Frequency Response

BY DR MIHAI CIOBOTARU

Overview

- **Frequency response**
  - Bode plots
- **Relation between closed-loop transient and <u>closed-loop</u> frequency responses**
  - Damping ratio and closed-loop frequency response
  - Response speed and closed-loop frequency response
- **Relation between closed-loop transient and <u>open-loop</u> frequency responses**
  - Damping ratio from phase margin
  - Response speed from open-loop frequency response
- **Systems with time delay**

Frequency Response (Bode plots)

Frequency Response

- The results of frequency response compensation techniques are not new or different from the results of root locus techniques.
- Frequency response methods (are older than the root locus method, which was discovered by Evans in 1948.
  - *Nyquist in 1932*
  - ***Bode in 1945***
  - *Nichols in 1947*

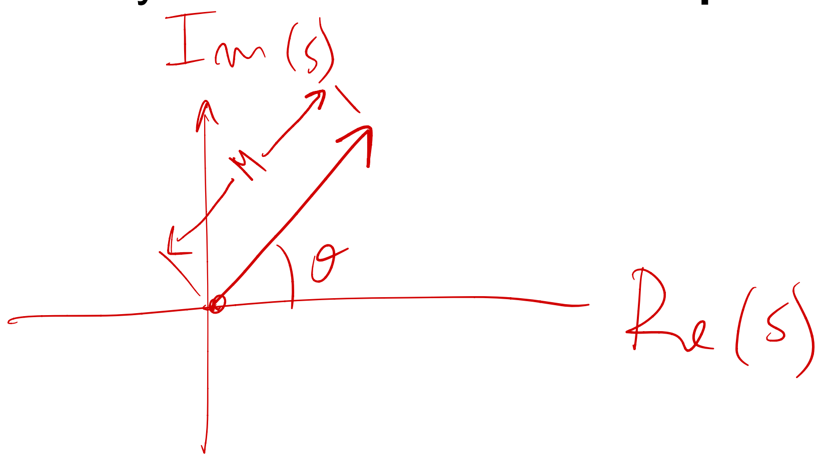
Frequency Response

- This course focuses on frequency response using **Bode plot**.
  - Frequency response method is not as intuitive as the root locus.
  - However, frequency response yields a new vantage point from which to view feedback control systems:
    - When modelling transfer functions from physical data
    - When designing lead compensators to meet a steady-state error requirement and a transient response requirement
    - When finding the stability of nonlinear systems
    - In settling ambiguities when sketching a root locus

Frequency Response

THE CONCEPT OF FREQUENCY RESPONSE

- In the steady state, sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency.
- These responses are of the same frequency as the input, but they differ in magnitude and phase angle from the input.
- These differences are functions of frequency.
- **Sinusoids representation**
  - sinusoids can be represented as complex numbers called **phasors**.
  - a system causes both the **amplitude and phase angle** of the input to be changed



Frequency Response (Bode plots)

- **Analytical expression for the frequency response** can be obtained from the transfer function G(s) of the system as

$$G(j\omega) = G(s)\,|_{s \to j\omega}$$

- **Plotting of the frequency response**

$$G(j\omega) = M_G(\omega)\ \angle\phi_G(\omega)$$

  – As a **function of frequency**, with separate magnitude and phase plots

Frequency Response (Bode plots)

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Asymptotic Approximations: Bode Plots

Bode plots or Bode diagrams: curves of the **log-magnitude** and **phase frequency** response as functions of log ω

- Consider:

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{N e^{j\angle N}}{D e^{j\angle D}} = \frac{N}{D} e^{j\angle N - D}$$

- Then, the magnitude response is given by:

$$|G(j\omega)| = \left.\frac{K|(s+z_1)||(s+z_2)|\cdots|(s+z_k)|}{|s^m||(s+p_1)||(s+p_2)|\cdots|(s+p_n)|}\right|_{s\to j\omega}$$

- Converting the **magnitude response into dB**, we obtain:

$$20\log|G(j\omega)| = 20\log K + 20\log|(s+z_1)| + 20\log|(s+z_2)| + \cdots - 20\log|s^m| - 20\log|(s+p_1)| - \cdots|_{s\to j\omega}$$

- **Phase frequency response**
  - is the sum of the phase frequency response curves of the **zero terms** minus the sum of the phase frequency response curves of the **pole terms**.

$$\angle G = \Sigma\theta_z - \Sigma\theta_p$$

Frequency Response (Bode plots)

Bode Plots for *G(s) = (s + a)*

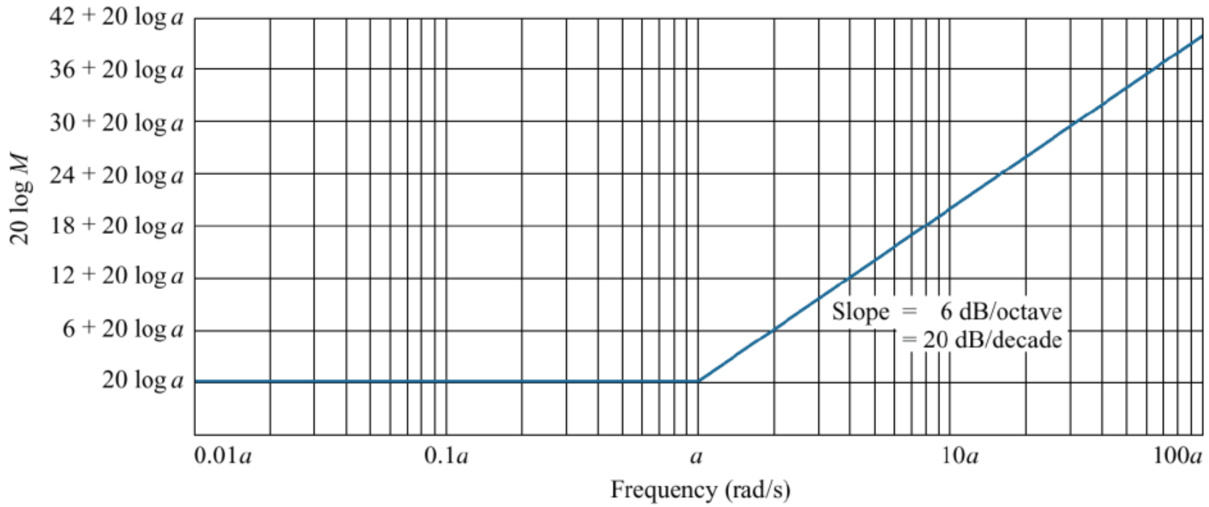
- *Low frequency asymptote*

$$G(s) = (s + a) \qquad s = j\omega$$

$$G(j\omega) = (j\omega + a) = a\left(j\frac{\omega}{a} + 1\right)$$

At low frequencies when ω approaches zero: $G(j\omega) \approx a$

The magnitude response in dB is: $20 \log M = 20 \log a$

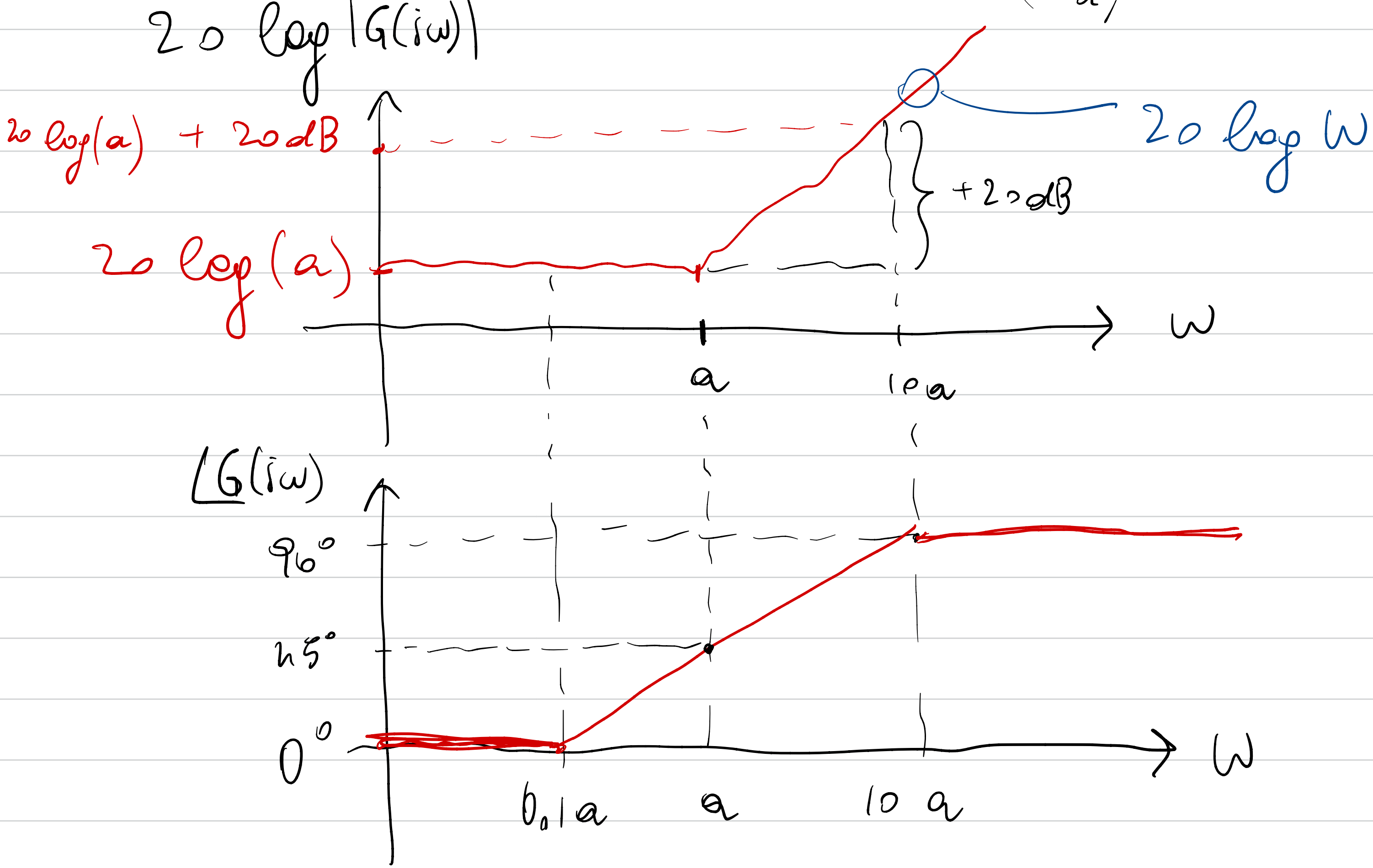


$$G(s) = s + a$$

$$G(j\omega) = j\omega + a = a\left(\frac{j\omega}{a} + 1\right)$$

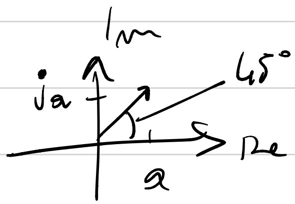
$$G(j\omega) = a\angle 0^\circ \quad (\omega \ll a)$$

$$G(j\omega) = a\left(\frac{j\omega}{a}\right) = \omega\angle +90^\circ \quad (\omega \gg a)$$



$\omega = a$

$$G(j\omega) = a(j + 1)$$



Frequency Response (Bode plots)

Bode Plots for *G(s) = (s + a)*

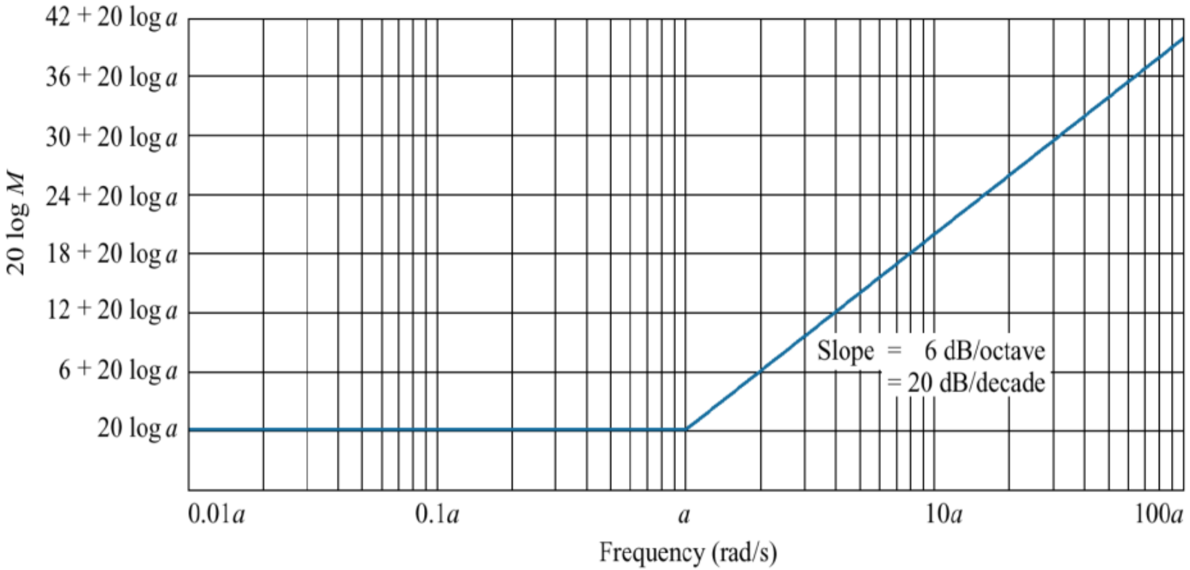
– *High frequency asymptote*

At high frequencies, where $\omega >> a$:

$$G(j\omega) = (j\omega + a) = a\left(j\frac{\omega}{a} + 1\right)$$

$$G(j\omega) \approx a\left(\frac{j\omega}{a}\right) = a\left(\frac{\omega}{a}\right)\angle 90^\circ = \omega\angle 90^\circ$$

$$20\log M = 20\log a + 20\log\frac{\omega}{a} = 20\log\omega$$



Frequency Response (Bode plots)

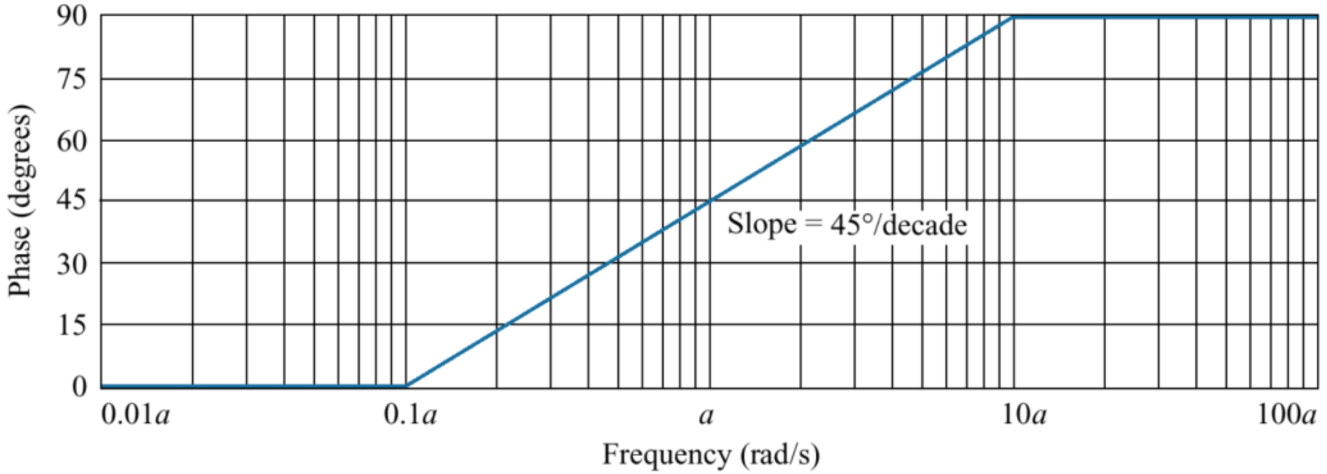
Bode Plots for ***G(s) = (s + a)***

– *Phase response*

$$G(j\omega) = (j\omega + a) = a\left(j\frac{\omega}{a} + 1\right)$$

From the above equation:

- at the break frequency, a, phase is 45^0
- at low frequencies the phase is 0^0
- at high frequencies the phase is 90^0



Frequency Response (Bode plots)

- It is often convenient to ***normalize the magnitude*** and ***scale the frequency*** so that the log-magnitude plot will be 0 dB at a break frequency of unity.

- Normalization and scaling makes it easier to add components to obtain the Bode plot of a function such as

$$|G(j\omega)| = \left. \frac{K|(s+z_1)||(s+z_2)|\cdots|(s+z_k)|}{|s^m||(s+p_1)||(s+p_2)|\cdots|(s+p_n)|} \right|_{s\to j\omega}$$

- To normalize $(s + a)$, factor out the quantity a and form $a[(s/a)+ 1]$.

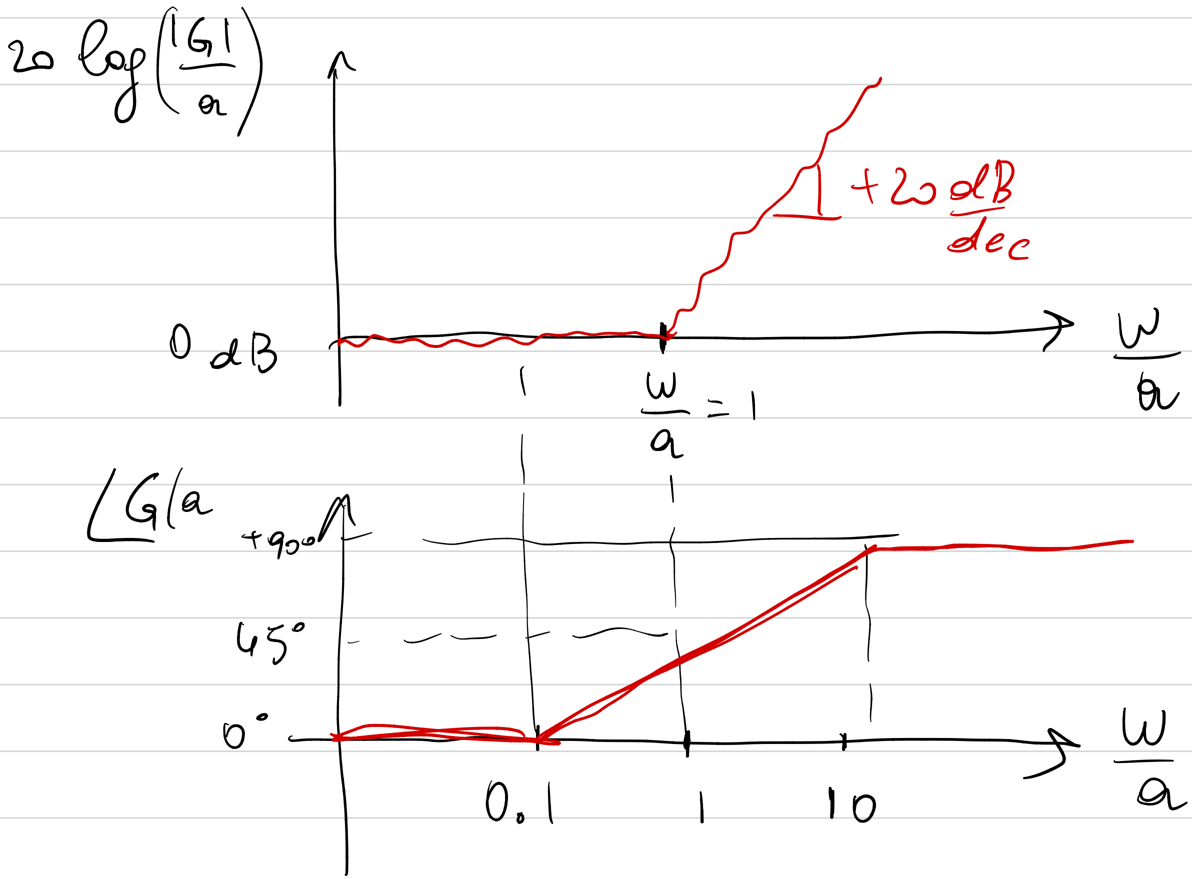
- The frequency is scaled by defining a new frequency variable, $s_1 = s/a$.

- The magnitude is divided by the quantity a to yield 0 dB at the break frequency.

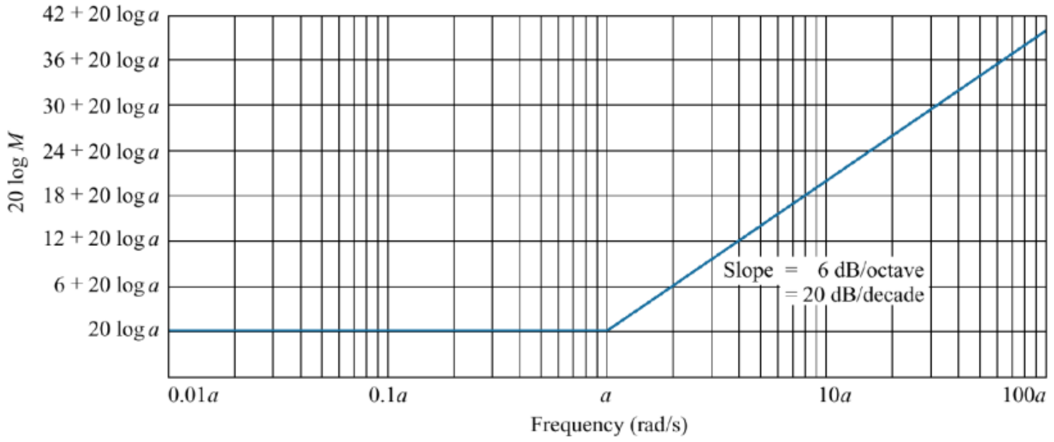
- Hence, the normalized and scaled frequency function is $(s_1 + 1)$.

$$G(j\omega) = a\left(\frac{j\omega}{a} + 1\right)$$

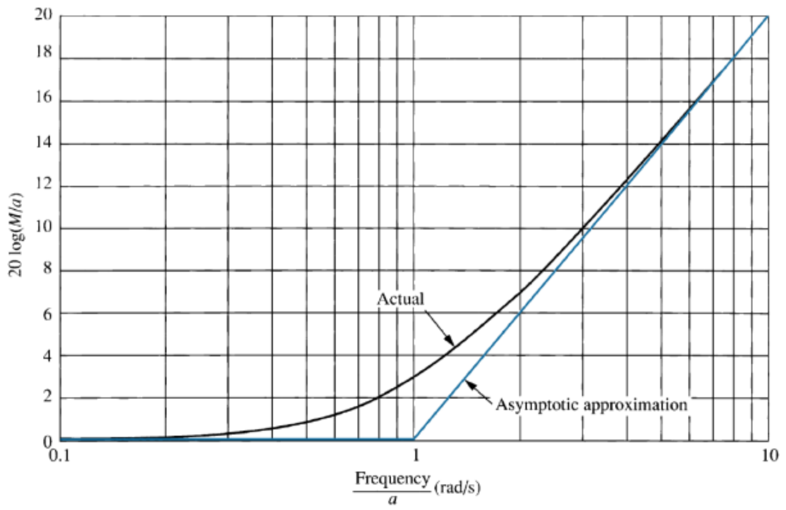
$$\frac{G(j\omega)}{a} = j\frac{\omega}{a} + 1 = \begin{cases} 1\angle 0^\circ & (\omega \ll a) \\ j\frac{\omega}{a} = \frac{\omega}{a}\angle +90^\circ & (\omega \gg a) \end{cases}$$



Frequency Response (Bode plots)

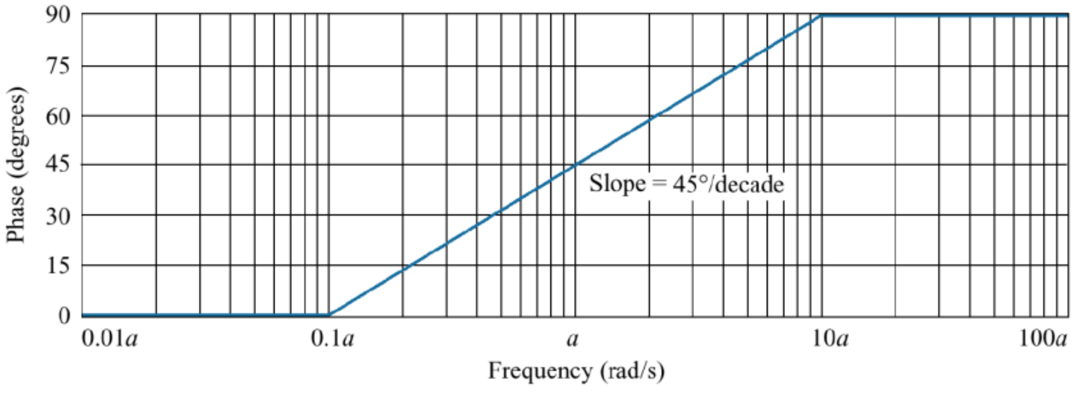


… original magnitude response



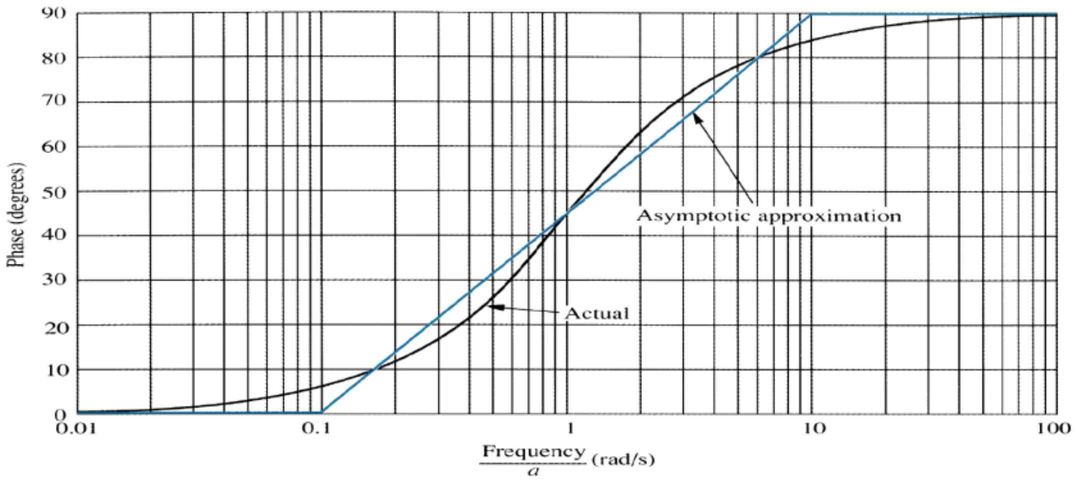
… normalized and scaled magnitude response

Frequency Response (Bode plots)



(b)

… original
phase response

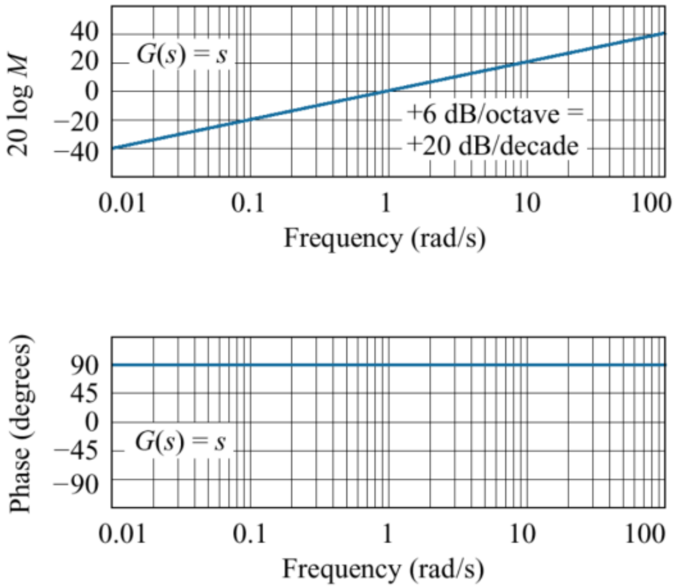


… normalized and scaled
phase response

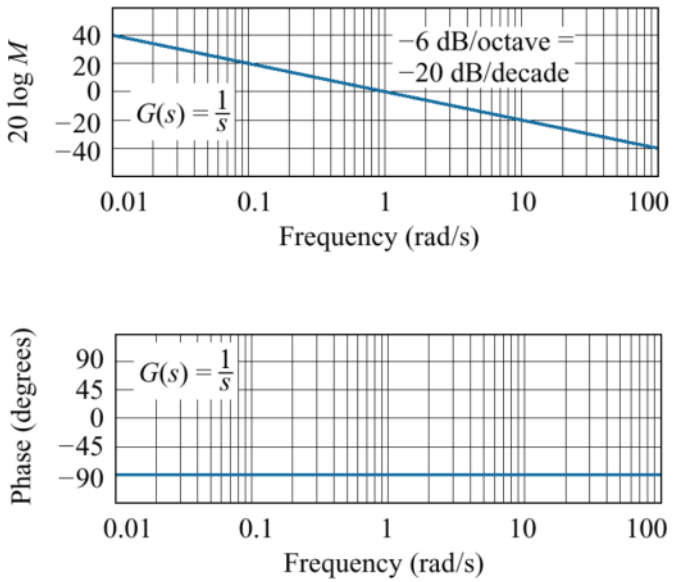
Frequency Response (Bode plots)

Similarly, we can find the normalized and **scaled Bode plots** for:

$\boxed{G(s) = s}$

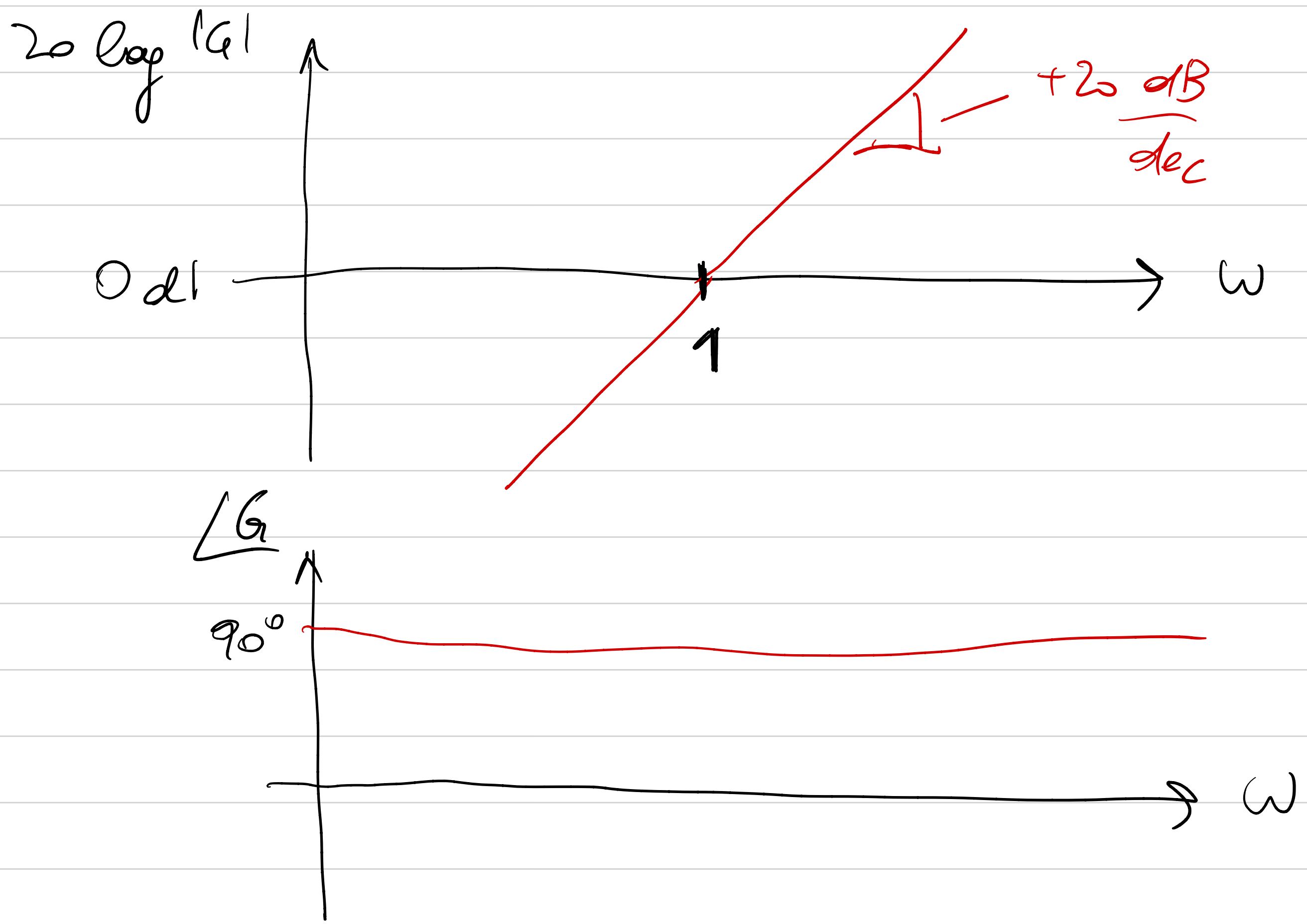


$\boxed{G(s) = 1/s}$



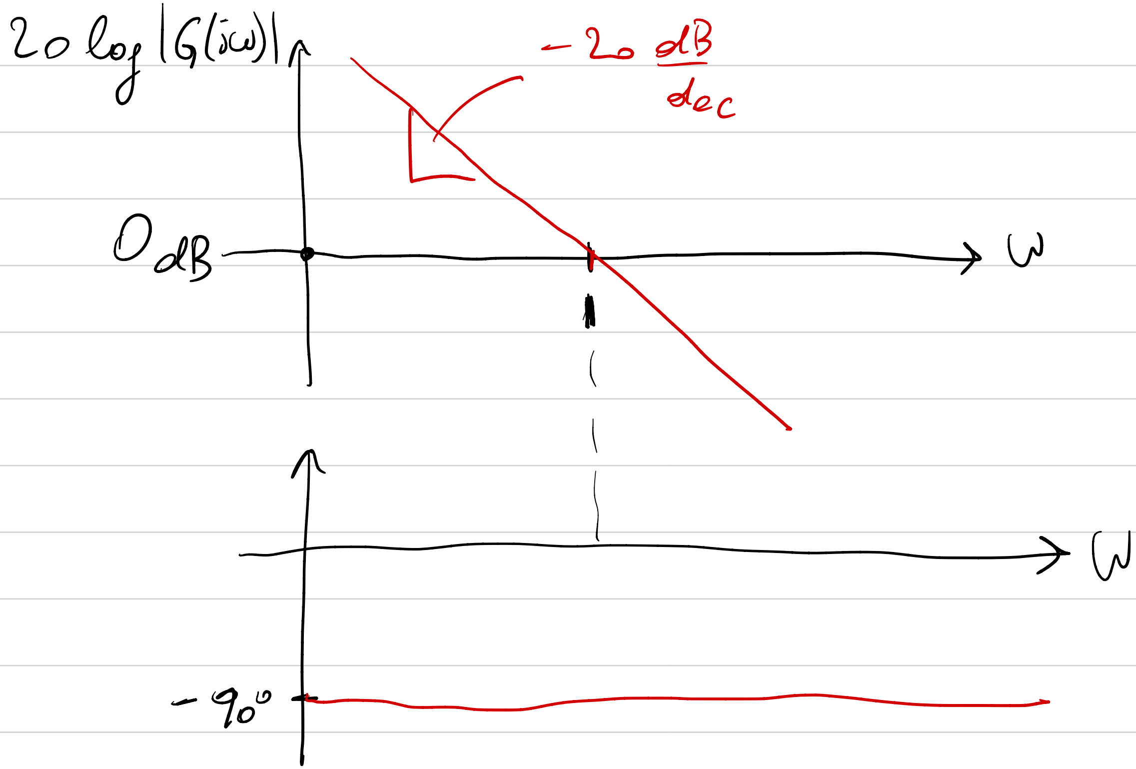
$G(s) = s$

$G(j\omega) = j\omega = \omega \angle +90^\circ$



$$G(s) = \frac{1}{s}$$

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} j^{-1} = -j\frac{1}{\omega} = \frac{1}{\omega}\angle -90^\circ$$

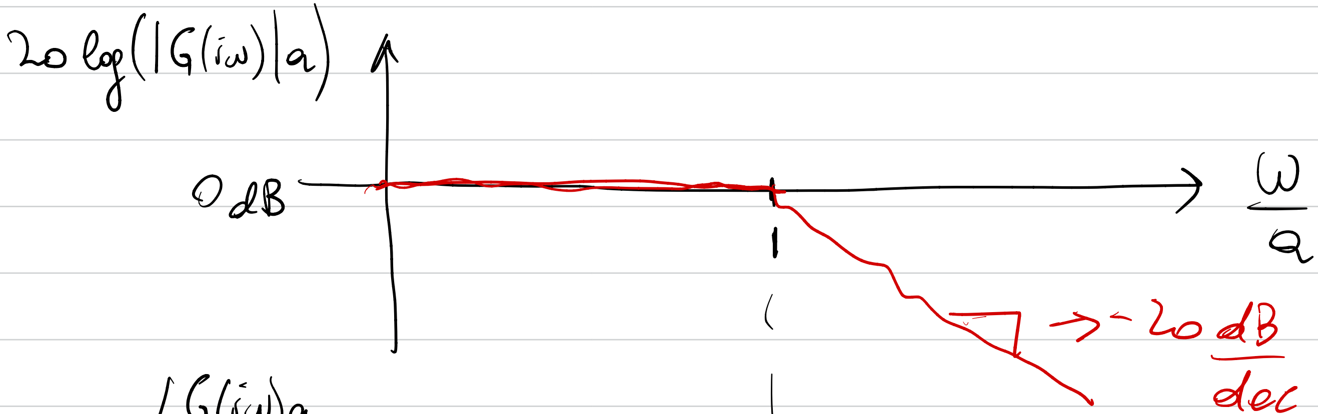


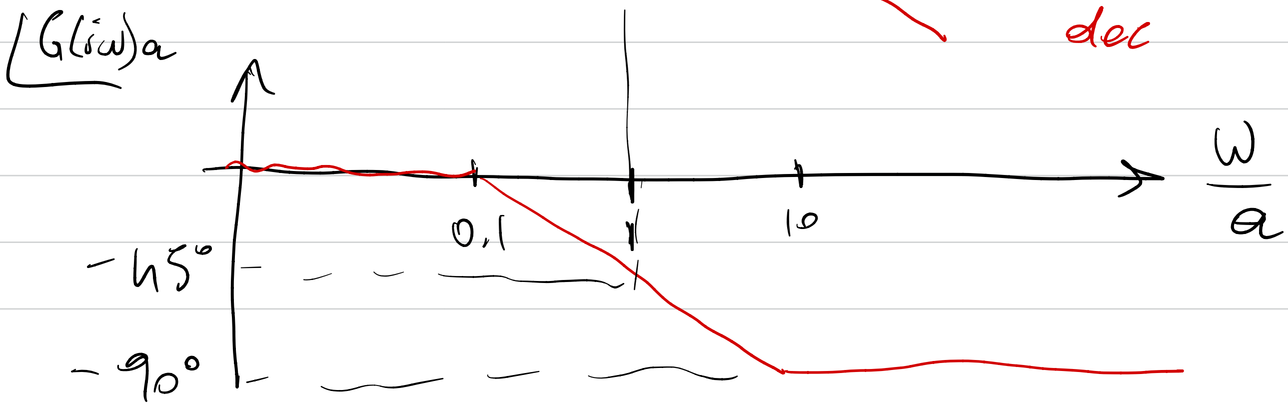
$$G(s) = \frac{1}{s+a}$$

$$G(j\omega) = \frac{1}{j\omega + a} = \frac{1}{a\left(\frac{j\omega}{a}+1\right)}$$

$$\frac{G(j\omega)}{\frac{1}{a}} = G(j\omega)\cdot a = \frac{1}{j\frac{\omega}{a}+1}$$

- $1 \angle 0^\circ \qquad \frac{\omega}{a} \ll 1$
- $\frac{1}{j\frac{\omega}{a}} = \frac{1}{\frac{\omega}{a}} \angle -90^\circ \qquad \frac{\omega}{a} \gg 1$

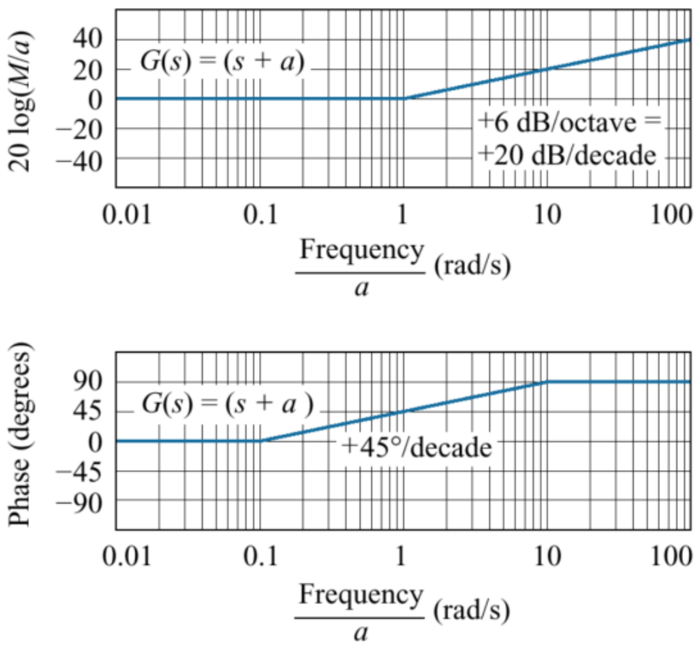




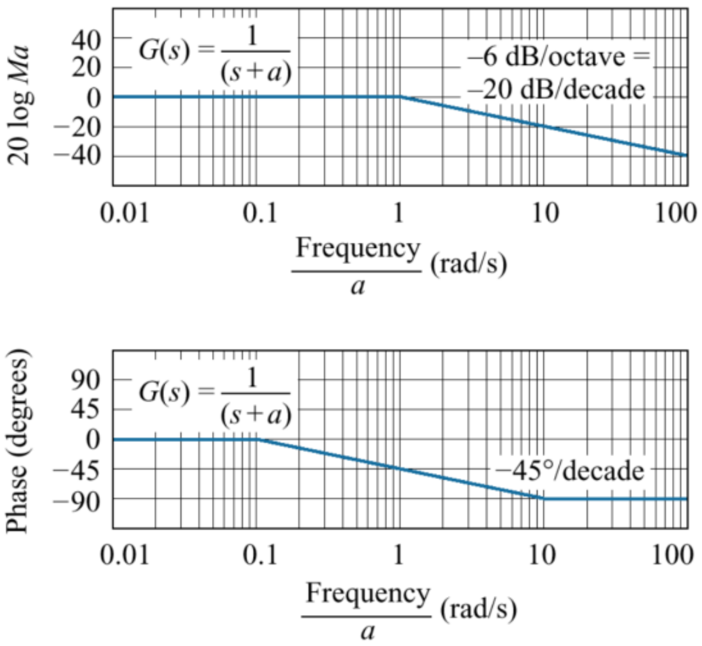
Frequency Response (Bode plots)

Similarly, we can find the normalized and **scaled Bode plots** for:

$\boxed{G(s) = (s + a)}$



$\boxed{G(s) = 1/(s + a)}$



Frequency Response (Bode plots)

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Bode Plots for $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = \omega_n^2\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)$$

At low frequencies: $G(s) \approx \omega_n^2 = \omega_n^2\angle 0°$

$$20\log M = 20\log|G(j\omega)| = 20\log \omega_n^2$$

At high frequencies: $G(s) \approx s^2$

$$G(j\omega) \approx -\omega^2 = \omega^2\angle 180°$$

$$20\log M = 20\log|G(j\omega)| = 20\log\omega^2 = 40\log\omega$$

Break frequency: ω_n

After scaling and normalization $\boxed{G(s_1)/\omega_n^2 = s_1^2 + 2\zeta s_1 + 1 \qquad s_1 = s/\omega_n}$

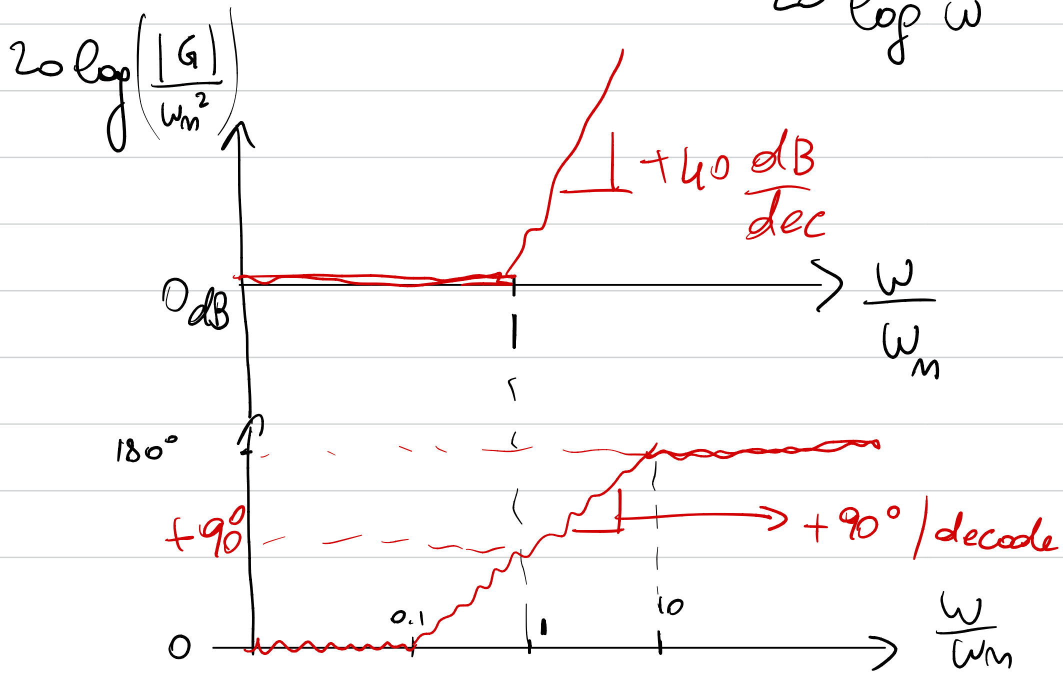
$$G(s) = s^2 + 2\xi\omega_n s + \omega_n^2$$

$$= \omega_n^2\left(\frac{s^2}{\omega_n^2} + \frac{2\xi s}{\omega_n} + 1\right)$$

$$G(j\omega) = \omega_n^2\left[\frac{(j\omega)^2}{\omega_n^2} + \frac{2\xi j\omega}{\omega_n} + 1\right] = \begin{cases} \omega_n^2 \angle 0^\circ & \frac{\omega}{\omega_n} \ll 1 \quad (\omega \ll \omega_n) \\ (j\omega)^2 = -\omega^2 = \omega^2 \angle 180^\circ & \frac{\omega}{\omega_n} \gg 1 \quad (\omega \gg \omega_n) \end{cases}$$

Magnitude in dB:

$$20\log|G(j\omega)| = \begin{cases} 20\log_{10}\omega_n^2 = 40\log\omega_n & (\omega \ll \omega_n) \\ 20\log\omega^2 = 40\log\omega & (\omega \gg \omega_n) \end{cases}$$



$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\boxed{\omega = \omega_n}$$

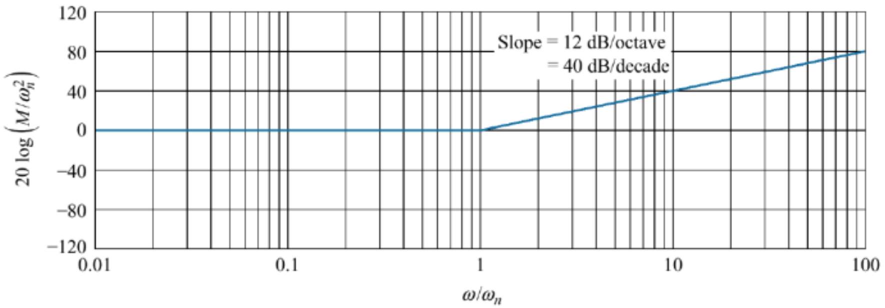
$$G(j\omega) = (j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2$$

$$= (\omega_n^2 - \omega^2) + j\,2\zeta\omega_n\omega = j\,2\zeta\omega_n^2$$

$$\left|\frac{G(j\omega)}{\omega_n^2}\right| = 2\zeta$$

Frequency Response (Bode plots)

Bode Plots for $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$



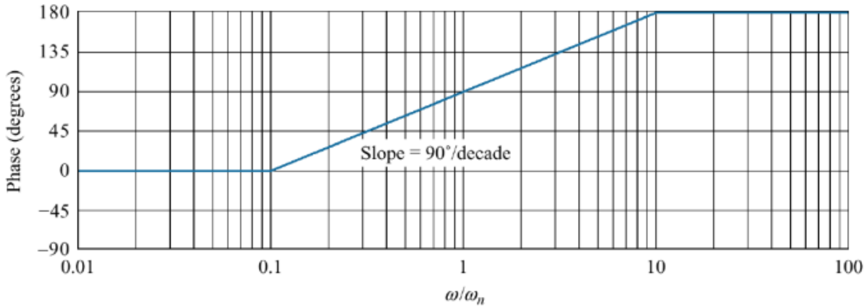
At low frequencies: Phase = 0^0 $\quad G(s) \approx \omega_n^2 = \omega_n^2 \angle 0°$

At high frequencies: Phase = 180^0 $\quad G(j\omega) \approx -\omega^2 = \omega^2 \angle 180°$

Phase at ω_n: $G(j\omega) = s^2 + 2\zeta\omega_n s + \omega_n^2|_{s \to j\omega} = (\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega$

$$G(j\omega_n) = j2\zeta\omega_n^2$$

- hence phase at ω_n is 90^0



Frequency Response (Bode plots)

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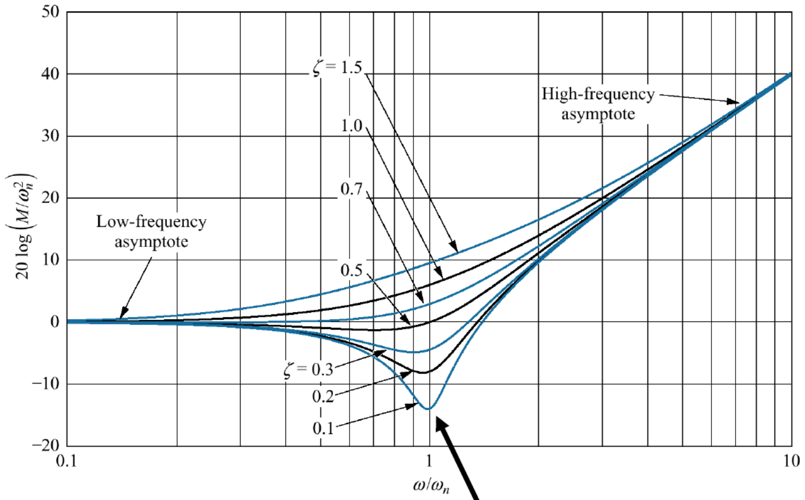
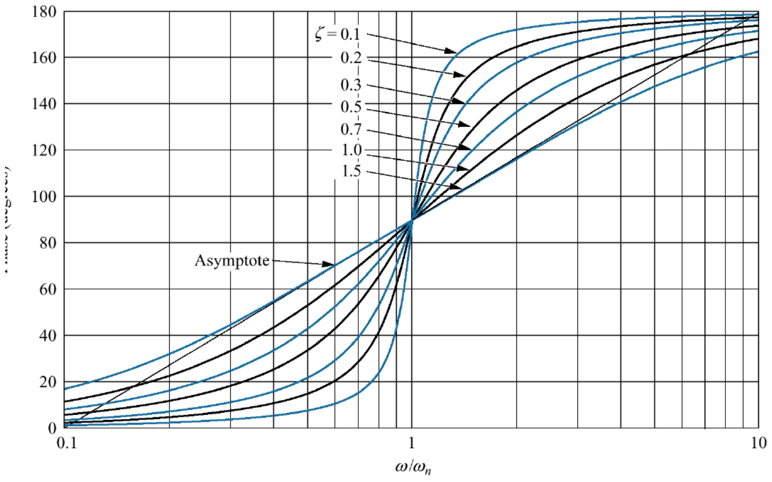
Corrections to second-order Bode Plots $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$

- there is an error between the actual response and the asymptotic approximation of the second-order polynomial

Actual response: $$M = \sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$\text{Phase} = \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}$$

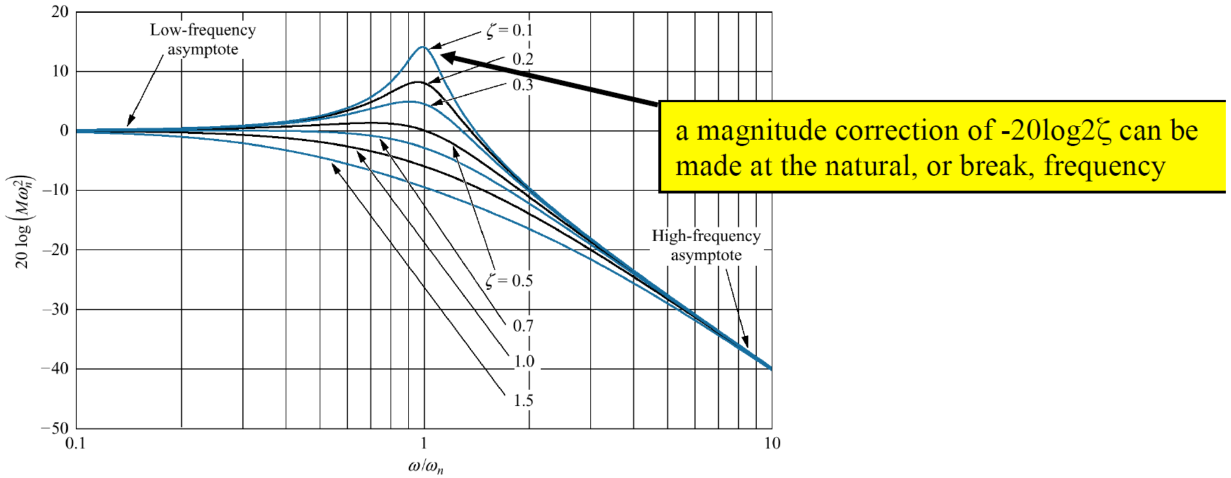


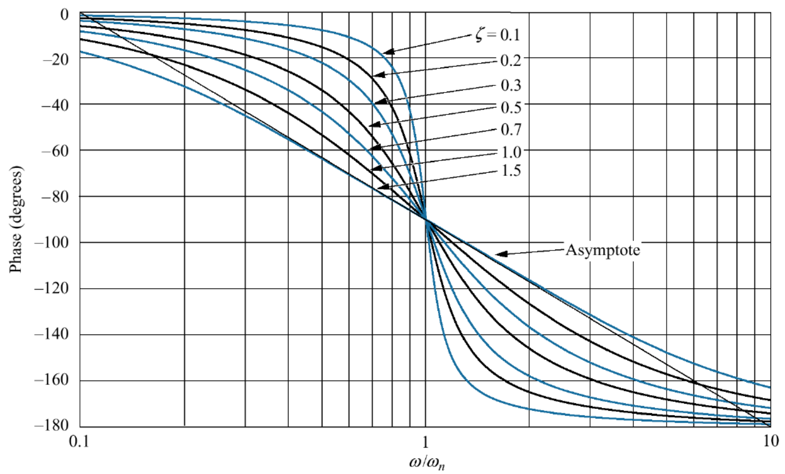


a magnitude correction of $+20\log 2\zeta$ can be made at the natural, or break, frequency

Frequency Response (Bode plots)

Bode Plots for $G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$





Frequency Response (Bode plots)

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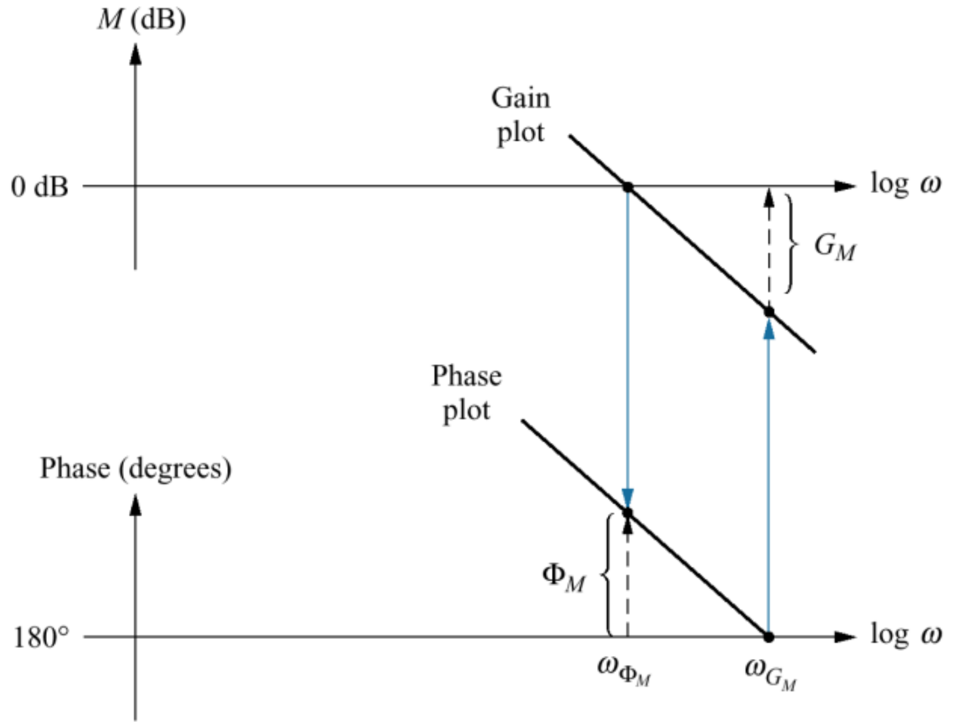
Stability, gain margin, and phase margin via Bode plots

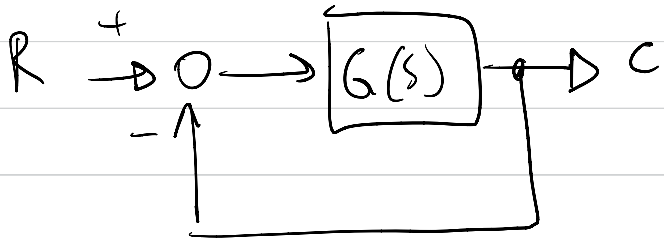
- ***Gain margin**, G_M* **,** is the change in open-loop gain, expressed in decibels (dB), required at 180° of phase shift to make the closed-loop system unstable.
- ***Phase margin**, Φ_M* **,** is the change in open-loop phase shift required at unity gain to make the closed-loop system unstable.



Always keep away from

$G(j\omega) = 1\angle 180°$

Assume that close to -1, $G(s) = G(j\omega)$

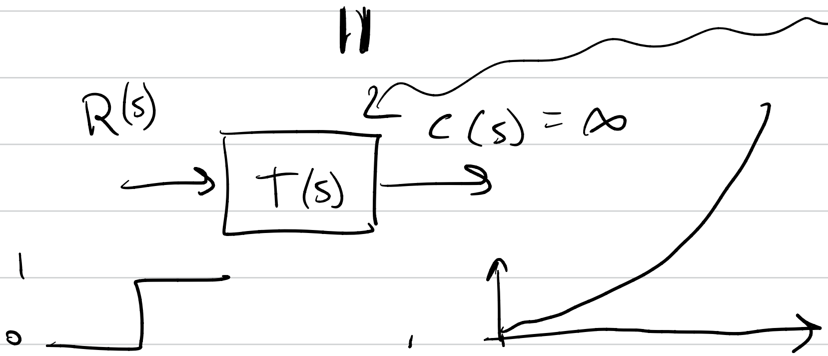


$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

if $G(s) = -1 \Rightarrow T(s) = \infty$

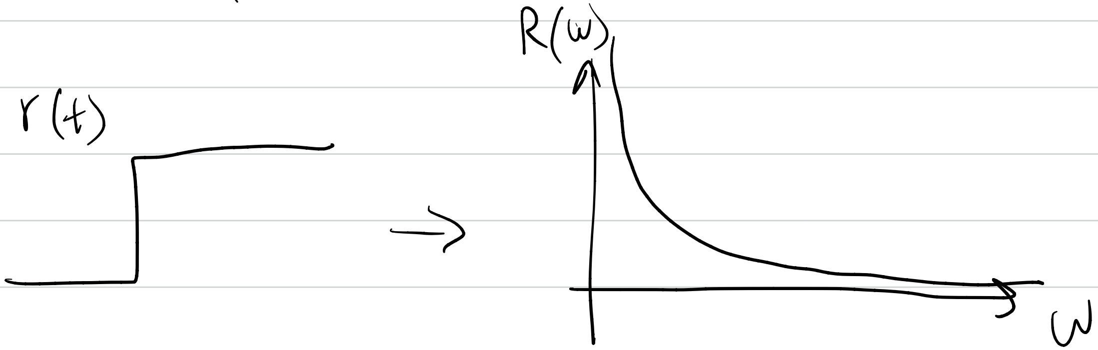
$\downarrow s = j\omega$

$$G(j\omega) = -1 = 1\angle -180^\circ$$



if $G(s) = -1$

$T(s) = \infty$

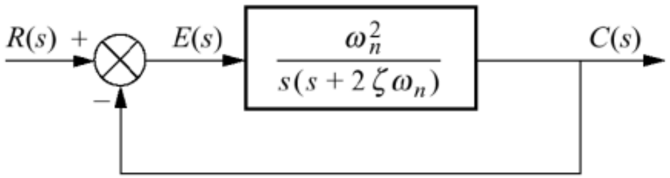


Relation between closed-loop transient (time) and closed-loop frequency responses

Relation between closed-loop transient and closed-loop frequency responses

Damping Ratio and Closed-Loop Frequency Response

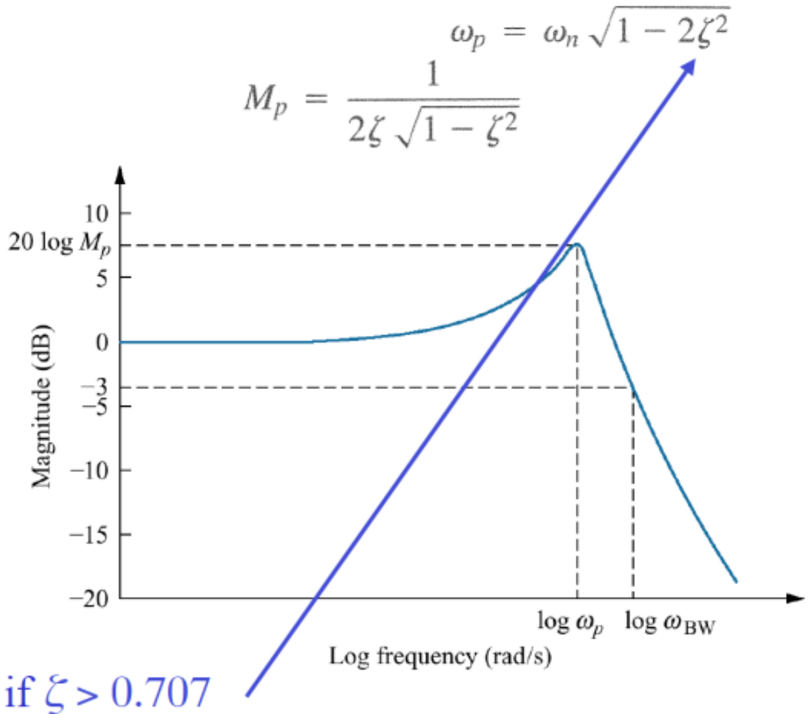
$\zeta > \frac{1}{\sqrt{2}}$



$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$M = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$



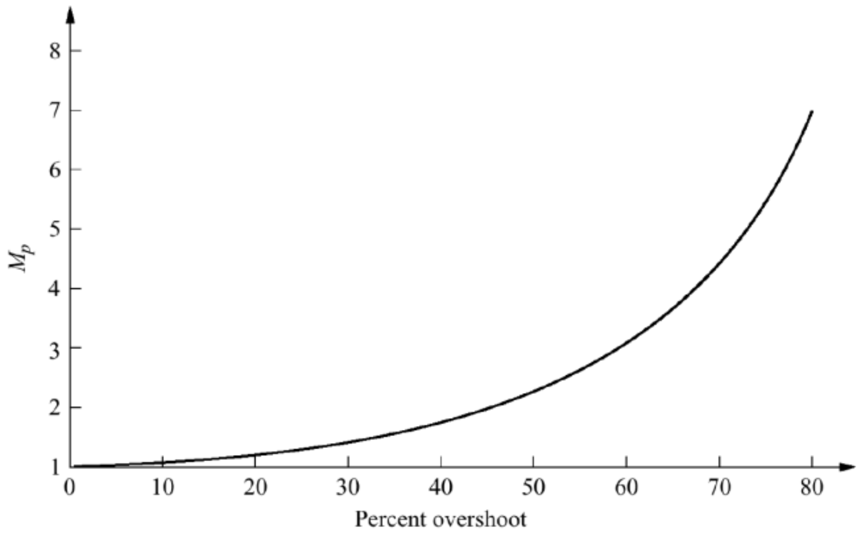
- There will be **no peak** at frequencies above zero if $\zeta > 0.707$

Relation between closed-loop transient and closed-loop frequency responses

Damping Ratio and Closed-Loop Frequency Response

Since ζ is related to percent overshoot (OS), we can plot M_p vs. percent overshoot

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} \qquad M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$



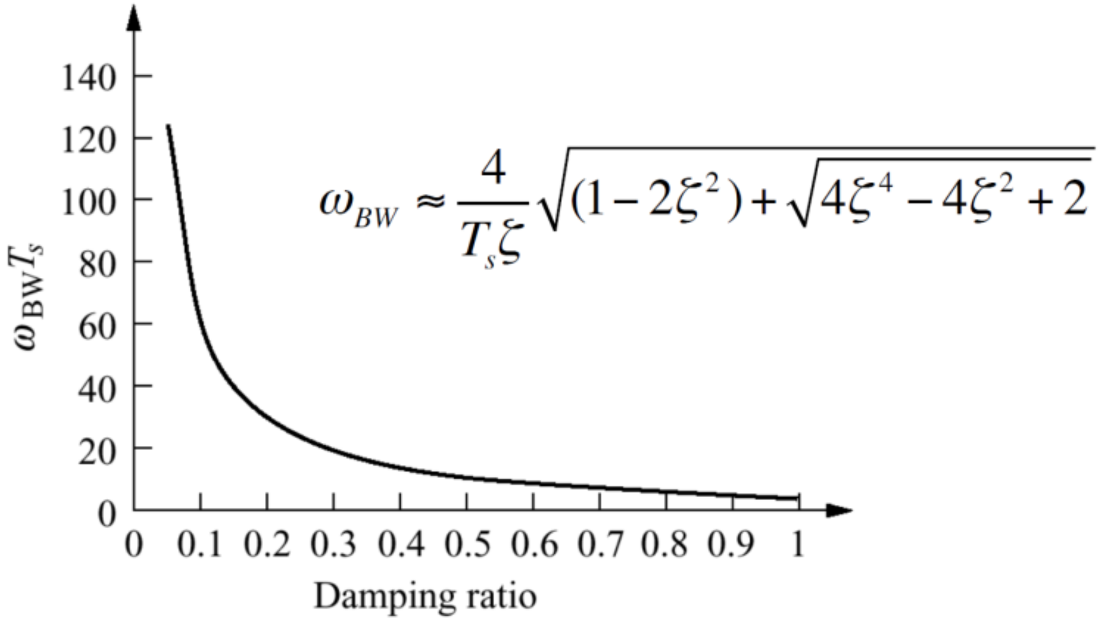
Closed-loop frequency response peak vs percent overshoot for a two-pole system

Relation between closed-loop transient and closed-loop frequency responses

Response Speed and Closed-Loop Frequency Response

Normalised bandwidth vs. damping ratio for:

- Settling time $\boldsymbol{T_S}$



Relation between closed-loop transient and closed-loop frequency responses

Response Speed and Closed-Loop Frequency Response

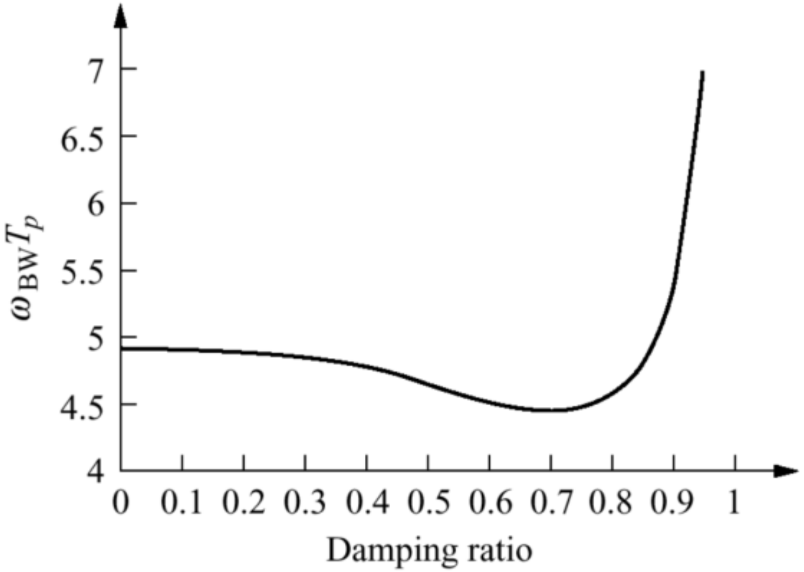
Normalised bandwidth vs. damping ratio for:

- Peak time T_p

$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_n = \pi / \left(T_p \sqrt{1-\zeta^2}\right)$$

$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$



Relation between closed-loop transient and closed-loop frequency responses

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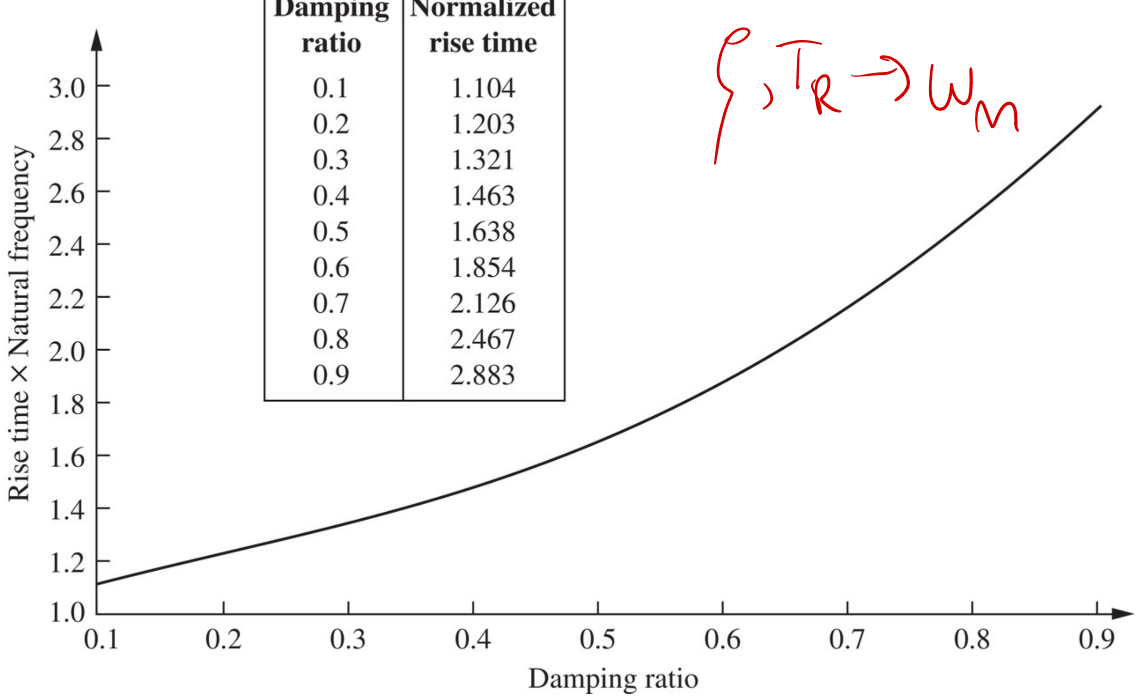
Response Speed and Closed-Loop Frequency Response

Normalised bandwidth vs. damping ratio for:

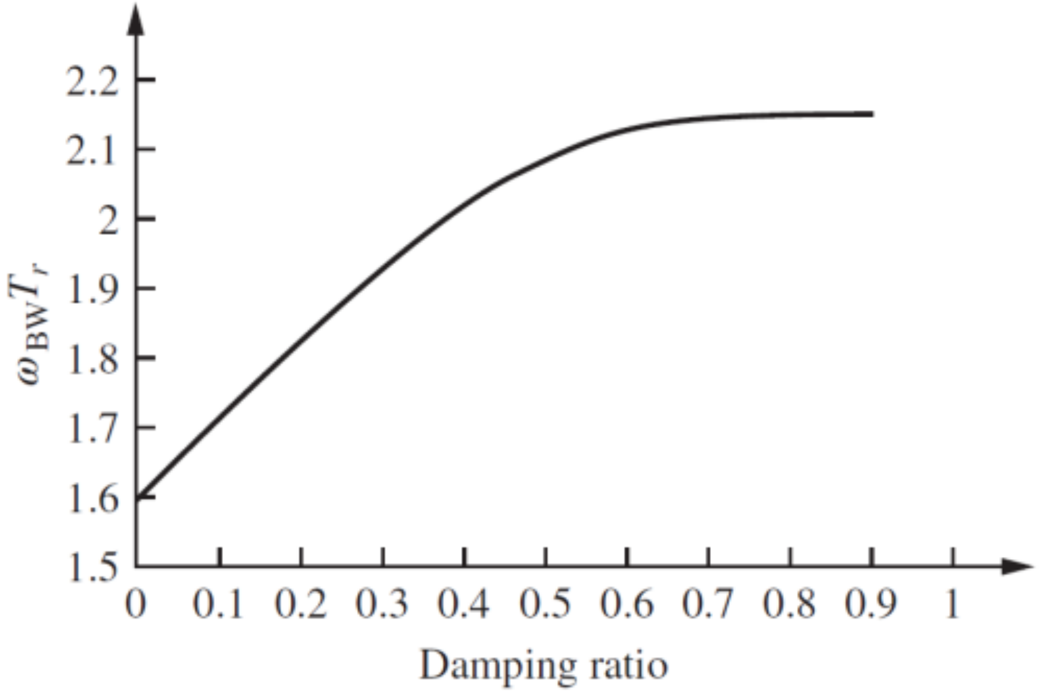
- Rise time $\boldsymbol{T_r}$
- To relate the bandwidth to rise time, $\boldsymbol{T_r}$, we can use Figure 4.16 (N4.6), knowing the desired ζ and T_r.

$\zeta, T_R \to \omega_M$



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Figure 4.16



ω_{BW} by T_r product as a function of ζ

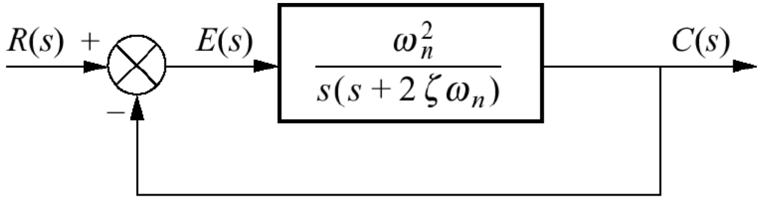
time

Relation between closed-loop transient and open-loop frequency responses

Relation between closed-loop transient and open-loop frequency responses

Damping ratio from phase margin

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \qquad \frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$



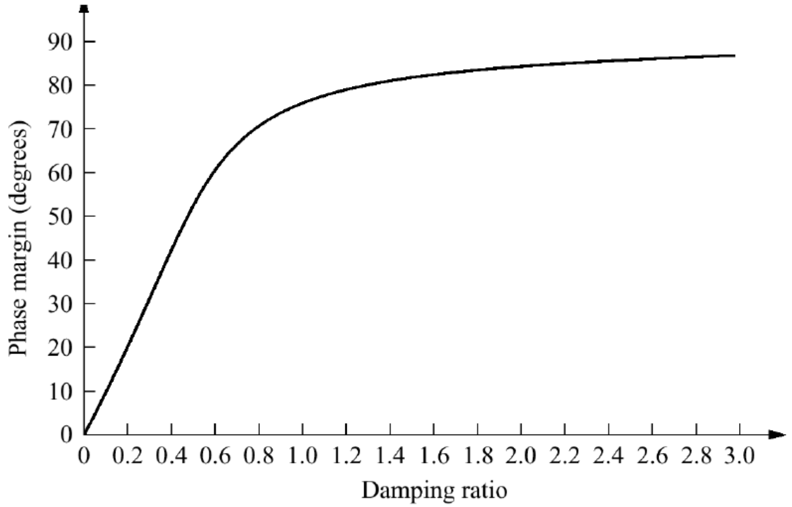
$$G(j\omega) = \frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)} = \frac{\omega_n^2}{-\omega^2+j2\zeta\omega_n\omega}$$

$$|G(j\omega)| = \frac{\omega_n^2}{|-\omega^2+j2\zeta\omega_n\omega|} = 1$$

$$\omega_1 = \omega_n\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}$$

$$\angle G(j\omega) = -90 - \tan^{-1}\frac{\omega_1}{2\zeta\omega_n}$$

$$= -90 - \tan^{-1}\frac{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}}{2\zeta}$$



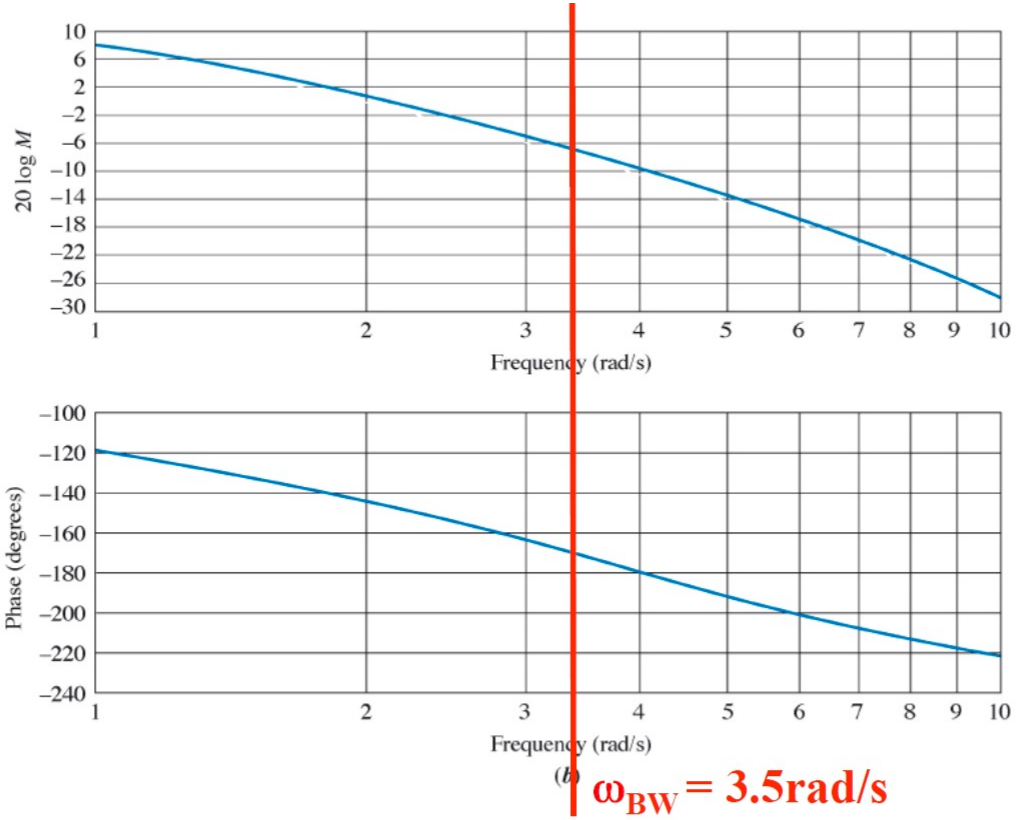
$$\Phi_M = 90 - \tan^{-1}\frac{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}{2\zeta} = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$$

- a phase margin of 65.52° ($\zeta = 0.707$) or larger is required from the *open-loop* frequency response to ensure there is no peaking in the *closed-loop* frequency response

Relation between closed-loop transient and open-loop frequency responses

Response speed from open-loop frequency response

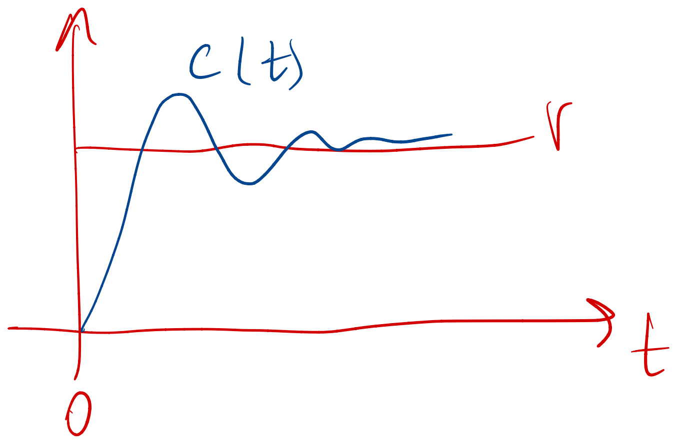
- The closed-loop bandwidth can be estimated from the open-loop frequency response
- It can be shown that, approximately, the closed-loop bandwidth, ω_{BW}, equals the frequency at which the open-loop magnitude response is between -6 and -7.5 dB if the open-loop phase response is between -135° and -225°

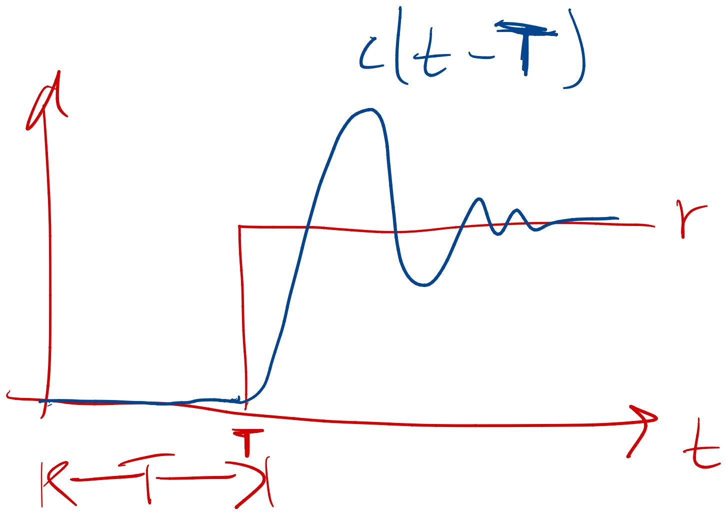


Systems with time delay

3. Systems with time delay

- **Time delay**
  - Delay between the commanded response (reference signal) and the start of the output response
  - Note: this is **not the same** as the transient response
  - **Nothing happens at the output**, during the time delay
  - Example: heating systems





3. Systems with time delay

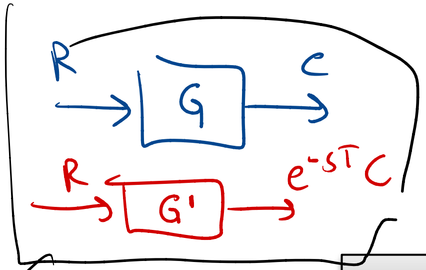
MACQUARIE University

time shift theorem

$$\mathcal{L}[c(t-T)] = e^{-sT}\mathcal{L}[c(t)] = e^{-sT}C(s)$$

Modeling time delay

- Assume: $R(s)$ – input, $G(s)$ – system, $C(s)$ – output
- If $G'(s)$ delays the output by T seconds, the output response is $c(t-T)$
- Laplace transform of $c(t-T)$ is $e^{-sT}C(s)$
- System without delay: $C(s)=R(s)G(s)$
- System with delay: $e^{-sT}C(s)=R(s)G'(s)$
- Dividing the two equations above: $G'(s)/G(s)=e^{-sT}$



$\rightarrow G'(s) = e^{-sT}G(s)$

$$C = R \cdot G$$

$$e^{-sT}C = RG'$$

divide second expression by the first one

$$\frac{e^{-sT}\cancel{C}}{\cancel{C}} = \frac{\cancel{R}G'}{\cancel{R}G} \Rightarrow e^{-sT} = \frac{G'}{G} \Rightarrow \boxed{G' = G \cdot e^{-sT}}$$

3. Systems with time delay

Modeling time delay

- Frequency response perspective ($s=j\omega$)
- Time delay:
  - **does not affect the magnitude**
  - it subtracts a linearly increasing phase shift (ωT) from the phase of $G(j\omega)$
  - **time reduces the phase-margin**
  - yields a reduced damping ratio for C/L system
  - **also reduces the gain-margin**
- **Summary:**
  - time delay **moves the system close to instability**

$$\boxed{G'(s) = e^{-sT} G(s)}$$

$$\boxed{G'(j\omega) = e^{-j\omega T} G(j\omega) = |G(j\omega)| \angle\{-\omega T + \angle G(j\omega)\}}$$