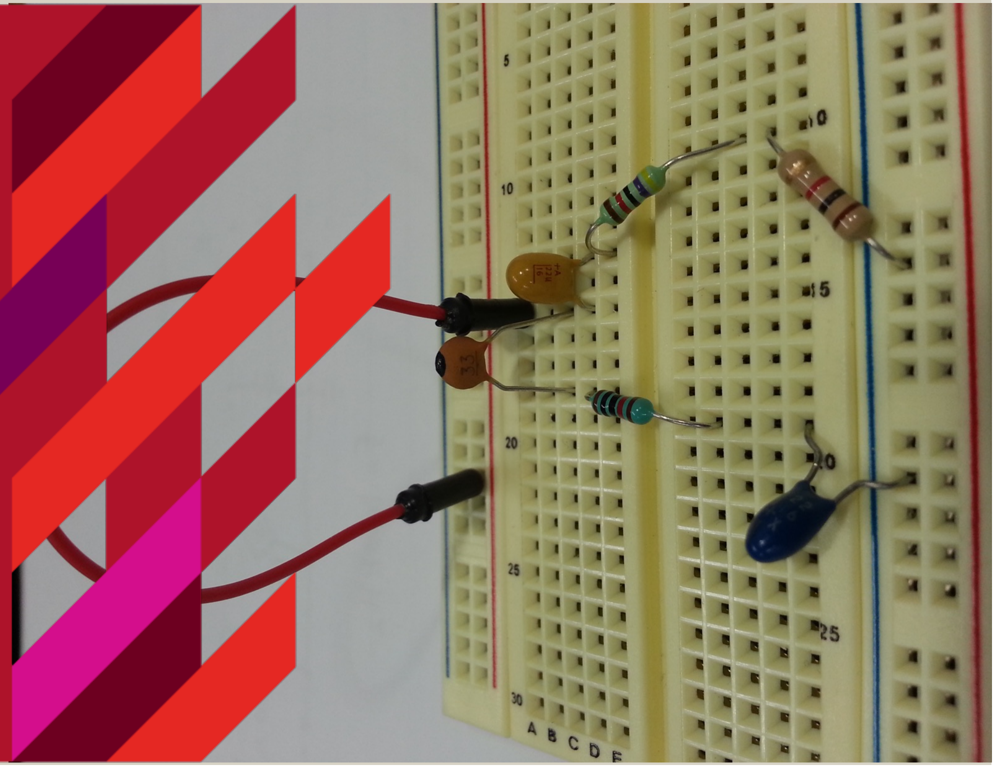
ELEC2070 Circuits and Devices

Week 4: The complete response of 1st order (RL and RC) circuits

Stuart Jackson



The “Big Ideas” in Chap. 8

- RL and RC circuits are examples of first-order circuits. A first-order circuit has effectively one energy storage element (one inductor or one capacitor). A first-order circuit can be represented by a first-order differential equation.

- How do we find the complete response of a first-order circuit to abrupt changes? Abrupt change may be a change of switch position or a change (i.e. discontinuity) in an input voltage/current.

The time-dependent response of simple circuits

What we know so far..

Resistor:

$$v = Ri$$

Capacitor:

$$i(t) = C\frac{d}{dt}v(t)$$

Inductor:

$$v(t) = L\frac{d}{dt}i(t)$$

Now we use these relationships with Kirchhoff's Current and Voltage Laws to examine simple circuits.

The initial analysis method

Simplify your circuit:

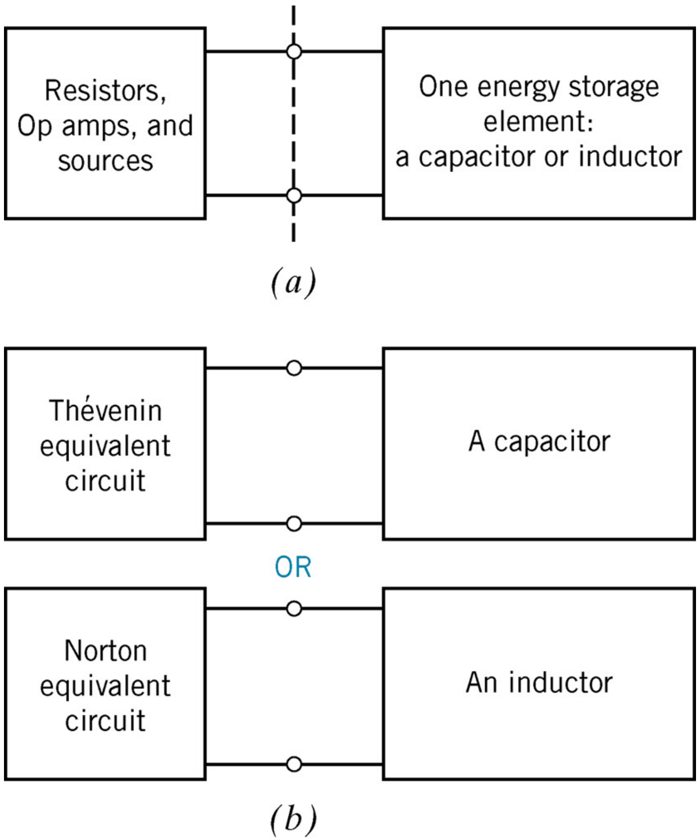
1. **Combine your capacitors (or inductors) to one side**
2. **Resistors and sources on the other side**

Find:

- **Thévenin equivalent circuit as seen by the combined capacitor**

or:

- **Norton equivalent circuit as seen by the combined inductor**

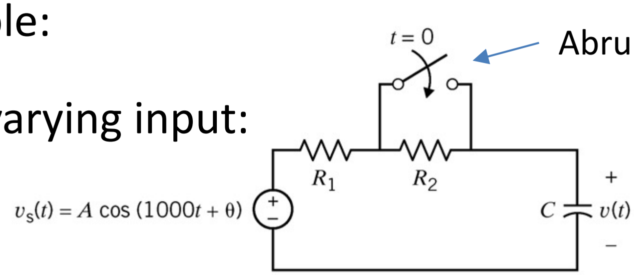


A circuit containing one inductor (and no capacitors) or one capacitor (and no inductors) is called a first order circuit.

The general time-dependent response – before we start

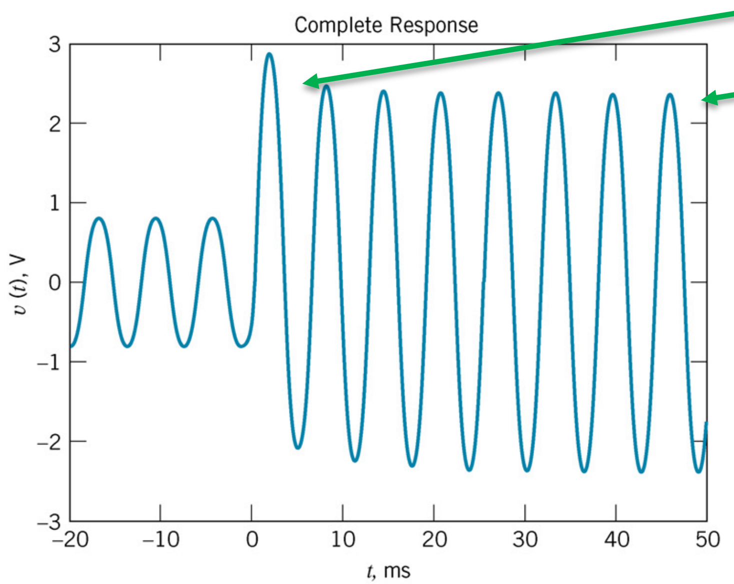
Example:

Time varying input:



(a)

complete response =
transient part + steady-state part



(b)

The response can be written as
(we show this later):

$$v(t) = Ke^{-t/\tau} + M\cos(1000t + \varphi)$$

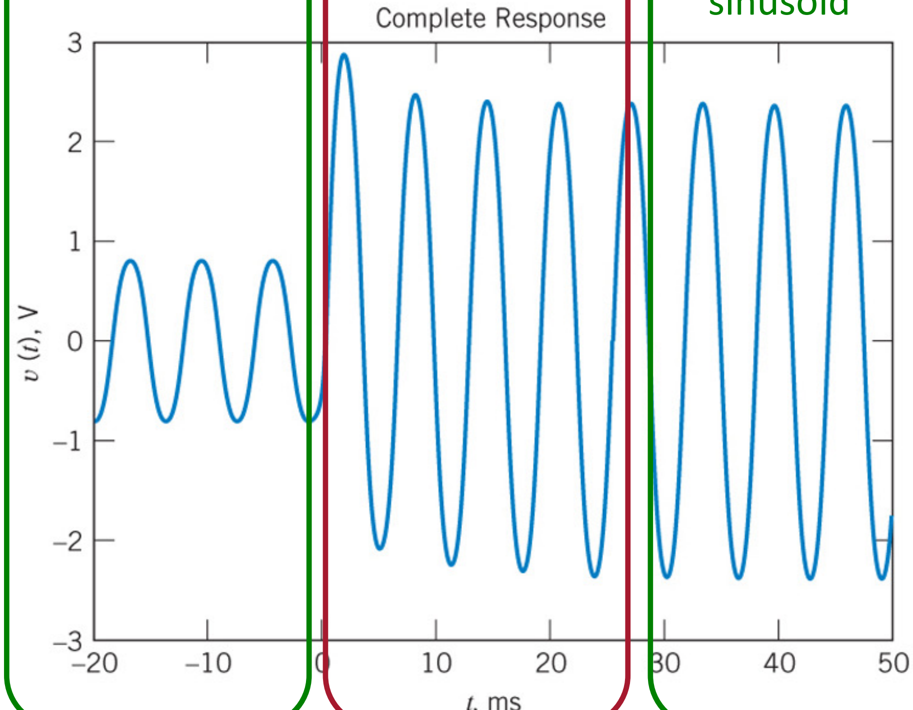
Transient part eventually dies out, as t -> ∞.
Steady-state part stays until current or voltage source stops.

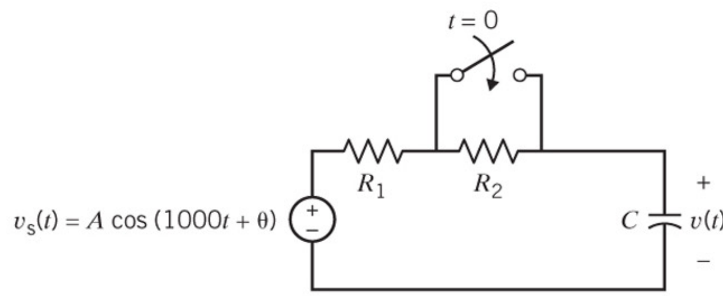
General description of circuit response

forced response: sinusoid

natural + forced responses: exponentially decaying sinusoid

forced response: sinusoid





complete response

=

forced response

+

natural response

A linear combination !

The differential equations

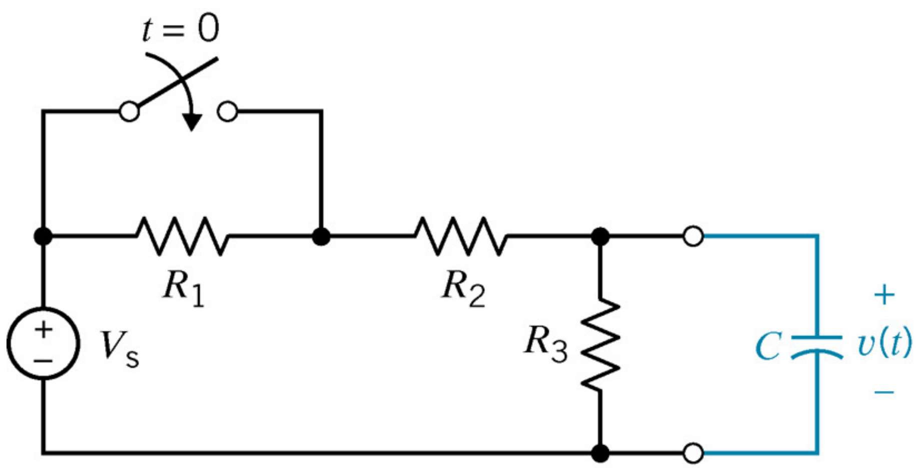
Describing the behaviour of first order (RC and RL) circuits

A constant input (source) to the circuit

The source could be voltage or current

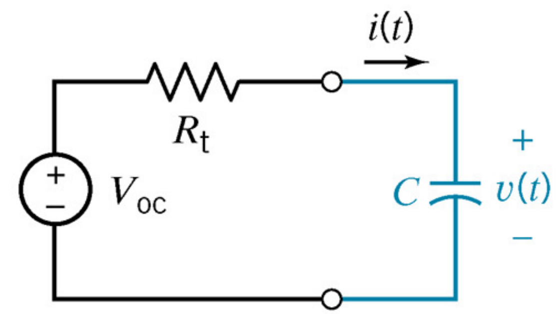
1. RC circuit with a DC source

t<0



(a)

t>0



(b)

1. Assume circuit is at steady state at t = 0^-.
2. Abrupt change at t = 0 → disturbs circuit
3. We know how to determine currents and voltages before disturbance (i.e., t<0).
4. How can we determine them for t>0?

Answer: Derive a 1st order differential equation and solve it.

Create Thévenin circuit FIRST

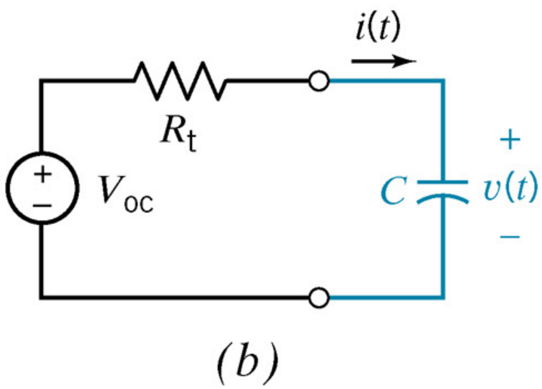
Thévenin equivalent circuit for t>0

$$V_{oc} = \frac{R_3}{R_2+R_3} V_s,\ R_t = \frac{R_2 R_3}{R_2+R_3}$$

Response for t > 0

What is the response of a first-order circuit to a constant input?

Our simplified circuit:



"Response" usually means the voltage across the capacitor (or current in the inductor)

We can apply KVL: $$V_{oc} = R_t i(t) + v(t) = R_t \left(C \frac{dv(t)}{dt} \right) + v(t)$$

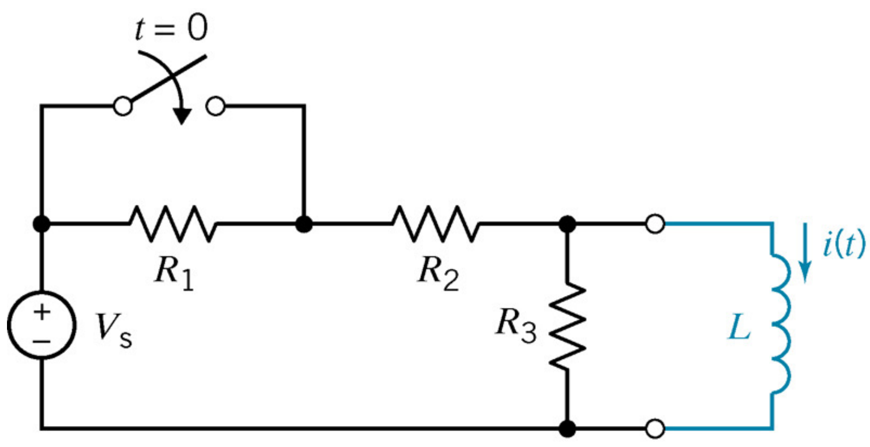
Re-arrange: $$\frac{dv(t)}{dt} + \frac{v(t)}{R_t C} = \frac{V_{oc}}{R_t C}$$

This is the standard differential equation for ANY 1st order RC circuit with a constant input.

First order (RC) circuit $\rightarrow$ involves a first order differential equation (or DE)

2. RL circuit with a DC source

t<0



(a)

1. Assume that the circuit is at steady state at t = 0^-.
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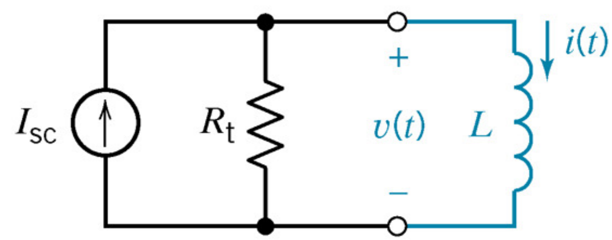
4. How can we determine them for t>0?

Answer: Derive a 1st order differential equation and solve it.

t>0

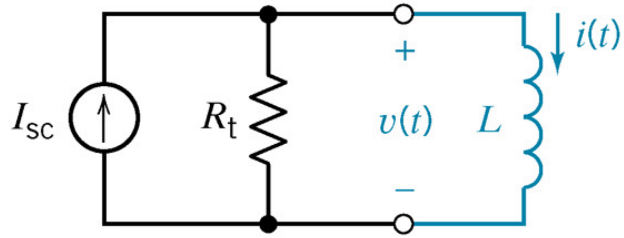


(b)

Create Norton circuit FIRST

$$I_{sc} = \frac{V_s}{R_2} \quad \text{and} \quad R_t = \frac{R_2 R_3}{R_2 + R_3}$$

for t>0



(b)

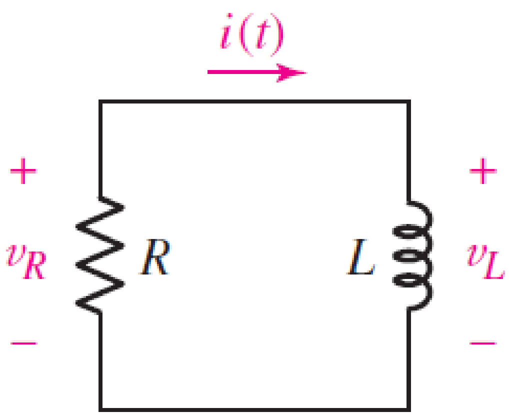
KCL (at top node): $$I_{sc} = \frac{v(t)}{R_t} + i(t) = \frac{L\frac{di(t)}{dt}}{R_t} + i(t)$$

Re-arrange: $$\frac{di(t)}{dt} + \frac{R_t}{L}i(t) = \frac{R_t}{L}I_{sc}$$

The standard differential equation for ANY 1st order RL circuit with a steady state input.

First order (RL) circuit → involves a first order differential equation

Source free RL circuit (pure natural response)



KVL:

$$Ri + v_L = Ri + L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

SOLUTION

$$i(t) = I_0 e^{-Rt/L}$$

Accounting for the energy

Power dissipated by the resistor:

$$p_R = i^2 R = I_0^2 R e^{-2Rt/L}$$

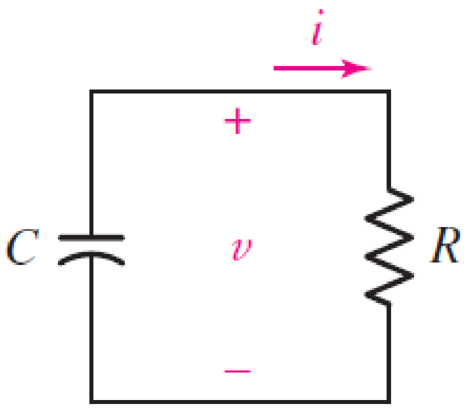
Total energy turned into heat:

$$w_R = \int_0^\infty p_R \, dt = I_0^2 R \int_0^\infty e^{-2Rt/L} \, dt$$

$$= I_0^2 R \left(\frac{-L}{2R}\right) e^{-2Rt/L}\Big|_0^\infty = \frac{1}{2} L \, I_0^2$$

Energy originally stored in capacitor

Source free RC circuit



KCL: $$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

SOLUTION

$$v(t) = v(0)e^{-t/RC} = V_0 e^{-t/RC}$$

Accounting for the energy - HOMEWORK

The general differential equation for DC input

Standard differential equation for ANY 1st order circuit with a constant input:

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = K$$

Where the "time constant" $\tau = R_t C \text{ or } L/R_t$

and K is a CONSTANT

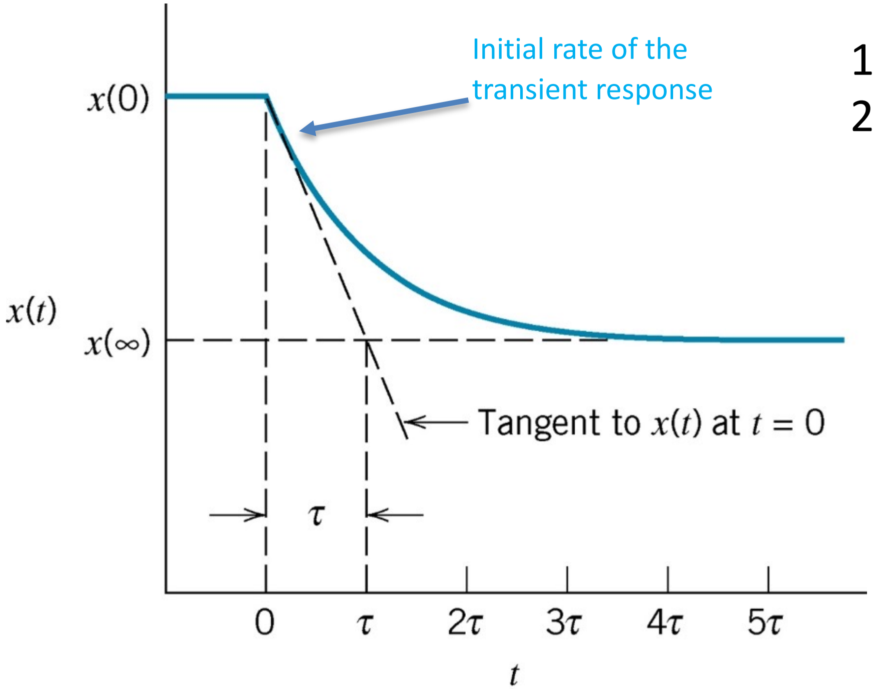
How do we solve this 1st order differential Eqn? It's in the book (Dorf). The solution is:

$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{-\frac{t}{\tau}}$$

Value of the long-term capacitor voltage or inductor current (i.e., when t -> infinity, if there are no other abrupt changes).

Initial condition = Value of capacitor voltage or inductor current at t =0 (when the abrupt change happens).

The time constant of a 1^{st} order circuit



1. Take the slope of the tangent at t=0
2. Where the tangent intercepts with the long-term value of x(t) or x(∞) is called the TIME CONSTANT, τ

$$\tau = R_t C \text{ or } L/R_t$$

More generally:

$$\tau = R_{\text{eq}} C_{\text{eq}}$$

τ is the time constant of the first-order circuit – the time the response would take to reach the final value if the (transient) response occurs at the initial rate of change.

It is an indication of how fast the response changes in a given 1^{st} order circuit.

The voltage or current at $t = \tau$

x(t) could be v or i

The general solution: $$x(t) = x(\infty) + [x(0) - x(\infty)]\, e^{-t/\tau}$$

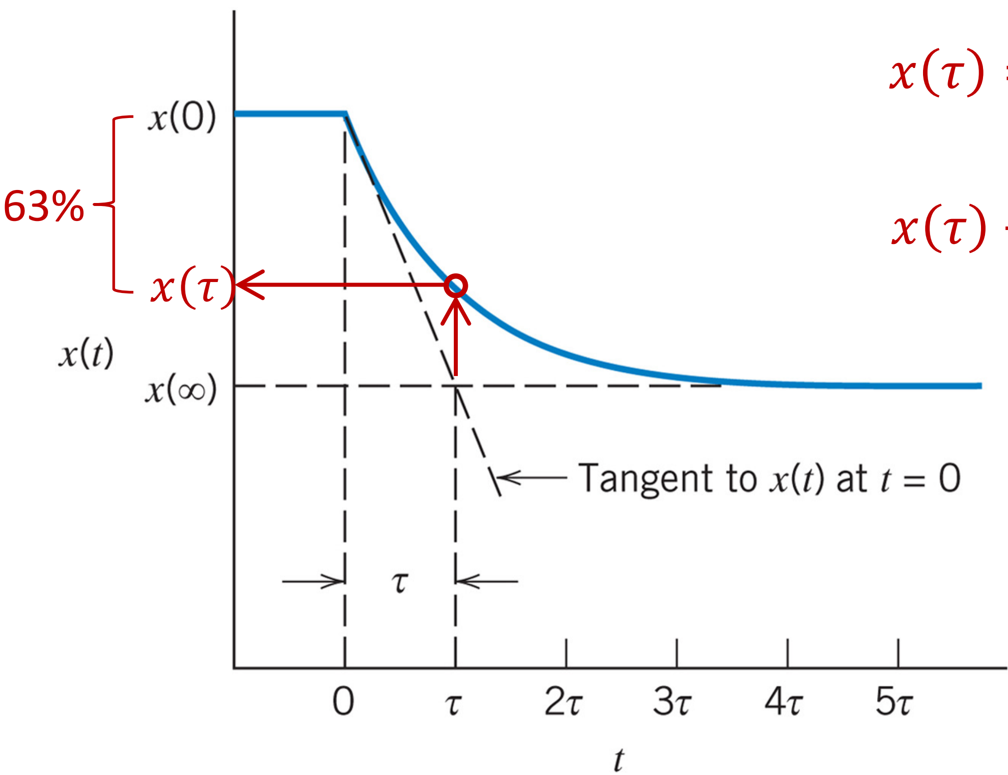
When t = τ:

$$x(\tau) = x(\infty) + [x(0) - x(\infty)]\, e^{-\tau/\tau}$$

$$x(\tau) = x(\infty) + [x(0) - x(\infty)]\, 0.368$$

$$\downarrow$$

$$x(\tau) - x(0) = 0.632\, [x(\infty) - x(0)]$$



When the time after an abrupt change to circuit is equal to the **time constant**, the value of the voltage or current is 63% **lower** than the value at the time the abrupt change occurs.

Summary of first order circuits with a DC source

First order RC and RL circuits: constant input

Standard differential equation for ANY 1st order RC circuit with a constant input:

$$\frac{dv(t)}{dt} + \frac{v(t)}{R_t C} = \frac{V_{oc}}{R_t C}$$

Standard differential equation for ANY 1st order RL circuit with a constant input:

$$\frac{di(t)}{dt} + \frac{R_t}{L} i(t) = \frac{R_t}{L} I_{sc}$$

The response of ANY RC circuit with a constant input is:

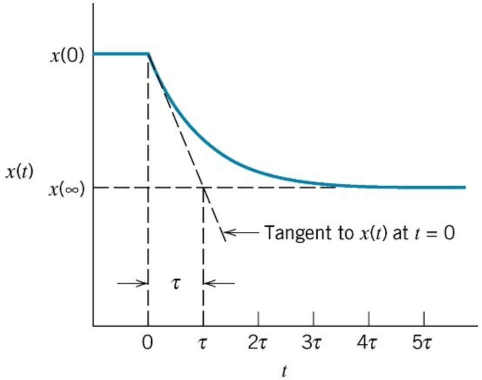
$$v(t) = V_{oc} + (v(0) - V_{oc}) e^{\frac{-t}{(R_t C)}}$$

The response of ANY RL circuit with a constant input is:

$$i(t) = I_{sc} + (i(0) - I_{sc}) e^{-(\frac{R_t}{L})t}$$

These template solutions is all you have to remember when finding the response of ANY 1st order circuit with a constant (DC) input .

Summary



$$v(t) = \boxed{V_{oc}} + \boxed{(v(0) - V_{oc})e^{\frac{-t}{(R_t C)}}}$$

$$i(t) = \boxed{I_{sc}} + \boxed{(i(0) - I_{sc})e^{-(\frac{R_t}{L})t}}$$

Steady-state part of the response or the Steady-State Response: This is what is left when *t* tends to infinity.

This is also the Forced Response of the circuit.

This part disappears when *t* tends to infinity so this is the transient part of the response or the Transient Response.

This is also the Natural Response of the circuit.

Sometimes the natural response is not transient

This description is appropriate only for those circuits in which the natural response eventually dies out. This always occurs in physical circuits where some resistance is associated with every element, but there are a number of "pathologic" circuits in which the natural response is nonvanishing as time becomes infinite. Those circuits in which trapped currents circulate around inductive loops, or voltages are trapped in series strings of capacitors, are examples.

Switching an RC or RL circuit

1. Use the equivalent circuits before the switch is activated to calculate the initial conditions ($t = t_0$)

2. Use the equivalent circuits after the switch is activated, to calculate the final conditions ($t = \infty$)

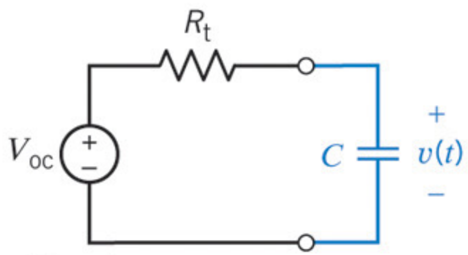
   It is always easier to determine the differential equations using equivalent circuits.

How to solve a switching problem

For RC circuits

Circuit for

$t > t_0$



NOTE: Circuit for

$t < t_0$

may be different

or the source

may be off

1. **Calculate the time constant**

$$\tau = R_t C$$

2. **Want capacitor voltage for t ≥ t_0, t_0= abrupt change**

$$v_C(t) = v_C(\infty) + [\, v_C(t_0) - v_C(\infty)]\, e^{-(t-t_0)/\tau}$$

(Initial and final conditions)

$$v_C(t_0^-) = v_C(t_0^+)$$

$$v_C(\infty) = V_{OC}$$

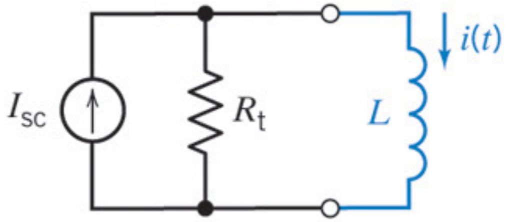
3. **Determine other voltages and currents in the circuit using the calculated capacitor voltage**

How to solve a switching problem

For RL circuits

Circuit for

$t > t_0$



Circuit for

$t < t_0$

may be different or the source may be off

1. **Calculate the time constant**

$$\tau = L/R_t$$

2. **Inductor current for $t \geq t_0$, t_0= abrupt change**

$$i_L(t) = i_L(\infty) + [\, i_L(t_0) - i_L(\infty)]\, e^{-(t-t_0)/\tau}$$

(Initial and final conditions)

$$i_L(t_0^-) = i_L(t_0^+)$$

$$i_L(\infty) = I_{SC}$$

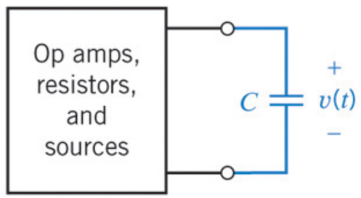
3. **Determine other voltages and currents in the circuit using the calculated inductor current**

Summary

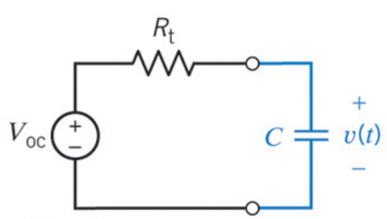
Table 8.12-1

***Suitable for first order circuits with dc sources turning “on” at t=0

FIRST-ORDER CIRCUIT CONTAINING A CAPACITOR



Replace the circuit consisting of op amps, resistors, and sources by its Thévenin equivalent circuit:



The capacitor voltage is:

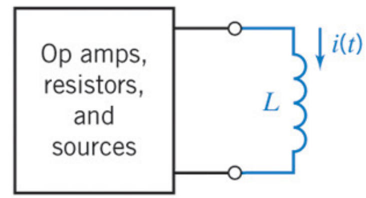
$$v(t) = V_{oc} + (v(0) - V_{oc})e^{-t/\tau}$$

where the time constant τ is

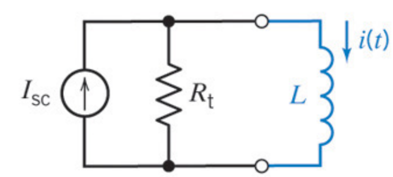
$$\tau = R_t C$$

and the initial condition $v(0)$ is the capacitor voltage at time $t = 0$.

FIRST-ORDER CIRCUIT CONTAINING AN INDUCTOR



Replace the circuit consisting of op amps, resistors, and sources by its Norton equivalent circuit:



The inductor current is

$$i(t) = I_{sc} + (i(0) - I_{sc})e^{-t/\tau}$$

where the time constant τ is

$$\tau = \frac{L}{R_t}$$

and the initial condition $i(0)$ is the inductor current at time $t = 0$.

Example 8.3-1

First order RC circuit

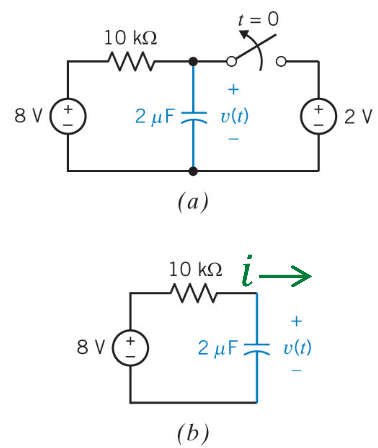
Initial conditions:

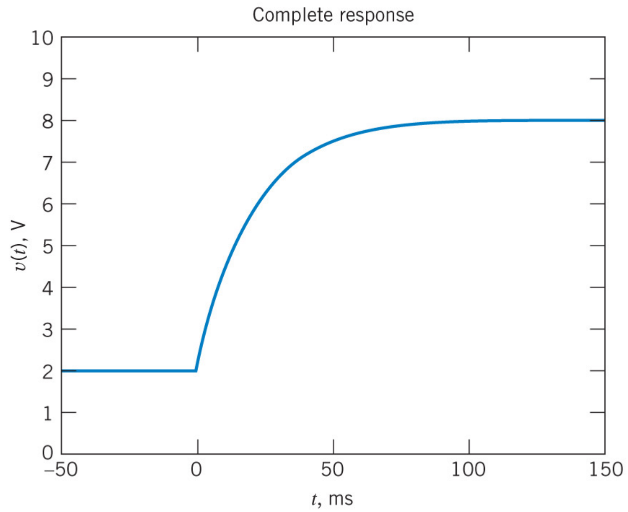
For $t \leq 0$ we have for the cap.:

$v(0^-) = v(0^+) = 2\ \mathrm{V}$

$i(0^-) = 0\ \mathrm{A}$

$$i(0^+) = \frac{8 - v(0^+)}{10\times 10^3} = 0.6\ \mathrm{mA}$$





For $t > 0$ find:

$v(t) = ?$

and $v(50\times 10^{-3}) = ?$

$i(t) = ?$

For this problem we use:

$$\frac{d}{dt}v(t) + \frac{v(t)}{RC} = \frac{V_s}{RC}$$

First-order differential equation

Example 8.3-1 - solution

First order RC circuit

Initial and final conditions:

$$v(0^-) = v(0^+) = v(0) = 2\ \mathrm{V}$$

$$v(\infty) = 8\ \mathrm{V}$$

$$\frac{d}{dt}v(t) + \frac{v(t)}{RC} = \frac{V_s}{RC}$$

First-order differential equation

Time constant

$$\tau = RC = 10\times10^{3}\times2\times10^{-6} = 20\times10^{-3}\mathrm{s}$$

Capacitor voltage

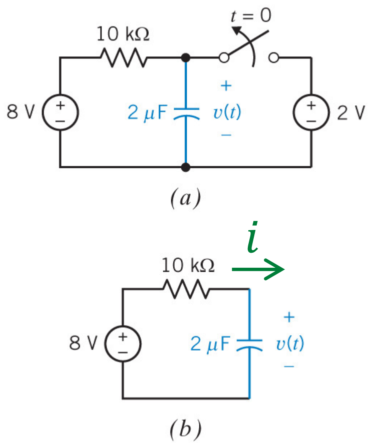
$$v(t) = v(\infty) + [v(0) - v(\infty)]\, e^{-t/\tau}$$

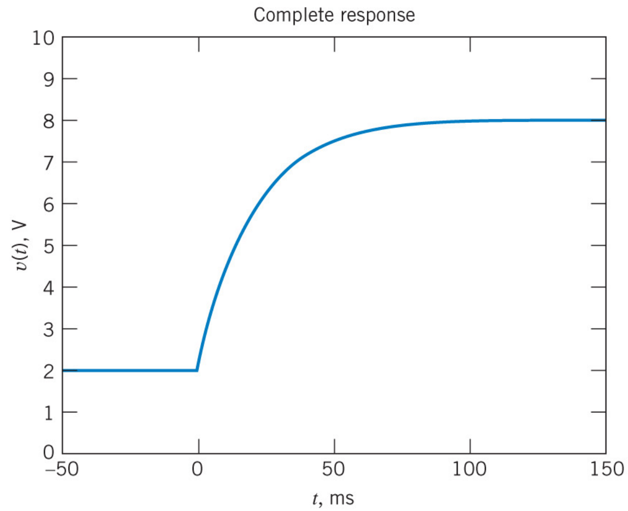
$$v(t) = 8 - 6e^{-t/(20\times10^{-3})}\ \mathrm{V}$$

$$v(50\times10^{-3}) = 7.51\ \mathrm{V}$$

Example 8.3-1

First order RC circuit





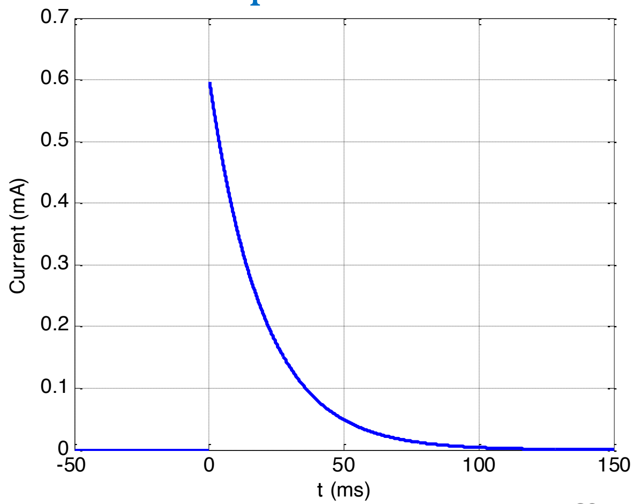
$t < 0$

$v(t) = 2\text{ V}$

$t > 0$

$v(t) = 8 - 6e^{-t/(20\times10^{-3})}\text{ V}$

Matlab plot:



Capacitor current:

$t < 0 \quad i(t) = 0$

$$t > 0 \quad i(t) = \frac{8 - v(t)}{10\times10^3} = 0.6\, e^{-t/(20\times10^{-3})}\ \textbf{mA}$$

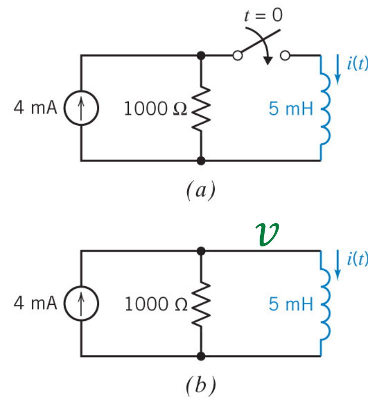
Note units

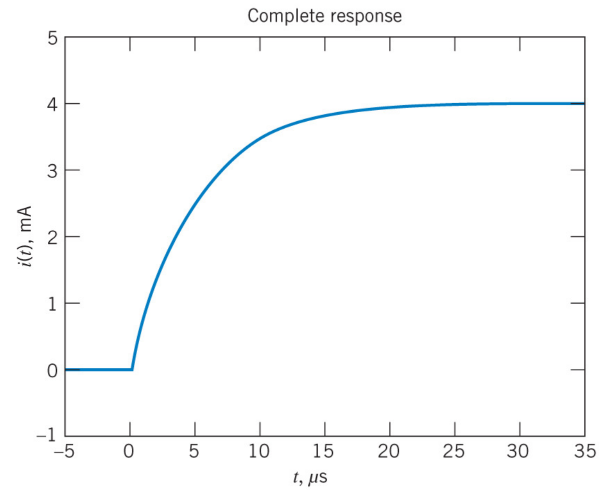
Example 8.3-2

First order RL circuit

Initial conditions:

$$i(0^-) = i(0^+) = 0\text{ A}$$
$$v(0^-) = 0\text{ V}$$
$$v(0^+) = 4\text{ V}$$





For $t > 0$ find: $i(t) = ?$ $\quad i(?) = 2\text{ mA}$ $\quad v(t) = ?$

It is easy to write the differential equation using the Norton equivalent:

KCL: $$I_s = \frac{v(t)}{R} + i(t) = \frac{L}{R}\frac{d}{dt}i(t) + i(t)$$

$$v(0^-) = 0\ V$$
$$v(0^+) = 1000 \times 4\text{ mA} = 4\text{ V}$$

$$\boxed{\frac{d}{dt}i(t) + \frac{R}{L}i(t) = \frac{R}{L}I_s}$$

First-order differential equation

$$i(t) = i(\infty) + [i(0) - i(\infty)]\, e^{-t/\tau}$$

Example 8.3-2

First order RL circuit

Initial and final conditions:

$i(0^-) = i(0^+) = 0\ \mathrm{A}$

$i(\infty)$ = 4 mA

Time constant

$$\tau = \frac{L}{R} = \frac{5\times10^{-3}}{10^3} = 5\times10^{-6}\mathrm{s}$$

Inductor current

$$i(t) = i(\infty) + [i(0) - i(\infty)]\ e^{-t/\tau}$$

$t > 0$ ⇨

$$i(t) = 4 - 4\ e^{-t/5\times10^{-6}}\ \mathrm{mA}$$

What time will the current be 2 mA?

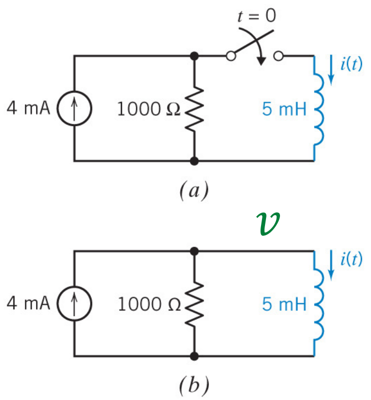
$i(?) = 2\ \mathrm{mA}$:

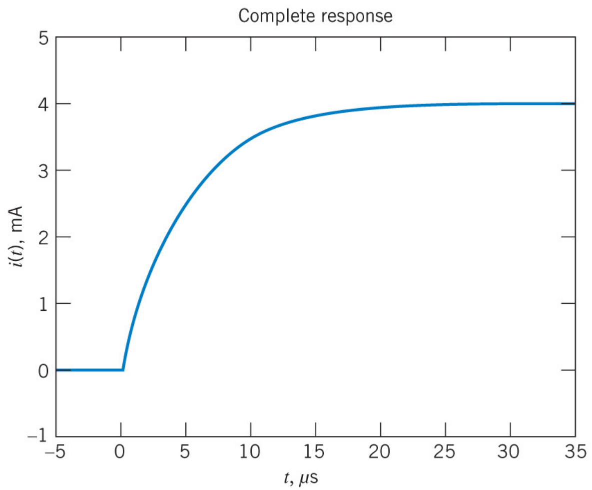
$$2 = 4 - 4\ e^{-t/5\times10^{-6}} \Rightarrow 4\ e^{-t/5\times10^{-6}} = 2 \Rightarrow -\frac{t}{5\times10^{-6}} = \ln\frac{2}{4}$$

$$t = 3.47\ \mu\mathrm{s}$$

Example 8.3-2

First order RL circuit





$t < 0$

$i(0) = 0 \text{ mA}$

$t > 0$

$i(t) = 4 - 4\, e^{-t/5\times10^{-6}} \text{ mA}$

Inductor voltage: $v(t) = ?$

$t < 0 \quad v(t) = 0$

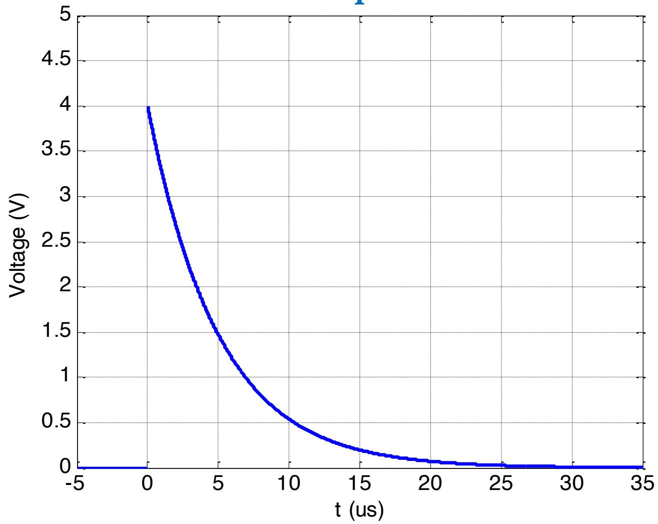
$t > 0$

Could apply: $v(t) = L\frac{d}{dt}i(t)$

Node analysis easier:

$$v(t) = 1000\big(4\,mA - i(t)\big) = 4\, e^{-t/(5\times10^{-6})} \text{ V}$$

Matlab plot:



The complete response

Here we simply add the natural (transient) response to the forced (steady state) response

The general differential equation involving forced and natural responses

General DE:

$$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = y(t)$$

Time constant

Function to be solved (voltage or current)

Forcing function due to source (remember = K for a constant source)

This type of differential equation may be solved by separating the variables and integrating. (Book Section 8.3)

General Solution:

$$x(t) = x_n(t) + x_f(t)$$

Complete response

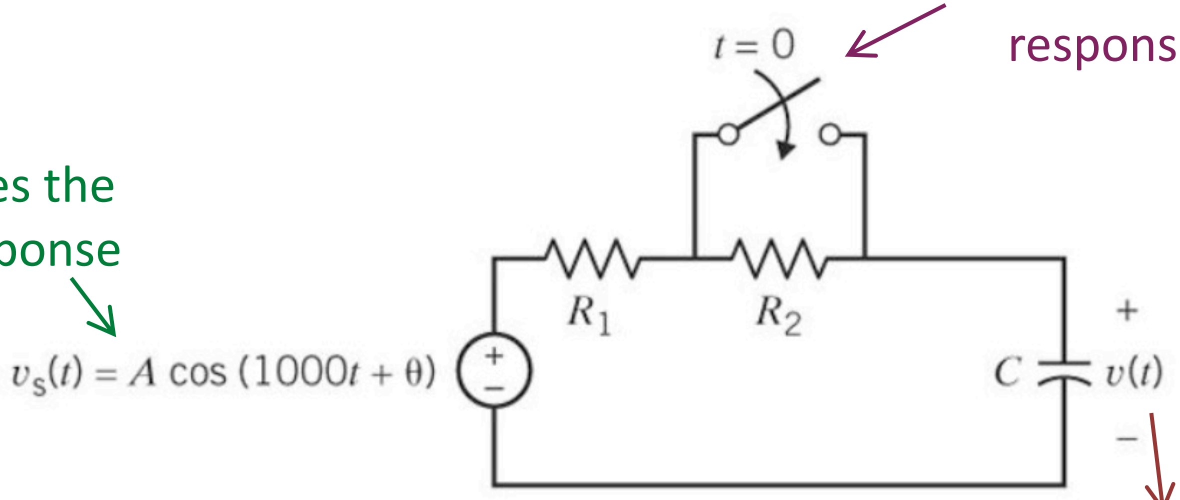
Natural response

Forced response

The forced and natural response

This initiates the natural response

This creates the forced response



Need to find the complete response

Natural response

Solution to the Homogeneous Differential Equation

Since the solution looks like this:
(superposition of natural and forced response)

$$x(t) = x_n(t) + x_f(t)$$

We can determine each part SEPARATELY (since they are just added together)!

IGNORING the forcing function means the general DE becomes:

$$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = 0$$

This is the **homogeneous** form of the differential equation

Has the general solution:

$$x_n(t) = Ke^{-(t-t_0)/\tau}$$

depends on the initial conditions

exponential

time when switched

time constant

Forced response

Response to the forcing function only

FORCING FUNCTION, $y(t)$	FORCED RESPONSE, $x_f(t)$
1. Constant	
$y(t) = M$	$x_f = N$, a constant
2. Exponential	
$y(t) = Me^{-bt}$	$x_f = Ne^{-bt}$
3. Sinusoid	
$y(t) = M \sin(\omega t + \theta)$	$x_f = A \sin \omega t + B \cos \omega t$

We have done this (DC)

Forcing functions and forced responses have a similar time dependence but they have different magnitudes.

Solution for a constant input (forcing function)

Assume: $x_f(t) = N$

Calculate the magnitude using the actual differential equation

Must satisfy the differential equation

$$\frac{dx_f(t)}{dt} + \frac{1}{\tau} x_f(t) = M$$

Forcing function from table

$$0 + \frac{1}{\tau} N = M$$

$$x_f(t) = M\tau$$

Example

$$\frac{dv_C(t)}{dt} + \frac{1}{0.32} v_C(t) = 11.25$$

$$v_{Cf}(t) = 11.25 \times 0.32 = 3.6 \text{ V}$$

The forced response

Time constant = 0.32 s

Forcing function = 11.25 V

Complete solution for constant input

Must satisfy the initial conditions

The complete response for constant input in general is given by:

$$x(t) = x_n(t) + x_f(t) = Ke^{-(t-t_0)/\tau} + M\tau$$

For our previous example we have:

$$v_C(t) = v_{Cn}(t) + v_{Cf}(t) = Ke^{-(t-0)/0.32} + 3.6\ \text{V}$$

Where the initial conditions are:

$$t_0 = 0$$
$$v_C(0) = 0$$

Solving for K based on the initial conditions:

$$v_C(0) = 0 = Ke^0 + 3.6 \Rightarrow K = -3.6$$

Complete response:

$$\boxed{v_C(t) = -3.6\, e^{-t/0.32} + 3.6\ \text{V}}$$

Mathematical approach for solving first order circuits

MACQUARIE University

Start with the differential equation in this form and an initial condition

$$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = y(t)$$

This is related to the source to the circuit

$$x(t_1)$$

Find [from y(t)] the correct form of the forced response from the table

$$x_f(t)$$

Use this forced response in the differential equation to find its unknown coefficients

$$\frac{dx_f(t)}{dt} + \frac{1}{\tau}x_f(t) = y(t)$$

Forced response is fully solved

Write the correct form of the natural response, add to the forced response

$$x(t) = x_n(t) + x_f(t) = Ke^{-(t-t_0)/\tau} + x_f(t)$$

Use the initial condition with the complete response to find the unknown coefficient of the natural response

$$x(t_1) = Ke^{-(t_1-t_0)/\tau} + x_f(t_1)$$

$$t_1 \geq t_0$$

Natural response is fully solved

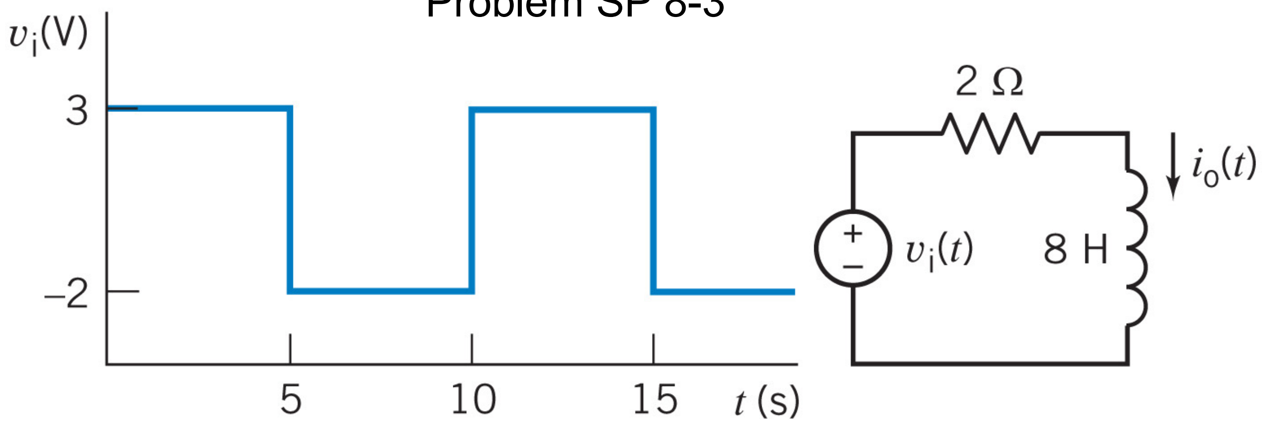
Sequential Switching

Sequential Switching

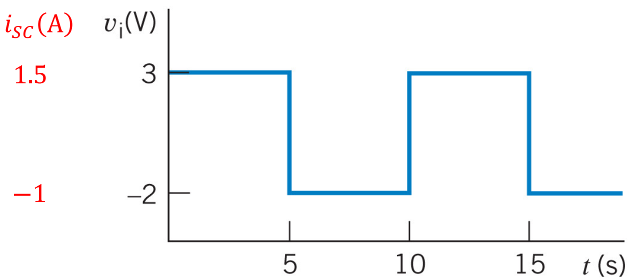
Our example: square wave input

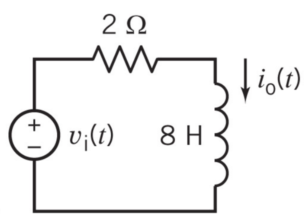
- **Initial conditions need to be calculated for each switching event**
- **The circuit may <u>not</u> have reached steady state before each switching event**
- **Each cycle may or may not have the same time constant – the switching process may involve new circuit elements**

Problem SP 8-3



Problem SP 8-3 (Dorf ed. 9) SP 8-2 (ed. 8)

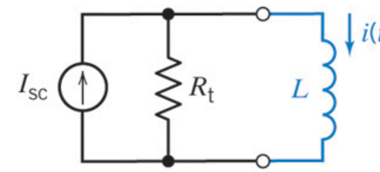




Norton Equivalent

$$i_{sc}(t) = \frac{v_i(t)}{R_t}$$

$$R_t = 2\ \Omega$$



Time constant

$$\tau = \frac{L}{R_t} = 4\ \text{s}$$

Solution at each half period

$$i_o(t) = i_o(\infty) + [\, i_o(t_0) - i_o(\infty)]\, e^{-(t-t_0)/\tau}$$

Solution for the first half period $\quad 0 < t < 5$

$$\boxed{\text{ASSUME: } v_i(t<0) = -2\ \text{V}}$$

$$i_o(0) = -\frac{2}{2} = -1\ \text{A}$$

$$i_o(\infty) = \frac{3}{2} = 1.5\ \text{A}$$

$$i_o(t) = 1.5 + [-1 - 1.5]\, e^{-t/4}$$

$$i_o(t) = 1.5 - 2.5\, e^{-t/4}\ \text{A}$$

$$i_o(5) = 1.5 - 2.5\, e^{-5/4} = 0.7837\ \text{A}$$

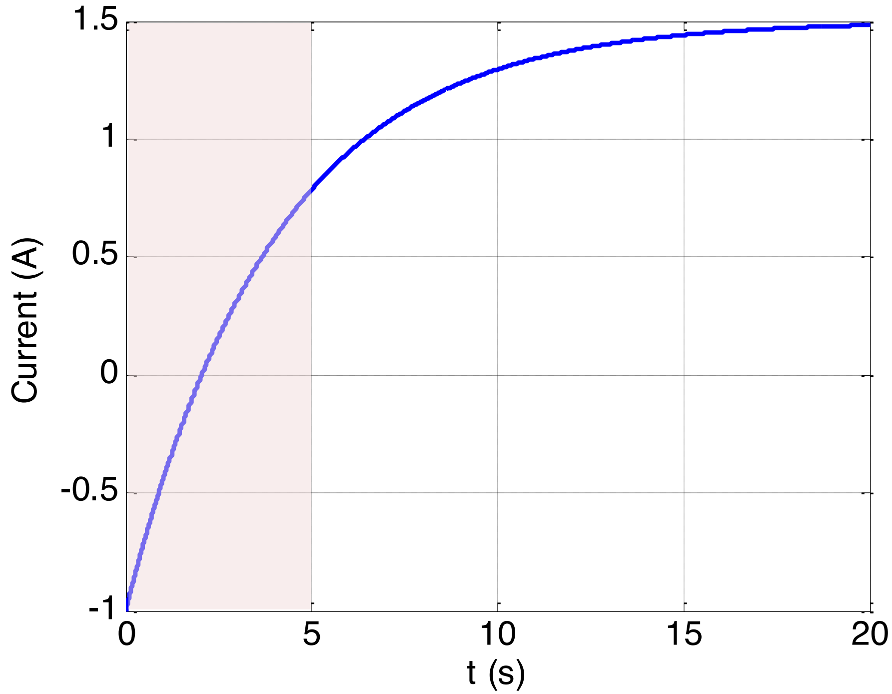
For 0 < t

$$i_o(0) = -\frac{2}{2} = -1\ \mathrm{A}$$

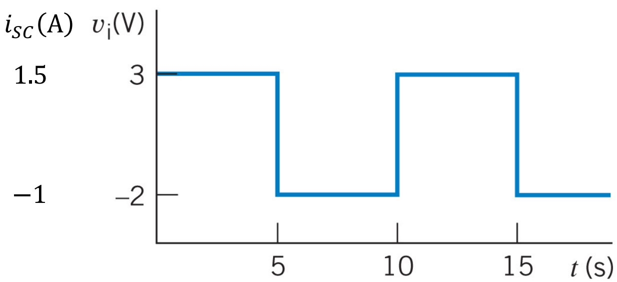
$$i_o(\infty) = \frac{3}{2} = 1.5\ \mathrm{A}$$

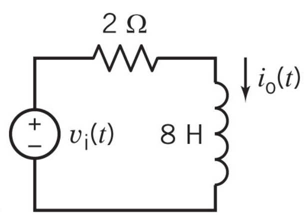
$$\boxed{i_o(t) = 1.5 - 2.5\ e^{-t/4}\ \mathrm{A}}$$

$$i_o(5) = 1.5 - 2.5\ e^{-5/4} = 0.7837\ \mathrm{A}$$



For 5 < t < 10

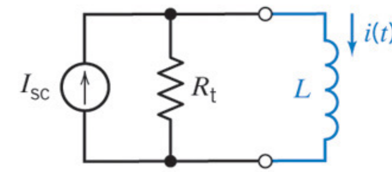




Norton Equivalent

$$i_{sc}(t) = \frac{v_i(t)}{R_t}$$

$$R_t = 2\ \Omega$$



Time constant

$$\tau = \frac{L}{R_t} = 4\ \mathrm{s}$$

Solution at each half period

$$i_o(t) = i_o(\infty) + [\, i_o(t_0) - i_o(\infty)]\, e^{-(t-t_0)/\tau}$$

Solution for the second half period $\quad 5 < t < 10$

$$i_o(5) = 0.7837\ \mathrm{A}$$

$$i_o(\infty) = -\frac{2}{2} = -1\ \mathrm{A}$$

$$i_o(t) = -1 + [0.7837 + 1]\, e^{-(t-5)/4}$$

$$i_o(t) = -1 + 1.7837\, e^{-(t-5)/4}\ \mathrm{A}$$

$$i_o(10) = -1 + 1.7837\, e^{-5/4} = -0.4890\ \mathrm{A}$$

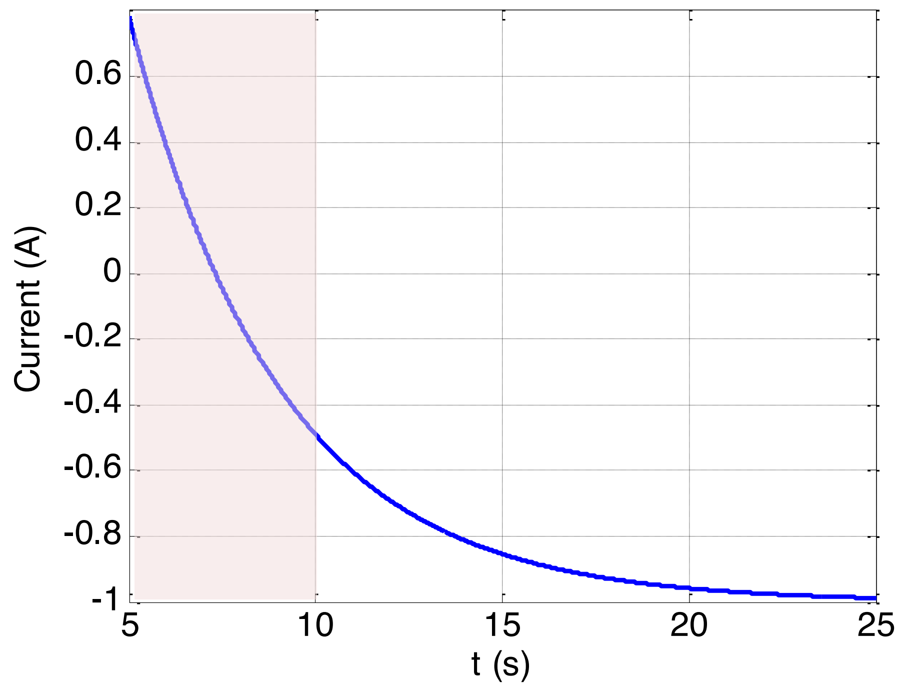
For 5 < t

$$i_o(5) = 0.7837 \text{ A}$$

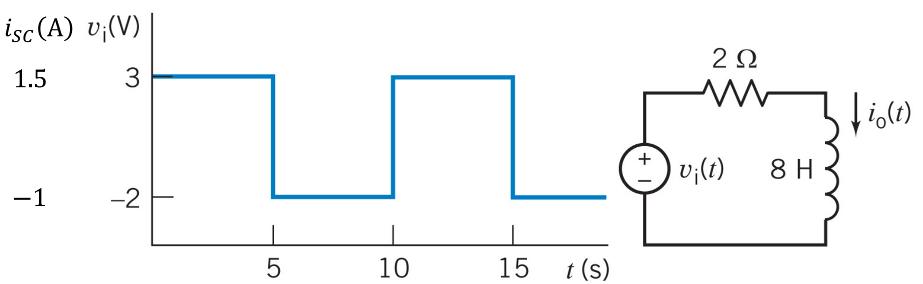
$$i_o(\infty) = -\frac{2}{2} = -1 \text{ A}$$

$$i_o(t) = -1 + 1.7837\, e^{-(t-5)/4} \text{ A}$$

$$i_o(10) = -1 + 1.7837\, e^{-5/4} = -0.4890 \text{ A}$$



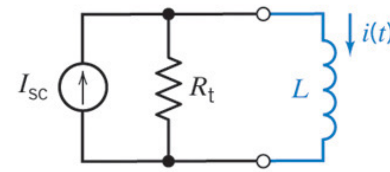
For 10 < t < 15



Norton Equivalent

$$i_{sc}(t) = \frac{v_i(t)}{R_t}$$

$$R_t = 2\ \Omega$$



Time constant

$$\tau = \frac{L}{R_t} = 4\ \text{s}$$

Solution at each half period

$$i_o(t) = i_o(\infty) + [\, i_o(t_0) - i_o(\infty)]\, e^{-(t-t_0)/\tau}$$

Solution for the third half period $\quad 10 < t < 15$

$$i_o(10) = -0.489\ \text{A}$$

$$i_o(\infty) = 1.5\ \text{A}$$

$$i_o(t) = 1.5 + [-0.489 - 1.5]\, e^{-(t-10)/4}$$

$$i_o(t) = 1.5 - 1.989\, e^{-(t-10)/4}\ \text{A}$$

$$i_o(15) = 1.5 - 1.989\, e^{-5/4} = 0.9301\ \text{A}$$

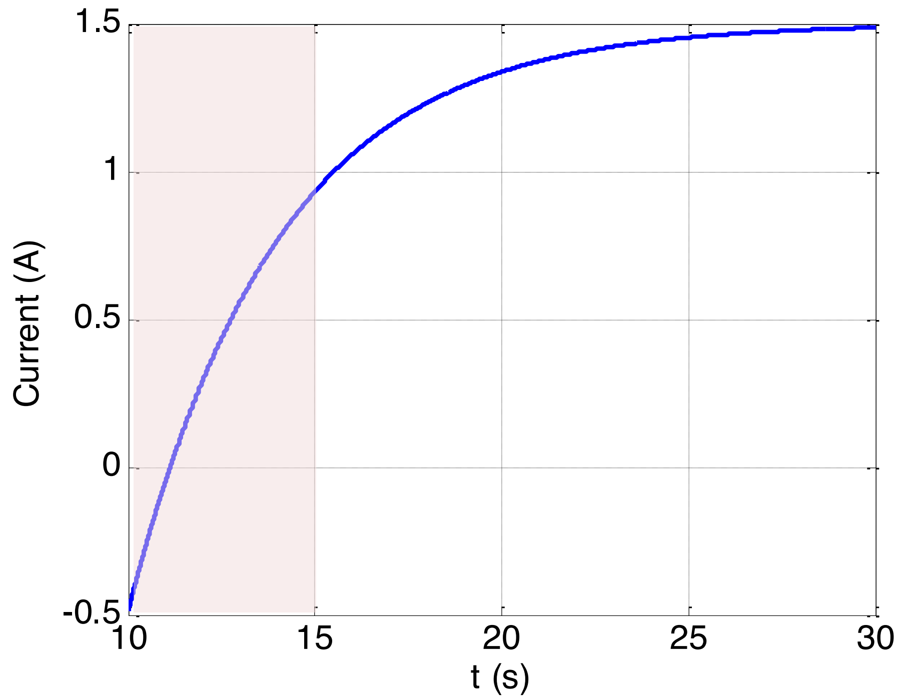
For 10 < t

$i_o(10) = -0.489\ \text{A}$

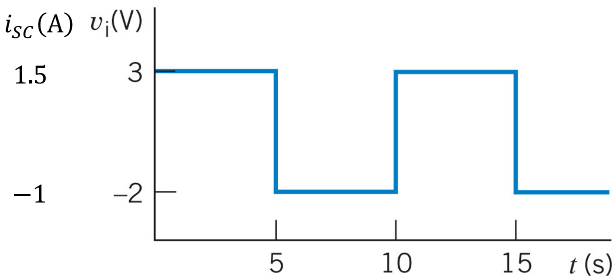
$i_o(\infty) = 1.5\ \text{A}$

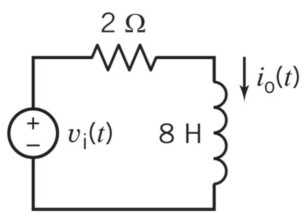
$$i_o(t) = 1.5 - 1.989\, e^{-(t-10)/4}\ \text{A}$$

$i_o(15) = 1.5 - 1.989\, e^{-5/4} = 0.9301\ \text{A}$



For 15 < t

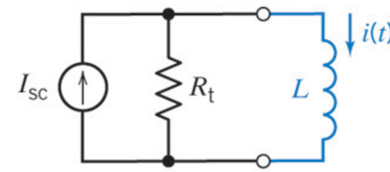




Norton Equivalent

$$i_{sc}(t) = \frac{v_i(t)}{R_t}$$

$$R_t = 2\ \Omega$$



Time constant

$$\tau = \frac{L}{R_t} = 4\ \mathrm{s}$$

Solution at each half period

$$i_o(t) = i_o(\infty) + [\, i_o(t_0) - i_o(\infty)]\, e^{-(t-t_0)/\tau}$$

Solution for the last cycle $15 < t$ (assuming switching stops after this)

$$i_o(15) = 0.9301\ \mathrm{A}$$

$$i_o(\infty) = -1\ \mathrm{A}$$

$$i_o(t) = -1 + [0.9301 + 1]\, e^{-(t-15)/4}$$

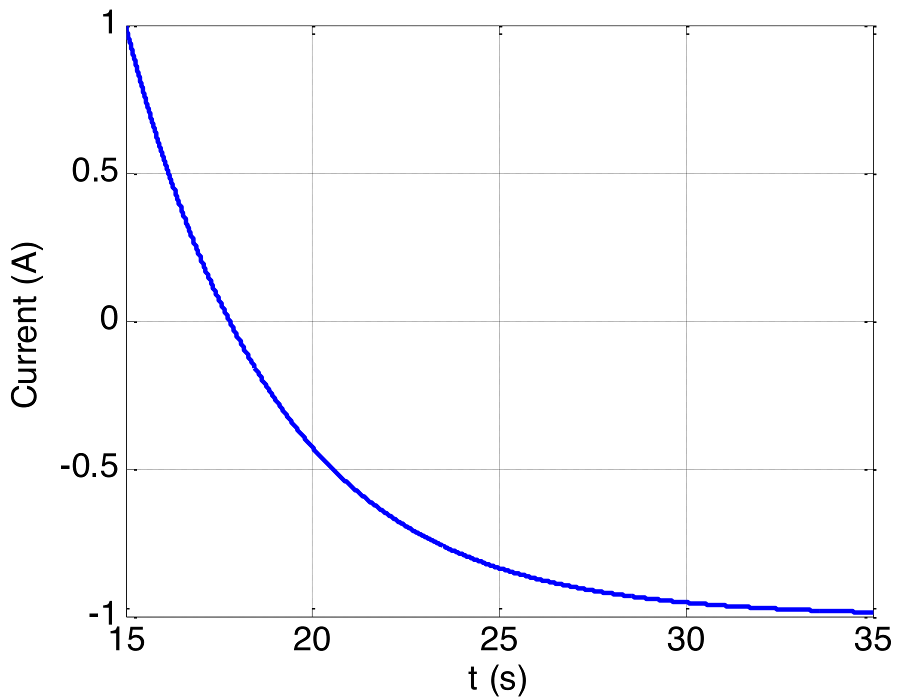
$$i_o(t) = -1 + 1.9301\, e^{-(t-15)/4}\ \mathrm{A}$$

For 15 < t

$$i_o(10) = 0.9301 \text{ A}$$

$$i_o(t) = -1 + 1.9301\, e^{-(t-15)/4} \text{ A}$$

$$i_o(\infty) = -1 \text{ A}$$

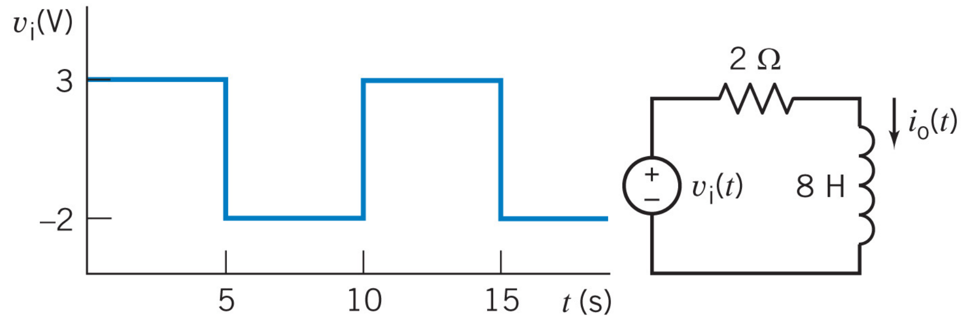


Problem SP 8-3

Solution summary

Find the current through the inductor

Assuming $v_i(t < 0) = -2$ V



We have the solution:

$0 < t < 5$ $\quad i_o(t) = 1.5 - 2.5\, e^{-t/4}$ A

$5 < t < 10$ $\quad i_o(t) = -1 + 1.7837\, e^{-(t-5)/4}$ A

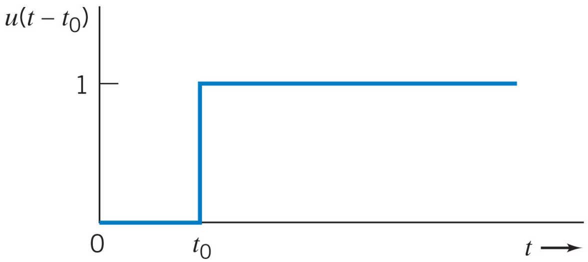
$10 < t < 15$ $\quad i_o(t) = 1.5 - 1.989\, e^{-(t-10)/4}$ A

$15 < t$ $\quad i_o(t) = -1 + 1.9301\, e^{-(t-15)/4}$ A

Unit Step Function

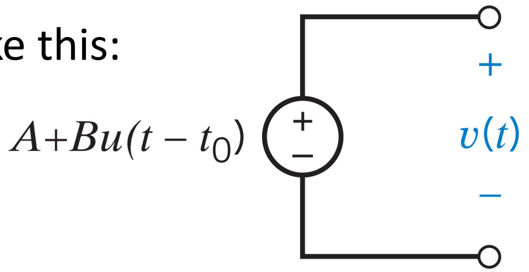
Unit step function

Creates a mathematical representation of switching



$$u(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$

If an independent voltage sources provides a voltage written like this:

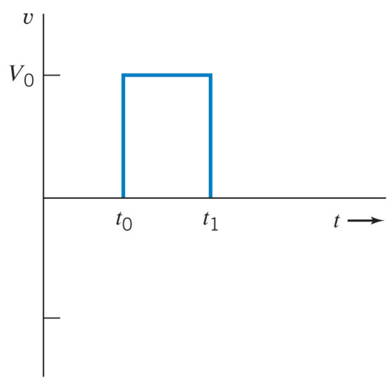


It means:

$$v(t) = A + B\,u(t-t_0) = \begin{cases} A & t < t_0 \\ A+B & t > t_0 \end{cases}$$

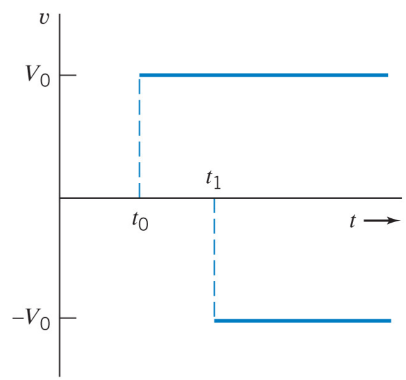
A rectangular pulse written mathematically

A rectangular pulse:



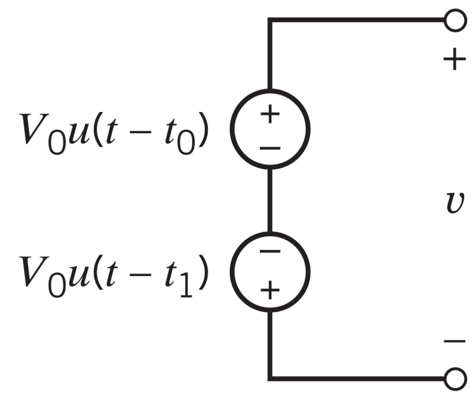
(a)

Is comprised of 2 step functions:



(b)

A circuit representation of a rectangular pulse:

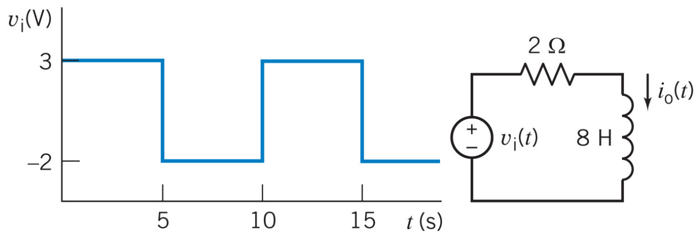


$$v(t) = V_0\,u(t-t_0) - V_0\,u(t-t_1) = \begin{cases} 0 & t < t_0 \\ V_0 & t_0 < t < t_1 \\ 0 & t_1 < t \end{cases}$$

Can we write the solution for SP8-3 using the unit step function?

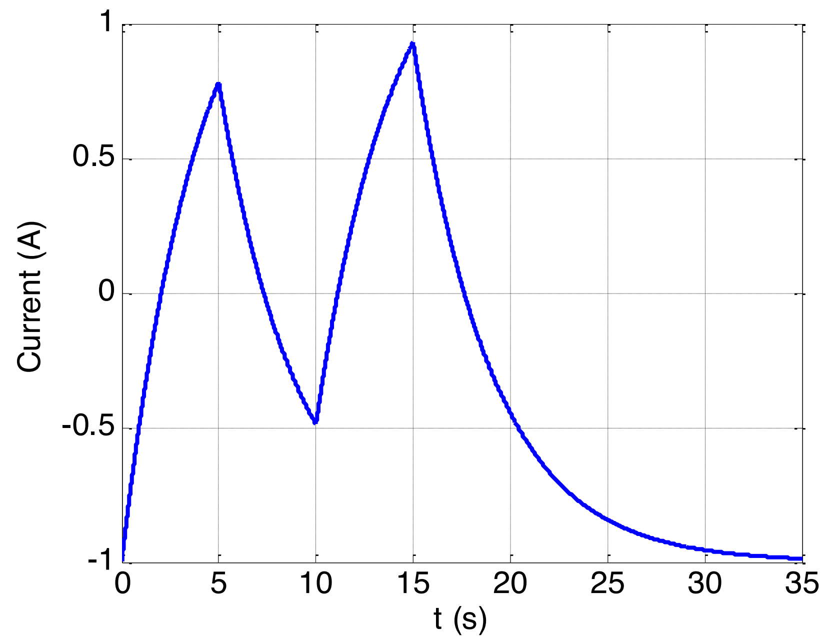
Problem SP 8-3

Solution Summary



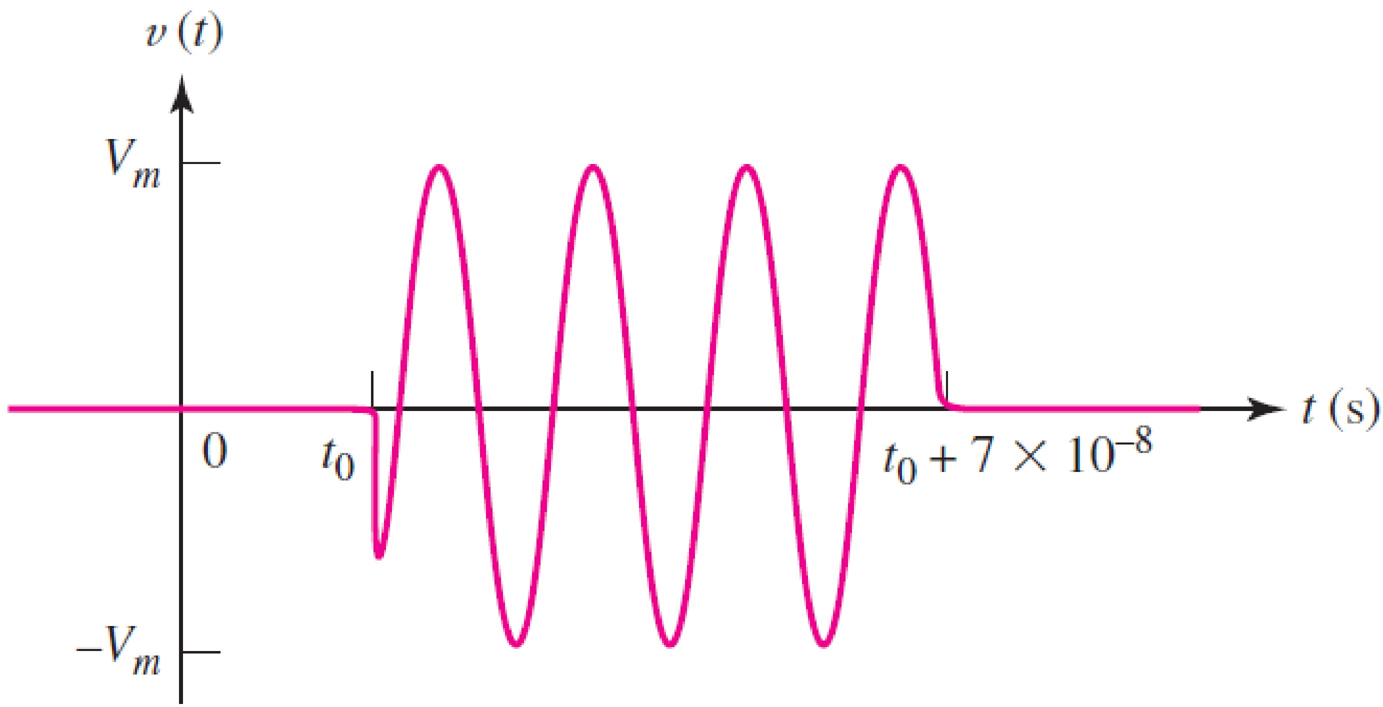
$$i_o(t) = [u(t-0) - u(t-5)]\left(1.5 - 2.5\, e^{-t/4}\right)$$
$$+ [u(t-5) - u(t-10)]\left(-1 + 1.7837\, e^{-(t-5)/4}\right)$$
$$+ [u(t-10) - u(t-15)]\left(1.5 - 1.989\, e^{-(t-10)/4}\right)$$
$$+ u(t-15)\left(-1 + 1.9301\, e^{-(t-15)/4}\right) \text{ A}$$

Useful for more mathematical approach, for example for plotting the solution in Matlab.



Another example

$$v(t) = V_m[u(t - t_0) - u(t - t_0 - 7 \times 10^{-8})] \sin(295 \times 10^6 t)$$



Important step functions

$$u(t) = \begin{cases} 0 \ for \ t < 0 \\ 1 \ for \ t > 0 \end{cases}$$

$$u(-t) = \begin{cases} 1 \ for \ t < 0 \\ 0 \ for \ t > 0 \end{cases}$$

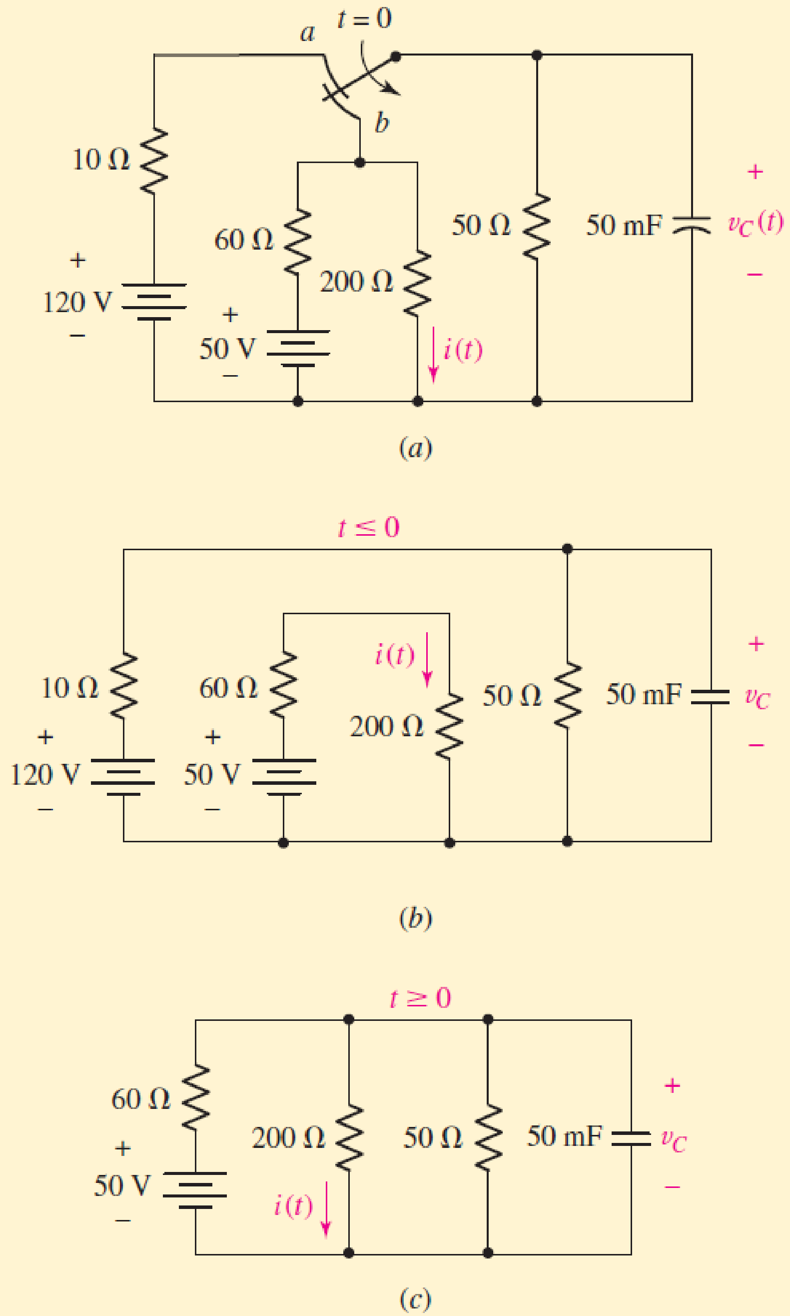
Driven RL circuits

1. With all independent sources zeroed out, simplify the circuit to determine R_{eq}, L_{eq}, and the time constant $\tau = L_{\text{eq}}/R_{\text{eq}}$.
2. Viewing L_{eq} as a short circuit, use dc analysis methods to find $i_L(0^-)$, the inductor current just prior to the discontinuity.
3. Again viewing L_{eq} as a short circuit, use dc analysis methods to find the forced response. This is the value approached by $f(t)$ as $t \to \infty$; we represent it by $f(\infty)$.
4. Write the total response as the sum of the forced and natural responses: $f(t) = f(\infty) + Ae^{-t/\tau}$.
5. Find $f(0^+)$ by using the condition that $i_L(0^+) = i_L(0^-)$. If desired, L_{eq} may be replaced by a current source $i_L(0^+)$ [an open circuit if $i_L(0^+) = 0$] for this calculation. With the exception of inductor currents (and capacitor voltages), other currents and voltages in the circuit may change abruptly.
6. $f(0^+) = f(\infty) + A$ and $f(t) = f(\infty) + [f(0^+) - f(\infty)]\, e^{-t/\tau}$, or total response $=$ final value $+$ (initial value $-$ final value) $e^{-t/\tau}$.

Driven RC circuits

1. With all independent sources zeroed out, simplify the circuit to determine R_{eq}, C_{eq}, and the time constant $\tau = R_{\text{eq}}C_{\text{eq}}$.
2. Viewing C_{eq} as an open circuit, use dc analysis methods to find $v_C(0^-)$, the capacitor voltage just prior to the discontinuity.
3. Again viewing C_{eq} as an open circuit, use dc analysis methods to find the forced response. This is the value approached by $f(t)$ as $t \to \infty$; we represent it by $f(\infty)$.
4. Write the total response as the sum of the forced and natural responses: $f(t) = f(\infty) + Ae^{-t/\tau}$.
5. Find $f(0^+)$ by using the condition that $v_C(0^+) = v_C(0^-)$. If desired, C_{eq} may be replaced by a voltage source $v_C(0^+)$ [a short circuit if $v_C(0^+) = 0$] for this calculation. With the exception of capacitor voltages (and inductor currents), other voltages and currents in the circuit may change abruptly.
6. $f(0^+) = f(\infty) + A$ and $f(t) = f(\infty) + [f(0^+) - f(\infty)]e^{-t/\tau}$, or total response = final value + (initial value − final value) $e^{-t/\tau}$.

Find the capacitor voltage $v_C(t)$ and the current $i(t)$ in the 200 Ω resistor of Fig. 8.42 for all time.



$$v_C(0) = \frac{50}{50+10}(120) = 100 \text{ V}$$

The corresponding circuit has been redrawn in Fig. 8.42*c* for convenience. The form of the natural response is obtained by replacing the 50 V source by a short circuit and evaluating the equivalent resistance to find the time constant (in other words, we are finding the Thévenin equivalent resistance "seen" by the capacitor):

$$R_{\text{eq}} = \frac{1}{\frac{1}{50} + \frac{1}{200} + \frac{1}{60}} = 24\ \Omega$$

Thus,

$$v_{Cn} = Ae^{-t/R_{\text{eq}}C} = Ae^{-t/1.2}$$

In order to evaluate the forced response with the switch at *b*, we wait until all the voltages and currents have stopped changing, thus treating the capacitor as an open circuit, and use voltage division once more:

$$v_{C_f} = 50\left(\frac{200 \parallel 50}{60 + 200 \parallel 50}\right)$$

$$= 50\left(\frac{(50)(200)/250}{60 + (50)(200)/250}\right) = 20\ \text{V}$$

Consequently,

$$v_C = 20 + Ae^{-t/1.2} \qquad \text{V}$$

and from the initial condition already obtained,

$$100 = 20 + A$$

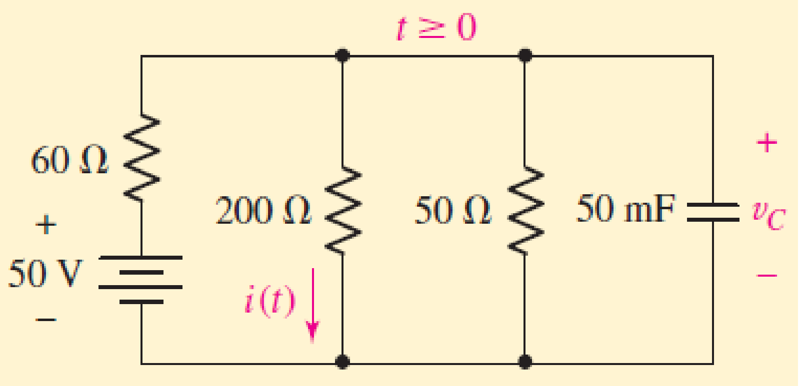
or

$$v_C = 20 + 80e^{-t/1.2} \qquad \text{V} \qquad t \geq 0$$

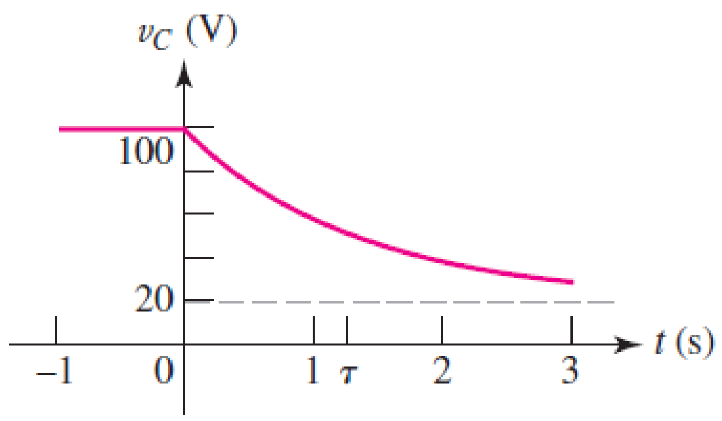
and

$$v_C = 100\ \text{V} \qquad t < 0$$

This response is sketched in Fig. 8.43*a*; again the natural response is seen to form a transition from the initial to the final response.



(*c*)



(*a*)

Next we attack $i(t)$. This response need not remain constant during the instant of switching. With the contact at a, it is evident that $i = 50/260 = 192.3$ milliamperes. When the switch moves to position b, the forced response for this current becomes

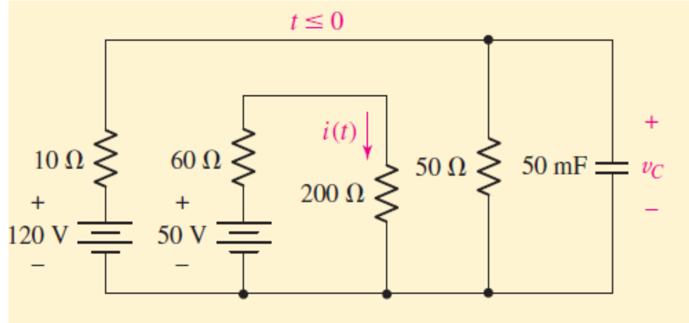
$$i_f = \frac{50}{60 + (50)(200)/(50 + 200)} \left(\frac{50}{50 + 200} \right) = 0.1 \text{ ampere}$$

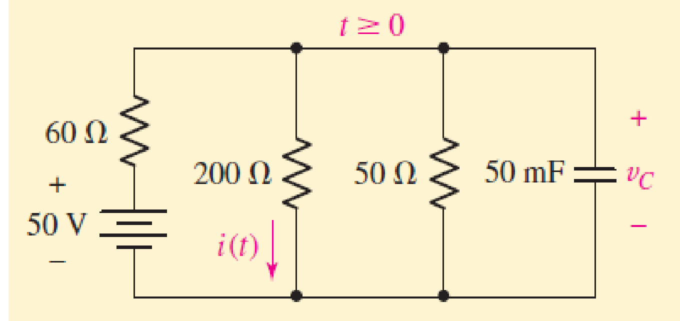
The form of the natural response is the same as that which we already determined for the capacitor voltage:

$$i_n = Ae^{-t/1.2}$$

Combining the forced and natural responses, we obtain

$$i = 0.1 + Ae^{-t/1.2} \qquad \text{amperes}$$





To evaluate A, we need to know $i(0^+)$. This is found by fixing our attention on the energy-storage element (the capacitor). The fact that v_C must remain 100 V during the switching interval is the governing condition which establishes the other currents and voltages at $t = 0^+$. Since $v_C(0^+) = 100$ V, and since the capacitor is in parallel with the 200 Ω resistor, we find $i(0^+) = 0.5$ ampere, $A = 0.4$ ampere, and thus

$$i(t) = 0.1923 \text{ ampere} \qquad t < 0$$

$$i(t) = 0.1 + 0.4e^{-t/1.2} \qquad \text{ampere} \qquad t > 0$$

or

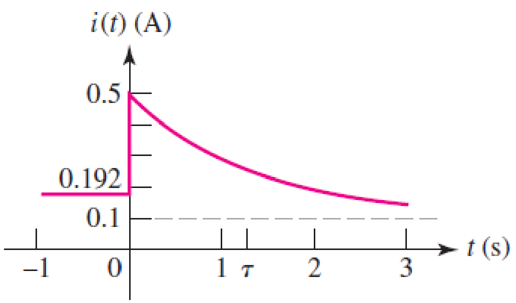
$$i(t) = 0.1923 + (-0.0923 + 0.4e^{-t/1.2})u(t) \qquad \text{amperes}$$

where the last expression is correct for all t.

The complete response for all t may also be written concisely by using $u(-t)$, which is unity for $t < 0$ and 0 for $t > 0$. Thus,

$$i(t) = 0.1923u(-t) + (0.1 + 0.4e^{-t/1.2})u(t) \qquad \text{amperes}$$

This response is sketched in Fig. 8.43*b*. Note that only four numbers are needed to write the functional form of the response for this single-energy-storage-element circuit, or to prepare the sketch: the constant value prior to switching (0.1923 ampere), the instantaneous value just after switching (0.5 ampere), the constant forced response (0.1 ampere), and the time constant (1.2 s). The appropriate negative exponential function is then easily written or drawn.



(*b*)