Coupled SHM

Prof Dave Spence

Driven or forced SHM

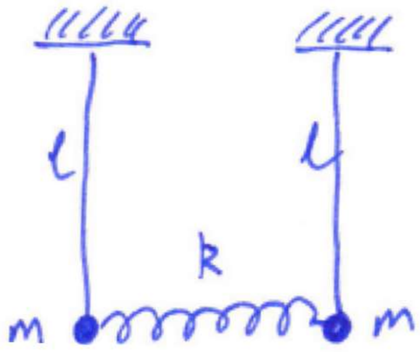
- We've looked at
  - undamped motion
  - (decaying) damped motion
  - forced motion
- Now we look at coupled motion
  - Multiple oscillators, 'coupled' in some way so they may exchange energy
- Identify *normal modes*
  - Simple patterns of vibration
  - General solution: superposition of normal modes

Coupled Oscillators

Coupled pendulums (minus the bagpipes)

Prototype system

- Consider two pendula coupled by a spring
  - define displacement for each pendulum
  - calculate forces given displacement
  - find equation of motion for each pendulum

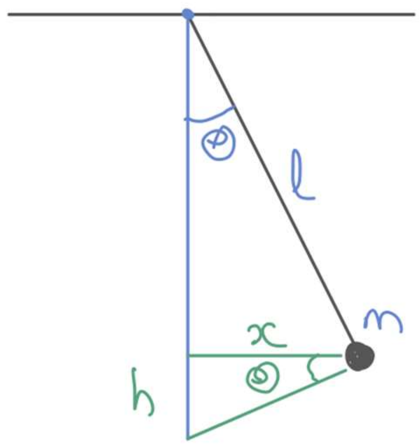


- New element: spring force depends on *both* displacements
  - motion of each pendulum depends on the other
  - equation of motion of each depends on position of both
- For small angles, we can neglect the vertical component of the motion……

Prototype system: two pendula coupled by a spring

- Use horizontal displacement rather than angle

For small angles, pendulum motion is almost horizontal…



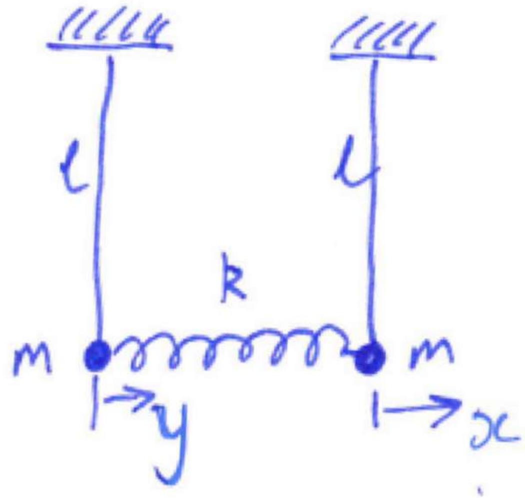
$$x = l\sin\theta \doteqdot l\theta$$

$$h = x\tan\theta \doteqdot l\theta^2$$

$$\therefore h \ll x$$

… with restoring force $-(mg/l)x$

$$\Longrightarrow \ddot{x} + (g/l)x = 0$$



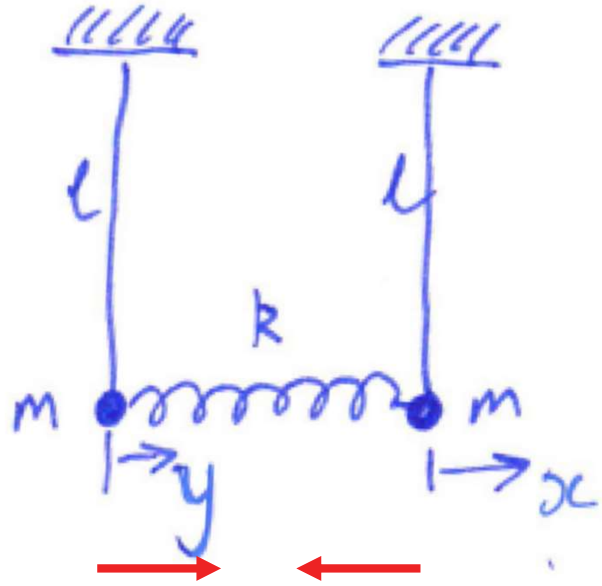
Define horizontal displacements y and x

Extension of spring is $x–y$

$$F_S = k(x - y)$$

Prototype system

- calculate forces given displacement
- find equation of motion for each pendulum



$$m\ddot{x} = -(mg/l)x - k(x - y)$$

$$m\ddot{y} = -(mg/l)y + k(x - y)$$

Coupled equations

Spring extension is $(x - y)$
Gives +ve force on y mass
Gives negative force on x mass

Decoupling the equations

$$m\ddot{x} = -(mg/l)x - k(x - y) \qquad \text{(A)}$$

$$m\ddot{y} = -(mg/l)y + k(x - y) \qquad \text{(B)}$$

$$(A) + (B): \quad m(\ddot{x} + \ddot{y}) = -(mg/l)(x + y)$$

$$\therefore \quad \ddot{X} = -(g/l)\,X \quad \text{where} \quad X = x + y$$

$$(A) - (B): \quad m(\ddot{x} - \ddot{y}) = -(mg/l)(x - y) - 2k(x - y)$$

$$\therefore \quad \ddot{Y} = -(g/l + 2k/m)\,Y \quad \text{where} \quad Y = x - y$$

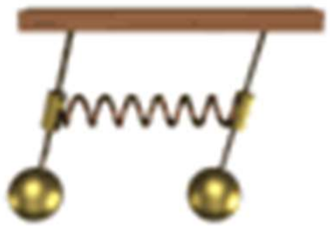
- New variables X and Y undergo *uncoupled SHM*!

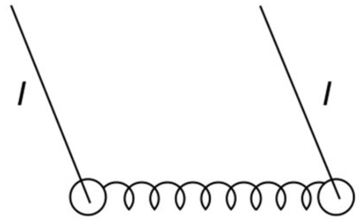
Writing $X = x + y$: $\ddot{X} = -(g/l)\,X$

Writing $Y = x - y$: $\ddot{Y} = -(g/l + 2k/m)\,Y$

These are the *normal modes.*

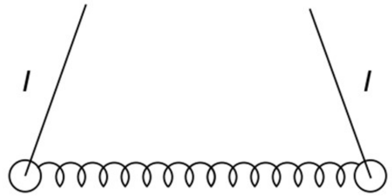
- If $Y, \dot{Y} = 0$, then $x = y$, we have a pure oscillation of the $X = x + y$ normal mode





In-phase X mode

- If $X, \dot{X} = 0$, then $x = -y$, and we have a pure oscillation of the $Y = x - y$ normal mode



Out-of-phase Y mode

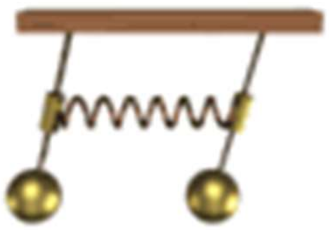
$$\ddot{X} = -(g/l)X \qquad \ddot{Y} = -(g/l + 2k/m)Y$$

$$\omega_0^X = (g/l)^{1/2} \qquad \omega_0^Y = (g/l + 2k/m)^{1/2}$$

$$X(t) = A^X \cos(\omega_0^X t + \phi^X) \qquad Y(t) = A^Y \cos(\omega_0^Y t + \phi^Y)$$

$$\dot{X}(t) = -A^X \omega_0^X \sin(\omega_0^X t + \phi^X) \qquad \dot{Y}(t) = -A^Y \omega_0^Y \sin(\omega_0^Y t + \phi^Y)$$

- Different modes have different frequency
  - The lowest frequency mode has oscillators in phase
  - The highest frequency mode has oscillators out of phase



Independence of normal modes

$$\ddot{X} = -(g/l)\,X$$

$$\ddot{Y} = -(g/l + 2k/m)\,Y$$

- Note that if $Y, \dot{Y} = 0$ at any instant then this will remain the case forever

- Equally, if $X, \dot{X} = 0$, then this will remain the case forever
- Further, the equation for the X oscillation do not depend on Y or $\dot{Y}$, and vice-versa

The normal modes are entirely independent

They do not exchange energy

- Total energy is $(aX^2 + b\dot{X}^2) + (cY^2 + d\dot{Y}^2)$

General solution

- A pure X or Y oscillation continues forever (if no damping!)

- General solution: superposition of X and Y normal modes
  - magnitude of the X and Y contributions doesn't change with time
  - beating occurs as the normal modes have different frequencies!
  - initial conditions (four!) determine the amplitudes and phases of the X and Y contributions
    - express the initial conditions in the normal coordinates (X and Y here)
    - find the correct superposition of the normal modes
    - retrieve the corresponding solution for the physical variables

Example: energy transfer between coupled pendula

Find the behaviour of two coupled pendula if $x(0) = 2a$ $y(0) = 0$

$\dot{x}(0) = 0$ $\dot{y}(0) = 0$

Find the initial conditions for the normal modes $X = x + y$ and $Y = x - y$

$X(0) = 2a$ $Y(0) = 2a$

$\dot{X}(0) = 0$ $\dot{Y}(0) = 0$

Find the particular solution for the normal modes

$$X(t) = A^X \cos(\omega_0^X t + \phi^X) \implies X(t) = 2a\cos(\omega_0^X t)$$

$$Y(t) = A^Y \cos(\omega_0^Y t + \phi^Y) \implies Y(t) = 2a\cos(\omega_0^Y t)$$

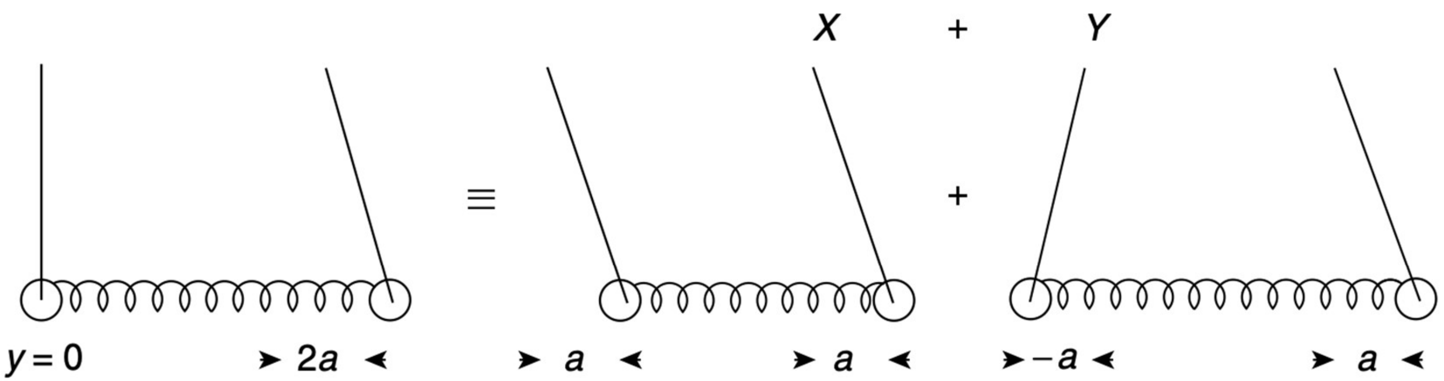


Figure 4.3 The displacement of one pendulum by an amount $2a$ is shown as the combination of the two normal coordinates $X + Y$

Example (cont)

Translate back to the physical displacements x and y

$$x = (X+Y)/2$$

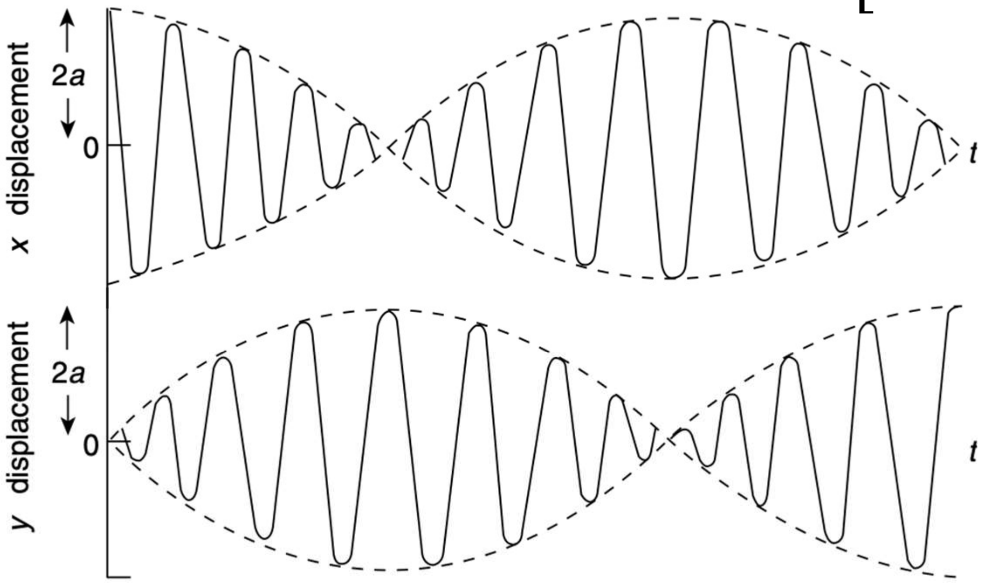
$$y = (X-Y)/2$$

$$X(t) = 2a\cos(\omega_0^X t)$$

$$Y(t) = 2a\cos(\omega_0^Y t)$$

$$x(t) = a\cos(\omega_0^X t) + a\cos(\omega_0^Y t) = 2a\cos\left[\frac{1}{2}(\omega_0^X + \omega_0^Y)t\right]\cos\left[\frac{1}{2}(\omega_0^X - \omega_0^Y)t\right]$$

$$y(t) = a\cos(\omega_0^X t) - a\cos(\omega_0^Y t) = -2a\sin\left[\frac{1}{2}(\omega_0^X + \omega_0^Y)t\right]\sin\left[\frac{1}{2}(\omega_0^X - \omega_0^Y)t\right]$$



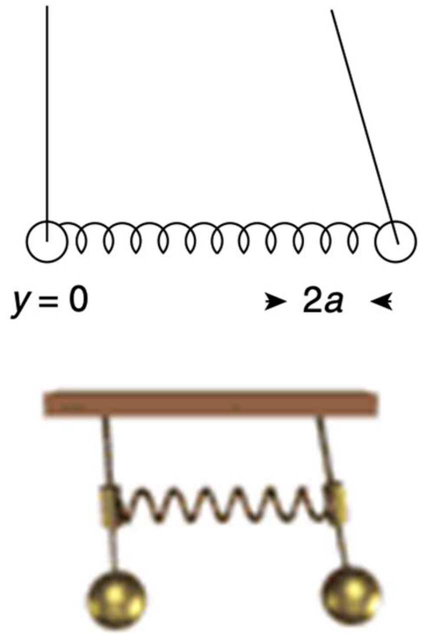


Figure 4.4 Behaviour with time of individual pendulums, showing complete energy exchange between the pendulums as x decreases from $2a$ to zero whilst y grows from zero to $2a$

Properties of normal modes

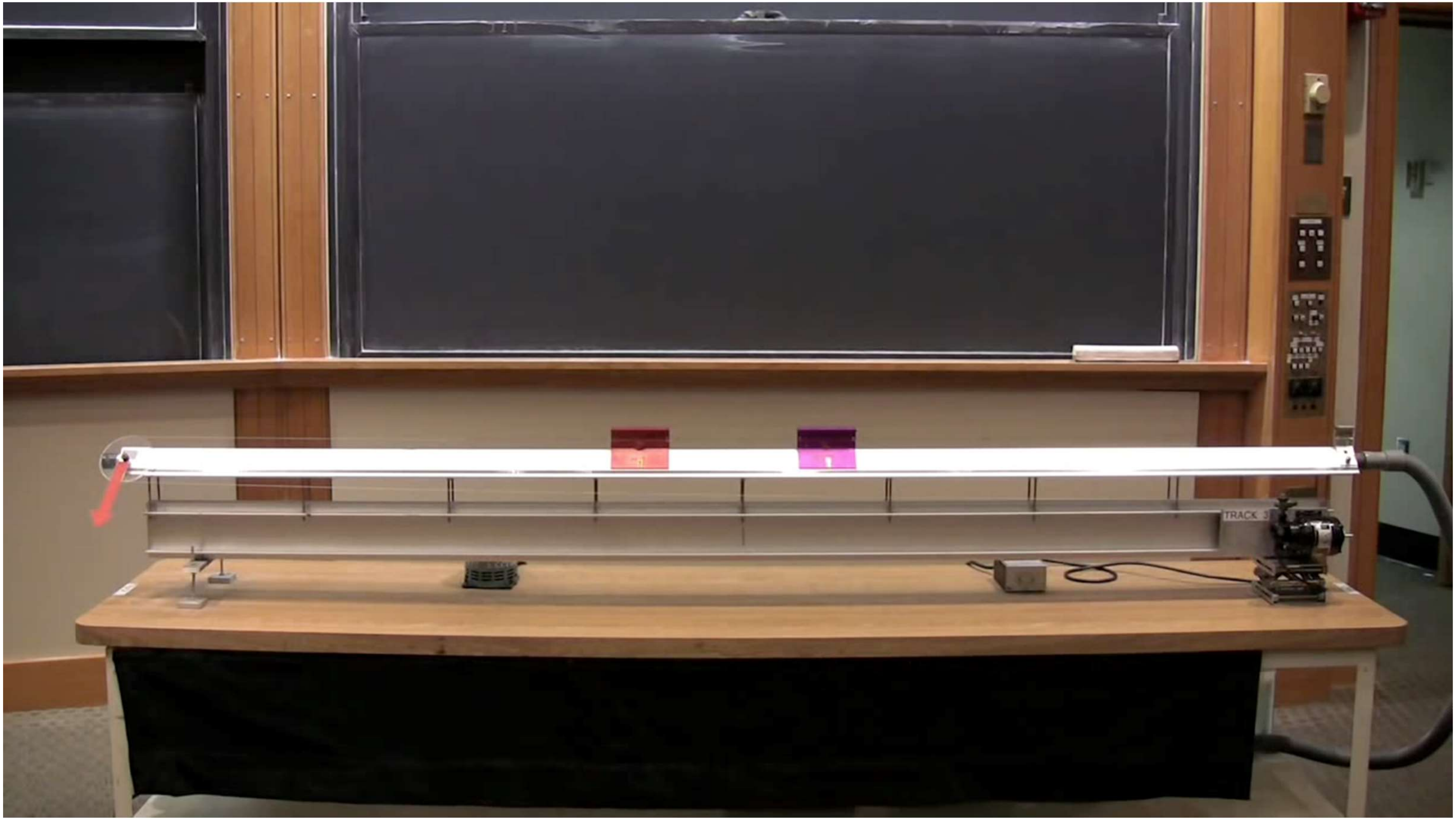
- For a normal mode
  - Oscillation is independent of other normal modes coordinates
  - Energy is not transferred to other normal modes
  - All oscillators have a fixed phase relation
  - All oscillators have fixed amplitude ratio
  - All oscillators share the same frequency

- Specifying a normal mode
  - Normal mode *dependent variables* X, Y
  - Normal *coordinates* $X, \dot{X}, Y, \dot{Y}$
  - Normal (mode) frequency ω_0^X, ω_0^Y

How many normal modes?

- The number of normal modes reflects the number of oscillators, n
  - n normal modes, each with
    - an associated normal variable X, Y, Z…. that is a linear combination of displacements x,y,z….
    - a characteristic frequency
    - 2 initial conditions

Coupled air carts - normal modes

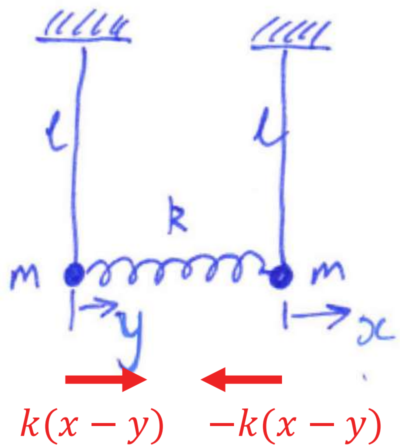


General method to find normal modes: the coupled pendulum

- Difficult/impossible to decouple equations "by eye"
- Instead, seek solutions that oscillate with frequency ω
- For the coupled pendulum, $x = A\,e^{i\omega t}$ $\quad y = B\,e^{i\omega t}$

$$\ddot{x} = -(g/l)x - (k/m)(x-y)$$

$$\Longrightarrow \quad A\omega^2 = (g/l + k/m)A - (k/m)B$$



$$\ddot{y} = -(g/l)y + (k/m)(x-y)$$

$$\Longrightarrow \quad B\omega^2 = (g/l + k/m)B - (k/m)A$$

Solve for ω to find normal frequencies

Finding ω

$$A\omega^2 = (g/l + k/m)A - (k/m)B$$

$$B\omega^2 = (g/l + k/m)B - (k/m)A$$

- Two equations, linear in A and B
  - system determines the coefficients on the right hand side
    - g, l, m, k — gravity, pendulum length and mass, spring
  - but there are three unknowns - A, B, ω
  - random choice of $\omega \implies$ only solution is $A = B = 0$!!
  - certain choices of ω: equations are *not* independent
    - i.e. 2nd equation = multiple of the first
    - either equation $\implies$ proportionality between A and B
    - g, l, m, k determine form and frequency of oscillations, but *not* the overall amplitude

Finding ω

$$A\omega^2 = (g/l + k/m)A - (k/m)B \qquad B\omega^2 = (g/l + k/m)B - (k/m)A$$

$$(\omega^2 - g/l - k/m)A/B = -k/m \qquad (\omega^2 - g/l - k/m) = -(k/m)A/B$$

$$\frac{A}{B} = \frac{-k/m}{\omega^2 - g/l - k/m} \qquad \frac{\omega^2 - g/l - k/m}{-k/m} = \frac{A}{B}$$

$$\therefore (\omega^2 - g/l - k/m)^2 = (k/m)^2$$

$$\omega^2 - g/l - k/m = \pm k/m$$

$$\omega^2 = g/l, g/l + 2k/m$$

$$\omega = \sqrt{g/l}, \sqrt{g/l + 2k/m}$$

(require $\omega \geq 0$)

Solving with matrices

$$A\omega^2 = (g/l + k/m)A - (k/m)B$$

$$B\omega^2 = (g/l + k/m)B - (k/m)A$$

Rewrite in matrix form as

$$\begin{bmatrix} g/l + k/m & -k/m \\ -k/m & g/l + k/m \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

(The general form $\boldsymbol{M}\,\mathbf{v} = \lambda\,\mathbf{v}$ is called an *eigenvalue equation.*)

$$\begin{bmatrix} g/l + k/m - \omega^2 & -k/m \\ -k/m & g/l + k/m - \omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

"Reminder": Matrices

$$\underbrace{\begin{pmatrix} & \end{pmatrix}}_{M}\underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{v} = 0$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Delta = ad - bc$$

$$M^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$MM^{-1} = M^{-1}M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Mv = 0$$

$$\underbrace{M^{-1}M}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} v = M^{-1}0 = 0$$

$$\therefore \underline{v = 0}$$

General solution with matrices…

$$\begin{bmatrix} g/l + k/m - \omega^2 & -k/m \\ -k/m & g/l + k/m - \omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If determinant is nonzero, matrix has an inverse and A=B=0, so instead we want

$$\begin{vmatrix} g/l + k/m - \omega^2 & -k/m \\ -k/m & g/l + k/m - \omega^2 \end{vmatrix} = 0$$

$$(g/l + k/m \ - \omega^2)^2 - (k/m)^2 = 0$$

$$(g/l + k/m - \omega^2) \quad = \pm(k/m)$$

$$\Rightarrow \ \omega^2 = g/l, \mathrm{or}\ (g/l + 2k/m)$$

General solution with matrices…

$$\omega^2 = g/l, \text{or } (g/l + 2k/m)$$

…substituting ω^2 back

$$A\omega^2 = (g/l + k/m)A - (k/m)B$$
$$B\omega^2 = (g/l + k/m)B - (k/m)A$$

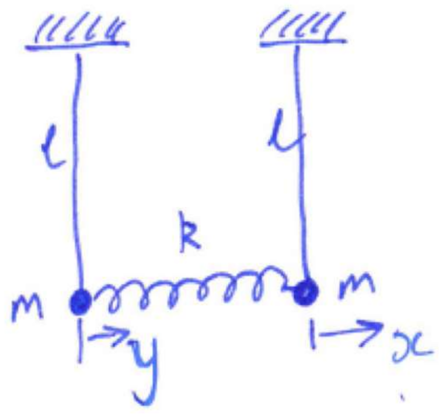
$$A\omega^2 = (g/l + k/m)A - k/m.B$$

$$A = B \text{ for } \omega^2 = g/l, \quad \text{and } A = -B \text{ for } \omega^2 = (g/l + 2k/m)$$

Eigenvectors describe relationship between normal and physical variables

Summary: Finding normal modes with matrices

$$m\ddot{x} = -(mg/l)x - k(x-y)$$
$$m\ddot{y} = -(mg/l)y + k(x-y)$$



- Seek solutions that oscillate with frequency ω

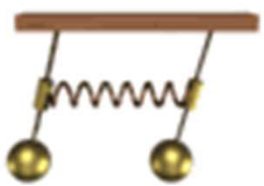
$$x = A\,e^{i\omega t} \qquad y = B\,e^{i\omega t}$$

$$\begin{bmatrix} g/l + k/m & -k/m \\ -k/m & g/l + k/m \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

Eigenvalue equation

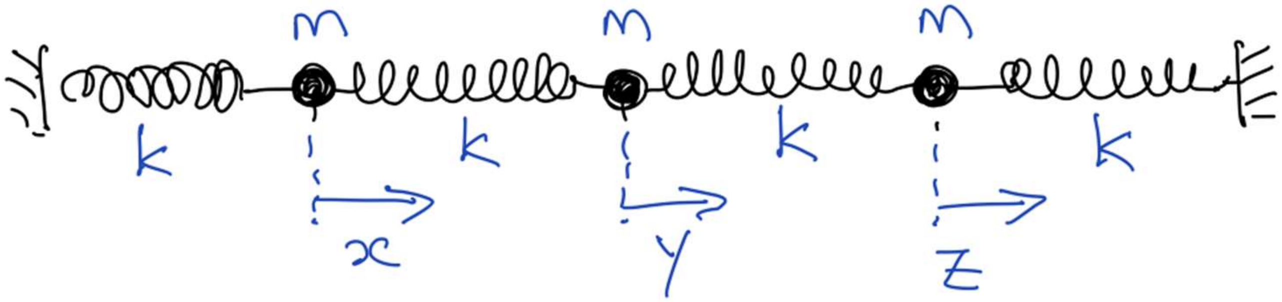
eigenvector

eigenvalue



$$\omega^2 = g/l \qquad\qquad g/l + 2k/m$$
$$A = B \qquad\qquad A = -B$$

Example — two walls, three masses and four springs

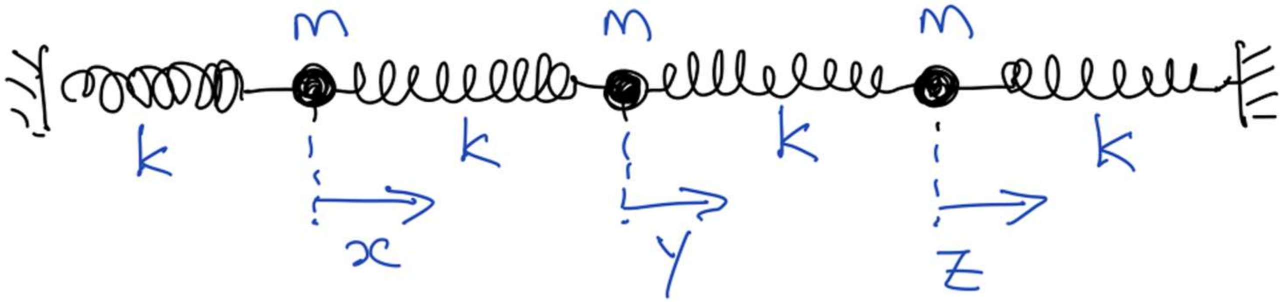


$$m\ddot{x} = -kx + k(y-x) = -2kx + ky$$
$$m\ddot{y} = -k(y-x) + k(z-y) = kx - 2ky + kz$$
$$m\ddot{z} = -k(z-y) - kz = ky - 2kz$$

Now assume $x = Ae^{i\omega t}$, $y = Be^{i\omega t}$, $z = Ce^{i\omega t}$

Then $\ddot{x} = -\omega^2 x$, $\ddot{y} = -\omega^2 y$, $\ddot{z} = -\omega^2 z$

Example — two walls, three masses and four springs



$$m\ddot{x} = -2kx + ky \qquad \ddot{x} = -\omega^2 x \qquad x = Ae^{i\omega t}$$

$$m\ddot{y} = kx - 2ky + kz \qquad \ddot{y} = -\omega^2 y \qquad y = Be^{i\omega t}$$

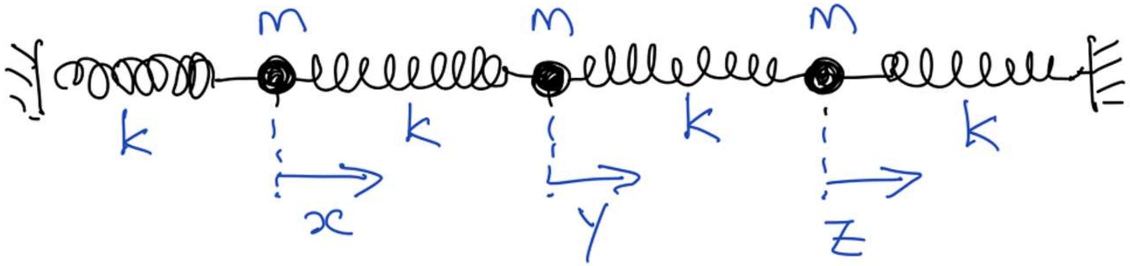
$$m\ddot{z} = ky - 2kz \qquad \ddot{z} = -\omega^2 z \qquad z = Ce^{i\omega t}$$

$$-\omega^2 \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\text{Put } \omega^2 = \lambda (k/m) \implies \begin{pmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

Example — two walls, three masses and four springs

$$m\ddot{x} = -2kx + ky$$
$$m\ddot{y} = kx - 2ky + kz$$
$$m\ddot{z} = ky - 2kz$$



$$x = Ae^{i\omega t}$$
$$y = Be^{i\omega t} \qquad \lambda = \omega^2/(k/m)$$
$$z = Ce^{i\omega t}$$

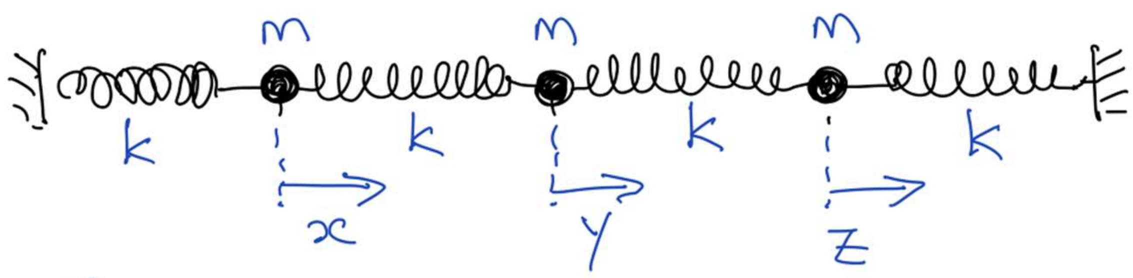
$$\begin{pmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

$$0 = \begin{vmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{vmatrix} = (\lambda-2)\left[(\lambda-2)^2-1\right]-(\lambda-2) = (\lambda-2)\left[(\lambda-2)^2-2\right]$$

$$\therefore \lambda = 2,\ 2\pm\sqrt{2} \qquad \text{i.e. } \omega^2 = 2k/m,\ (2\pm\sqrt{2})k/m$$

Example — two walls, three masses and four springs

$$m\ddot{x} = -2kx + ky$$
$$m\ddot{y} = kx - 2ky + kz$$
$$m\ddot{z} = ky - 2kz$$



$$x = Ae^{i\omega t}$$
$$y = Be^{i\omega t} \qquad \lambda = \omega^2/(k/m)$$
$$z = Ce^{i\omega t}$$

$$\begin{pmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

$$\therefore \lambda = 2,\ 2\pm\sqrt{2}$$

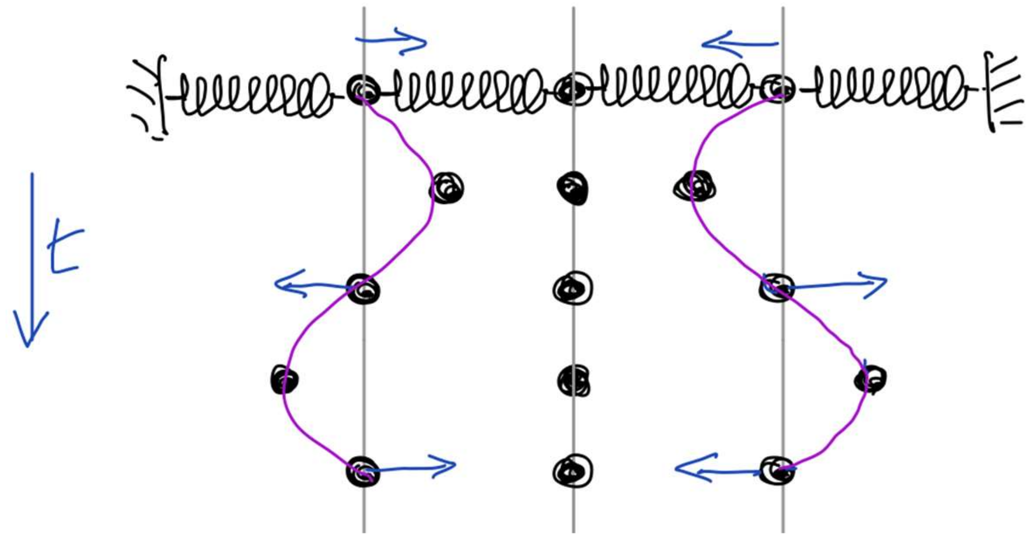
i.e. $\omega^2 = 2k/m,\ (2\pm\sqrt{2})k/m$

$\underline{\lambda = 2}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

$$\omega^2 = 2k/m$$

$$B = 0 \qquad C = -A$$



Example — two walls, three masses and four springs

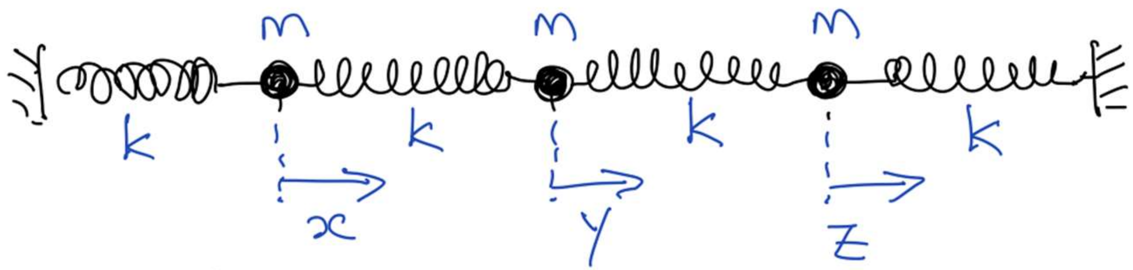
$$m\ddot{x} = -2kx + ky$$
$$m\ddot{y} = kx - 2ky + kz$$
$$m\ddot{z} = ky - 2kz$$

$$x = Ae^{i\omega t}$$
$$y = Be^{i\omega t} \qquad \lambda = \omega^2/(k/m)$$
$$z = Ce^{i\omega t}$$

$$\begin{pmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\lambda = 2,\ 2\pm\sqrt{2}$$

$$\omega^2 = 2\omega_0^2,\ (2\pm\sqrt{2})\omega_0^2$$

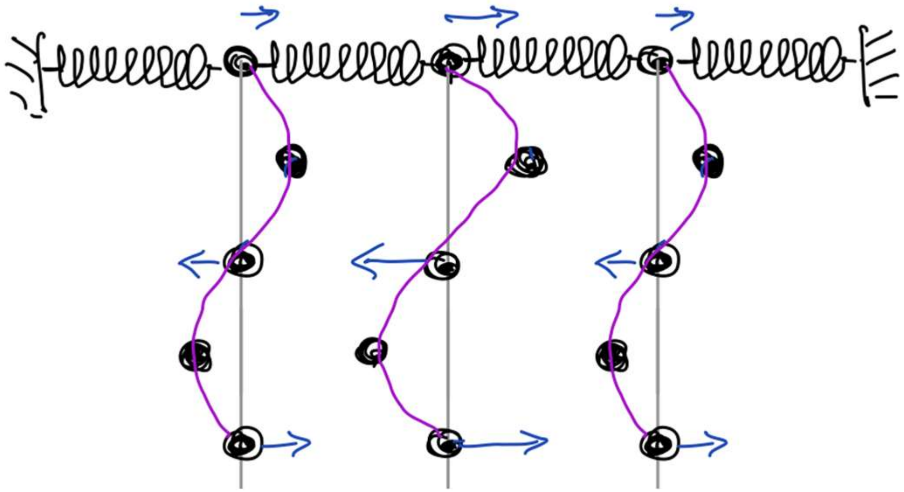


$$\underline{\lambda = 2-\sqrt{2}}$$

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

$$\omega^2 = (2-\sqrt{2})\,k/m$$

$$B = \sqrt{2}\,A$$
$$C = A$$



Example — two walls, three masses and four springs

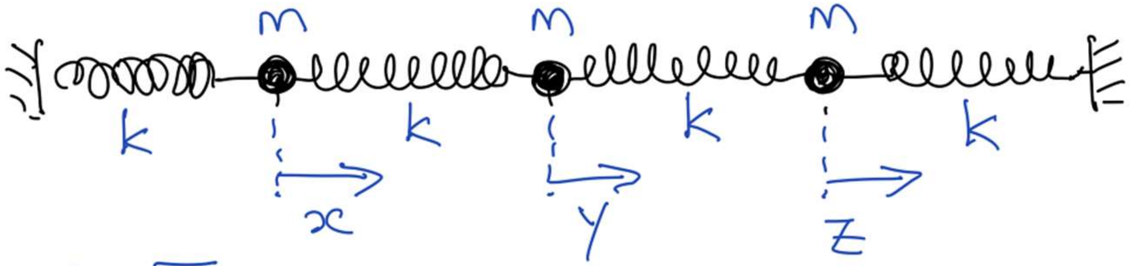
$$m\ddot{x} = -2kx + ky$$
$$m\ddot{y} = kx - 2ky + kz$$
$$m\ddot{z} = ky - 2kz$$

$$x = Ae^{i\omega t}$$
$$y = Be^{i\omega t} \qquad \lambda = \omega^2/(k/m)$$
$$z = Ce^{i\omega t}$$

$$\begin{pmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\lambda = 2,\ 2\pm\sqrt{2}$$

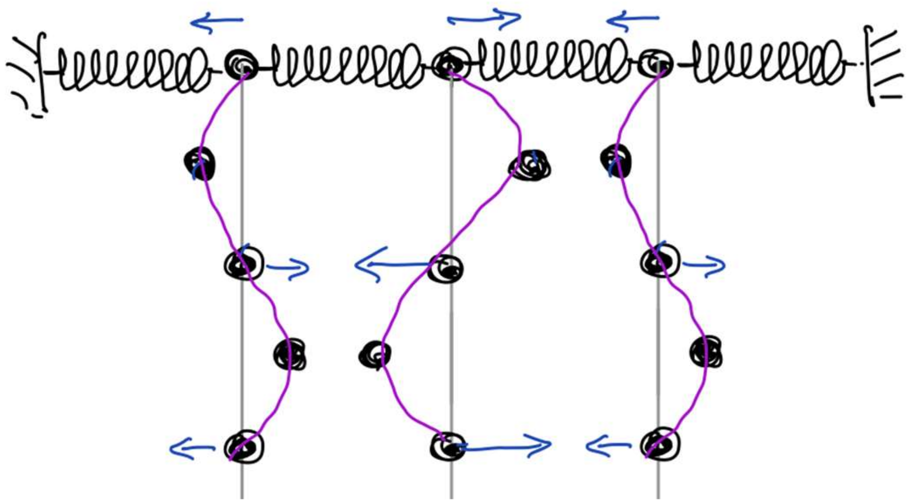
$$\omega^2 = 2\omega_0^2,\ (2\pm\sqrt{2})\omega_0^2$$



$$\underline{\underline{\lambda = 2+\sqrt{2}}}$$

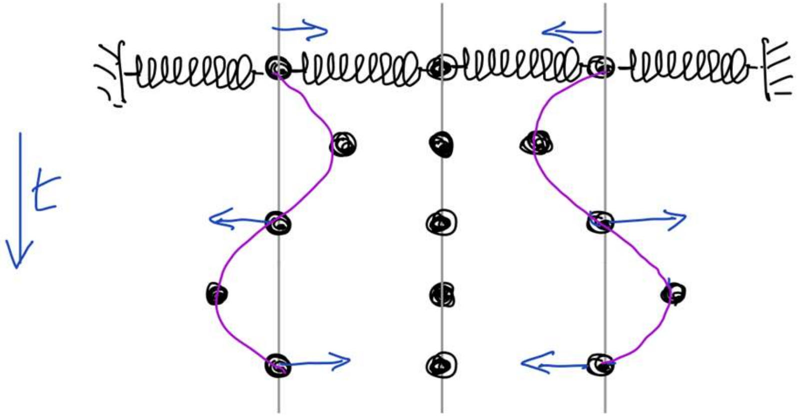
$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

$$\omega^2 = (2+\sqrt{2})\,k/m$$
$$B = -\sqrt{2}A$$
$$C = A$$

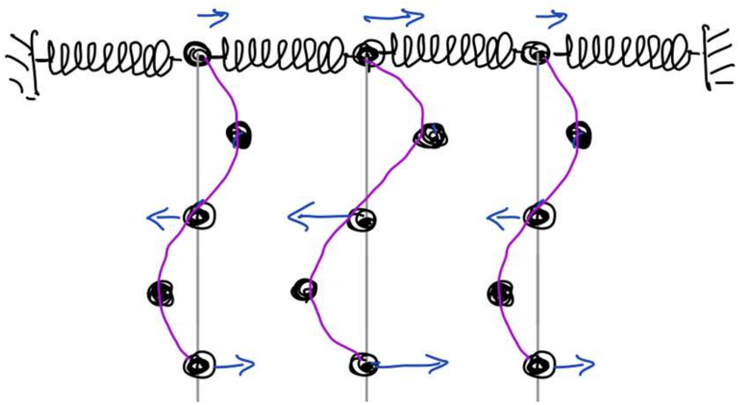


Example — two walls, three masses and four springs

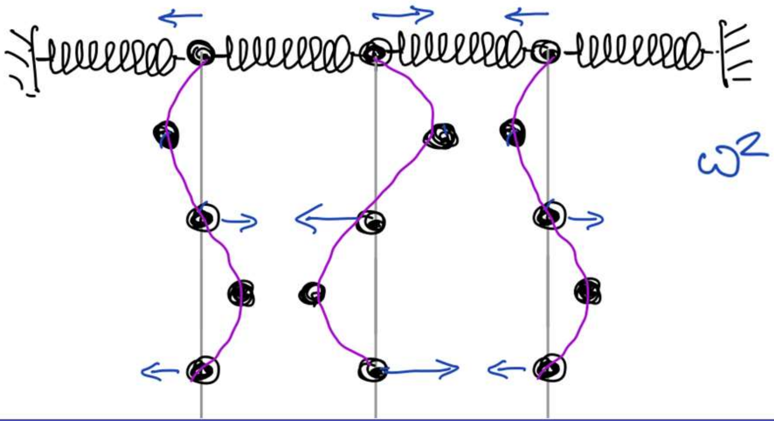
$$x = Ae^{i\omega t}$$
$$y = Be^{i\omega t}$$
$$z = Ce^{i\omega t}$$



$$\omega^2 = 2k/m$$
$$B = 0$$
$$C = -A$$



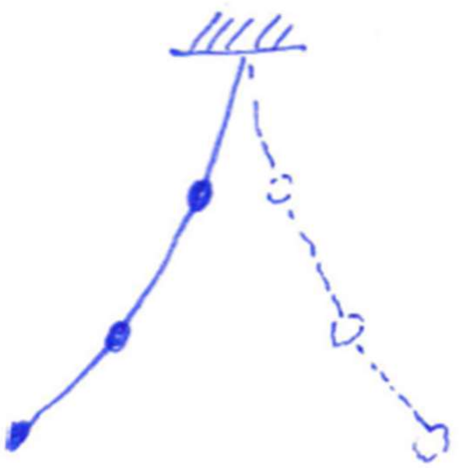
$$\omega^2 = (2-\sqrt{2})\,k/m$$
$$B = \sqrt{2}\,A$$
$$C = A$$

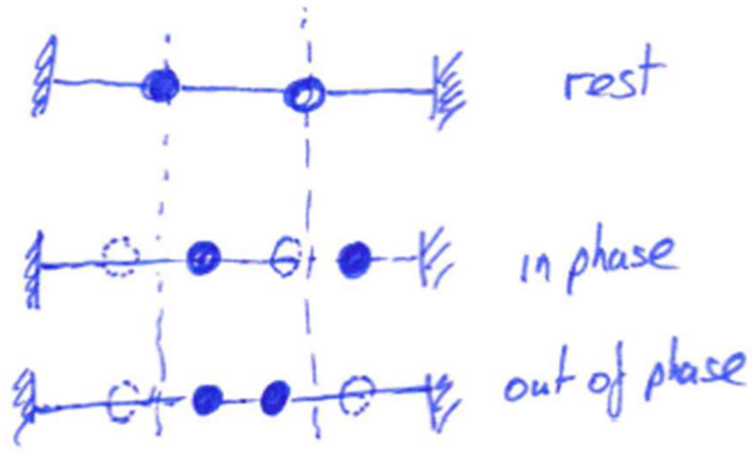


$$\omega^2 = (2+\sqrt{2})\,k/m$$
$$B = -\sqrt{2}\,A$$
$$C = A$$

- Number of modes = number of oscillators
- Lowest frequency mode: all masses are in-phase
- Highest frequency mode: adjacent masses are out-of-phase

Examples – more!





- Number of modes =
  number of oscillators
- Lowest frequency mode:
  all masses are in-phase
- Highest frequency mode: adjacent
  masses are out-of-phase