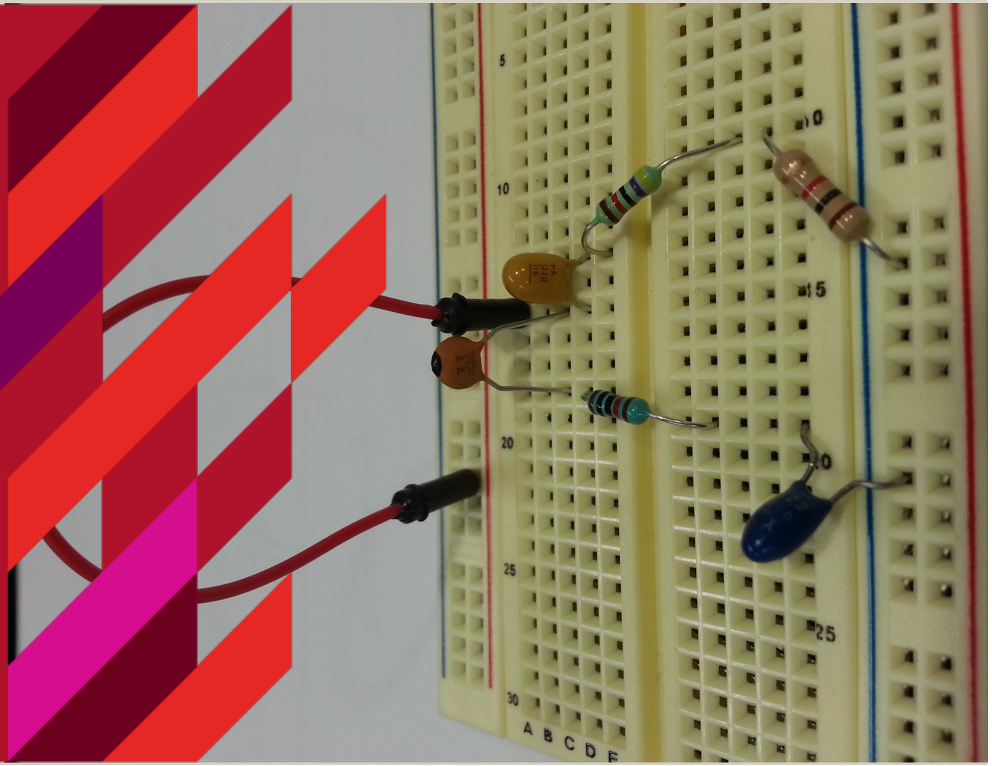
MACQUARIE University

ELEC2070 Linear Circuits and Devices

Week 9: Complex power and complex frequency

Stuart Jackson



Power

Instantaneous power: dc + abrupt change

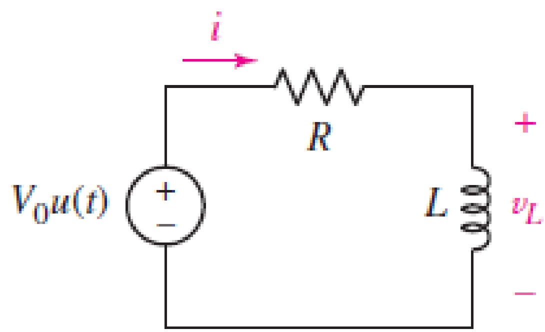
Instantaneous power in a resistor: $$p(t) = v(t)i(t) = i^2(t)R = \frac{v^2(t)}{R}$$

Instantaneous power in a inductor: $$p(t) = v(t)i(t) = Li(t)\frac{di(t)}{dt} = \frac{1}{L}v(t)\int_{-\infty}^{t} v(t')\,dt'$$

Instantaneous power in a capacitor: $$p(t) = v(t)i(t) = Cv(t)\frac{dv(t)}{dt} = \frac{1}{C}i(t)\int_{-\infty}^{t} i(t')\,dt'$$

EXAMPLE

Consider the following circuit:



The current i(t) will be given by: $$i(t) = \frac{V_0}{R}(1 - e^{-Rt/L})u(t)$$

Instantaneous power: an example

Instantaneous power from the source: $p(t) = v(t)i(t) = \frac{V_0^2}{R}(1 - e^{-Rt/L})u(t)$

Instantaneous power delivered to the resistor: $p_R(t) = i^2(t)R = \frac{V_0^2}{R}(1 - e^{-Rt/L})^2 u(t)$

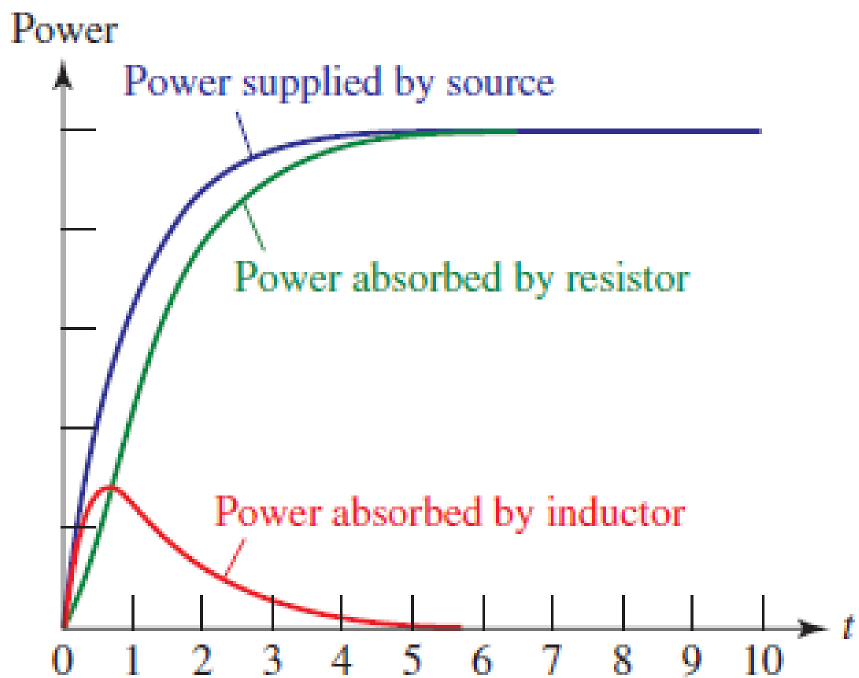
To obtain the instantaneous power delivered to the inductor, we need its voltage $v_L(t)$:

$$\begin{aligned} v_L(t) &= L\frac{di(t)}{dt} \\ &= V_0 e^{-Rt/L}u(t) + \frac{LV_0}{R}(1 - e^{-Rt/L})\frac{du(t)}{dt} \\ &= V_0 e^{-Rt/L}u(t) \end{aligned}$$

This is zero for t>0

Instantaneous power delivered to the inductor: $p_L(t) = v_L(t)i(t) = \frac{V_0^2}{R}e^{-Rt/L}(1 - e^{-Rt/L})u(t)$

Instantaneous power: graphed

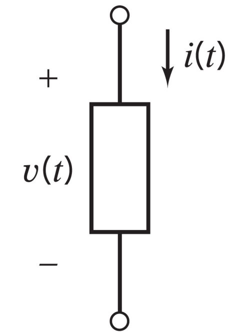


And of course: $p(t) = p_R(t) + p_L(t)$

Instantaneous power is conserved in a circuit

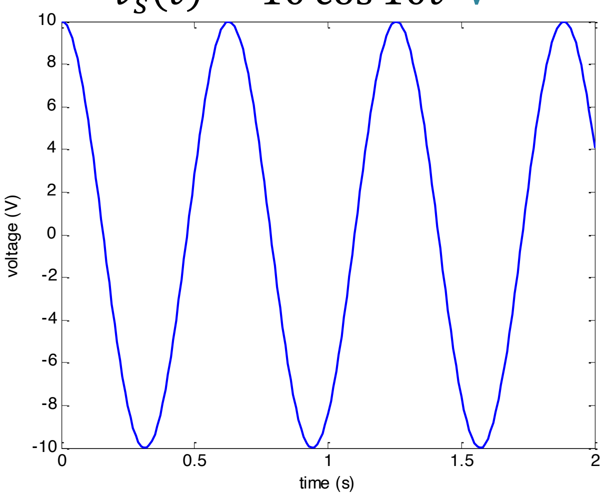
Instantaneous power in the time domain: AC example

$$p(t) = v(t) \cdot i(t) \quad \mathrm{W}$$

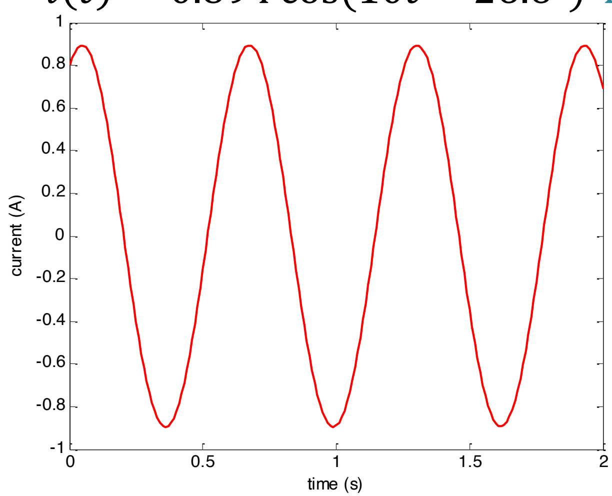


Example 10.6-3 in Dorf

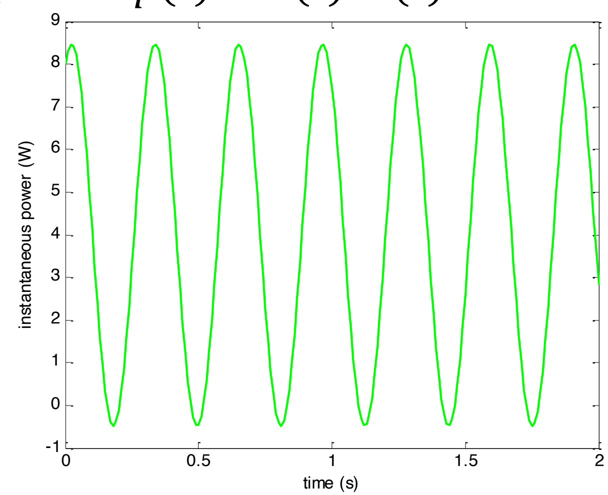
$v_s(t) = 10 \cos 10t$ V



$i(t) = 0.894 \cos(10t - 26.6°)$ A



$p(t) = v(t) \cdot i(t)$ W



Average power received by ideal components

Average power

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt \quad \mathrm{W}$$

Instantaneous ac power:

$$p(t) = \frac{V_m I_m}{2}[\underbrace{\cos(\theta_V - \theta_I)}_{\text{constant in time}} + \underbrace{\cos(2\omega t + \theta_V + \theta_I)}_{\text{periodic in time: average is zero}}]$$

constant in time

periodic in time: **average is zero**

Average ac power:

$$\boxed{P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I)}$$

Average ac power delivered to a <u>resistor</u>

$$\theta_V = \theta_I$$

$$P = \frac{V_m I_m}{2} = V_{rms} I_{rms}$$

Average ac power delivered to an <u>inductor</u>

$$\theta_V - \theta_I = 90$$

$$P = 0$$

Average ac power delivered to a <u>capacitor</u>

$$\theta_V - \theta_I = -90$$

$$P = 0$$

A problem involving average power

Find the average power being delivered to an impedance $\mathbf{Z}_L = 8 - j11\ \Omega$ by a current $\mathbf{I} = 5\underline{/20^\circ}$ A.

Average power: answer

We may find the solution quite rapidly by using Eq. [12]. Only the 8 Ω resistance enters the average-power calculation, since the $j11$ Ω component will not absorb any *average* power. Thus,

$$P = \tfrac{1}{2}(5^2)8 = 100 \text{ W}$$

RECALL:

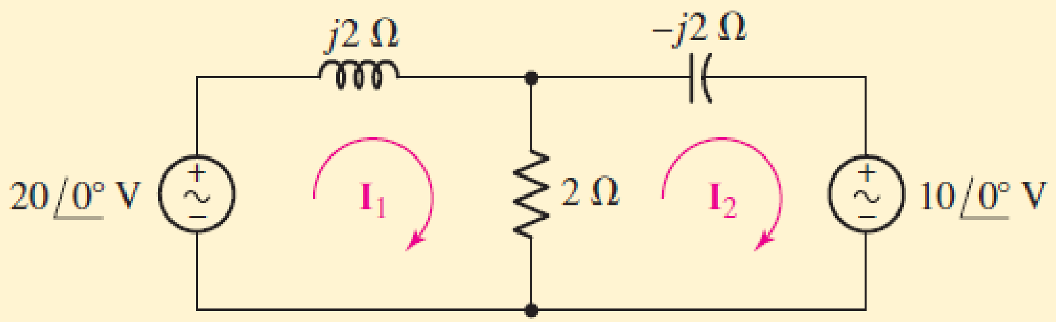
$$P_R = \tfrac{1}{2} V_m I_m \cos 0 = \tfrac{1}{2} V_m I_m$$

$$P_R = \tfrac{1}{2} I_m^2 R$$

$$P_R = \frac{V_m^2}{2R}$$

Example problem

Find the average power absorbed by each of the three passive elements in Fig. 11.5, as well as the average power supplied by each source.



Solution

Without even analyzing the circuit, we already know that the average power absorbed by the two reactive elements is zero.

The values of $\mathbf{I}_1$ and $\mathbf{I}_2$ are found by any of several methods, such as mesh analysis, nodal analysis, or superposition. They are

$$\mathbf{I}_1 = 5 - j10 = 11.18\underline{/-63.43^\circ}\ \mathrm{A}$$

$$\mathbf{I}_2 = 5 - j5 = 7.071\underline{/-45^\circ}\ \mathrm{A}$$

The downward current through the 2 Ω resistor is

$$\mathbf{I}_1 - \mathbf{I}_2 = -j5 = 5\underline{/-90^\circ}\ \mathrm{A}$$

so that $I_m = 5$ A, and the average power absorbed by the resistor is found most easily by Eq. [12]:

$$P_R = \tfrac{1}{2} I_m^2 R = \tfrac{1}{2}(5^2)2 = 25\ \mathrm{W}$$

Check

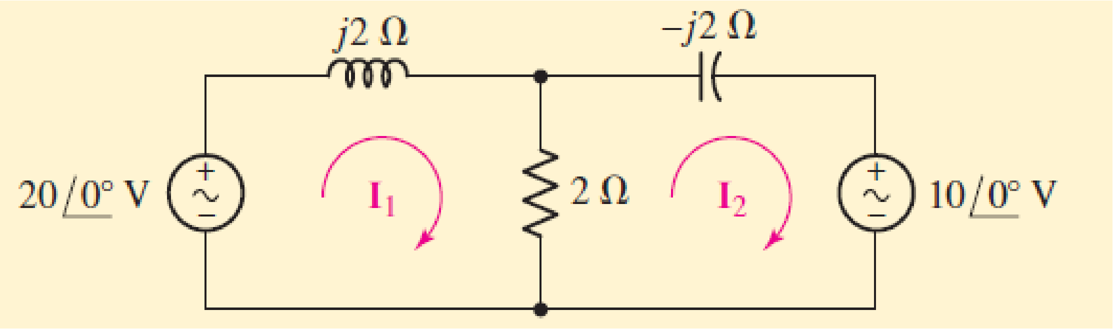
This result may be checked by using Eq. [11] or Eq. [13]. We next turn to the left source. The voltage $20\angle 0^\circ$ V and associated current $\mathbf{I}_1 = 11.18\angle -63.43^\circ$ A satisfy the *active* sign convention, and thus the power *delivered* by this source is

$$P_{\text{left}} = \tfrac{1}{2}(20)(11.18)\cos[0^\circ - (-63.43^\circ)] = 50 \text{ W}$$

In a similar manner, we find the power *absorbed* by the right source using the *passive* sign convention,

$$P_{\text{right}} = \tfrac{1}{2}(10)(7.071)\cos(0^\circ + 45^\circ) = 25 \text{ W}$$

Since $50 = 25 + 25$, the power relations check.



Apparent power and power factor

Apparent power and power factor

Recall that average power for sinusoidal forcing functions is:

Phase angle of voltage
Phase angle of current
(from other text book)

$$P = \frac{1}{2} V_m I_m \cos(\theta - \phi)$$

In terms of effective values: $$P = V_{\text{eff}} I_{\text{eff}} \cos(\theta - \phi)$$

The **apparent power** is the equivalent DC average power and is given by:

$$\frac{1}{2} V_m I_m \quad \text{or} \quad V_{\text{eff}} I_{\text{eff}}$$

Units: VA

The ratio of the real or average power to the apparent power is the **power factor**, PF:

$$\text{PF} = \frac{\text{average power}}{\text{apparent power}} = \frac{P}{V_{\text{eff}} I_{\text{eff}}}$$

For the sinusoidal case, this is just:

$$\cos(\theta_V - \theta_I) \quad \text{OR} \quad \cos(\theta - \phi)$$

Power factor

For a purely resistive load, the voltage and current are in phase - power factor (PF) is 1

Thus the average power and the apparent power are EQUAL.

(Note a network with inductors and capacitors may have PF = 1 if the element values and operating frequency are carefully chosen to create and input impedance with zero phase angle.)

Purely reactive load: PF = 0

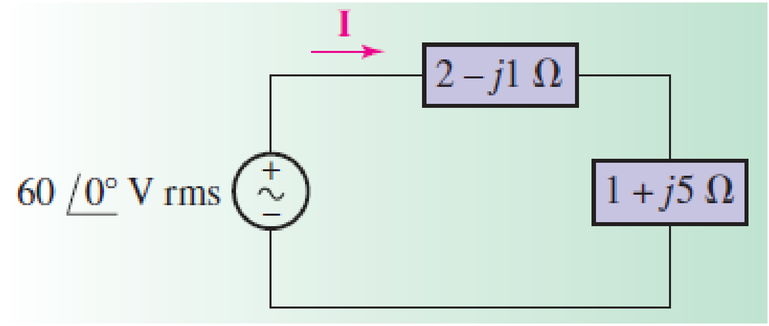
In between, e.g., PF = 0.5 means $\theta_V - \theta_I = 60^o \; or \; -60^o$

$\theta_V - \theta_I = 60^o$ means an inductive load (voltage LEADS the current)

$\theta_V - \theta_I = -60^o$ means an capacitive load (current LEADS the voltage)

Example

Calculate values for the average power delivered to each of the two loads shown in Fig. 11.13, the apparent power supplied by the source, and the power factor of the combined loads.



Solution – what we know

The average power refers to the power drawn by the resistive components of the load elements; the apparent power is the product of the effective voltage and the effective current of the load combination.

The effective voltage is 60 V rms, which appears across a combined load of $2 - j + 1 + j5 = 3 + j4\ \Omega$.

The average power P supplied to a load is given by

$$P = I_{\text{eff}}^2 R$$

where R is the real part of the load impedance. The apparent power supplied by the source is $V_{\text{eff}} I_{\text{eff}}$, where $V_{\text{eff}} = 60$ V rms.

The power factor is calculated as the ratio of these two quantities:

$$\text{PF} = \frac{\text{average power}}{\text{apparent power}} = \frac{P}{V_{\text{eff}} I_{\text{eff}}}$$

Solution

We require I_{eff}:

$$\mathbf{I} = \frac{60\underline{/0^\circ}}{3 + j4} = 12\underline{/-53.13^\circ} \text{ A rms}$$

so $I_{\text{eff}} = 12$ A rms, and $ang\ \mathbf{I} = -53.13^\circ$.

The average power delivered to the top load is given by

$$P_{\text{upper}} = I_{\text{eff}}^2 R_{\text{top}} = (12)^2(2) = 288 \text{ W}$$

and the average power delivered to the right load is given by

$$P_{\text{lower}} = I_{\text{eff}}^2 R_{\text{right}} = (12)^2(1) = 144 \text{ W}$$

The source itself supplies an apparent power of $V_{\text{eff}} I_{\text{eff}} = (60)(12) = 720$ VA.

Solution

Finally, the power factor of the combined loads is found by considering the voltage and current associated with the combined loads. This

power factor is, of course, identical to the power factor for the source. Thus

$$\mathrm{PF} = \frac{P}{V_{\text{eff}} I_{\text{eff}}} = \frac{432}{60(12)} = 0.6 \ \textit{lagging}$$

since the combined load is *inductive*.

Complex power

Chapter 11 of Dorf and Svoboda

Definitions

In the frequency domain, the current and voltage are in general given by:

$$\mathbf{I}(\omega) = I_{\mathrm{m}} \angle \theta_{\mathrm{I}} \quad \text{and} \quad \mathbf{V}(\omega) = V_{\mathrm{m}} \angle \theta_{\mathrm{V}}$$

The **complex power** delivered to the element is defined (in polar form) as

$$\mathbf{S} = \frac{\mathbf{VI}^*}{2} = \frac{(V_{\mathrm{m}} \angle \theta_{\mathrm{V}})\left(I_{\mathrm{m}} \angle -\theta_{\mathrm{I}}\right)}{2} = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \angle \theta_{\mathrm{V}} - \theta_{\mathrm{I}}$$

Alternatively: $\mathbf{S} = \mathbf{V}_{\mathrm{eff}} \mathbf{I}^*_{\mathrm{eff}}$

The **apparent power** is the magnitude of the complex power: $|\mathbf{S}| = \dfrac{V_{\mathrm{m}} I_{\mathrm{m}}}{2}$

Why complex power?

The complex power allows us to calculate the contributions to the total power in a clean and precise way.

Converting the complex power from polar form to rectangular form we get: $\mathbf{S} = P + jQ$

$$\mathbf{S} = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \cos\left(\theta_{\mathrm{V}} - \theta_{\mathrm{I}}\right) + j \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \sin\left(\theta_{\mathrm{V}} - \theta_{\mathrm{I}}\right)$$

Average power from previous time domain analysis (in Watts)

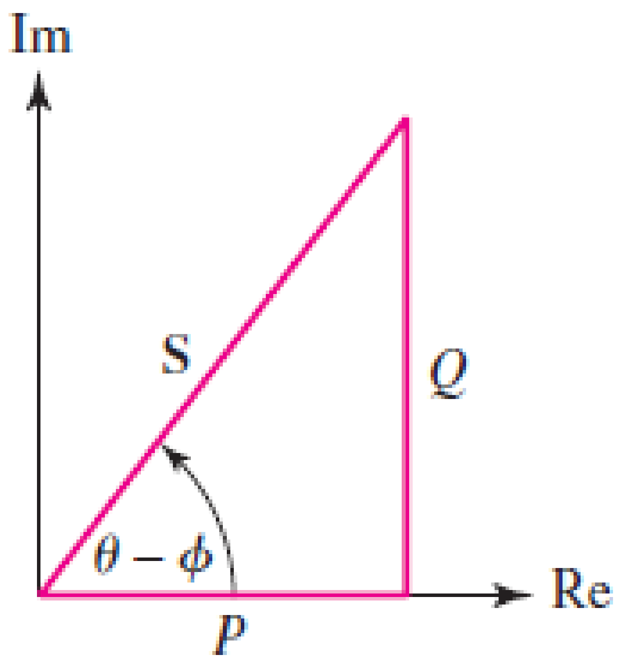
Reactive power

Then we have:

$$P = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \cos\left(\theta_{\mathrm{V}} - \theta_{\mathrm{I}}\right) \qquad Q = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \sin\left(\theta_{\mathrm{V}} - \theta_{\mathrm{I}}\right)$$

The power triangle

The power triangle:



In this case:

θ Is the phase angle of the voltage
ϕ Is the phase angle of the current

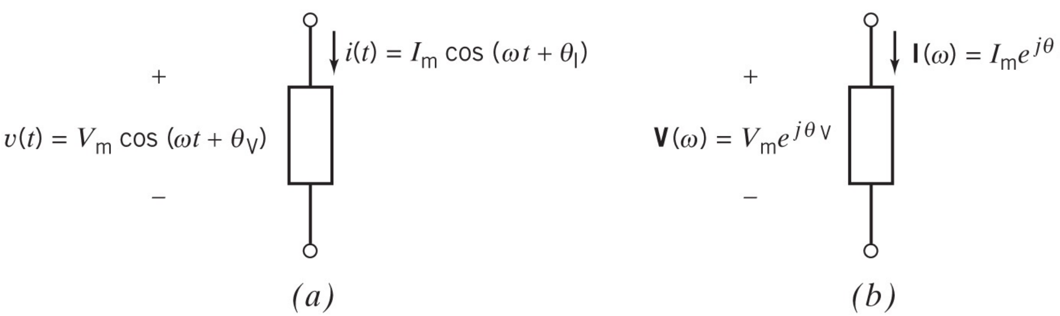
Reactive power, Q

The dimensions of reactive power are the same as the real power P, the complex power **S** and the apparent power I**S**I. To avoid confusion, with these other quantities, the unit of Q is **volt-ampere-reactive**.

The physical interpretation of reactive power is the time rate of energy flow back and forth between the source and the reactive components of the load. These components alternatively charge and discharge, which leads to current flow from the source and to the source respectively.

Complex power wrt impedance

Units: Complex power: VA
Average power: W
Reactive power: VAR (R means “reactive”)



The impedance is given by:

$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = \frac{V_m \angle \theta_V}{I_m \angle \theta_I} = \frac{V_m}{I_m} \angle \theta_V - \theta_I$$

Which can be converted to rectangular form: $\mathbf{Z}(\omega) = \frac{V_m}{I_m} \cos(\theta_V - \theta_I) + j \frac{V_m}{I_m} \sin(\theta_V - \theta_I)$

Or: $\boldsymbol{Z}(\omega) = R + jX$

Where: $R = \frac{V_m}{I_m} \cos(\theta_V - \theta_I)$ and $X = \frac{V_m}{I_m} \sin(\theta_V - \theta_I)$

Complex power wrt impedance

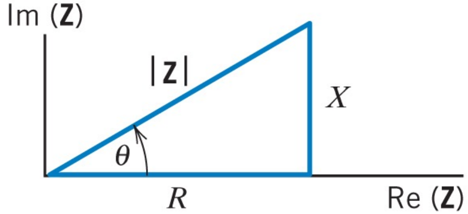
Recall that the complex power is given by: $\mathbf{S} = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + j\frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$

Which can be re-written as: $\boldsymbol{S} = \frac{I_m^2}{2}\frac{V_m}{I_m}\cos(\theta_V - \theta_I) + j\frac{I_m^2}{2}\frac{V_m}{I_m}\sin(\theta_V - \theta_I)$

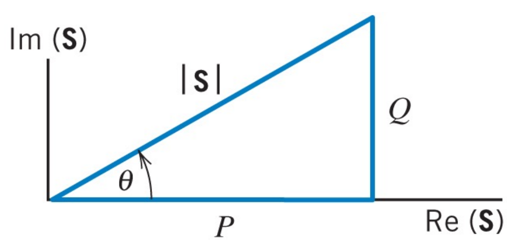
(multiplying by $\frac{I_m}{I_m}$)

Which is: $\boldsymbol{S} = \frac{I_m^2}{2}Re(\boldsymbol{Z}) + j\frac{I_m^2}{2}Im(\boldsymbol{Z})$

Hence the average (true) power is: $\mathrm{P} = \frac{I_m^2}{2}Re(\boldsymbol{Z})$ and the reactive power: $\mathrm{Q} = \frac{I_m^2}{2}Im(\boldsymbol{Z})$



(a)



(b)

Power in the frequency domain

QUANTITY	RELATIONSHIP USING PEAK VALUES	RELATIONSHIP USING rms VALUES	UNITS
Element voltage, $v(t)$	$v(t) = V_m \cos(\omega t + \theta_v)$	$v(t) = V_{rms}\sqrt{2}\cos(\omega t + \theta_V)$	V
Element current, $i(t)$	$i(t) = I_m \cos(\omega t + \theta_I)$	$i(t) = I_{rms}\sqrt{2}\cos(\omega t + \theta_I)$	A
Complex power, $\mathbf{S}$	$\mathbf{S} = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + j\frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$	$\mathbf{S} = V_{rms} I_{rms}\cos(\theta_V - \theta_I) + jV_{rms} I_{rms}\sin(\theta_V - \theta_I)$	VA
Apparent power, $\lvert\mathbf{S}\rvert$	$\lvert\mathbf{S}\rvert = \frac{V_m I_m}{2}$	$\lvert\mathbf{S}\rvert = V_{rms} I_{rms}$	VA
Average power, P	$P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I)$	$P = V_{rms} I_{rms}\cos(\theta_V - \theta_I)$	W
Reactive power, Q	$Q = \frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$	$Q = V_{rms} I_{rms}\sin(\theta_V - \theta_I)$	VAR

Conservation of power in the frequency domain

Section 11.5 in Dorf

Complex power is conserved in a circuit

$$\sum_{\substack{all\\ elements}} \frac{\mathbf{V}_k \mathbf{I}_k^*}{2} = 0$$

Average power is conserved in a circuit

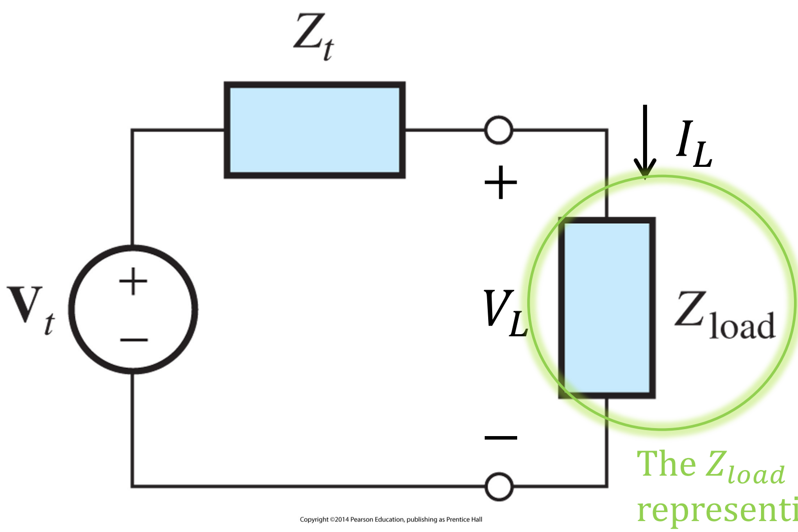
$$\sum_{\substack{all\\ elements}} \mathrm{Re}\left(\frac{\mathbf{V}_k \mathbf{I}_k^*}{2}\right) = 0$$

Reactive power is conserved in a circuit

$$\sum_{\substack{all\\ elements}} \mathrm{Im}\left(\frac{\mathbf{V}_k \mathbf{I}_k^*}{2}\right) = 0$$

Maximum power transfer

Complex power delivered to a load



The Z_{load} may be representing another circuit

The complex power delivered to the load:

$$S_L = \frac{V_L \cdot I_L^*}{2}$$

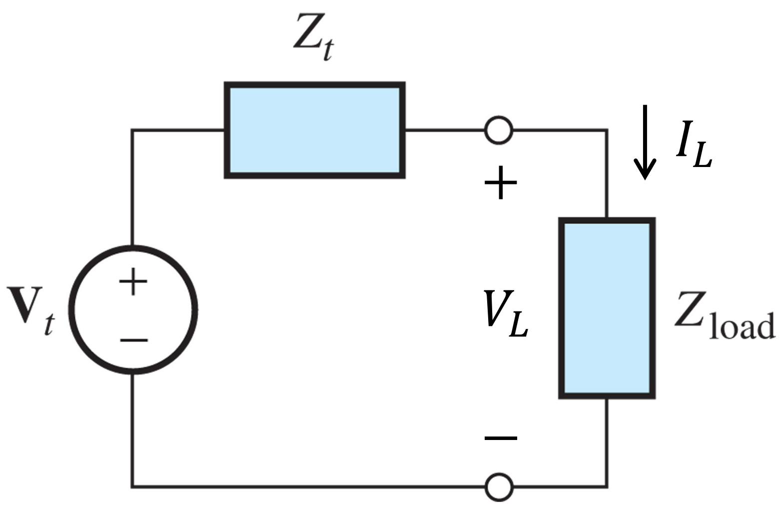
You need to calculate V_L and I_L first, then calculate S_L:

$$I_L = \frac{V_t}{(Z_t + Z_{load})} \qquad V_L = Z_{load} \cdot I_L \qquad S_L = \frac{Z_{load} \cdot I_L \cdot I_L^*}{2} = \frac{Z_{load} \cdot |I_L|^2}{2}$$

This version is useful in the laboratory:

$$S_L = \frac{V_L \cdot V_L^*}{2 \cdot Z_{load}^*} = \frac{|V_L|^2}{2 \cdot Z_{load}^*}$$

Average power delivered to load



Copyright ©2014 Pearson Education, publishing as Prentice Hall

The complex power transferred to the load:

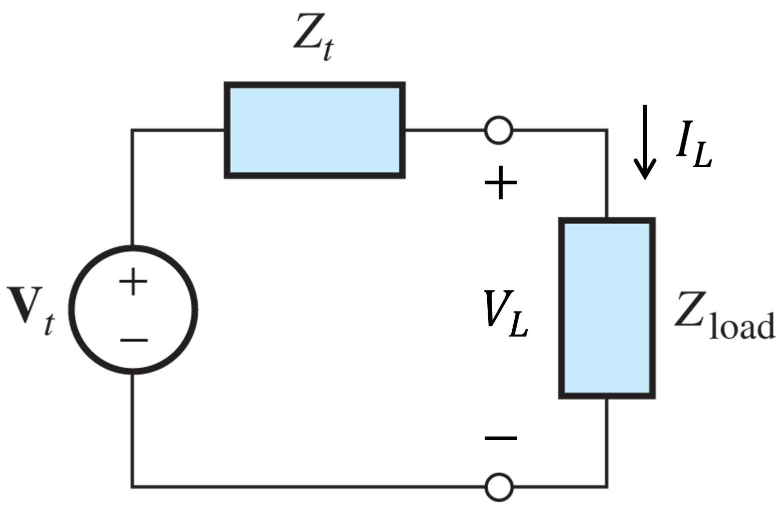
$$S_L = \frac{Z_{load} \cdot |I_L|^2}{2}$$

The average power transferred to the load:

$$P_L = \mathrm{Re}\{S_L\} = \frac{\mathrm{Re}\{Z_{load}\} \cdot |I_L|^2}{2} = \frac{|V_L|^2}{2\,\mathrm{Re}\{Z_{load}\}}$$

What is the reactive power transferred to the load?

Maximum AVERAGE power transfer



The average power transferred from a fixed source circuit [with a Thévenin equivalent] to a load

is maximum when

$$Z_{load} = Z_t^*$$

$$Z_t = R_t + jX_t$$

$$Z_{load} = R_t - jX_t$$

When $Z_{load} = Z_t^*$, the total impedance seen by the Thévenin source is purely resistive (the load reactance cancels the reactance of the Thévenin impedance).

$$Z_t + Z_{load} = 2R_t$$

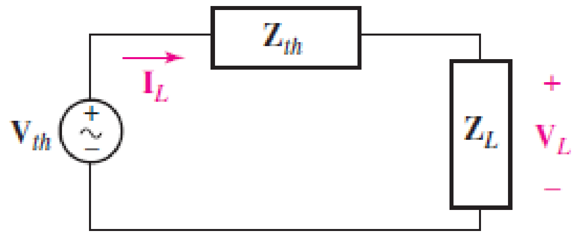
This is the same condition as for the maximum <u>DC</u> power transfer !!

Maximum average power transfer theorem for the sinusoidal steady state

An independent voltage source in *series* with an impedance $\mathbf{Z}_{th}$ or an independent current source in *parallel* with an impedance $\mathbf{Z}_{th}$ delivers a ***maximum average power*** to that load impedance $\mathbf{Z}_L$ which is the conjugate of $\mathbf{Z}_{th}$, or $\mathbf{Z}_L = \mathbf{Z}_{th}^*$.

Proof

For the circuit:

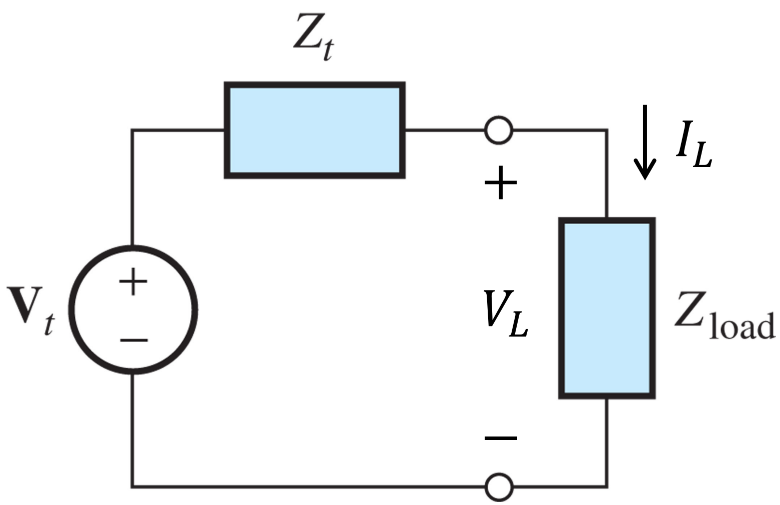


The average power delivered to Z_L can be shown to be: $$P = \frac{\frac{1}{2}|\mathbf{V}_{th}|^2\sqrt{R_L^2 + X_L^2}}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}\cos\left(\tan^{-1}\left(\frac{X_L}{R_L}\right)\right)$$

Need to differentiate wrt R_L and X_L and equate to zero. (Homework for the keen student)

Maximum average power transfer

Simply replace Z_{load} with Z_t^* and calculate the power



$$\boldsymbol{Z_{load} = Z_t^*}$$

$$Z_t = R_t + jX_t$$

$$Z_{load} = R_t - jX_t$$

$$Z_t + Z_{load} = 2R_t$$

$$S_L = \frac{Z_{load} \cdot |I_L|^2}{2}$$

$$I_L = \frac{V_t}{2R_t} \qquad V_L = (R_t - jX_t)I_L$$

$$\boxed{P_{L\,max} = \frac{R_t \cdot |I_L|^2}{2} = \frac{|V_t|^2}{8R_t}}$$

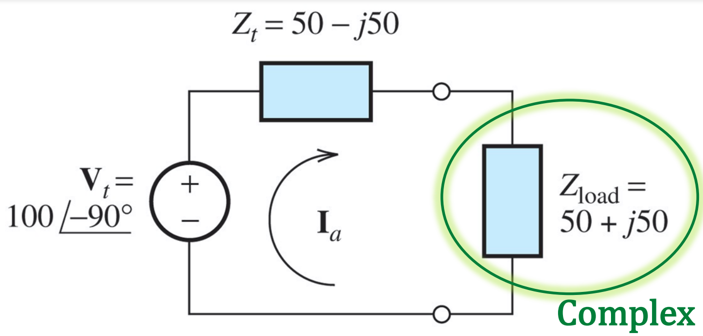
This is the maximum available average power from the source circuit !!

Why? because it only depends on the parameters of the source circuit only.

Example

MACQUARIE University

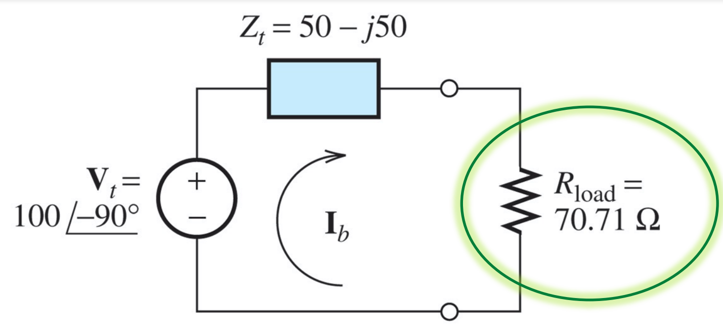
Maximum available average power for two loads



Complex load impedance

$$I_a = \frac{V_t}{2R_t} = 1\angle -90° \text{ A}$$

$$P_{a\,max} = \frac{50 \cdot |I_a|^2}{2} = 25 \text{ W}$$



Purely resistive load

$$I_b = \frac{V_t}{Z_t + R_{load}} = \frac{V_t}{50 - j50 + 70.71}$$

$$I_b = 0.7654\angle -67.50° \text{ A}$$

$$P_{b\,max} = \frac{70.71 \cdot |I_b|^2}{2} = 20.71 \text{ W}$$

If $Z_{load} = 50\ \Omega$ $\quad I_c = \frac{100\angle -90°}{100 - j50} = 0.8944\angle -63.43° \text{ A}$ $\quad P_c = \frac{50 \cdot |I_c|^2}{2} = 20 \text{ W}$

Summary of AC power terms

Term	Symbol	Unit	Description
Instantaneous power	$p(t)$	W	$p(t) = v(t)i(t)$. It is the value of the power at a specific instant in time. It is *not* the product of the voltage and current phasors!
Average power	P	W	In the sinusoidal steady state, $P = \frac{1}{2} V_m I_m \cos(\theta - \phi)$, where θ is the angle of the voltage and ϕ is the angle of the current. Reactances do not contribute to P.
Effective or rms value	V_{rms} or I_{rms}	V or A	Defined, e.g., as $I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt}$; if $i(t)$ is sinusoidal, then $I_{\text{eff}} = I_m/\sqrt{2}$.
Apparent power	$\lvert\mathbf{S}\rvert$	VA	$\lvert\mathbf{S}\rvert = V_{\text{eff}} I_{\text{eff}}$, and is the maximum value the average power can be; $P = \lvert\mathbf{S}\rvert$ only for purely resistive loads.
Power factor	PF	None	Ratio of the average power to the apparent power. The PF is unity for a purely resistive load, and zero for a purely reactive load.
Reactive power	Q	VAR	A means of measuring the energy flow rate to and from reactive loads.
Complex power	$\mathbf{S}$	VA	A convenient complex quantity that contains both the average power P and the reactive power Q: $\mathbf{S} = P + jQ$.

Complex frequency

Enables us to combine together ALL our forcing functions into one neat package

The damped sinusoid

Consider the damped sinusoidal voltage forcing function (which is real)

$$v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)$$

Normally σ is negative (damped) but sometimes it can be positive.

The equation above is very powerful because it captures all the forcing functions that we have been studying into ONE function

1. $\sigma = \omega = 0$. We have a DC forcing function for the voltage.
2. $\sigma = 0$. Now we have a sinusoidal forcing function for the voltage
3. $\omega = 0$. Now we have an exponential forcing function for the voltage.

This function captures all the important functions of time for both current and voltage.

The variable σ has the units frequency (since it multiplies with *t* to give a dimensionless number).

A mathematical definition

Let us start with a **general** mathematical definition of this function:

$$f(t) = \mathbf{K}e^{st}$$

Here **K** and s are complex numbers and s (= σ + jω) is known as **the complex frequency** (since it multiplies with *t* to give a *dimensionless value* – it therefore has units of s^{-1}). Now we can say a few things:

1. The complex frequency of a DC voltage or current is s = 0.
2. The complex frequency of an exponential voltage or current is s = σ (ω=0)
3. The complex frequency of a sinusoidal voltage or current could be s = jω (σ =0, remember Euler's identity)

For this last case, we need to do more work.

The sinusoidal case

Let us look at the sinusoidal case more closely

Let's start with a sinusoidal voltage which is given by: $v(t) = V_m \cos(\omega t + \theta)$

Euler:

$e^{ix} = \cos x + i \sin x$

Using Euler's identity, this function can be converted to exponential functions only:

$$
\begin{aligned}
v(t) &= \tfrac{1}{2} V_m [e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}] \\
&= \left(\tfrac{1}{2} V_m e^{j\theta}\right) e^{j\omega t} + \left(\tfrac{1}{2} V_m e^{-j\theta}\right) e^{-j\omega t}
\end{aligned}
$$

Since: $\cos(\omega t + \theta) = \tfrac{1}{2}[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}]$

The general form

This latter equation can be put into our general form: $v(t) = \mathbf{K}_1 e^{\mathbf{s}_1 t} + \mathbf{K}_2 e^{\mathbf{s}_2 t}$

$$\left(\tfrac{1}{2}V_m e^{j\theta}\right) e^{j\omega t} + \left(\tfrac{1}{2}V_m e^{-j\theta}\right) e^{-j\omega t}$$

Where: $\mathbf{s} = \mathbf{s}_1 = j\omega$

And: $\mathbf{s} = \mathbf{s}_2 = -j\omega$

Hence we have the conjugates: $\mathbf{s}_2 = \mathbf{s}_1^*$

Likewise: $\mathbf{K}_1 = \frac{1}{2}V_m e^{j\theta}$

And: $\mathbf{K}_2 = \mathbf{K}_1^* = \frac{1}{2}V_m e^{-j\theta}$

The K terms are also complex conjugates (they must be since their sum is a real quantity, i.e., the sinusoidal voltage!)

The exponentially damped sinusoid

Now were are are ready to create the general form of the exponentially damped sinusoid.

Our previous exponentially damped sinusoid is a **real** function of time which is now given by:

$$v(t) = V_m e^{\sigma t}\cos(\omega t + \theta)$$
$$= \tfrac{1}{2} V_m e^{\sigma t}[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}]$$

This can be expanded to give:

$$v(t) = \tfrac{1}{2} V_m e^{j\theta} e^{j(\sigma+j\omega)t} + \tfrac{1}{2} V_m e^{-j\theta} e^{j(\sigma-j\omega)t}$$

Now we find that the complex conjugate of the complex frequencies is used to describe the exponentially damped sinusoid:

$$\mathbf{s}_1 = \sigma + j\omega$$

$$\mathbf{s}_2 = \mathbf{s}_1^* = \sigma - j\omega$$

Understanding the complex frequency, s

!. A positive and real value for s: $\mathbf{s} = 5 + j0$

Implies that the function is positively increasing: $\mathbf{K}e^{+5t}$

Here **K** MUST be real in order for the (forcing) function to be real.

2. A positive but imaginary value for s = $j\,10$

Implies that the forcing function is: $\mathbf{K}e^{j\,10t}$

Or: $\mathbf{K}(\cos 10t + j\,\sin 10t)$ → We cannot have a part of the forcing function or response that is imaginary

So, to create a real function, we need the complex conjugate: $\mathbf{s}_{1,2} = \pm j\,10$

Each one being associated with the respective complex conjugates of **K**

Complex conjugates can create real values.

The complex conjugates

We may identify either of the complex frequencies with a sinusoidal voltage but in practice, we don't always include both complex conjugates in our analysis. It is just understood that they are both involved.

Physical nature of the complex frequency

So now our real sinusoidal forcing function is:

$$v(t) = \mathbf{K}_1 e^{\mathbf{s}_1 t} + \mathbf{K}_2 e^{\mathbf{s}_2 t} \qquad \mathbf{s}_2 = \mathbf{s}_1^* \qquad \text{and} \qquad \mathbf{K}_2 = \mathbf{K}_1^*$$

For example: $\mathbf{s}_1 = j10$ and $\mathbf{K}_1 = 6 - j8$, $\longrightarrow$ $K_1 = \frac{1}{2} V_m e^{j\theta} = 10 < -53.1^0$

We get the forcing function, the **real** sinusoid: $20\cos(10t - 53.1^\circ)$

In general, the complex frequency *s* is associated with a real function when it is accompanied by its complex conjugate. The complex frequency contains all the parameters we need to understand the time behaviour of the voltage

The specific phase and amplitude of a general exponentially damped sinusoid is contained in **K** and its complex conjugate. These are the **NON** time dependent parameters.

Features of the complex frequency

1. The complex frequency describes an exponentially varying sinusoidal function
2. The real part of s is associated with the exponential variation
    - If it is negative, the sinusoid decays as t increases
    - If it is positive, the sinusoid increases as t increases
    - If it is zero, the sinusoidal amplitude is constant
3. The larger the magnitude of the real part of s, the greater the rate of exponential decrease or increase
4. The imaginary part of s describes the sinusoidal part of the variation
5. A large value for the imaginary part of s indicates a more rapidly varying sinusoidal function of time.

The complex frequency, s

It is customary to use the letter σ to designate the real part of s and ω (NOT $j\omega$) to designate the imaginary part.

$$\boxed{\mathbf{s} = \sigma + j\omega}$$

The parts of the complex frequency are:

1. σ is the neper frequency (in nepers per second)
2. ω is the angular frequency (in radians per second)
3. $f = \omega/2\pi$ is the cycle frequency
4. The complex frequency s is measured in complex nepers per second or complex radians per second

Example

14.1 Identify all the complex frequencies present in these real functions: (*a*) $(2e^{-100t} + e^{-200t}) \sin 2000t$; (*b*) $(2 - e^{-10t}) \cos(4t + \phi)$; (*c*) $e^{-10t} \cos 10t \sin 40t$.

Ans: 14.1: $-100 + j2000, -100 - j2000, -200 + j2000, -200 - j2000\ \mathrm{s}^{-1}$; $j4, -j4, -10 + j4, -10 - j4\ \mathrm{s}^{-1}$; $-10 + j30, -10 - j30, -10 + j50, -10 - j50\ \mathrm{s}^{-1}$;

$$\sin \alpha \cdot \cos \beta = \frac{\sin(\alpha - \beta) + \sin(\alpha + \beta)}{2}$$

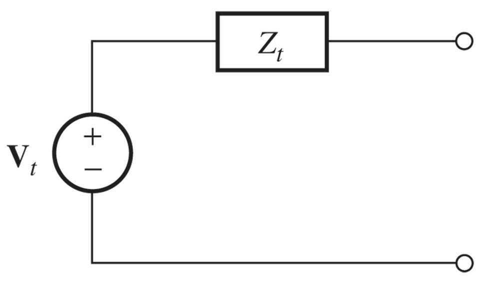
A final note about equivalent circuits

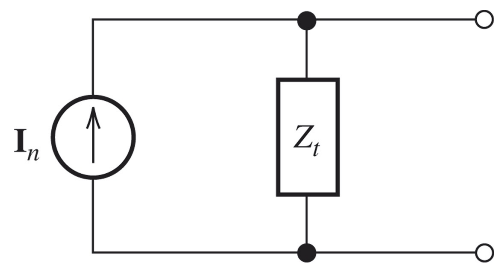
Thévenin and Norton equivalent circuits in the frequency domain

Section 10.7 in Dorf

When are the Thévenin and Norton equivalent circuits appropriate?

Only when we don't care what is inside a (two-terminal) circuit [with sources] and we want to simplify it to use with other (two-terminal) circuits.





$$V_t = V_{oc}$$

$$I_n = I_{sc}$$

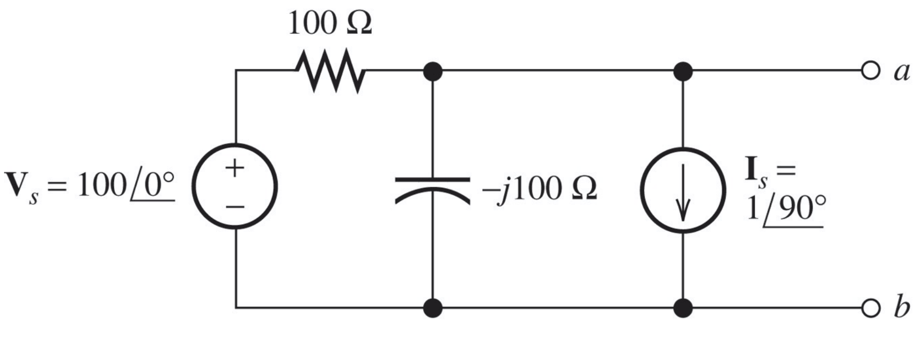
$$Z_t = \frac{V_{oc}}{I_{sc}} = \frac{V_t}{I_n}$$

Good and bad things about the Thévenin and Norton equivalent circuits

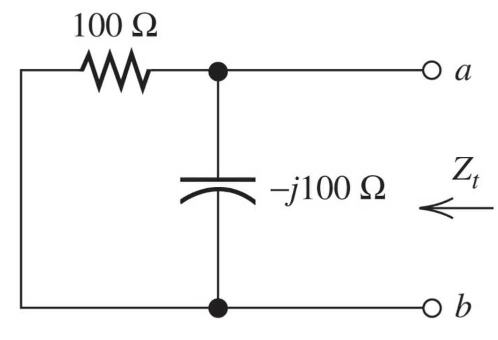
MACQUARIE University

Good things:	Bad things:
They are simpler than the original circuit.	The original circuit becomes a black box. We won't know what is really inside.
The voltage and current relationships at the terminals can be correctly predicted for any new component connected to it.	We can't calculate how much power the real source circuit dissipates (wastes).
	If we are not happy with the V_t or Z_t or I_t we can't fix it.

Example



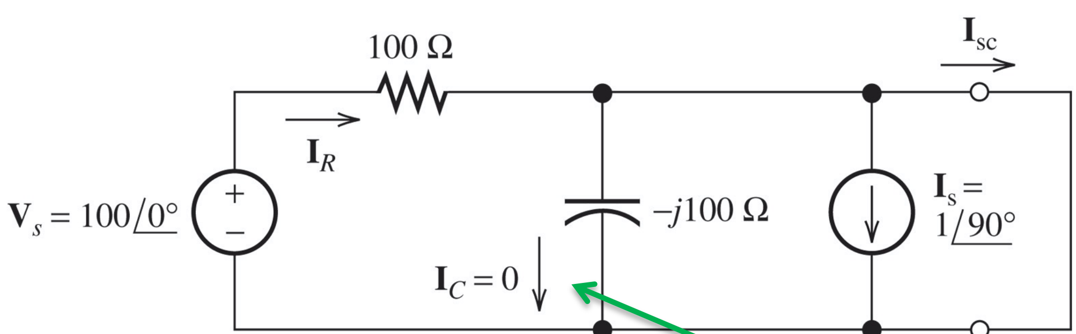
(a) Original circuit



(b) Circuit with the sources zeroed

When is it OK to do this?

It is not good to use this method if there are dependent sources.

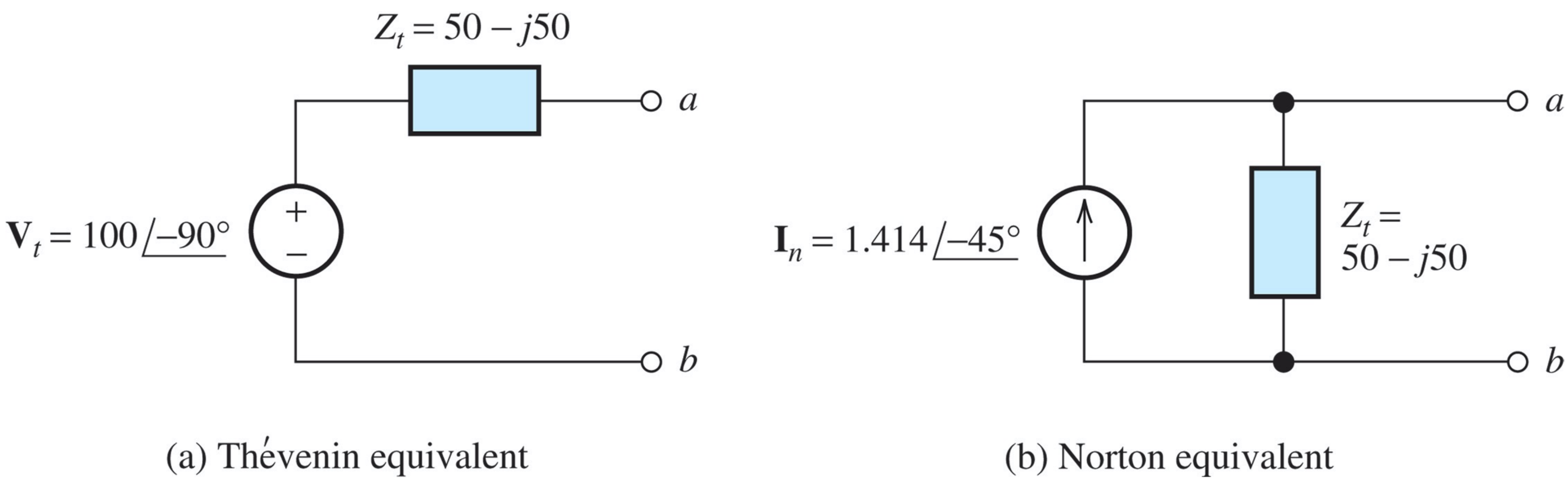


(c) Circuit with a short circuit

Why is $I_C = 0$?

What is I_R ? $I_R = I_s + I_{sc}$

Example



$Z_t = 50 - j50$
a
$\mathbf{V}_t = 100\angle{-90^\circ}$
+
–
b
(a) Thévenin equivalent
a
$\mathbf{I}_n = 1.414\angle{-45^\circ}$
$Z_t =$
$50 - j50$
b
(b) Norton equivalent