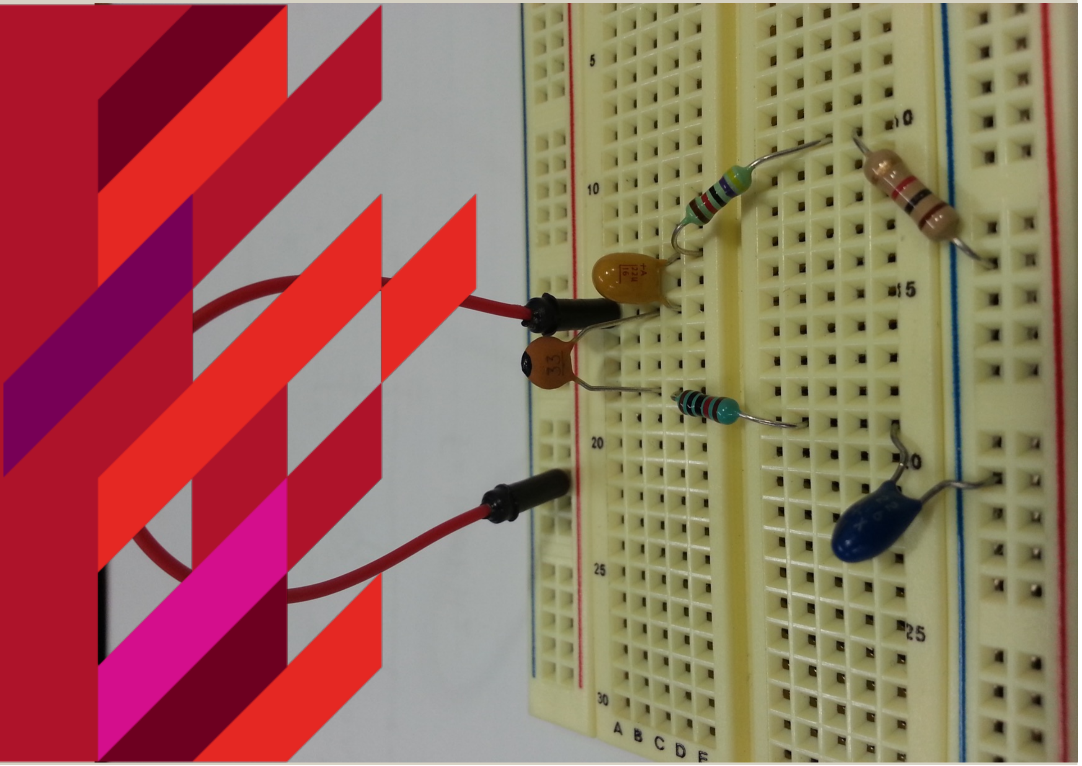
ELEC2070 Circuits and Devices

Week 6: Review of complete response of RL, RC and RLC circuits + Stability

Stuart Jackson



Complete Response for First Order Circuits

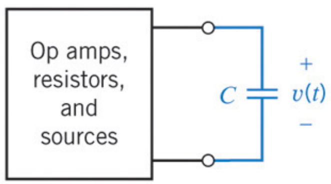
Chapter 8 in Dorf

With abrupt changes (switching) and DC sources

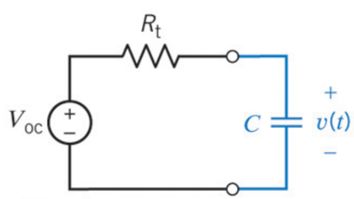
Table 8.12-1 in Dorf (this is an engineering approach)

Suitable for first order circuits with DC sources turning on at t=0.
If turning on at t_0, replace t with $(t-t_0)$ and 0 with t_0.

FIRST-ORDER CIRCUIT CONTAINING A CAPACITOR



Replace the circuit consisting of op amps, resistors, and sources by its Thévenin equivalent circuit:



$v(\infty) = V_{OC}$

The capacitor voltage is:

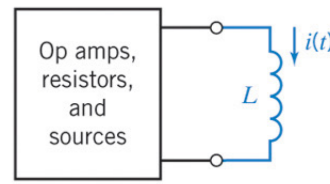
$$v(t) = V_{oc} + (v(0) - V_{oc})e^{-t/\tau}$$

where the time constant τ is

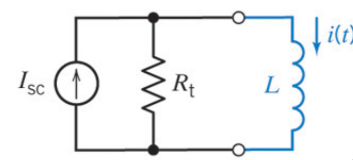
$$\tau = R_t C$$

and the initial condition $v(0)$ is the capacitor voltage at time $t = 0$.

FIRST-ORDER CIRCUIT CONTAINING AN INDUCTOR



Replace the circuit consisting of op amps, resistors, and sources by its Norton equivalent circuit:



$i(\infty) = I_{SC}$

The inductor current is

$$i(t) = I_{sc} + (i(0) - I_{sc})e^{-t/\tau}$$

where the time constant τ is

$$\tau = \frac{L}{R_t}$$

and the initial condition $i(0)$ is the inductor current at time $t = 0$.

These are complete solutions!

2 approaches to first order circuits

1. **Engineering approach** (useful for DC only): use the formula involving initial and final voltages (or currents) and the time constant

2. **Mathematical approach** (more general and needed for time varying sources): Find the differential equation using Kirchoff's laws and Ohm's Law

(Of course only the mathematical approach is used for second order circuits)

With time-varying sources

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Be careful with Formula Sheet (need to be more mathematical)

First Order Circuits in Time Domain: Forced response

Differential Equation

$$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = y(t)$$

Complete Response

$$x(t) = x_n(t) + x_f(t)$$

Natural Response

$$x_n(t) = Ke^{-(t-t_0)/\tau}$$

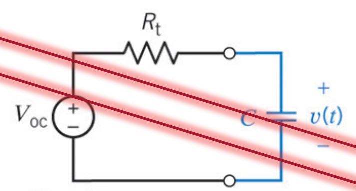
Complementary Solution

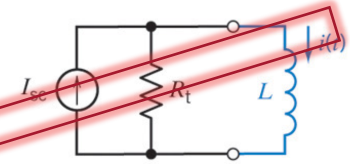
Particular Solution

Forcing Function $y(t)$	Forced Response $x_f(t)$
Constant: $y(t) = K$	$x_f(t) = A$
Exponential: $y(t) = Ke^{-bt}$	$x_f(t) = Ae^{-bt}$
Sinusoidal: $y(t) = K\sin(\omega t + \theta)$	$x_f(t) = A\sin\omega t + B\cos\omega t$

Don't use this part – this is for DC only

Summary of complete response for simple RC and RL circuits for $t > t_0$ if dc source turns on at $t = t_0$





$$v(t) = V_{OC} + [v(t_0) - V_{OC}]e^{-(t-t_0)/\tau}$$

$$\tau = R_tC$$

$$i(t) = I_{SC} + [i(t_0) - I_{SC}]e^{-(t-t_0)/\tau}$$

$$\tau = L/R_t$$

Finding the Complete Response

$x(t) = x_n(t) + x_f(t)$ (First order circuits)

1. **Find the differential equation using Kirchhoff's and Ohm's laws. Need an initial condition for the complete response.**

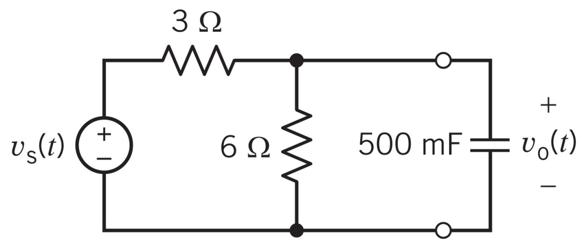
2. Find forced response (including coefficients) using the differential equation (no initial conditions are needed)

3. Find the time constant; then write the natural response (with one unknown coefficient)

4. Add two responses together (the complete response now has one unknown coefficient)

5. Use the initial condition to find the unknown coefficient

P 8.6-2



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$$v_s(t) = 3 + 3u(t)\ \text{V}$$

Use the mathematical approach this time

1. Find the differential equation and the initial condition

Thévenin circuit as seen by the capacitor: $v_{oc}(t) = \dfrac{6}{9} v_s(t) = 2 + 2u(t)\ \text{V}$

$$v_{oc}(t) = 2\ \text{V}, \qquad t < 0$$

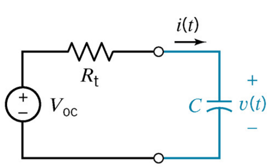
$$v_{oc}(t) = 4\ \text{V}, \qquad t > 0$$

$$R_{th} = \frac{6 \times 3}{6+3} = 2\ \Omega$$

$$\tau = CR_{th} = 1\ \text{s}$$

Initial condition: $v_o(t) = 2\ \text{V} \quad t < 0 \Rightarrow v_o(0) = 2\ \text{V}$

Equation for the current through capacitor: $for\ t > 0$



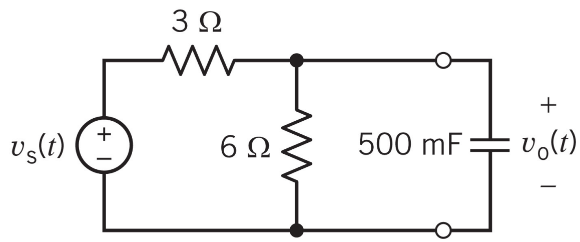
(b)

$$i_o(t) = C\frac{dv_o(t)}{dt} = \frac{v_{oc}(t) - v_o(t)}{R_{th}} \Rightarrow \frac{dv_o(t)}{dt} + \frac{v_o(t)}{CR_{th}} = \frac{v_{oc}(t)}{CR_{th}}$$

Capacitor current = Thevenin Resistor current

$$\frac{dv_o(t)}{dt} + v_o(t) = 4 \qquad (1)$$

P 8.6-2



$v_s(t) = 3 + 3u(t)$ V

2. Find the forced response

$$\frac{dv_{of}(t)}{dt} + v_{of}(t) = 4 \quad (1)$$

Assume constant forced response: $v_{of}(t) = N$ (2)

Put (2) in (1): $\frac{dN}{dt} + N = 4$ ⇒ $N = 4$ ⇒ $v_{o,f}(t) = 4$

3. Find the natural response

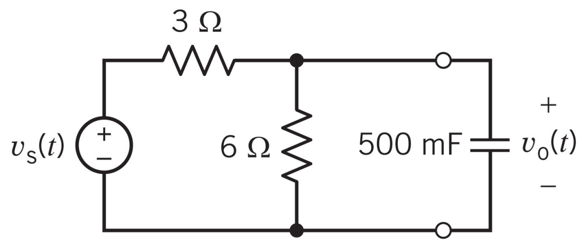
Zero the forcing function in (1): $\frac{dv_o(t)}{dt} + v_o(t) = 0$ (Homogenous equation)

Characteristic equation: $s + 1 = 0$ (Using differential operator approach)

Solution to characteristic equation: $s = -1$

Natural response $v_{o,n}(t) = Ae^{st} = Ae^{-t}$

P 8.6-2



$v_s(t) = 3 + 3u(t)$ V

4. Add forced and natural responses

$$v_o(t) = v_{of}(t) + v_{on}(t) = 4 + Ae^{-t}$$

5. Use the initial condition to find the unknown coefficient

$$v_o(0) = 4 + Ae^0 = 2 \Rightarrow A = -2$$

$$\boxed{v_o(t) = 4 - 2e^{-t}}$$

Complete response !

Check at home to see the engineering (formula) approach gives the same result

Reminder

You need to memorise these!
They are not in formula sheet.

Voltage and Current Relationships in the Time Domain (using derivatives)

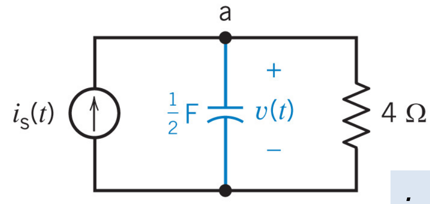
	Resistor	Capacitor	Inductor
Voltage	$R\,i_R$	v_C (no instantaneous changes in time)	$L\dfrac{di_L}{dt}$
Current	i_R	$C\dfrac{dv_C}{dt}$	i_L (no instantaneous changes in time)

Voltage and Current Relationships using s-operators

	Resistor	Capacitor	Inductor
Voltage	$R\,i_R$	$\dfrac{1}{sC}i_C$	$sL\,i_L$
Current	i_R	i_C	i_L

$$s = \frac{d}{dt}$$

Example 8.7-2



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$i_s(t) = 10\sin 2t\, u(t)$ A $v(0) = 0$ V

We must use the mathematical approach (since time varying source)

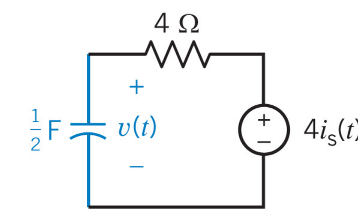
1. Find the differential equation

Thévenin circuit seen by the capacitor:

$$v_{oc}(t) = 4i_s(t) = 40\sin 2t \text{ V} \qquad t > 0$$

$$R_{th} = 4\ \Omega$$

$$\tau = CR_{th} = 2\text{ s}$$

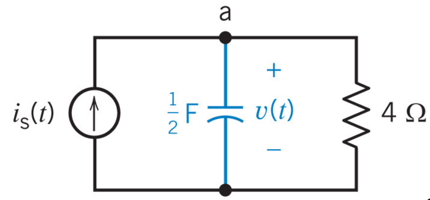


Differential equation (current through resistor):

$$i_C(t) = C\frac{dv(t)}{dt} = \frac{v_{oc}(t) - v(t)}{R_{th}} \quad \Rightarrow \quad \frac{dv(t)}{dt} + \frac{v(t)}{CR_{th}} = \frac{v_{oc}(t)}{CR_{th}} \qquad t > 0$$

$$\frac{dv(t)}{dt} + \frac{v(t)}{2} = 20\sin 2t \qquad (1)$$

Example 8.7-2



$i_s(t) = 10\sin 2t\,u(t)$ A $\quad v(0) = 0$ V

2. Find the forced response

$$\frac{dv_f(t)}{dt} + \frac{v_f(t)}{2} = 20\sin 2t \quad (1)$$

Assume sinusoidal forced response: $v_f(t) = A\sin 2t + B\cos 2t$ (2)

$$\Rightarrow \quad \frac{dv_f(t)}{dt} = 2A\cos 2t - 2B\sin 2t \quad (3)$$

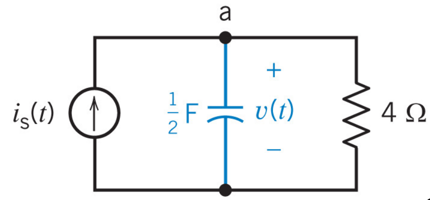
Put (2) and (3) in (1):

$$2A\cos 2t - 2B\sin 2t + \frac{A\sin 2t + B\cos 2t}{2} = 20\sin 2t$$

cosine terms: $2A + \frac{B}{2} = 0$ (4) sine terms: $-2B + \frac{A}{2} = 20$ (5)

Solve (4) and (5) together: $A = 2.35$, $B = -9.41$

Example 8.7-2



$i_s(t) = 10 \sin 2t \, u(t)$ A $\quad v(0) = 0$ V

2. Forced response

$$v_f(t) = 2.35 \sin 2t - 9.41 \cos 2t$$

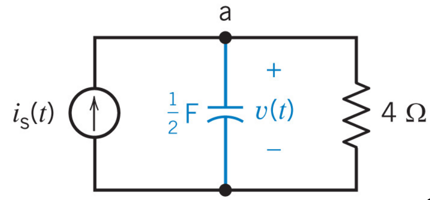
3. Find the natural response

$$\frac{dv(t)}{dt} + \frac{v(t)}{2} = 0 \Rightarrow s + \frac{1}{2} = 0 \Rightarrow s = -\frac{1}{2} \Rightarrow v_n(t) = De^{-t/2}$$

4. Add forced and natural responses

$$v(t) = v_f(t) + v_n(t) = 2.35 \sin 2t - 9.41 \cos 2t + De^{-t/2}$$

Example 8.7-2



$$i_s(t) = 10\sin 2t\, u(t)\ \text{A} \qquad v(0) = 0\ \text{V}$$

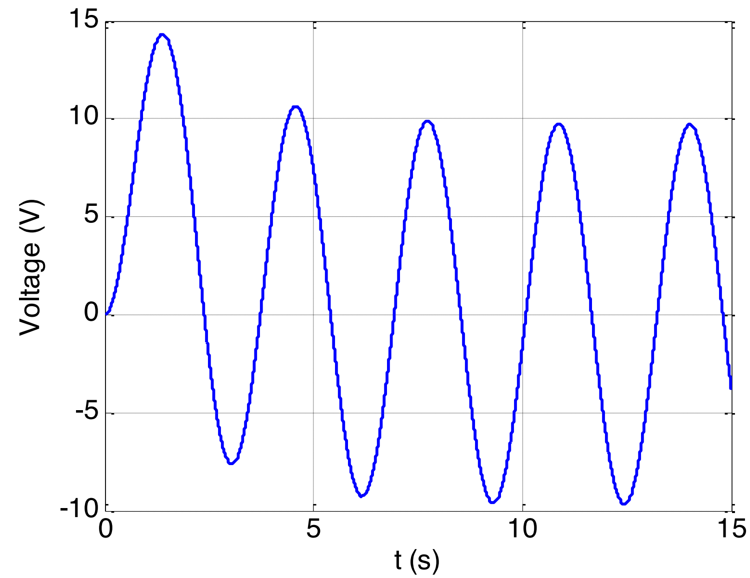
$$v(t) = v_f(t) + v_n(t) = 2.35\sin 2t - 9.41\cos 2t + De^{-t/2}$$

5. Use the initial condition to find the unknown coefficient

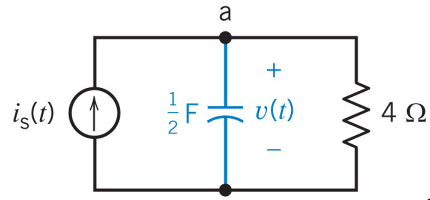
$$v(0) = 2.35\times 0 - 9.41\times 1 + D\times 1 = 0 \quad \Rightarrow \quad D = 9.41$$

Complete response !

$$\boxed{v(t) = 2.35\sin 2t - 9.41\cos 2t + 9.41e^{-t/2}}$$



Example 8.7-2



$i_s(t) = 10 \sin 2t \, u(t)$ A $\quad v(0) = 0$ V

$$v(t) = 2.35 \sin 2t - 9.41 \cos 2t + 9.41e^{-t/2}$$

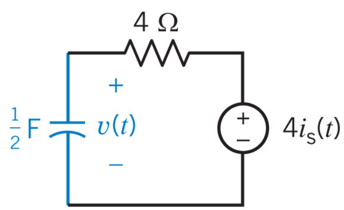
Find capacitor current

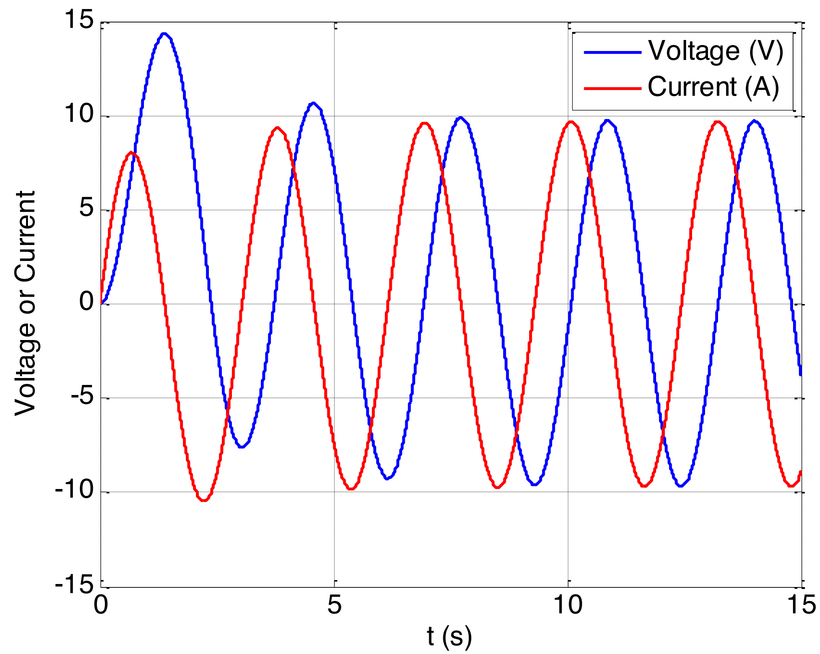
Going back our original current equation:

$$i(t) = C\frac{dv(t)}{dt} = \frac{v_{oc}(t) - v(t)}{R_{th}}$$

$v_{oc}(t) = 40 \sin 2t$ V

$R_{th} = 4\ \Omega$

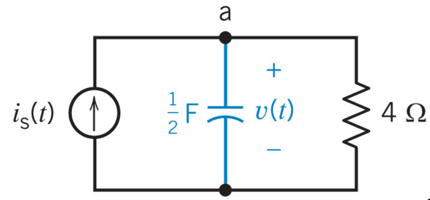




$$i(t) = 10 \sin 2t - 0.59 \sin 2t + 2.35 \cos 2t - 2.35e^{-t/2}$$

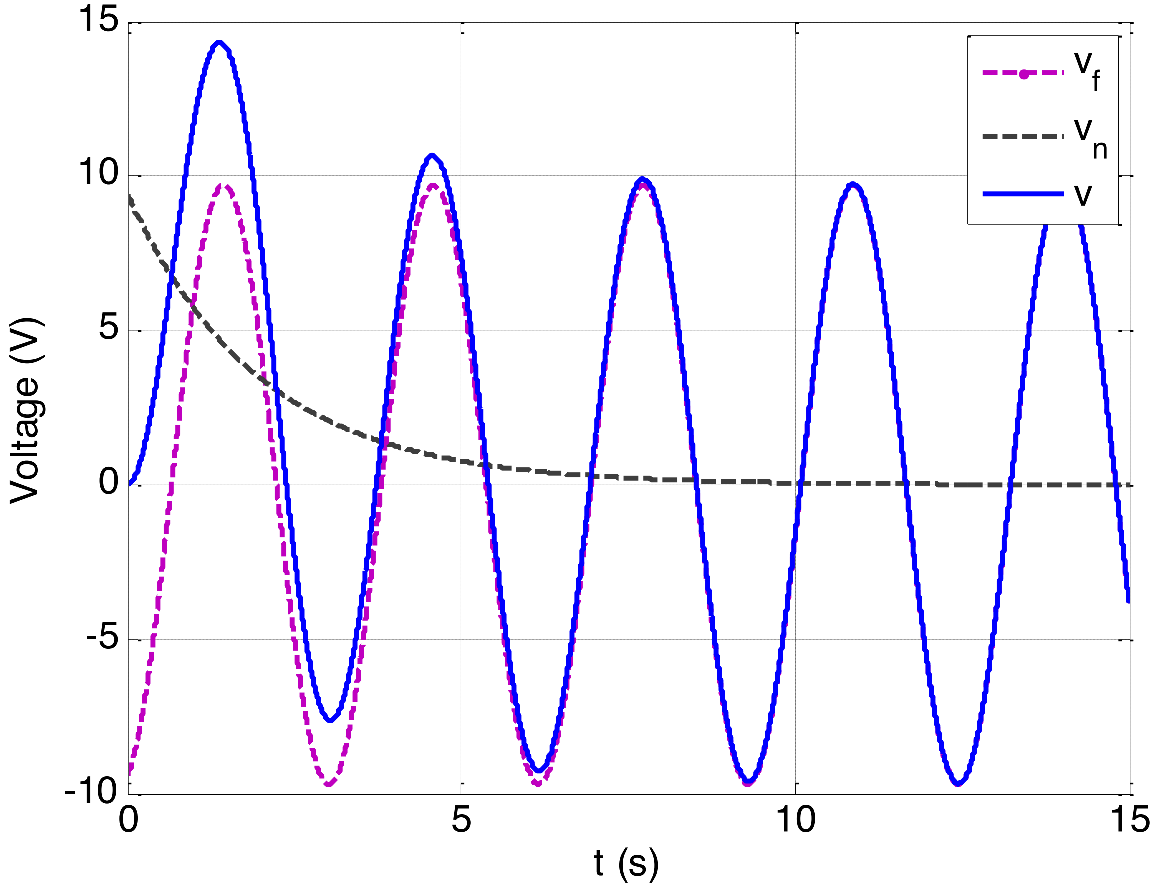
$$i(t) = 9.41 \sin 2t + 2.35 \cos 2t - 2.35e^{-t/2}$$

Example 8.7-2



$i_s(t) = 10 \sin 2t\, u(t)$ A $\quad v(0) = 0$ V

$$v(t) = 2.35 \sin 2t - 9.41 \cos 2t + 9.41 e^{-t/2}$$



A useful relationship

A cosine waveform and a sine waveform with the same frequency, when added together, make a cosine waveform with a phase shift.

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2+B^2}\cos(\omega t - \theta)$$

It is easy to prove this using Euler's formula

$$\theta = \tan^{-1}\frac{B}{A}$$

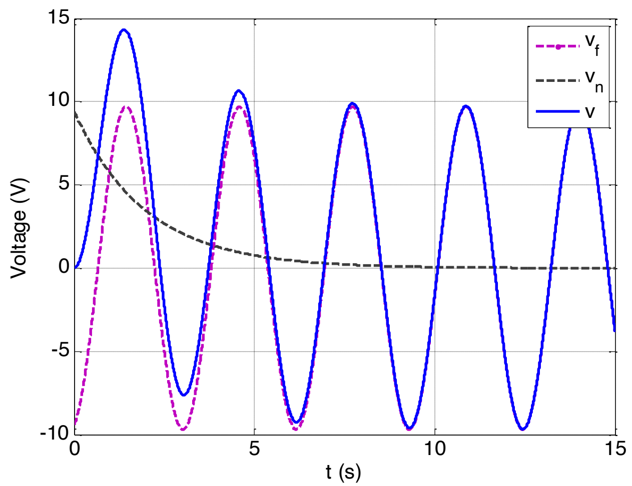
(phase shift is in radians)

$$v(t) = 2.35\sin 2t - 9.41\cos 2t + 9.41e^{-t/2}$$
$$= -(9.41\cos 2t + 2.35\sin 2t) + 9.41e^{-t/2}$$

$(Here, A = 9.41\ and\ B = 2.35)$

$$v(t) = -9.7\cos(2t + 0.245) + 9.41e^{-t/2}$$

$0.245\ \text{rad} = 14\,°$



Complete Response for Second Order Circuits

Chapter 9 in Dorf

Summary of Complete Responses

Second Order Circuits

You will be given these equations in the exam!

(Circuit specific) Differential equation: $\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = f(t)$

Characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$

CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$	$(A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$

	INPUT, *f(t)*	FORCED RESPONSE, x_f
Constant	K	A
Ramp	$K\,t$	$A + Bt$
Sinusoid	$K \cos \omega t$, $K \sin \omega t$, or $K \cos(\omega t + \theta)$	$A \cos \omega t + B \sin \omega t$
Exponential	Ke^{-bt}	Ae^{-bt}

Solving for Complete Response

$x(t) = x_n(t) + x_f(t)$

1. **Find the differential equation and the initial conditions using Kirchhoff's and Ohm's laws.**
2. Find forced response (including coefficients) from the differential equation (no initial conditions are needed)
3. Find natural frequencies; then write the natural response (with two unknown coefficients if second order)
4. Add together (complete response with two unknown coefficients if second order)
5. Use two initial conditions to find the remaining two unknown coefficients, for example

$$x(0) \text{ and } \frac{dx(0)}{dt}$$

Summary of source-free RLC circuits

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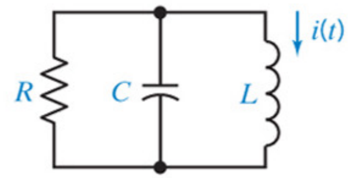
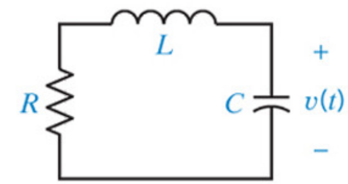
Natural Response	PARALLEL *RLC*	SERIES *RLC*
Circuit		
Differential equation	$\frac{d^2}{dt^2}i(t) + \frac{1}{RC}\frac{d}{dt}i(t) + \frac{1}{LC}i(t) = 0$	$\frac{d^2}{dt^2}v(t) + \frac{R}{L}\frac{d}{dt}v(t) + \frac{1}{LC}v(t) = 0$
Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$	$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$
Damping coefficient, rad/s	$\alpha = \frac{1}{2RC}$	$\alpha = \frac{R}{2L}$
Resonant frequency, rad/s	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
Damped resonant frequency, rad/s	$\omega_d = \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$	$\omega_d = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$
Natural frequencies: overdamped case	$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$ when $R < \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$ when $R > 2\sqrt{\frac{L}{C}}$
Natural frequencies: critically damped case	$s_1 = s_2 = -\frac{1}{2RC}$ when $R = \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1 = s_2 = -\frac{R}{2L}$ when $R = 2\sqrt{\frac{L}{C}}$
Natural frequencies: underdamped case	$s_1, s_2 = -\frac{1}{2RC} \pm j\sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$ when $R > \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1, s_2 = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$ when $R < 2\sqrt{\frac{L}{C}}$

Types of Natural Responses

Second Order Circuits

Differential equation: $\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = f(t)$

Characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$

CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped $\alpha^2 > \omega_0^2$	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped $\alpha^2 = \omega_0^2$	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped $\alpha^2 < \omega_0^2$	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_\mathrm{d}$	$(A_1 \cos \omega_\mathrm{d} t + A_2 \sin \omega_\mathrm{d} t)e^{-\alpha t}$

Damped resonant frequency: ω_d

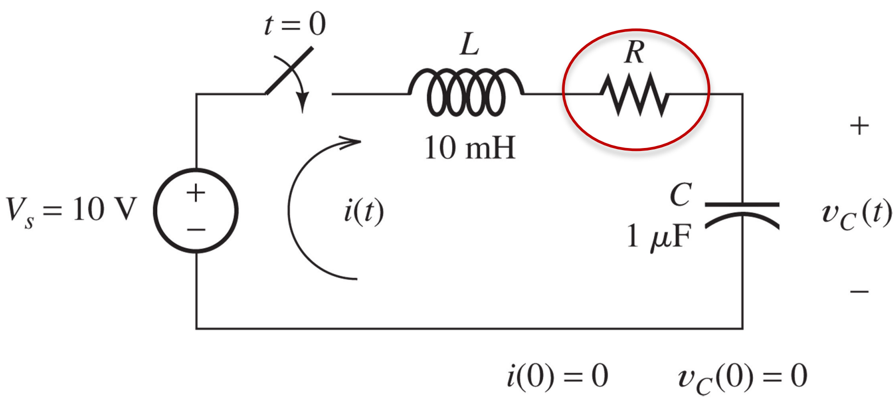
Alternatively, we could use: Damping ratio: $\zeta = \frac{\alpha}{\omega_0}$

ζ<1 $\rightarrow$underdamped, ζ=1 $\rightarrow$critically damped

ζ>1 $\rightarrow$overdamped

Types of Natural Responses

Example 4.5 in Hambley

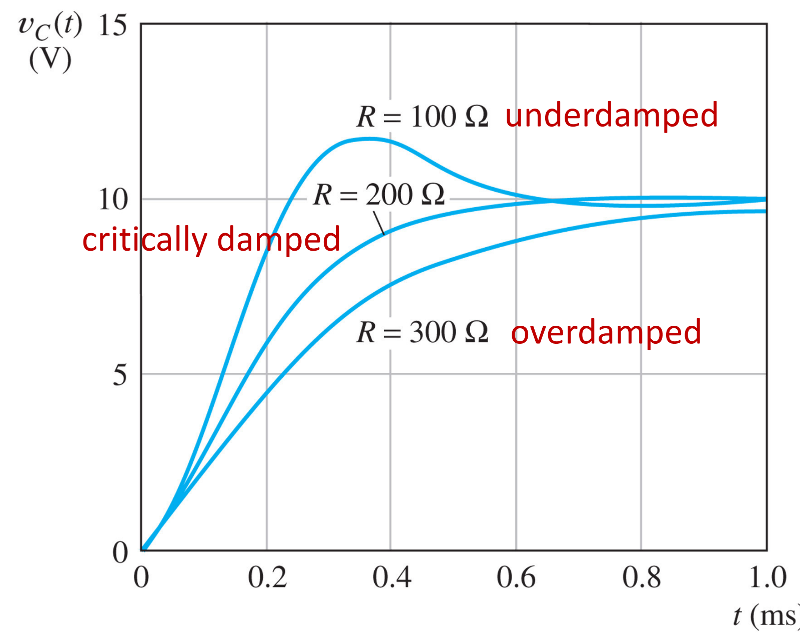


$For\ t > 0$:

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = \frac{V_s}{LC}$$

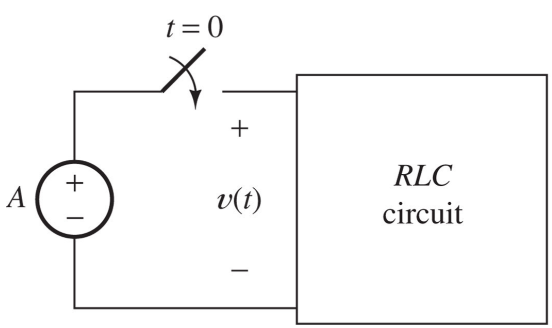
$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Type of natural response depends on the resistance value



More extreme damping ratios

Step response of RLC circuit (Section 4.5 in Hambley)



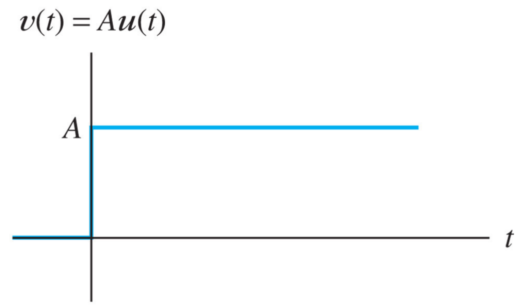
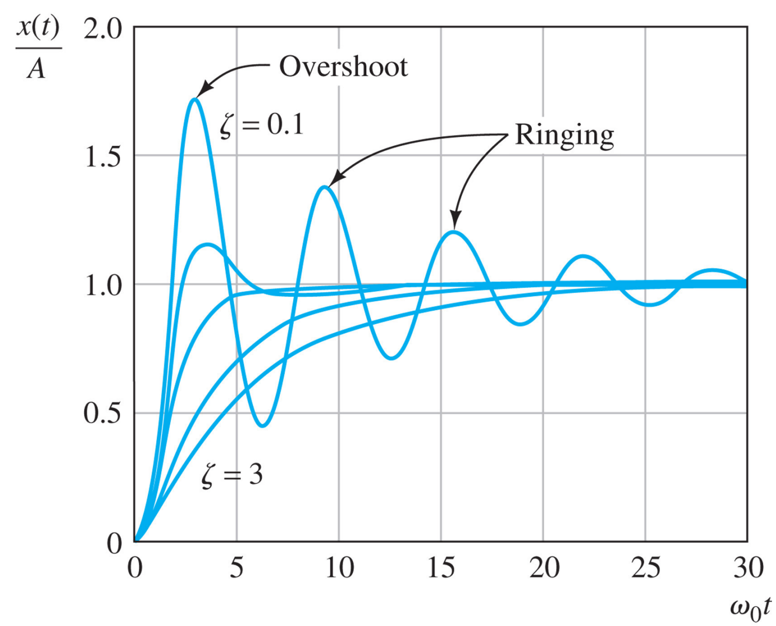


Figure 4.29 Normalized step responses for second-order systems described by Equation 4.99 with damping ratios of ζ = 0.1, 0.5, 1, 2, and 3. The initial conditions are assumed to be *x*(*0*) = 0 and *x'*(*0*) = 0.

$$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = Au(t)$$

$$\zeta = \frac{\alpha}{\omega_0}$$



Parallel RLC circuit problem

Consider a parallel *RLC* circuit having an inductance of 10 mH and a capacitance of 100 μF. Determine the resistor values that would lead to overdamped and underdamped responses.

We first calculate the resonant frequency of the circuit:

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(10 \times 10^{-3})(100 \times 10^{-6})}} = 10^3 \text{ rad/s}$$

An *overdamped* response will result if $\alpha > \omega_0$; an *underdamped* response will result if $\alpha < \omega_0$. Thus,

$$\frac{1}{2RC} > 10^3$$

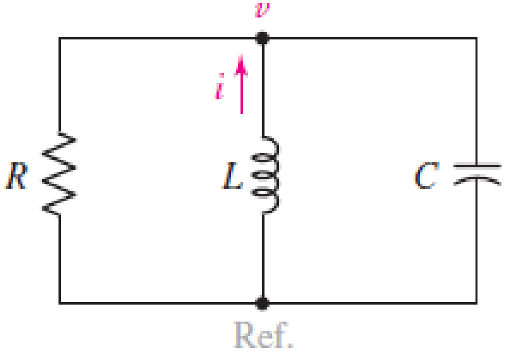
and so

$$R < \frac{1}{(2000)(100 \times 10^{-6})}$$

or

$$R < 5\ \Omega$$

leads to an overdamped response; $R > 5\ \Omega$ leads to an underdamped response.



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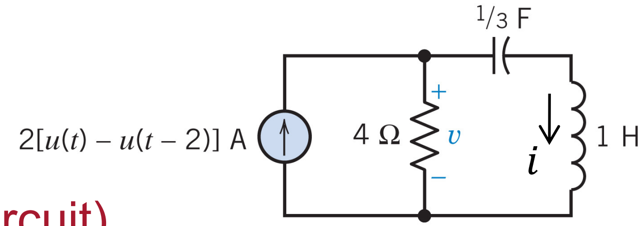


Dorf (Series RLC circuit)

We want the voltage across the resistor

TASK: Calculate $v(t)$ $t > 0$

$$i_s(t) = 2[u(t) - u(t-2)] \text{ A}$$

$i_s(t) = 0 \text{ A} \qquad t < 0$

$i_s(t) = 2 \text{ A} \qquad 0 < t < 2 \text{ s}$

$i_s(t) = 0 \text{ A} \qquad t > 2 \text{ s}$

Assume initial conditions:

$v_C(0) = 0 \qquad i_L(0) = 0$

It is easier to use <u>superposition</u> in this case:

$$i_s(t) = 2u(t) + [-2u(t-2)] \text{ A}$$

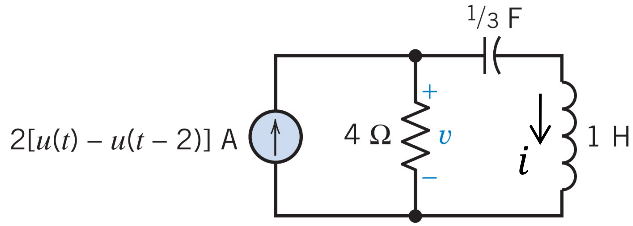
1. Find the differential equation (using the differential operator method):

$$v_C = \frac{1}{sC} i_C \qquad v_L = sLi_L \qquad i_C = i_L = i \qquad v = R(i_s - i) = \frac{1}{sC} i + sLi$$

$$R(i_s - i) = \frac{1}{sC} i + sLi \quad \Rightarrow \quad sRi_s = \frac{1}{C} i + s^2 Li + sRi \quad \Rightarrow \quad 4\frac{di_s}{dt} = 3i + \frac{d^2 i}{dt^2} + 4\frac{di}{dt}$$

Differential equation for $i_C = i_L = i$

P 9.8-5



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$$\frac{d^2 i}{dt^2} + 4\frac{di}{dt} + 3i = 4\frac{di_s}{dt} \quad (1) \qquad v_C(0) = 0 \qquad i_L(0) = 0$$

Standard form of the differential equation and the initial conditions

Superposition

$$i_s(t) = i_{s1}(t) + i_{s2}(t)$$

$$i_{s1}(t) = 2u(t)\ \mathrm{A}$$

$$i_{s2}(t) = -2u(t-2)\ \mathrm{A}$$

First part of superposition

First solve for: $i_{s1}(t) = 2u(t)\ \mathrm{A}$ $\quad t > 0 \quad \frac{di_{s1}}{dt} = 0 \Rightarrow \frac{d^2 i}{dt^2} + 4\frac{di}{dt} + 3i = 0 \quad (2)$

2. Find the forced response

Assume constant forced response: $i_{f1} = A$ ⇒ Put in (2): $i_{f1} = 0$

3. Find the natural response

$s^2 + 4s + 3 = 0$ ⇒ $s_1 = -1 \quad s_2 = -3$

Characteristic equation

$$i_{n1}(t) = A_1 e^{-t} + A_2 e^{-3t}$$

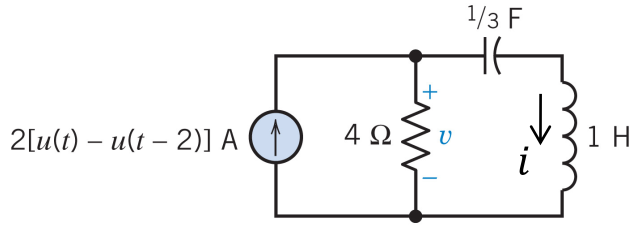
What type of damping?

4. Complete response

$$i_1(t) = A_1 e^{-t} + A_2 e^{-3t} \quad (3)$$

Forced response is zero.
Only the natural response!

P 9.8-5



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5. Find coefficients of the complete response

$$i_1(t) = A_1 e^{-t} + A_2 e^{-3t} \quad (3)$$

First initial condition: $i_L(0) = i_1(0) = i_1(0^+) = 0$ $\quad A_1 + A_2 = 0 \quad (a)$

Second initial condition: $v_C(0) = v_C(0^+) = 0$ We can use $\dfrac{di_1(0^+)}{dt}$ to find the second eq.

$t > 0 \quad i_{s1}(t) = 2\text{ A}$ ⇒ $v_C = v_1 - L\dfrac{di_1}{dt} = 4(i_{s1} - i_1) - L\dfrac{di_1}{dt} = 8 - 4i_1 - \dfrac{di_1}{dt}$ KVL

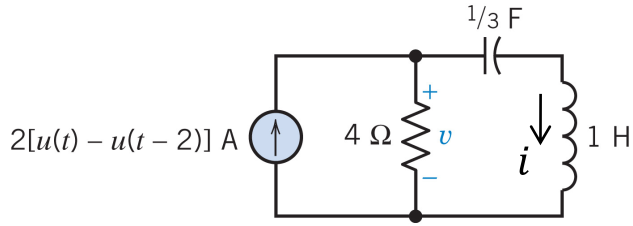
$at\ t = 0^+ \quad v_C(0^+) = 8 - 4i_1(0^+) - \dfrac{di_1(0^+)}{dt}$ ⇒ $\dfrac{di_1(0^+)}{dt} = 8$

(3) ⇒ $\dfrac{di_1}{dt} = -A_1 e^{-t} - 3A_2 e^{-3t}$ ⇒ $8 = -A_1 - 3A_2 \quad (b)$

Solve (a) and (b) together: $A_1 = 4 \quad A_2 = -4$ $\quad i_1(t) = 4e^{-t} - 4e^{-3t} \quad t > 0$

Now we can find the voltage across the resistor: $v_1(t) = 4(2 - i_1) = u(t)(8 - 16e^{-t} + 16e^{-3t})$

P 9.8-5



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Second part of superposition (this is easy now)

Now solve for: $i_{s2}(t) = -2u(t-2)$ A $\quad v_C(2) = 0 \quad i_L(2) = 0$

Note that we assume this part happens in isolation from the first part

Same characteristic equation means: $i_2(t) = A_1 e^{-(t-2)} + A_2 e^{-3(t-2)}$

First initial condition: $i_L(2) = 0 \qquad A_1 + A_2 = 0 \quad (a)$

Second initial condition: $v_C(2) = 0 \qquad i_{s2}(t) = -2$ A $\qquad t > 2$

$$v_C = v_2 - L\frac{di_2}{dt} = 4(i_{s2} - i_2) - L\frac{di_2}{dt} = -8 - 4i_2 - \frac{di_2}{dt}$$

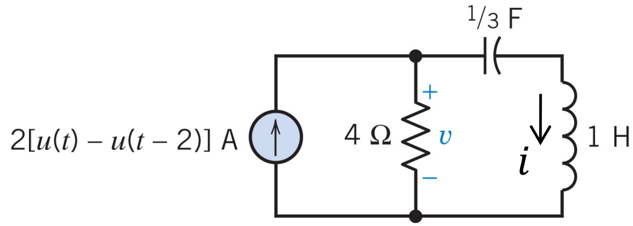
$$v_C(2^+) = -8 - 4i_2(2^+) - \frac{di_2(2^+)}{dt} \Rightarrow 8 = A_1 + 3A_2 \quad (b)$$

Solve (a) and (b) together: $A_1 = -4 \quad A_2 = 4$ $\qquad i_2(t) = -4e^{-(t-2)} + 4e^{-3(t-2)} \qquad t > 2$

$$v_2(t) = 4(-2 - i_2) = u(t-2)\left(-8 + 16e^{-(t-2)} - 16e^{-3(t-2)}\right)$$

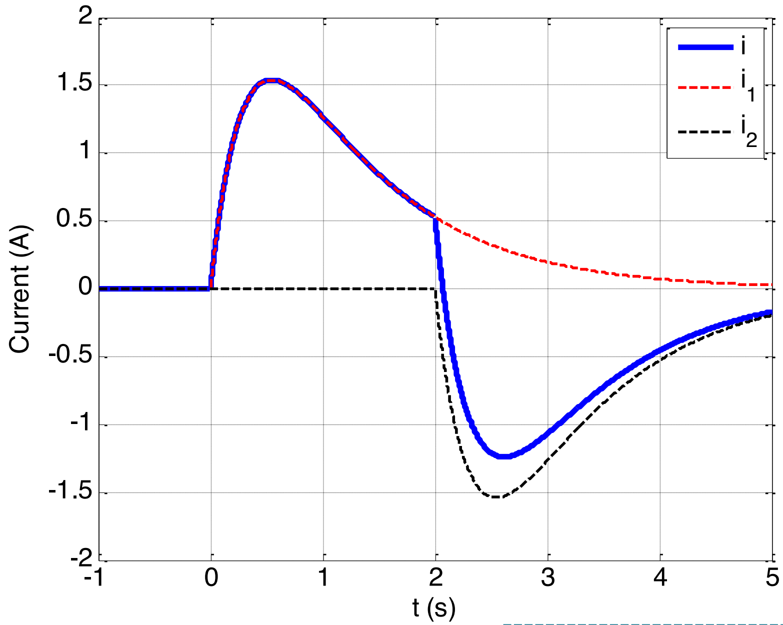
P 9.8-5

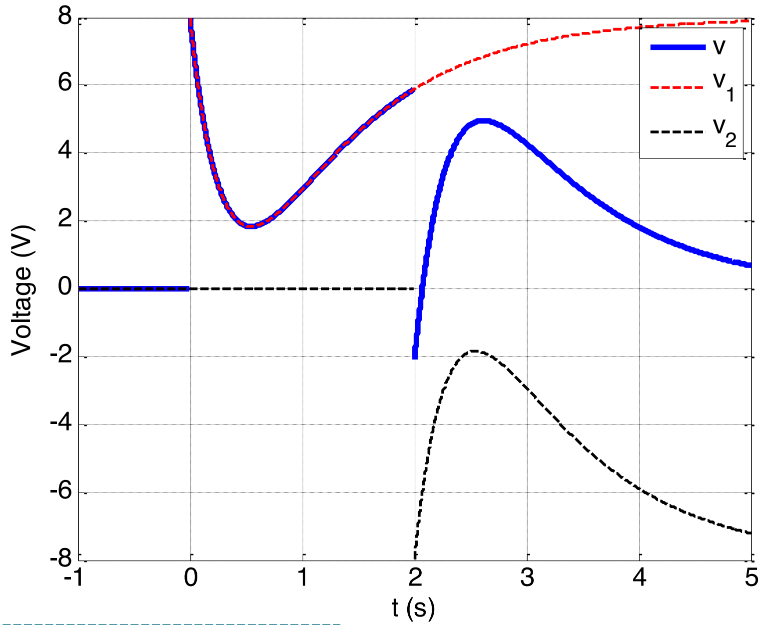
Dorf



$$i(t) = u(t)(4e^{-t} - 4e^{-3t}) + u(t-2)\left(-4e^{-(t-2)} + 4e^{-3(t-2)}\right)$$

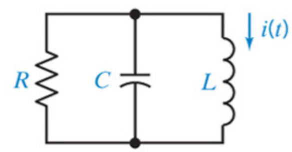
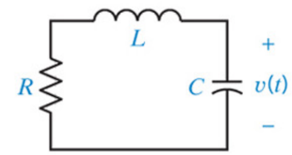
$$v(t) = u(t)(8 - 16e^{-t} + 16e^{-3t}) + u(t-2)\left(-8 + 16e^{-(t-2)} - 16e^{-3(t-2)}\right)$$





Superposition of currents and voltages

Useful information for the laboratory session

	PARALLEL *RLC*	SERIES *RLC*
Circuit		
Differential equation	$\frac{d^2}{dt^2}i(t) + \frac{1}{RC}\frac{d}{dt}i(t) + \frac{1}{LC}i(t) = 0$	$\frac{d^2}{dt^2}v(t) + \frac{R}{L}\frac{d}{dt}v(t) + \frac{1}{LC}v(t) = 0$
Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$	$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$
Damping coefficient, rad/s	$\alpha = \frac{1}{2RC}$	$\alpha = \frac{R}{2L}$
Resonant frequency, rad/s	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
Damped resonant frequency, rad/s	$\omega_d = \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$	$\omega_d = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$
Natural frequencies: overdamped case	$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$ when $R < \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$ when $R > 2\sqrt{\frac{L}{C}}$
Natural frequencies: critically damped case	$s_1 = s_2 = -\frac{1}{2RC}$ when $R = \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1 = s_2 = -\frac{R}{2L}$ when $R = 2\sqrt{\frac{L}{C}}$
Natural frequencies: underdamped case	$s_1, s_2 = -\frac{1}{2RC} \pm j\sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$ when $R > \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1, s_2 = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$ when $R < 2\sqrt{\frac{L}{C}}$

OVERDAMPED

Find an expression for $v_C(t)$ valid for $t > 0$ in the circuit of Fig. 9.3*a*.

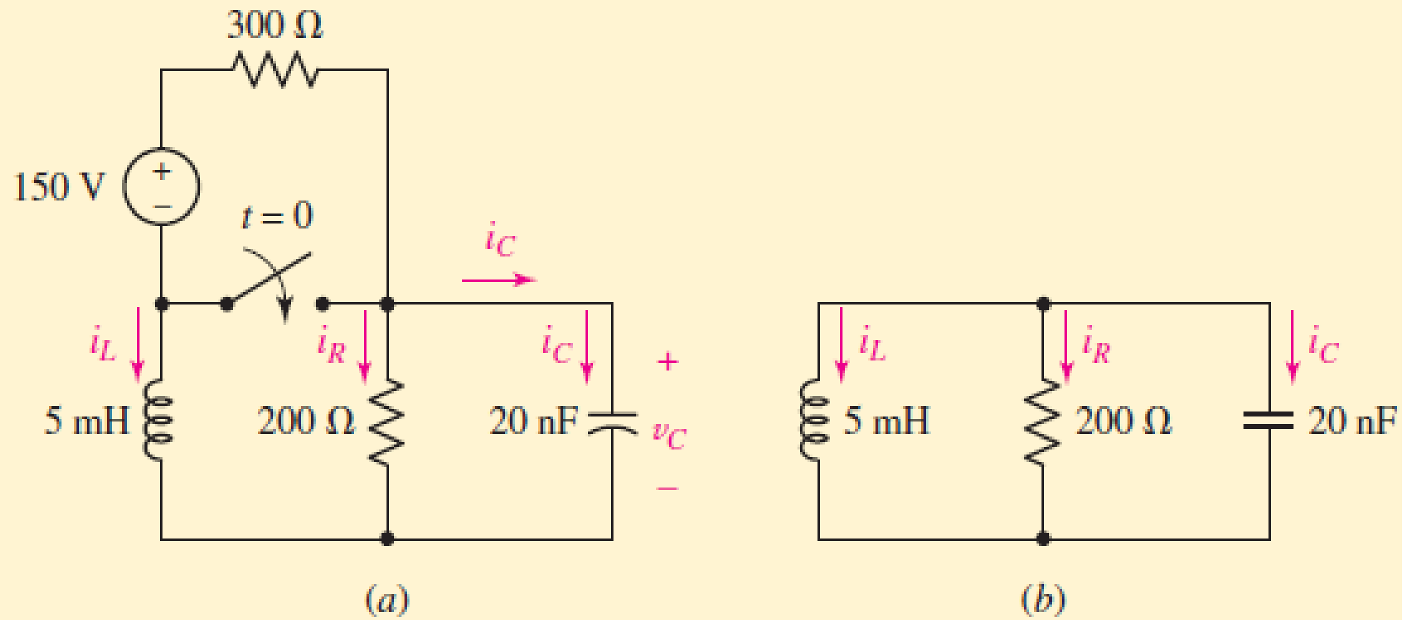


FIGURE 9.3 (*a*) An *RLC* circuit that becomes source-free at $t = 0$. (*b*) The circuit for $t > 0$, in which the 150 V source and the 300 Ω resistor have been shorted out by the switch, and so are of no further relevance to v_C.

Identify the goal of the problem.

We are asked to find the capacitor voltage after the switch is thrown. This action leads to no sources remaining connected to either the inductor or the capacitor.

Collect the known information.

After the switch is thrown, the capacitor is left in parallel with a 200 Ω resistor and a 5 mH inductor (Fig. 9.3*b*). Thus, $\alpha = 1/2RC = 125{,}000\ \text{s}^{-1}$, $\omega_0 = 1/\sqrt{LC} = 100{,}000$ rad/s, $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -50{,}000\ \text{s}^{-1}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -200{,}000\ \text{s}^{-1}$.

Devise a plan.

Since $\alpha > \omega_0$, the circuit is overdamped and so we expect a capacitor voltage of the form

$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

We know s_1 and s_2; we need to obtain and invoke two initial conditions to determine A_1 and A_2. To do this, we will analyze the circuit at $t = 0^-$ (Fig. 9.4*a*) to find $i_L(0^-)$ and $v_C(0^-)$. We will then analyze the circuit at $t = 0^+$ with the assumption that neither value changes.

CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$	$(A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$

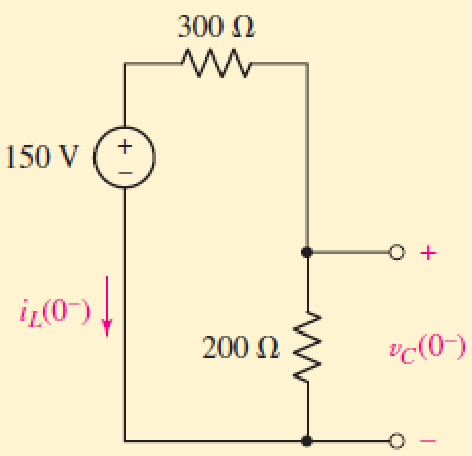
Construct an appropriate set of equations.

From Fig. 9.4*a*, in which the inductor has been replaced with a short circuit and the capacitor with an open circuit, we see that

$$i_L(0^-) = -\frac{150}{200 + 300} = -300 \text{ mA}$$

and

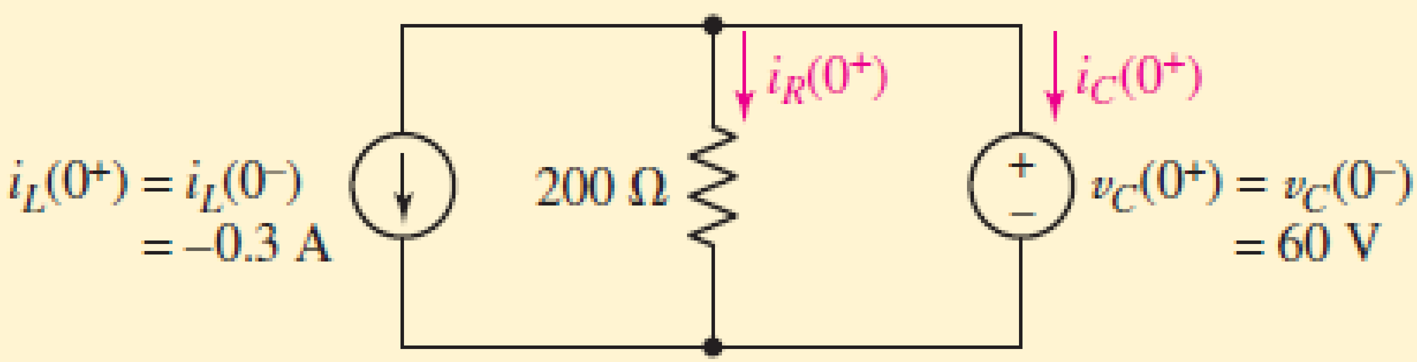
$$v_C(0^-) = 150\frac{200}{200 + 300} = 60 \text{ V}$$



(*a*)

Remember how we defined i_L for t>0

In Fig. 9.4*b*, we draw the circuit at $t = 0^+$, representing the inductor current and capacitor voltage by ideal sources for simplicity. Since neither can change in zero time, we know that $v_C(0^+) = 60$ V.



(*b*)

Determine if additional information is required.

We have an equation for the capacitor voltage: $v_C(t) = A_1 e^{-50{,}000t} + A_2 e^{-200{,}000t}$. We now know $v_C(0) = 60$ V, but a third equation is still required. Differentiating our capacitor voltage equation, we find

$$\frac{dv_C}{dt} = -50{,}000 A_1 e^{-50{,}000t} - 200{,}000 A_2 e^{-200{,}000t}$$

which can be related to the capacitor current as $i_C = C(dv_C/dt)$. Returning to Fig. 9.4b, KCL yields

$$i_C(0^+) = -i_L(0^+) - i_R(0^+) = 0.3 - [v_C(0^+)/200] = 0$$

Attempt a solution.

Application of our first initial condition yields

$$v_C(0) = A_1 + A_2 = 60$$

and application of our second initial condition yields

$$i_C(0) = -20 \times 10^{-9}(50{,}000A_1 + 200{,}000A_2) = 0$$

Solving, $A_1 = 80$ V and $A_2 = -20$ V, so that

$$v_C(t) = 80e^{-50{,}000t} - 20e^{-200{,}000t} \text{ V}, \qquad t > 0$$

Verify the solution. Is it reasonable or expected?

At the very least, we can check our solution at $t = 0$, verifying that $v_C(0) = 60$ V. Differentiating and multiplying by 20×10^{-9}, we can also verify that $i_C(0) = 0$. Also, since we have a source-free circuit for $t > 0$, we expect that $v_C(t)$ must eventually decay to zero as t approaches ∞, which our solution does.

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CRITICAL DAMPING

Current source:

5 A for t<0
0 A for t>=0

Select a value for R_1 such that the circuit of Fig. 9.12 will be characterized by a critically damped response for $t > 0$, and a value for R_2 such that $v(0) = 2$ V.

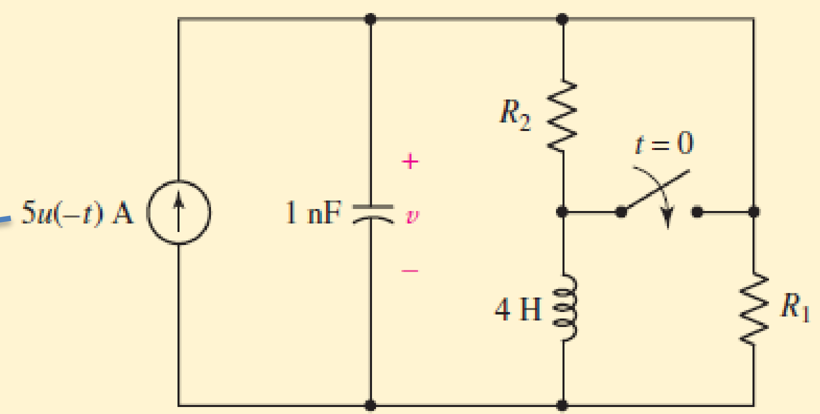


FIGURE 9.12 A circuit that reduces to a parallel *RLC* circuit after the switch is thrown.

We note that at $t = 0^-$, the current source is on, and the inductor can be treated as a short circuit. Thus, $v(0^-)$ appears across R_2, and is given by

$$v(0^-) = 5R_2$$

and a value of 400 mΩ should be selected for R_2 to obtain $v(0) = 2$ V.

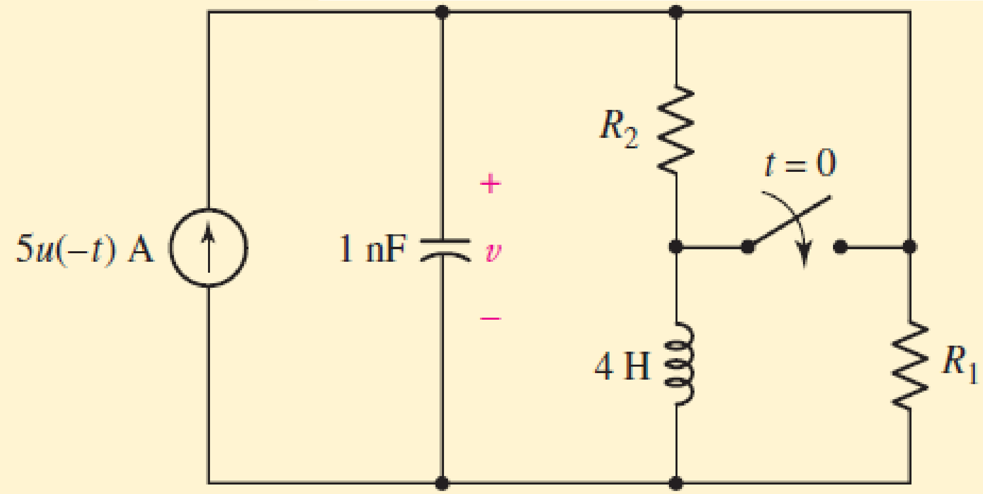
After the switch is thrown, the current source has turned itself off and R_2 is shorted. We are left with a parallel RLC circuit comprised of R_1, a 4 H inductor, and a 1 nF capacitor.

We may now calculate (for $t > 0$)

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 10^{-9} R_1}$$

and

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times 10^{-9}}} = 15{,}810 \text{ rad/s}$$



Therefore, to establish a critically damped response in the circuit for $t > 0$, we need to set $R_1 = 31.63$ kΩ. (*Note: since we have rounded to four significant figures, the pedantic can rightly argue that this is still not* ***exactly*** *a critically damped response—a difficult situation to create.*)

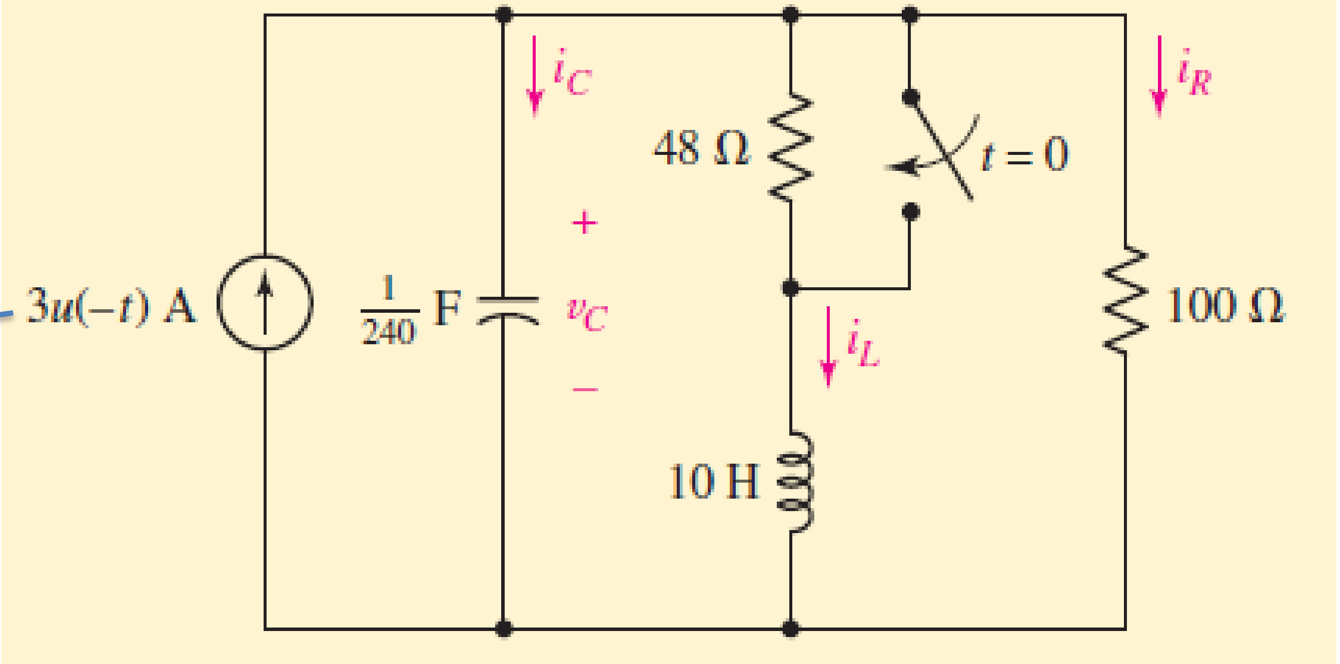
UNDER DAMPING

Determine $i_L(t)$ for the circuit of Fig. 9.17*a*, and plot the waveform.

Current source:

3 A for t<0

0 A for t>=0

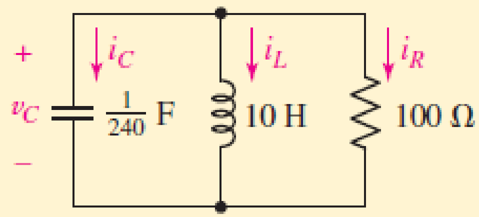


(*a*)

At $t = 0$, both the 3 A source and the 48 Ω resistor are removed, leaving the circuit shown in Fig. 9.17*b*. Thus, $\alpha = 1.2\ \text{s}^{-1}$ and $\omega_0 = 4.899$ rad/s. Since $\alpha < \omega_0$, the circuit is *underdamped*, and we therefore expect a response of the form

$$i_L(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \qquad [28]$$

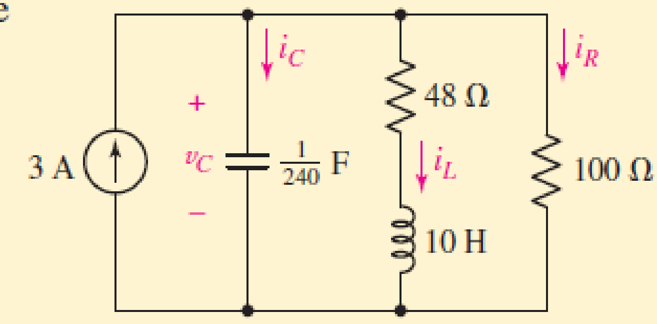
where $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 4.750$ rad/s. The only remaining step is to find B_1 and B_2.



(*b*)

CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$	$(A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$

Figure 9.17c shows the circuit as it exists at $t = 0^-$. We may replace the inductor with a short circuit and the capacitor with an open circuit; the result is $v_C(0^-) = 97.30$ V and $i_L(0^-) = 2.027$ A. Since neither quantity can change in zero time, $v_C(0^+) = 97.30$ V and $i_L(0^+) = 2.027$ A.



(c)

Substituting $i_L(0) = 2.027$ into Eq. [28] yields $B_1 = 2.027$ A. To determine the other constant, we first differentiate Eq. [28]:

$$\frac{di_L}{dt} = e^{-\alpha t}(-B_1\omega_d \sin\omega_d t + B_2\omega_d \cos\omega_d t) - \alpha e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) \qquad [29]$$

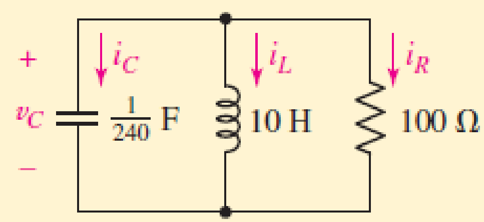
and note that $v_L(t) = L(di_L/dt)$. Referring to the circuit of Fig. 9.17b, we see that $v_L(0^+) = v_C(0^+) = 97.3$ V. Thus, multiplying Eq. [29] by $L = 10$ H and setting $t = 0$, we find that

$$v_L(0) = 10(B_2\omega_d) - 10\alpha B_1 = 97.3$$

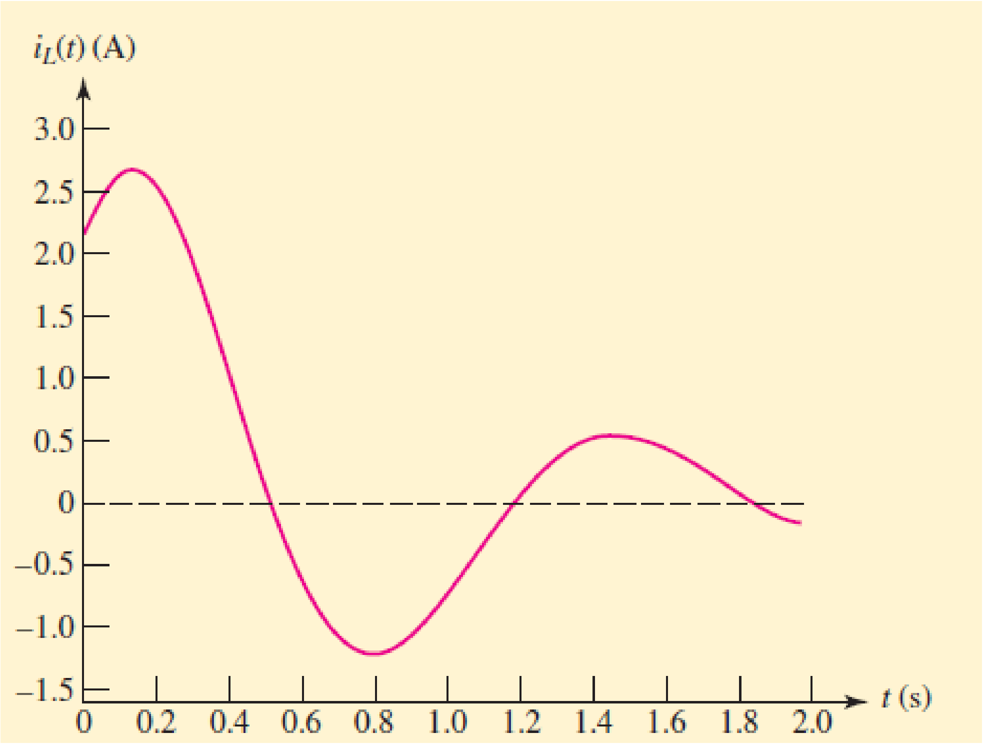
Solving, $B_2 = 2.561$ A, so that

$$i_L = e^{-1.2t}(2.027 \cos 4.75t + 2.561 \sin 4.75t) \qquad \text{A}$$

which we have plotted in Fig. 9.18.



(b)



$i_L(t)$ (A)
3.0
2.5
2.0
1.5
1.0
0.5
0
–0.5
–1.0
–1.5
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
t (s)

TABLE 9.1 Summary of Relevant Equations for Source-Free *RLC* Circuits

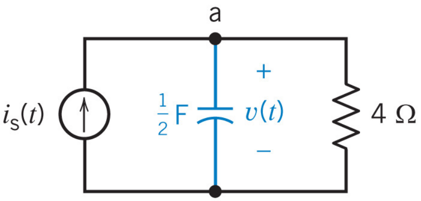
Type	Condition	Criteria	α	ω_0	Response
Parallel	Overdamped	$\alpha > \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$, where $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$
Series			$\frac{R}{2L}$		
Parallel	Critically damped	$\alpha = \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(A_1 t + A_2)$
Series			$\frac{R}{2L}$		
Parallel	Underdamped	$\alpha < \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Series			$\frac{R}{2L}$		

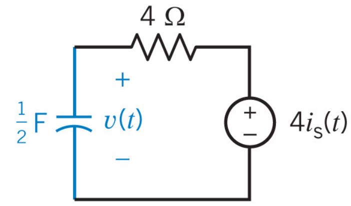
It is evident that if we work in terms of the parameters α, ω_0, and ω_d, the mathematical forms of the responses for the dual situations are identical. An increase in α in either the series or parallel circuit, while keeping ω_0 constant, tends toward an overdamped response. The only caution that we need exert is in the computation of α, which is $1/2RC$ for the parallel circuit and $R/2L$ for the series circuit; thus, α is ~~increased~~ decreased by increasing the series resistance or decreasing the parallel resistance. The key equations for parallel and series *RLC* circuits are summarized in Table 9.1 for convenience.

Stability of circuits with capacitors and inductors

Sections 8.5 and 9.10 in Dorf

Complete response, 1st order





$i_s(t) = 10 \sin 2t\, u(t)$ A

$v(0) = 0$ V

$R_t = 4\ \Omega$

$v_f(t) = 2.35 \sin 2t - 9.41 \cos 2t$

$v_f(t) = -9.7 \cos(2t + 0.245)$

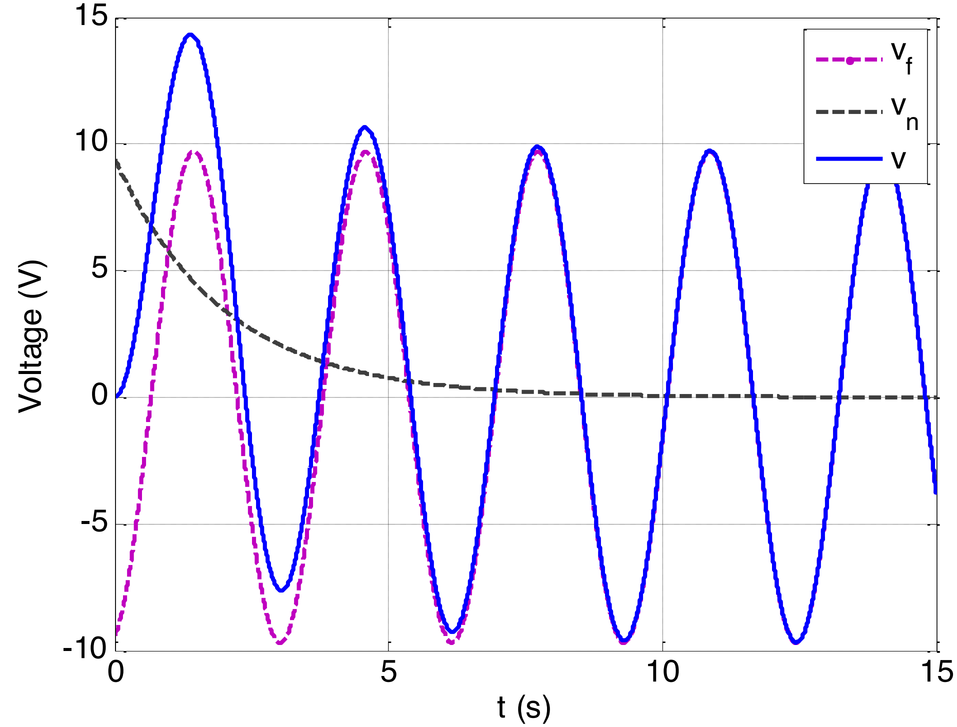
Sinusoid

$\tau = R_t C = 2$ s

$v_n(t) = 9.41 e^{-t/2}$

Decaying Exponential

$v(t) = 2.35 \sin 2t - 9.41 \cos 2t + 9.41 e^{-t/2}$



Is this a stable circuit?

Stability of First Order Circuits

stable $R_t > 0$ Exponentially decaying natural response

unstable $R_t \leq 0$ Exponentially increasing natural response

$$x_n(t) = Ae^{-t/\tau} \qquad \tau = R_t C \quad \text{or} \quad \tau = L/R_t$$

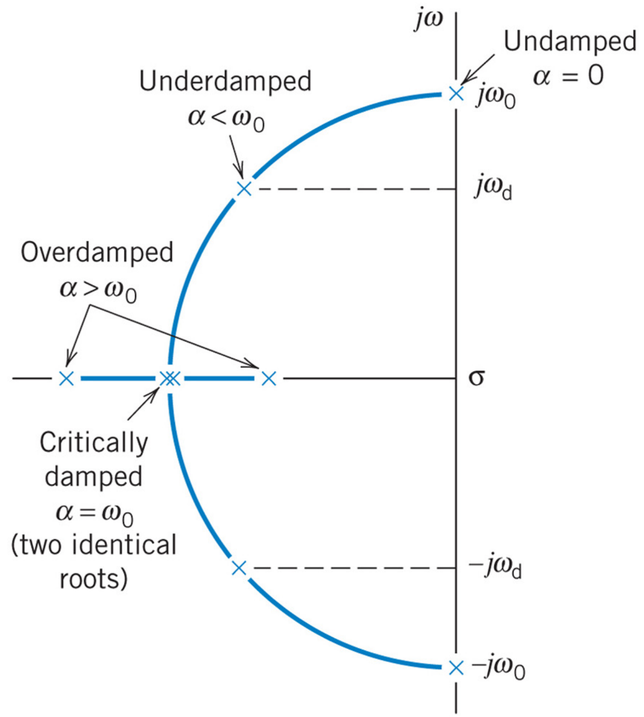
Stability of Second Order Circuits

CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$	$(A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$

s-plane

stable

$\sigma = -\alpha < 0$

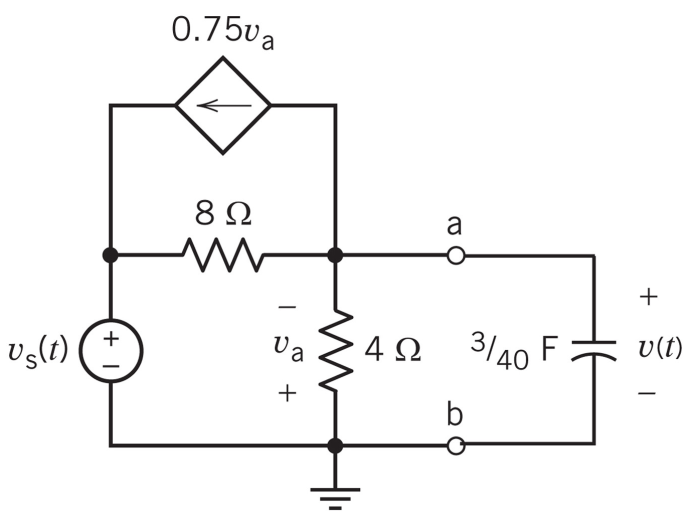


unstable

$\sigma = -\alpha > 0$

Unstable circuit example

Dorf Problem P8.6-12



$v_s(t) = 12 - 6u(t)$ V   Find $v(t)$ for $t \geq 0$

FIRST: Find initial condition: $v(t)$ for $t < 0$

$-v_a = v(t)$ (Note polarity)

Constant voltage means capacitor can be assumed open circuit

Source voltage: $v_s(t) = 12$ V   for $t < 0$

(Note polarity)

KCL at node a:

$$\frac{v(t)}{4} + \frac{v(t) - 12}{8} - 0.75v(t) = 0$$

$$\frac{v(t)}{4} + \frac{v(t) - 12}{8} - \frac{3v(t)}{4} = 0$$

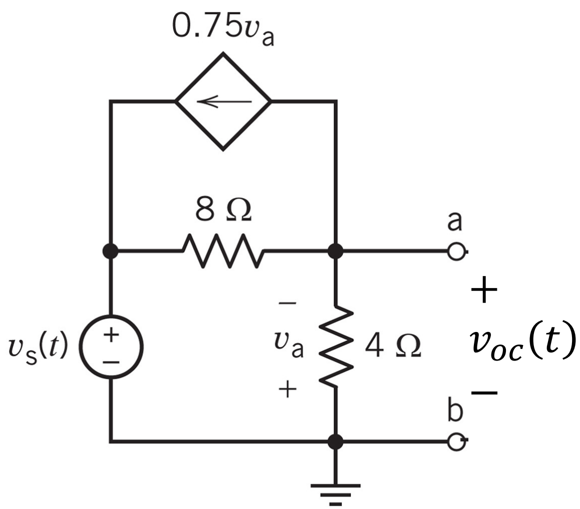
$$2v(t) + v(t) - 12 - 6v(t) = 0$$

$$-3v(t) = 12$$

$v(t) = -4$ V for $t < 0$ and for $t = 0^+$

Unstable circuit example

Dorf Problem P8.6-12 (Thévenin equivalent)



$v_s(t) = 12 - 6u(t)$ V   Find $v(t)$ for $t \geq 0$

Find Thevenin equivalent for $t > 0$

$-v_a = v_{oc}(t)$ open circuit voltage

$v_s(t) = 6$ V

KCL at node a:

$$\frac{v_{oc}(t)}{4} + \frac{v_{oc}(t) - 6}{8} - \frac{3v_{oc}(t)}{4} = 0$$

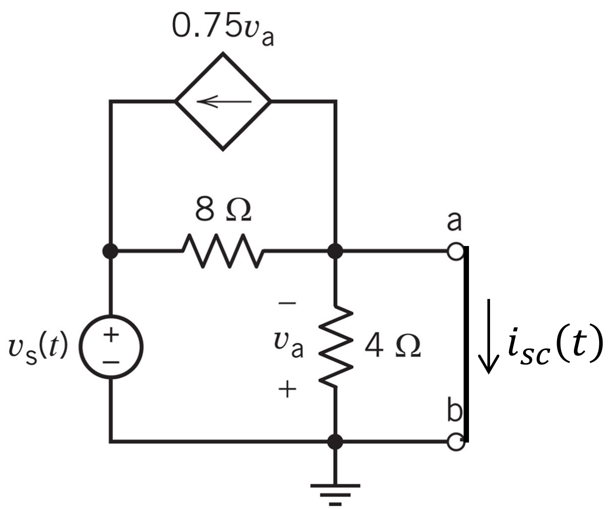
$$2v_{oc}(t) + v_{oc}(t) - 6 - 6v_{oc}(t) = 0$$

$$-3v_{oc}(t) = 6$$

$v_{oc}(t) = -2$ V for $t > 0$

Unstable circuit example

Dorf Problem P8.6-12 (Thévenin equivalent)



$v_s(t) = 12 - 6u(t)$ V Find $v(t)$ for $t \geq 0$

Find short circuit current for $t > 0$

$-v_a = 0$ (short circuit voltage)

$v_s(t) = 6$ V

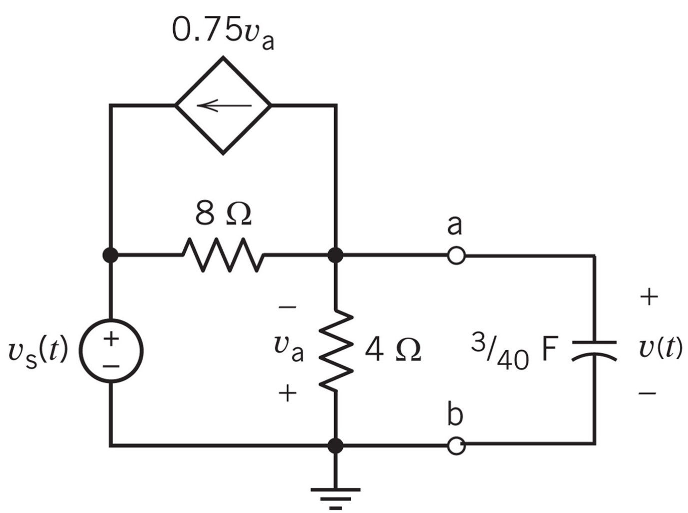
short circuit current: $i_{sc}(t) = \frac{6}{8}$ A for $t > 0$

$$R_t = \frac{v_{oc}(t)}{i_{sc}(t)} = \frac{-2 \times 8}{6} = \frac{-8}{3} \Omega \text{ for } t > 0$$

Take home message: Whilst in reality no resistor value can be negative, we can calculate a negative Thevénin resistance! (Comes from dependent sources in circuit.)

Unstable circuit example

Dorf Problem P8.6-12 (time constant)



$v_s(t) = 12 - 6u(t)$ V Find $v(t)$ for $t \geq 0$

$$R_t = \frac{-8}{3}\ \Omega \text{ for } t > 0$$

$$\tau = R_t C = \frac{-8}{3} \times \frac{3}{40} = -\frac{1}{5} \text{ s for } t > 0$$

$v(t) = -4$ V for $t < 0$ and for $t = 0^+$

$v_{oc}(t) = -2$ V for $t > 0$

$$v(t) = -2 + [-4 + 2]\, e^{5t}$$

$$v(t) = -2 - 2\, e^{5t} \text{ V for } t \geq 0$$

increases exponentially