1.3 EXPONENTIAL & SINUSOIDAL SIGNALS (P14)

C.T.

$$\boxed{x(t) = C e^{at}}$$ C, a complex

$$x[n] = C \alpha^n$$ C, α complex

CASES : ① C & a REAL VALUED

② $C = Ae^{j\phi}$, $a = j\omega_0$

$\Rightarrow x(t) = A\cos(\omega_0 t + \phi) + j A\sin(\omega_0 t + \phi)$

③ $C = Ae^{j\phi}$, $a = r + j\omega_0$

$\Rightarrow x(t) = Ae^{rt}\cos(\omega_0 t + \phi) + j Ae^{rt}\sin(\omega_0 t + \phi)$

D.T. ① C & α REAL

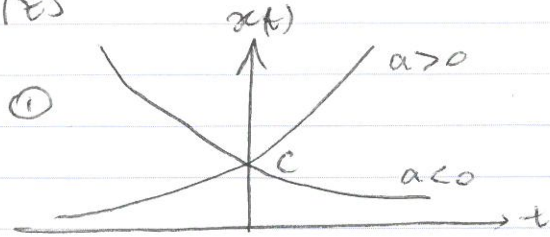
② $C = Ae^{j\phi}$, $\alpha = e^{j\omega_0}$

③ $C = Ae^{j\phi}$, $\alpha = |\alpha| e^{j\omega_0}$

LECTURE 2 NOTES

L2.1

CASE ①



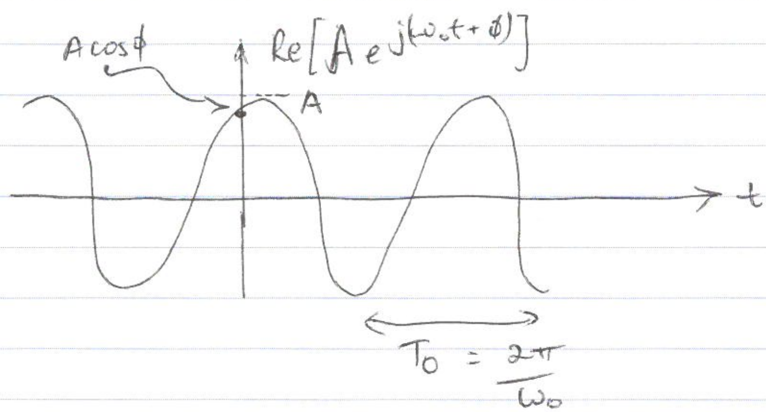
CASE ② IS PERIODIC BECAUSE

$$A e^{j[\omega_0 t + \phi]} = A e^{j[\omega_0 (t+T) + \phi]} \quad \forall t$$

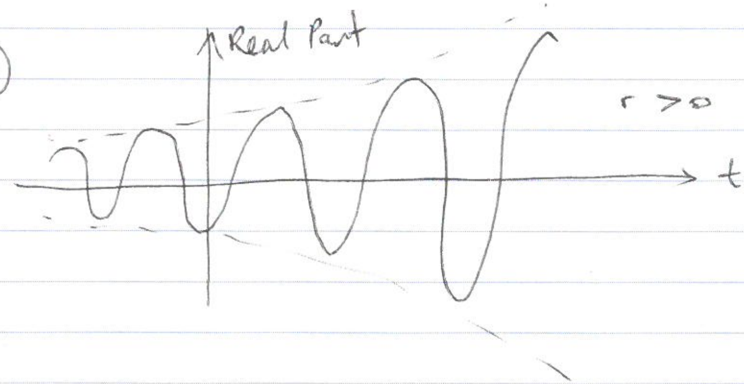
$$\Rightarrow \quad 1 = e^{j\omega_0 T} \quad \forall t$$

$$\Rightarrow \quad 1 = \cos(\omega_0 T) + j\sin(\omega_0 T)$$

$$\Rightarrow \quad \omega_0 T = \pm 2\pi \times \text{integer}$$

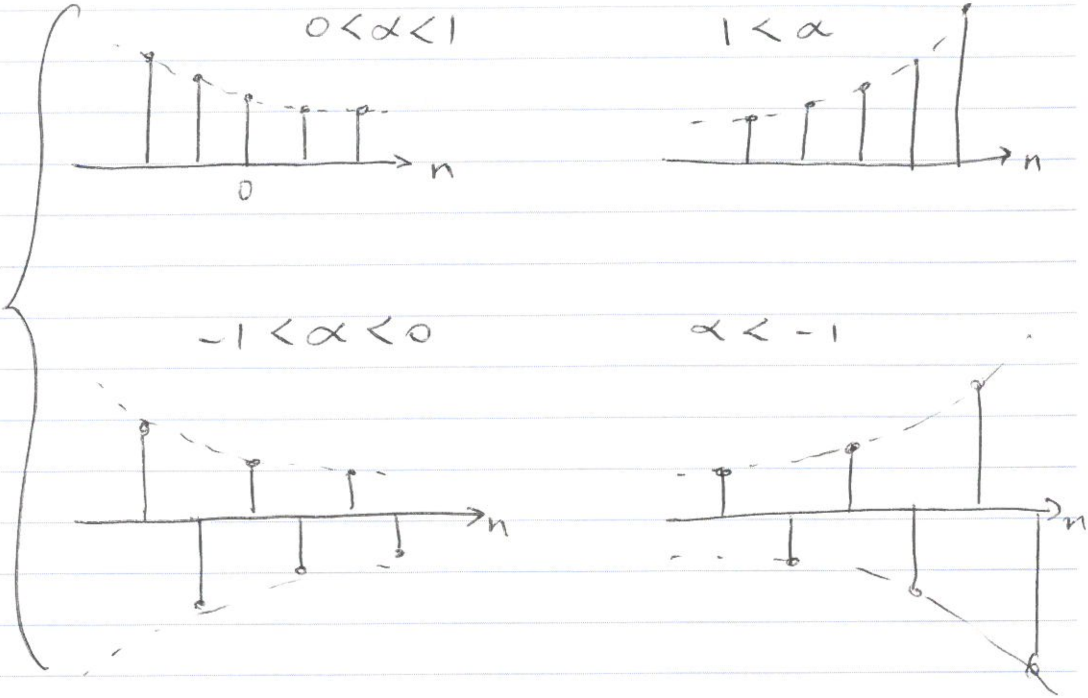


CASE ③

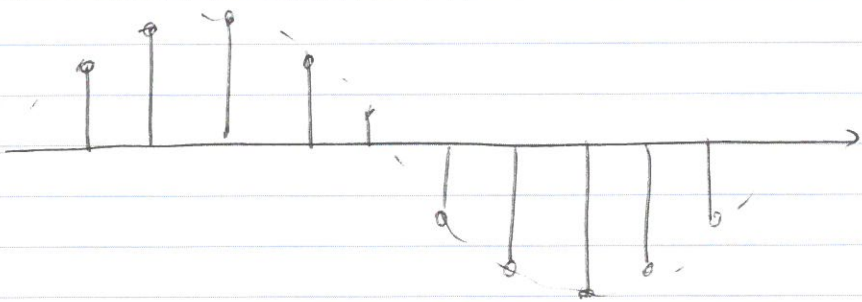


D.T.

case ①



CASE ② $x[n] = A\left[\cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi)\right]$



PERIODIC ?

PERIODICITY PROPERTIES OF D.T. COMPLEX EXP

(P 26)

C.T. $x(t) = e^{j\omega_0 t}$ is always periodic

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$

can always be solved, and the solⁿ is independent of t. (ie. for all t)

D.T. $x[n] = e^{j\omega_0 n}$ is NOT always periodic!

This is because n is an integer rather than a continuous variable (like t).

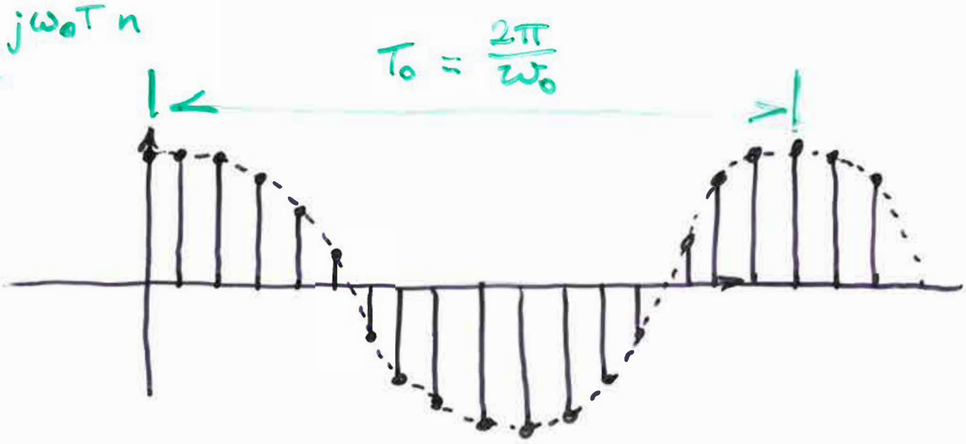
SUPPOSE WE SAMPLE THE C.T. SIGNAL

$x(t) = e^{j\omega_0 t}$ AT TIMES $t = 0, T, 2T, 3T \ldots$

TO GENERATE

$x[n] = e^{j\omega_0 T n}$

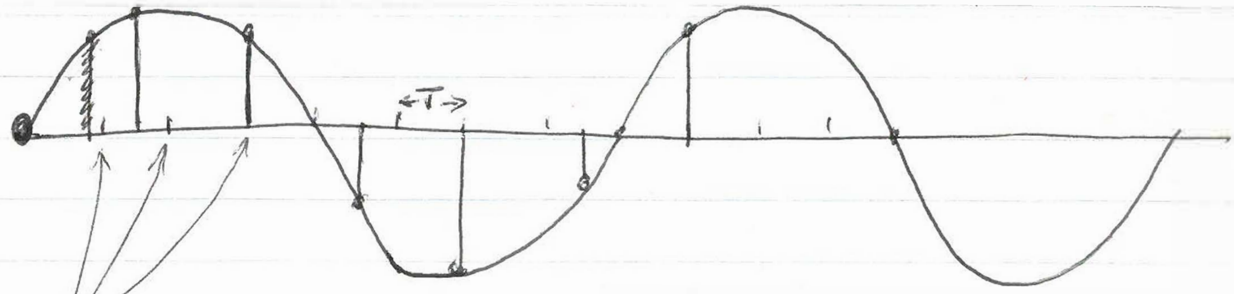
LOOKS LIKE:



WHEN T IS SMALL (ie. fast sampling)

BUT WILL THIS BE PERIODIC ?

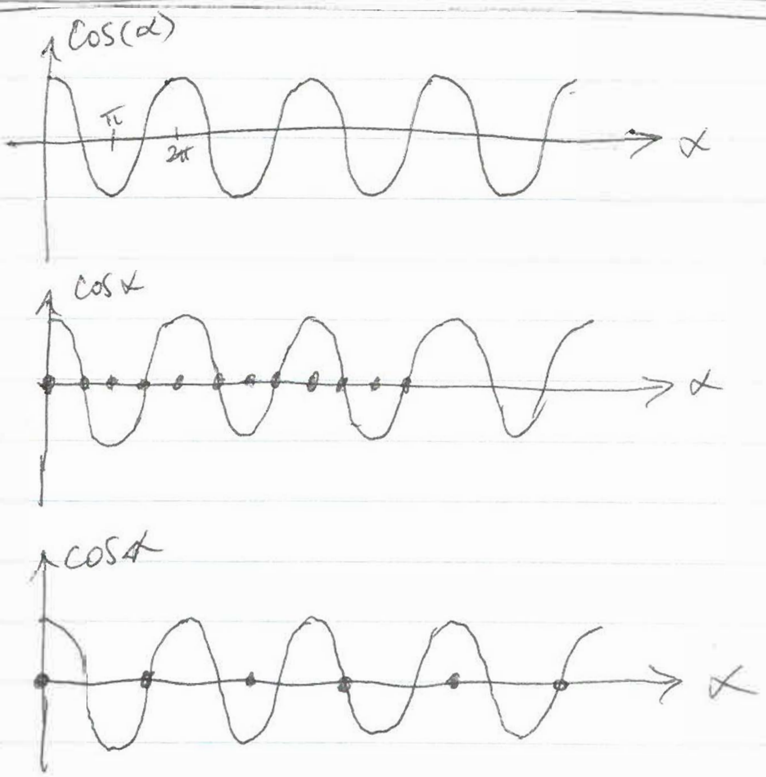
ONLY IF YOU CHOOSE T s.t. YOU SAMPLE THE **SAME** BIT OF THE CURVE ON IT'S NEXT CYCLE

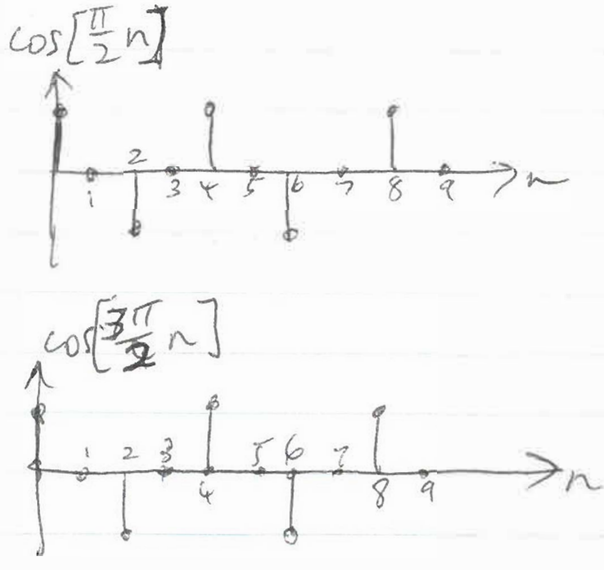


would be periodic
if sampled here & with
same period as the sine wave

NOT same period if
sampled at 1.5 T
actually 2wice the
period

Not periodic if sampled
at 1.5123 T





$\boxed{\text{frequencies between } -\pi \text{ \& } \pi \text{ repeat}}$

Same!

$x[n] = \cos(\omega_0 n)$ for various $\omega_0 = 2\pi f_0$ $(T=1)$

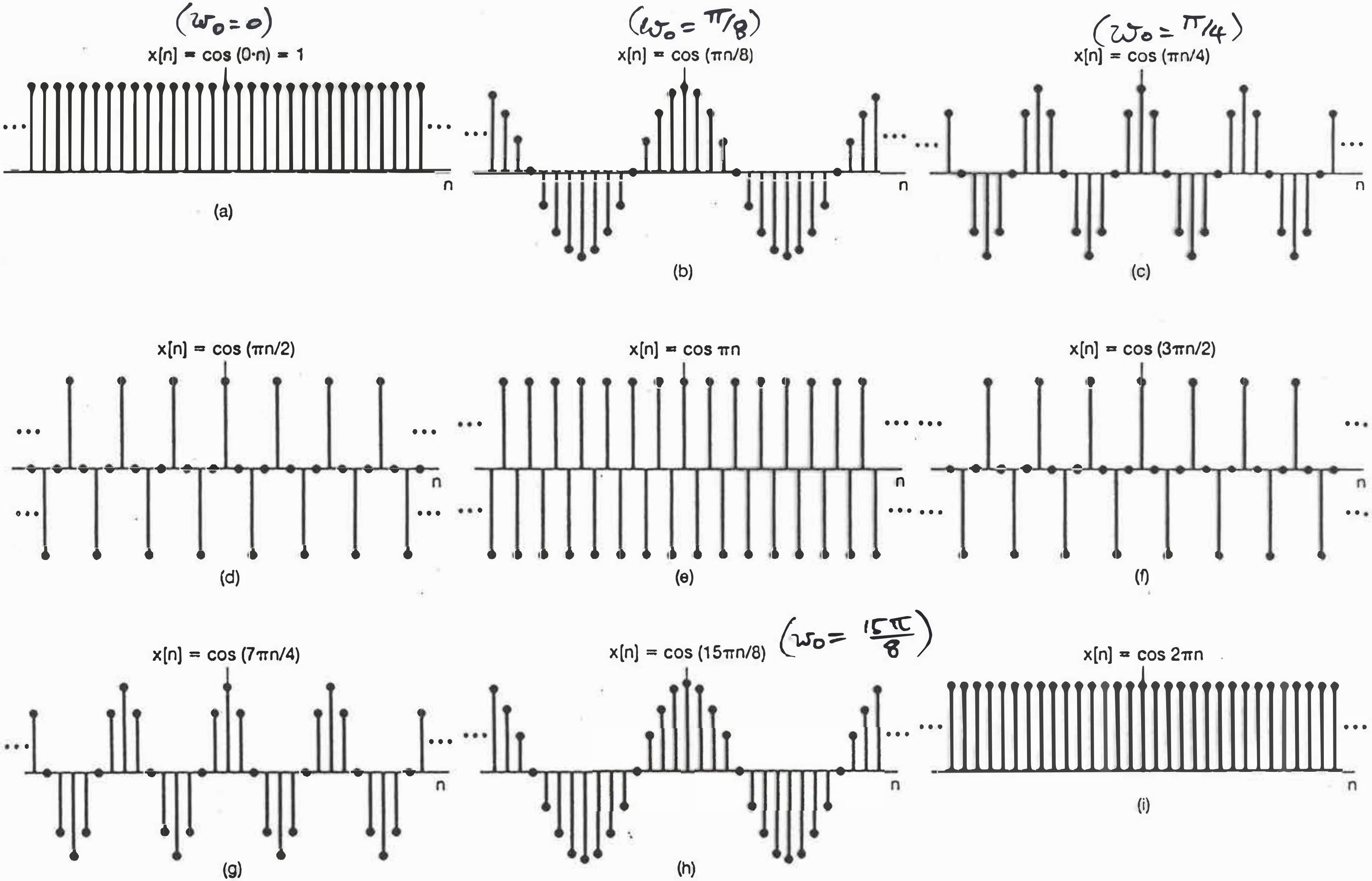


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

Oppenheim & Willsky
p 27

PERIODIC?

$$e^{j\omega_0 T n} \stackrel{?}{=} e^{j\omega_0 T(n+N)} \quad \forall n$$

$$1 \stackrel{?}{=} e^{j\omega_0 T N} = \cos(\omega_0 T N) + j\sin(\omega_0 T N)$$

$$\Rightarrow \quad \omega_0 T N \stackrel{?}{=} 2\pi \times \text{integer}$$

$\Rightarrow$ PERIODIC IF SAMPLING TIME T IS CHOSEN s.t. $N = \dfrac{2\pi(\text{integer})}{\omega_0 T}$ IS AN INTEGER!

EXAMPLE if $T = \dfrac{2\pi}{10\,\omega_0}$

$$\Rightarrow \quad x[n] = e^{j\omega_0 T n}$$

$$= e^{j\frac{2\pi}{10} n} \quad \text{IS PERIODIC}$$

EXAMPLES

① $x[n] = e^{j2n}$

② $x[n] = e^{j\frac{3}{4}\pi n}$

$\Big\}$ PERIODIC ?

ANSWERS

① NO. $j2n \Rightarrow \omega_0 T = 2$

$$\Rightarrow N = \frac{2\pi(\text{integer})}{2}$$

which can never be an integer

$\therefore$ NOT PERIODIC!

② YES $N = \dfrac{2\pi(\text{integer})}{\frac{3}{4}\pi} \quad \therefore N_0 = 8$

UNIT IMPULSE & STEP FUNCTIONS (P30)

D.T.

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

DIFFERENCE EQUATIONS (cf. DIFFERENTIAL EQNS IN C.T.)

eg.

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

OR

$$= \sum_{k=0}^{\infty} \delta[n-k]$$

SAMPLING PROPERTY

$$x[n]\,\delta[n] = x[0]\,\delta[n]$$

$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0]$$

THE DISCRETE TIME FOURIER TRANSFORM

(p 358)

DT FT FOR APERIODIC SIGNAL

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$$

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$

$\int_{2\pi}$ MEANS INTEGRATE OVER ANY 2π RANGE

EXAMPLE : [p 362]

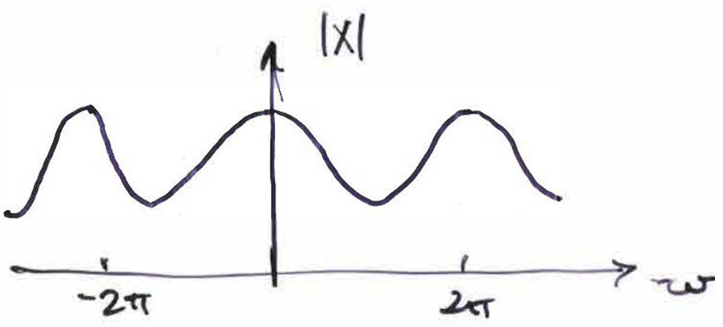
$$x[n] = a^n u[n] \qquad |a| < 1$$

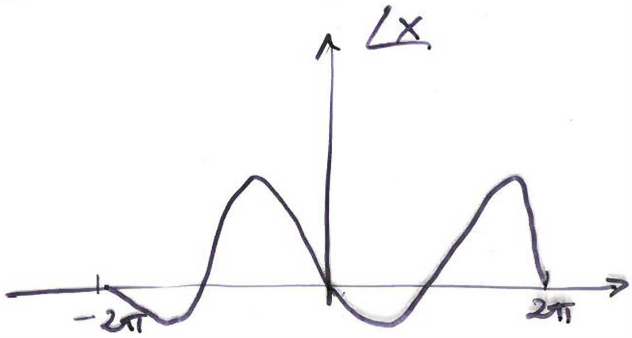
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

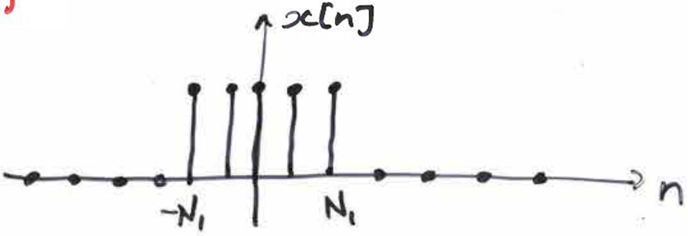
$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{1}{1 - a e^{-j\omega}}$$





EXAMPLE [p 365]



($N_1 = 2$)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= e^{j\omega N_1} + e^{j\omega(N_1-1)} + \ldots\ldots + e^{-j\omega N_1}$$

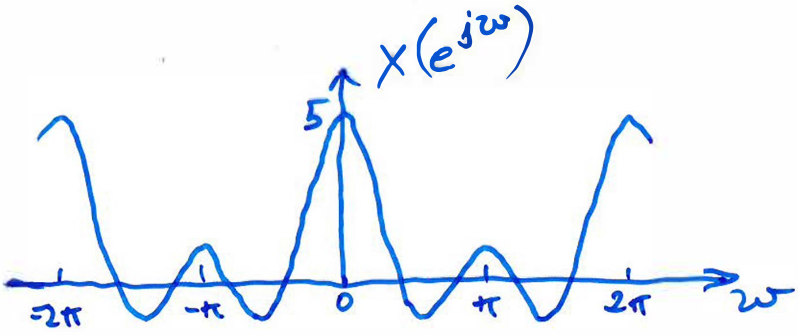
$$= e^{j\omega N_1}\left[1 + e^{-j\omega} + (e^{-j\omega})^2 + \ldots + (e^{-j\omega})^{2N_1}\right]$$

$$= e^{j\omega N_1}\ \frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{j\omega}}$$

$$= \frac{e^{j\omega N_1} - e^{-j\omega(N_1+1)}}{1 - e^{-j\omega}}$$

$$= \frac{e^{j\omega(N_1+\frac{1}{2})} - e^{-j\omega(N_1+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= \frac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\omega/2)}$$



($N_1 = 2$)

EXAMPLE:

$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1 \quad \forall \omega$$

DT FT FOR PERIODIC SIGNALS [p 367]

$$X(e^{j\omega}) = \boxed{\sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0)}$$

EXAMPLE :

$$x[n] = \cos(\omega_0 n) \qquad \omega_0 = 2\pi/N$$

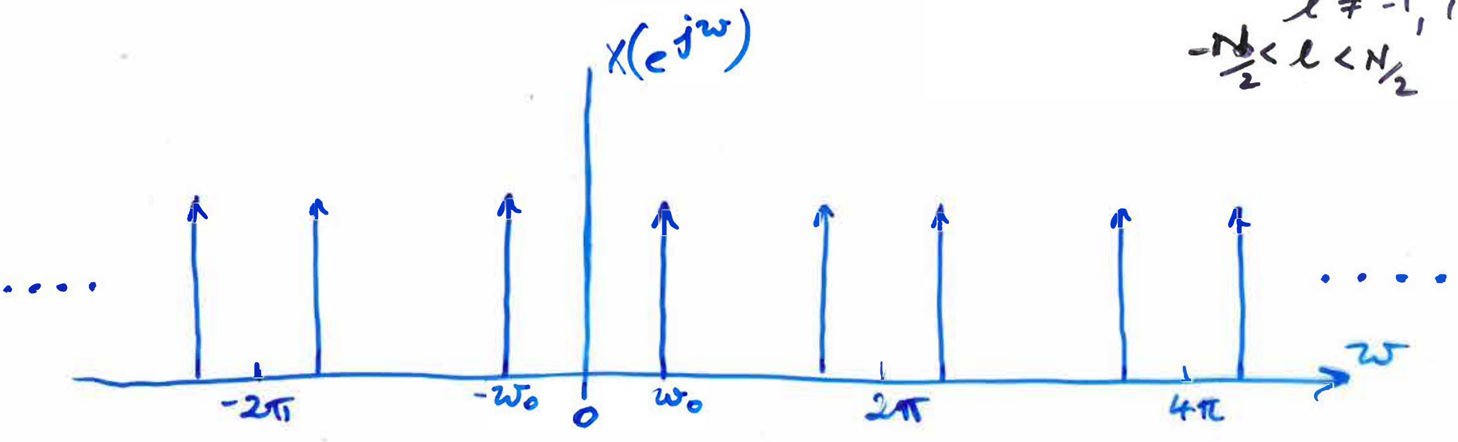
DTFT IS :

$$x[n] = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\Rightarrow \quad a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2}, \quad a_\ell = 0$$

$\ell \neq -1, 1$

$-\frac{N}{2} < \ell < N/2$



$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi\, \delta(\omega - \omega_0 - 2\pi l)$$

$$+ \sum_{l=-\infty}^{\infty} \pi\, \delta(\omega + \omega_0 - 2\pi l)$$

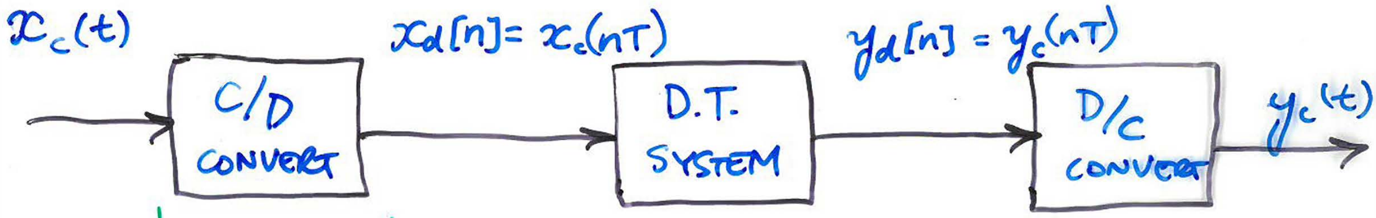
<u>OR</u>

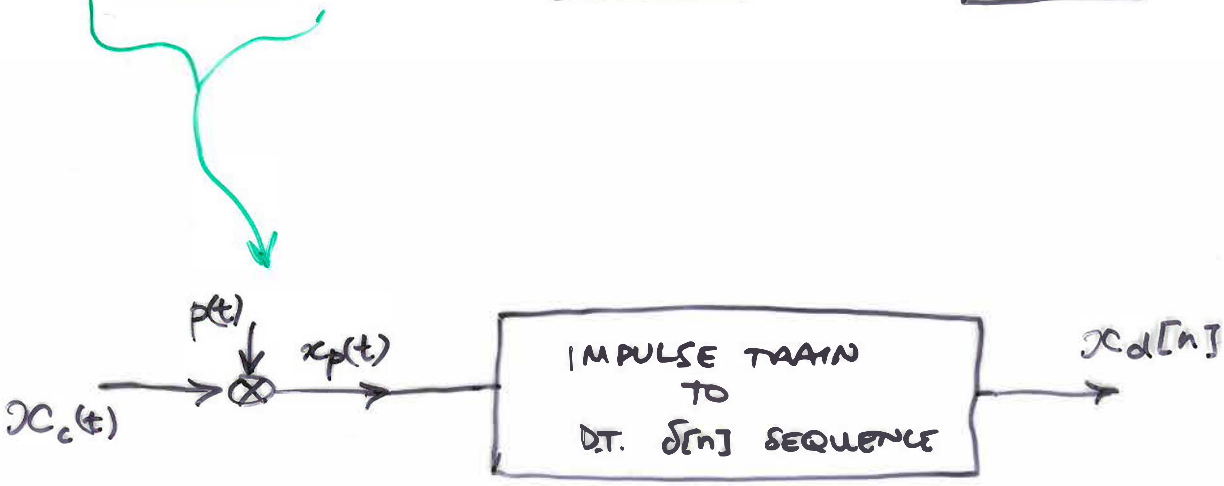
$$X(e^{j\omega}) = \pi\, \delta(\omega - \omega_0) + \pi\, \delta(\omega + \omega_0)$$

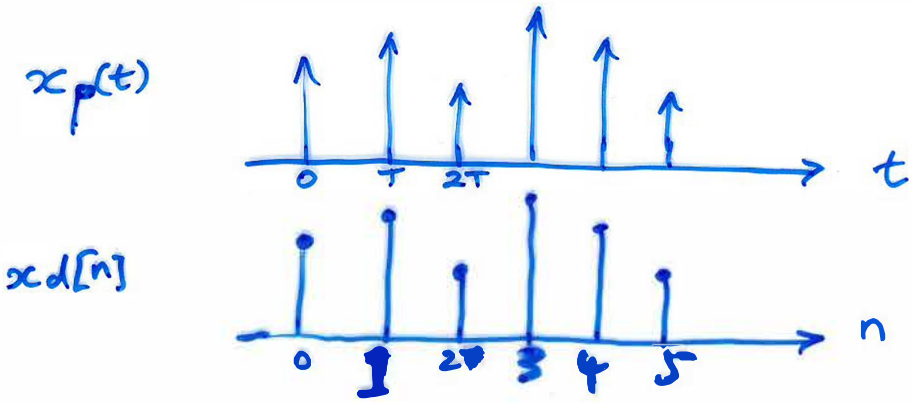
for $-\pi < \omega < \pi$

& 2π PERIODIC

D.T. PROCESSING OF C.T. SIGNALS







Q. So, HOW DO $\underbrace{X_p(j\omega)}_{\text{CT FT OF } x_p(t)}$ & $\underbrace{X_d(e^{j\Omega})}_{\text{DT FT OF } x_d[n]}$ RELATE?

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT)$$

$$\therefore\ X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\omega nT}$$

$$\& \quad X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]\, e^{-j\Omega n}$$

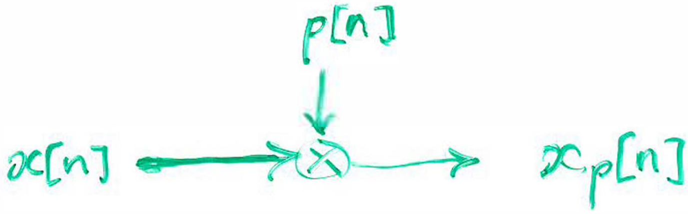
$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\Omega n}$$

$$\therefore \boxed{\text{C.T. FREQ } \omega = \frac{\text{DISC. TIME FREQ } \Omega}{T}}$$

FOR $x_p(t)$ TO HAVE SAME FREQ PROPERTIES AS $x_d[n]$

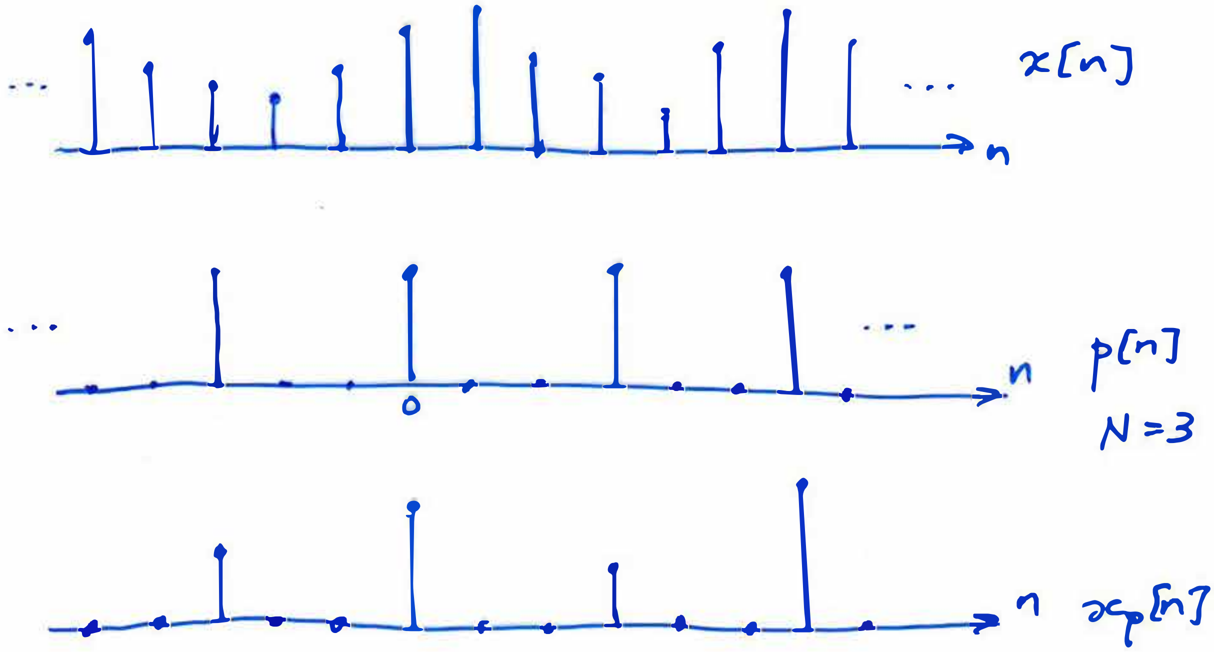
SAMPLING OF D.T. SIGNALS [p 545]

GIVEN $x[n]$, SAMPLE TO PRODUCE $x_p[n] = \begin{cases} x[n], & n = mN \\ 0, & \text{other} \end{cases}$



$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

$$x_p[n] = x[n]\, p[n]$$



Q. GIVEN $X(e^{j\omega})$, WHAT IS $X_p(e^{j\omega})$?

A. $p[n]$ IS PERIODIC, PERIOD 3

$$a_k = \frac{1}{3} \sum_{n=0}^{2} p[n]\, e^{-jk\frac{2\pi}{3}n} = \frac{1}{3}$$

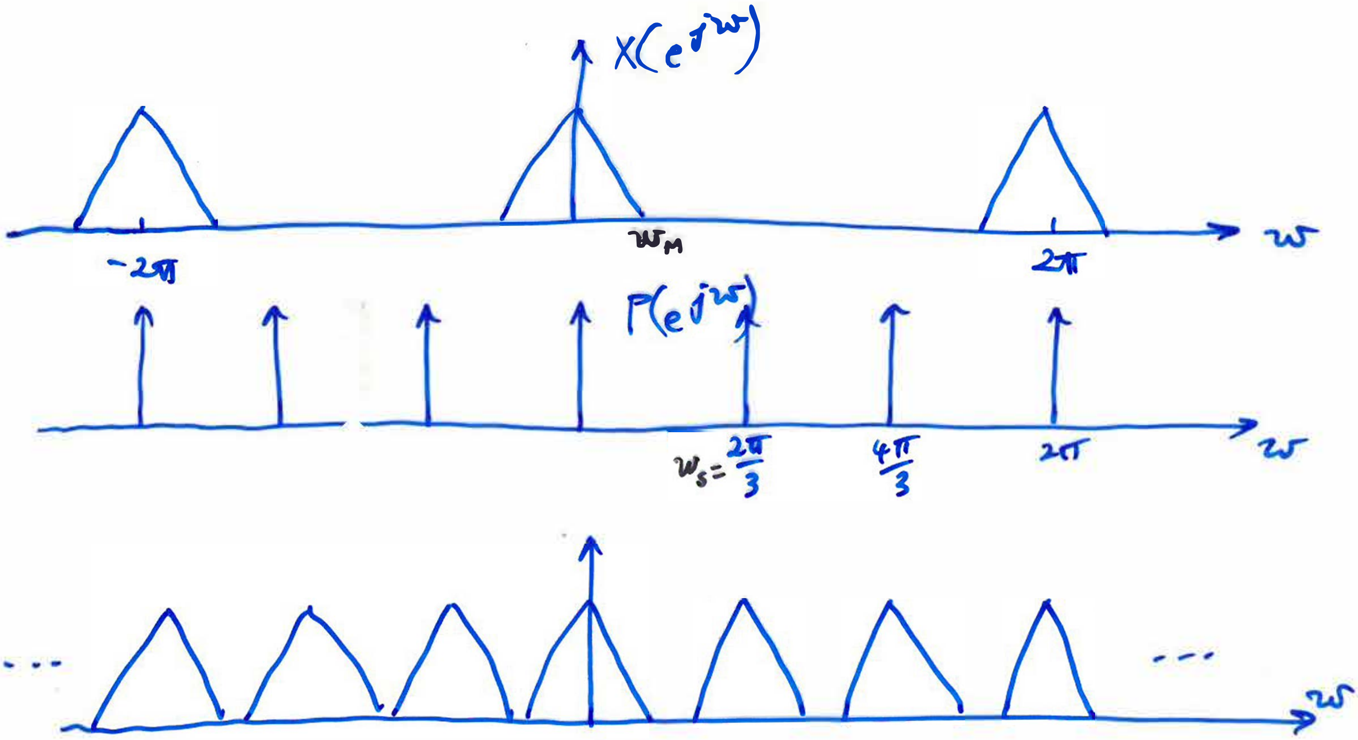
$$P(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{3}\,\delta(\omega - k\,\omega_s)$$

$$\omega_s = \frac{2\pi}{3}$$

$$\Rightarrow \quad X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})\, X\left(e^{j(\omega-\theta)}\right) d\theta$$

$$= \frac{1}{2\pi}\int_{2\pi} \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} \delta\left(\theta - \frac{2\pi}{3}k\right) X\left(e^{j(\omega-\theta)}\right) d\theta$$

$$= \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j\left(\omega - \frac{2\pi}{3}k\right)}\right)$$

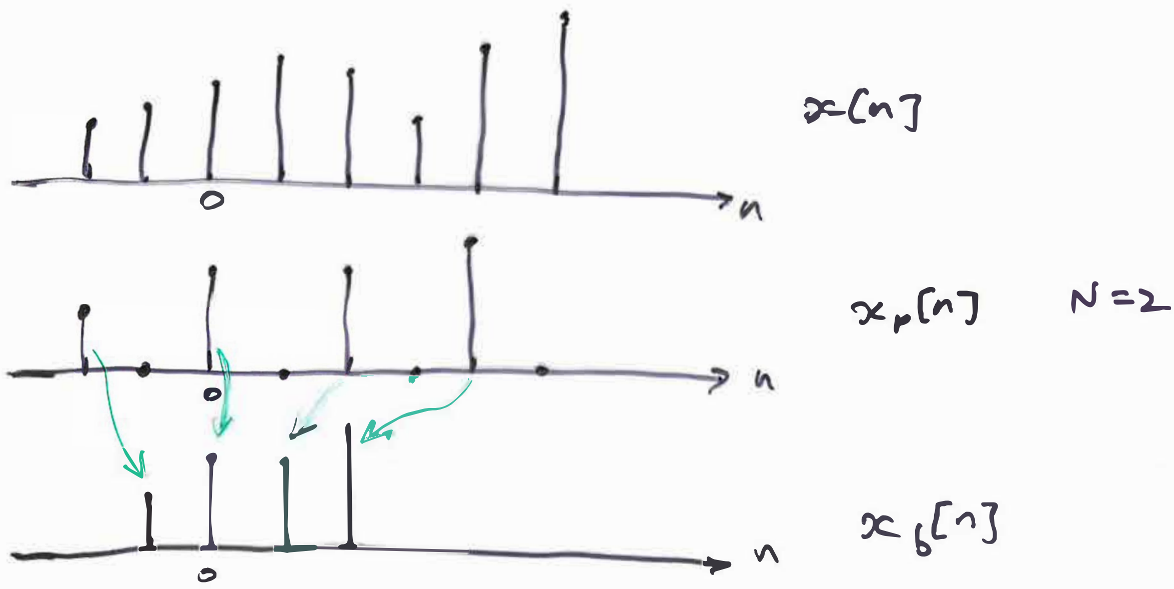


CAN RECOVER ORIGONAL, IF

$$\boxed{\omega_s = \frac{2\pi}{N} > 2\,\omega_M}$$

D.T. DECIMATION (DOWN SAMPLING)

SAMPLING, THEN REMOVE ZEROS.



$$x_b[n] = x_p[nN]$$

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x_b[k]\, e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x_p[kN]\, e^{-j\omega k}$$

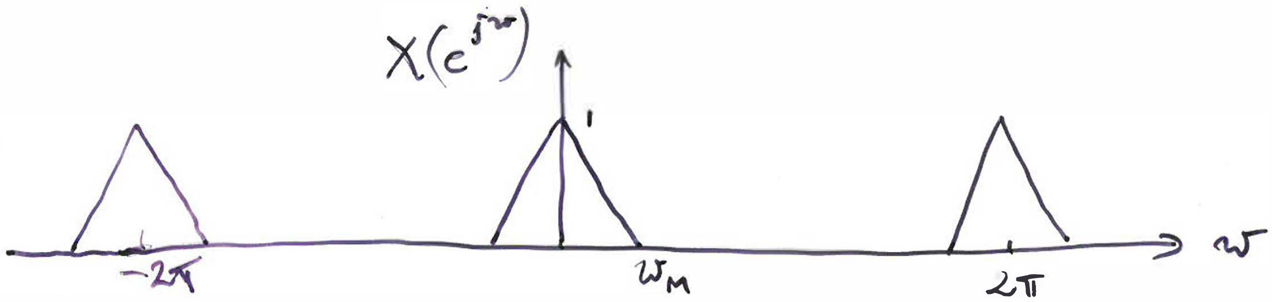
put $kN = n$

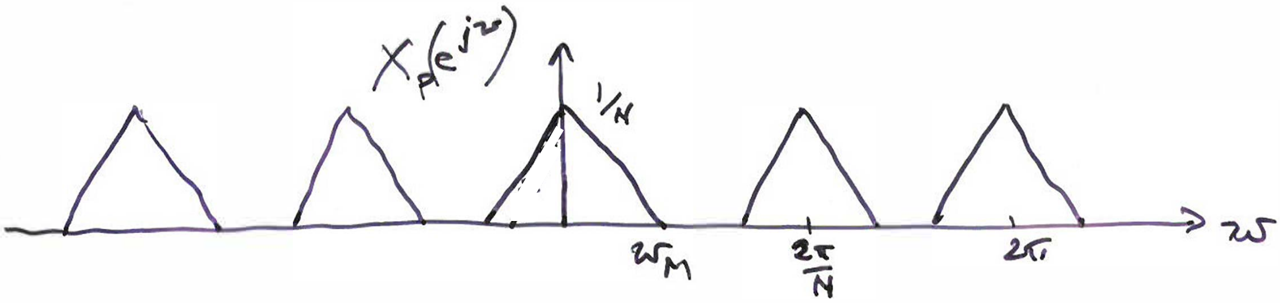
$$= \sum_{n = \text{multiple of } N} x_p[n]\, e^{-j\omega n/N}$$

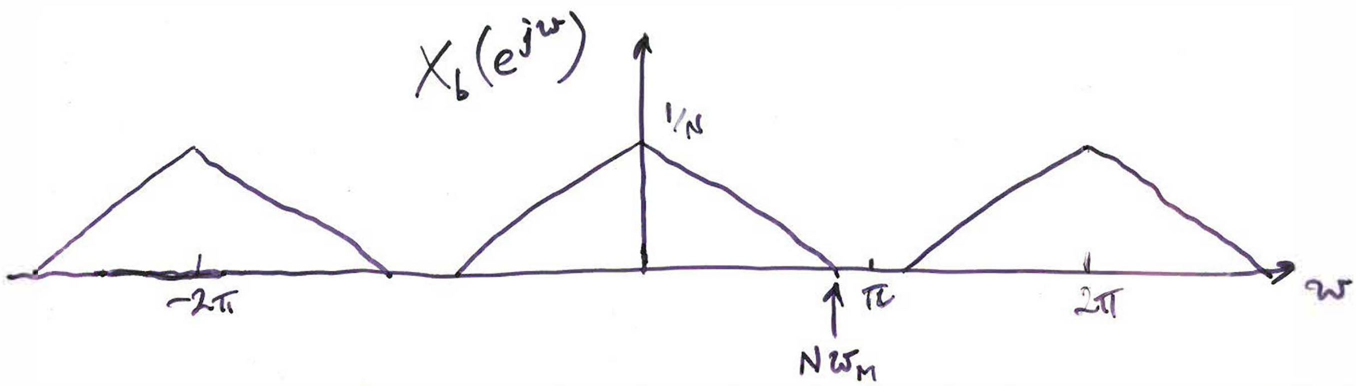
$$= \sum_{n=-\infty}^{\infty} x_p[n]\, e^{-j\omega n/N}$$ because other values $=0$

$$\therefore \quad X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

(COMPARE WITH TIME EXPANSION L14.2)

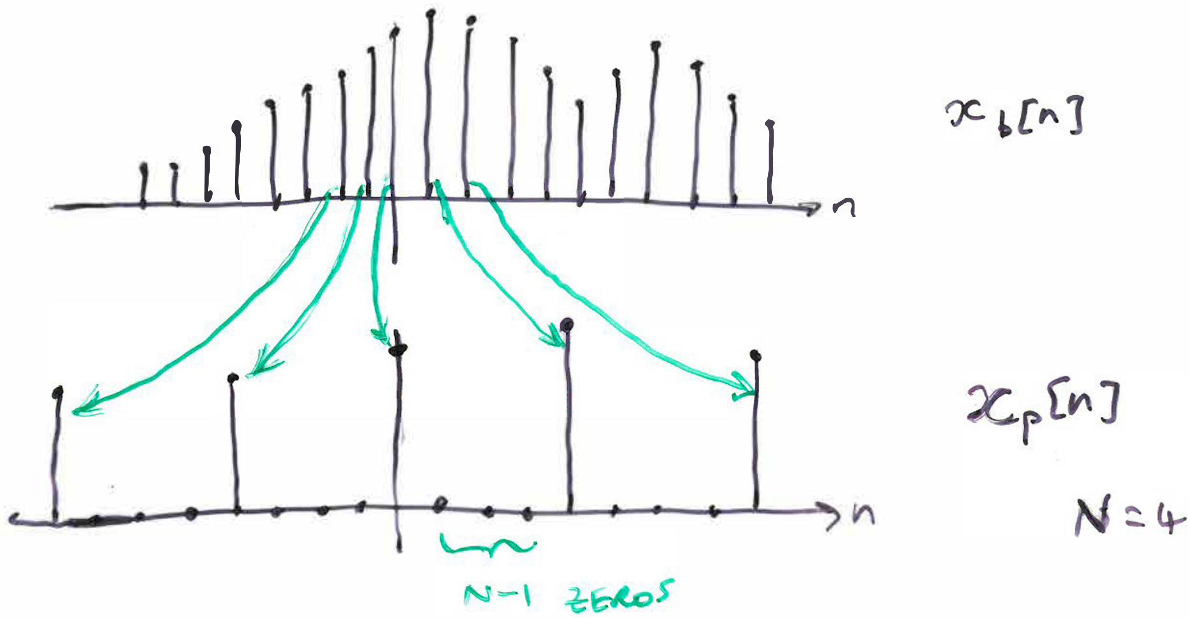


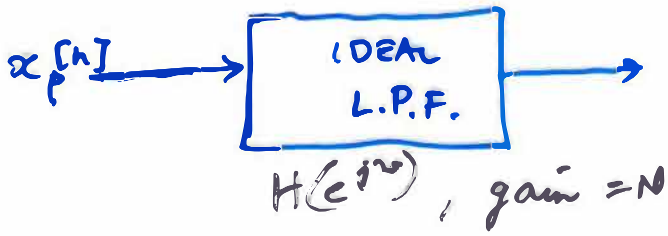


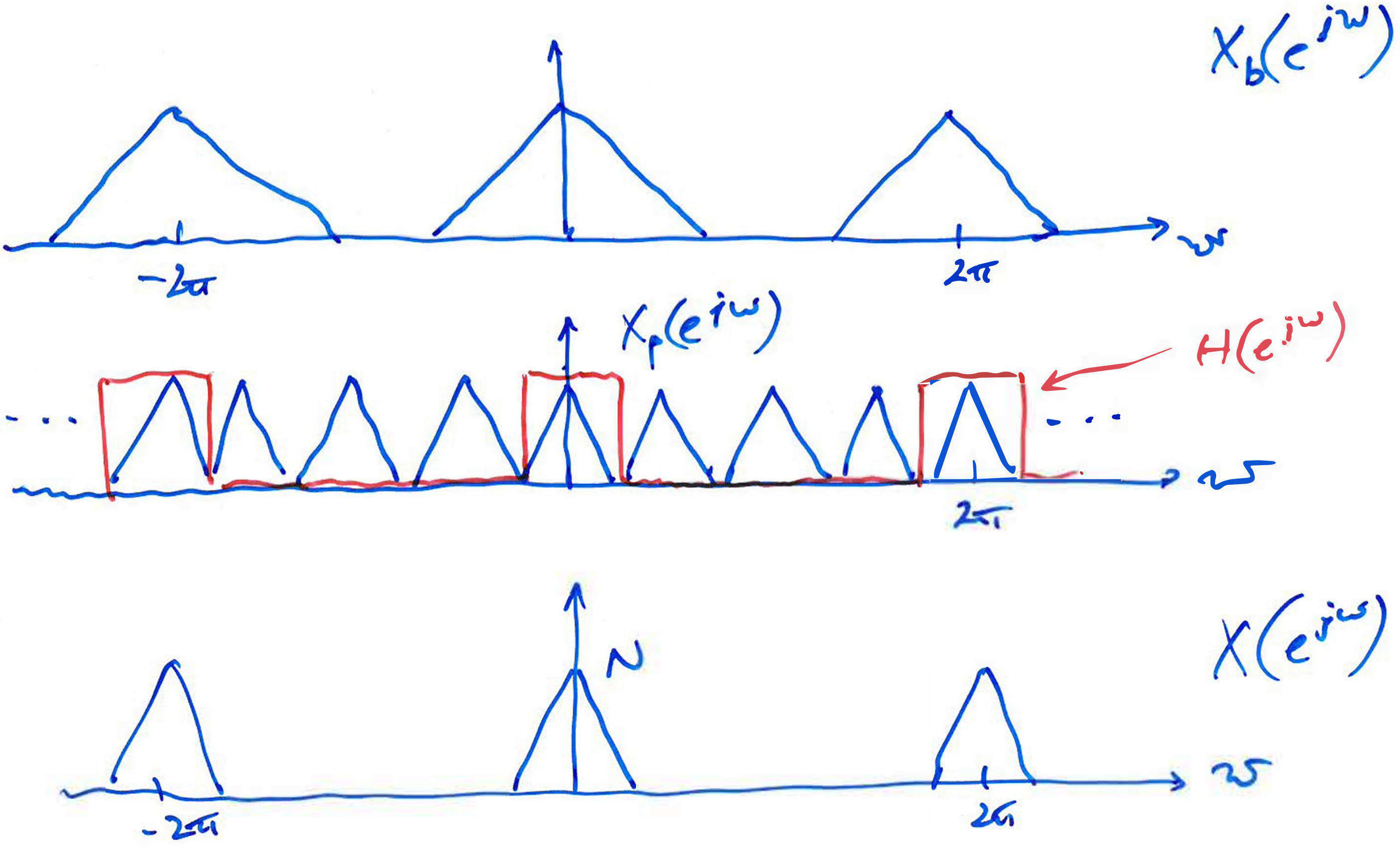


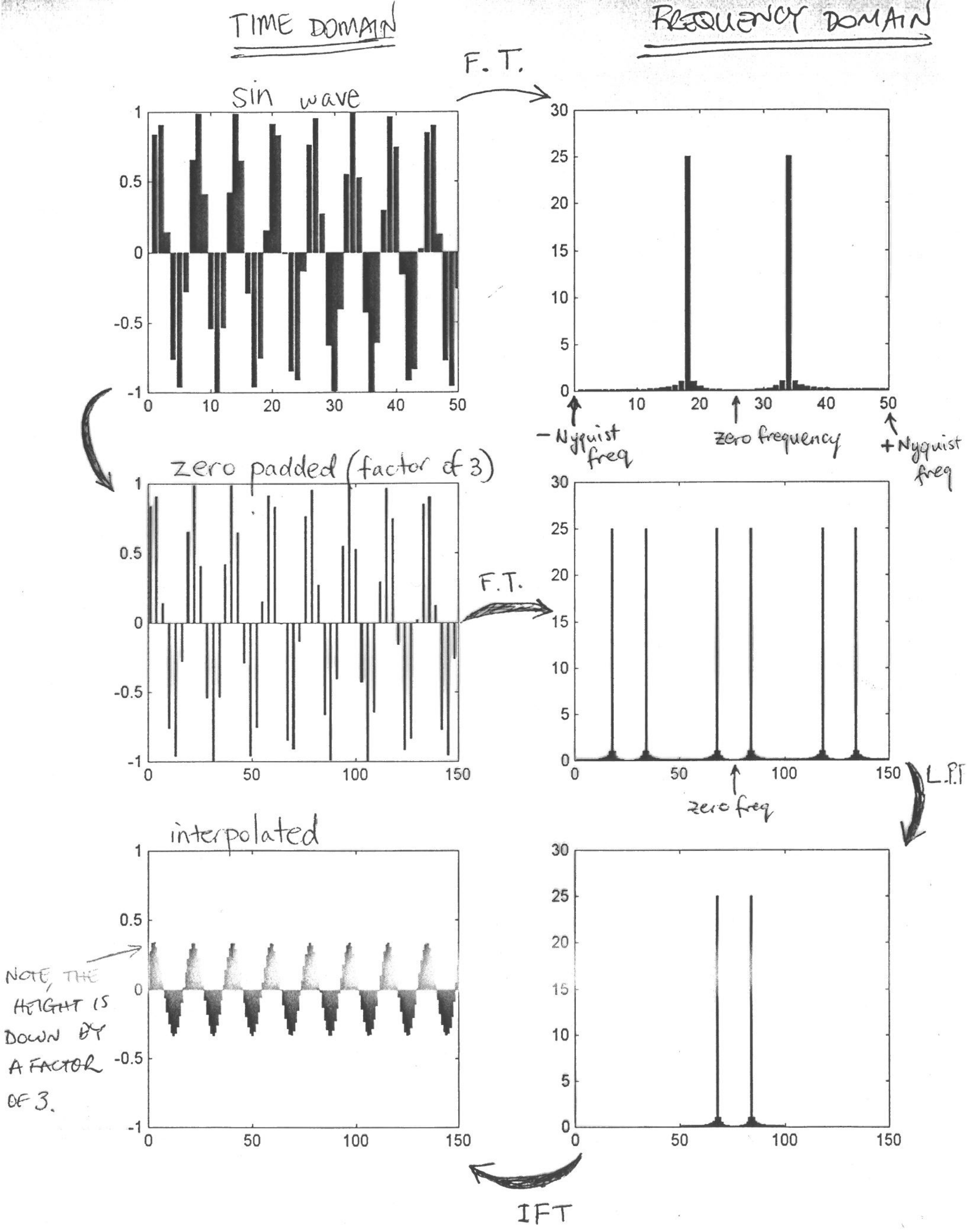
INTERPOLATION (UPSAMPLING)

ADDING SAMPLES (RATHER THAN REMOVING THEM)









~~PAD WITH TOO MANY ZEROS, YOU WOULD GET ALIASING~~