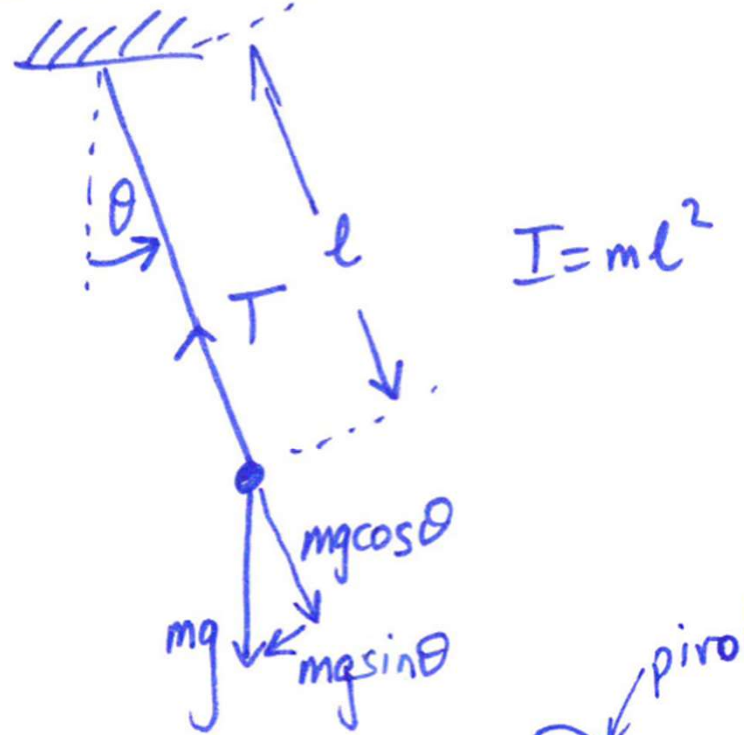
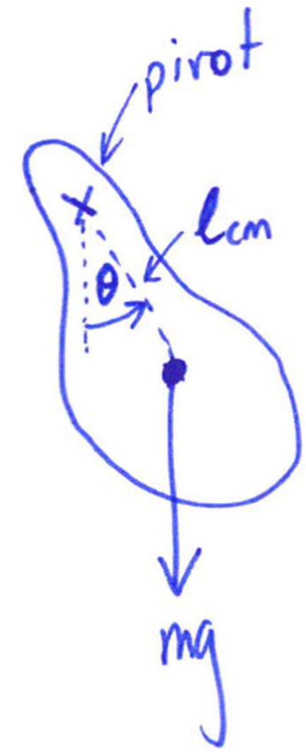
Pendulum

- Using $\tau = I\ddot{\theta}$ for a point mass
  - $-mgl\sin\theta = ml^2\ddot{\theta}$
  - $\ddot{\theta} = -(g/l)\theta$ for small angles
  - SHM with $\boldsymbol{\omega} = \sqrt{\boldsymbol{g/l}}$

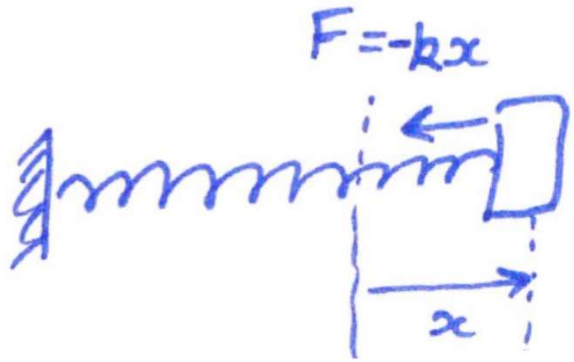


- For arbitrary shapes, we need to know:
  - mass m
  - distance from pivot to centre of mass l_{CM}
  - moment of inertia about the pivot I
  - $\boldsymbol{\omega} = \sqrt{\dfrac{\boldsymbol{mgl_{CM}}}{\boldsymbol{I}}}$



The energy route to SHM equation

$KE = \frac{1}{2} m\dot{x}^2$ and $PE = \frac{1}{2} kx^2$



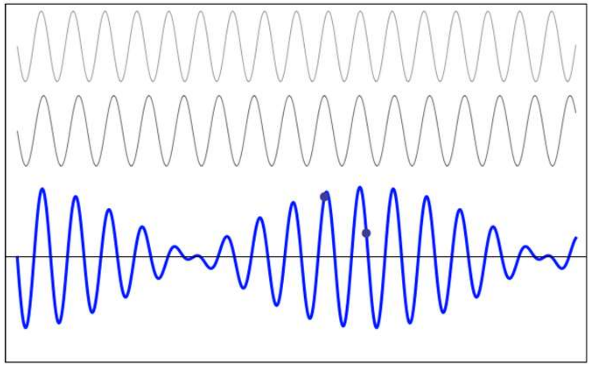
- $KE + PE = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$
- Differentiate wrt time:
    - $\frac{1}{2} m \, 2\dot{x} \, \ddot{x} + \frac{1}{2} k \, 2x \, \dot{x} = 0$ (chain rule!)
    - $m\dot{x}\ddot{x} + kx\dot{x} = 0$
    - $\ddot{x} = -(k/m)x,$ and so $\boldsymbol{\omega} = \sqrt{k/m}$

Superposition

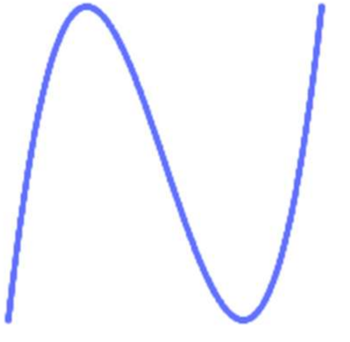
Prof David Spence

Simultaneous oscillations

- Sometimes a system vibrates in more than one frequency, and/or in more than one dimension
  - Oscillations at the same frequency in the same dimension
  - Oscillations at different frequency in the same dimension
    - BEATING

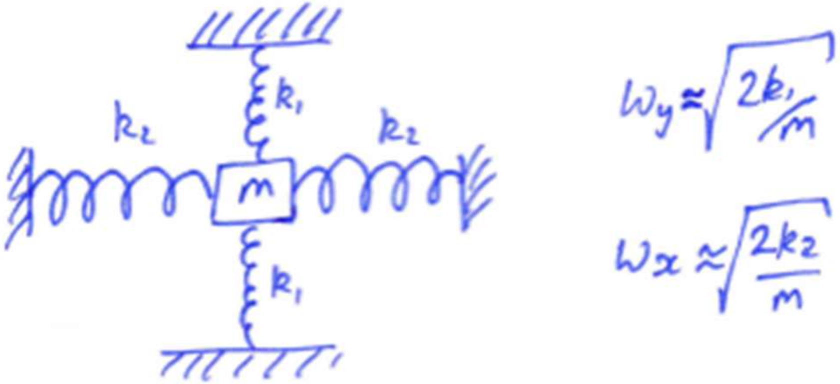


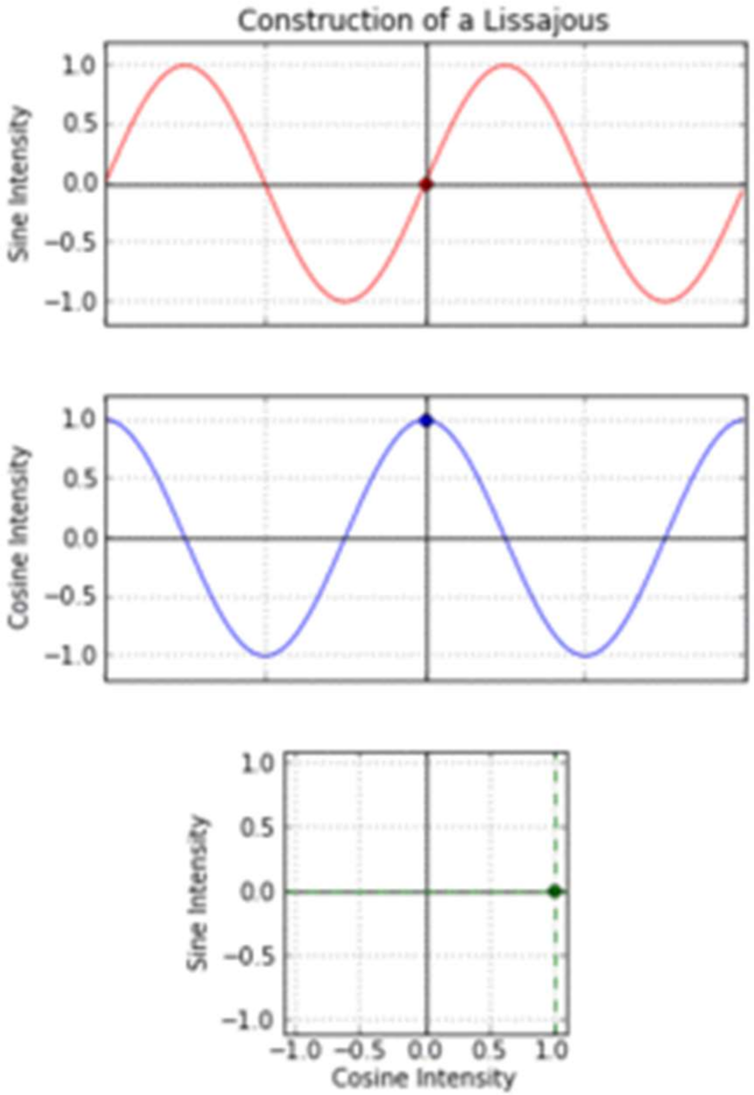
  - Oscillations in different dimensions
    - LISSAJOUS FIGURES



- SHM in two orthogonal directions creates patterns of motion, *Lissajous figures*, depending on
  - Ratio of frequencies
  - Phase offset (for rational ratios)

  - $x = a\sin(\omega_x t)$
  - $y = b\sin(\omega_y t + \delta)$

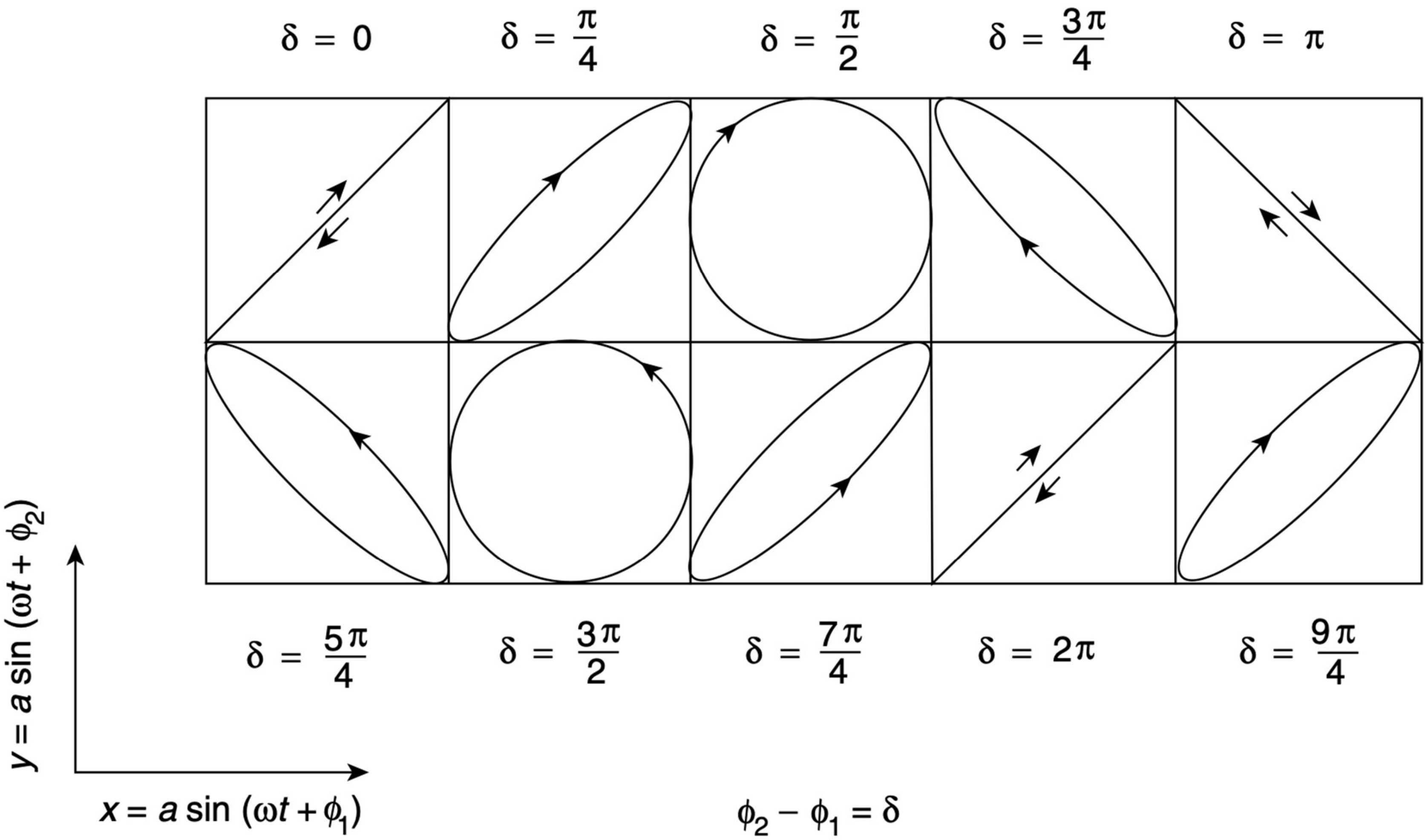




2D SHM with the same frequency

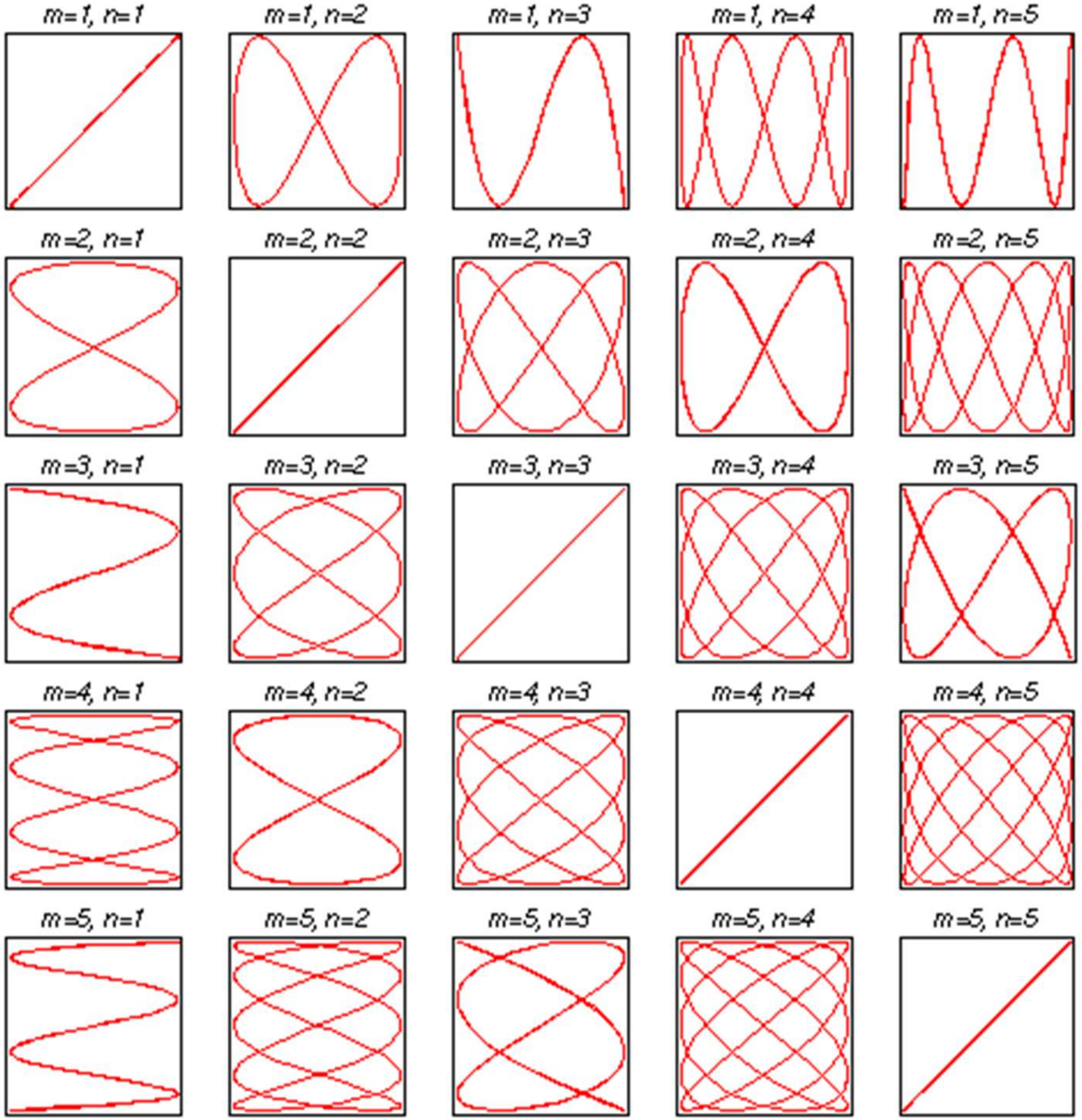
- Gives circle, ellipse, or line according to relative phase

$$x = a\sin(\omega t + \phi_1) \quad y = a\sin(\omega t + \phi_2) \quad \delta = \phi_2 - \phi_1$$



2D SHM with different frequencies

- Static figures when ratio of frequencies is a rational number
  - ‘rotation’ of the figure depends on phase
  - Ratio can be read off by counting peaks in each dimension



http://functions.wolfram.com/ElementaryFunctions/Sin/visualizations/11/ShowAll.html

Many animations on the web…

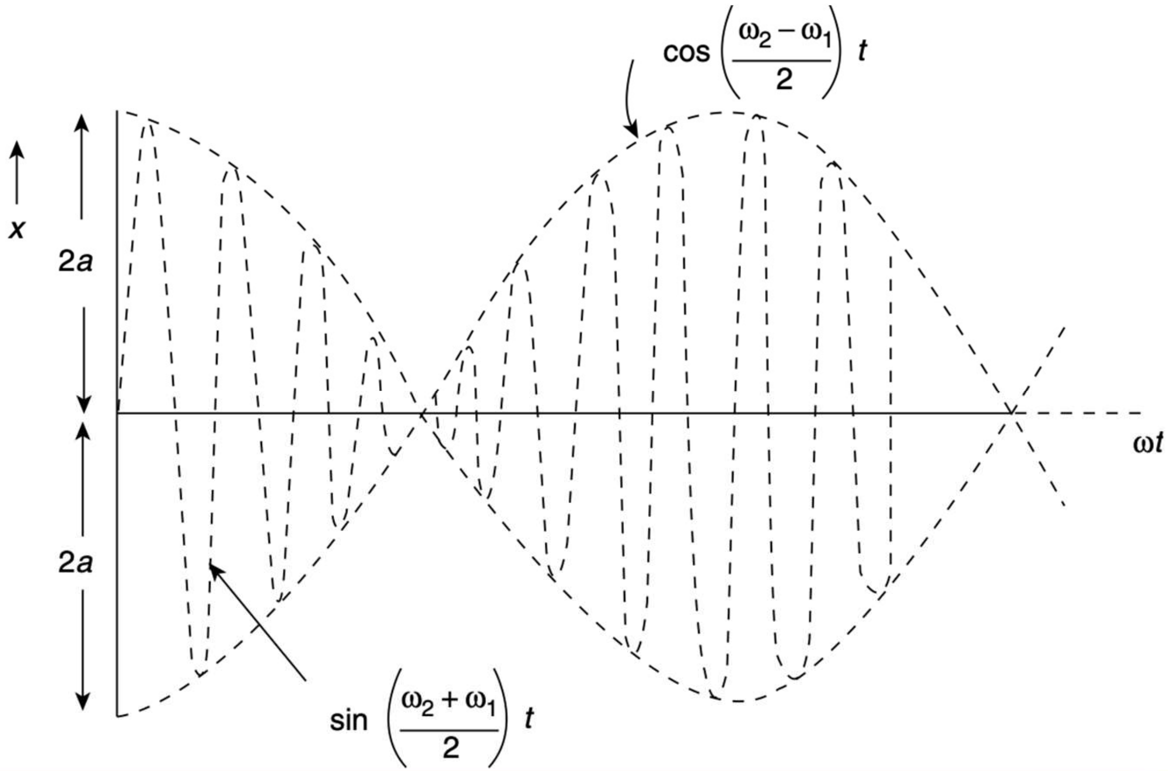
https://codepen.io/kotwgarnku/full/dMqKZG

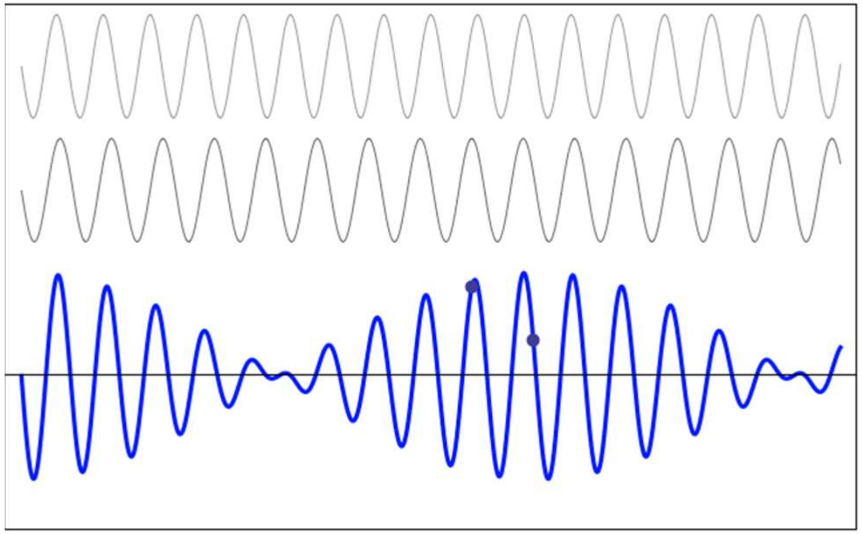
another example: http://jsxgraph.uni-bayreuth.de/wiki/index.php/Lissajous_curves

Beating

- Two simultaneous oscillations along the same axis create beating at the *difference frequency* $\omega_2 - \omega_1$
  - $x = a\sin(\omega_1 t) + a\sin(\omega_2 t)$

$$= 2a\sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right)t\right]\cos\left[\left(\frac{\omega_2 - \omega_1}{2}\right)t\right]$$

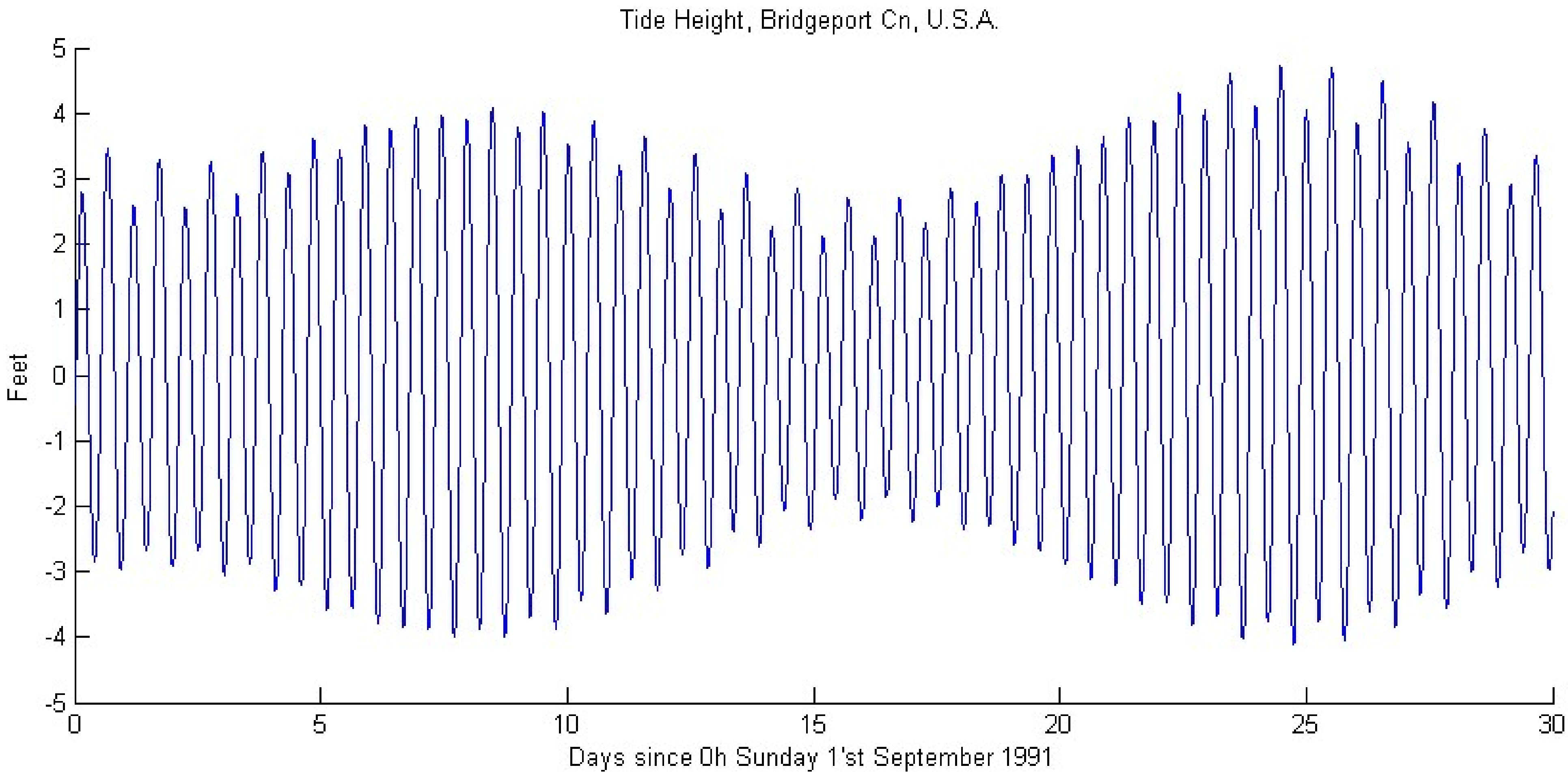




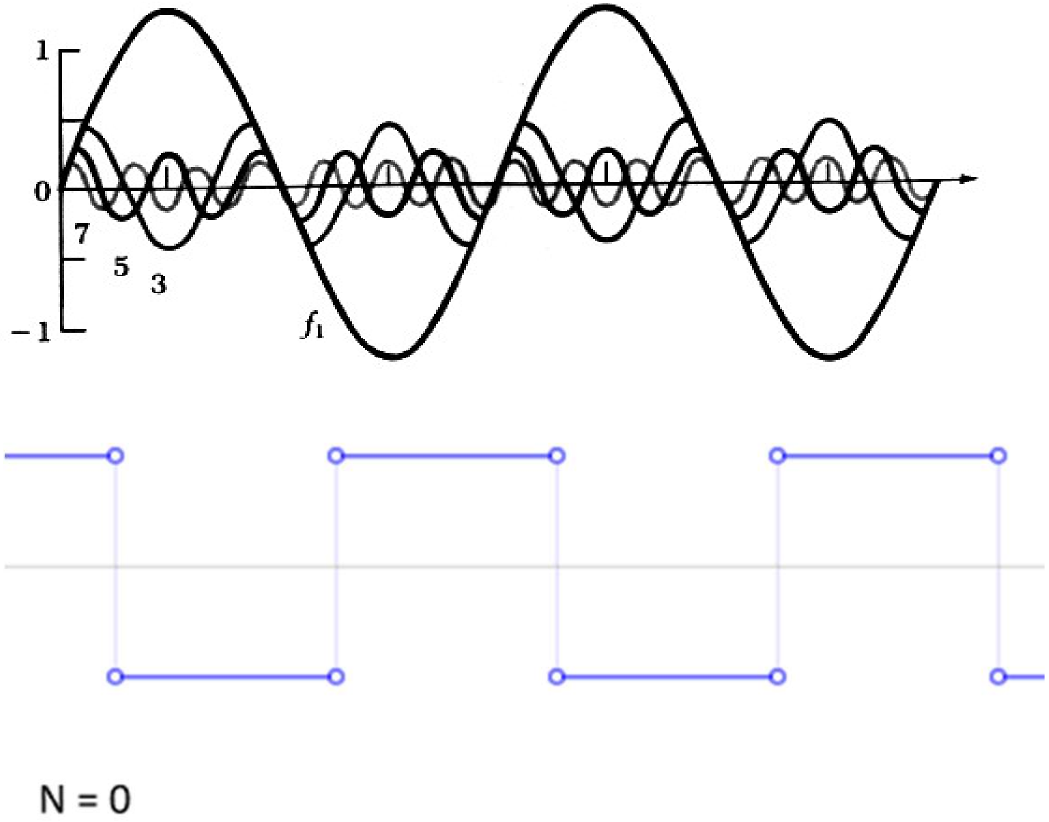
http://www.acs.psu.edu/drussell/Demos/superposition/superposition.html

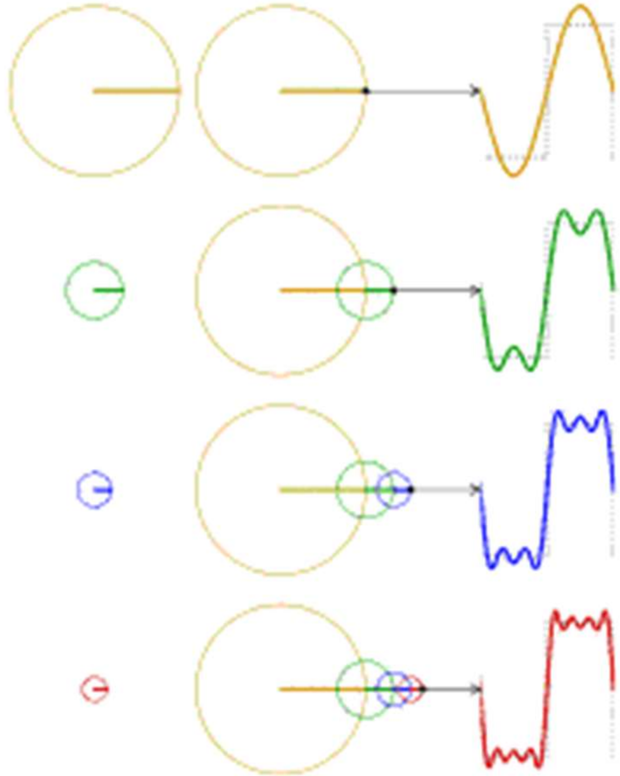
Tides show beating of sun and moon tides

- Two moon tides every ≈25 hours
- Two (smaller) sun tides every 24 hours



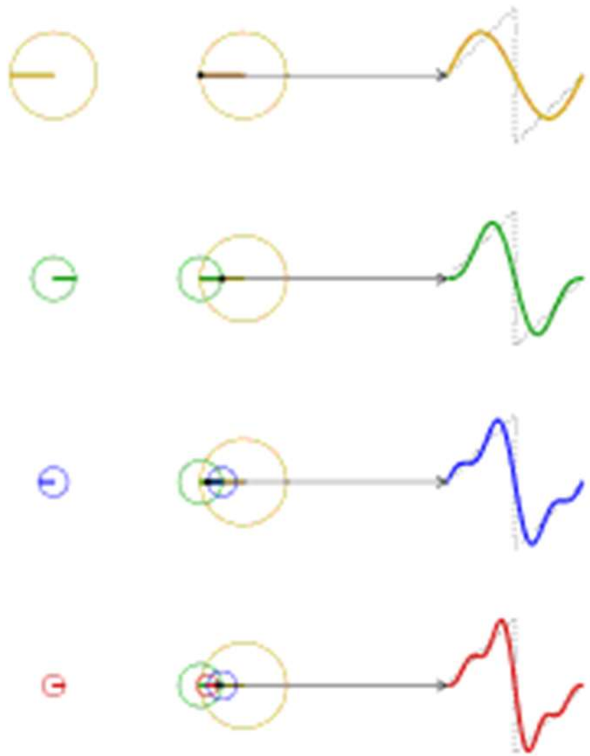
http://en.wikipedia.org/wiki/Arthur_Thomas_Doodson





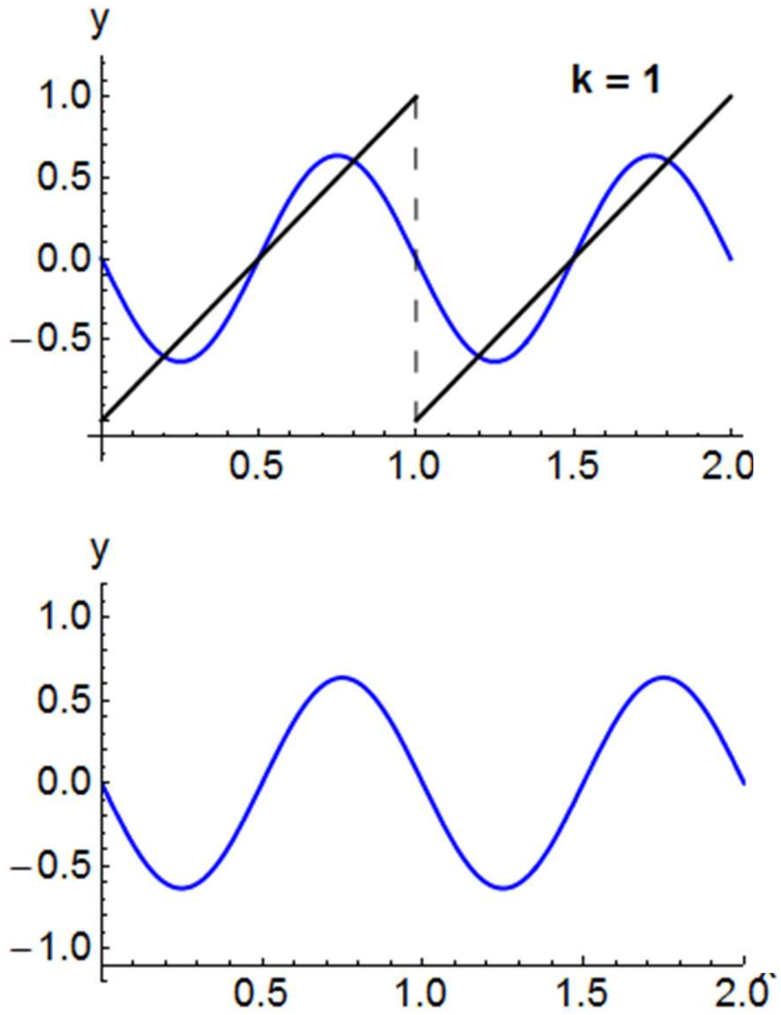
http://commons.wikimedia.org/

- Any periodic function can be represented by a sum of harmonic sinusoids with frequencies $N\omega = N.2\pi/T$
  - $f(t) = \frac{1}{2}a_0 + \sum_{N=1}^{\infty} a_N \cos N\omega t \; + \sum_{N=1}^{\infty} b_N \sin N\omega t$
  - e.g. for a square wave $f_{SQ}(t) = \frac{4}{\pi}\sum_{N=1,3,5..}^{\infty} \frac{1}{N} \sin N\omega t$



e.g. for a sawtooth wave

$$f_{SAW}(t) = \frac{1}{\pi}\sum_{N=1}^{\infty}\frac{1}{N}\sin N\omega t$$



http://commons.wikimedia.org/wiki/File:Sawtooth_Fourier_Animation.gif

Damped SHM

Prof David Spence

Drag

- Drag forces are complex resistive forces caused by motion through a fluid
  - Form drag
  - Skin drag
- Depend on shape, size, velocity, temperature….

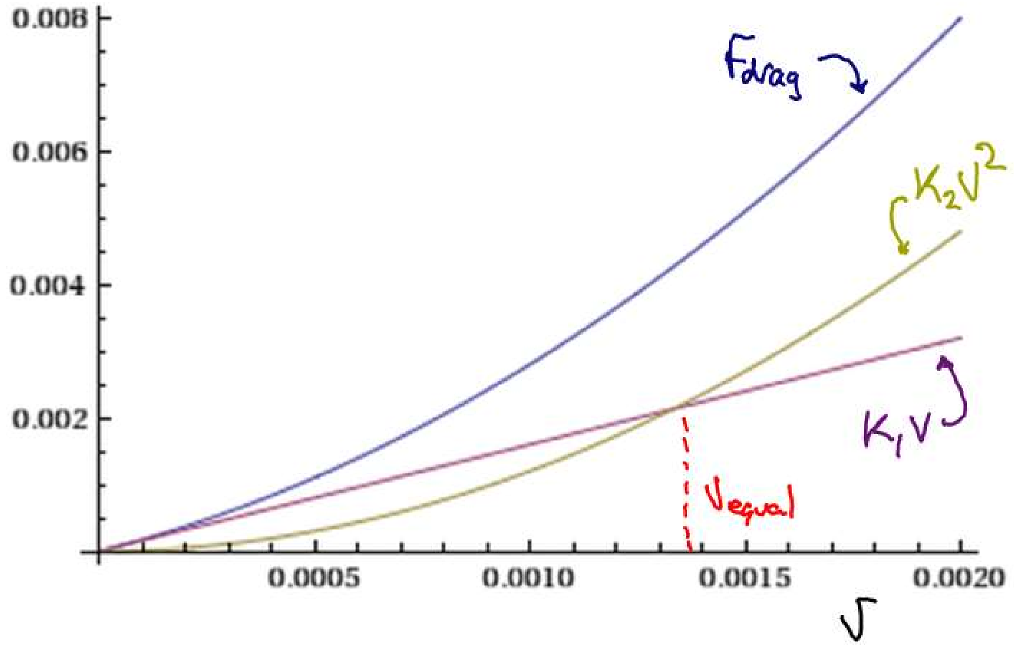
http://giphy.com/gifs/

Felix Baumgartner 2012

Drag depends on Reynolds number…

- Has a *viscous* term and a *pressure* term
  - Drag force $\vec{F}_{drag} = -k_1\vec{v} - k_2 v^2 \hat{v}$
  - Dissipated power $d/dt\left(-\vec{F}_{drag}.\vec{x}\right) = P_{drag} = k_1 v^2 + k_2 v^3$

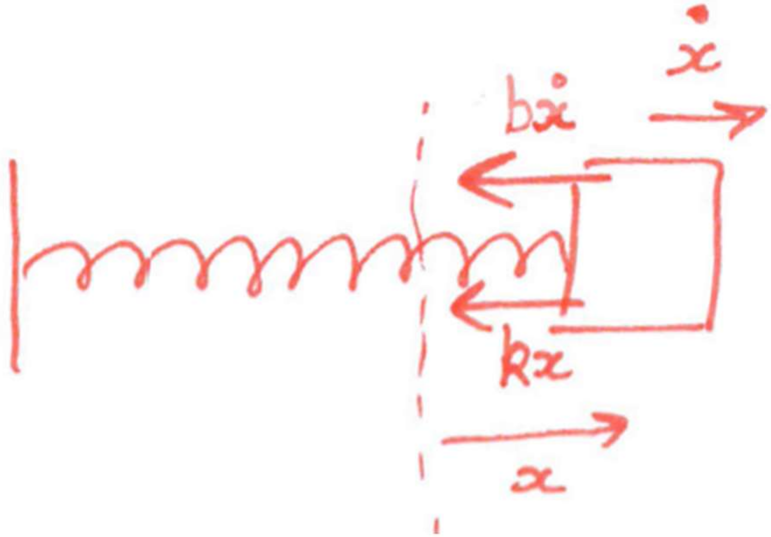
- Pressure term dominates at 'high' speed
  - Drag on a car on freeway or skydiver
- Viscous term dominates at 'low' speed
  - Falling through oil



Let's add a viscous drag term…

- Restoring force, $-kx$, resists *displacement*
- Drag force, $-b\dot{x}$, resists *velocity*

$$m\ddot{x} = -kx - b\dot{x}$$



$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

ω_0 is the SHM frequency the system would have if damping were absent

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

- Try a trial solution $x = Ce^{\alpha t}$

$\dot{x} = C\alpha e^{\alpha t}$ and $\ddot{x} = C\alpha^2 e^{\alpha t}$

- Substitute in to get $Ce^{\alpha t}(\alpha^2 + \gamma\alpha + \omega_0^2) = 0$
  - Trivial solution $C = 0$ or….
  - $\alpha^2 + \gamma\alpha + \omega_0^2 = 0$, with C an unknown constant

$$\Longrightarrow \quad \alpha = -\gamma/2 \pm (\gamma^2/4 - \omega_0^2)^{1/2}$$

$$x = Ce^{-\gamma t/2}\, e^{t\left(\gamma^2/4-\omega_0^2\right)^{1/2}} \text{ or } Ce^{-\gamma t/2}\, e^{-t\left(\gamma^2/4-\omega_0^2\right)^{1/2}}$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

$$x = \; C\, e^{-\gamma t/2}\, e^{t\left(\gamma^2/4-\omega_0^2\right)^{1/2}} \text{ or } \; C\, e^{-\gamma t/2}\, e^{-t\left(\gamma^2/4-\omega_0^2\right)^{1/2}}$$

- General solution is the sum $x = e^{-\frac{\gamma}{2}t}(A\, e^{\Omega t} + B\, e^{-\Omega t})$

  with A and B determined by initial conditions

- Behaviour depends on $\Omega = (\gamma^2/4 - \omega_0^2)^{1/2}$
  - Real, zero, or imaginary? $(\gamma > 2\omega_0\,,\ \gamma = 2\omega_0\,,\ \gamma < 2\omega_0)$

Case 1: $\gamma > 2\omega_0$ *overdamped*

- In this case $\Omega = (\gamma^2/4 - \omega_0^2)^{1/2}$ is real,

$$x = e^{-\frac{\gamma}{2}t}(A\, e^{\Omega t} + B\, e^{-\Omega t})$$

$\Omega < \gamma/2$ so both terms decay with time

A and B depend on boundary conditions

e.g., if $x(0) = 0$ and $\dot{x}(0) = v_0$ (passing through origin at $t = 0$) :

$$x(t) = Ae^{\left[-\frac{\gamma}{2}+\Omega\right]t} + Be^{\left[-\frac{\gamma}{2}-\Omega\right]t} \qquad \dot{x}(t) = A\left[-\frac{\gamma}{2}+\Omega\right]e^{\left[-\frac{\gamma}{2}+\Omega\right]t} + B\left[-\frac{\gamma}{2}-\Omega\right]e^{\left[-\frac{\gamma}{2}-\Omega\right]t}$$

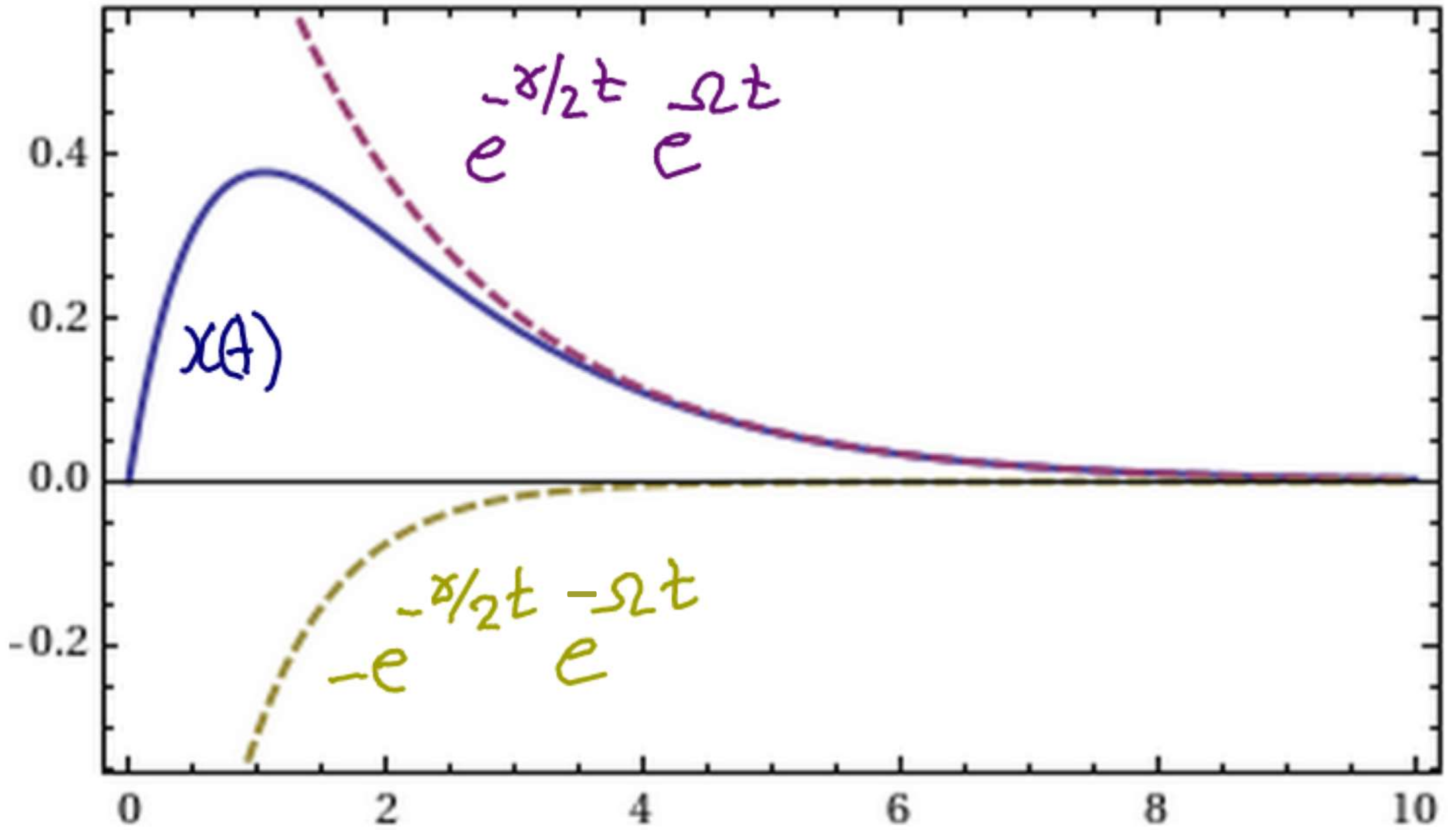
$$x(0) = A + B = 0 \qquad \dot{x}(0) = A\left[\cancel{-\frac{\gamma}{2}}+\Omega\right] + B\left[\cancel{-\frac{\gamma}{2}}-\Omega\right] = 2A\Omega = v_0$$

$$x(t) = \frac{v_0}{2\Omega}(e^{-\frac{\gamma}{2}t}\, e^{\Omega t} - e^{-\frac{\gamma}{2}t}e^{-\Omega t})$$

Overdamping and initial conditions

$$x(t) = \frac{v_0}{2\Omega}\left(e^{-\frac{\gamma}{2}t}\, e^{\Omega t} - e^{-\frac{\gamma}{2}t} e^{-\Omega t}\right)$$

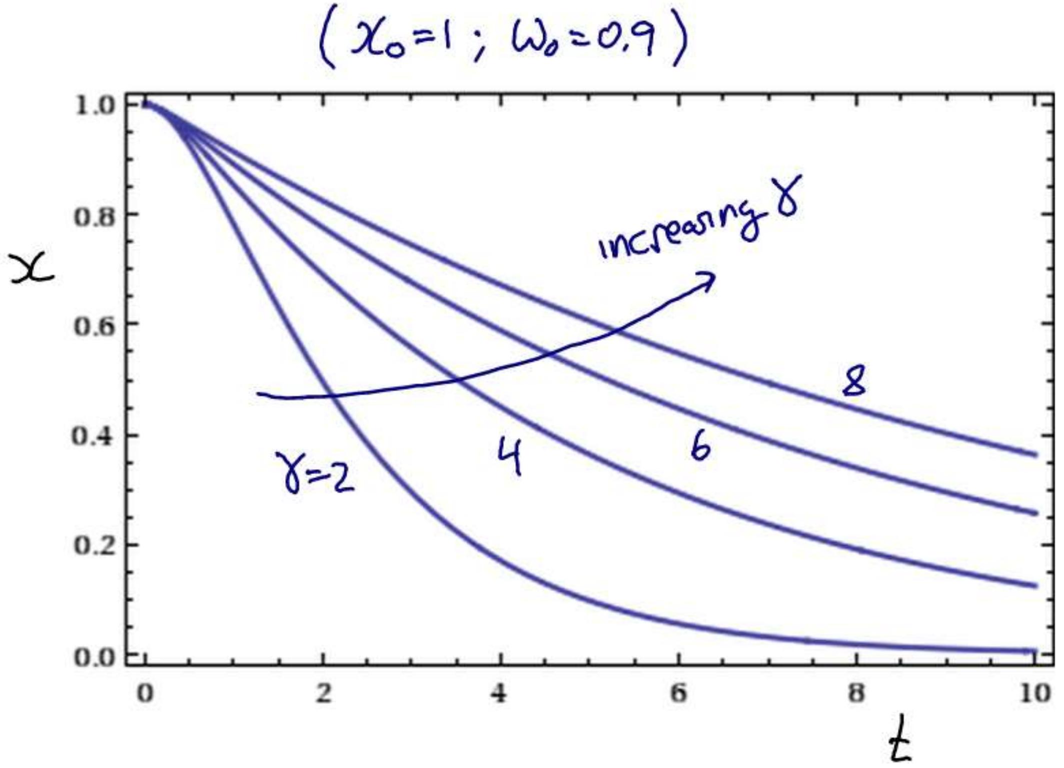
$(v_0 = 2\Omega\ ;\ \gamma = 2\ ;\ \omega_0 = 0{,}9)$



Overdamping and initial conditions

- If $x(0) = x_0$ and $\dot{x}(0) = 0$ (displaced, stationary)

$$x = x_0 e^{-\frac{\gamma}{2}t}\left(\left(\frac{1}{2}+\frac{1}{4}\frac{\gamma}{\Omega}\right)e^{\Omega t}+\left(\frac{1}{2}-\frac{1}{4}\frac{\gamma}{\Omega}\right)e^{-\Omega t}\right)$$



Critically damped SHM

Case 2: $\gamma = 2\omega_0$ *critically damped*

- In this case $\Omega = (\gamma^2/4 - \omega_0^2)^{1/2}$ is zero and the general solution becomes

$$x = Ae^{-\frac{\gamma}{2}t} + Be^{-\frac{\gamma}{2}t}$$

- This only has one free constant so can't be the general solution. In fact, it is

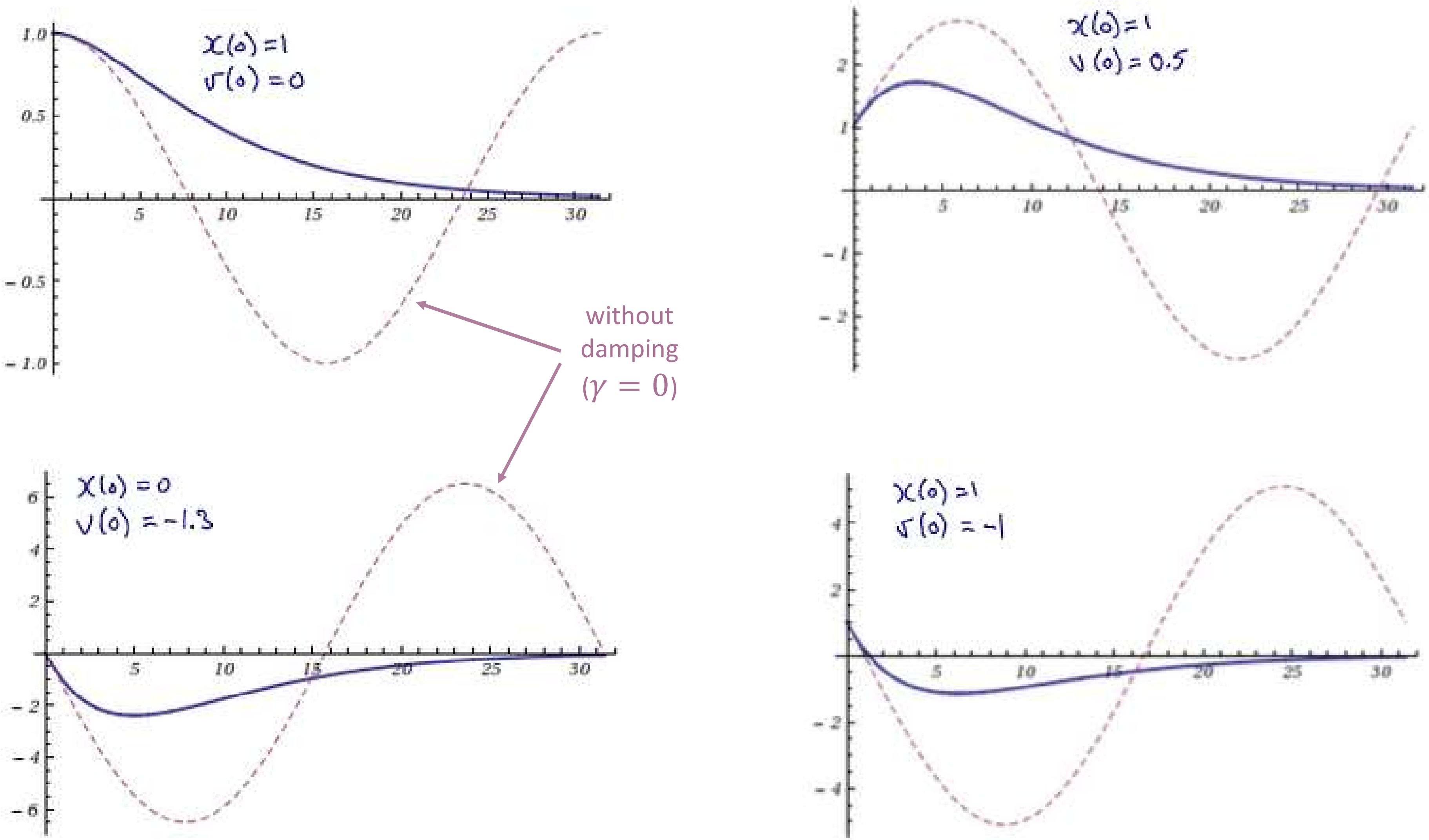
$$x = (A + Bt)\, e^{-\frac{\gamma}{2}t}$$

(try it!)

- A and B depend on initial conditions
  - $x(0) = A$, and $\dot{x}(0) = B - A\gamma/2$

Different critically damped motions

- E.g for $\gamma = 0.4$ and $\omega_0 = 0.2$:



Case 3: $\gamma < 2\omega_0$ *underdamped*

- In this case $\Omega = (\gamma^2/4 - \omega_0^2)^{1/2} = i\omega_d$ is imaginary, and the solution becomes

$$x = e^{-\frac{\gamma}{2}t}(A\, e^{\mathrm{i}\omega_d t} + B\, e^{-\mathrm{i}\omega_d t})$$

where $\omega_d = (\omega_0^2 - \gamma^2/4)^{1/2}$ is real

- The general solution needs to be real everywhere, constraining $B = A^*$

$$x = e^{-\frac{\gamma}{2}t}\mathrm{Re}[2A\, e^{\mathrm{i}\omega_d t}]$$

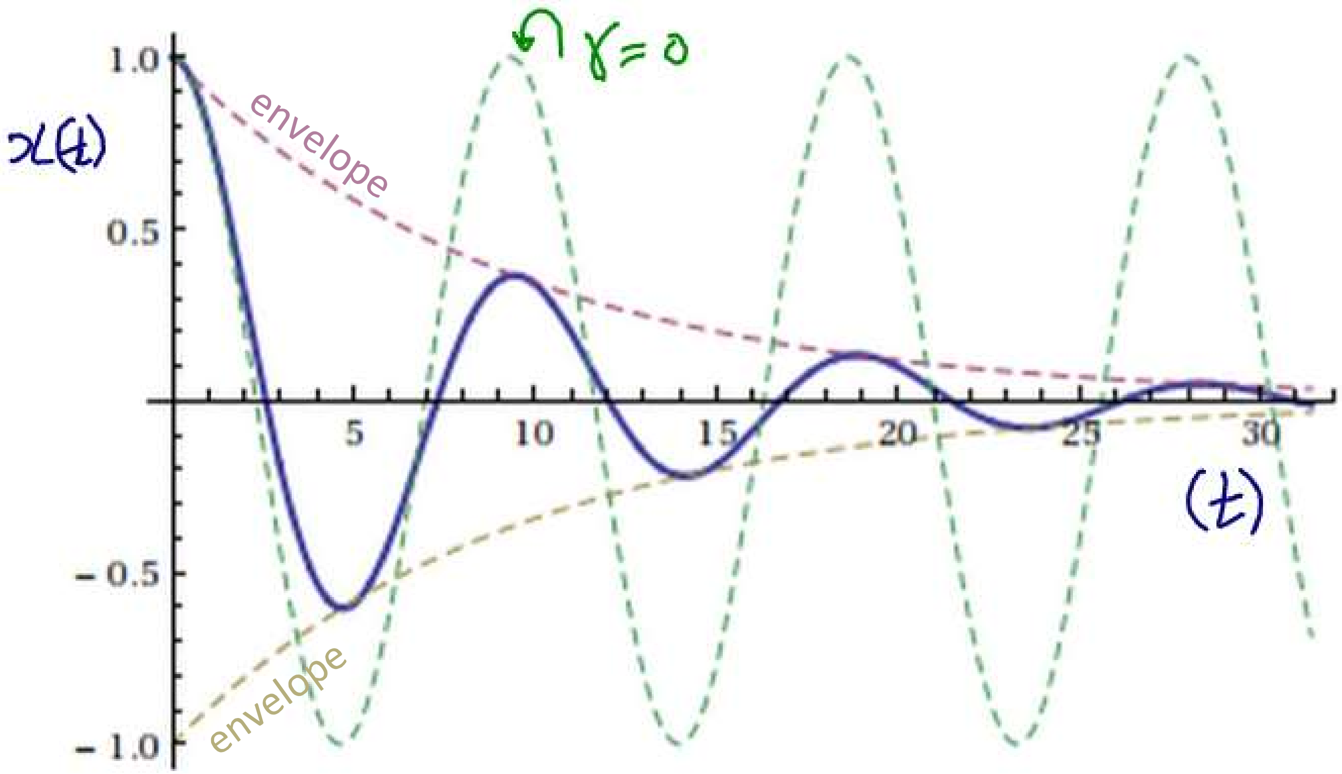
- A can be complex, representing amplitude and phase. Write $2A = Ce^{i\phi}$:

$$x = e^{-\frac{\gamma}{2}t}\mathrm{Re}[Ce^{\mathrm{i}(\omega_d t + \phi)}]$$

$$x = Ce^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi)$$

$$x = Ce^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi)$$

- $\omega_d = (\omega_0^2 - \gamma^2/4)^{1/2} < \omega_0$
- Constants C and ϕ determined by initial conditions



Demo: critical damping (d = 2 mm)

Background info: http://xmdemo.wordpress.com/068

https://youtu.be/99ZE2RGwqSM

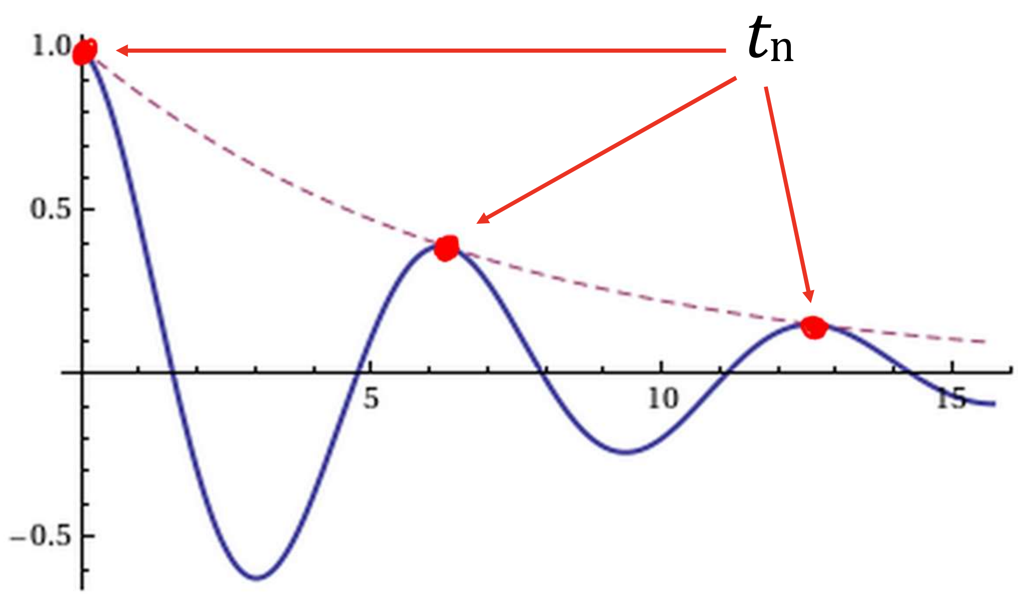
Logarithmic decrement

- Let's consider motion $x(t) = x_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t)$
- We can look at successive values where the cosine is maximised, at $t_n = n\ 2\pi/\omega_d = nT_d$

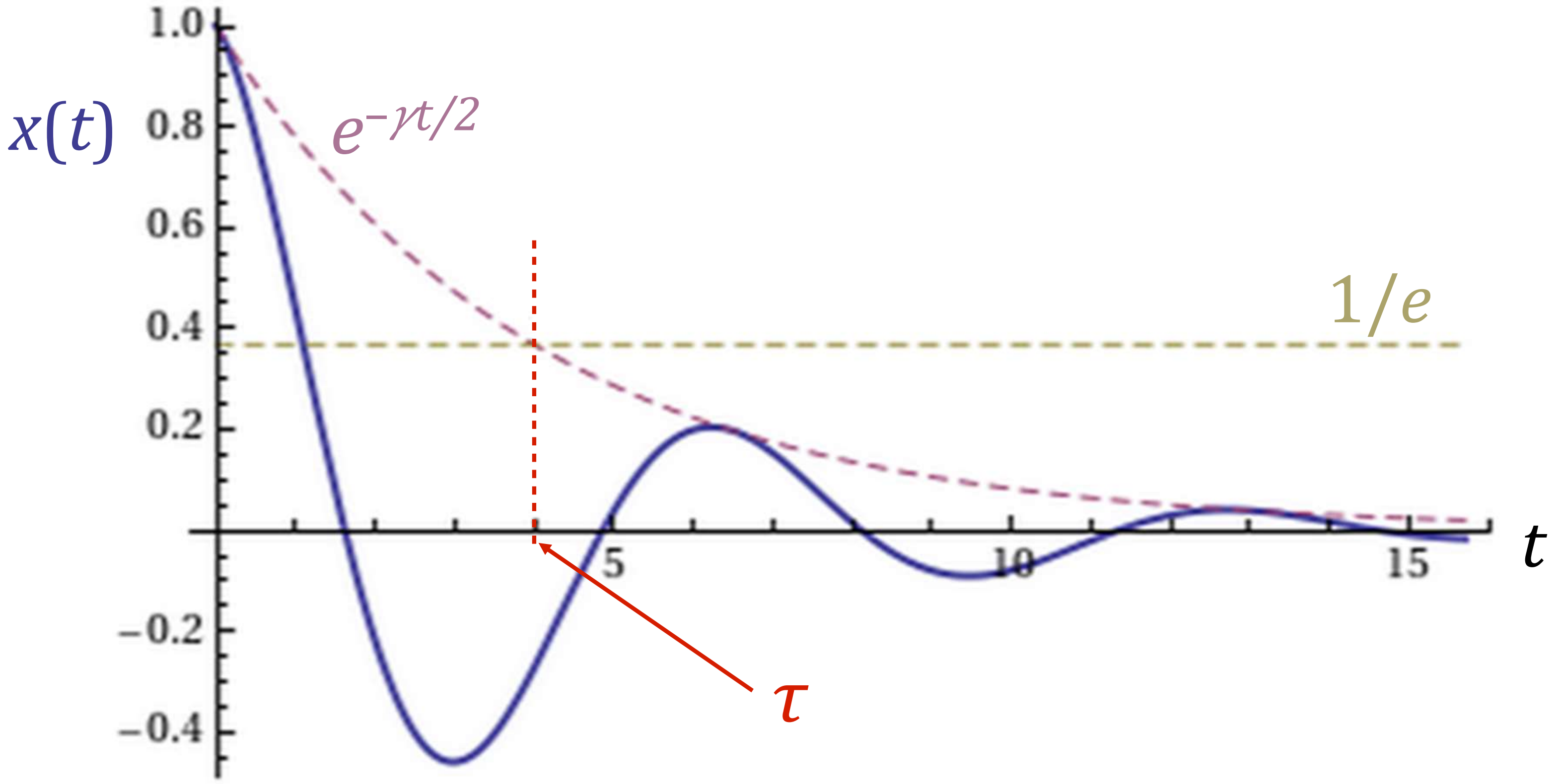
$$x(t_n) = x_0 e^{-(\gamma t_n/2)} = x_0 e^{-(\gamma n T_d/2)}$$

- Ratio between peaks n and $n+1$ $= e^{\delta} = e^{(\gamma T_d/2)}$

- δ is the *logarithmic decrement*



- Another way to characterise is the time τ taken for the amplitude to decay by a factor $e^{-1} \approx 0.37$



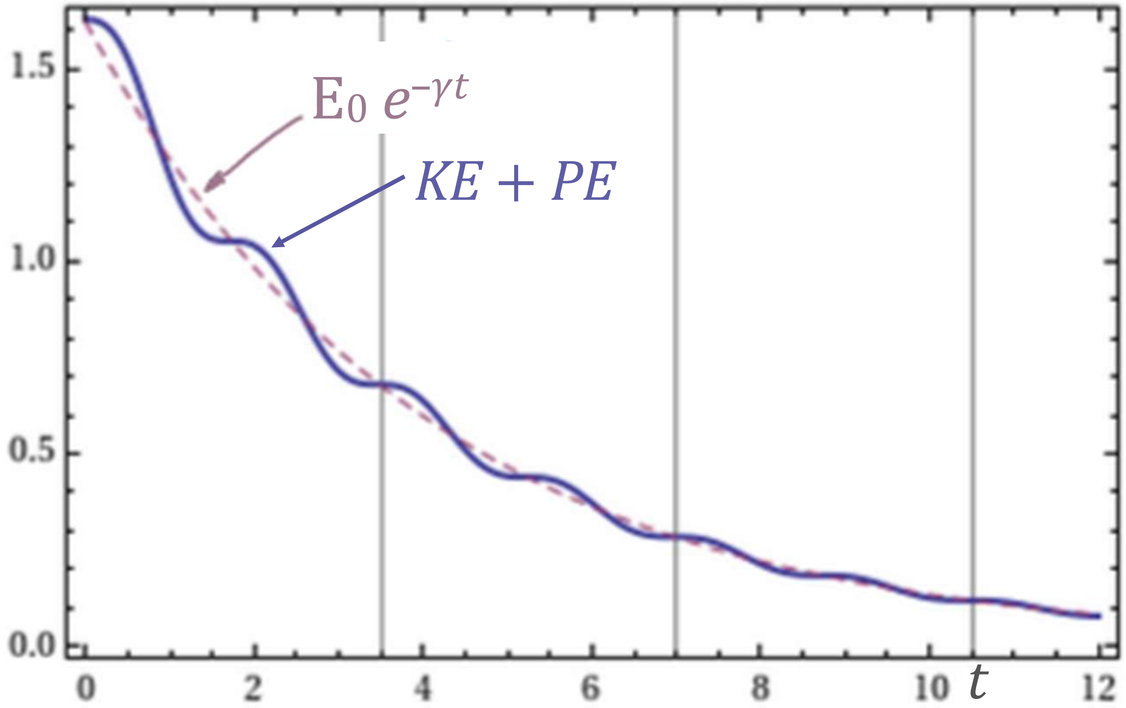
- From $x(t) = x_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t)$, we see $\tau = 2/\gamma$

Energy in an underdamped oscillator

- Consider $x(t) = x_0 e^{-\frac{\gamma}{2}t}\sin(\omega_d t)$

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2 e^{-\gamma t}\sin^2\omega_d t$$

$$KE = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mx_0^2 e^{-\gamma t}\left(\omega_d \cos\omega_d t - \frac{1}{2}\gamma \sin\omega_d t\right)^2$$



Energy and weak damping

$$x(t) = x_0 e^{-\frac{\gamma}{2}t} \sin(\omega_d t)$$

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2 e^{-\gamma t} \sin^2\omega_d t$$

$$KE = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mx_0^2 e^{-\gamma t}\left(\omega_d \cos\omega_d t - \frac{1}{2}\gamma \sin\omega_d t\right)^2$$

$$\approx \frac{1}{2}m\omega_d^2 x_0^2\, e^{-\gamma t} \cos^2\omega_d t \quad \text{for weak damping}$$

$$\approx \frac{1}{2}kx_0^2\, e^{-\gamma t} \cos^2\omega_d t$$

- Total mechanical energy $KE + PE = \frac{1}{2}kx_0^2\, e^{-\gamma t}$
  - Note decay constant γ not $\gamma/2$!
- Rate of energy loss is $dE(t)/dt = -\gamma E(t)$
  - Normalised loss rate $\frac{dE(t)/dt}{E(t)} = -\gamma$ is constant

Q factor

- Q-factor is a common way to quantify energy loss in periodic systems (we say 'high Q' or 'low Q')

- Q = Number of radians for energy to decay to $E_0 e^{-1}$
  - If $E(t) = E_0 e^{-\gamma t}$, then $E(\tau) = E_0 e^{-1}$ for $\tau = 1/\gamma$
  - If system oscillates with ω=$\omega_d \Rightarrow Q = \omega_d/\gamma$

- Alternatively Q = $2\pi \dfrac{Energy\ stored\ in\ system}{Energy\ lost\ per\ cycle}$
  - After one cycle, $t = 2\pi/\omega_d$, and $E = E_0 e^{-2\pi\gamma/\omega_d}$
  - Energy lost per cycle $dE \approx E_0 2\pi\gamma/\omega_d$ for small damping

$$Q = \omega_d/\gamma$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

$$x = Ae^{-\gamma t/2}\, e^{\Omega t} + Be^{-\gamma t/2}\, e^{-\Omega t}, \text{where } \Omega = (\gamma^2/4 - \omega_0^2)^{1/2}$$

Underdamped: $\gamma < 2\omega_0$

$$x = e^{-\frac{\gamma}{2}t}(A\, e^{i\omega_d t} + B\, e^{-i\omega_d t})$$

$\omega_d = (\omega_0^2 - \gamma^2/4)^{1/2}$ is real

Overdamped: $\gamma > 2\omega_0$

$$x = e^{-\frac{\gamma}{2}t}(A\, e^{\Omega t} + B\, e^{-\Omega t})$$

$\Omega < \gamma/2$ so both terms decay with time

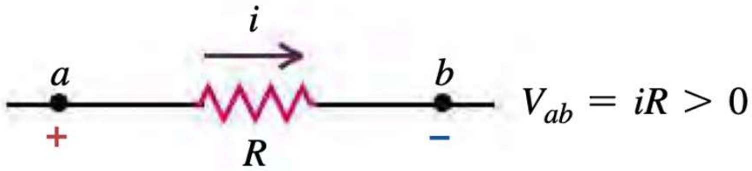
Critical damping: $\gamma = 2\omega_0$

$$x = (A + Bt)\, e^{-\frac{\gamma}{2}t}$$

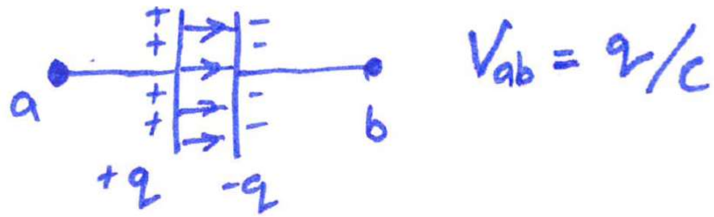
L-C-R circuits revision

- Resistor R obeys $V = IR \Rightarrow V = \dot{q}R$

(a) Resistor with current i flowing from a to b: potential drops from a to b.



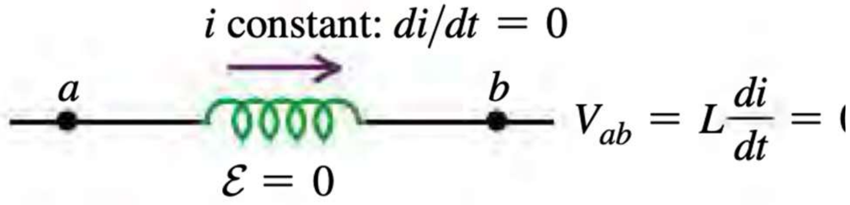
- Capacitor obeys $q = CV \Rightarrow V = q/C$
  - Stores electric potential energy $\frac{1}{2}q^2/C$



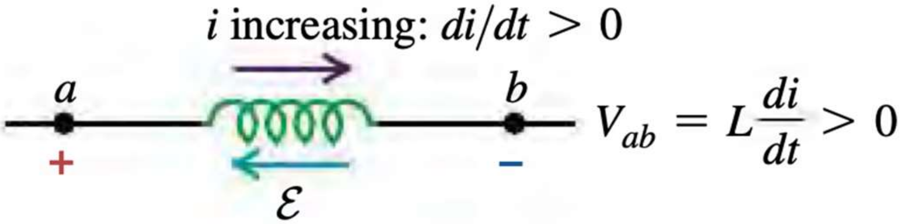
- Inductor obeys $V = LdI/dt$

  $\Rightarrow V = L\ddot{q}$
  - Stores magnetic potential energy $\frac{1}{2}L\dot{q}^2$

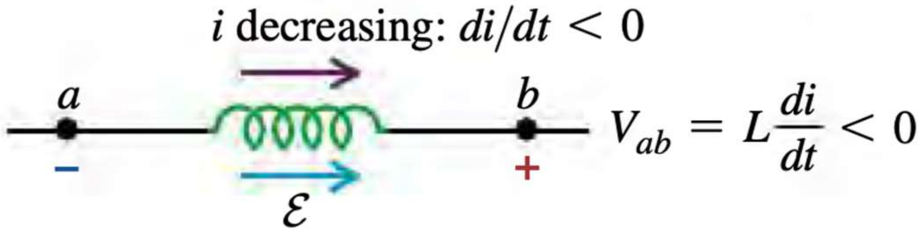
(b) Inductor with *constant* current i flowing from a to b: no potential difference.



(c) Inductor with *increasing* current i flowing from a to b: potential drops from a to b.

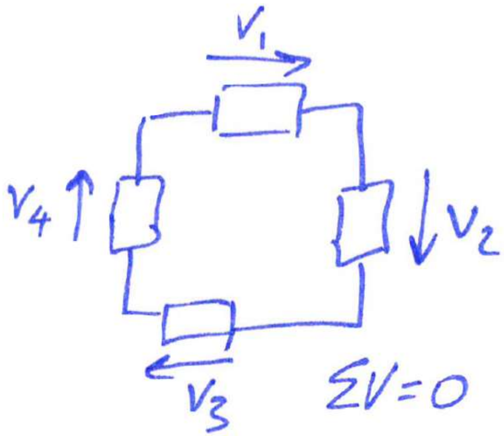


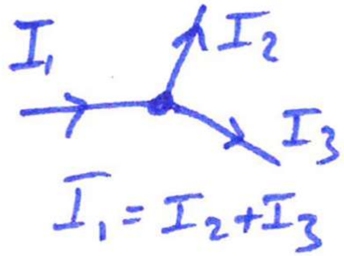
(d) Inductor with *decreasing* current i flowing from a to b: potential increases from a to b.



Electronic SHM

- Circuits obey Kirchoff's Laws
  - $\sum V = 0$ round a loop
  - $\sum I = 0$ into a node





- LCR circuits undergo simple harmonic motion

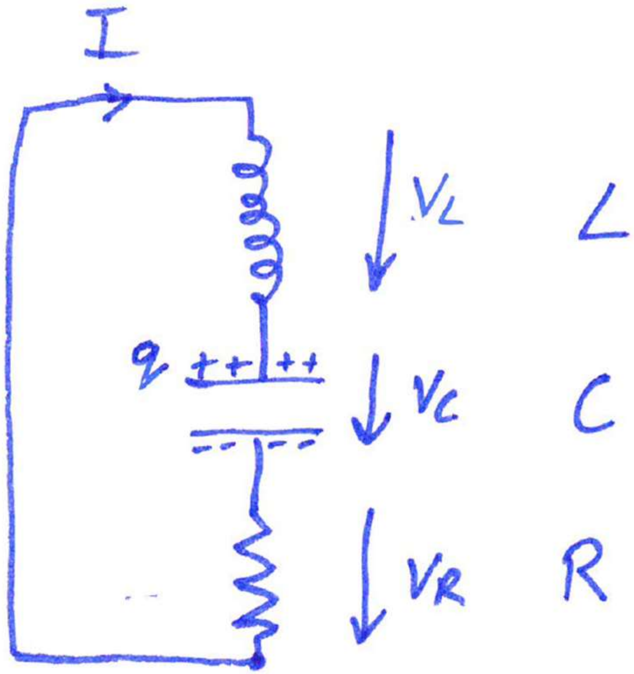
$$L\ddot{q} + \dot{q}R + q/C = 0$$

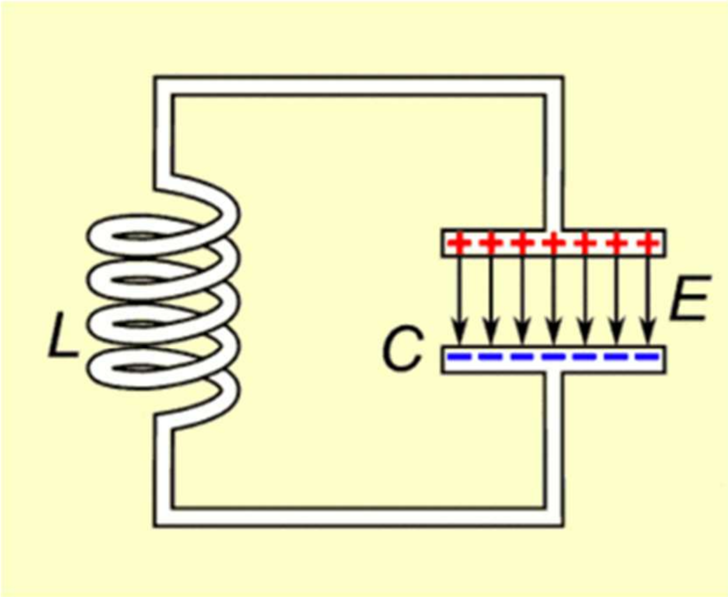
$$\Rightarrow \ddot{q} + \dot{q}R/L + q/(LC) = 0$$

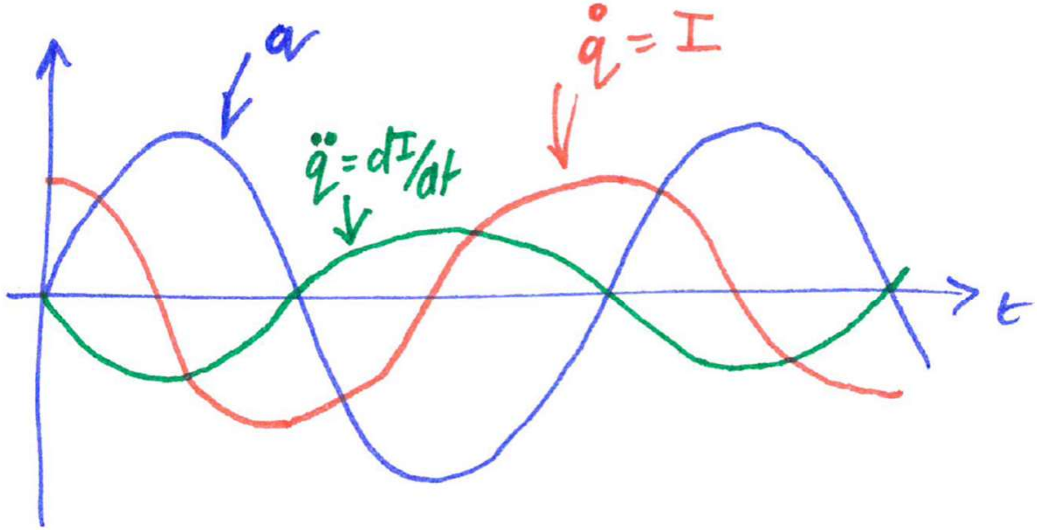
cf. $\ddot{x} + \gamma\dot{x} + \omega_0^2\, x$

Charge q oscillates with SHM,

$\omega_0 = (LC)^{-1/2}$, and damping $\gamma = R/L$







- $\ddot{q} + q/LC = 0$

$$\omega_0^2 = 1/LC$$

$$E = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}q^2/C$$

Magnetic and electric potential

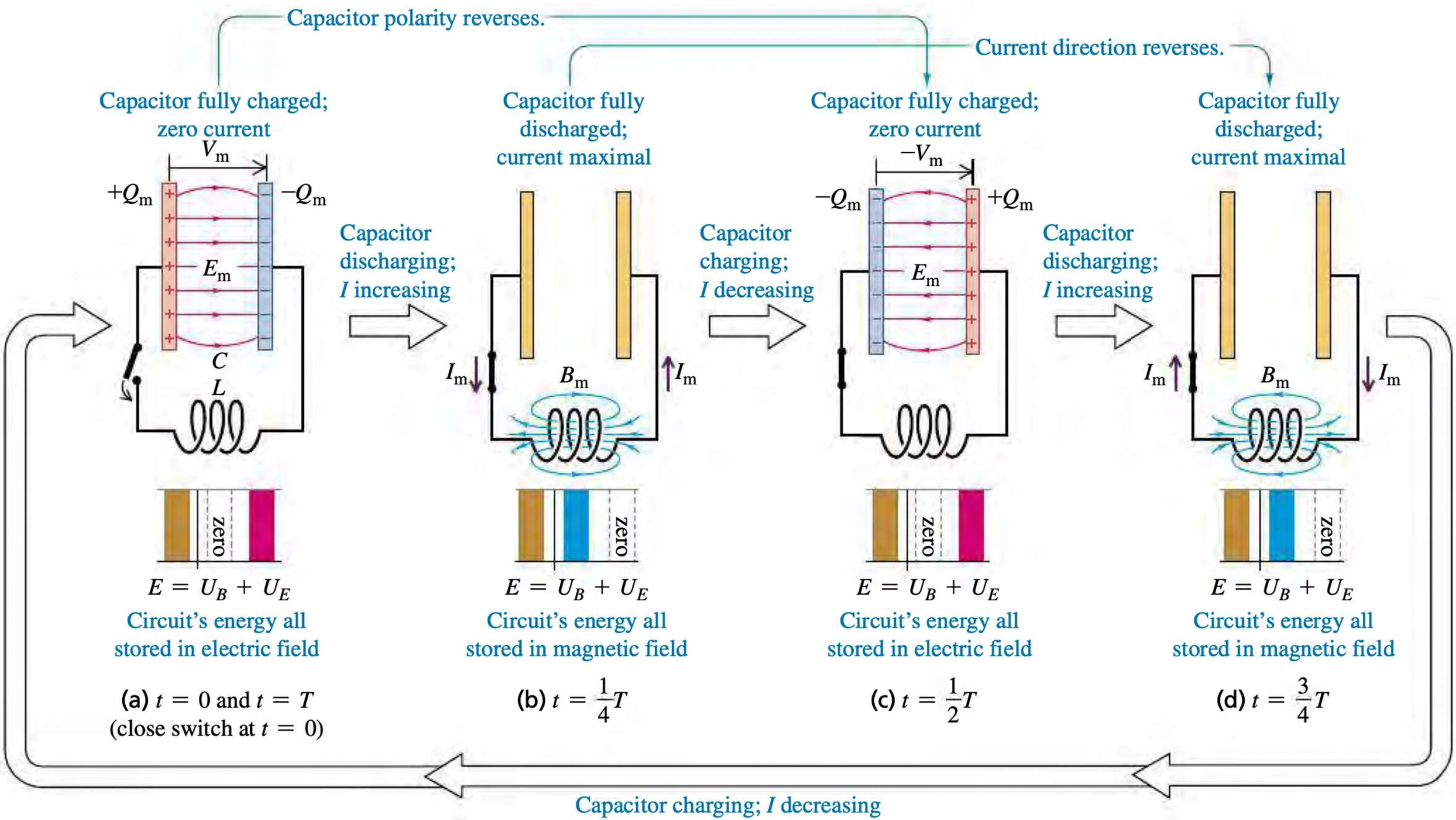
- $\ddot{x} + (k/m)x = 0$

$$\omega_0^2 = k/m$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

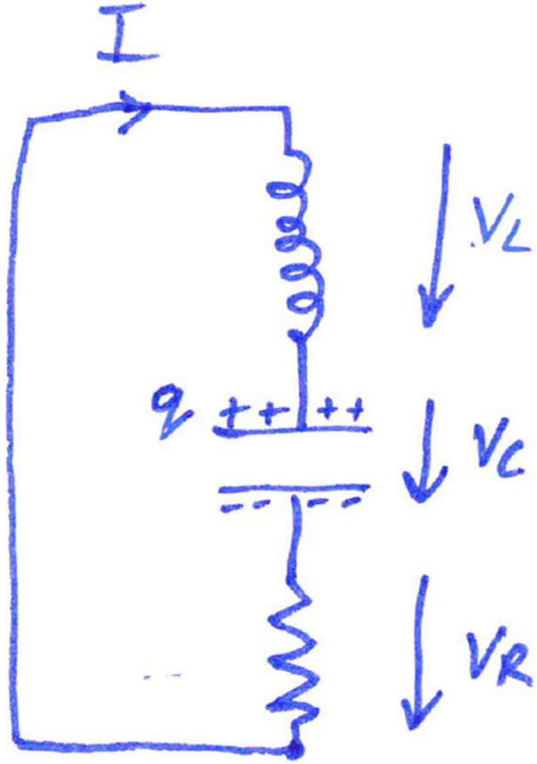
Kinetic and potential

$$q \equiv x, L \equiv m, 1/C \equiv k, \text{voltage} \equiv \text{force}$$



Young and Freedman, *University Physics*

Resonant LCR circuit



$$\ddot{q} + \dot{q}R/L + q/LC = 0$$

$$\Rightarrow \omega_0^2 = 1/LC \text{ and } \gamma = R/L,$$

$$q = Ae^{-Rt/2L}\, e^{\Omega t} + Be^{-Rt/2L}\, e^{-\Omega t}$$

Overdamped - $\Omega = (1/4.\, R^2/L^2 - 1/LC)^{1/2}$

Critical - $R/L = 2\omega_0,\, q = (A + Bt)e^{-\left(\frac{R}{2L}t\right)}$

Underdamped - $\omega_d = (1/LC - 1/4.\, R^2/L^2)^{\frac{1}{2}}$

amplitude relaxation time 2L/R

energy relaxation time L/R