $g \circ f$ is continuous. Applying this inductively yields continuity of F^n for all n. □

Remark 1 (Why observed irregularity can still appear)**.** The proposition above shows that *iteration alone does not break continuity* on the intrinsic space $\mathcal{X}$. Observed "irregularity" typically arises from the projection/observation map π. Two failure modes are especially important:

1. **Loss of single-valuedness (non-injectivity).** If there exist $u \neq v$ such that $\pi(u) = \pi(v) = z$ but $\pi(F(u)) \neq \pi(F(v))$, then a deterministic map $G : \mathcal{Z} \to \mathcal{Z}$ satisfying $z_{n+1} = G(z_n)$ is not well-defined at z. In this case, the reduced dynamics becomes set-valued, and "continuity of the observed evolution" is not the right question—the evolution is not a function.

2. ε–δ **mismatch under projection.** Even when G is approximately defined, local folding/rank loss of π (or of $\pi \circ F$) can cause neighborhoods in $\mathcal{X}$ to collapse in $\mathcal{Z}$, degrading the ε–δ correspondence in observation space.

Bridge to Toy Example 0. Even when the intrinsic recurrence is continuous, discontinuity-like patterns may appear after projection onto a restricted observation domain. Toy Example 2.2 illustrates this mechanism in its simplest form via branch-cut wrapping.

Role of the discriminant. The discriminant DISCR used in this paper is therefore not a test of whether F is continuous on $\mathcal{X}$. Instead, it is a *representation-validity test*: it checks whether the chosen projection/observation π yields a stable, single-valued (or approximately single-valued) evolution in the reduced space $\mathcal{Z}$ over the regime of interest.

2.2 Toy Example 0: Domain-induced discontinuity via branch-cut wrapping

We begin with an elementary one-dimensional recurrence that exhibits *apparent* discontinuity purely due to domain restriction (chart selection), rather than intrinsic loss of continuity in the underlying evolution.

Recurrence. Define

$$y_{n+1} = G(y_n), \qquad G(y) = \arctan(\tan y). \tag{18}$$

The function $\arctan(\cdot)$ returns values in the principal interval

$$I := \Big(-\frac{\pi}{2}, \frac{\pi}{2} \Big), \tag{19}$$

so $G : \mathbb{R} \to I$ "wraps" any input back into a single chart.

Underlying continuity vs. observed jumps. On the covering space $\mathbb{R}$, the phase y evolves continuously. However, the observation map implicit in G uses a *single* chart I with branch points at

$$\partial I = \Big\{ -\frac{\pi}{2}, \frac{\pi}{2} \Big\} \pmod{\pi}. \tag{20}$$

Whenever y crosses $\frac{\pi}{2} + k\pi$ or $-\frac{\pi}{2} + k\pi$, the wrapped value $G(y)$ exhibits a finite jump. Thus, the discontinuity is *domain-induced*: it arises from projecting a continuous phase onto a chart with a branch cut.

Interpretation (projection heuristic). This toy example captures, in the simplest possible form, the principle used throughout this paper: a smooth evolution in an intrinsic (or lifted) space can appear irregular after projection to a reduced observation space. The "irregularity" here is not a PDE singularity; it is a coordinate artifact induced by chart selection.

A minimal discriminant. A simple PASS/FAIL test can be defined by distance to the branch set. Let $d_I(y)$ be the distance from y to the nearest branch point (mod π):

$$d_I(y) := \min_{k\in\mathbb{Z}} \left| y - \left(\frac{\pi}{2} + k\pi\right) \right|. \tag{21}$$

For a tolerance $\varepsilon > 0$, declare

$$\mathsf{DISCR}_0(y) = \mathbf{1}\{\, d_I(y) \geq \varepsilon \,\}, \tag{22}$$

so that $\mathsf{DISCR}_0(y) = 1$ indicates a "safe" regime away from the branch cut, while $\mathsf{DISCR}_0(y) = 0$ flags potential representation-induced jumps.

3 Experiments: Burgers Recurrence and Fourier Top-k Projection

3.1 Purpose

We provide a fully reproducible, CPU-only demonstration of the projection–recurrence mechanism using the viscous Burgers equation on a periodic domain.

3.2 Model

We consider

$$u_t + u\,u_x = \nu u_{xx}, \qquad x \in [0,1],\ t \geq 0, \tag{23}$$

with periodic boundary conditions and initial condition

$$u(x,0) = \sin(2\pi x) + \tfrac{1}{2}\sin(4\pi x). \tag{24}$$

Spatial derivatives are computed using Fourier spectral differentiation. Time stepping is treated as a recurrence $u^{n+1} = F_{\Delta t,\nu}(u^n)$ with explicit convection and semi-implicit diffusion in Fourier space.

3.3 Fourier top-k projection

Let $\hat{u}$ denote the discrete Fourier transform. We define $\hat{u}_k(m) = \hat{u}(m)$ for $|m| \leq k$ and $\hat{u}_k(m) = 0$ otherwise, and reconstruct $u_k = \mathcal{F}^{-1}(\hat{u}_k)$. This serves as a concrete projection $\pi(u) = u_k$.

3.4 Metrics and discriminants

Note on ε–δ vs. empirical proxies. Our ε–δ correspondence is a representation-level notion; in the experiments we do not compute it directly. Instead, we use information-loss proxies (high-frequency energy ratio and reconstruction error) as sufficient indicators that the observation map becomes unstable under aggressive truncation.

We report a topology-like signature (zero-crossings + extrema count) for u and u_k and flag `TopoChanged` if they differ. We compute the high-frequency energy ratio $\mathrm{HF\_ratio}$ beyond the top-k band and the relative reconstruction error $\mathrm{ReconError} = \|u - u_k\|_2 / \|u\|_2$. We set `FoldFlag` as a *degeneracy proxy* if $\mathrm{HF\_ratio}$ exceeds a threshold or $\mathrm{ReconError}$ exceeds tolerance τ (it is not a geometric fold computed from a Jacobian). Finally, `DISCR` is PASS iff `TopoChanged=False`, `FoldFlag=False`, and $\mathrm{ReconError} \leq \tau$.

3.5 Reproducibility

A Docker container is provided to reproduce the parameter sweep over $\nu \in \{0.10, 0.05, 0.02, 0.01, 0.005\}$ and $k \in \{4, 8, 16, 32\}$ at $(N, \Delta t, T) = (256, 10^{-3}, 0.5)$, producing `output/results.csv` and `output/example.png`.

4 Results

We report a minimal set of outputs designed for rapid verification by reviewers and for reproducibility in automated evaluation. All experiments are CPU-first and packaged as a Docker container (see supplementary archive).

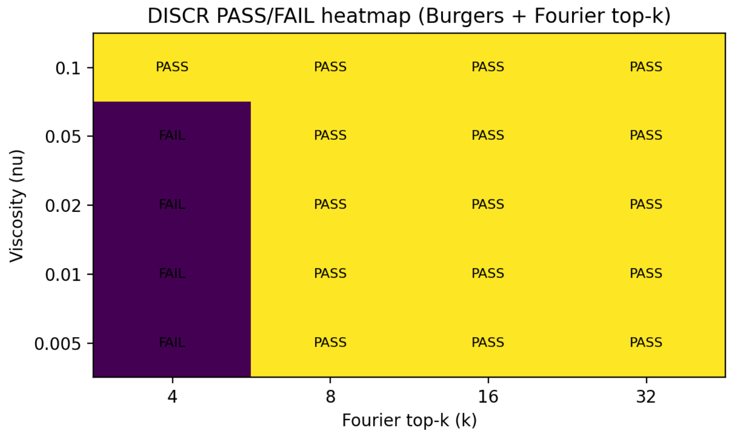

Figure 2: PASS/FAIL heatmap for the Burgers recurrence under Fourier top-k projection.

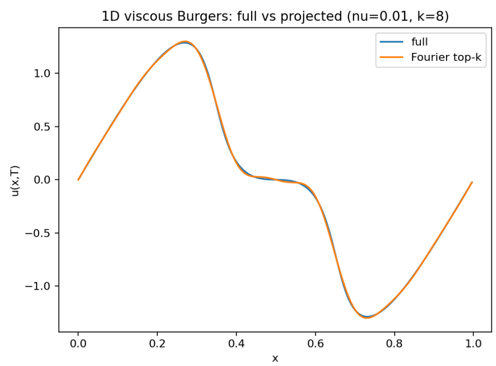

Figure 3: Representative run: 1D viscous Burgers solution (full) vs Fourier top-k projection ($\nu = 0.01$, $k = 8$).

Interpretation. The heatmap and Table 1 illustrate a clear "trust region": coarse truncation ($k = 4$) fails as viscosity decreases, while moderate truncations ($k \geq 8$) remain stable across the tested regime. This behavior is consistent with the representation-level hypothesis: projection can induce instability (or apparent irregularity) when too much high-frequency content is discarded.

5 Discussion

What the criterion is (and is not). Our ε–δ discriminant is a *representation-level* validity gate: it evaluates whether an observed/reduced evolution can be treated as effectively single-valued and stable under a chosen projection. It is not a proof of global Navier–Stokes regularity, and `FAIL` is not a necessary condition for physical singularity.

Relation to map viewpoints and projection literature. Viewing Navier–Stokes evolution through a time-Δt map $u_{n+1} = F_{\Delta t}(u_n)$ connects naturally to classical dynamical-systems treatments (time-T/Poincaré maps) and to modern recurrent-solution descriptions of turbulence. Separately, ROM/LES work has long emphasized that projection and filtering can induce representation-dependent effects (e.g., commutator errors, closure/stability issues). Our contribution complements these lines by offering a compact PASS/FAIL gate that can be checked in a reproducible setting before deploying low-dimensional models.

On proxies and sufficiency. In the minimal Burgers+Fourier top-k demo, we do not compute observation-space ε–δ correspondence directly. Instead, we use information-loss proxies (high-

nu \k	4	8	16	32
0.1	PASS	PASS	PASS	PASS
0.05	FAIL	PASS	PASS	PASS
0.02	FAIL	PASS	PASS	PASS
0.01	FAIL	PASS	PASS	PASS
0.005	FAIL	PASS	PASS	PASS

Table 1: DISCR PASS/FAIL over viscosity ν and Fourier top-k truncation.

frequency energy ratio and reconstruction error) as sufficient indicators that aggressive truncation makes the observation map unstable. This conservative choice favors low false-positive `PASS` outcomes, consistent with the sufficient-condition interpretation.

Extensions. Two immediate extensions are (i) replacing proxies with geometric diagnostics when available (e.g., Jacobian/rank-loss estimates for a chosen observation map), and (ii) moving beyond periodic settings to realistic boundary conditions and higher-dimensional turbulent datasets, where the same validity-gate philosophy can be applied with domain-appropriate diagnostics.

6 Conclusion

6.1 Summary of contributions

This work introduced a topology–recurrent viewpoint for interpreting observed irregularities in Navier–Stokes-related flow fields. Rather than treating every irregular pattern as an intrinsic analytic breakdown of the underlying dynamics, we emphasized the role of *representation*—in particular, *projection* and *chart selection*—in producing discontinuity-like phenomena. The central message is that smooth evolution in an extended or intrinsic representation can appear irregular after projection into a real-valued observation space.

Concretely, we contributed:

- a **recurrence-first perspective** $u_{n+1} = F(u_n)$ that aligns practical computation with a dynamical-systems formulation;
- a **projection-induced irregularity hypothesis**, where apparent "singularity-like" events are associated with local non-invertibility or rank-loss of an observation/projection map composed with the recurrence;
- a **phase/complex-lift heuristic** explaining how chart cuts and hidden phase dimensions can create jumps in real-valued coordinates without violating continuity on the covering space;
- a **2D reproducible demonstration and discriminants** that separates intrinsic dynamics from representation artifacts via topology signatures, Jacobian/rank-loss tests (or their empirical proxies), and PASS/FAIL criteria.

6.2 Scope statement (what this paper does not claim)

This paper does not claim to resolve the Clay Mathematics Institute Millennium Problem on global Navier–Stokes regularity. Our results are presented as a *representation-level* program: we provide mechanisms and tests indicating when observed irregularities can be explained by projection and chart effects, and we outline how these mechanisms may be used to build reliable low-dimensional models and validation criteria.

6.3 Result interpretation: topology-dependent regularity

A key outcome is the separation of two notions of "regularity": (i) intrinsic regularity of the underlying evolution in an appropriate state space, and (ii) observed regularity after projection into a reduced representation. In this framework, topology signatures and local degeneracy indicators (e.g., Jacobian/rank loss or their empirical proxies or their empirical proxies) serve as practical tools to detect when an observed field ceases to admit stable ε–δ correspondence under the chosen representation. This provides a concrete rationale for why certain automated modeling or reduced-order pipelines succeed in one regime and fail in another.

6.4 Outlook and research program

The proposed framework naturally suggests several next steps:

1. **Quantify fold energy**: formalize and estimate a fold-energy term $\mathcal{E}_{\text{fold}}$ from measurable Jacobian/rank statistics and relate it to intermittency indicators.
2. **Atlas-based extensions**: implement chart atlases (e.g., piecewise-affine charts) where degeneracy sets can be localized and tested with controlled tolerances.
3. **Low-dimensional model validity**: develop a certified criterion linking PASS/FAIL discriminants to the stability of reduced recurrences $z_{n+1} = \tilde{F}(z_n)$, and characterize failure modes when projection degeneracy increases.
4. **Noisy observation and data assimilation**: study robustness of topology/Jacobian tests under measurement noise and partial observation, clarifying which irregularities are physical versus representational.

6.5 Reproducibility and the role of AI assistance

A guiding objective of this project is reproducibility: the core claims are designed to be testable first in simple 2D settings and to be executed in a CPU-only environment (e.g., via Docker). We also employed AI assistance for drafting and structuring technical text and for accelerating implementation scaffolding; however, all mathematical claims, criteria, and conclusions were verified by the human researcher, who remains the accountable author of the work.

6.6 Closing remark

By elevating "quantity" and "irregularity" from mere aggregation and visual inspection to topology- and degeneracy-aware discriminants, this work aims to provide a mathematically honest pathway toward reliable low-dimensional modeling of complex flows and toward auditable computational pipelines in applied settings.

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A Weak/Energy Form of Incompressible Navier–Stokes via Integration by Parts

This appendix derives the standard weak formulation of the incompressible Navier–Stokes equations by multiplying against a divergence-free test function and applying integration by parts. The derivation is included for completeness; the main text uses only the resulting weak/energy identities.

A.1 Strong form

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a domain and consider

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, \tag{25}$$
$$\nabla \cdot u = 0, \tag{26}$$

where $u : \Omega \times [0, T] \to \mathbb{R}^d$ is the velocity field, p is pressure, $\nu > 0$ is viscosity, and f is an optional forcing term.

Boundary conditions. To remove boundary terms cleanly, one may assume either (i) a periodic box, or (ii) no-slip Dirichlet conditions $u|_{\partial\Omega} = 0$ together with test functions $\varphi|_{\partial\Omega} = 0$. Other boundary conditions are possible but require keeping the corresponding boundary integrals.

A.2 Test and integrate

Let φ be a smooth divergence-free test function ($\nabla \cdot \varphi = 0$). Multiply the momentum equation by φ and integrate over Ω:

$$\int_\Omega \partial_t u \cdot \varphi + \int_\Omega (u \cdot \nabla)u \cdot \varphi + \int_\Omega \nabla p \cdot \varphi - \nu \int_\Omega \Delta u \cdot \varphi = \int_\Omega f \cdot \varphi. \tag{27}$$

A.3 Integration by parts

(A) Viscous term. Using integration by parts componentwise,

$$-\int_\Omega \Delta u \cdot \varphi = \int_\Omega \nabla u : \nabla \varphi - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \varphi, \tag{28}$$

where $\frac{\partial u}{\partial n}$ denotes the normal derivative and $A : B = \sum_{i,j} A_{ij} B_{ij}$. Under periodic boundary conditions, or under Dirichlet conditions with $\varphi|_{\partial\Omega} = 0$, the boundary term vanishes and the viscous contribution becomes $\nu \int_\Omega \nabla u : \nabla \varphi$.

(B) Pressure term. Similarly,

$$\int_\Omega \nabla p \cdot \varphi = -\int_\Omega p\,(\nabla \cdot \varphi) + \int_{\partial\Omega} p\,\varphi \cdot n. \tag{29}$$

For divergence-free test functions ($\nabla \cdot \varphi = 0$) and periodic or compatible boundary conditions (so that the boundary term vanishes), the pressure term drops out of the weak formulation.

A.4 Weak form

Collecting terms gives the standard weak form:

$$\langle \partial_t u, \varphi \rangle + \langle (u \cdot \nabla)u, \varphi \rangle + \nu \langle \nabla u, \nabla \varphi \rangle = \langle f, \varphi \rangle, \qquad \forall \varphi \in V, \tag{30}$$

where $\langle a, b \rangle = \int_\Omega a \cdot b$ denotes the L^2 pairing and V is the space of admissible divergence-free test functions (with boundary conditions consistent with the problem setting).

A.5 Energy identity (formal)

Taking $\varphi = u$ (formally) yields the familiar energy balance/inequality

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \nu \|\nabla u\|_2^2 = \langle f, u \rangle, \tag{31}$$

highlighting that viscosity damps high-frequency components and stabilizes the discrete-time map viewpoint used in the main text.

AI Co-Scientist Challenge Korea Paper Checklist

Claims

Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

Answer: [Yes]

Justification: The abstract and introduction explicitly state the paper's core claim that observed irregularities in recurrent Navier–Stokes dynamics can arise from projection- or representation-induced effects rather than intrinsic analytic singularities. These claims are developed and supported in Sections 1, 1.1, and 2, and summarized in Section 5.1 (Summary of contributions).

Guidelines:

- The answer NA means that the abstract and introduction do not include the claims made in the paper.
- The abstract and/or introduction should clearly state the claims made, including the contributions made in the paper and important assumptions and limitations. A No or NA answer to this question will not be perceived well by the reviewers.
- The claims made should match theoretical and experimental results, and reflect how much the results can be expected to generalize to other settings.
- It is fine to include aspirational goals as motivation as long as it is clear that these goals are not attained by the paper.

Limitations

Question: Does the paper discuss the limitations of the work performed by the authors?

Answer: [Yes]

Justification: The limitations and scope of the work are clearly discussed in Section 5.2 (Scope statement), where the paper explicitly states that it does not resolve global Navier–Stokes regularity and focuses on representation-level phenomena. Additional practical limitations and future extensions are discussed in Section 5.4 (Outlook and research program).

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