

# Epipolar Geometry with Fundamental Matrix

CS484 Introduction to Computer Vision

Homework 3 supplementary slides

- Demosaic the raw image using the following three methods

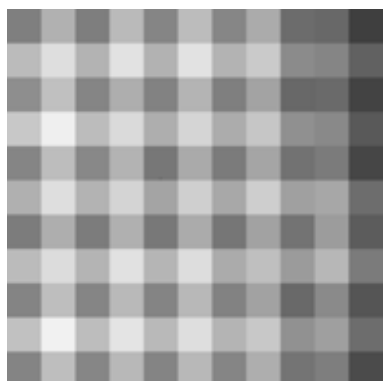
1. Down-sampling
2. Linear interpolation
3. Bicubic interpolation



Raw image



Color image  
(reference)



Bayer pattern  
(RGGB)



Demosaic image

- Bicubic interpolation

$$f(x, y) = \sum_{i=0}^3 \sum_{j=0}^3 a_{ij} x^i y^j \quad x, y \in [0, 1] \times [0, 1]$$

- 16 unknown coefficients  $a_{ij}$

- 16 known equations

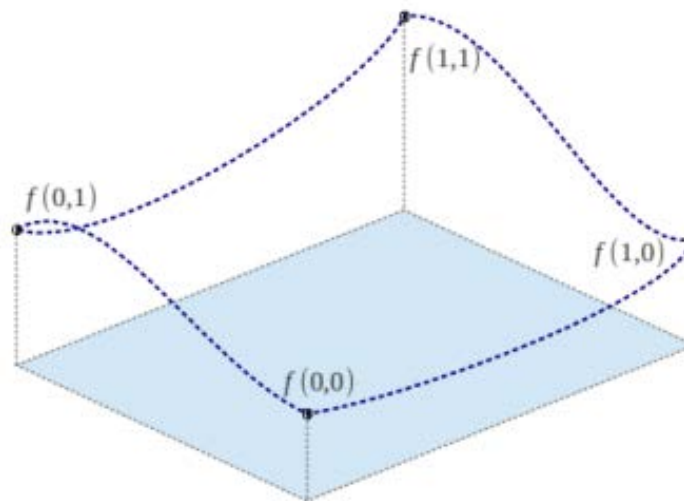
- $f(0, 0), f(0, 1), f(1, 0), f(1, 1)$
- $f_x(0, 1), f_x(0, 0), f_x(1, 0), f_x(1, 1)$
- $f_y(0, 1), f_y(0, 0), f_y(1, 0), f_y(1, 1)$
- $f_{xy}(0, 1), f_{xy}(0, 0), f_{xy}(1, 0), f_{xy}(1, 1)$

- Difference approximation

$$f_x(x, y) = [f(x+1, y) - f(x-1, y)]/2$$

$$f_y(x, y) = [f(x, y+1) - f(x, y-1)]/2$$

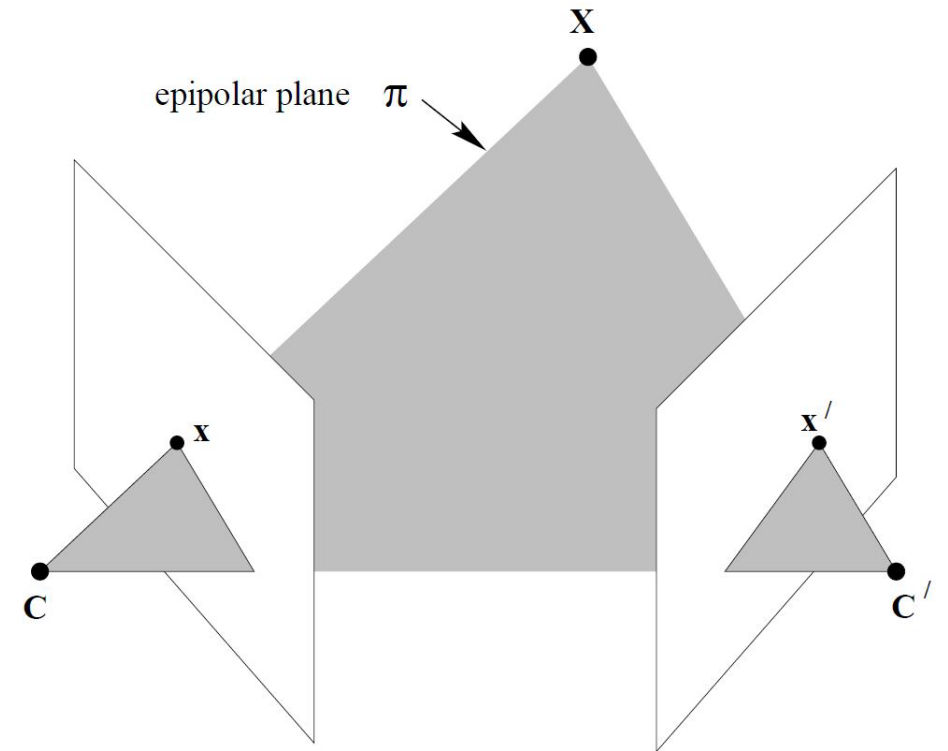
$$f_{xy}(x, y) = [f(x+1, y+1) - f(x-1, y-1) + f(x+1, y-1) - f(x-1, y+1)]/4$$



# Epipolar geometry

World coordinate  $\mathbf{X}$  projects to image coordinate  $\mathbf{x}$  and  $\mathbf{x}'$

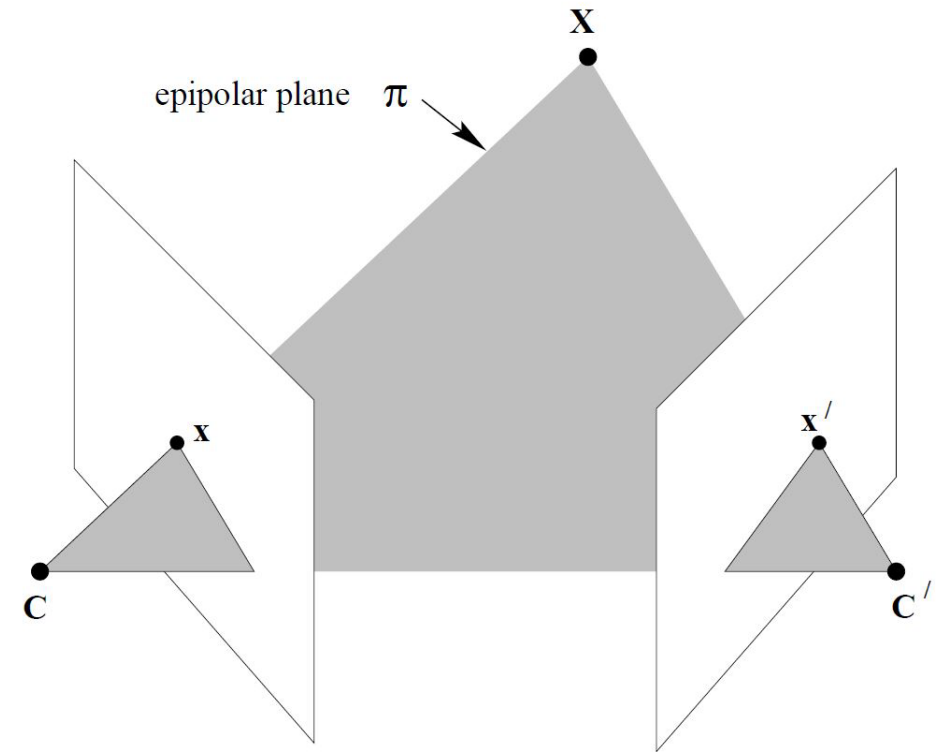
What is the relation between  $\mathbf{x}$  and  $\mathbf{x}'$ ?



# Epipolar geometry

The camera centers  $\mathbf{C}$  and  $\mathbf{C}'$ , a 3D point  $\mathbf{X}$ , and its image  $\mathbf{x}$  and  $\mathbf{x}'$  lie in a common plane  $\pi$ .

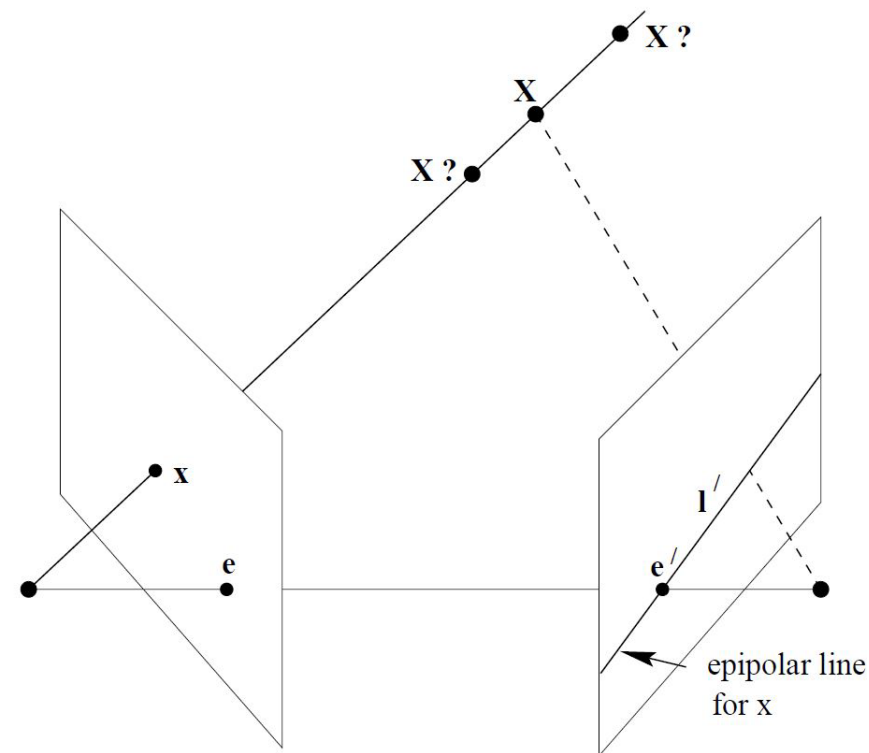
The plane  $\pi$  is **epipolar plane**.



# Epipolar geometry

World coordinate  $\mathbf{X}$  projects to image coordinate  $\mathbf{x}$ , but It can't distinguish with dots on the ray from  $\mathbf{C}$  to  $\mathbf{X}$ .

The projection of the ray from  $\mathbf{C}$  to  $\mathbf{X}$  on the image plane 2 is the line  $\mathbf{l}'$ .

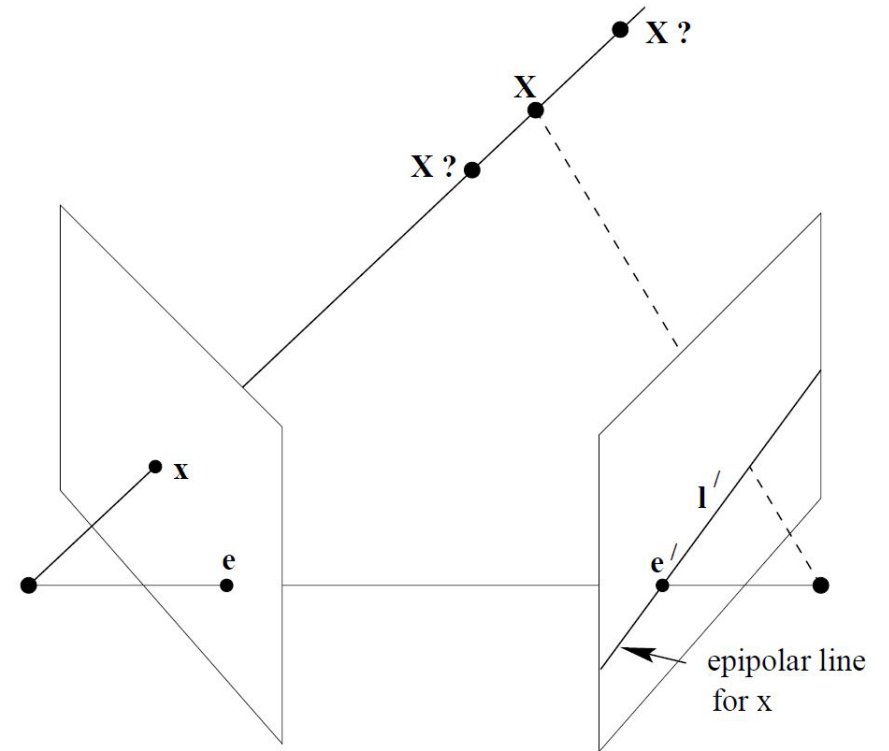


# Epipolar geometry

The line  $\mathbf{l}'$  is the **epipolar line**.

The projection of  $\mathbf{X}$  should be on the line  $\mathbf{l}'$ .

It is also the intersection of epipolar plane and image plane.



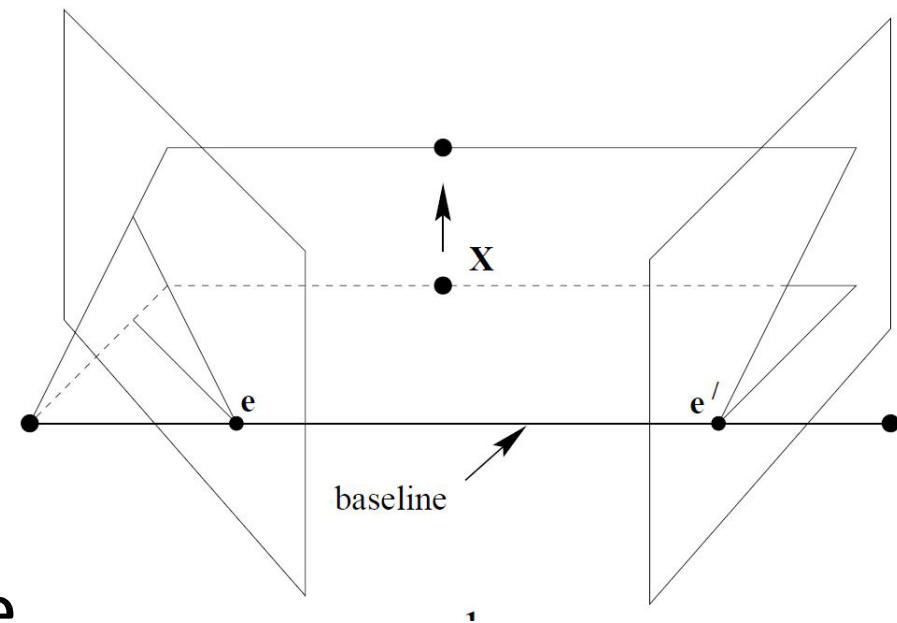
# Epipolar geometry

Intersection of the epipolar planes is **baseline**.

**C** projects to  **$e'$** , that every epipolar line cross. The point  **$e'$**  is **epipole**.

The epipole is the intersection of the baseline and the image plane.

Every epipolar line intersect on the epipole.





# Fundamental matrix

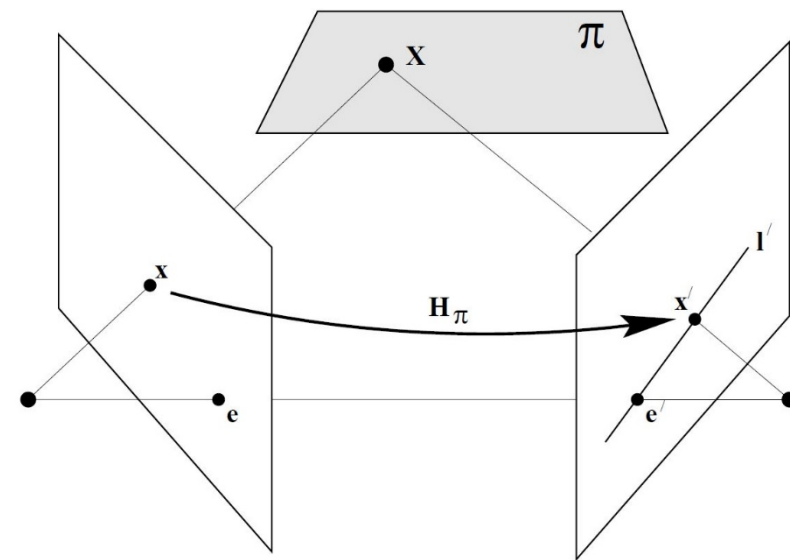
We want to know the relation between  $\mathbf{x}$  and  $\mathbf{l}'$ .

The line  $\mathbf{l}'$  can be represented by

$$a'x' + b'y' + c' = 0$$

$\mathbf{l}'$  can be define as  $\mathbf{l}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$   $\mathbf{l}'^T \mathbf{x}' = \mathbf{x}'^T \mathbf{l}' = 0$

The scale of  $\mathbf{l}'$  can be changed.



# Fundamental matrix

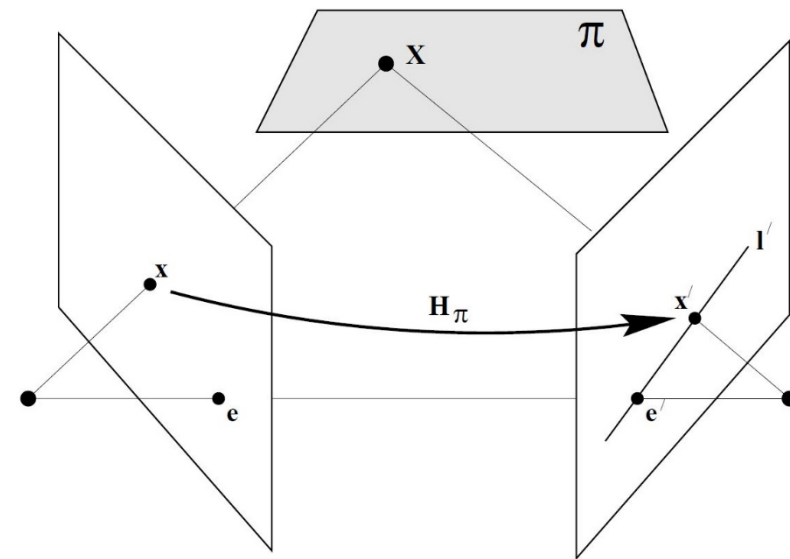
$\mathbf{l}'$  pass through  $\mathbf{x}'$  and  $\mathbf{e}'$ .

$\mathbf{l}'$  is perpendicular to both  $\mathbf{x}'$  and  $\mathbf{e}'$

$\mathbf{l}'$  can be written as  $\mathbf{l}' = \mathbf{e}' \times \mathbf{x}'$

Cross product can be represent by multiplication with a skew-symmetric matrix

$$[\mathbf{e}']_{\times} = \begin{bmatrix} 0 & -e'_3 & e'_2 \\ e'_3 & 0 & -e'_1 \\ -e'_2 & e'_1 & 0 \end{bmatrix} \quad \mathbf{e}' \times \mathbf{x}' = [\mathbf{e}']_{\times} \mathbf{x}'$$



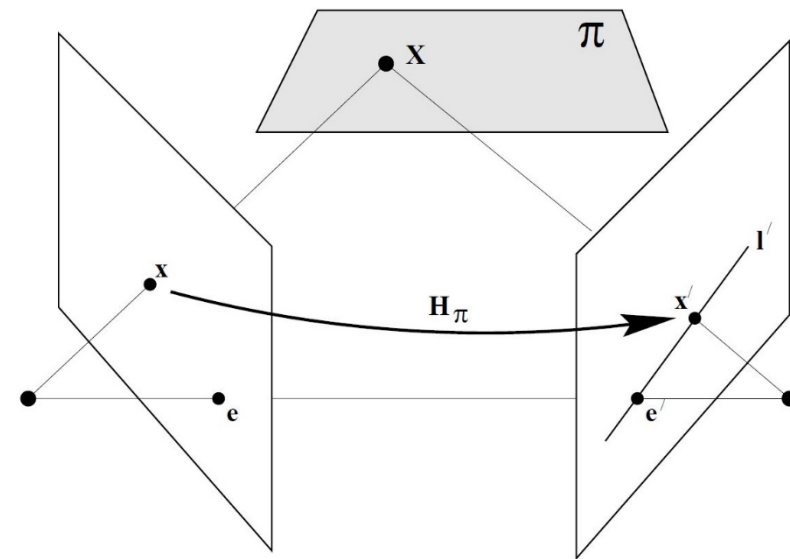
# Fundamental matrix

$\mathbf{x}$  can project to **any** plane  $\pi$ . The projected point is  $\mathbf{X}$ .

The transformation from a 2D plane to another 2D plane is homography.

Homography can be represented by 3x3 non-singular matrix.

Again,  $\mathbf{X}$  can project to the image plane. The projected point is  $\mathbf{H}_\pi \mathbf{x}$ , where  $\mathbf{H}_\pi$  is the homography from the image plane through plane  $\pi$  to another image plane.



# Fundamental matrix

$\mathbf{H}_\pi \mathbf{x}$  should be on the epipolar line  $\mathbf{l}'$   
whether  $\mathbf{H}_\pi \mathbf{x}$  is not same with  $\mathbf{x}'$ .

Then,  $\mathbf{l}'$  can be written as

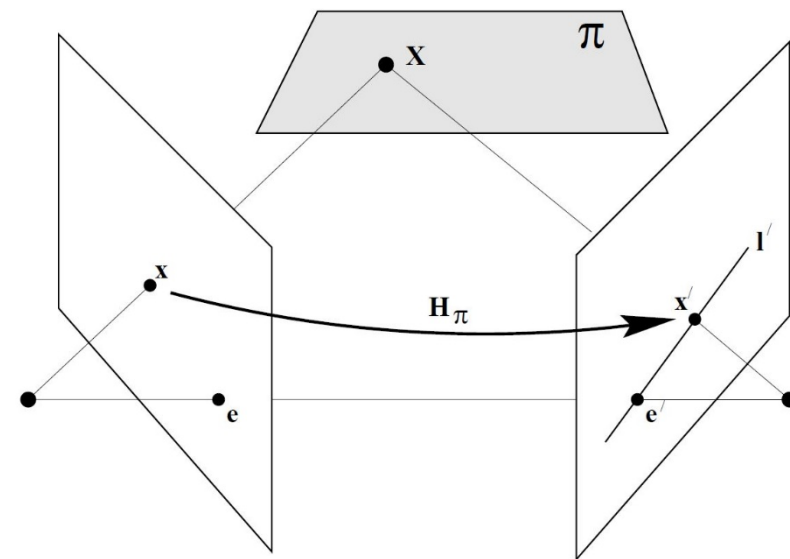
$$\mathbf{l}' = \mathbf{e}' \times \mathbf{H}_\pi \mathbf{x} = [\mathbf{e}']_\times \mathbf{H}_\pi \mathbf{x}$$

The fundamental matrix  $\mathbf{F}$  is

$$\mathbf{F} = [\mathbf{e}']_\times \mathbf{H}_\pi$$

The relation between  $\mathbf{x}$  and  $\mathbf{x}'$  is

$$\mathbf{x}'^T \mathbf{F} \mathbf{x} = \mathbf{x}'^T \mathbf{l}' = 0$$



# The properties of fundamental matrix

- $F$  is a rank 2 homogeneous matrix with 7 degrees of freedom.
- **Point correspondence:** If  $\mathbf{x}$  and  $\mathbf{x}'$  are corresponding image points, then  $\mathbf{x}'^T F \mathbf{x} = 0$ .
- **Epipolar lines:**
  - ◇  $\mathbf{l}' = F \mathbf{x}$  is the epipolar line corresponding to  $\mathbf{x}$ .
  - ◇  $\mathbf{l} = F^T \mathbf{x}'$  is the epipolar line corresponding to  $\mathbf{x}'$ .
- **Epipoles:**
  - ◇  $F \mathbf{e} = 0$ .
  - ◇  $F^T \mathbf{e}' = 0$ .
- **Computation from camera matrices  $P, P'$ :**
  - ◇ General cameras,  
 $F = [\mathbf{e}']_{\times} P' P^+$ , where  $P^+$  is the pseudo-inverse of  $P$ , and  $\mathbf{e}' = P' \mathbf{C}$ , with  $P \mathbf{C} = 0$ .
  - ◇ Canonical cameras,  $P = [I \mid \mathbf{0}]$ ,  $P' = [M \mid \mathbf{m}]$ ,  
 $F = [\mathbf{e}']_{\times} M = M^{-T} [\mathbf{e}]_{\times}$ , where  $\mathbf{e}' = \mathbf{m}$  and  $\mathbf{e} = M^{-1} \mathbf{m}$ .
  - ◇ Cameras not at infinity  $P = K[I \mid \mathbf{0}]$ ,  $P' = K'[R \mid \mathbf{t}]$ ,  
 $F = K'^{-T} [\mathbf{t}]_{\times} R K^{-1} = [K' \mathbf{t}]_{\times} K' R K^{-1} = K'^{-T} R K^T [K R^T \mathbf{t}]_{\times}$ .

# Eight-point algorithm

We want to get a fundamental matrix from two images in different view.

In the images, at least 8 corresponding points are given.

**Left image**



**Right image**



# Eight-point algorithm

We want to get a fundamental matrix from two images in different view.

In the images, at least 8 corresponding points are given.

If there are  $m$  correspondences, they satisfy

$$\mathbf{x}_i'^T \mathbf{F} \mathbf{x}_i = 0 \quad i = 1, \dots, m \quad \text{where} \quad \mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} \quad \mathbf{x}_i' = \begin{bmatrix} x_i' \\ y_i' \\ 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}$$

# Eight-point algorithm

It can be represented by 9 unknown linear system.

$$\mathbf{A}\mathbf{f} = 0$$

where

$$\mathbf{A} = \begin{bmatrix} xx' & xy' & x & yx' & yy' & y & x' & y' & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ xx' & xy' & x & yx' & yy' & y & x' & y' & 1 \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix}$$



# Eight-point algorithm

The only nonzero solution of  $\mathbf{A}\mathbf{x} = 0$  can exist if  $rank(\mathbf{A}) = 9 - 1 = 8$

Each correspondence make one equation (a row of  $\mathbf{A}$ )

It need eight points!

# Eight-point algorithm: Implementation

1.  $\mathbf{f} \leftarrow$  the eigenvector of  $\mathbf{A}^T \mathbf{A}$  corresponding the smallest eigenvalue.
2.  $\mathbf{F}$  ( $3 \times 3$  fundamental matrix)  $\leftarrow \text{reshape}(\mathbf{f})$
3.  $\mathbf{F} = \mathbf{U}\mathbf{S}\mathbf{V}^T$  where  $\mathbf{U}$ ,  $\mathbf{S}$ ,  $\mathbf{V}$  are a singular value decomposition for  $\mathbf{F}$ .
4. Make the minimum singular value for  $\mathbf{S}$  become zero. (the diagonal entries of  $\mathbf{S}$  are the singular values for  $\mathbf{S}$ .)
5.  $\mathbf{F} \leftarrow \mathbf{U}\mathbf{S}\mathbf{V}^T$  by using modified  $\mathbf{S}$  at 4.

# Eight-point algorithm: Proof

If there are more than 8 correspondences, we should get an approximation.

$$\min_{\mathbf{f}} \|\mathbf{A}\mathbf{f}\|^2 \quad \text{subject to} \quad \|\mathbf{f}\|^2 = 1$$

$$\mathbf{g}(\mathbf{f}) = \|\mathbf{A}\mathbf{f}\|^2 = (\mathbf{A}\mathbf{f})^T (\mathbf{A}\mathbf{f}) = \mathbf{f}^T \mathbf{A}^T \mathbf{A} \mathbf{f}$$

$$\mathbf{h}(\mathbf{f}) = 1 - \|\mathbf{f}\|^2 = 1 - \mathbf{f}^T \mathbf{f}$$

# Eight-point algorithm: Proof

Make the Lagrangian of the optimization.

$$L(\mathbf{f}, \lambda) = \mathbf{g}(\mathbf{f}) - \lambda \mathbf{h}(\mathbf{f}) = \mathbf{f}^T \mathbf{A}^T \mathbf{A} \mathbf{f} - \lambda (1 - \mathbf{f}^T \mathbf{f})$$

$$\begin{array}{ccc} \min_{\mathbf{f}} \|\mathbf{A}\mathbf{f}\|^2 & \longrightarrow & \min_{\mathbf{f}} L(\mathbf{f}, \lambda) \\ s.t. \quad \|\mathbf{f}\|^2 = 1 & & \end{array}$$

# Eight-point algorithm: Proof

Take derivatives of the Lagrangian.

$$\partial_{\mathbf{f}} L(\mathbf{f}, \lambda) = \mathbf{A}^T \mathbf{A} \mathbf{f} - \lambda \mathbf{f} = 0$$

$$\partial_{\lambda} L(\mathbf{f}, \lambda) = 1 - \mathbf{f}^T \mathbf{f} = 0$$

$\mathbf{f}$  is normalized eigenvector of  $\mathbf{A}^T \mathbf{A}$

# Eight-point algorithm: Proof

Let  $\mathbf{e}_\lambda$  is an eigenvector with eigenvalue  $\lambda$ .

$$\mathbf{g}(\mathbf{e}_\lambda) = \mathbf{e}_\lambda^T \mathbf{A}^T \mathbf{A} \mathbf{e}_\lambda = \mathbf{e}_\lambda^T \lambda \mathbf{e}_\lambda = \lambda$$

The eigenvector with the smallest eigenvalue is the result.

# Eight-point algorithm: Proof

Is the result  $F$  have rank 2?

It is not guaranteed.

We should reduce the dimension by singular value decomposition.

- Get SVD of  $F$   $\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$   
 $\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] \quad \mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] \quad \mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$
- Set the smallest singular values to 0  
 $\hat{\mathbf{\Sigma}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- Recompute  $F$   $\hat{\mathbf{F}} = \mathbf{U}\hat{\mathbf{\Sigma}}\mathbf{V}^T$

# Normalized eight-point algorithm

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels
- Use the eight-point algorithm to compute  $F$  from the normalized points
- Enforce the rank-2 constraint (for example, take SVD of  $F$  and throw out the smallest singular value)
- Transform fundamental matrix back to original units: if  $T$  and  $T'$  are the normalizing transformations in the two images, then the fundamental matrix in original coordinates is  $T'^T F T$



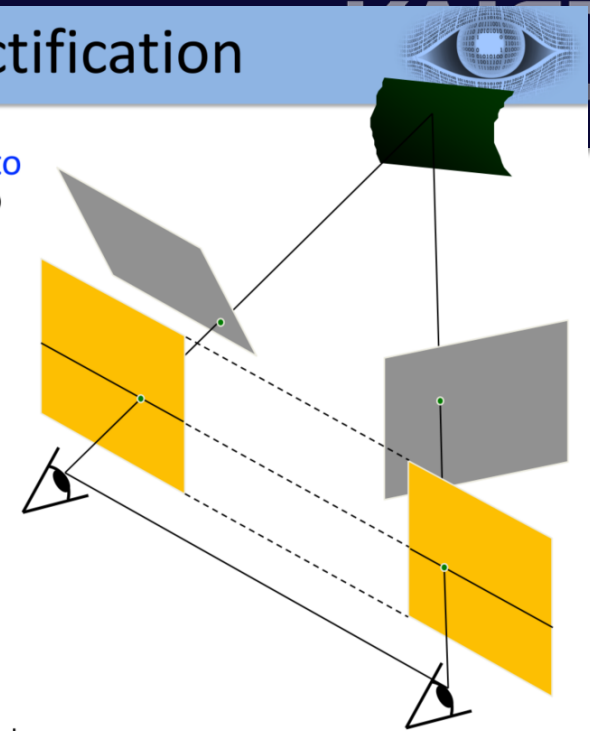
# Rectification

- Left and right image should be reprojected onto the common plane parallel to the line between camera centers  
➔ **Defined by homography matrices  $H$  and  $H'$**

- Calculating  $H$  and  $H'$  when  $F$  and corresponding points are known is already implemented as the matlab native function.
- You need to simply **apply**  $H$  and  $H'$  to the left and right images.

## Stereo image rectification

- Reproject image planes onto a common plane parallel to the line between camera centers
  - Pixel motion is horizontal after this transformation
  - Rectification: Two homographies (3x3 transforms), one for each input image reprojection
- C. Loop and Z. Zhang. [Computing Rectifying Homographies for Stereo Vision](#). IEEE Conf. Computer Vision and Pattern Recognition, 1999.



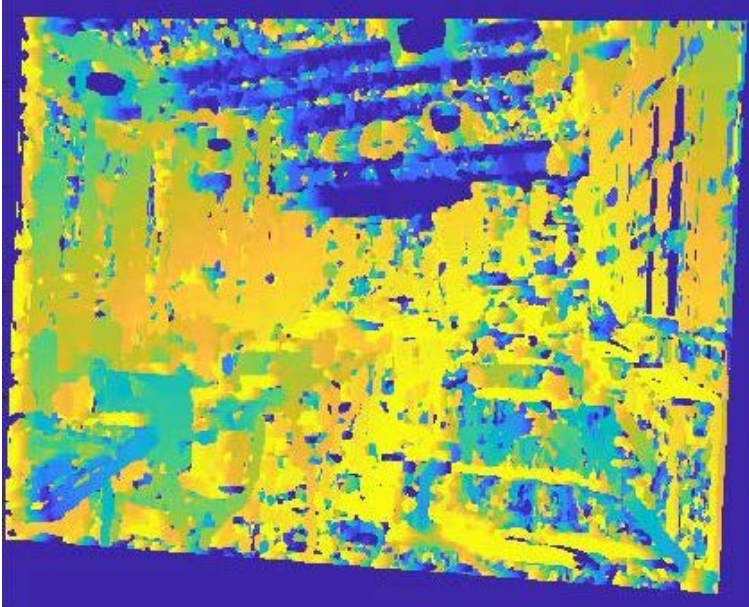
Lecture 10: Stereo Vision (2024) CS484: Introduction to Computer Vision

Slide credit: Kristen Grauman



- When you implement the part applying  $H$  and  $H'$  to the left and right images, following matlab functions might be helpful:
  - `imwarp()`  
<https://www.mathworks.com/help/images/ref/imwarp.html>
  - `projective2d()`  
<https://www.mathworks.com/help/images/ref/projective2d.html>
  - `transformPointsForward()`  
<https://www.mathworks.com/help/images/ref/affine2d.transformpointsforward.html>
- Your rectified images should be aligned well as shown in the previous page.

# Disparity Map



- This is one example of disparity map result.
- It may poorly works in some region due to the inherent limitation of uncalibrated stereo problem.
- Disparity map can be improved by using
  - Cost aggregation with box filter (+5 pts)
  - More sophisticated cost aggregation (+10 pts)
  - Calibrated cameras (not for this homework)