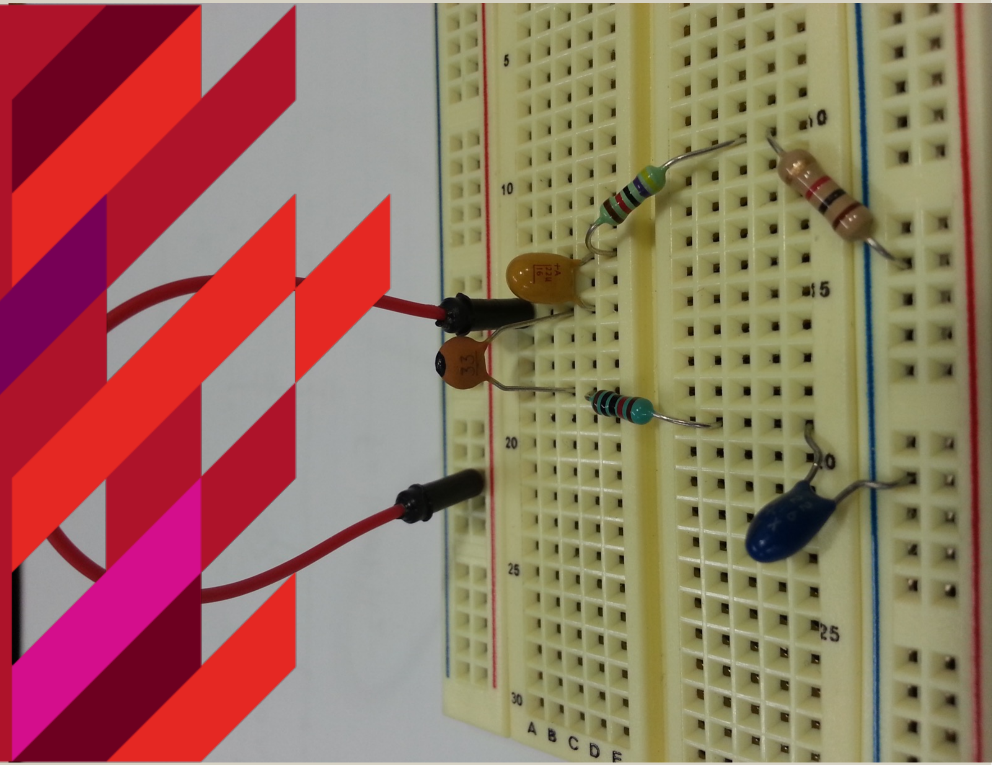
ELEC2070 Circuits and Devices

Week 2: Power transfer + Energy storage elements (capacitors + inductors)

Stuart Jackson



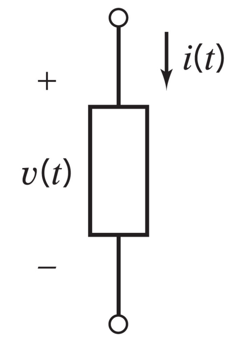
Maximum power transfer

Instantaneous Power

The definition is correct for any circuit element

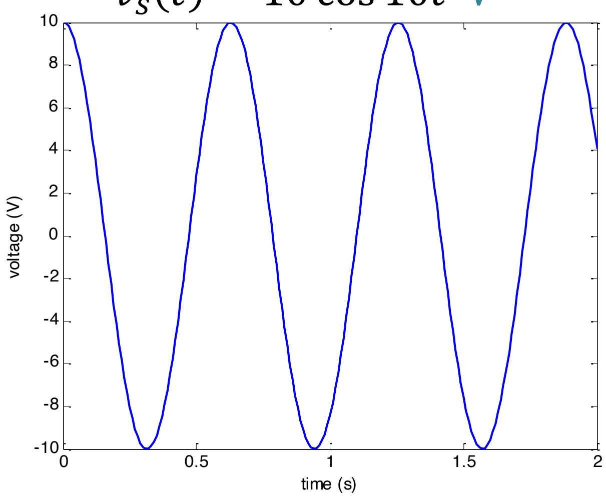
Instantaneous power

$$p(t) = v(t) \cdot i(t) \quad \text{W}$$

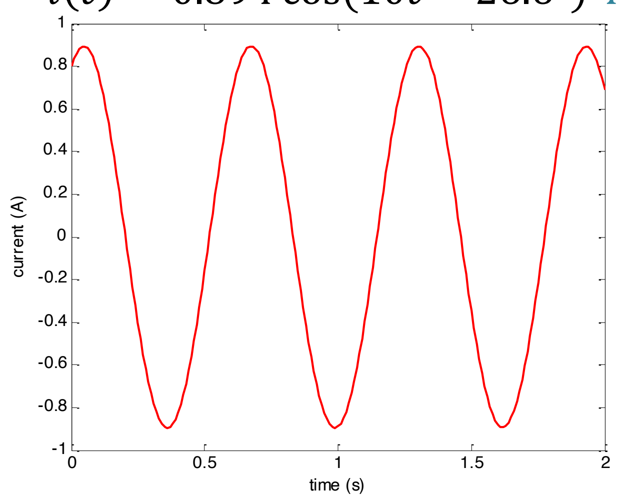


Example 10.6-3 voltage and current

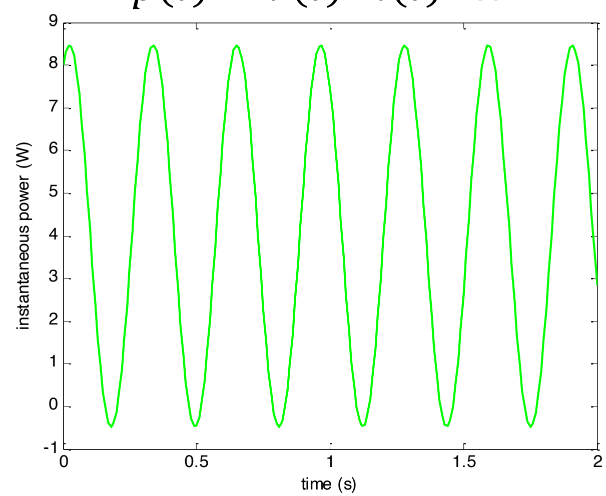
$$v_s(t) = 10 \cos 10t \quad \text{V}$$



$$i(t) = 0.894 \cos(10t - 26.6°) \quad \text{A}$$

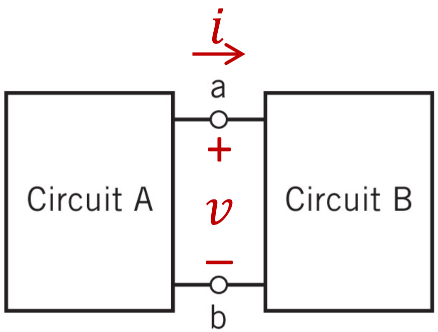


$$p(t) = v(t) \cdot i(t) \quad \text{W}$$



Power has direction

Because current has direction (relative to voltage)



$$p(t) = v(t) \cdot i(t) \quad \mathrm{W}$$

In this example power is transferred from left to right (assuming voltage and current are positive in the direction shown).

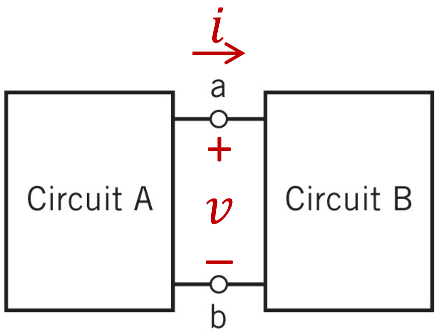
Circuit A is supplying it and Circuit B is receiving it.

Instantaneous power is also conserved. Total power (in a closed circuit) is zero.

$$p(t) > 0 \qquad \text{power received}$$

$$p(t) < 0 \qquad \text{power supplied}$$

Power transfer



$$p(t) = v(t) \cdot i(t) \quad \mathrm{W}$$

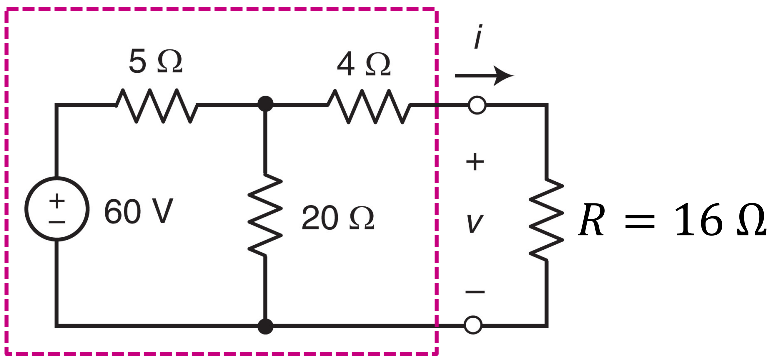
Circuit A and Circuit B are arbitrary.

(They can also be the two parts of the same circuit.)

We can calculate power transferred from any part to any other part if we know the voltage and the current at the intersection between the 2 circuits.

Employing our equivalent circuits

Is a simpler circuit with the same terminal voltage and current



$$p(t) = v(t) \cdot i(t) \ \ \mathrm{W}$$

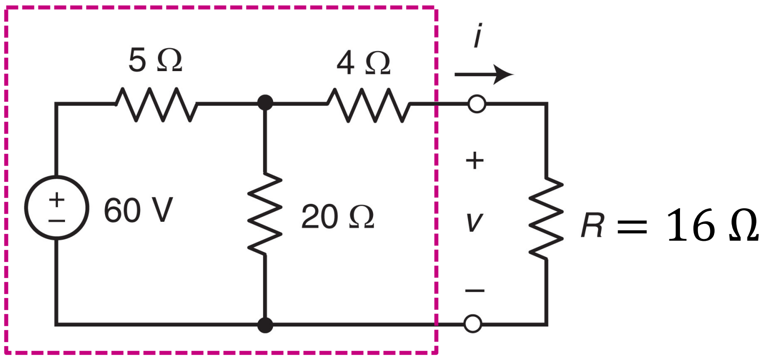
To find the power received by R we need to know its voltage and current but **not the other voltages and currents in the circuit**.

We can use an equivalent circuit for the rest of the circuit as long as we don't change the voltage across R and the current through R.

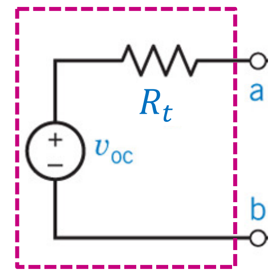
If there are current or voltage sources in the rest of the circuit, we can use Thevenin or Norton equivalent circuits.

Transferred power - example

Question: Find the power transferred to R (=16 Ω) (we will use $p = v \times i$)



1. First find the Thévenin euiv. of LHS:



$v_{oc} = 48\ \text{V}$

$R_t = 8\ \Omega$

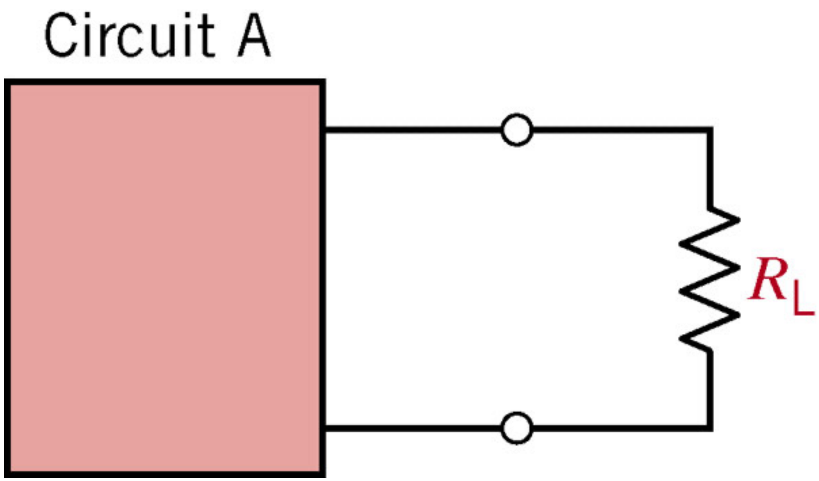
2. Now connect the equivalent circuit to R

Use voltage divider rule: $v = \frac{R}{R+R_t} v_{oc} = 32\ \text{V}$

Next use Ohm's Law to find current: $i = \frac{v_{oc}}{R+R_t} = 2\ \text{A}$

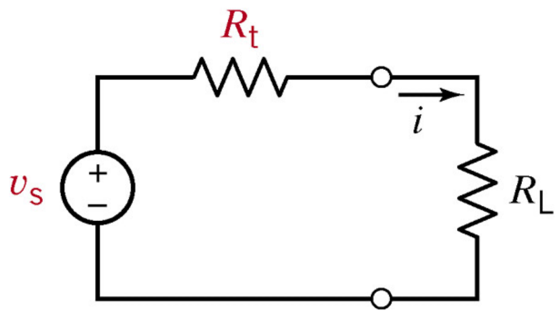
Therefore the transferred power is: $p = 64\ \text{W}$

Maximum power transfer



We can calculate the maximum power available to R_L, without knowing the value of R_L.

Replace circuit A with Thévenin



Want to find the value of R_L that results in the maximum power delivered to it.

Now, the power dissipated by R_L is $p = i^2 R_L$

But the current is $i = v_s/(R_t + R_L)$ therefore the power in R_L is: $p = \left(\frac{v_s}{R_t+R_L}\right)^2 R_L$

Assuming v_s and R_t are constant:

Then the change in power (in R_L) with the change in R_L or $\frac{dp}{dR_L} = v_s^{\,2}\,\frac{(R_t+R_L)^2-2(R_t+R_L)R_L}{(R_t+R_L)^4}$

Now the maximum power is when $\frac{dp}{dR_L} = 0$ or $(R_t + R_L)^2 - 2(R_t + R_L)R_L = 0$

or $R_t + R_L - 2R_L = 0$ or

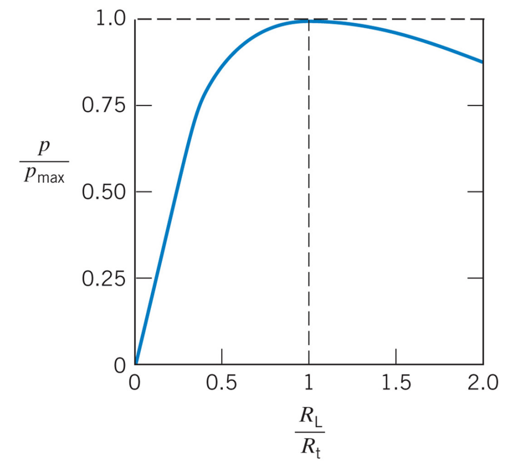
$$R_L = R_t$$

This says: a resistor connected to a circuit A will receive the maximum power from A when its resistance is equal to the Thévenin resistance of A.

Maximum power transfer

Since the power is given by: $p = (\frac{v_s}{R_t+R_L})^2 R_L$

Substituting $R_L = R_t$ we get the maximum power $p_{max} = \frac{v_s^2}{4R_t}$



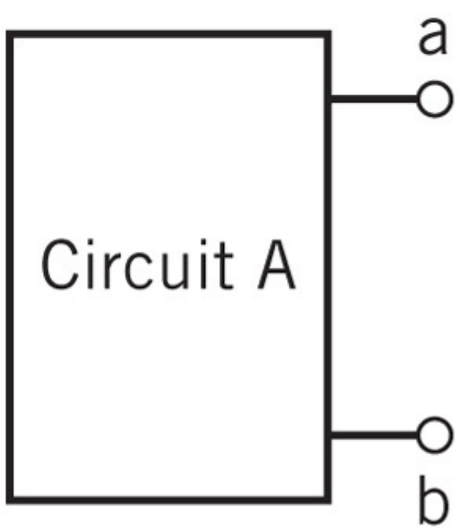
Using: $R_L = R_t$ and $R_t = \frac{v_{oc}}{i_{sc}}$

You can calculate all the following (and more) using this condition:

$$p_{max} = \left(\frac{v_{oc}}{2}\right)^2 \frac{1}{R_t} \qquad p_{max} = \left(\frac{i_{sc}}{2}\right)^2 R_t \qquad p_{max} = \frac{v_{oc} i_{sc}}{4}$$

Homework: Check if formulas are correct!

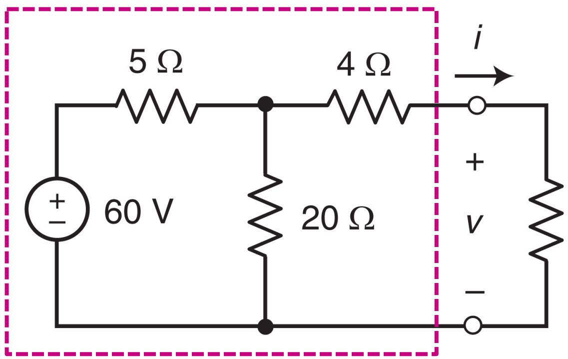
Maximum available power



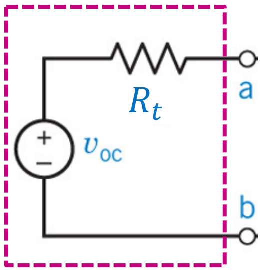
Maximum available power from any circuit (with sources) is the power that could be delivered by that circuit when it is terminated by a resistance that is equal to its Thévenin resistance.

"Maximum available power" may also be called "available power"

Previous example problem



$R = 16\ \Omega$



$v_{oc} = 48\ \text{V}$

$R_{th} = 8\ \Omega$

The maximum available power can only be transferred if: $R = R_t = 8\ \Omega$

So calculating the max. power transfer: $(= v \times i)$

The voltage across R: $v = \dfrac{R}{R + R_t} v_{oc} = \dfrac{8}{8 + 8} \times 48 = 24\ \text{V}$

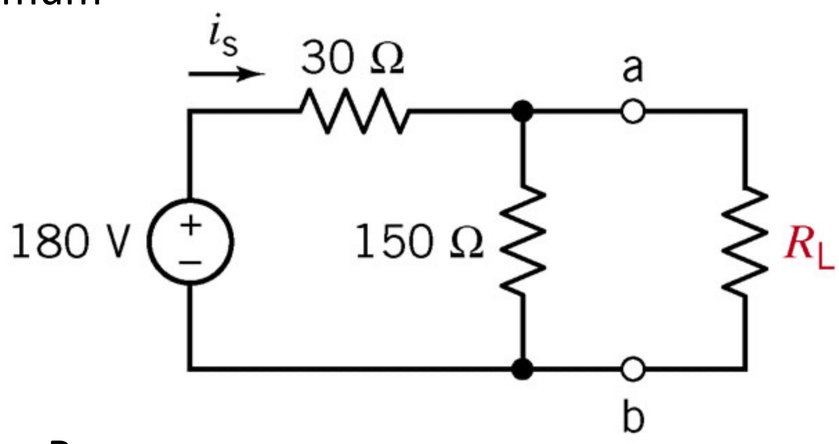
The current through R: $i = \dfrac{v_{oc}}{R + R_t} = 3\ \text{A}$

Max. power transferred: $p_{max} = 72\ \text{W}$

Compared to 64 W when R = 16 Ω

Example problem 1

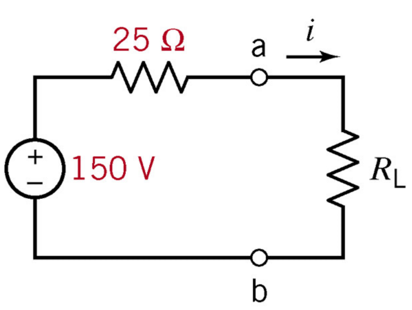
Find the resistance R_L that will result in the maximum power delivered.



Find Thévenin equivalent circuit of LHS

v_{oc} can be found by voltage divider rule: v_{oc} = 180x(150/180) = 150 V

R_t can be found by switching off the 180 V source, R_t = 30x150/180 = 25 Ω



Max. power when R_L = 25 Ω

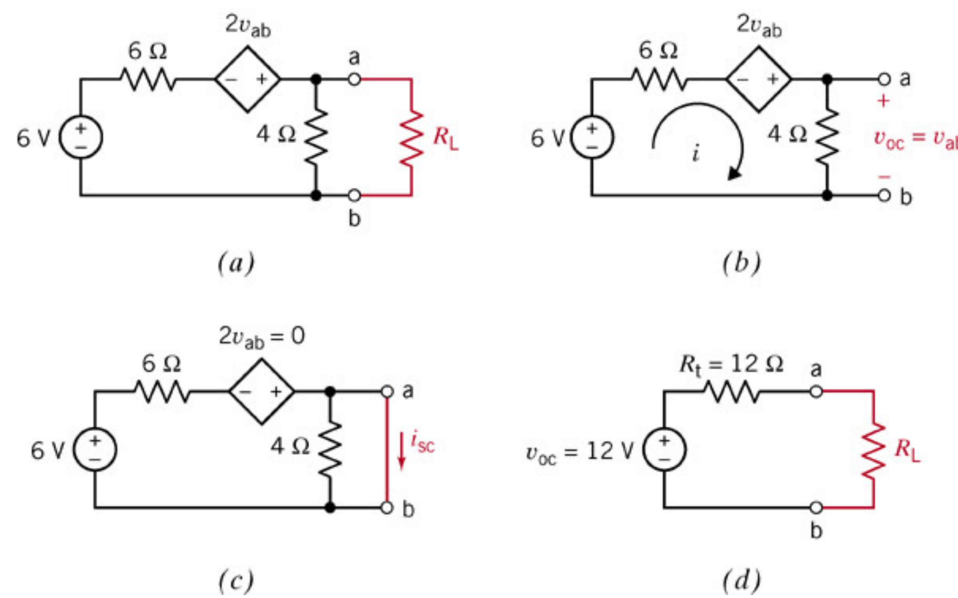
Power = v_{ab} x i = 75 x (150/50) = 225 W

Check: $p_{max} = \frac{v_{oc}^2}{4R_t} = \frac{150^2}{4x25} = 225\ W$

Example problem 2

Find R_L in circuit (a) that allows maximum power transfer and find that power.

(Need to find Thévenin equiv. of LHS)



Use fig. (b) to find v_{oc}:

Using KVL we have:

$$-6 + (6+4)i - 2v_{ab} = 0$$

But using Ohm's Law: $v_{ab} = 4i$

Therefore: $-6 + 10i - 8i = 0$

Or: $i = 3$ A

Hence: $v_{oc} = v_{ab} = 12$ V

Use fig. (c) to find i_{sc} (note $v_{ab} = 0$)

KVL around loop: $-6 + 6i_{isc} = 0$

Hence: $i_{sc} = 1$ A.

Therefore $R_t = 12/1 = 12\ \Omega$

$R_L = R_t = 12\ \Omega$

Max. power delivered = $v_{oc}{}^2/4R_t$

$= (12)^2 / 4 \times 12 = 3$ W

ENERGY STORAGE ELEMENTS

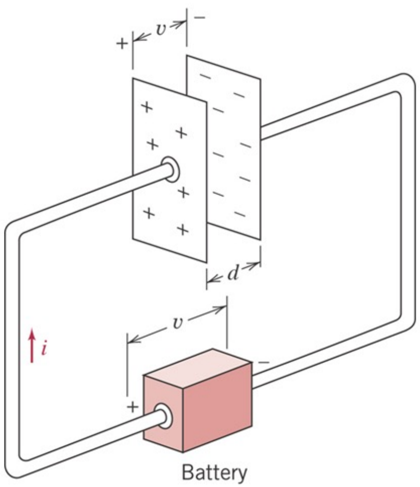
The “Big Ideas” in Chap. 7

- **Capacitors** and **inductors** store energy (as opposed to resistors, which dissipate energy) – they are PASSIVE elements
- While they are classed as linear elements, the current-voltage relationships are a function of time
- Their general behaviour in a circuit is represented by differential equations (as opposed to resistors, which are simply related by Ohm’s Law).
- A set of series/parallel capacitors (or inductors) can be reduced to an equivalent capacitor (or inductor). The formulas for inductors are similar to those for resistances. The formulas for capacitors are similar to those of conductances.
- In a DC circuit (or in any circuit when currents and voltages do not change with time), **capacitors act like open circuits**, and **inductors act like short circuits**.
- Signals can be integrated or differentiated using circuits that contain capacitors or inductors.

Capacitors

Pass current ONLY when the voltage is changing

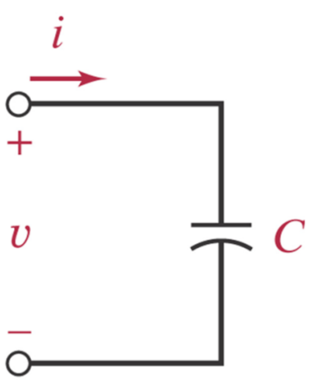
Capacitor basics



A capacitor is a two terminal element which can be modelled as two parallel conducting plates separated by a non-conducting material, called a *dielectric*. Electric charge is stored on the plates.

The **capacitance** (C) is directly proportional to the area (A) of the dielectric and inversely proportional to the spacing (d) between the plates

$$C = \frac{k\varepsilon_0 A}{d}$$

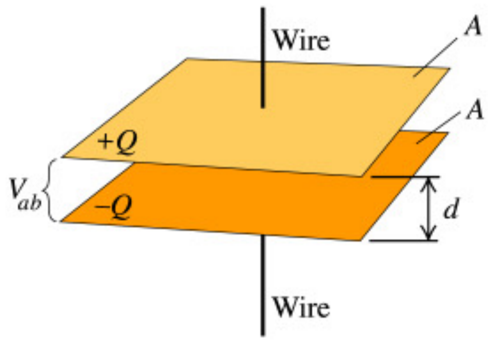


Where C = the capacitance in **farads** (F)
k = is the relative permittivity (dielectric constant)
ε_0 = permittivity of free space
d = distance between the plates

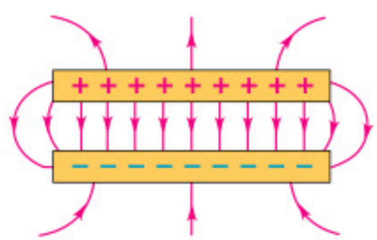
$$\varepsilon = \varepsilon_0 = 8.854 \text{ pF/m.}$$

Energy is stored in the electric field between the plates

MACQUARIE University



(a)



(b)

$$E = \frac{V}{d}$$

Electric field

$$C = \frac{Q}{V}$$

Electric charge

Voltage

Coulombs per volt

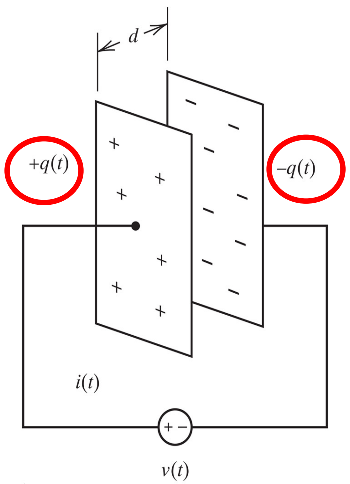
1 Farad=1 C per V

http://www.physics.sjsu.edu/becker/physics51/capacitors.htm

Self study:

Learn what a "displacement current" is

The important equations



The charge stored by a capacitor: $q(t) = Cv(t)$

The capacitor current is given by: $i(t) = \frac{d}{dt}q(t)$

Hence capacitor current is: $$i(t) = C\frac{d}{dt}v(t)$$

The capacitor voltage is given by: $$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

Types of capacitors

Integrated circuit capacitors:

Practical capacitance values on CMOS wafers range from 0.5 pF to a few tens of picofarads.

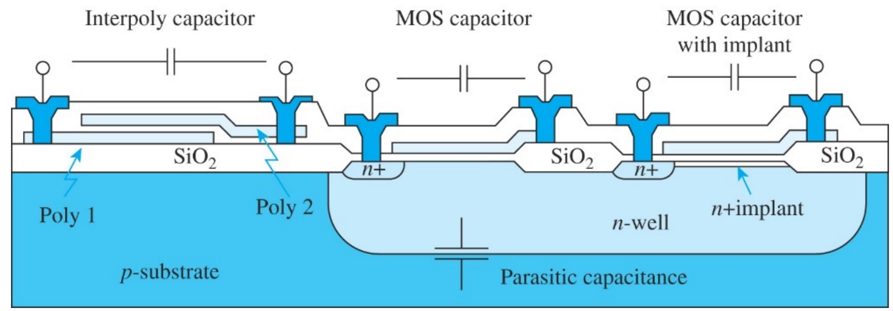
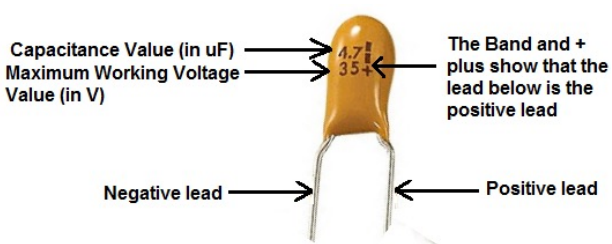


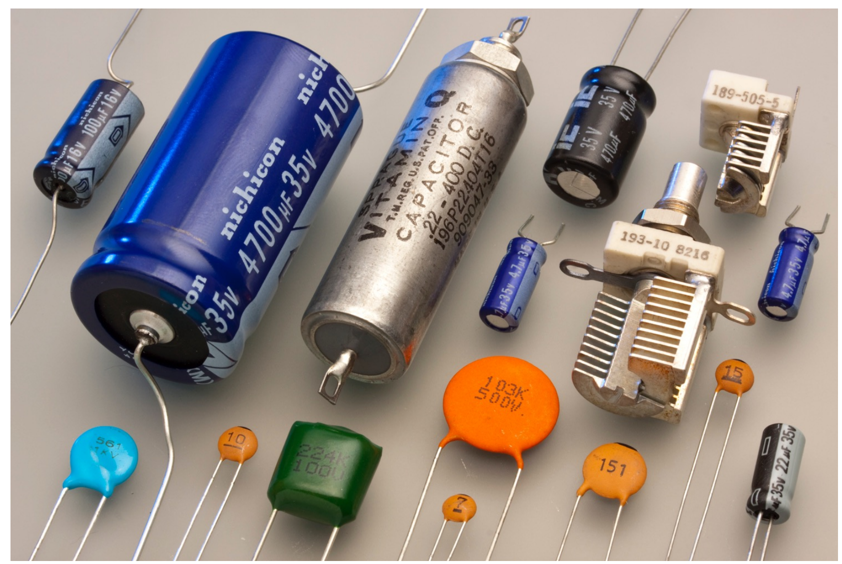
Figure A.8 Interpoly and MOS capacitors in an n-well CMOS process.

SEDRA/SMITH "Microelectronic Circuits" 6th Edition, Oxford University Press

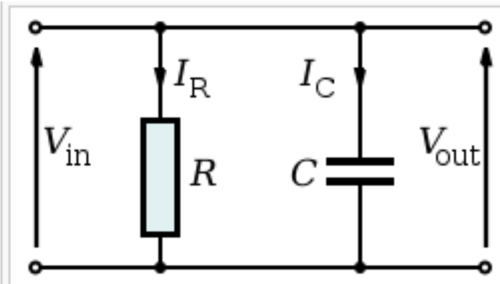
Discrete capacitors:

http://www.learningaboutelectronics.com/images/How-to-read-a-tantalum-capacitor.png





“Real” capacitors



Parallel RC circuit

Physical capacitors may be modelled by a resistance in parallel with the capacitor. The dielectric material between the plates is not a perfect insulator. The effect of the dielectric material is modelled as a high value resistor. Because the value of this resistance is quite large, the resistance is often ignored in circuit analysis.

Points to note

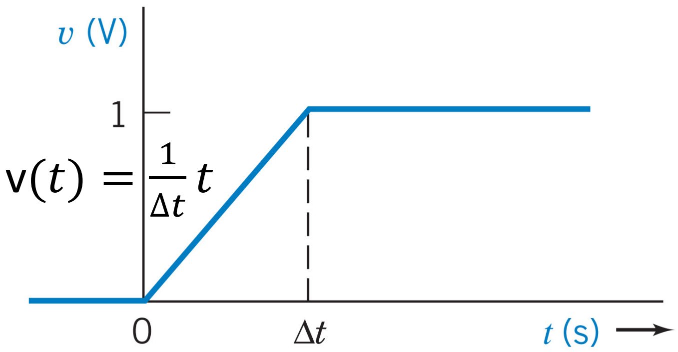
Any two conducting surfaces not in direct contact with each other may be characterised by a nonzero (usually very small) capacitance.

A capacitance of several hundred microfarads (μF is considered "large")

Implications of the equations

Since current is proportional to the time rate of change of voltage, a CONSTANT voltage applied across a capacitor will result in NO current flow. This means capacitors are “open circuit to DC”

Can we have an instantaneous voltage across a capacitor?



A voltage ramp will cause a constant current.

A constant current will cause a voltage ramp across the capacitor

Using $i(t) = C\frac{d}{dt}v(t)$ we have:

$$i(t) = \begin{cases} 0 & t < 0 \\ \frac{C}{\Delta t} & 0 < t < \Delta t \\ 0 & t > \Delta t \end{cases}$$

Instantaneous voltage means Δt=0 which implies $i(t) = \frac{C}{\Delta t} = \infty$

→ Voltage of across a capacitor cannot change instantaneously

Example problem

Determine the current i flowing through the capacitor of Fig. 7.1 for the two voltage waveforms of Fig. 7.3 if $C = 2$ F.

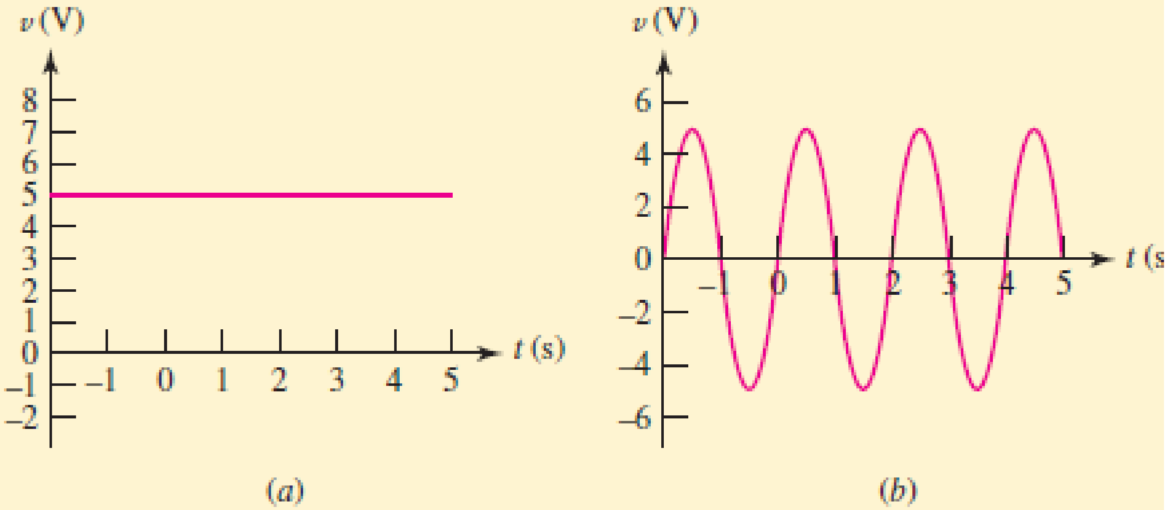


FIGURE 7.3 (*a*) A dc voltage applied to the terminals of the capacitor. (*b*) A sinusoidal voltage waveform applied to the capacitor terminals.

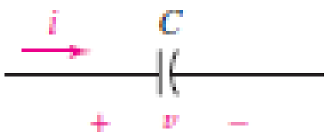


FIGURE 7.1 Electrical symbol and current-voltage conventions for a capacitor.

Solution

The current i is related to the voltage v across the capacitor by Eq. [1]:

$$i = C\frac{dv}{dt}$$

For the voltage waveform depicted in Fig. 7.3a, $dv/dt = 0$, so $i = 0$; the result is plotted in Fig. 7.4a. For the case of the sinusoidal waveform of Fig. 7.3b, we expect a cosine current waveform to flow in response, having the same frequency and twice the magnitude (since $C = 2$ F). The result is plotted in Fig. 7.4b.

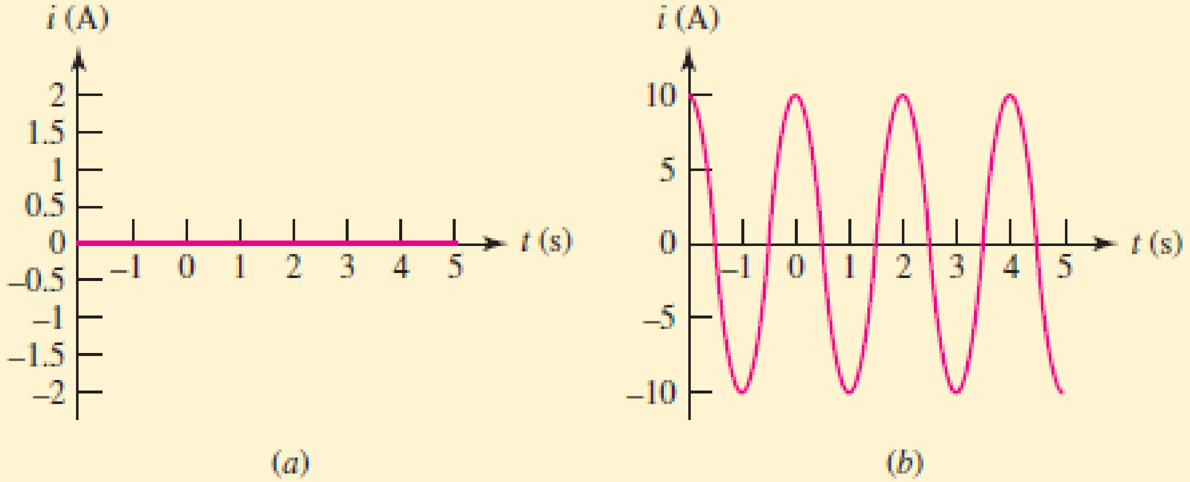


FIGURE 7.4 (a) $i = 0$ as the voltage applied is dc. (b) The current has a cosine form in response to a sine wave voltage.

Initial conditions for voltage

The voltage can be found from : $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$

Not that practical!!

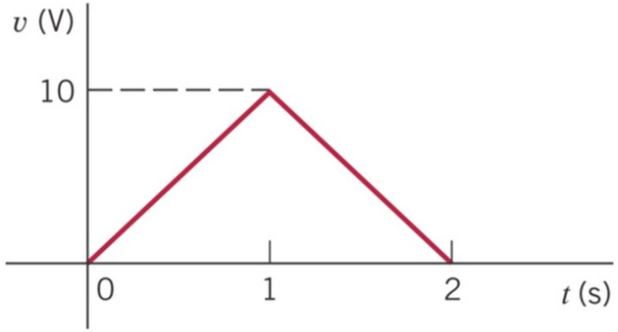
Separate the time frames: $v(t) = \frac{1}{C}\int_{-\infty}^{t_0} i(\tau)d\tau + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau$

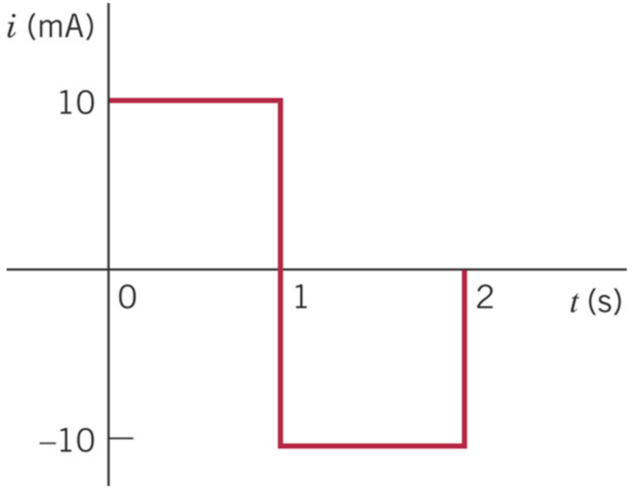
We get: $v(t) = \boxed{v(t_0)} + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau$

The initial voltage condition: the voltage across the capacitor at time = t_0

Example problem

Find the current through the capacitor, $C = 1$ mF, when the voltage across the capacitor is given by,





Solution

$$v(t) = \begin{cases} 0, & t \le 0, \\ 10t, & 0 < t \le 1, \\ 20 - 10t, & 1 < t \le 2, \\ 0, & t > 2. \end{cases}$$

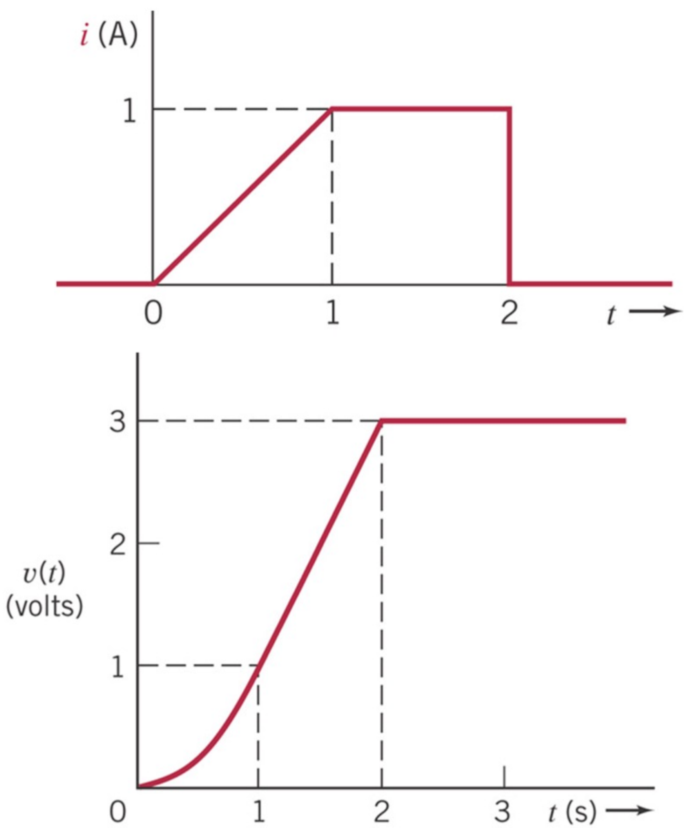
$$i = C\frac{dv}{dt} = 10^{-3}\frac{dv}{dt}$$

$$= \begin{cases} 0, & t \le 0, \\ 10^{-2}, & 0 < t < 1, \\ -10^{-2}, & 1 < t < 2, \\ 0, & t > 2. \end{cases}$$

Note that v is a continuous function of t and that i has discontinuities.

Example problem

Find the voltage across the capacitor, $C = \frac{1}{2}$ F, when the current through the capacitor is given by,



Solution

$$i = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 1, \\ 1, & 1 < t. \end{cases}$$

For $t < 0, v = 0$

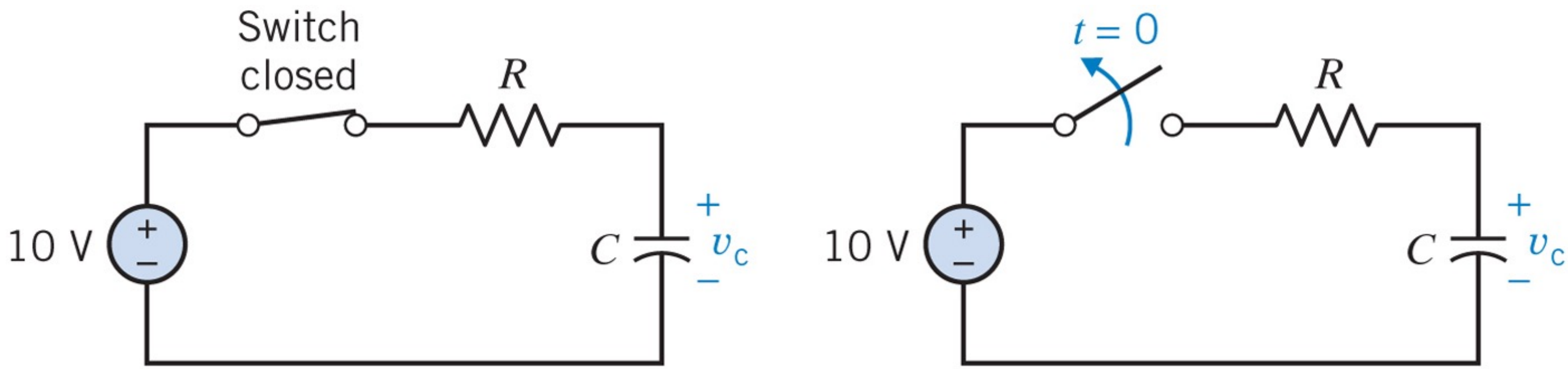
For $t > 0,$

$$v = \frac{1}{C}\int_0^t i d\tau = \begin{cases} \frac{1}{C}\int_0^t \tau d\tau = \frac{t^2}{2C}, & t < 1, \\ \frac{1}{C}\int_0^1 \tau d\tau + \frac{1}{C}\int_1^t 1 d\tau = v(1) + \frac{t}{C} = \frac{1}{2C} + \frac{t-1}{C}, & 1 \le t < 2 \\ v(2) + \frac{1}{C}\int_2^t 0 d\tau = v(2) = \frac{3}{2C}, & 2 \le t. \end{cases}$$

$$v = \begin{cases} 0, & t < 0 \\ t^2, & 0 \le t < 1, \\ 2t - 1, & 1 \le t < 2, \\ 3, & \end{cases}$$

Putting in the value for C

Capacitor circuits with a switch



Voltage of a capacitor cannot change instantaneously:

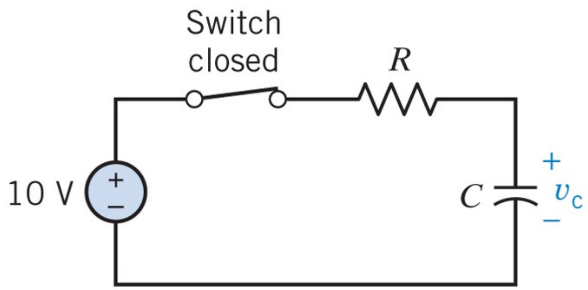
Therefore the voltage across
the capacitor in the above circuit: $v_c(0^-) = v_c(0^+) = 10\text{ V}$

$$v_c(t_0^-) = v_c(t_0^+)$$

The voltage across a capacitor just before t_0 is the SAME as just after t_0

Energy storage in a capacitor

Energy storage in a capacitor



The definition of energy is given by: $w(t) = \int_{-\infty}^{t} p(\tau)d\tau$

When we insert the definition of power we get: $w_c(t) = \int_{-\infty}^{t} v_c(\tau)i(\tau)d\tau$

But for a capacitor: $i(t) = C\frac{d}{dt}v_c(t)$

$$w_c(t) = \int_{-\infty}^{t} Cv_c\frac{dv_c}{dt}d\tau \quad = C\int_{v_c(-\infty)}^{v_c(t)} v_c dv_c \quad = \frac{1}{2}Cv_c^2\Big|_{v_c(-\infty)}^{v_c(t)} \quad = \frac{1}{2}Cv_c^2(t) \text{ Joules}$$

$$\boldsymbol{w_c(t) = \frac{1}{2}Cv_c^2(t)}$$

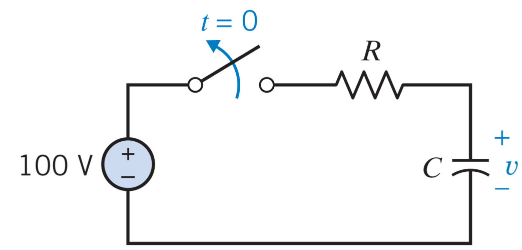
Energy stored in the capacitor at time *t*

Energy is stored!

We have: $w_c(t) = \frac{1}{2} C v^2(t)$ J

Or knowing C=q/v, then:

$$w_c(t) = \frac{1}{2C} q^2(t) \text{ J}$$

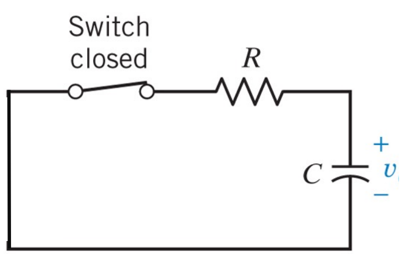


Example:

$C = 10$ mF

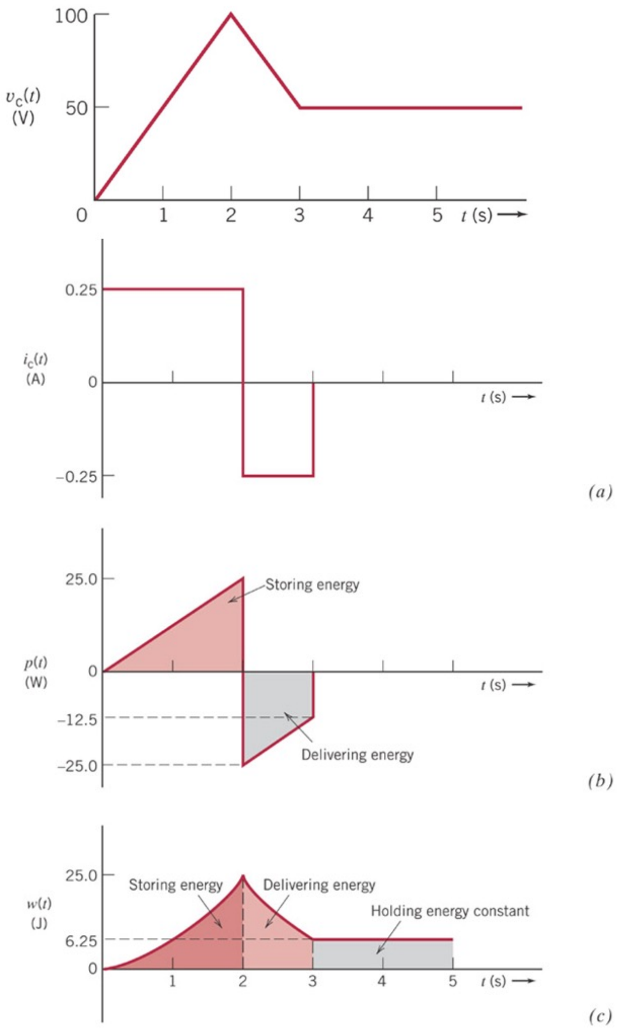
$v_c(0^-) = v_c(0^+) = 100$ V

$w_c(t) = \frac{1}{2} 10^{-2} 100^2 = 50$ J



If we later remove the voltage source and close the switch, all this energy will be dissipated in the resistor.

Example problem



The voltage across a 5 mF capacitor varies as a function of time as shown. Determine and plot the capacitor current, power, and energy.

Solution

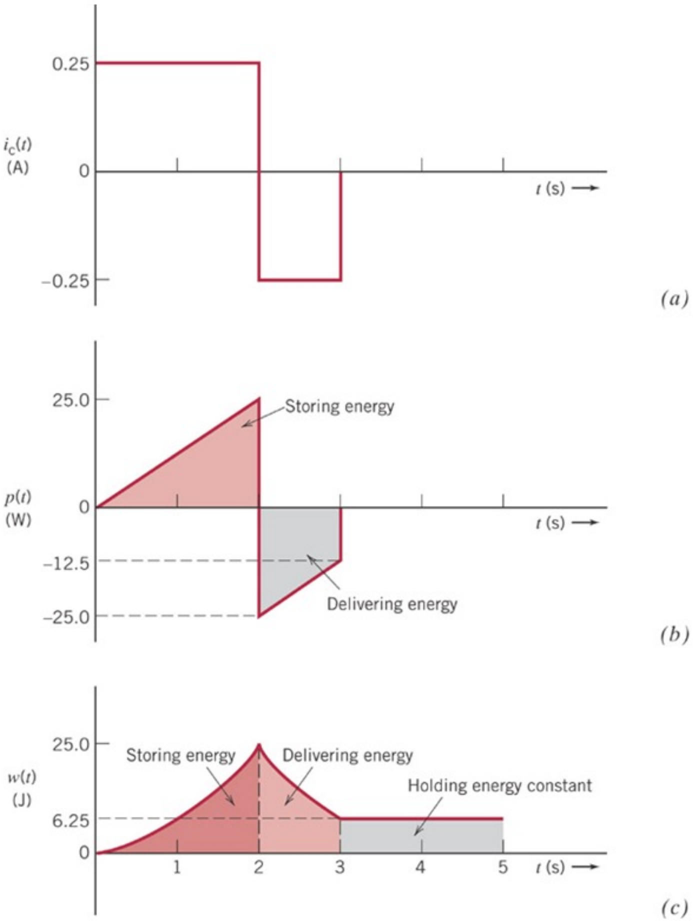
$$i = C\frac{dv_c(t)}{dt}$$

$$P = v_c i = Cv_c\frac{dv_c(t)}{dt}$$

$$w_c = \int_0^t P(\tau)d\tau = \int_0^t Cv_c\frac{dv_c(t)}{dt}d\tau$$

(Note that w_c is the energy under the P curve.)

Solution



$$v=\begin{cases}0, & t<0,\\ 50t, & 0\le t<2,\\ 200-50t, & 2\le t\le 3,\\ 0, & t>3\end{cases}$$

C=0.005 F

$$i=C\frac{dv}{dt}=\begin{cases}0, & t<0,\\ 0.25, & 0\le t<2,\\ -0.25, & 2\le t\le 3,\\ 0, & t>3\end{cases}$$

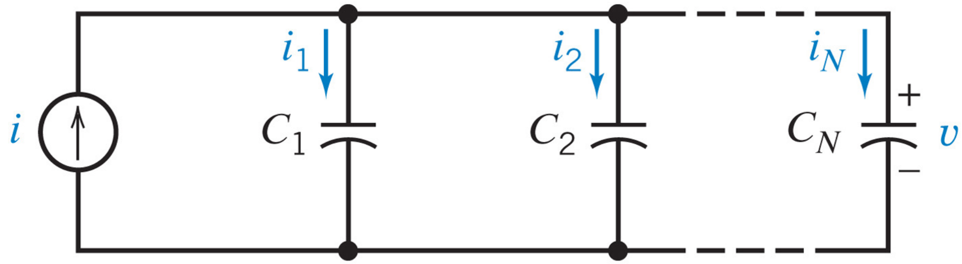
$$p=vi=\begin{cases}0, & t<0,\\ 12.5t, & 0\le t<2,\\ 12.5t-50, & 2\le t\le 3,\\ 0, & t>3\end{cases}$$

$$w_c=\int_0^t p\,d\tau=\begin{cases}0, & t<0,\\ 6.25t^2, & 0\le t<2,\\ 25+\int_2^t(200-50\tau)(-0.25)d\tau=6.25t^2-50t+100, & 2\le t\le 3,\\ \int_0^3 p\,d\tau=6.25 & t>3\end{cases}$$

Series and parallel combinations

Capacitors

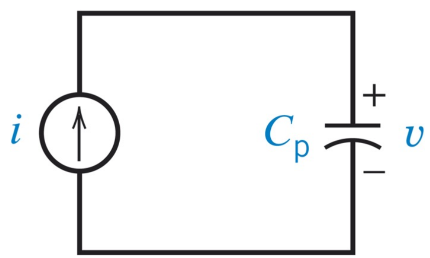
Parallel capacitors



KCL:

$$i = i_1 + i_2 + \cdots + i_N$$

$$i_n = C_n \frac{dv}{dt}$$



$$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt}$$

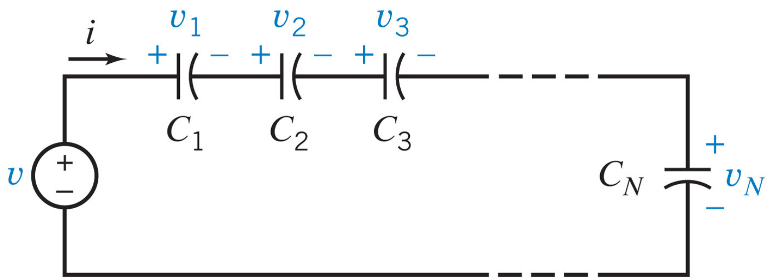
$$i = (C_1 + C_2 + \cdots + C_N) \frac{dv}{dt}$$

$$C_p = (C_1 + C_2 + \cdots + C_N)$$

The capacitors all have the same initial condition [v(0)]

Remember conductance?

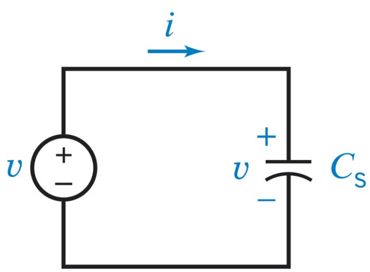
Series capacitors



KVL:

$$v = v_1 + v_2 + \cdots + v_N$$

$$v_n(t) = v_n(t_0) + \frac{1}{C_n}\int_{t_0}^{t} i(\tau)d\tau$$



$$v = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0) + \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(\tau)d\tau$$

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

$$\frac{1}{C_s} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)$$

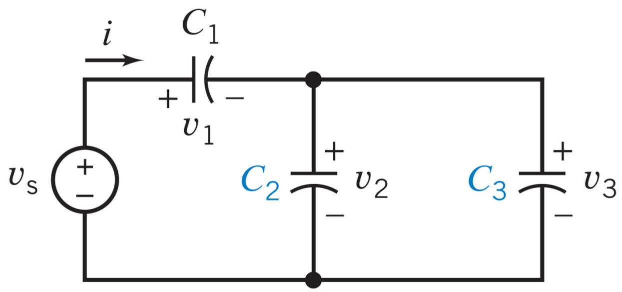
Voltage:

$$v = \left(\sum_{n=1}^{N} \frac{1}{C_n}\right) \int_{t_0}^{t} i \, d\tau + v(t_0)$$

For 2 series capacitors:

$$C_s = \frac{C_1 C_2}{C_1 + C_2}$$

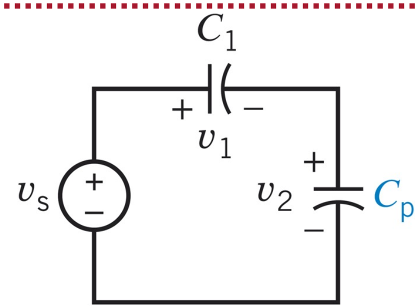
Example problem



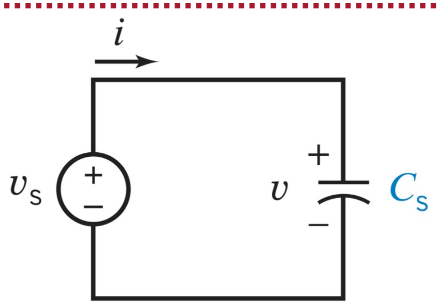
$$C_1 = C_2 = C_3 = 2\ \text{mF}$$

$$v_1(0) = 10\ \text{V}$$

$$v_2(0) = v_3(0) = 20\ \text{V}$$



$$C_p = C_2 + C_3 = 4\ \text{mF}$$



$$C_s = \frac{C_1 C_p}{C_1 + C_p} = \frac{2\times10^{-3}\times4\times10^{-3}}{2\times10^{-3} + 4\times10^{-3}} = \frac{8}{6}\text{mF}$$

$$v_s(0) = v_1(0) + \ v_2(0) = 30\ \text{V}$$

Summary

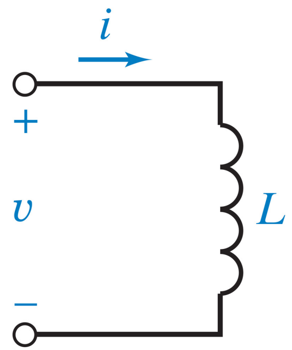
Important Characteristics of an Ideal Capacitor

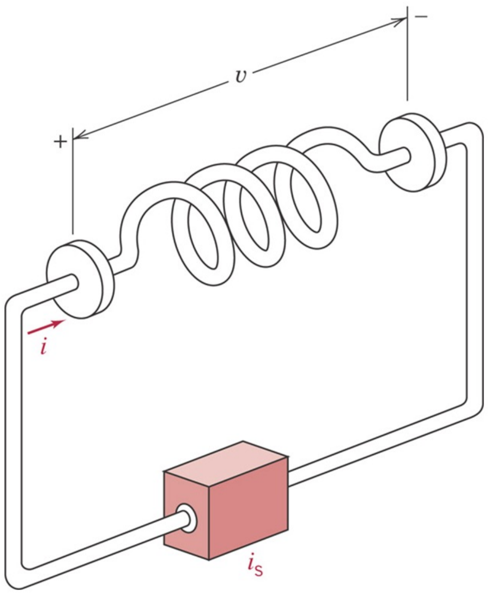
1. There is no current through a capacitor if the voltage across it is not changing with time. A capacitor is therefore an *open circuit to dc*.
2. A finite amount of energy can be stored in a capacitor even if the current through the capacitor is zero, such as when the voltage across it is constant.
3. It is impossible to change the voltage across a capacitor by a finite amount in zero time, as this requires an infinite current through the capacitor. (A capacitor resists an abrupt change in the voltage across it in a manner analogous to the way a spring resists an abrupt change in its displacement.)
4. A capacitor never dissipates energy, but only stores it. Although this is true for the *mathematical model*, it is not true for a *physical* capacitor due to finite resistances associated with the dielectric as well as the packaging.

Inductors

Maintain a voltage ONLY when the current is changing

The inductor circuit element





The inductor equations:

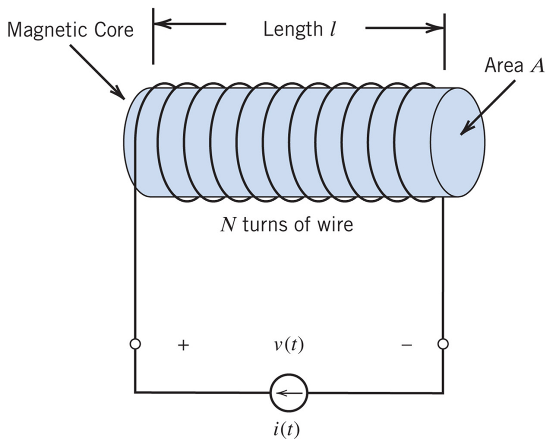
$$v(t) = L\frac{d}{dt}i(t)$$

A constant current produces no voltage.
A time-varying current produces an induced voltage.

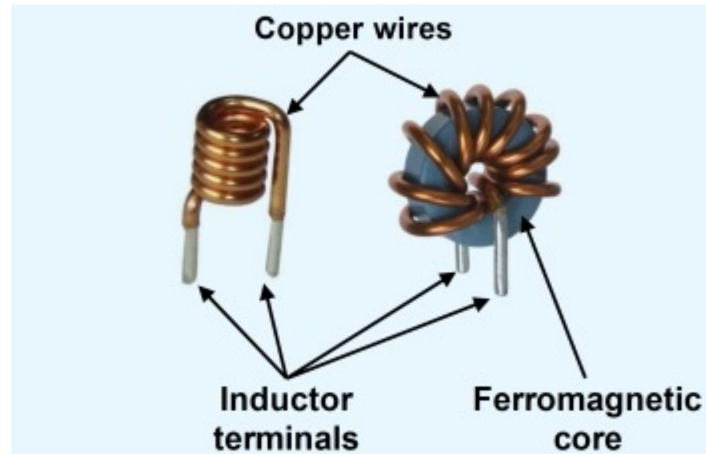
Current through an inductor cannot change instantaneously (but its voltage can)

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

Inductors



Solenoid (single layer coil)



http://www.hobbytronica.eu/electrical-resistance-capacitance-and-inductance/

For inductors wound into a coil:

Magnetic constant (permeability) (H/m)

Cross-sectional area of core (m²)

$$L = \frac{\mu N^2 A}{l}$$

Inductance (Henry)

Length of core (m)

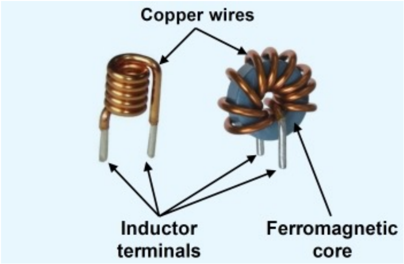
Inductance is a measure of the ability of a device to store energy in the form of a magnetic field

Inductors are expensive to make

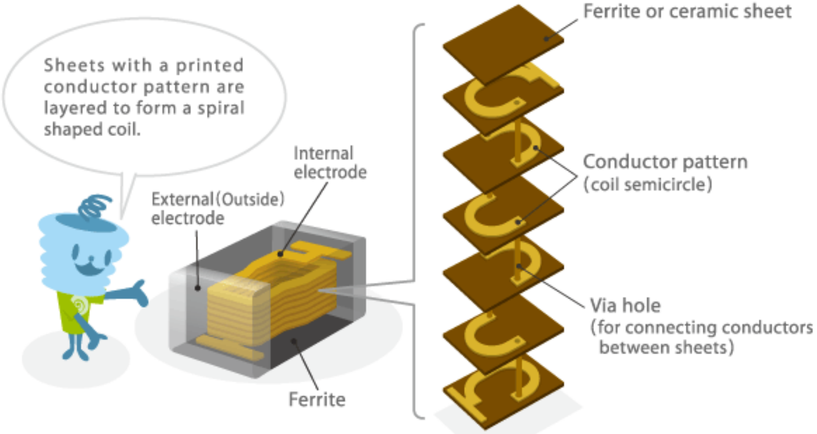
Because magnetic materials are expensive

$$L = \frac{\mu N^2 A}{l}$$

$$\mu = \mu_r \cdot \mu_0$$



How a multilayer chip inductor is made



(Illustration shows a simplified version of sheet type multilayer construction. Other types using printed thick-film as multilayer material also exist.)

http://www.global.tdk.com/techmag/inductive/vol3/index.htm

Permeability of air

$\mu_0 = 4\pi\times10^{-7}$ H/m

Relative permeability of iron

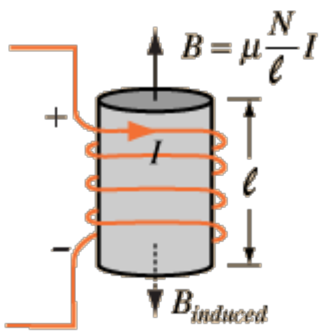
$\mu_r =$ varies

Some representative relative permeabilities:

magnetic iron	200	at a magnetic flux density of 0.002 W/m^2
nickel	100	
permalloy (78.5% nickel, 21.5% iron)	8,000	
mumetal (75% nickel, 2% chromium, 5% copper, 18% iron)	20,000	

The magnetic field

$$L = \frac{\mu N^2 A}{l}$$



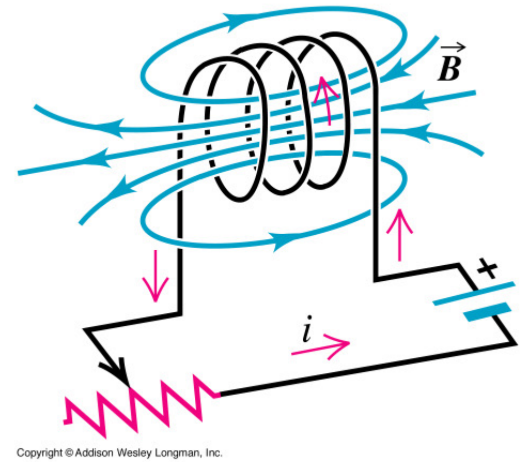
http://hyperphysics.phy-astr.gsu.edu/hbase/electric/indeng.html

$$B = \mu \frac{N}{l} I$$

Magnetic field

Current

Energy is stored in the magnetic field.



http://www.physics.sjsu.edu/becker/physics51/induction.htm

Iron cores have a higher permeability than and CONCENTRATE the magnetic field – the inductance is therefore increased.

Example problem

Given the waveform of the current in a 3 H inductor as shown in Fig. 7.12*a*, determine the inductor voltage and sketch it.

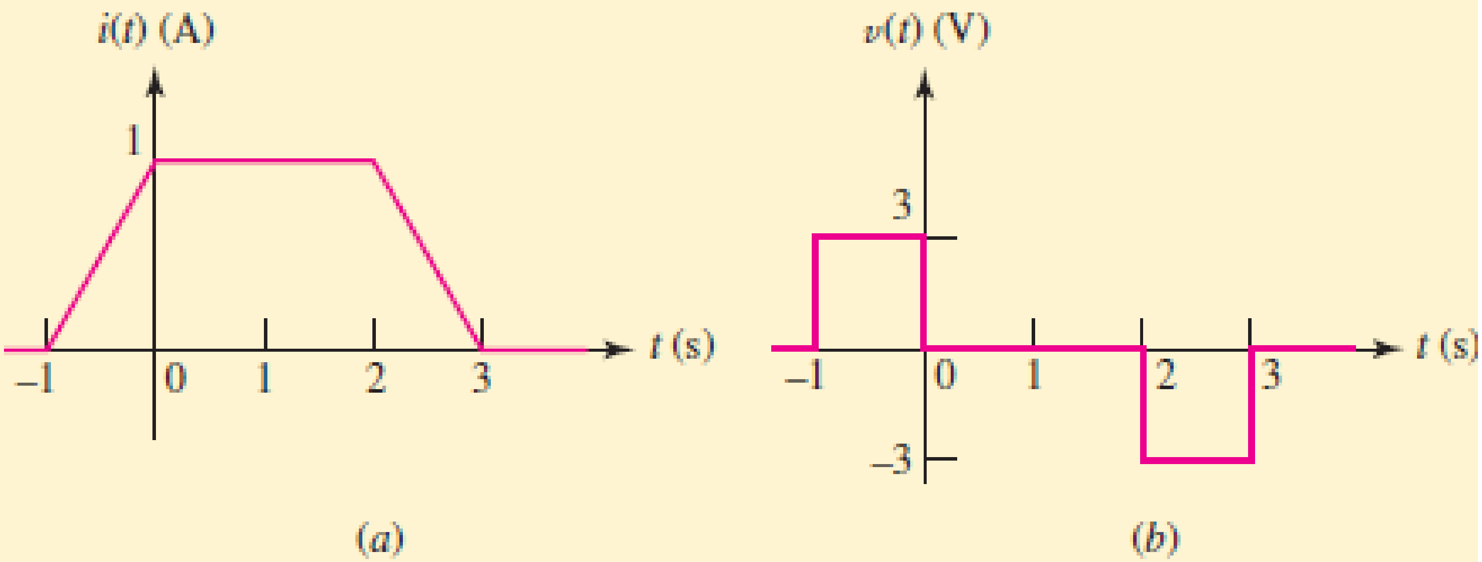


FIGURE 7.12 (*a*) The current waveform in a 3 H inductor. (*b*) The corresponding voltage waveform, $v = 3\, di/dt$.

Solution

Defining the voltage v and the current i to satisfy the passive sign convention, we may obtain v from Fig. 7.12*a* using Eq. [5]:

$$v = 3\frac{di}{dt}$$

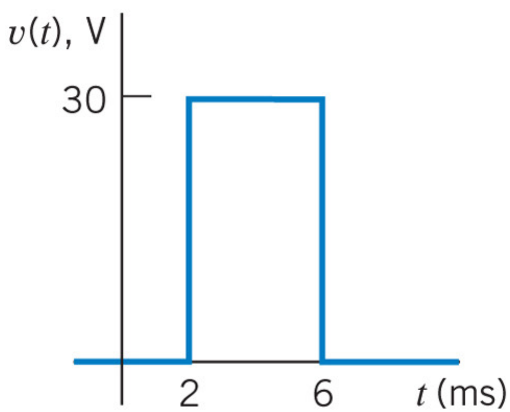
Since the current is zero for $t < -1$ s, the voltage is zero in this interval. The current then begins to increase at the linear rate of 1 A/s, and thus a constant voltage of $L\, di/dt = 3$ V is produced. During the following 2 s interval, the current is constant and the voltage is therefore zero. The final decrease of the current results in $di/dt = -1$ A/s, yielding $v = -3$ V. For $t > 3$ s, $i(t)$ is a constant (zero), so that $v(t) = 0$ for that interval. The complete voltage waveform is sketched in Fig. 7.12*b*.

Initial current condition

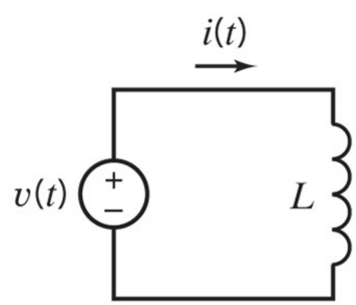
The current in an inductor: $i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$

This can be separated into 2 parts:

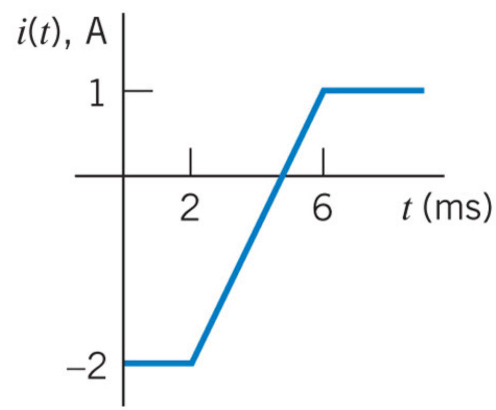
$$i(t) = \frac{1}{L}\int_{-\infty}^{t_0} v(\tau)d\tau + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau$$



VOLTAGE

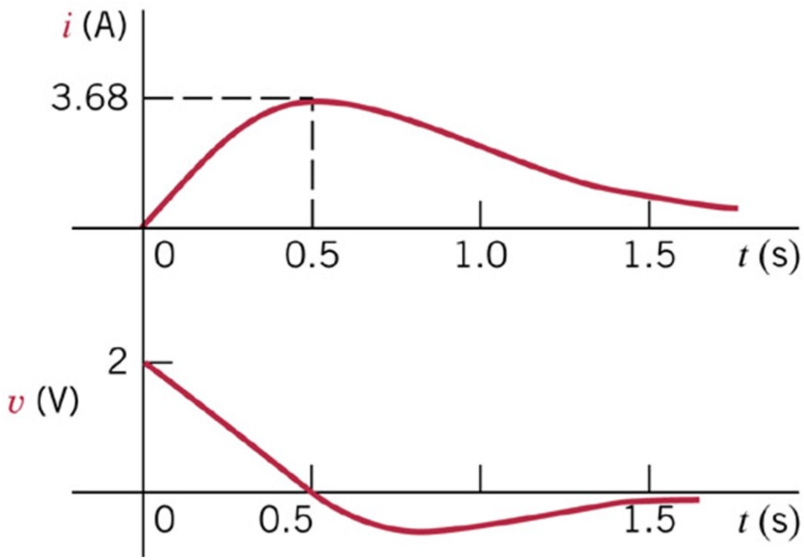


CIRCUIT



CURRENT

Example problem 1



Find the voltage across an $L = 0.1$ H inductor when the current in the inductor is, $i = 20te^{-2t}$ A, for $t \geq 0$ and $i(0) = 0$.

Solution

$$v = L\frac{di}{dt} = (0.1)\frac{d}{dt}\left(20te^{-2t}\right)$$

$$= 2\frac{d}{dt}\boxed{\left(te^{-2t}\right)}$$

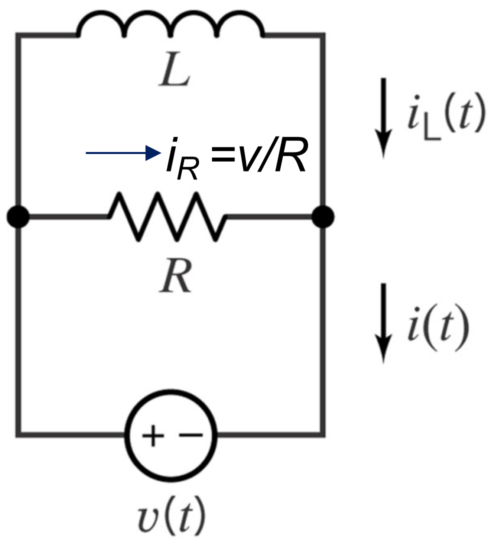
$$= 2\left[t(-2)e^{-2t} + e^{-2t}\right]$$

$$= 2[1-2t]e^{-2t} \quad \text{V}$$

$$d(uv) = u\,dv + v\,du,$$

Recall: differentiation of a product of two functions (*u* and *v*)

Example problem 2



The input to the circuit is, $v(t) = 4e^{-20t}$ V for $t > 0$,

The output current is, $i(t) = -1.2e^{-20t} - 1.5$ A for $t > 0$.

The initial conductor current is, $i(0) = -3.5$ A

Determine the inductance, *L*, and the resistance, *R*.

Solution

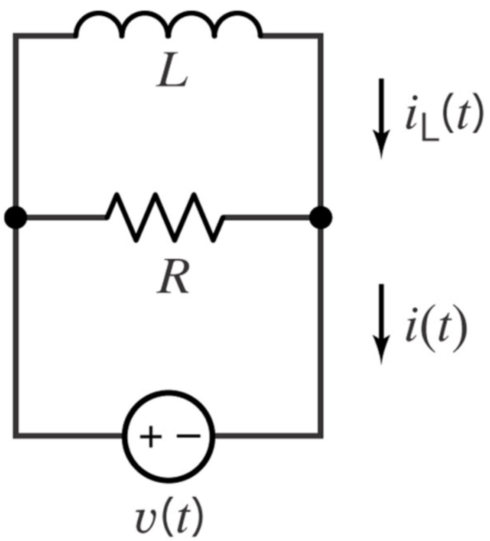
Apply the KCL, $i(t) - i_L(t) - \dfrac{V}{R} = 0$

$$i(t) = i_L(t) + \frac{V}{R}$$

$$= \left[\frac{1}{L}\int_0^t v(\tau)\,d\tau + i_L(0)\right] + \frac{V(t)}{R}$$

$$-1.2e^{-20t} - 1.5 = \left[\frac{1}{L}\int_0^t 4e^{-20\tau}\,d\tau - 3.5\right] + \frac{4e^{-20t}}{R}$$

Solution



$$-1.2e^{-20t} - 1.5 = \left[\frac{1}{L}\int_0^t 4e^{-20\tau}d\tau - 3.5\right] + \frac{4e^{-20t}}{R}$$

$$-1.2e^{-20t} - 1.5 = \frac{4}{L(-20)}e^{-20\tau}\Big|_0^t - 3.5 + \frac{4e^{-20t}}{R}$$

$$\boxed{-1.2e^{-20t} - 1.5 = \frac{1}{(-5L)}[e^{-20t} - 1] - 3.5 + \left(\frac{4}{R}\right)e^{-20t}}$$

Therefore, $\quad -1.5 = \frac{1}{(5L)} - 3.5, \text{ and } \; L = 0.1 \;\; \text{H}$

and, $\quad -1.2e^{-20t} = \frac{1}{(-5L)}e^{-20t} + \left(\frac{4}{R}\right)e^{-20t}$

$$-1.2 = \frac{1}{(-5L)} + \left(\frac{4}{R}\right), \qquad -1.2 = -2 + \frac{4}{R}$$

$$R = \frac{4}{.8} = 5\,\Omega$$

Energy storage in an inductor

Energy storage

The energy: $$w(t) = \int_{-\infty}^{t} p(\tau) d\tau$$

Which is: $$w(t) = \int_{-\infty}^{t} v(\tau) i(\tau) d\tau$$

Noting that: $$v(t) = L \frac{di(t)}{dt}$$

We get: $$w(t) = \int_{-\infty}^{t} iL \frac{di}{dt} d\tau = L \int_{i(-\infty)}^{i(t)} i di = \frac{1}{2} Li^2 \bigg|_{i(-\infty)}^{i(t)} = \frac{1}{2} Li^2(t) \text{ J}$$

Remember:

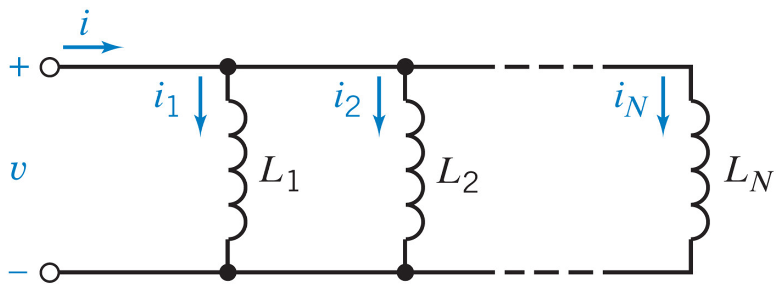
An inductor does not generate or dissipate energy

Inductors and capacitors are both passive elements

Series and parallel combinations

INDUCTORS

Parallel inductors

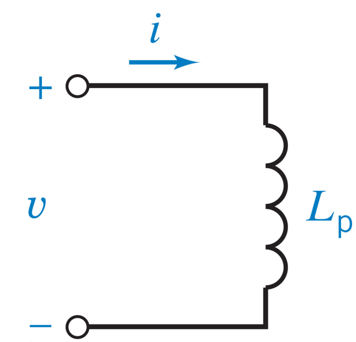


Appling KCL: $i = i_1 + i_2 + \cdots + i_N$

But: $i_n(t) = i_n(t_0) + \dfrac{1}{L_n}\displaystyle\int_{t_0}^{t} v(\tau)d\tau$

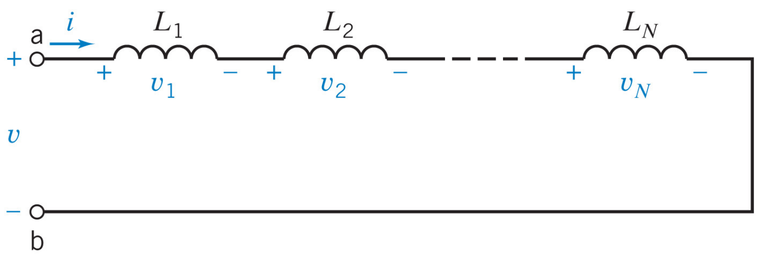
Hence: $i = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0) + \left(\dfrac{1}{L_1} + \dfrac{1}{L_2} + \cdots + \dfrac{1}{L_N}\right)\displaystyle\int_{t_0}^{t} v(\tau)d\tau$

Therefore: $$\frac{1}{L_p} = \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)$$



With the initial current: $$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

Series inductors



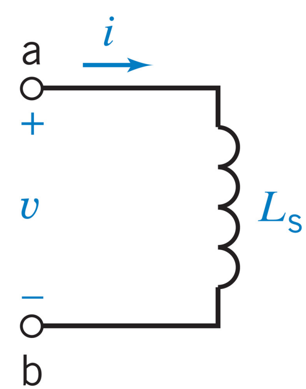
Use KVL: $v = v_1 + v_2 + \cdots + v_N$

But: $v_n = L_n \dfrac{di}{dt}$

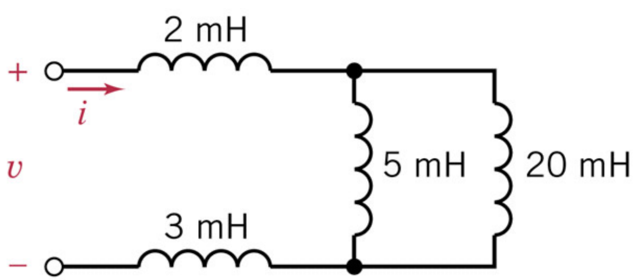
Therefore: $v = L_1 \dfrac{di}{dt} + L_2 \dfrac{di}{dt} + \cdots + L_N \dfrac{di}{dt}$

$$v = (L_1 + L_2 + \cdots + L_N)\frac{di}{dt}$$

$$\boxed{L_s = (L_1 + L_2 + \cdots + L_N)}$$



Example problem



Find the equivalent inductance of the circuit. All the inductor circuits are zero at t_o.

Solution

The equivalent inductance of the 5 mH and 20 mH inductances in parallel, L_p is,

$$L_p = \frac{1}{\dfrac{1}{20\times10^{-3}} + \dfrac{1}{5\times10^{-3}}} = 4\times10^{-3}$$

$$L_{eq} = 2\times10^{-3} + 4\times10^{-3} + 3\times10^{-3} = 9\times10^{-3}$$

$$= 9 \text{ mH}$$

Summary

Important Characteristics of an Ideal Inductor

1. There is no voltage across an inductor if the current through it is not changing with time. An inductor is therefore a *short circuit to dc*.
2. A finite amount of energy can be stored in an inductor even if the voltage across the inductor is zero, such as when the current through it is constant.
3. It is impossible to change the current through an inductor by a finite amount in zero time, for this requires an infinite voltage across the inductor. (An inductor resists an abrupt change in the current through it in a manner analogous to the way a mass resists an abrupt change in its velocity.)
4. The inductor never dissipates energy, but only stores it. Although this is true for the *mathematical* model, it is not true for a *physical* inductor due to series resistances.

Initial conditions of switched circuits

Circuit analysis with switches

1. Analyse the circuit before the switch changes position at $t = t_0$ (steady-state)
2. Recognise that capacitor voltages and inductor currents cannot change instantaneously

$$v_C(t_0^-) = v_C(t_0^+)$$

$$i_L(t_0^-) = i_L(t_0^+)$$

Initial conditions

3. Analyse the circuit (differential equations) for $t > t_0$

Steady state DC circuits with capacitors and inductors

1. Voltage across an inductor with DC current is zero

$$v(t) = L\frac{di(t)}{dt} = 0$$

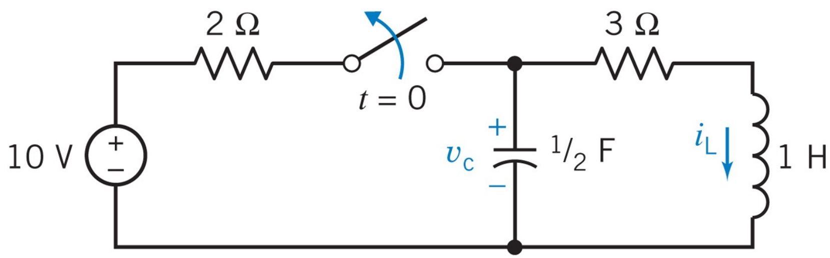
2. Replace inductors with short circuits

3. Current through a capacitor with DC voltage is zero

$$i(t) = C\frac{dv(t)}{dt} = 0$$

4. Replace capacitors with open circuits

Example problem

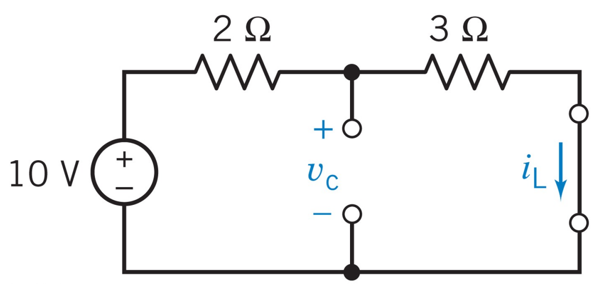


Switch OPENS at t=0

$$v_c(0^+) = ?$$

$$i_L(0^+) = ?$$

$t < 0$:



$$i_L(0^-) = \frac{10}{5} = 2\text{ A}$$

$$i_L(0^+) = i_L(0^-) = 2\text{ A}$$

$$v_c(0^-) = 10 \times \frac{3}{5} = 6\text{ V}$$

$$v_c(0^+) = v_c(0^-) = 6\text{ V}$$

Summary of temporal response

VARIABLE	INDUCTORS	CAPACITORS
Passive sign convention	$\xrightarrow{i}$ L, $+\ v\ -$	$\xrightarrow{i}$ C, $+\ v\ -$
Voltage	$v = L\frac{di}{dt}$	$v = \frac{1}{C}\int_0^t i d\tau + v(0)$
Current	$i = \frac{1}{L}\int_0^t v d\tau + i(0)$	$i = C\frac{dv}{dt}$
Power	$p = Li\frac{di}{dt}$	$p = Cv\frac{dv}{dt}$
Energy	$w = \frac{1}{2}Li^2$	$w = \frac{1}{2}Cv^2$
An instantaneous change is not permitted for the element's	Current	Voltage
Will permit an instantaneous change in the element's	Voltage	Current
This element acts as a (see note below)	Short circuit to a constant current into its terminals	Open circuit to a constant voltage across its terminals

Note: Assumes that the element is in a circuit with steady-state condition.

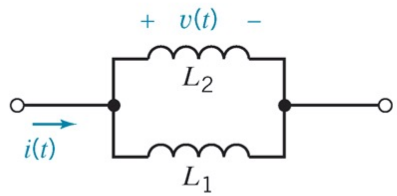
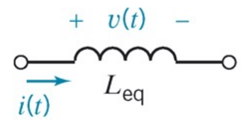
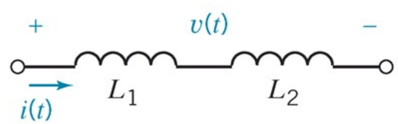
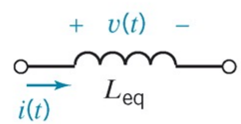
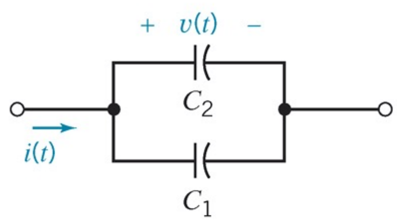
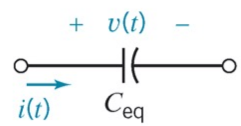
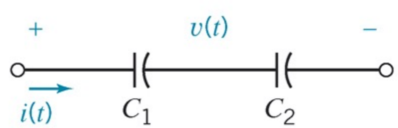
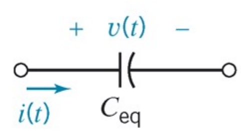
You must remember these!

None of these formulas will be provided in the exam !!!

These are good to remember too but you can infer them from the equations.

Summary of series and parallel circuits

Table 7.13-2 Parallel and Series Capacitors and Inductors

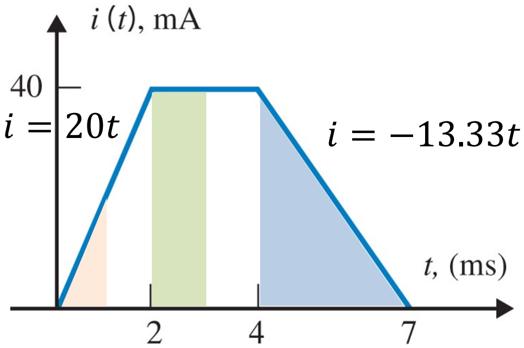
SERIES OR PARALLEL CIRCUIT	EQUIVALENT CIRCUIT	EQUATION
		$L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2}}$
		$L_{eq} = L_1 + L_2$
		$C_{eq} = C_1 + C_2$
		$C_{eq} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$

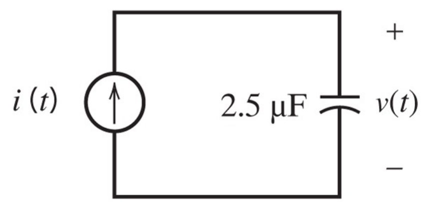
These formulas will NOT be provided in the exam.

Cf. Resistance

Cf. Conductance

Problem 7.2-22





$$v(0) = -20 \text{ V}$$

$$v(1\ ms) = ?$$

$$v(3\ ms) = ?$$

$$v(7\ ms) = ?$$

$$v(t) = v(0) + \frac{1}{C}\int_0^t i(\tau)d\tau \quad \Rightarrow \quad v(t) = -20 + \frac{1}{2.5\times10^{-6}} X \text{ "area under curve"}$$

$$v(1\ ms) = -20 + \frac{1}{2.5\times10^{-6}}\left(\frac{1\times10^{-3}\times20\times10^{-3}}{2}\right) = -20 + 4 = -16 \text{ V}$$

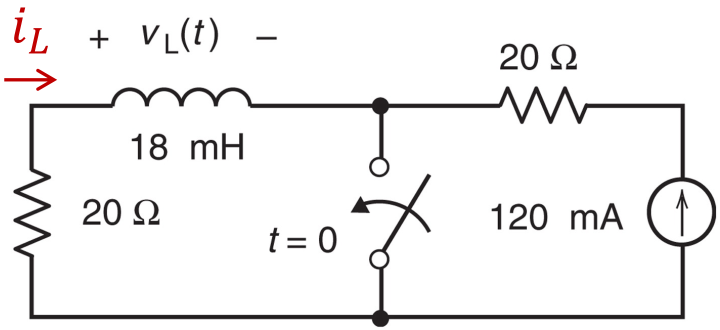
$$v(2\ ms) = -20 + \frac{1}{2.5\times10^{-6}}\left(\frac{2\times10^{-3}\times40\times10^{-3}}{2}\right) = -20 + 16 = -4 \text{ V}$$

$$v(3\ ms) = v(2ms) + \frac{1}{2.5\times10^{-6}}(1\times10^{-3}\times40\times10^{-3}) = -4 + 16 = 12 \text{ V}$$

$$v(4\ ms) = v(3ms) + 16 = 28 \text{ V}$$

$$v(7\ ms) = v(4ms) + \frac{1}{2.5\times10^{-6}}\left(\frac{3\times10^{-3}\times40\times10^{-3}}{2}\right) = 52 \text{ V}$$

Problem 7.8-6

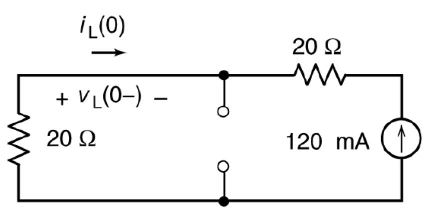


$v_L(0^-) =?$

$v_L(0^+) =?$

Also write the differential equation for i_L

t < 0

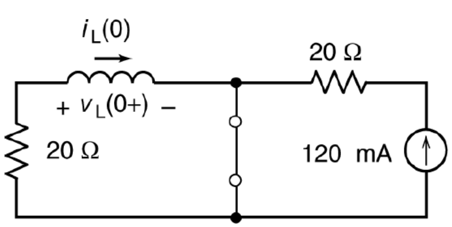


$v_L(0^-) = 0$ (when DC inductors act like short circuits)

$i_L(0^-) = -120\text{ mA}$ (note polarity!)

$i_L(0^+) = -120\text{ mA}$ (inductor currents do not change rapidly)

t > 0

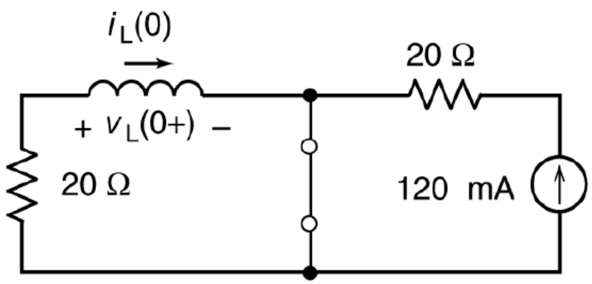


$$v_L(t) = L\frac{di_L(t)}{dt}$$

KVL (LH mesh): $v_L(t) + 20i_L(t) = 0 \Rightarrow v_L(0^+) + 20i_L(0^+) = 0$

$v_L(0^+) = 2.4\ V$

Problem 7.8-6



Differential equation:

We use: $v_L(t) = L\frac{di_L(t)}{dt}$

For $t > 0$

KVL of LH mesh : $v_L(t) + 20\, i_L(t) = 0$

$$18\times10^{-3}\frac{di_L}{dt} + 20\, i_L = 0 \qquad \frac{di_L}{dt} + \frac{20}{18\times10^{-3}} i_L = 0$$

Differential equation of i_L

This part is the inverse of "time constant"

$\frac{di_L}{dt} + \frac{1}{\tau} i_L = f(t)$	General form of first order differential equation