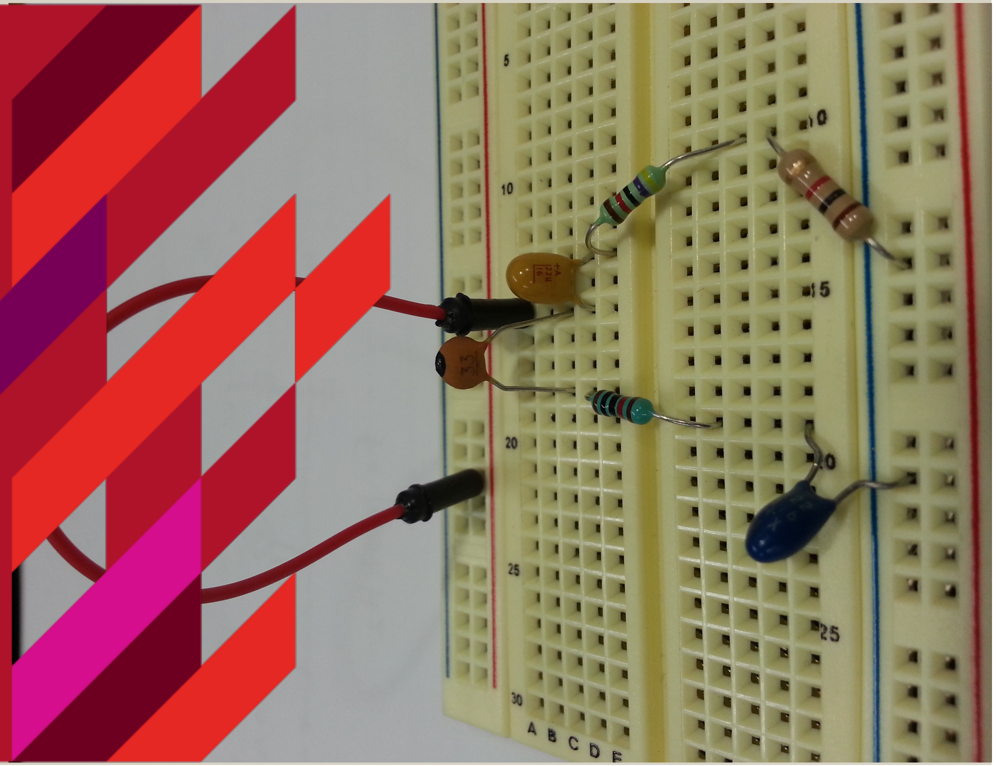
ELEC2070 Circuits and Devices

Week 8: AC Thévenin + Norton Equivalents + Superposition + AC power

Stuart Jackson



Unit in review

Our job: We are concerned with applying one or more sources of electrical energy (a source defined by some forcing function) to a circuit and then quantitatively determining the consequent response throughout that circuit.

First Module: We studied the forced (steady state) response of a circuit to a steady state dc input

Second Module: We found the natural (transient) response when energy storage elements are involved and assumed the form of the forced response in order to obtain the complete response

Third module: We are going to concentrate on the forced response when the steady state forcing function is **sinusoidal**.

Why study sinusoidal functions?

We found sinusoidal behaviour when the natural response was underdamped. In fact, sinusoidal responses are quite common:

1. Motion of a pendulum
2. Bouncing of a ball
3. Vibration of a guitar string

Negative exponentials are also very common (as we have already found) and we show later that when we combine a exponential with a sinusoid, this leads us to the concept of the **complex frequency**.

Any periodic function can be composed as a sum of sinusoidal functions with frequency that are integral multiples of f_0 (Fourier Transform)

Also, the differentials and integrals of sinusoids are sinusoids (this is also true for exponentials) making the analysis easier.

Lastly, sinusoids are found in the power industry.

Review of last lecture

Voltage and Current Relationships in Frequency Domain

	Resistor	Capacitor	Inductor
Voltage	$R\ I_R$	$\frac{1}{j\omega C} I_C$	$j\omega L\ I_L$
Current	I_R	I_C	I_L

IMPEDANCE:

Capacitor

$$\mathbf{Z}_C(\omega) = \frac{\mathbf{V}_C(\omega)}{\mathbf{I}_C(\omega)} = \frac{1}{j\omega C} \quad \Omega$$

Inductor

$$\mathbf{Z}_L(\omega) = \frac{\mathbf{V}_L(\omega)}{\mathbf{I}_L(\omega)} = j\omega L \quad \Omega$$

Resistor

$$\mathbf{Z}_R(\omega) = \frac{\mathbf{V}_R(\omega)}{\mathbf{I}_R(\omega)} = R \quad \Omega$$

Kirchoff’s Laws (from last lecture)

Once a circuit has been converted to the frequency domain:

1. Ohms Law applies with the voltage and current related through the impedance
2. Analyse the circuit using node analysis or mesh equations (Cramer's Rule can help here too)

Now we want to examine the other tools we have learnt to obtain the circuit response with a dc forcing function:

1. Thevenin’s and Norton’s theorems
2. Superposition
3. Source transformations

OVERALL: All the shortcuts and methods used for DC forcing functions are applicable for ac forcing functions.

Impedance can be treated like resistance

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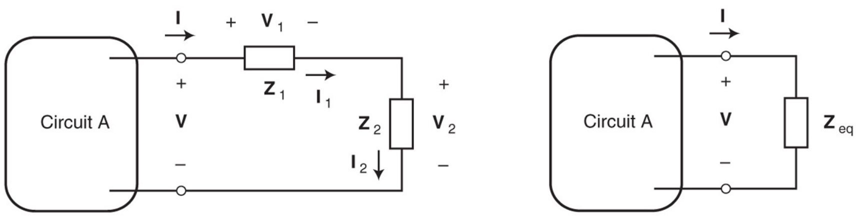
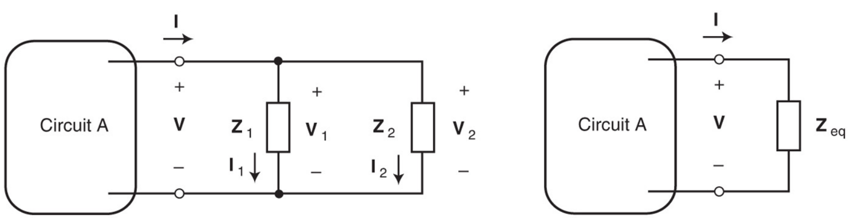
Series orientated impedances



(a) (b)

$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n$$

Parallel orientated impedances

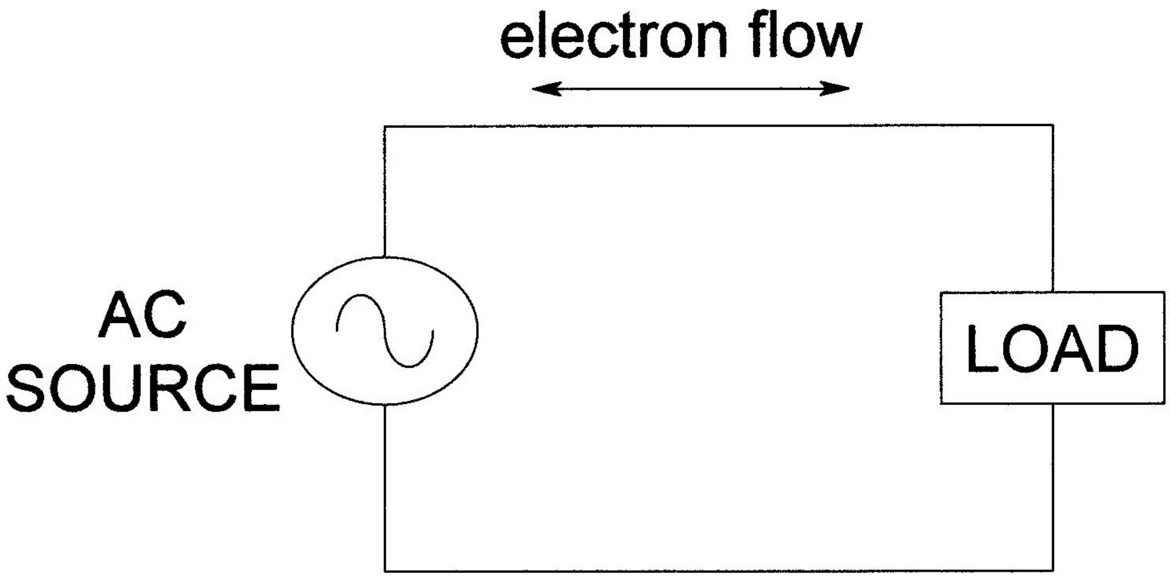


(a) (b)

$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n}}$$

Ohm's and Kirchhoff's Laws will always apply to impedances

Circuit symbol

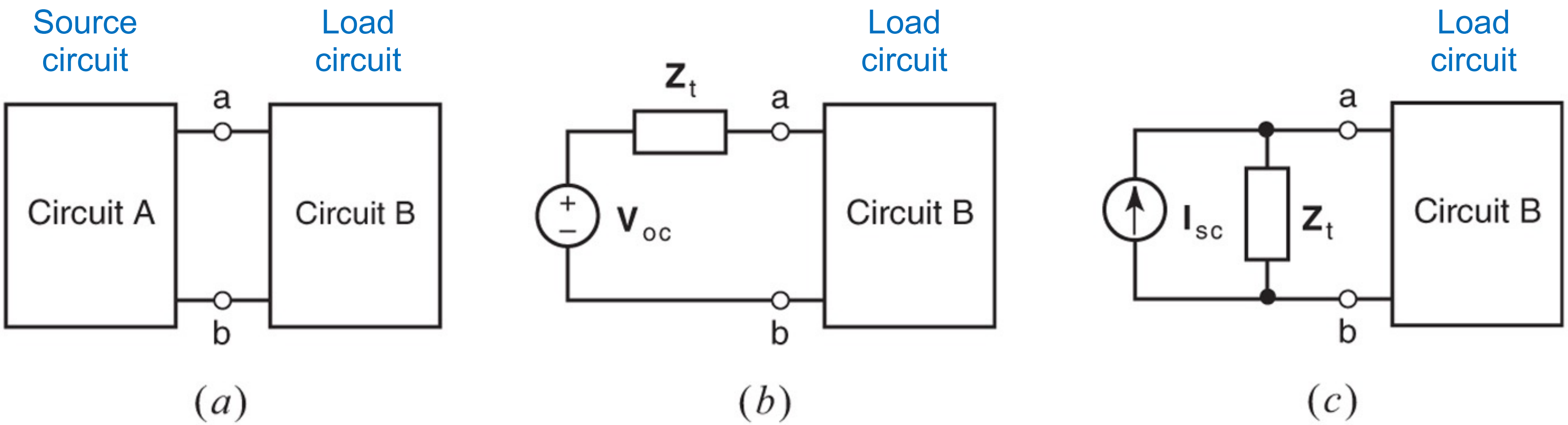


Thévenin & Norton equivalent circuits in the frequency domain

Chapter 10

Equivalents for source circuits

Similar to dc case but replace resistance with impedance



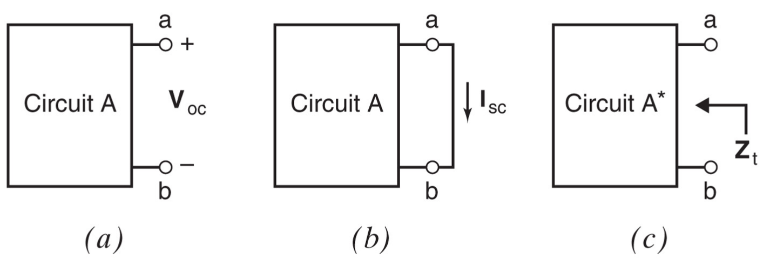
Thévenin equivalent for the source circuit

Norton equivalent for the source circuit

How to calculate the equivalent circuit in the frequency domain

The Thévenin or Norton equivalent circuit of circuit A involves three parameters: the open-circuit voltage V_{oc}, the short-circuit current I_{sc}, and the Thévenin impedance Z_t

$$\mathbf{V}_{oc} = \mathbf{Z}_t \mathbf{I}_{sc}$$



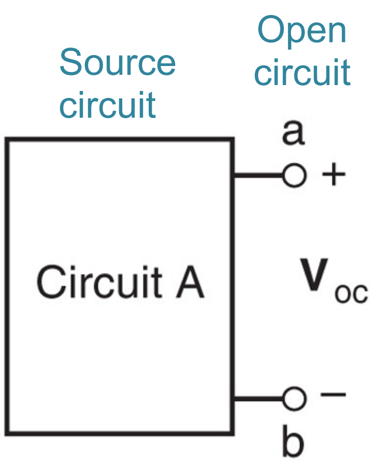
(a) an open circuit is connected across the terminals of circuit A - the voltage across that open circuit is the open-circuit voltage V_{oc}.

(b) a short circuit is connected across the terminals of circuit A - the current in that short circuit is the short-circuit current, I_{sc}.

(c) Z_t, is the equivalent impedance of circuit A* which is formed by replacing all the independent voltage (or current) sources by short (or open) circuits. Dependent sources are not replaced with open or short circuits.

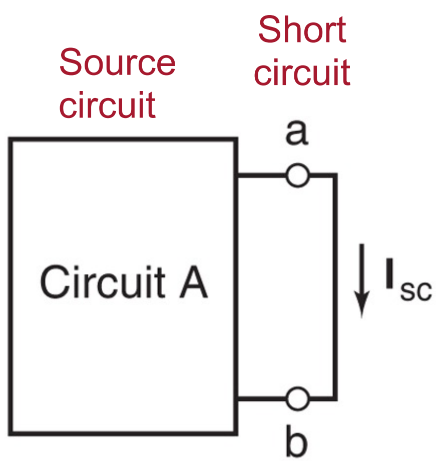
How to find the circuit equivalents?

Step 1:



Find V_{oc}

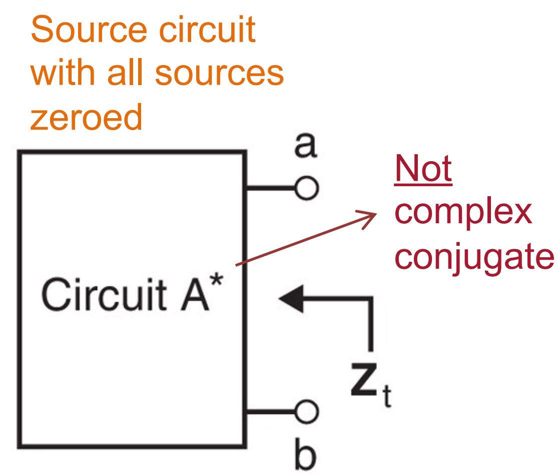
Step 2:



Find I_{sc}

$$Z_{th} = V_{oc}/I_{sc}$$

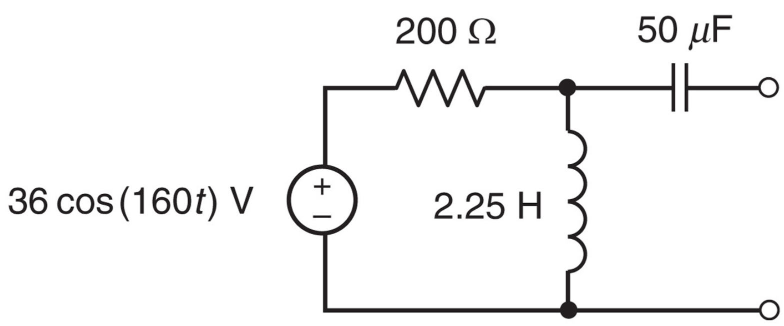
Step 2 alternative:



Find equivalent impedance = Z_t

Example 10.7-1

Find frequency-domain Thévenin equivalent circuit

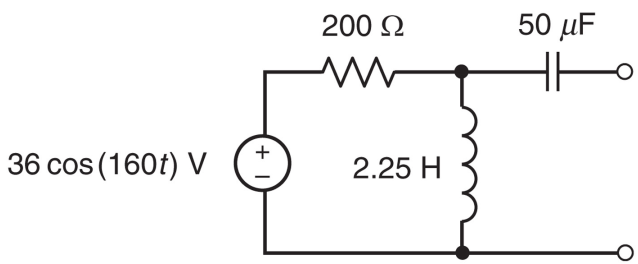


What do we do first?

Example 10.7-1

Find frequency-domain Thévenin equivalent circuit

We redraw the circuit in the frequency domain:



$$\omega = 160 \text{ rad/s}$$

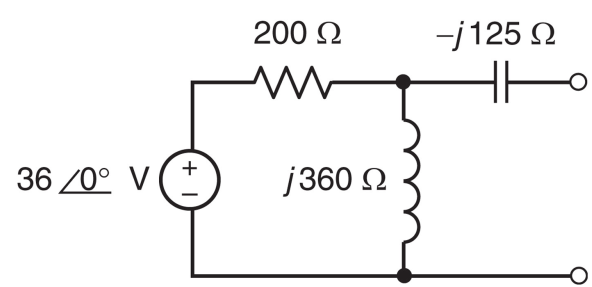
$$\boldsymbol{V}_s = 36\angle 0° = 36 \text{ V}$$

$$\boldsymbol{Z}_L(\omega) = j\omega L \qquad \boldsymbol{Z}_C(\omega) = 1/j\omega C$$

$$\boldsymbol{Z}_L = j160 \cdot 2.25 = j360\ \Omega$$

$$\boldsymbol{Z}_C = \frac{1}{j160 \times 50 \times 10^{-6}} = -j125\ \Omega$$

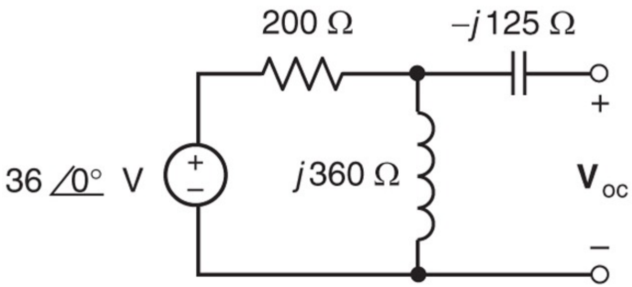
Now we have the frequency domain circuit:



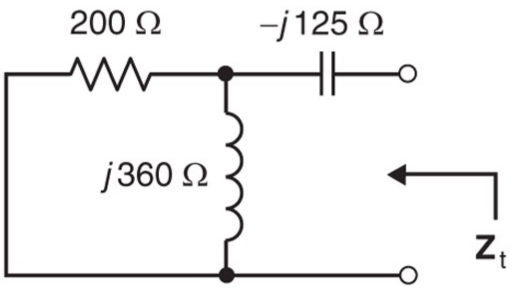
Example 10.7-1

Find frequency-domain Thévenin equivalent circuit

Step 1:



Step 2:



"Switch off" the voltage source

$$\boldsymbol{V}_{oc} = \frac{j360}{200 + j360}36$$

(no current through capacitor)

Convert to phasors

$$\boldsymbol{V}_{oc} = 31.47\ \angle 29.1°\ \ \text{V}$$

$$\boldsymbol{Z}_t = -j125 + j360||200$$

$$\boldsymbol{Z}_t = -j125 + \frac{200 \cdot j360}{200 + j360}$$

$$\boldsymbol{Z}_t = -j125 + 152.83 + j84.91$$

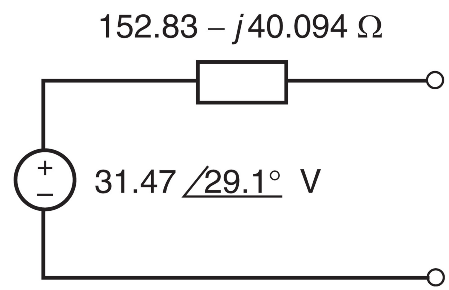
$$\boldsymbol{Z}_t = 158\angle(-14.7°)\ \Omega$$

Example 10.7-1

Find frequency-domain Thévenin equivalent circuit

$$\boldsymbol{V}_{oc} = 31.47\ \angle 29.1^\circ\ \text{V}$$

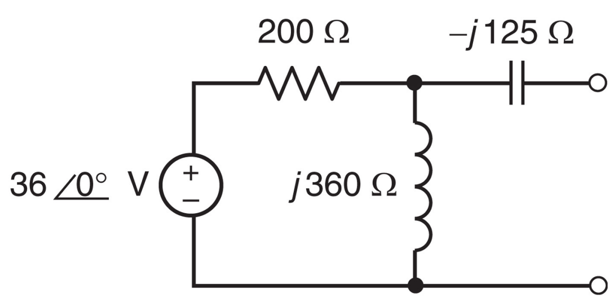
$$\boldsymbol{Z}_t = 158\angle(-14.7^\circ)\ \Omega$$



Is this a capacitive or inductive circuit?

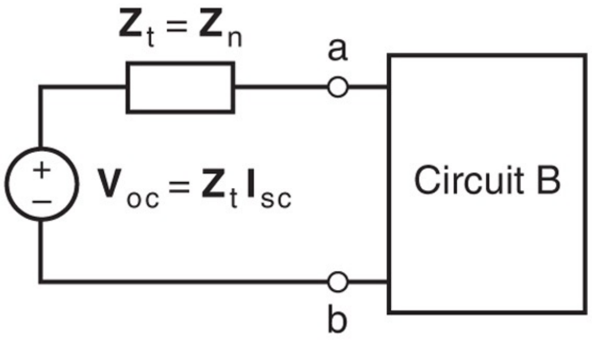
Is it possible to convert this back to the time domain?

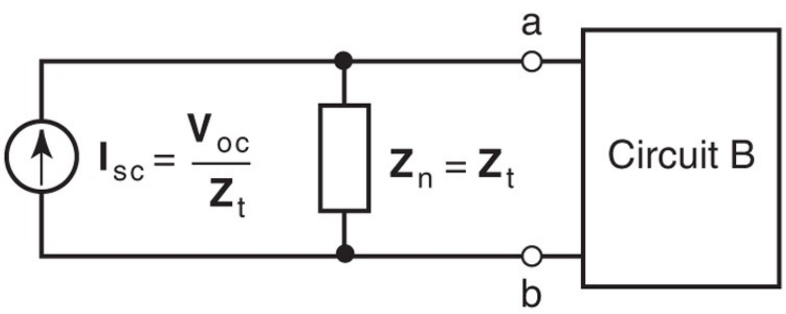
These are equivalent circuits



Source transformations in the frequency domain

Similar to dc case but replace resistance with impedance





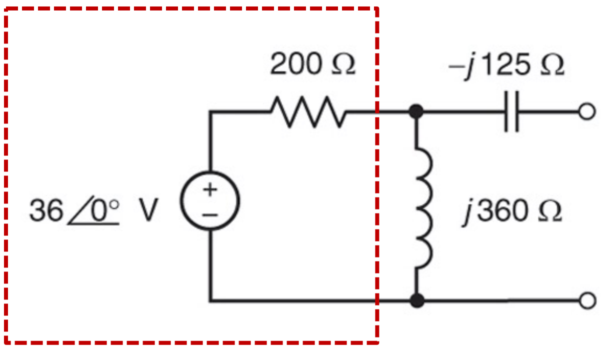
Thévenin equivalent for the source circuit

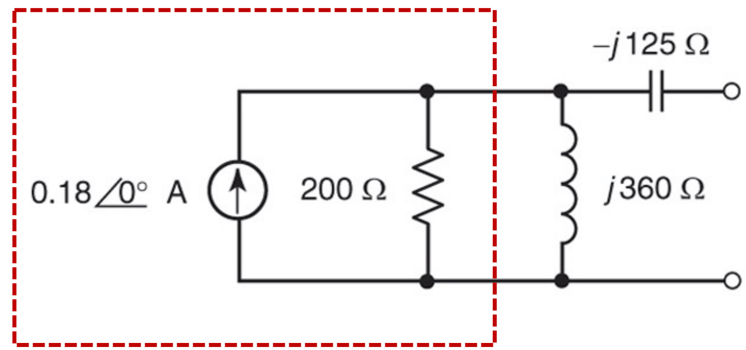
Norton equivalent for the source circuit

Convert both ways as necessary

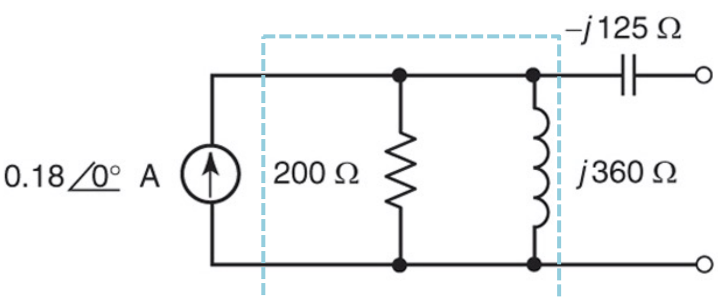
Example 10.7-3

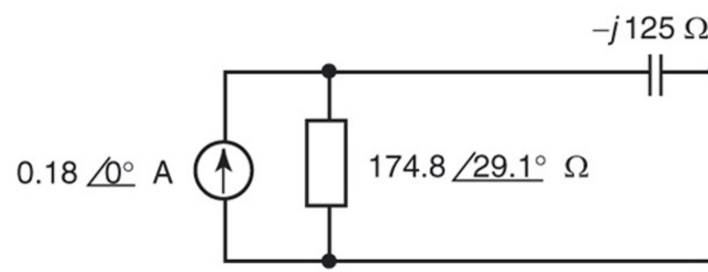
Redo previous example using source transformations





$$\boldsymbol{I}_{sc} = \frac{36\angle 0^\circ}{200} = 0.18\angle 0^\circ \text{ A}$$

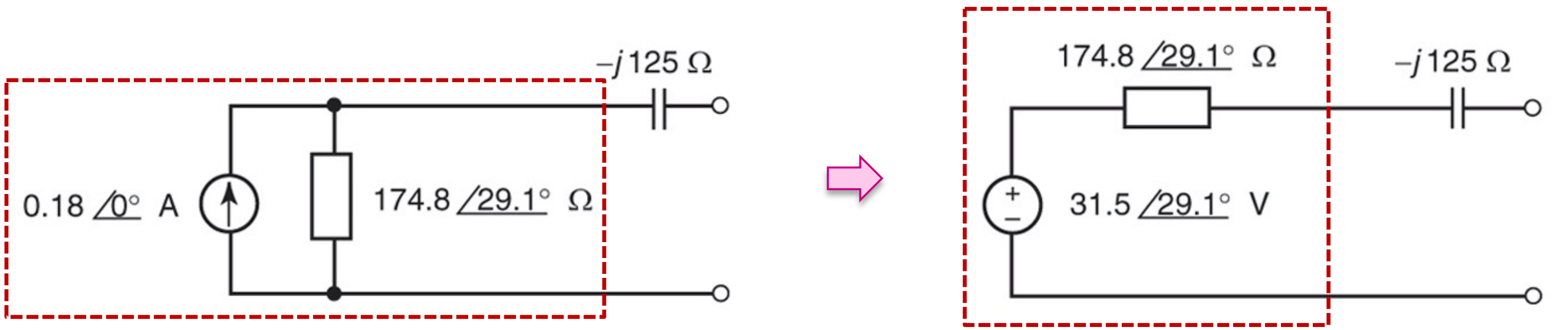




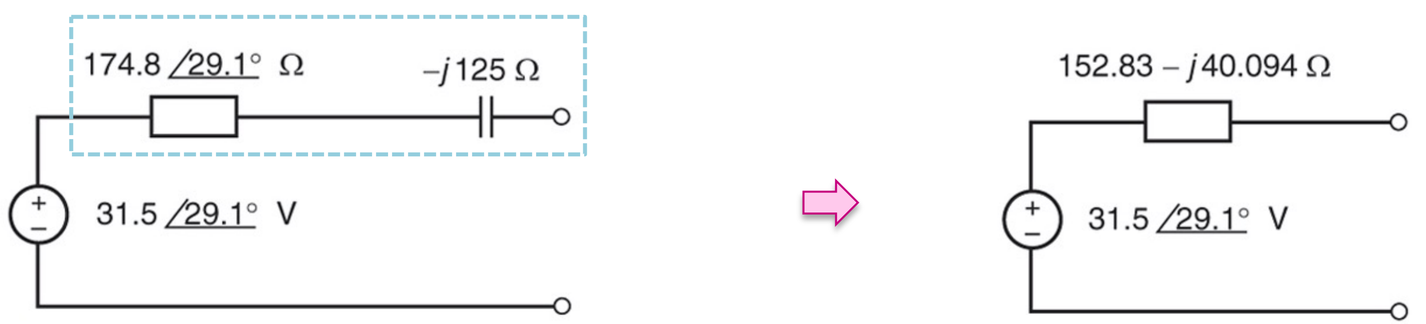
$$j360||200 = \frac{200 \cdot j360}{200 + j360} = 152.83 + j84.91 = 174.8\angle 29.1^\circ\ \Omega$$

Example 10.7-3

Redo previous example using source transformations



$$\boldsymbol{V}_{oc} = (174.8\angle 29.1°)(0.18\angle 0°) = 31.5\angle 29.1°\ \mathrm{V}$$



$$174.8\angle 29.1° - j125 = 152.83 + j84.91 - j125 = 152.83 - j40.094\ \Omega$$

Superposition in the frequency domain

Chapter 10

Superposition is important in ac circuits

Frequency domain circuits are analysed one frequency at a time. This means that all sources must be at the **same** frequency for a single calculation.

If there are sources at different frequencies, we use superposition to understand the overall behaviour of the circuit:

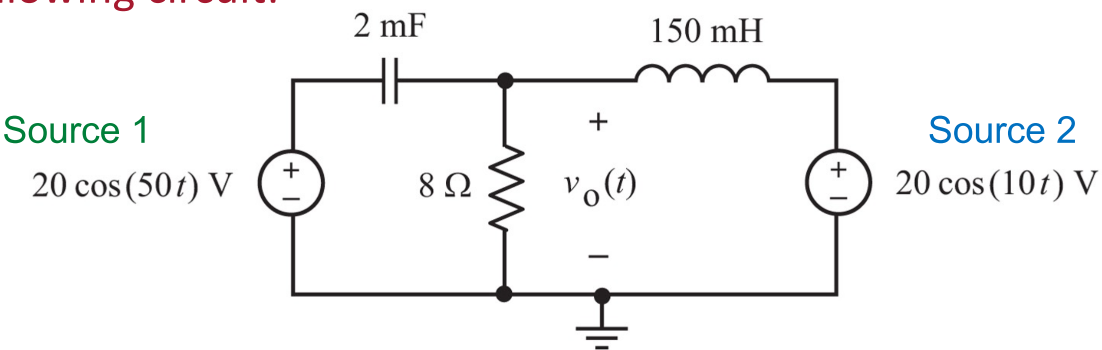
- The sources with the same frequency are analysed together while all the “other-frequency” sources are “zeroed”. (independent voltage sources = shorts, independent current sources = open)
- All the individual solutions corresponding to each frequency are then added up in the time domain to give the total time-domain solution.

There can also be sources at DC (or a zero frequency) – this solution can be added to the AC time-domain solution to obtain the total solution.

Example 10.8-1

Find v_o in the following circuit:

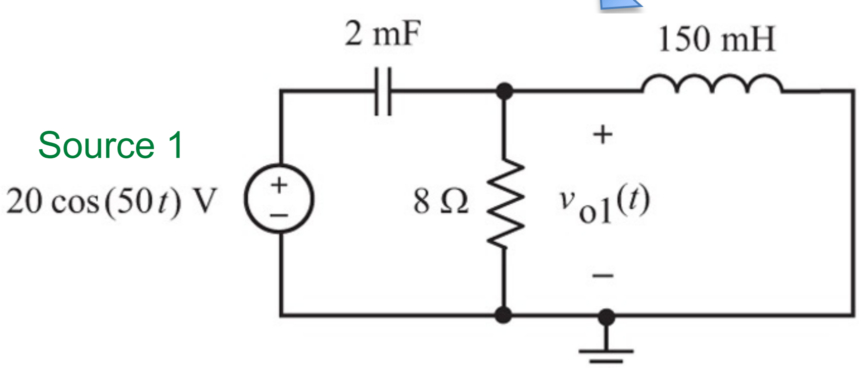
Source 1



Source 2

To find v_{o1} :

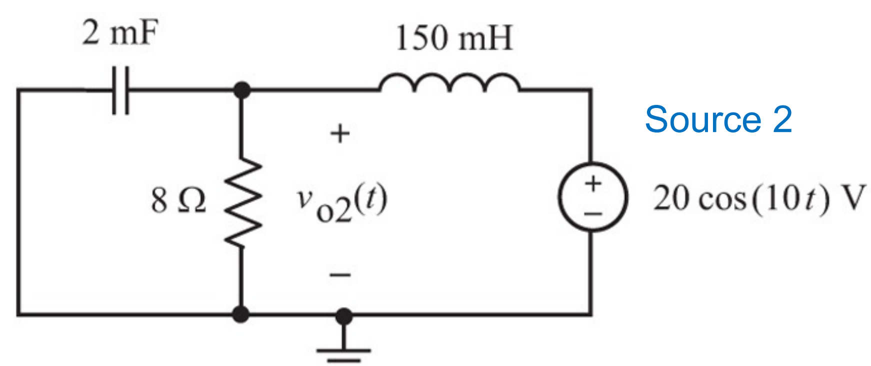
Source 1



Source 2
zeroed

To find v_{o2} :

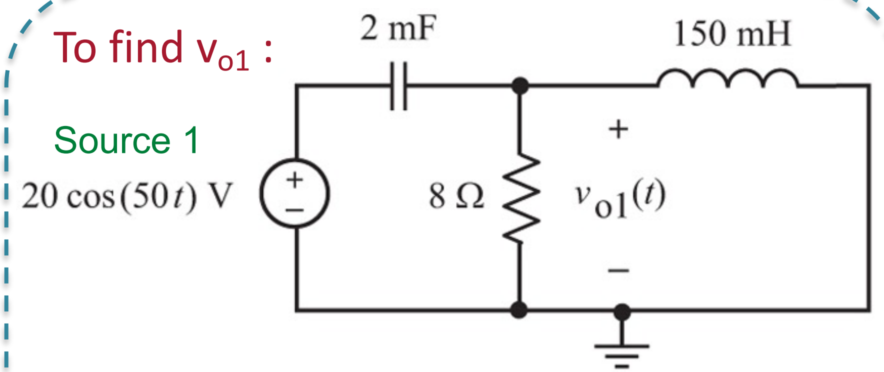
Source 1
zeroed



Source 2

To find v_{o1} :

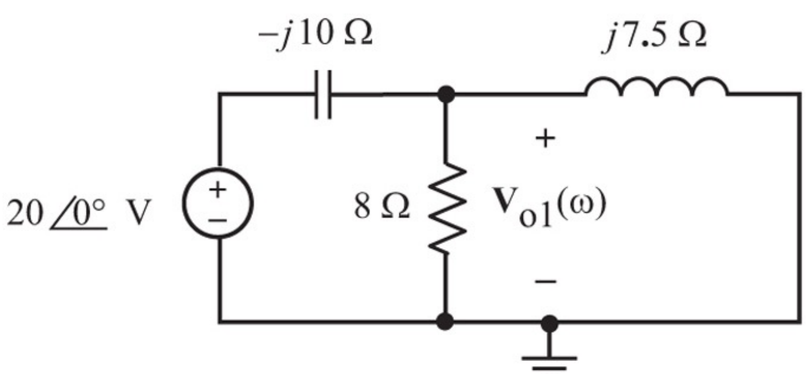
Source 1



$$\omega_1 = 50 \text{ rad/s}$$

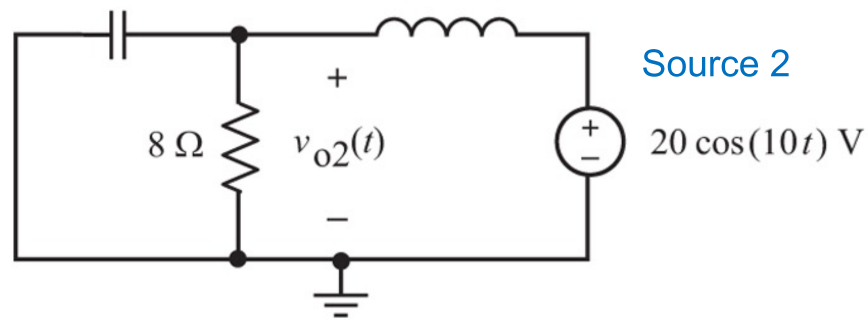
$$\boldsymbol{Z}_L(50) = j50\times0.15 = j7.5\ \Omega$$

$$\boldsymbol{Z}_C(50) = \frac{1}{j50\times2\times10^{-3}} = -j10\ \Omega$$



To find v_{o2} :

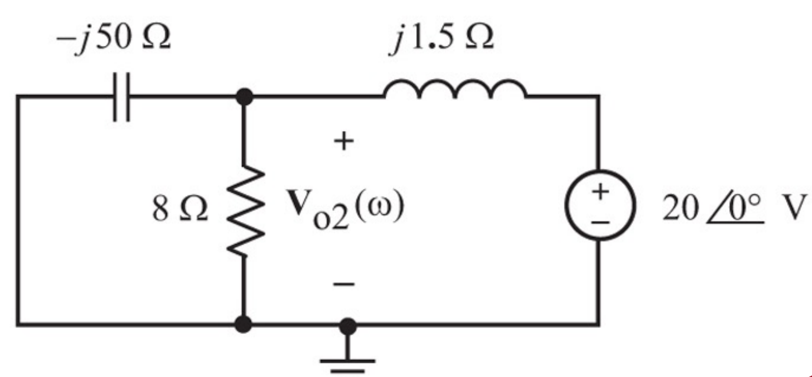
Source 2

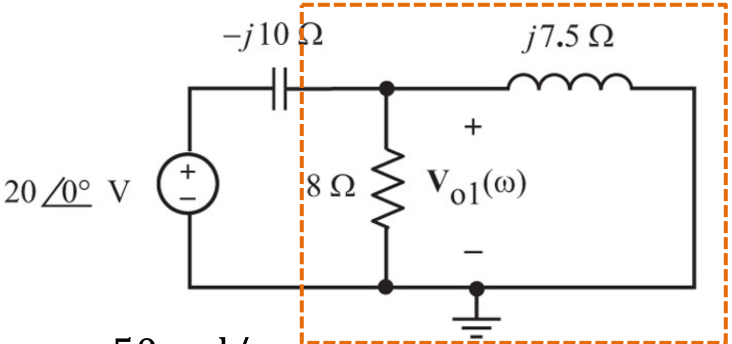


$$\omega_2 = 10 \text{ rad/s}$$

$$\boldsymbol{Z}_L(10) = j10\times0.15 = j1.5\ \Omega$$

$$\boldsymbol{Z}_C(10) = \frac{1}{j10\times2\times10^{-3}} = -j50\ \Omega$$





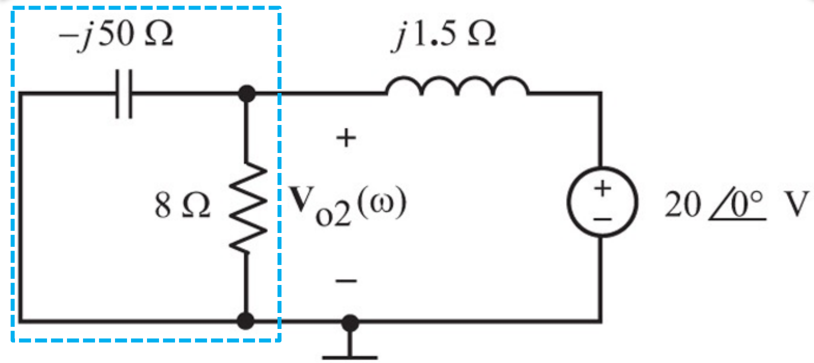
$\omega_1 = 50$ rad/s

$$j7.5||8 = \frac{8(j7.5)}{8 + j7.5} = 3.74 + j3.99$$

$$\boldsymbol{V}_{o1}(50) = \frac{3.74 + j3.99}{-j10 + 3.74 + j3.99} 20$$

$$\boldsymbol{V}_{o1}(50) = 15.46\,\angle 104.9°\ \text{V}$$

$$v_{o1}(t) = 15.46\cos(50t + 104.9°)\ \text{V}$$



$\omega_2 = 10$ rad/s

$$-j50||8 = \frac{-8(j50)}{8 - j50} = 7.8 - j1.25$$

$$\boldsymbol{V}_{o2}(10) = \frac{7.8 - j1.25}{j1.5 + 7.8 - j1.25} 20$$

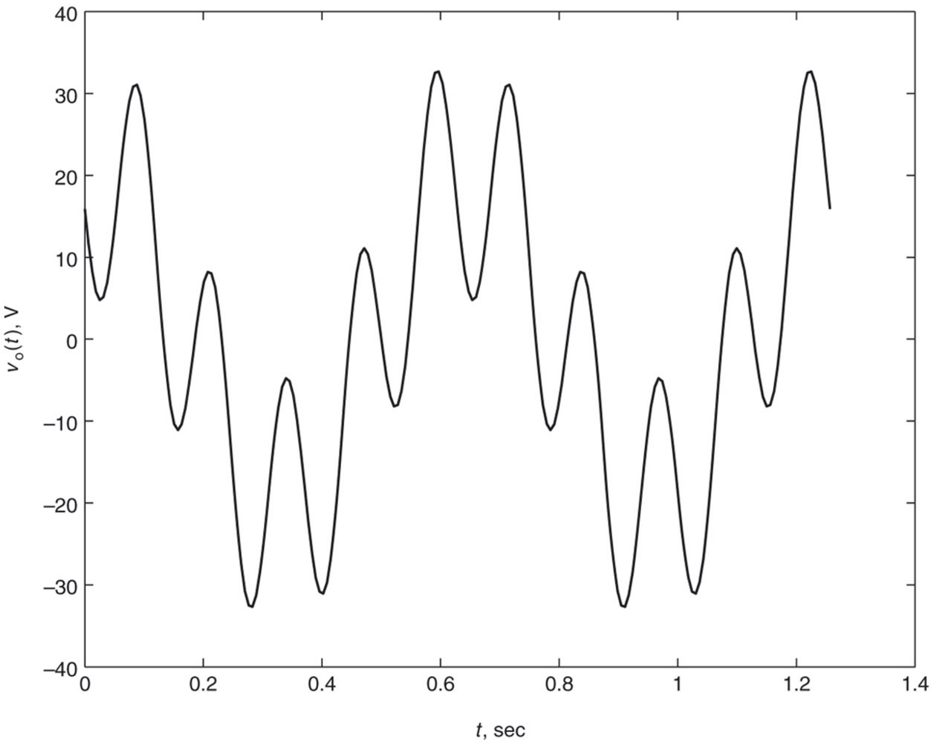
$$\boldsymbol{V}_{o2}(10) = 20.24\,\angle -10.94°\ \text{V}$$

$$v_{o2}(t) = 20.24\cos(10t + 10.94°)\ \text{V}$$

$$v_o(t) = 15.46\cos(50t + 104.9°) + 20.24\cos(10t + 10.94°)\ \text{V}$$

Example 10.8-1 – graphical

You will see this on an oscilloscope



What will you see on a spectrum analyser?

$$v_o(t) = 15.46\cos(50t + 104.9°) + 20.24\cos(10t + 10.94°)\ \text{V}$$

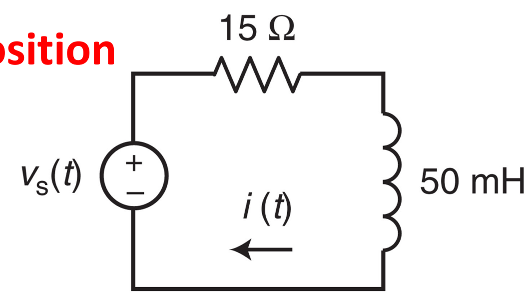
Problem 10.8-3

Find the steady state current *i(t)*

Here a single source has 2 components – use superposition

$$v_s(t) = \underbrace{8}_{\text{dc part}} + \underbrace{8\cos(400t - 135°)}_{\text{ac part}} \ \text{V}$$

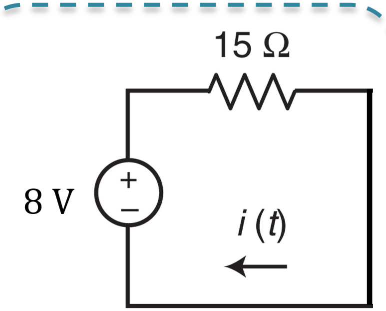
$\omega_1 = 0$ rad/s $\qquad\qquad \omega_2 = 400$ rad/s



$\boldsymbol{Z}_L(0) = j0\times 0.05 = 0\ \Omega$

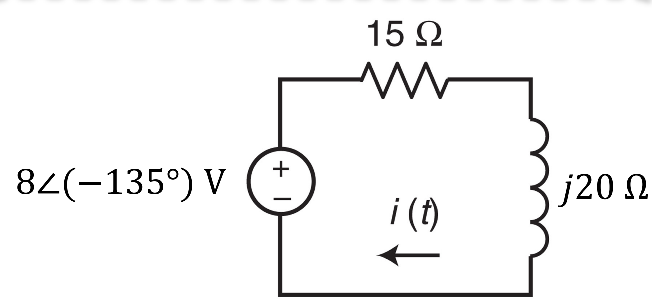
$\boldsymbol{Z}_L(400) = j400\times 0.05 = j20\ \Omega$

DC part:



$$\boldsymbol{I}(0) = \frac{8}{15} = 0.533\ \text{A}$$

AC part:



$$\boldsymbol{I}(400) = \frac{8\angle(-135°)}{15 + j20} = 0.32\angle(-188.1°)\ \text{A}$$

$$i(t) = \underbrace{0.533}_{\text{dc part}} + \underbrace{0.32\cos(400t - 188.1°)}_{\text{ac part}}\ \text{A}$$

Variation on Problem 10.8-3

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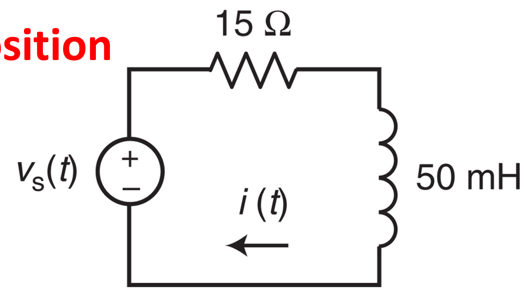
Find the steady state current *i(t)*

Here a single source has 3 components – use superposition

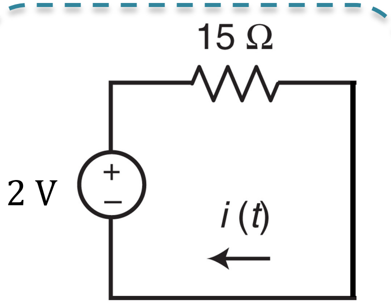
$$v_s(t) = \underbrace{2}_{\text{dc part}} + \underbrace{\cos(100t)}_{\text{ac part 1}} + \underbrace{4\cos(200t + 35°)}_{\text{ac part 2}}\ \mathrm{V}$$

$\omega_1 = 100\ \mathrm{rad/s}$ $\omega_2 = 200\ \mathrm{rad/s}$

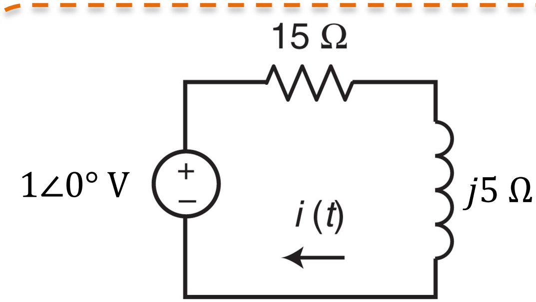


$$\boldsymbol{Z}_L(100) = j100 \times 0.05 = j5\ \Omega$$

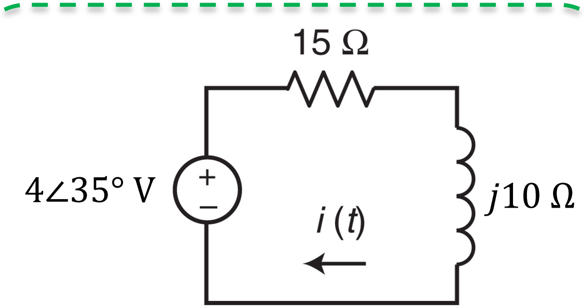
$$\boldsymbol{Z}_L(200) = j200 \times 0.05 = j10\ \Omega$$



$$\boldsymbol{I}(0) = \frac{2}{15} = 0.133\ \mathrm{A}$$



$$\boldsymbol{I}(100) = \frac{1}{15 + j5} = 0.063\angle -18.43°\ \mathrm{A}$$



$$\boldsymbol{I}(200) = \frac{4\angle 35°}{15 + j10} = 0.222\angle 1.31°\ \mathrm{A}$$

$$i(t) = \underbrace{0.133}_{\text{dc part}} + \underbrace{0.063\cos(100t - 18.43°)}_{\text{ac part 1}} + \underbrace{0.222\cos(200t + 1.31°)}_{\text{ac part 2}}\ \mathrm{A}$$

Table 10.6-1

Time domain and frequency domain relationships

ELEMENT	TIME DOMAIN	FREQUENCY DOMAIN
Current Source	$i(t) = A\cos(\omega t + \theta)$	$\mathbf{I}(\omega) = Ae^{j\theta}$
Voltage source	$v(t) = B\cos(\omega t + \phi)$	$\mathbf{V}(\omega) = Be^{j\phi}$
Resistor	R; $i(t)$; $v(t)$; $v(t) = R\,i(t)$	R; $\mathbf{I}(\omega)$; $\mathbf{V}(\omega)$; $\mathbf{V}(\omega) = R\,\mathbf{I}(\omega)$
Capacitor	C; $i(t)$; $v(t)$; $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$	$\frac{1}{j\omega C}$; $\mathbf{I}(\omega)$; $\mathbf{V}(\omega)$; $\mathbf{V}(\omega) = \frac{1}{j\omega C}\,\mathbf{I}(\omega)$
Inductor	L; $i(t)$; $v(t)$; $v(t) = L\frac{d}{dt}i(t)$	$j\omega L$; $\mathbf{I}(\omega)$; $\mathbf{V}(\omega)$; $\mathbf{V}(\omega) = j\omega L\,\mathbf{I}(\omega)$
CCVS	$i_c(t)$; $i(t)$; $v(t) = K\,i_c(t)$	$\mathbf{I}_c(\omega)$; $\mathbf{I}(\omega)$; $\mathbf{V}(\omega) = K\,\mathbf{I}_c(\omega)$

Phasor diagrams

What is a phasor diagram?

Phasors are complex numbers and can be graphed in the complex plane. The relationship between phasors when drawn on the complex plane is called a phasor diagram.

A phasor diagram is a graphical representation of phasors and their relationships on the complex plane.

Why a phasor diagram?

1. It provides a graphical method for solving certain problems when complex calculations are tedious
2. It serves as a check on more exact analytical solutions
3. Helps in certain symmetrical problems by enabling the symmetry to be recognised and helpfully applied.

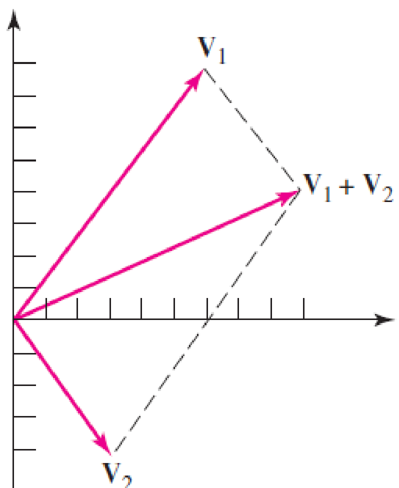
The phasor diagram

Since phasor voltages and currents are complex numbers, they can be identified as points on the complex plane

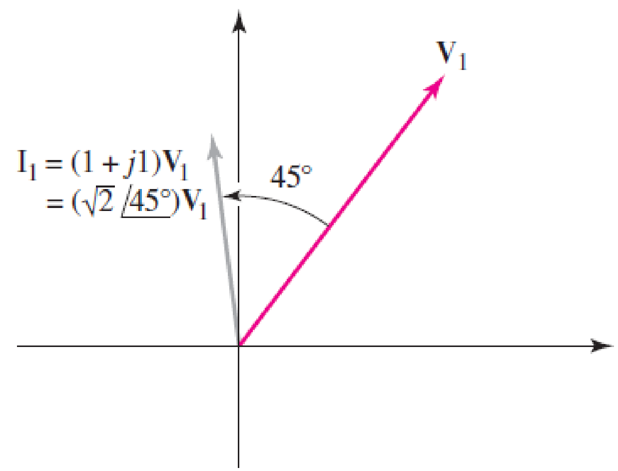
Since addition and subtraction are easy on the complex plane so phasor addition and subtraction is also easy.

Multiplication and division result in the addition and subtraction respectively of the phasor ANGLES. The amplitude will also change but depends on the scale of the diagram (e.g., current and voltage may have their own scale, but the angles will be the same).

Addition:



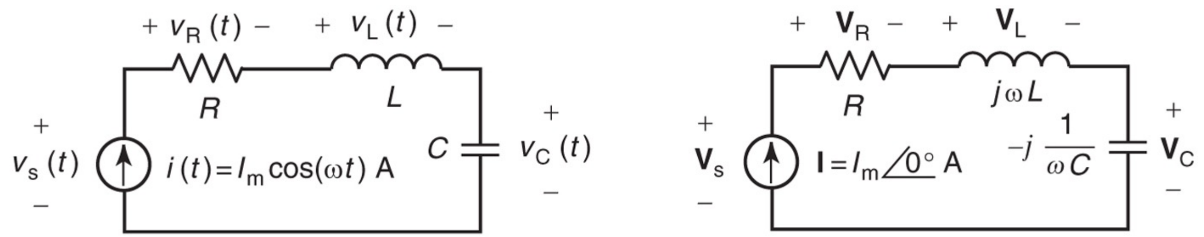
Multiplication:



Example problem

Find the voltages
across each circuit element

Assume current is: $\boldsymbol{I} = I_m \langle 0$



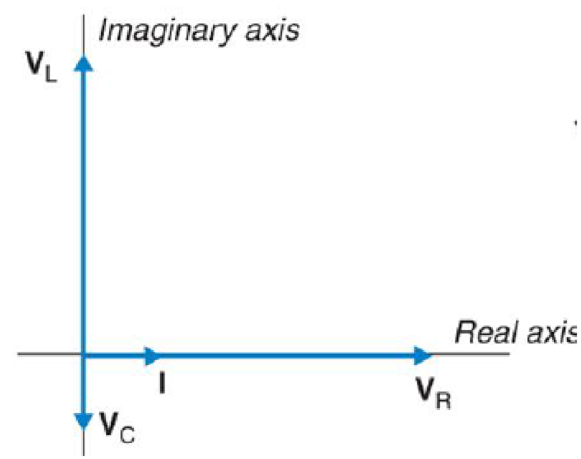
(a) (b)

Resistor voltage: $\boldsymbol{V_R} = RI_m \langle 0$

Inductor voltage: $\boldsymbol{V_L} = j\omega L I_m \langle 0 = \omega L I_m \langle 90^0$

Capacitor voltage: $\boldsymbol{V_C} = -j\frac{1}{\omega C} I_m \langle 0 = \frac{1}{\omega C} I_m \langle -90^0$

These are represented on the complex plane as:



The source voltage

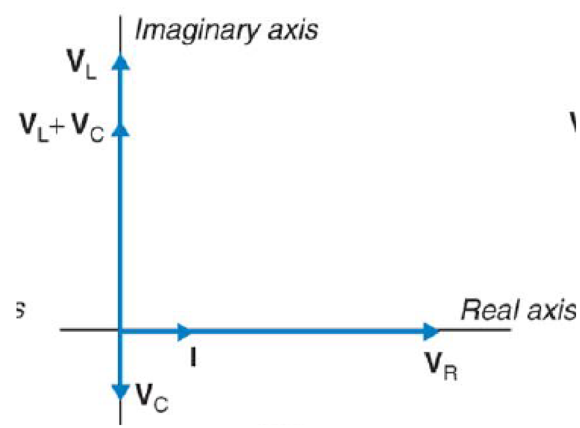
From KVL we find:

$$\mathbf{V}_S = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C = \mathbf{V}_R + (\mathbf{V}_L + \mathbf{V}_C)$$

The phasor $V_L + V_C$ is given by:

$$\mathbf{V}_L + \mathbf{V}_C = j\left(\omega L - \frac{1}{\omega C}\right)\left(I_m \angle 0^\circ\right) = \left(\omega L - \frac{1}{\omega C}\right)\left(I_m \angle 90^\circ\right)$$

On the phasor diagram, however, $V_L + V_C$ are easily added:

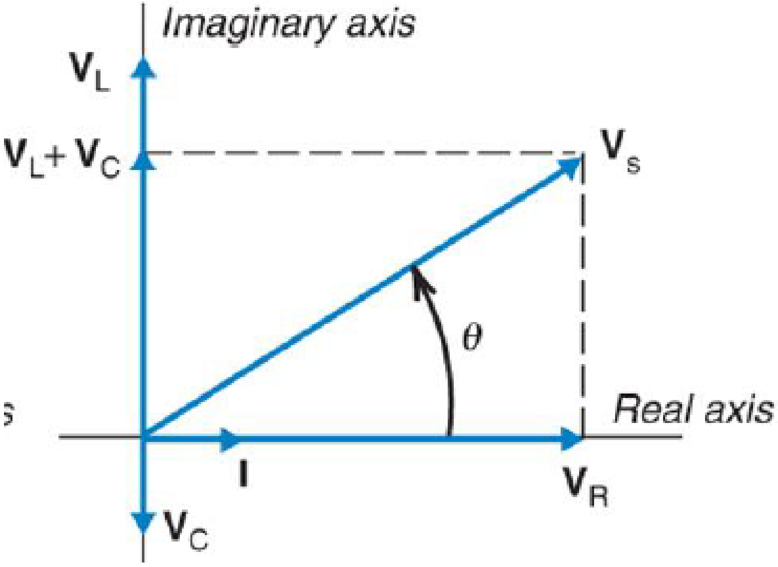


The source voltage

The source voltage is now (mathematically):

$$\mathbf{V}_{\mathrm{S}} = \mathbf{V}_{\mathrm{R}} + (\mathbf{V}_{\mathrm{L}} + \mathbf{V}_{\mathrm{C}}) = RI_m + j\left(\omega L - \frac{1}{\omega C}\right)(I_m)$$

Again, this is easily determined from the phasor diagram:



The natural response in AC circuits

In order to find the complete response

Finding the complete response of a switched AC circuit

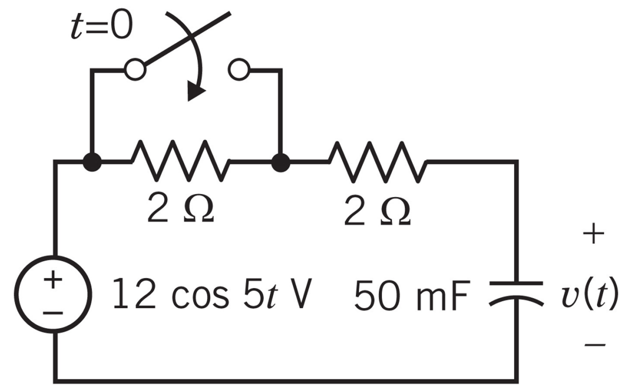
Generally, to find the complete response of an AC circuit with switches:

1. Represent the required (complete) response by a differential equation.
2. Find the general solution of the homogeneous differential equation. This solution is the natural response $v_n(t)$. The natural response will contain unknown constants that will be evaluated later.
3. Find a solution of the differential equation due to the AC source. This solution is the forced response $v_f(t)$.
4. Represent the complete response of the circuit as $v(t) = v_n(t) + v_f(t)$.
5. Use the initial conditions, for example, the initial values of the currents in inductors and the voltages across capacitors to evaluate the unknown constants.

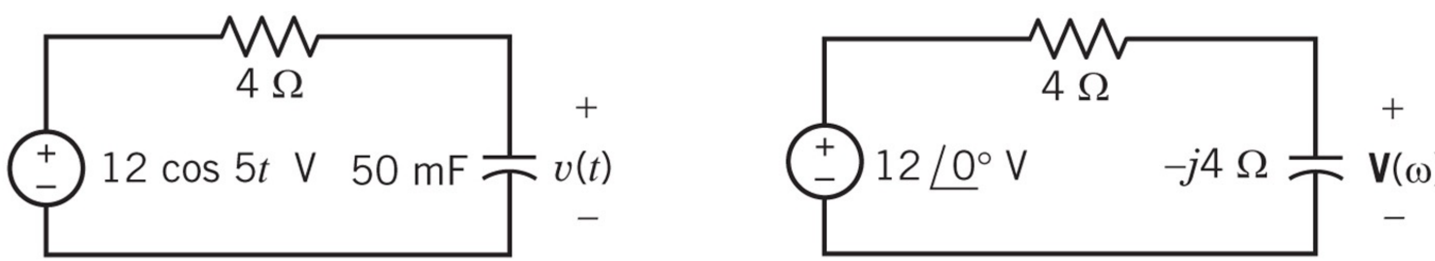
This process is not difficult using phasors

Example 10.11-1

Consider the switched circuit problem:



Step 1: For t<0, the switch is open we have circuit (a) below. Circuit (b) is the same circuit drawn in the frequency domain



(a) (b)

Example 10.11-1

Voltage division gives us the voltage across the capacitor:

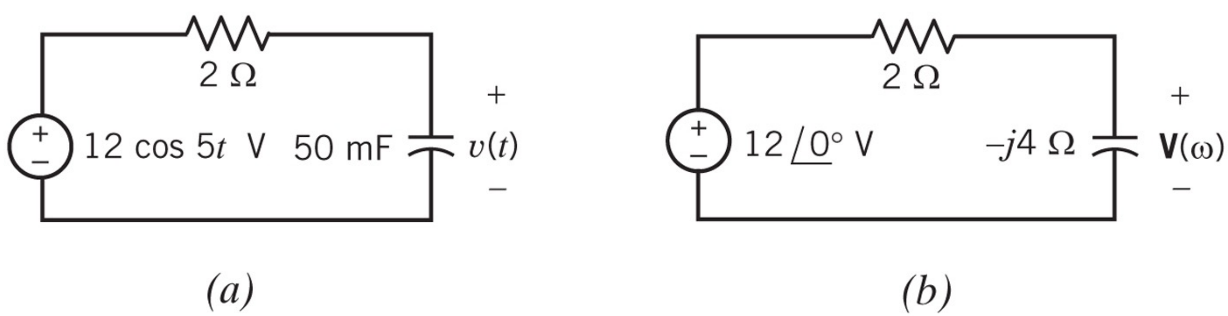
$$V(5) = \frac{-j4}{4-j4} 12\langle 0^o = \frac{48\langle -90^o}{5.66\langle -45^o} = 8.485\langle -45^o\ V$$

In the time domain, the forced response (steady state voltage) : $v(t) = 8.485\cos(5t - 45^o)\,V$

At t = 0 (or 0^-) the voltage across the capacitor is 8.485cos(0-45^0) = 6 V

Remember the capacitor voltage will be 6 V just after the switch closes (at t = 0^+)

Step 2: For t>0, we have a new circuit (a) or in the frequency domain (b):



Example 10.11-1

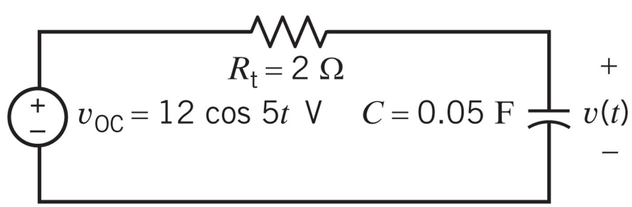
Voltage division gives us the voltage across the capacitor:

$$V(5) = \frac{-j4}{2-j4} 12\langle 0^o = \frac{48\langle -90^o}{4.47\langle -63.4^o} = 10.74\langle -26.6^o\ V$$

In the time domain, the forced response (steady state voltage) : $v(t) = 10.74\cos(5t - 26.6^o)\, V$

Step 3: Immediately after the switch is closed, the circuit is not in steady state and hence

the natural response will be: $v_n(t) = Ke^{-\frac{t}{R_t C}}$ (where R_t C = 2 x 50x10^{-3} = 0.1 s)



Therefore the complete response for t>0 is:

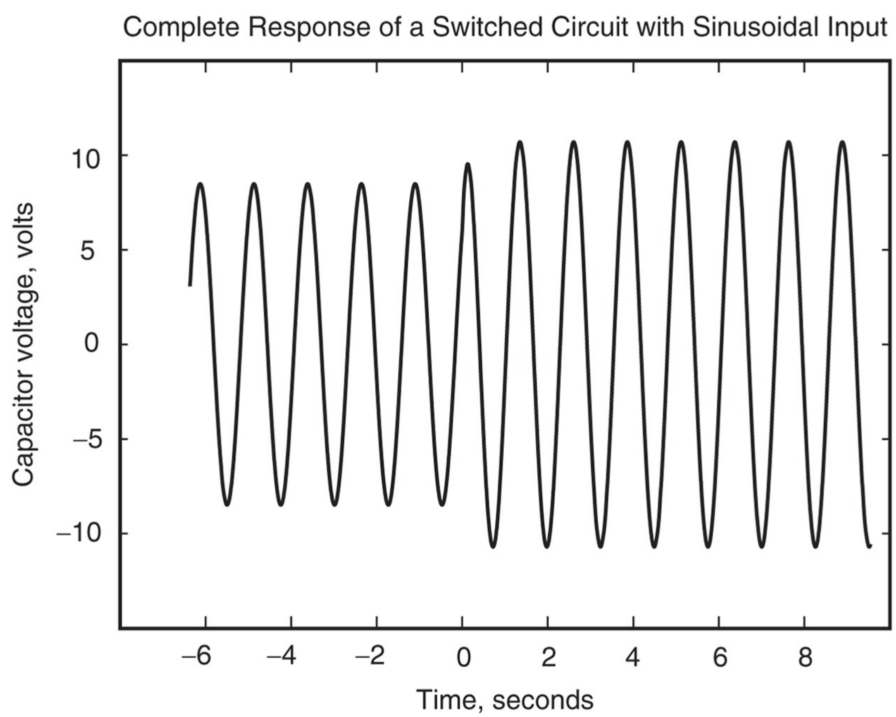
$$v(t) = v_n(t) + v_f(t) = Ke^{-10t} + 10.74\cos(5t - 26.6^0)$$

When t=0, v(t) = 6 V therefore: $6 = K + 9.6$ therefore K=-3.6

Example 10.11-1 - Solution

Complete response:

$$v(t) = \begin{cases} 8.458\cos(5t - 45^o)\,V\ for\ t < 0 \\ -3.6e^{-10t} + 10.74\cos(5t - 26.6^o)\,V\ for\ t > 0 \end{cases}$$



Power

Chapter 11 of Dorf and Svoboda

Instantaneous and average power

Chapter 11 of Dorf and Svoboda

Power

We are always interested in the circuit response BUT we are also interested in:

1. The amount of energy supplied from the source(s)
2. The amount of energy dissipated or stored within a circuit
3. The way in which energy is delivered to the points at which the responses are determined.

Primarily, we want the *rate* at which energy is being generated or stored, hence we are interested in **POWER** (Joules per second).

AC Power

Different kinds of power for different kinds of signals, i.e.,

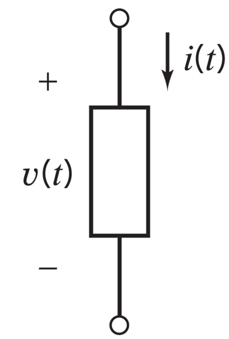
- **Instantaneous power** – relates to any signal
- **Average power** – relates to periodic signals
- **Complex power** – since AC currents and voltages are described using complex numbers (i.e., phasors)

Instantaneous power

This is only relevant to the time domain

Instantaneous power

$$p(t) = v(t) \cdot i(t) \quad \mathrm{W}$$



The ac currents and voltages (note they have the same ω) across any element (in general) is:

$$v(t) = V_m \cos(\omega t + \theta_V) \qquad i(t) = I_m \cos(\omega t + \theta_I)$$

The instantaneous power is given by: $p(t) = v(t) \cdot i(t)$

Hence: $p(t) = V_m\, I_m \cos(\omega t + \theta_V) \cos(\omega t + \theta_I)$

Using the trig. relation:

$$\cos\alpha \cdot \cos\beta = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}$$

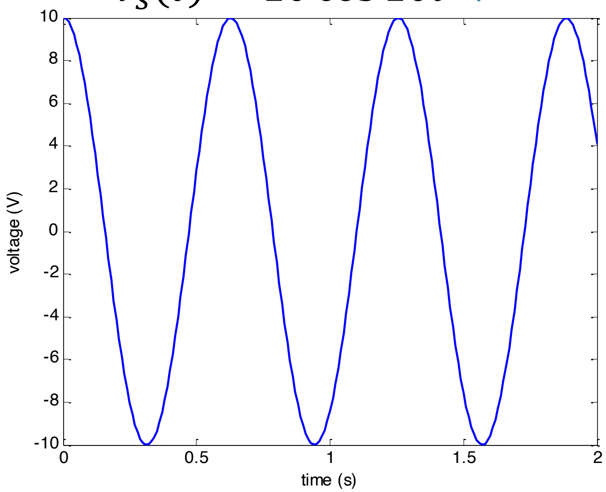
$$p(t) = \frac{V_m I_m}{2} [\underbrace{\cos(\theta_V - \theta_I)}_{\text{constant}} + \underbrace{\cos(2\omega t + \theta_V + \theta_I)}_{\substack{\text{periodic in time} \\ \text{(at twice the angular frequency)}}}]$$

Instantaneous power - plotted

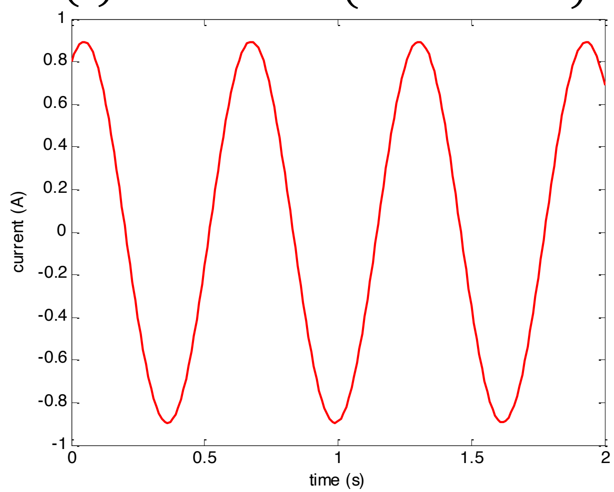
$$p(t) = \frac{V_m I_m}{2}\left[\underbrace{\cos(\theta_V - \theta_I)}_{\text{constant (not time dependent)}} + \underbrace{\cos(2\omega t + \theta_V + \theta_I)}_{\text{periodic in time (at twice the frequency)}}\right]$$

Example 10.6-3 voltage and current

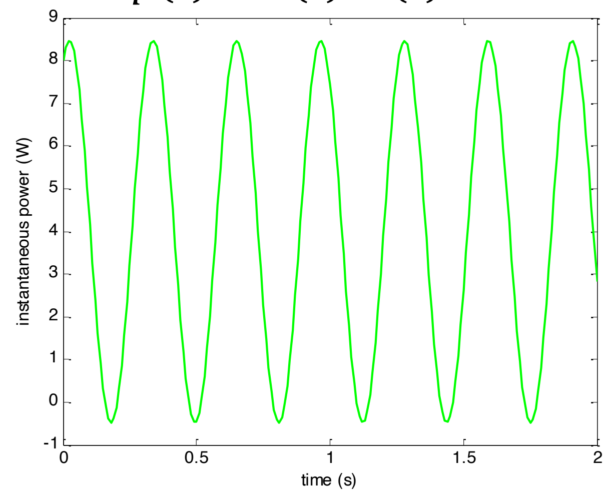
$v_s(t) = 10\cos 10t$ V



$i(t) = 0.894\cos(10t - 26.6°)$ A

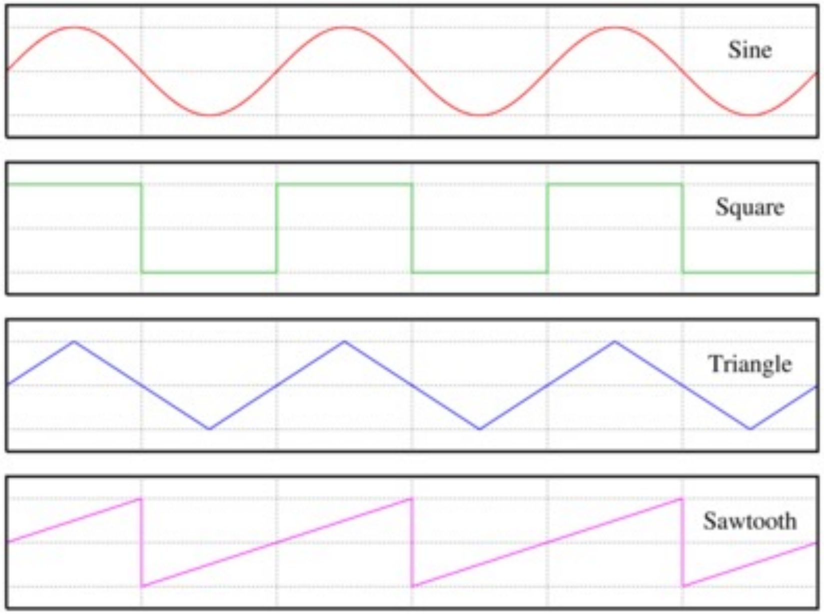


$p(t) = v(t) \cdot i(t)$ W



Power of periodic signals

Types of periodic signals:



We only need to examine the average power over one period!

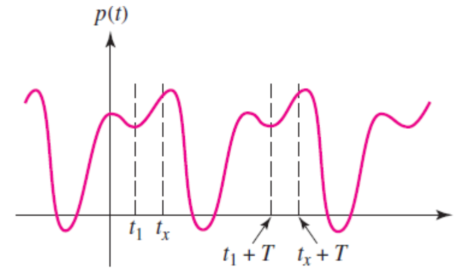
Average power

Average power

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt$$

Units: Watts

It is defined ONLY over one period, T (since it all repeats during the next T)



The average power may be computed by integrating the instantaneous power over ANY interval which is one period in length.

Using our instantaneous ac power:

$$p(t) = \frac{V_m I_m}{2}\left[\underbrace{\cos(\theta_V - \theta_I)}_{\text{constant in time}} + \underbrace{\cos(2\omega t + \theta_V + \theta_I)}_{\text{periodic in time; average over one period is zero}}\right]$$

constant in time

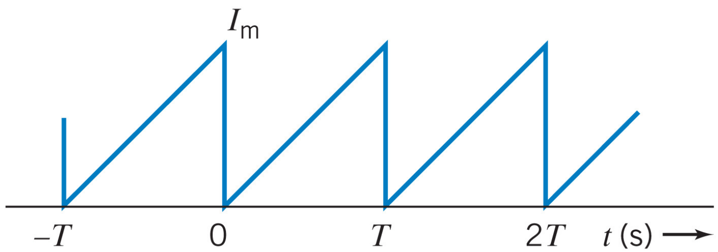
periodic in time
average over one period is zero

Average ac power

$$P = \frac{V_\mathrm{m} I_\mathrm{m}}{2}\cos\left(\theta_\mathrm{V} - \theta_\mathrm{I}\right)$$

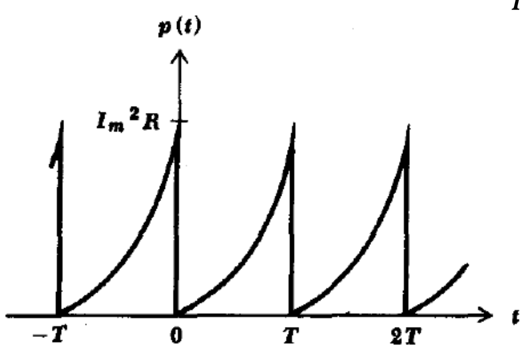
Example 11.3-1

Find the average power delivered to a resistor R when the current i(t) through the the resistor is a sawtooth:



The current can be defined mathematically as: $i(t) = \frac{I_m}{T} t$ for the period: $0 \leq t < T$

The instantaneous power within a period is $P(t) = i(t)^2 R = \frac{I_m^2 R}{T^2} t^2$ for $0 \leq t < T$



The average power is:

$$P = \frac{I_m^2 R}{T^3} \int_0^T t^2 dt = \frac{I_m^2 R}{T^3} \frac{T^3}{3} = \frac{I_m^2 R}{3} \text{ W}$$

Problem: average power for a sinusoidal function

Given the time-domain voltage $v = 4\cos(\pi t/6)$ V, find both the average power and an expression for the instantaneous power that result when the corresponding phasor voltage $\mathbf{V} = 4\underline{/0^\circ}$ V is applied across an impedance $\mathbf{Z} = 2\underline{/60^\circ}$ Ω.

The phasor current is $\mathbf{V}/\mathbf{Z} = 2\underline{/-60^\circ}$ A, and so the average power is

$$P = \tfrac{1}{2}(4)(2)\cos 60^\circ = 2 \text{ W}$$

Time dependent voltage is :

$$v(t) = 4\cos\frac{\pi t}{6} \qquad \text{V}$$

Time dependent current is :

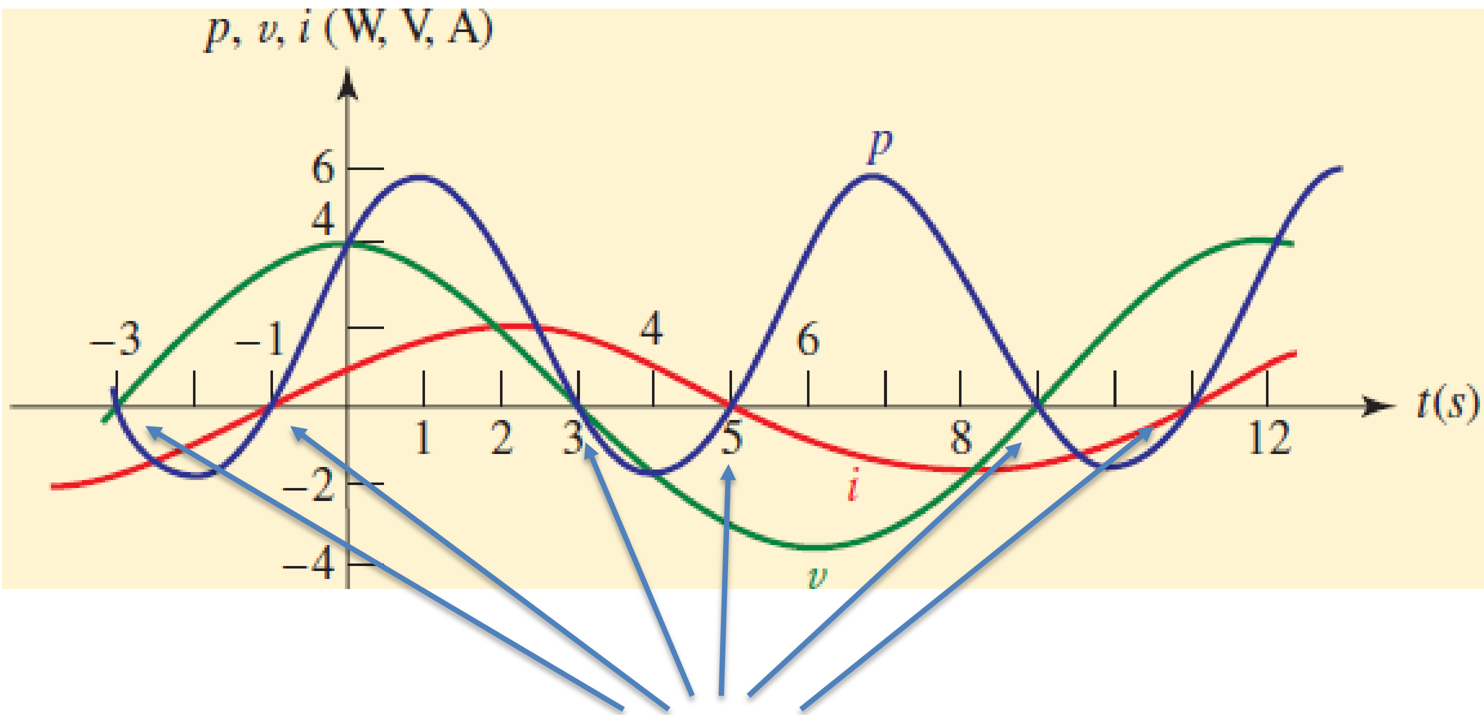
$$i(t) = 2\cos\left(\frac{\pi t}{6} - 60^\circ\right) \qquad \text{A}$$

The instantaneous power is:

$$\begin{aligned} p(t) &= 8\cos\frac{\pi t}{6}\cos\left(\frac{\pi t}{6} - 60^\circ\right) \\ &= 2 + 4\cos\left(\frac{\pi t}{3} - 60^\circ\right) \qquad \text{W} \end{aligned}$$

Power across the inductive load

GREEN = voltage
RED = current
BLUE = instantaneous power



Instantaneous power is zero when either v(t) or i(t) is zero

AC average power of the circuit elements

$$\boxed{P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I)}$$

Average AC power for a resistor

$$\theta_V = \theta_I$$

$$P = \frac{V_m I_m}{2}$$

Average AC power for an inductor

$$\theta_V - \theta_I = 90$$

$$P = 0$$

Average AC power for a capacitor

$$\theta_V - \theta_I = -90$$

$$P = 0$$

Ideal capacitors and inductors (i.e., that have zero internal resistance) do not consume power!

Resistance consumes power only!

Average power defined using impedance

Impedance can be defined as:

POLAR FORM: $$\mathbf{Z} = \frac{V_m}{I_m}\angle(\theta_V - \theta_I) = Z_m\angle\theta_Z$$

RECTANGULAR FORM: $$\mathbf{Z} = \underbrace{Z_m\cos\theta_Z}_{\text{Real part}} + j\underbrace{Z_m\sin\theta_Z}_{\text{Imaginary part}} = R + jX$$

resistance

reactance

We know the average ac power is given by: $$\boxed{P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I)}$$

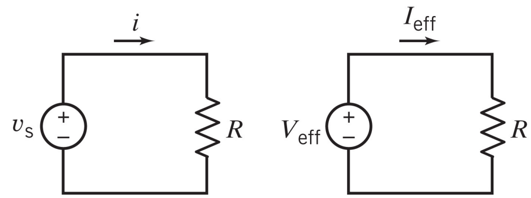
Substituting the impedance, the average AC power in terms of impedance is: $$\boxed{P = \frac{Z_m I_m^2}{2}\cos\theta_Z}$$

The effective value

The effective value of any periodic current (or voltage) is equal to the value of the direct current (or voltage) which, flowing through a resistance R, delivers the same power to R as the periodic current (or voltage) does.

Effective value of a periodic signal

Power delivered to a resistive load by a DC current is: $P_{DC} = i^2 R$



The same amount of average power will be delivered to a resistive load by an AC current if

$$P_{DC} = P_{AC}$$

Which is given by:

$$I_{eff}^2 R = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(t) R\, dt$$

or:

$$\boxed{I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\, dt}}$$

mean = average

root-mean-square

Define:

$$I_{\text{eff}} = I_{\text{rms}}$$

$$P = I_{\text{eff}}^2 R$$

Root-mean-square values

Similarly for an AC voltage $$V_{\text{eff}} = V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)\, dt}$$

For a sinusoidal current $I(t) = I_m \cos(\omega t)$

We have: $$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T I_{\text{m}}^2 \cos^2 \omega t\, dt} = \sqrt{\frac{I_{\text{m}}^2}{T}\int_0^T \frac{1}{2}(1+\cos 2\omega t)\, dt} = \frac{I_{\text{m}}}{\sqrt{2}}$$

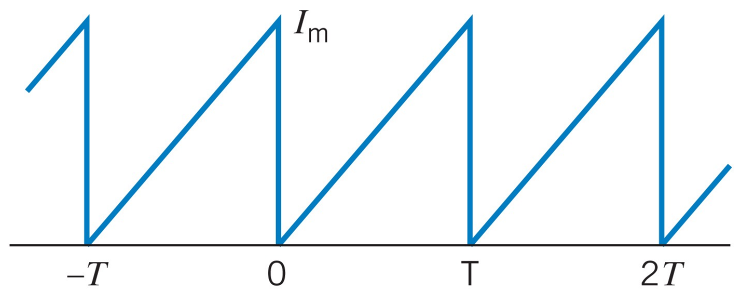
For sinusoidal voltages and currents

$$v(t) = V_m \cos(\omega t) \qquad V_{\text{rms}} = \frac{V_m}{\sqrt{2}}$$

$$i(t) = I_m \cos(\omega t) \qquad I_{\text{rms}} = \frac{I_m}{\sqrt{2}}$$

Example 11.4-1

For a current with a sawtooth waveform:



Find the rms value for the current!

The current can be defined mathematically as:

$$i(t) = \frac{I_m}{T} t \text{ for the period: } 0 \leq t < T$$

Therefore the effective value is:

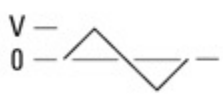
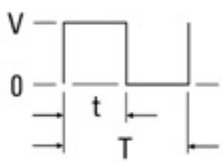
$$I_{rms}^2 = \frac{1}{T}\int_0^T i^2 dt = \frac{1}{T}\int_0^T \frac{I_m^2}{T^2} t^2 dt = \frac{I_m^2}{T^3}\left[\frac{t^3}{3}\right]_0^T = \frac{I_m^2}{3}$$

The rms current is: $I_{rms} = \frac{I_m}{\sqrt{3}}$

RMS of other periodic waveforms

You need to calculate the integral or use the look up tables in books

Peak value to the effective value

Waveform Shape	Crest Factor (C.F.)	AC RMS	AC + DC RMS	Average Responding Error
	1.414	$\frac{V}{1.414}$	$\frac{V}{1.414}$	Calibrated for 0 error
	1.732	$\frac{V}{1.732}$	$\frac{V}{1.732}$	-3.9%
	$\sqrt{\frac{T}{t}}$	$\frac{V}{C.F.} \times \sqrt{1-\left(\frac{1}{C.F.}\right)^2}$	$\frac{V}{C.F.}$	-46% for C.F. = 4

Average ac power:

$$P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_V - \theta_I)$$

For a real load, the DC voltage must be reduced by $\sqrt{2}$ to deliver the same average power as an AC voltage.

Complex power

Chapter 11 of Dorf and Svoboda

Definitions

In the frequency domain, the current and voltage are given by:

$$\mathbf{I}(\omega) = I_m \angle \theta_I \quad \text{and} \quad \mathbf{V}(\omega) = V_m \angle \theta_V$$

The **complex power** delivered to the element is defined as

$$\mathbf{S} = \frac{\mathbf{VI}^*}{2} = \frac{(V_m \angle \theta_V)\left(I_m \angle -\theta_I\right)}{2} = \frac{V_m I_m}{2} \angle \theta_V - \theta_I$$

The **apparent power** is defined as

$$|\mathbf{S}| = \frac{V_m I_m}{2}$$

(which is the magnitude of the complex power)

Or the apparent power is just $V_{eff} I_{eff}$

Reactive power

Converting the complex power from polar form to rectangular form we get:

$$\mathbf{S} = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \cos(\theta_{\mathrm{V}} - \theta_{\mathrm{I}}) + j \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \sin(\theta_{\mathrm{V}} - \theta_{\mathrm{I}})$$

Average power from previous time domain analysis (in Watts)

Reactive power

If we define: $\mathbf{S} = P + jQ$

Then we have:

$$P = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \cos(\theta_{\mathrm{V}} - \theta_{\mathrm{I}}) \qquad Q = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \sin(\theta_{\mathrm{V}} - \theta_{\mathrm{I}})$$

Explanation for the reactive power

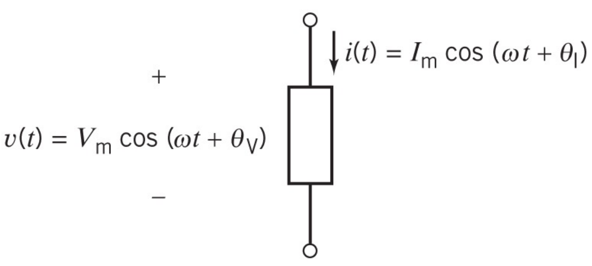
Purely reactive loads such as inductors and capacitors dissipate zero power, yet the fact that they drop voltage and draw current gives the *deceptive impression* that they actually do dissipate power. This "phantom power" is called reactive power, and it is measured in a unit called Volt-Amps-Reactive (VAR), rather than Watts. The mathematical symbol for reactive power is the capital letter Q. The actual amount of power being used, or dissipated, in a circuit is called the real (average) power, and it is measured in Watts (symbolized by the capital letter P, as always).

As a rule, real power is a function of a circuit's dissipative elements, usually resistances (R). Reactive power is a function of a circuit's reactance (X).

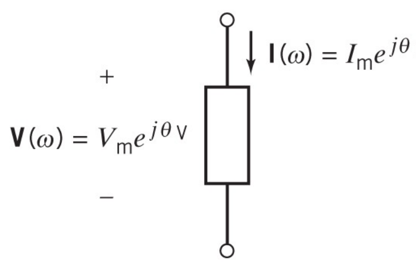
Complex power wrt impedance

Units:

Complex power: VA
Average power: W
Reactive power: VAR (R means “reactive”)



(a)



(b)

The impedance is given by:

$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = \frac{V_m \angle \theta_V}{I_m \angle \theta_I} = \frac{V_m}{I_m} \angle \theta_V - \theta_I$$

Which can be converted to rectangular form: $\mathbf{Z}(\omega) = \frac{V_m}{I_m} \cos(\theta_V - \theta_I) + j\frac{V_m}{I_m} \sin(\theta_V - \theta_I)$

Or: $\boldsymbol{Z}(\omega) = R + jX$

Where: $R = \frac{V_m}{I_m}\cos(\theta_V - \theta_I)$ and $X = \frac{V_m}{I_m}\sin(\theta_V - \theta_I)$

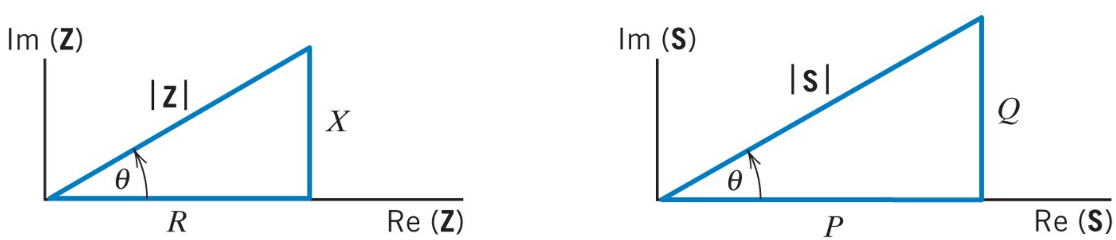
Complex power in terms of impedance

Recall that the complex power is given by: $\mathbf{S} = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + j\frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$

Which can be re-written as: $\boldsymbol{S} = \frac{I_m^2}{2}\frac{V_m}{I_m}\cos(\theta_V - \theta_I) + j\frac{I_m^2}{2}\frac{V_m}{I_m}\sin(\theta_V - \theta_I)$

Which is: $\boldsymbol{S} = \frac{I_m^2}{2}Re(\boldsymbol{Z}) + j\frac{I_m^2}{2}Im(\boldsymbol{Z})$

Hence the average (true) power is: $\mathrm{P} = \frac{I_m^2}{2}Re(\boldsymbol{Z})$ and the reactive power: $\mathrm{Q} = \frac{I_m^2}{2}Im(\boldsymbol{Z})$



(a) *(b)*

Power in the frequency domain

QUANTITY	RELATIONSHIP USING PEAK VALUES	RELATIONSHIP USING rms VALUES	UNITS
Element voltage, $v(t)$	$v(t) = V_m \cos(\omega t + \theta_V)$	$v(t) = V_{rms}\sqrt{2}\cos(\omega t + \theta_V)$	V
Element current, $i(t)$	$i(t) = I_m \cos(\omega t + \theta_I)$	$i(t) = I_{rms}\sqrt{2}\cos(\omega t + \theta_I)$	A
Complex power, $\mathbf{S}$	$\mathbf{S} = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + j\frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$	$\mathbf{S} = V_{rms} I_{rms}\cos(\theta_V - \theta_I) + jV_{rms} I_{rms}\sin(\theta_V - \theta_I)$	VA
Apparent power, $\lvert\mathbf{S}\rvert$	$\lvert\mathbf{S}\rvert = \frac{V_m I_m}{2}$	$\lvert\mathbf{S}\rvert = V_{rms} I_{rms}$	VA
Average power, P	$P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I)$	$P = V_{rms} I_{rms}\cos(\theta_V - \theta_I)$	W
Reactive power, Q	$Q = \frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$	$Q = V_{rms} I_{rms}\sin(\theta_V - \theta_I)$	VAR