MACQUARIE University

ELEC2070 Circuits and Devices

Week 12: Inverse Laplace Transform and circuits in the complex frequency domain

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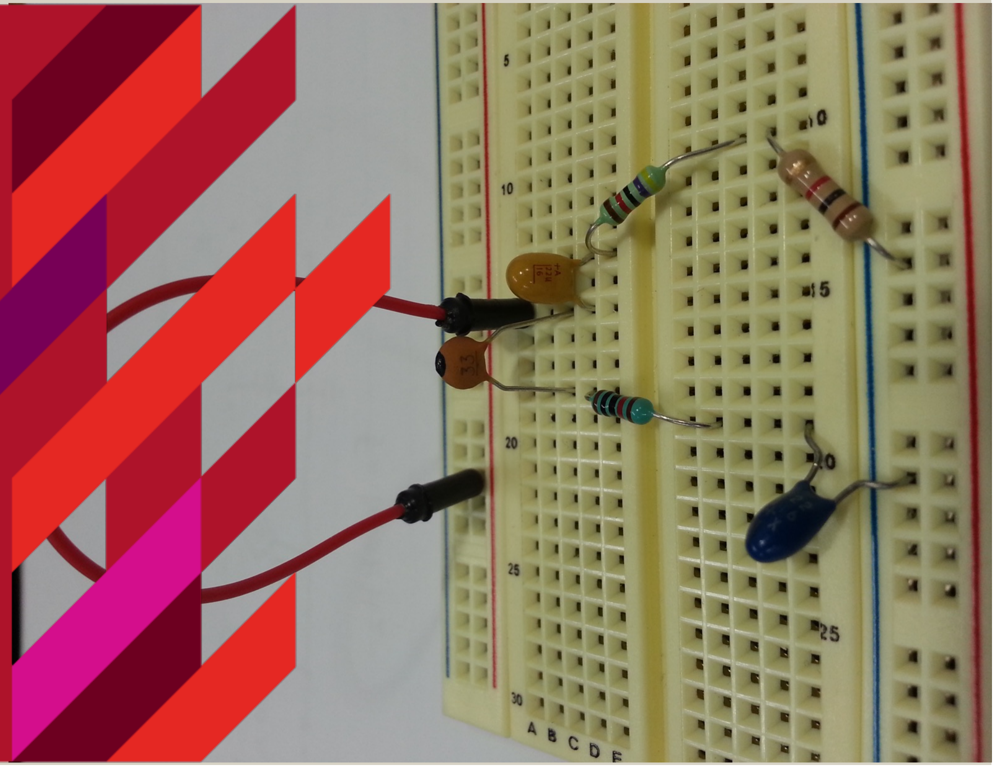
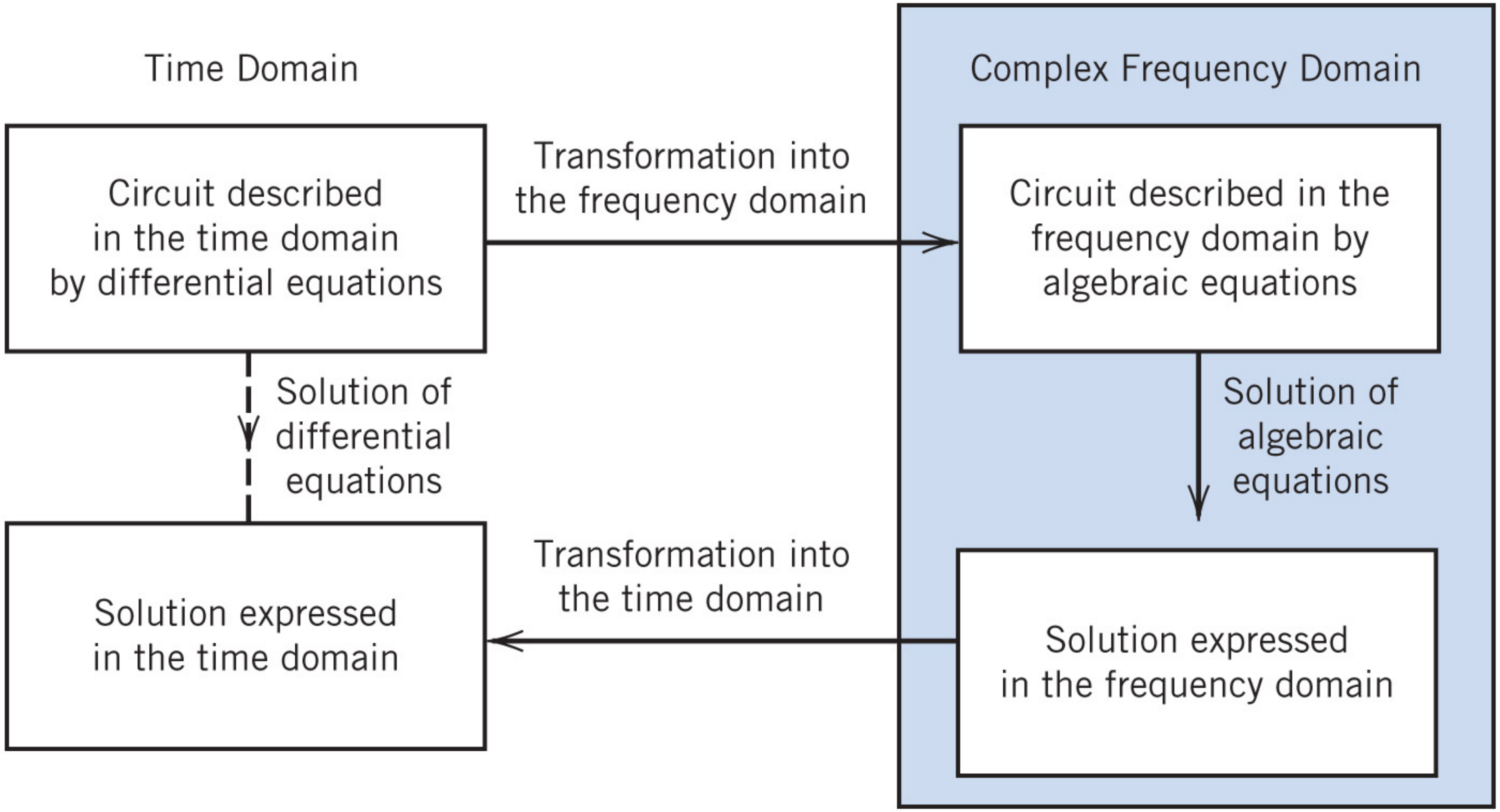


The process of using the transform



All functions that are physically possible have a Laplace Transform.

Table of Laplace Transforms

$f(t)$ for $t > 0$	$F(s) = \mathscr{L}[f(t)u(t)]$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at}t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Important Properties

PROPERTY	$f(t),\ t > 0$	$F(s) = \mathscr{L}[f(t)u(t)]$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time scaling	$f(at)$, where $a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time integration	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
Time differentiation	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - \left(sf(0^-) + \frac{df(0^-)}{dt}\right)$
	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1} f(0^-)}{dt^{k-1}}$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time convolution	$f_1(t)^* f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$	$F_1(s) F_2(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\lambda) d\lambda$
Frequency differentiation	$tf(t)$	$-\frac{dF(s)}{ds}$
Initial value	$f(0^+)$	$\lim_{s \to \infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s \to 0} sF(s)$

Method of Residues

Method of Residues

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

We can find the inverse Laplace transform of the proper rational function F(s) in a three step process.

Step 1:
Perform a partial fraction expansion of F(s) as a sum of simpler functions $F_i(s)$ i.e.,

$$F(s) = F_1(s) + F_2(s) + \cdots F_i(s) + \cdots + F_n\ (s)$$

Step 2:
We use the <u>transform pair</u> table and the <u>properties of transforms</u> table to find the inverse Laplace transform of each $F_i(s)$

Step 3:
Using the linearity property, we sum each inverse transform to obtain the final inverse Laplace transform of F(s)

How to find the partial fraction expansion of F(s)

If all the poles of F(s) are simple poles (that means that they only occur once) then the partial fraction expansion of F(s) is given by the formula:

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + \cdots + \frac{R_i}{s-p_i} + \cdots + \frac{R_n}{s-p_n}$$

Here only one term in the expansion exists for each pole.

The coefficients R_i are called **residues** and the residue R_i corresponds to the pole p_i

The residue corresponding to a real pole is a real number.

The residues corresponding to complex conjugate poles are also complex conjugates.

The residues are calculated from:

$$\boxed{R_i = (s-p_i)F(s)\big|_{s=p_i}}$$

Residues for simple poles

The formula for residues:

$$R_i = (s - p_i)F(s)\big|_{s=p_i}$$

The *i*th residue

Multiply this term with the original frequency domain function FIRST

The original frequency domain function (Laplace Transform)

Then replace ALL "s" variables with the pole p_i

Example

Determine the inverse transform of

$$\mathbf{P(s)} = \frac{7\mathbf{s}+5}{\mathbf{s}^2+\mathbf{s}}$$

We see that $\mathbf{P(s)}$ is a rational function (the degree of the numerator is *one*, whereas the degree of the denominator is *two*), so we begin by factoring the denominator and write:

$$\mathbf{P(s)} = \frac{7\mathbf{s}+5}{\mathbf{s}(\mathbf{s}+1)} = \frac{a}{\mathbf{s}} + \frac{b}{\mathbf{s}+1}$$

where our next step is to determine values for a and b. Applying the method of residues,

$$a = \left.\frac{7\mathbf{s}+5}{\mathbf{s}+1}\right|_{\mathbf{s}=0} = 5 \quad \text{and} \quad b = \left.\frac{7\mathbf{s}+5}{\mathbf{s}}\right|_{\mathbf{s}=-1} = 2$$

We may now write $\mathbf{P(s)}$ as

$$\mathbf{P(s)} = \frac{5}{\mathbf{s}} + \frac{2}{\mathbf{s}+1}$$

the inverse transform of which is simply $p(t) = [5 + 2e^{-t}]u(t)$.

Example 14.4-1

Find the inverse Laplace transform of $F(s) = \dfrac{s+3}{s^2+7s+10}$.

Solution

The given $F(s)$ is indeed a proper rational function. Factor the denominator and perform a partial fraction expansion.

$$F(s) = \frac{s+3}{s^2+7s+10} = \frac{s+3}{(s+2)(s+5)} = \frac{R_1}{s+2} + \frac{R_2}{s+5}$$

where $R_1 = (s+2)\left(\dfrac{s+3}{(s+2)(s+5)}\right)\Bigg|_{s=-2} = \dfrac{s+3}{s+5}\Bigg|_{s=-2} = \dfrac{-2+3}{-2+5} = \dfrac{1}{3}$

and $R_2 = (s+5)\left(\dfrac{s+3}{(s+2)(s+5)}\right)\Bigg|_{s=-5} = \dfrac{s+3}{s+2}\Bigg|_{s=-5} = \dfrac{-5+3}{-5+2} = \dfrac{2}{3}$

Then

$$F(s) = \frac{\frac{1}{3}}{s+2} + \frac{\frac{2}{3}}{s+5}$$

Using linearity and taking the inverse Laplace transform of each term gives

$$F(t) = \mathscr{L}^{-1}[F(s)] = \mathscr{L}^{-1}\left[\frac{\frac{1}{3}}{s+2} + \frac{\frac{2}{3}}{s+5}\right] = \frac{1}{3}\mathscr{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{2}{3}\mathscr{L}^{-1}\left[\frac{1}{s+5}\right] = \frac{1}{3}e^{-2t} + \frac{2}{3}e^{-5t} \text{ for } t \geq 0$$

Repeated poles

Consider the function: $$\mathbf{V}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{(\mathbf{s}-p)^n}$$

We want to expand this function into: $$\mathbf{V}(\mathbf{s}) = \frac{a_n}{(\mathbf{s}-p)^n} + \frac{a_{n-1}}{(\mathbf{s}-p)^{n-1}} + \cdots + \frac{a_1}{(\mathbf{s}-p)}$$

To determine each constant (residue), we multiply the non-expanded version of **V(s)** by $(s - p)^n$

The residue a_n (the leading term) is found by evaluating the resulting expression at s = p

The remaining residues are found by differentiating the expression $(s - p)^n$ **V(s)** the appropriate number of times before the evaluation at s = p and then dividing by a factorial term

For example, a_{n-2} is found by evaluating

$$\frac{1}{2!}\frac{d^2}{d\mathbf{s}^2}[(\mathbf{s}-p)^n\mathbf{V}(\mathbf{s})]_{\mathbf{s}=p}$$

the term a_{n-k} is found by evaluating

$$\boxed{\frac{1}{k!}\frac{d^k}{d\mathbf{s}^k}[(\mathbf{s}-p)^n\mathbf{V}(\mathbf{s})]_{\mathbf{s}=p}}$$

Example

Compute the inverse transform of the function

$$V(s) = \frac{2}{s^3 + 12s^2 + 36s}$$

We note that the denominator can be easily factored, leading to

$$V(s) = \frac{2}{s(s+6)(s+6)} = \frac{2}{s(s+6)^2}$$

As promised, we see that there are indeed three poles, one at $s = 0$, and two at $s = -6$. Next, we expand the function into

$$V(s) = \frac{a_1}{(s+6)^2} + \frac{a_2}{(s+6)} + \frac{a_3}{s}$$

and apply our new procedure to obtain the unknown constants a_1 and a_2; we will find a_3 using the previous procedure. Thus,

$$a_1 = \left[(s+6)^2 \frac{2}{s(s+6)^2}\right]_{s=-6} = \left.\frac{2}{s}\right|_{s=-6} = -\frac{1}{3}$$

and

$$a_2 = \frac{d}{ds}\left[(s+6)^2 \frac{2}{s(s+6)^2}\right]_{s=-6} = \left.\frac{d}{ds}\left(\frac{2}{s}\right)\right|_{s=-6} = \left.-\frac{2}{s^2}\right|_{s=-6} = -\frac{1}{18}$$

The remaining constant a_3 is found using the procedure for distinct poles

$$a_3 = \left.s\frac{2}{s(s+6)^2}\right|_{s=0} = \frac{2}{6^2} = \frac{1}{18}$$

Example

Thus, we may now write $\mathbf{V}(\mathbf{s})$ as

$$\mathbf{V}(\mathbf{s}) = \frac{-\frac{1}{3}}{(\mathbf{s}+6)^2} + \frac{-\frac{1}{18}}{(\mathbf{s}+6)} + \frac{\frac{1}{18}}{\mathbf{s}}$$

Using the linearity theorem, the inverse transform of $\mathbf{V}(\mathbf{s})$ can now be found by simply determining the inverse transform of each term. We see that the first term on the right is of the form

$$te^{-\alpha t}u(t) \Leftrightarrow \frac{1}{(\mathbf{s}+\alpha)^2}$$

$$\frac{K}{(\mathbf{s}+\alpha)^2}$$

and making use of Eq. [21], we find that its inverse transform is $-\frac{1}{3}te^{-6t}u(t)$. In a similar fashion, we find that the inverse transform of the second term is $-\frac{1}{18}e^{-6t}u(t)$, and that of the third term is $\frac{1}{18}u(t)$. Thus,

$$v(t) = -\frac{1}{3}te^{-6t}u(t) - \frac{1}{18}e^{-6t}u(t) + \frac{1}{18}u(t)$$

or, more compactly,

$$v(t) = \frac{1}{18}[1 - (1+6t)e^{-6t}]u(t)$$

Complex poles (formula method)

Now, if a pole is a complex conjugate pair, by this we mean:

$p_1 = -a + jb$ and $p_2 = -a - jb$

Then the residues will also be complex, and they will be given by:

$R_1 = c+jd$ and $R_2 = c-jd$

The partial expansion of F(s) is given by:

This is due to the other poles of F(s)

$$F(s) = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + F_3(s) = \frac{c+jd}{s-(-a+jb)} + \frac{c-jd}{s-(-a-jb)} + F_3(s)$$

Complex residues

Now we want to have a common denominator for the first 2 terms, we do this by:

$$
\begin{aligned}
F(s) &= \frac{c+jd}{s+a-jb} + \frac{c-jd}{s+a+jb} + F_3(s) \\
&= \frac{(c+jd)(s+a+jb)+(c-jd)(s+a-jb)}{(s+a-jb)(s+a+jb)} + F_3(s) \\
&= \frac{2c\,s + 2(a\,c - b\,d)}{s^2 + 2\,a\,s + a^2 + b^2} + F_3(s) \\
&= \frac{2\,c(s+a) - 2\,b\,d}{(s+a)^2 + b^2} + F_3(s) \\
&= 2c\frac{s+a}{(s+a)^2+b^2} - 2d\frac{b}{(s+a)^2+b^2} + F_3(s)
\end{aligned}
$$

Notice that the partial fraction expansion of F(s) can be expressed as:

$$F(s) = \frac{K_1 s + K_2}{s^2 + 2\,a\,s + a^2 + b^2} + F_3(s)$$

$K_1 = 2c$

$K_2 = 2(ac - bd)$

Taking the inverse Laplace Transform

Taking the inverse Laplace transform of these 2 terms gives:

Frequency shift of a

$$\mathscr{L}^{-1}\left[2\,c\,\frac{s+a}{(s+a)^2+b^2}\right] = 2\,c\,\mathscr{L}^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] = 2\,c\,e^{-at}\,\mathscr{L}^{-1}\left[\frac{s}{s^2+b^2}\right] = 2\,c\,e^{-at}\cos(bt)$$

and

$$\mathscr{L}^{-1}\left[2\,d\,\frac{b}{(s+a)^2+b^2}\right] = 2\,d\,\mathscr{L}^{-1}\left[\frac{b}{(s+a)^2+b^2}\right] = 2\,d\,e^{-at}\,\mathscr{L}^{-1}\left[\frac{b}{s^2+b^2}\right] = 2\,d\,e^{-at}\sin(bt)$$

Using linearity we get

$$\mathscr{L}^{-1}[F(s)] = 2\,c\,e^{-at}\cos(bt) - 2\,d\,e^{-at}\sin(bt) + \mathscr{L}^{-1}[F_3(s)]$$

Complex poles lead to sinusoidal functions of time.

Example 14.4-2 (using simple pole method)

Find the inverse Laplace transform of $F(s) = \dfrac{10}{(s^2+6s+10)(s+2)}$.

Solution

The roots of the quadratic $(s^2+6s+10)$ are complex, and we may write $F(s)$ as

$$F(s) = \frac{10}{(s+3-j)(s+3+j)(s+2)}$$

Using a partial fraction expansion, we have

$$F(s) = \frac{10}{(s+3-j)(s+3+j)(s+2)} = \frac{R_1}{s-(-3+j)} + \frac{R_2}{s-(-3-j)} + \frac{R_3}{s+2}$$

Using Eq. 14.4-3,

$$R_i = (s-p_i)F(s)\big|_{s=p_i}$$

$$R_1 = (s+3-j)\left(\frac{10}{(s+3-j)(s+3+j)(s+2)}\right)\Bigg|_{s=-3+j}$$

$$= \frac{10}{(s+3+j)(s+2)}\Bigg|_{s=-3+j} = \frac{10}{(-3+j+3+j)(-3+j+2)} = -\frac{5}{2}+j\frac{5}{2}$$

Comparing to Eq. 14.4-4, we see that $a=3$, $b=1$, $c=-2.5$, and $d=2.5$. Next,

$$F(s) = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + F_3(s) = \frac{c+jd}{s-(-a+jb)} + \frac{c-jd}{s-(-a-jb)} + F_3(s)$$

$$R_2 = (s+3+j)\left(\frac{10}{(s+3-j)(s+3+j)(s+2)}\right)\Bigg|_{s=-3-j}$$

$$= \frac{10}{(s+3-j)(s+2)}\Bigg|_{s=-3-j} = \frac{10}{(-3-j+3-j)(-3-j+2)} = -\frac{5}{2}-j\frac{5}{2}$$

and

$$R_3 = (s+2)\left(\frac{10}{(s+3-j)(s+3+j)(s+2)}\right)\Bigg|_{s=-2} = \frac{10}{s^2+6s+10}\Bigg|_{s=-2} = 5$$

Finally, using Eq. 14.4-6,

$$\mathscr{L}^{-1}[F(s)] = 2\,c\,e^{-at}\cos(bt) - 2\,d\,e^{-at}\sin(bt) + \mathscr{L}^{-1}[F_3(s)]$$

$$f(t) = \mathscr{L}^{-1}\left[\frac{10}{(s^2+6s+10)(s+2)}\right] = 2\,c\,e^{-at}\cos(bt) - 2\,d\,e^{-at}\sin(bt) + \mathscr{L}^{-1}\left[\frac{5}{s+2}\right]$$

$$= 2(-2.5)e^{-3t}\cos(1t) - 2(2.5)e^{-3t}\sin(1t) + 5e^{-2t}$$

$$= -5e^{-3t}\cos(t) - 5e^{-3t}\sin(t) + 5\,e^{-2t} \quad \text{for } t \geq 0$$

Example 14.4-2 (using formula method)

Using Eq. 14.4-5, we can express $F(s)$ as

$$F(s)=\frac{10}{(s^2+6s+10)(s+2)}=\frac{K_1\,s+K_2}{s^2+6s+10}+F_3(s)=\frac{K_1 s+K_2}{s^2+6s+10}+\frac{R_3}{s+2}$$

Using Eq. 14.4-3, we calulate

$$R_3=(s+2)\left(\frac{10}{(s^2+6s+10)(s+2)}\right)\bigg|_{s=-2}=\frac{10}{s^2+6s+10}\bigg|_{s=-2}=5$$

Then

$$\frac{10}{(s^2+6s+10)(s+2)}=\frac{K_1 s+K_2}{s^2+6s+10}+\frac{5}{s+2} \qquad (14.4\text{-}7)$$

Multiplying both sides of this equation by the denominator of $F(s)$ gives

$$10=(K_1+5)s^2+(2K_1+K_2+30)s+2K_2+50$$

The coefficients of s^2, s^1, and s^0 on the right side of this equation must each be equal to the corresponding coefficients on the left side. (The coefficients of s^2 and s^1 on the left side are zero.) Equating corresponding coefficients gives

$$0=K_1+5, \quad 0=2K_1+K_2+30 \quad \text{and} \quad 10=2K_2+50$$

Solving these equations gives $K_1=-5$ and $K_2=-20$. Substituting into Eq 14.4-7 gives

$$\frac{10}{(s^2+6s+10)(s+2)}=\frac{-5s-20}{s^2+6s+10}+\frac{5}{s+2}$$

Next,

$$\frac{-5s-20}{s^2+6s+10}=\frac{-5s-20}{(s^2+6s+9)+1}=\frac{-5s-20}{(s+3)^2+1}=\frac{-5(s+3)-5}{(s+3)^2+1}=-5\left(\frac{s+3}{(s+3)^2+1}\right)-5\left(\frac{1}{(s+3)^2+1}\right)$$

Then

$$\mathscr{L}^{-1}\left[\frac{-5s-20}{s^2+6s+10}\right]=-5\mathscr{L}^{-1}\left[\frac{s+3}{(s+3)^2+1}\right]-5\mathscr{L}^{-1}\left[\frac{1}{(s+3)^2+1}\right]$$
$$=-5e^{-3t}\cos(t)-5e^{-3t}\sin(t)$$

Using superposition,

$$f(t)=\mathscr{L}^{-1}\left[\frac{10}{(s^2+6s+10)(s+2)}\right]=-5e^{-3t}\cos(t)-5e^{-3t}\sin(t)+5e^{-2t}\text{ for } t\ge 0$$

as before.

Initial and final value theorems

Initial and final value theorems

The **initial value** of a function is the value of the function at t = 0.

If the function is discontinuous at t = 0, then the initial value is the value of the function just on the positive side of 0, i.e., 0^+

The initial value of a function can be found from the **initial value theorem**:

$$f(0+) = \lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s)$$

The **final value** of a function is the value of the function at t = ∞.

The final value theorem is given by:

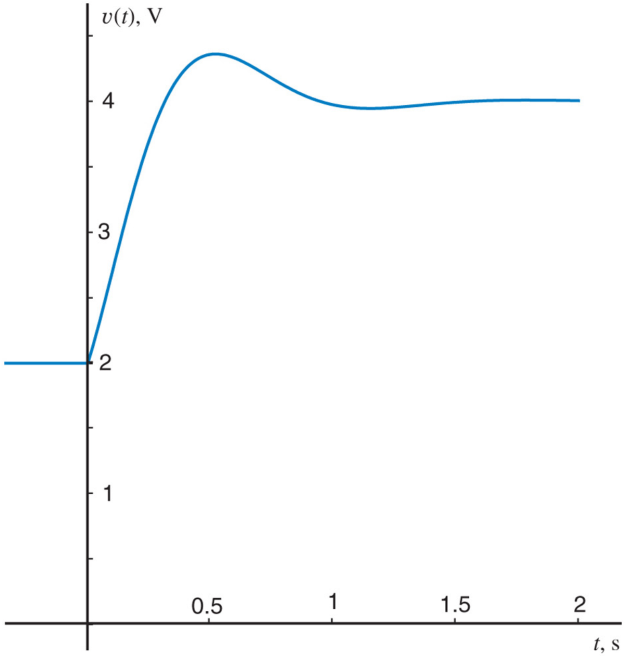
$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Example 14.5-1

Consider the situation in which we build a circuit in the laboratory and analyze the same circuit, using Laplace transforms. Figure 14.5-1 shows a plot of the circuit output $v(t)$ obtained by laboratory measurement. Suppose our circuit analysis gives

$$V(s) = \mathscr{L}[v(t)] = \frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)} \tag{14.5-3}$$

Does the circuit analysis agree with the laboratory measurement?



Example 14.5-1

Determining the inverse Laplace transform of $V(s)$ requires a partial fraction expansion. Before we do that work, let's use the initial- and final value theorems to see whether it is possible that $V(s)$, given in Eq. 14.5-3, can be the Laplace transform $v(t)$ shown in Figure 14.5-1.

From Figure 14.5-1, we see that the initial and final values are

$$v(0+) = \lim_{t \to 0+} v(t) = 2\text{ V} \quad \text{and} \quad v(\infty) = \lim_{t \to \infty} v(t) = 4\text{ V} \tag{14.5-4}$$

Next, we calculate

$$v(0) = \lim_{s \to \infty} s\left(\frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)}\right) = \lim_{s \to \infty} \frac{2s^2 + 30s + 136}{s^2 + 9s + 34} = \lim_{s \to \infty} \frac{\frac{2s^2}{s^2} + \frac{30s}{s^2} + \frac{136}{s^2}}{\frac{s^2}{s^2} + \frac{9s}{s^2} + \frac{34}{s^2}} = \frac{2}{1} = 2\text{ V}$$

and

$$v(\infty) = \lim_{s \to 0} s\left(\frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)}\right) = \lim_{s \to 0} \frac{2s^2 + 30s + 136}{s^2 + 9s + 34} = \frac{136}{34} = 4\text{ V}$$

Because these initial and final values agree, it is possible that $V(s)$, given in Eq. 14.5-3, can be the Laplace transform of $v(t)$ shown in Figure 14.5-1. It is now appropriate to determine the inverse Laplace transform of $V(s)$.

We can express $V(s)$ as

$$V(s) = \frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)} = \frac{K_1 s + K_2}{s^2 + 9s + 34} + \frac{R_3}{s}$$

where

$$R_3 = s\left(\frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)}\right)\bigg|_{s=0} = \frac{2s^2 + 30s + 136}{s^2 + 9s + 34}\bigg|_{s=0} = 4$$

Then

$$V(s) = \frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)} = \frac{K_1 s + K_2}{s^2 + 9s + 34} + \frac{4}{s}$$

Example 14.5-1

Multiplying both sides $s(s^2 + 9s + 34)$ gives

$$2s^2 + 30s + 136 = s(K_1 s + K_2) + 4(s^2 + 9s + 34) = (K_1 + 4)s^2 + (K_2 + 36)s + 136$$

Equating the coefficients of s^2 and s^1 gives $K_1 = -2$ and $K_2 = -6$. Then,

$$V(s) = \frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)} = \frac{4}{s} - \frac{2s + 6}{s^2 + 9s + 34} = \frac{4}{s} - \frac{2(s+3)}{(s+3)^2 + 25}$$

Taking the inverse Laplace transform gives

$$v(t) = \mathscr{L}^{-1}\left[\frac{4}{s} - \frac{2(s+3)}{(s+3)^2 + 25}\right] = 4 - 2\,e^{-3t}\cos(5t) \quad \text{for } \geq 0$$

which is indeed the equation representing the function shown in Figure 14.5-1.

Transients by the Laplace Transform Method

Switched DC circuits involving solution to the differential equations

The procedure

We can solve the set of differential equations describing a circuit using the following method:

1. Use Kirchoff's Laws and the element equations to obtain the differential equation or equations that describe the circuit.

2. Transform each differential equation into an algebraic equation by taking the Laplace transform of both sides of the equation.

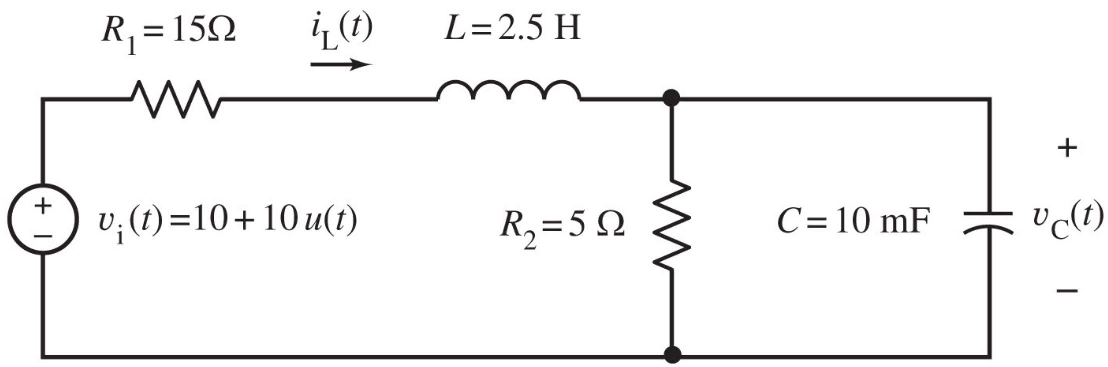
3. Solve the algebraic equations to obtain the Laplace transform of the output of the circuit.

4. Take the inverse Laplace transform to obtain the output of the circuit itself.

Time domain $\rightarrow$ LT $\rightarrow$ Frequency domain $\rightarrow$ ILT $\rightarrow$ Time domain

Example 14.6-1: Switched DC circuit

Consider the circuit:



We want to find $v_C(t)$ and we can calculate that $i_L(0^-) = 0.5$ A and $v_C(0^-) = 2.5$ V

Applying KCL at the top node:

$$i_L(t) = \frac{v_C(t)}{R_2} + C\frac{d\,v_C(t)}{dt}$$

Applying KVL at the left mesh:

$$v_i(t) = R_1 i_L(t) + L\frac{di_L(t)}{dt} + v_C(t)$$

Example 14.6-1

Now remembering that: $\dfrac{df}{dt} \leftrightarrow sF(s) - f(0^-)$

We take the Laplace transform of the KCL equation:

$$I_L(s) = \frac{V_C(s)}{R_2} + C(V_C(s)s - v_C(0-))$$

We take the Laplace transform of the KVL equation:

$$V_i(s) = R_1 I_L(s) + L(I_L(s)s - i_L(0-)) + V_C(s)$$

Now we are entirely in the frequency domain.

Example 14.6-1

We place the relationship for $I_L(s)$ into the equation for $V_i(s)$ to give:

$$V_\mathrm{i}(s) = \left(LCs^2 + \left(\frac{L}{R_2} + R_1C\right)s + 1 + \frac{R_1}{R_2}\right)V_\mathrm{C}(s) - (LCs + R_1C)v_\mathrm{C}(0-) - Li_\mathrm{L}(0-)$$

Noticing that $v_\mathrm{i} = 20$ V for $t > 0$, we determine $V_\mathrm{i}(s) = \mathscr{L}[20] = \frac{20}{s}$. Then, using the given values of the initial conditions and of the circuit parameters, we obtain

$$\frac{20}{s} = (s^2 + 26s + 160)V_\mathrm{C}(s) - (s+6)(2.5) - 2.5(0.5)$$

Solving for $V_\mathrm{C}(s)$ gives

$$V_\mathrm{C}(s) = \frac{2.5s^2 + 65s + 800}{s(s^2 + 26s + 160)} = \frac{2.5s^2 + 65s + 800}{s(s+10)(s+16)}$$

Performing partial fraction expansion gives

$$V_\mathrm{C}(s) = \frac{2.5s^2 + 65s + 800}{s(s+10)(s+16)} = \frac{5}{s} + \frac{4.17}{s+16} - \frac{6.67}{s+10}$$

Taking the inverse Laplace transform gives

$$v_\mathrm{C}(t) = 5 + 4.17e^{-16t} - 6.67e^{-10t}\ \mathrm{V}\ \text{for}\ t > 0$$

Complex frequency (s) domain impedance

Voltage and current for capacitors and inductors in the complex frequency domain

Circuit elements

Method: We convert each element of a circuit to the frequency (s) domain using the Laplace transform and then do the algebra.

Circuit Elements

Resistor:

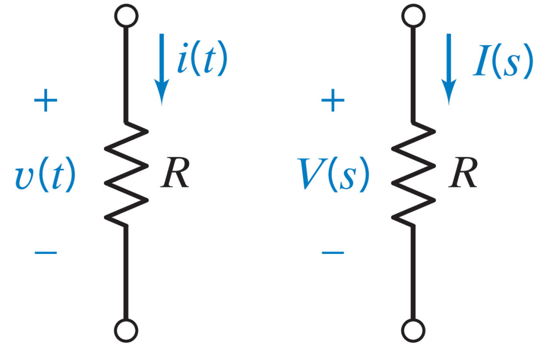
$$v(t) = i(t)R$$

The Laplace transform of this equation is:

$$V(s) = I(s)R$$

This can be seen in this figure

(a) = time domain
(b) = frequency domain



(a) (b)

Complex frequency (s) domain impedance

The impedance of an element is defined as: $$Z(s) = \frac{V(s)}{I(s)}$$

This impedance is defined in the complex frequency domain.

Of course, for a resistor, the impedance is equal to the resistance in the time domain.

Impedance of a capacitor

In the time domain, the voltage across a capacitor is given by:

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0)$$

We want to use Z(s)=V(s)/I(s) in order to find Z(s)

The Laplace transform of this equation (using the table) is given by:

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

This is the voltage across a capacitor in the frequency domain

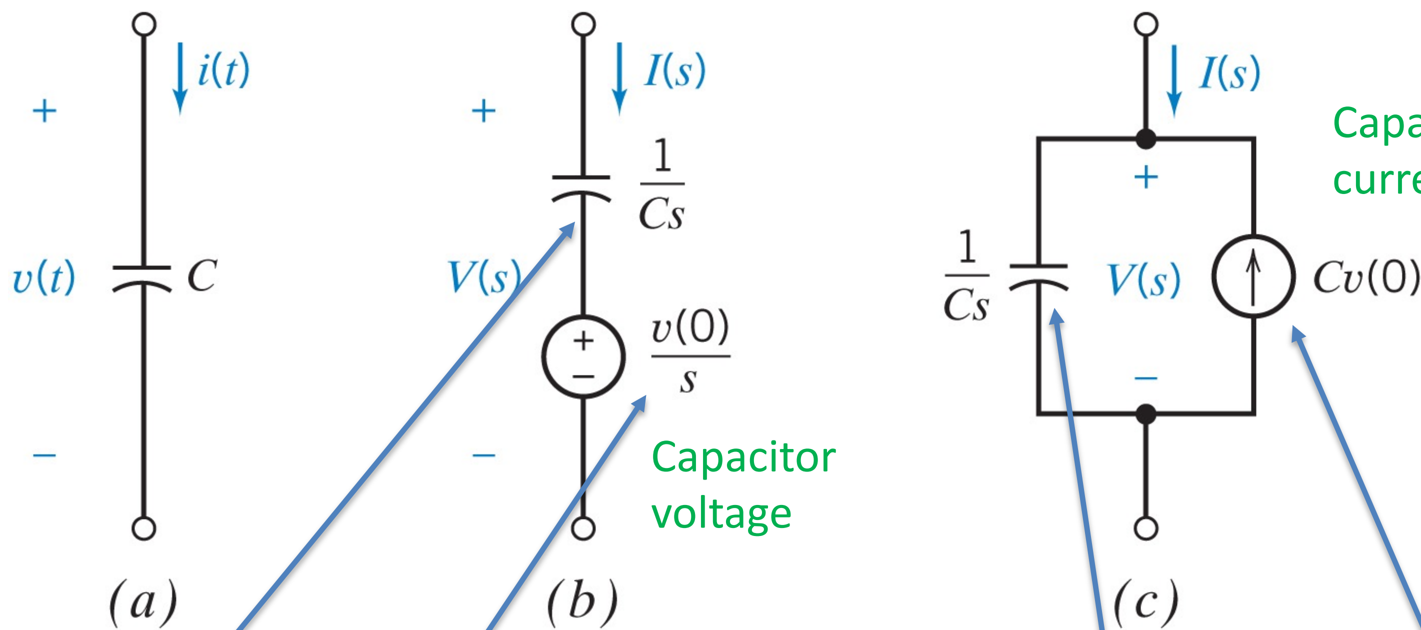
To determine the impedance, we set the initial voltage (i.e., initial condition) to zero and we obtain:

$V(s) = \frac{I(s)}{Cs}$ hence the frequency domain impedance is $Z_C(s) = \frac{1}{Cs}$

Capacitors in the complex frequency domain

Time domain:

Frequency domain:



Capacitor voltage

Capacitor current

$$V(s) = \frac{1}{Cs} I(s) + \frac{v(0)}{s}$$

Solving for I(s) gives: $I(s) = CsV(s) - Cv(0)$

In **series** because it is the sum of 2 voltages

In **parallel** because it is the sum of 2 currents (note polarity of second term)

Impedance of an inductor

In the time domain, the voltage across an inductor is given by:

$$v(t) = L\frac{d}{dt}i(t)$$

The Laplace transform of this equation is given by:

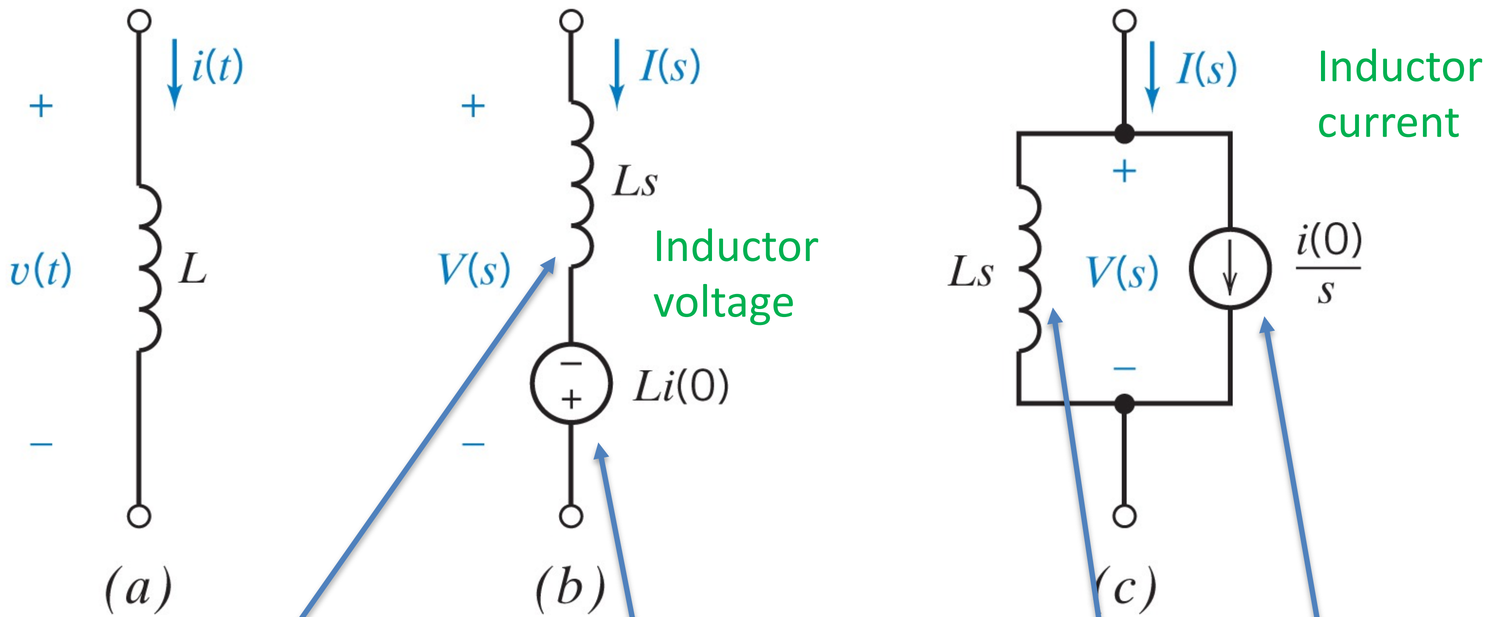
$$V(s) = LsI(s) - Li(0)$$

To determine the impedance, we set the initial current to zero and we obtain:

$V(s) = LsI(s)$ hence the impedance is $Z_L(s) = Ls$

Inductors in the frequency domain

Time domain: Frequency domain:



$$V(s) = LsI(s) - Li(0)$$

Solving for I(s) gives:

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

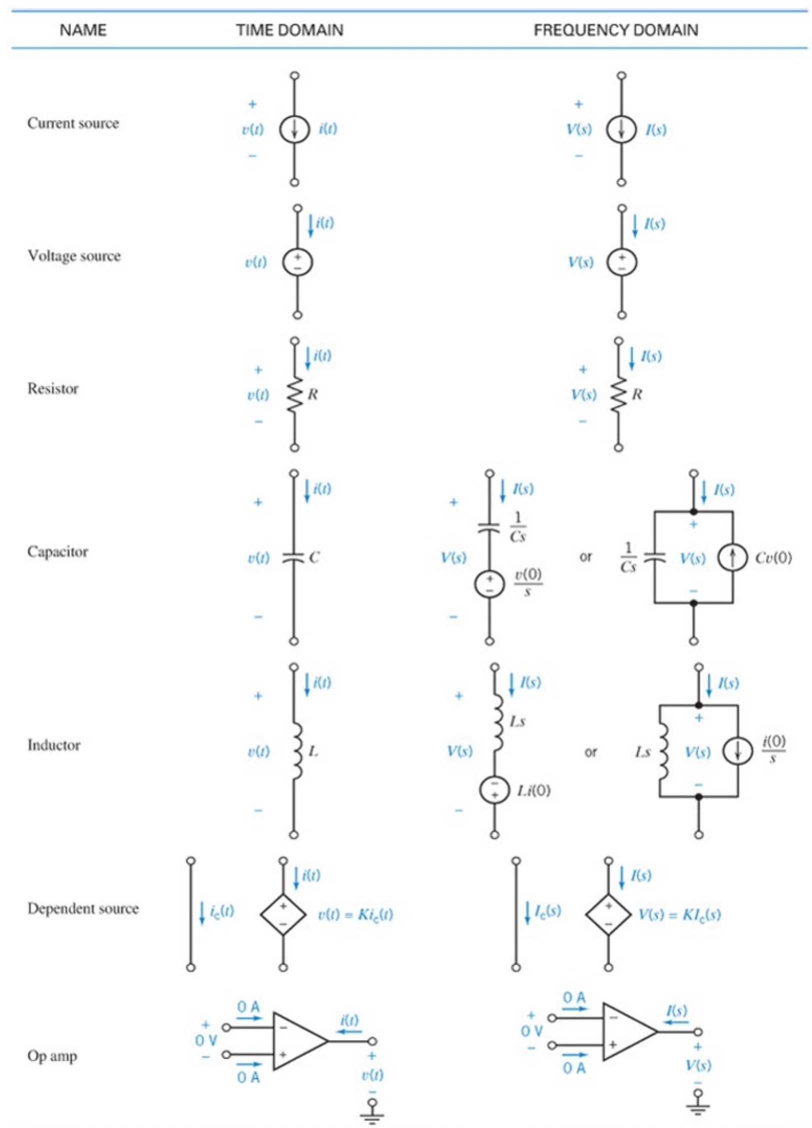
In **series** because it is the sum of 2 voltages (note polarity of second voltage term)

In **parallel** because it is the sum of 2 currents

Time and frequency domain of circuit elements

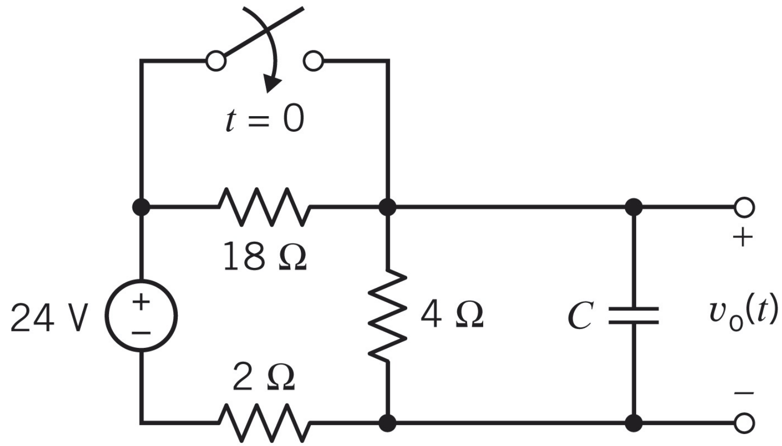
The **in series form** of capacitors and inductors is used when applying KVL (Mesh Equations)

The **in parallel form** of capacitors and inductors is used when applying KCL (Node Analysis)



Example 14.7-1

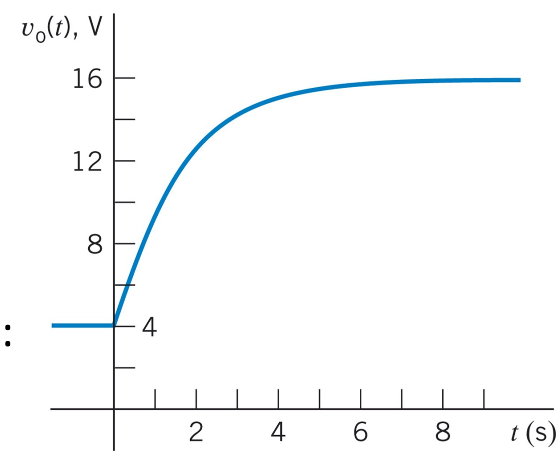
We have a circuit:



The input to the circuit is 24 V and the voltage across the capacitor is:

$$v_o(t) = 16 - 12e^{-0.6t}\ \text{V} \quad \text{when } t > 0$$

The voltage $v_c(t)$ looks like:

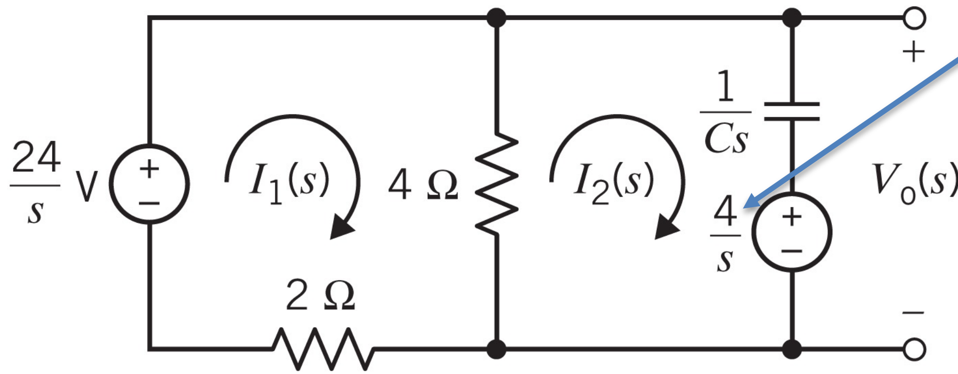


What is the value of capacitance C?

Example 14.7-1

How does the value for capacitance effect the capacitor voltage? (for t<0, $v_0(0)=4V$)

First, examine the circuit in the frequency domain for t>0:



Apply KVL to the left mesh to get

$$4(I_1(s) - I_2(s)) + 2I_1(s) - \frac{24}{s} = 0$$

Solving for $I_1(s)$ gives

$$I_1(s) = \frac{2}{3}I_2(s) + \frac{4}{s}$$

Apply KVL to the right mesh to get

$$\frac{1}{Cs}I_2(s) + \frac{4}{s} - 4(I_1(s) - I_2(s)) = 0$$

Collecting the terms involving $I_2(s)$ gives

$$\left(\frac{1}{Cs} + 4\right)I_2(s) = -\frac{4}{s} + 4I_1(s)$$

Substituting the expression for $I_1(s)$ from Eq. 14.7-11 gives

$$\left(\frac{1}{Cs} + 4\right)I_2(s) = -\frac{4}{s} + 4\left(\frac{2}{3}I_2(s) + \frac{4}{s}\right) = \frac{12}{s} + \frac{8}{3}I_2(s)$$

Example 14.7-1

Collecting the terms involving $I_2(s)$ gives

$$\left(\frac{1}{Cs}+\frac{4}{3}\right)I_2(s)=\frac{12}{s}$$

Multiply both sides of this equation by $\frac{3}{4}s$ to get

$$\left(s+\frac{3}{4C}\right)I_2(s)=9$$

Solving for $I_2(s)$ gives

$$I_2(s)=\frac{9}{s+\frac{3}{4C}}$$

The capacitor voltage is given by:

$$V_o(s)=\frac{1}{Cs}I_2(s)+\frac{4}{s}$$

Substituting the expression for $I_2(s)$ from Eq. 14.7-12 gives

$$V_o(s)=\left(\frac{1}{Cs}\right)\frac{9}{s+\frac{3}{4C}}+\frac{4}{s}=\frac{\frac{9}{C}}{s\left(s+\frac{3}{4C}\right)}+\frac{4}{s}$$

Partial fraction expansion gives:

$$V_o(s)=\frac{12}{s}-\frac{12}{s+\frac{3}{4C}}+\frac{4}{s}=\frac{16}{s}-\frac{12}{s+\frac{3}{4C}}$$

Taking the Laplace transform of the original voltage function:

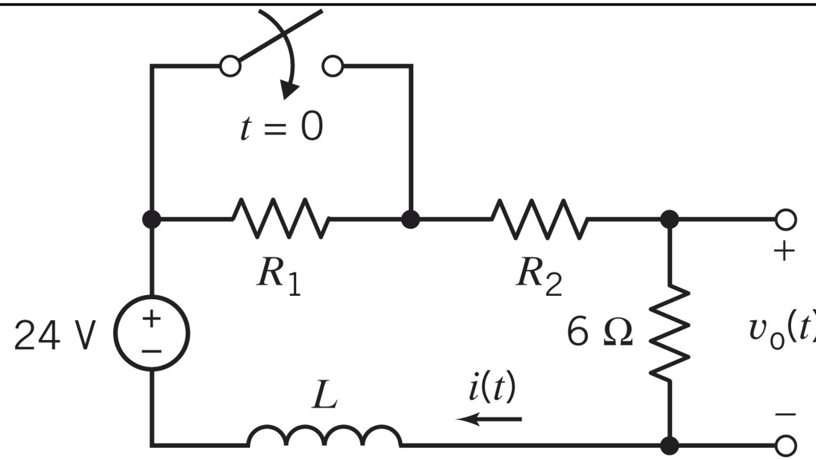
$$V_o(s)=\mathscr{L}[v_o(t)]=\mathscr{L}\left[\left(16-12e^{-0.6t}\right)u(t)\right]=\frac{16}{s}-\frac{12}{s+0.6}$$

Thus equating like terms gives:

$$0.6=\frac{3}{4C}\quad\Rightarrow\quad C=1.25\text{ F}$$

Example 14.7-2

Consider the circuit :



The voltage across the 6 Ω resistor is given by: $v_o(t) = 12 - 6\,e^{-0.35t}\ \text{V} \quad \text{when } t > 0$

Determine the resistances R_1 and R_2 and the inductance L.

For t<0, the inductor is a short circuit and the $v_0(t)$ equation above gives: $v_0(0) = 6$ V

Note also that voltage division tells us that $R_1 + R_2 = 18\ \Omega$

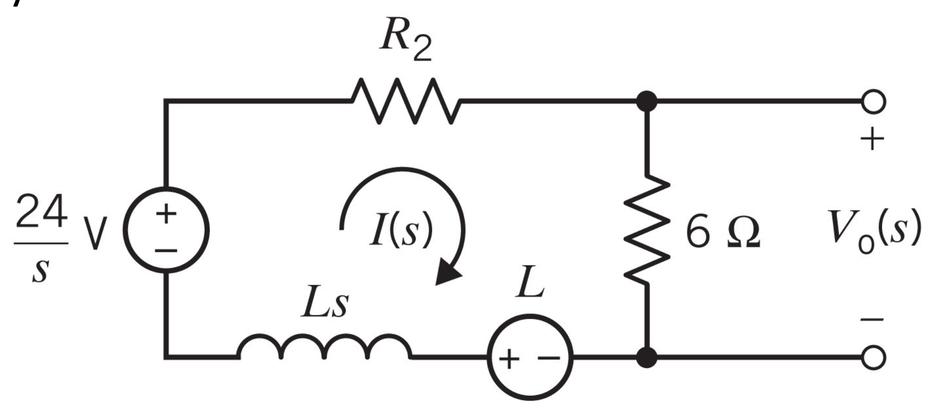
Example 14.7-2

For t>0, the current in the mesh is also the current through the 6 Ω resistor or

$$i(t) = \frac{v(t)}{6} = \frac{12 - 6e^{-0.35t}}{6} = 2 - e^{-0.35t}\ \text{A} \quad \text{when } t > 0$$

Hence, for t <= 0, i(0) = 1 A (but we knew that anyway)

For t>0, in the frequency domain the circuit looks like:



Since we will be applying KVL, we use the series form of the frequency domain inductor, hence we need the additional voltage source for the inductor.
The voltage for this source (at t=0) is:

$$Li(0) = (L)(1) = L$$

Example 14.7-2

Applying KVL gives:

$$(R_2 + 6 + Ls)I(s) = L + \frac{24}{s}$$

Solving for I(s) gives:

$$I(s) = \frac{L + \dfrac{24}{s}}{Ls + R_2 + 6} = \frac{s + \dfrac{24}{L}}{s\left(s + \dfrac{R_2 + 6}{L}\right)}$$

Using Ohm's Law gives:

$$V_o(s) = 6I(s) = \frac{6s + \dfrac{(6)(24)}{L}}{s\left(s + \dfrac{R_2 + 6}{L}\right)}$$

Partial fraction expansion gives:

$$V_o(s) = \frac{\dfrac{(6)(24)}{R_2 + 6}}{s} - \frac{\dfrac{6(18 - R_2)}{R_2 + 6}}{s + \dfrac{R_2 + 6}{L}}$$

Example 14.7-2

Taking the Laplace transform of the original voltage function gives:

$$V_o(s) = \mathscr{L}[v_o(t)] = \mathscr{L}\left[\left(12 - 6\,e^{-0.35t}\right)u(t)\right] = \frac{12}{s} - \frac{6}{s+0.35}$$

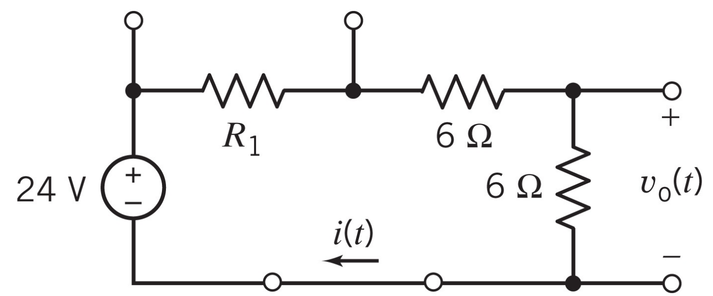
Comparing like terms gives:

$$\frac{(6)(24)}{R_2+6} = 12 \quad \Rightarrow \quad R_2 = 6\ \Omega$$

$$0.35 = \frac{R_2+6}{L} = \frac{12}{L} \quad \Rightarrow \quad L = \frac{12}{0.35} = 34.29\ \mathrm{H}$$

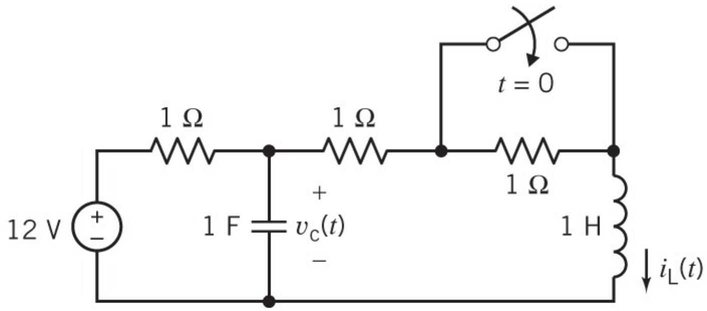
Since we have already determined $R_1 + R_2 = 18\ \Omega$

Then $R_1 = 12\ \Omega$



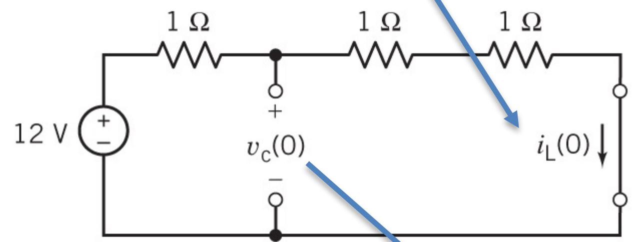
Example 14.7-3

Consider the circuit:



(a)

For t<0, $i_L(0)$ = 4A



(b)

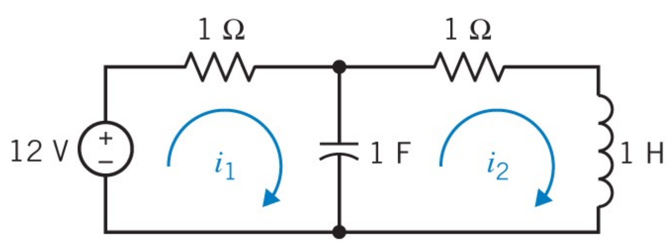
$v_c(0)$=8 V

We need both $v_c(0)$ AND $i_L(0)$ since we have a capacitor and an inductor in the circuit

Determine the inductor current $i_L(t)$ for t>0

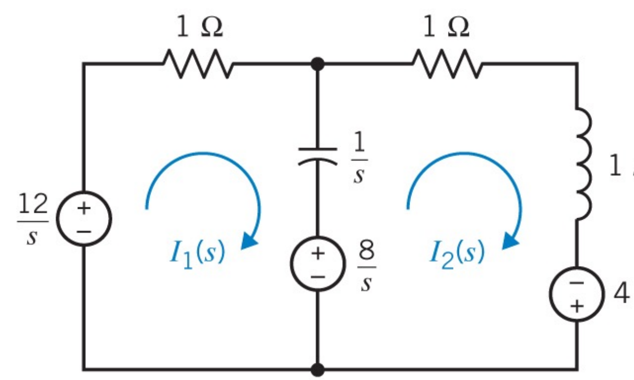
Since we will be using the mesh equations, we will use the **in series** version for the inductor and capacitor.

For t>0:



(a)

In the time domain



(b)

In the frequency domain

Example 14.7-3

The mesh equations are:

$$\left(1+\frac{1}{s}\right)I_1(s)-\frac{1}{s}I_2(s)=\frac{12}{s}-\frac{8}{s}$$

$$-\frac{1}{s}I_1(s)+\left(1+s+\frac{1}{s}\right)I_2(s)=4+\frac{8}{s}$$

Solving or $I_2(s)$:

$$I_2(s)=\frac{4(s^2+3s+3)}{s(s^2+2s+2)}$$

The partial fraction expansion:

$$\frac{I_2(s)}{4}=\frac{s^2+3s+3}{s(s^2+2s+2)}=\frac{A}{s}+\frac{Bs+D}{s^2+2s+2}$$

Calculating the residues gives:

$A=1.5$, $B=-0.5$, and $D=0$. Then, we can state

$$\frac{I_2(s)}{4}=\frac{1.5}{s}+\frac{-0.5s}{(s+1)^2+1}$$

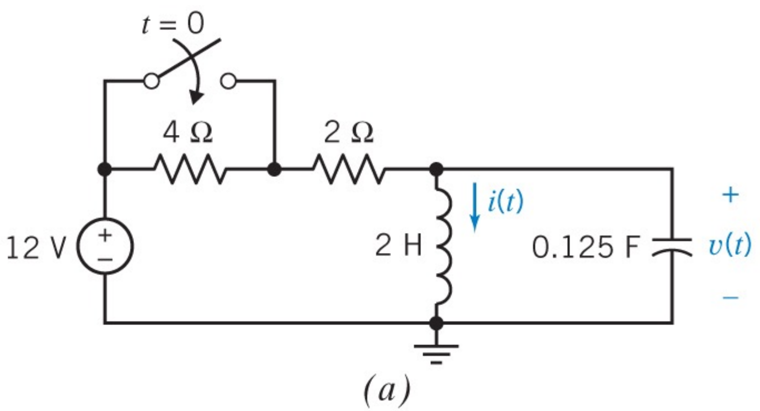
The Inverse Laplace transform gives:

$$i_L(t)=i_2(t)=\{6+2\sqrt{2}e^{-t}\sin(t-45^\circ)\}\text{ A}\quad\text{for}\quad t>0$$

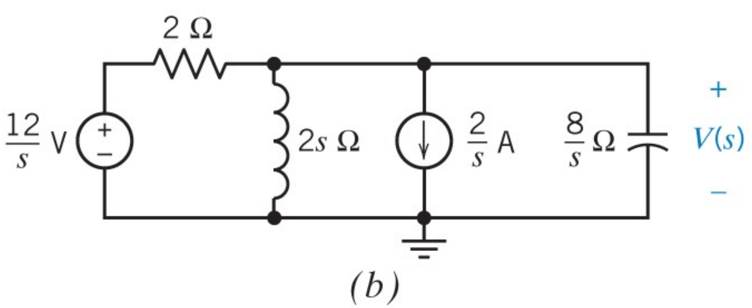
Checking: $i_2(0) = 6-2 = 4$ A and $i_2(\infty) = 6+0 = 6$ A

Example 14.7-4

Determine the voltage v(t) after the switch closes



For t>0 in the frequency domain:



For this example we will use **Node Analysis**, hence we use the **in parallel s domain impedance** form that involves currents for the inductor and capacitor.

For t<0, of course the current through the capacitor $i_c(0) = 0$ (since it is an open circuit) and the current through the inductor i(0) = 2 A (since it is a short circuit)

Example 14.7-4

KCL at the top node gives:

$$\frac{V(s)-\frac{12}{s}}{2}+\frac{V(s)}{2s}+\frac{2}{s}+\frac{V(s)}{\frac{8}{s}}=0$$

Solving for V(s) gives:

$$V(s)=\frac{32}{s^2+4s+4}=\frac{32}{(s+2)^2}$$

Taking the Laplace transform gives:

$$v(t)=\mathscr{L}^{-1}\left[\frac{32}{(s+2)^2}\right]=32te^{-2t}u(t)\text{ V}$$