Driven SHM

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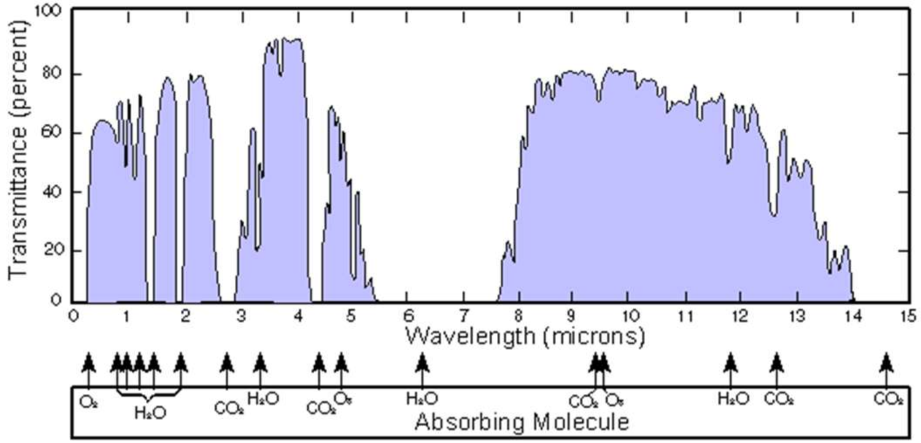
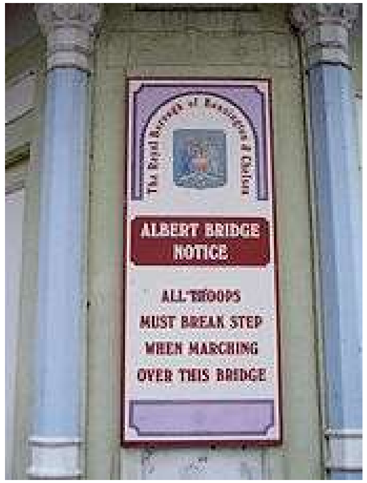
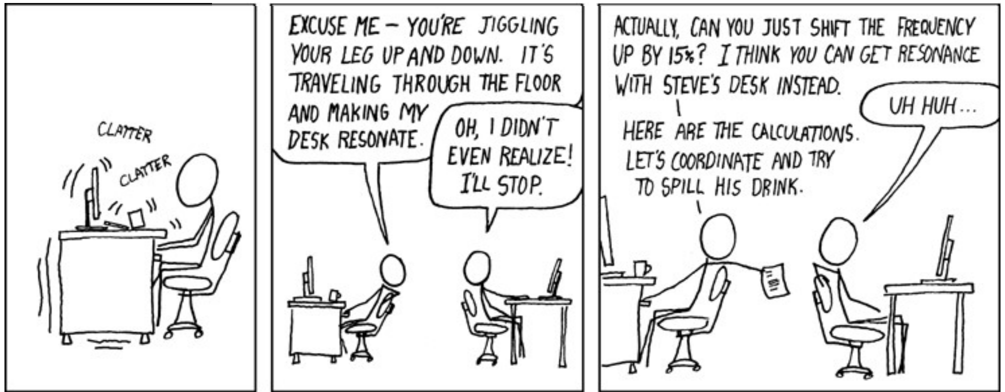
Driven or forced SHM

- We've looked at
  - undamped motion
  - (decaying) damped motion
- Now consider forced motion
  - sinusoidal driving forces of different frequencies
    - sum responses to describe arbitrary driving forces
    - find: resonances



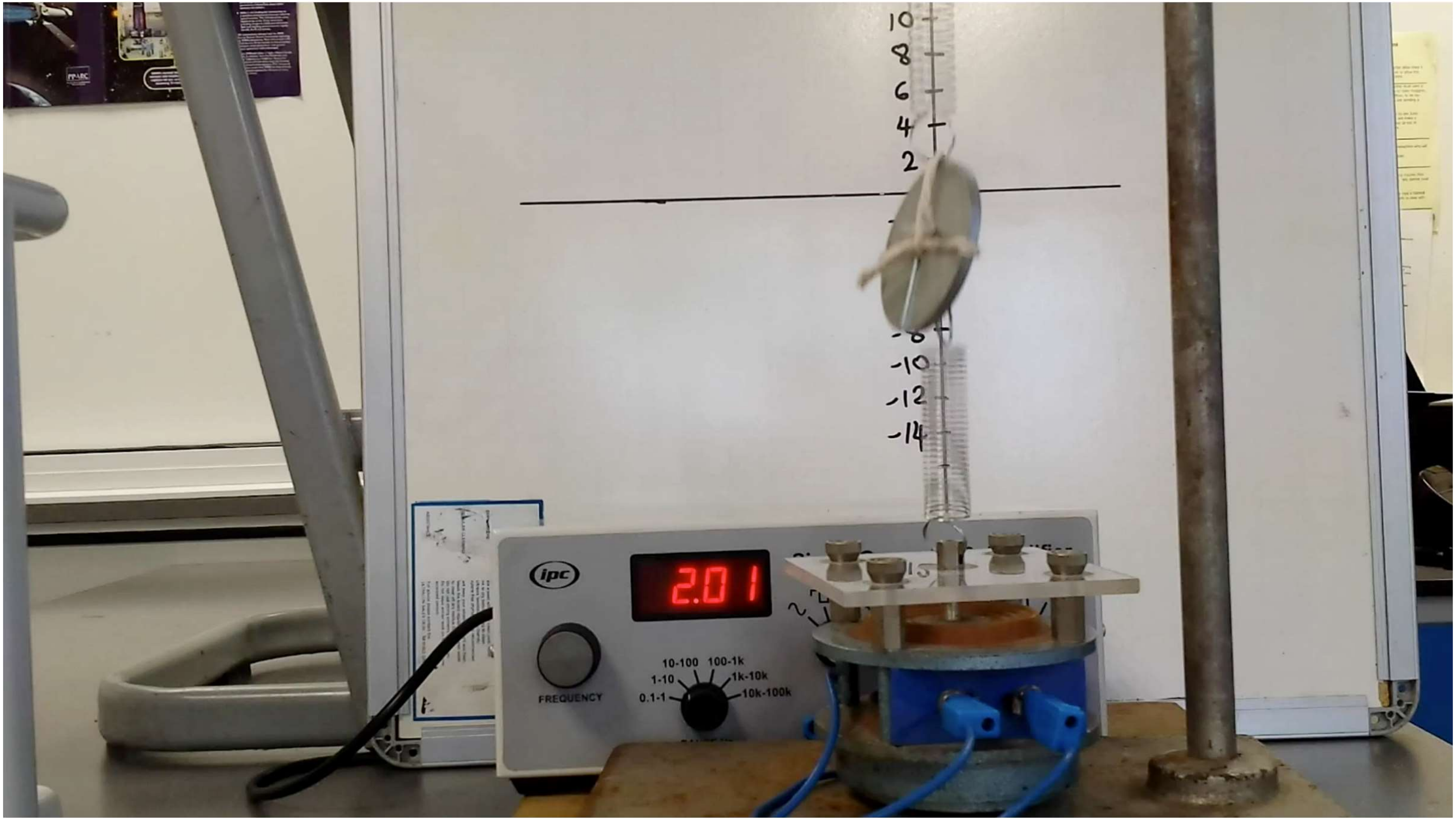


- Innumerable examples:
  - Tuned electrical circuits
  - Optical absorption
  - Mechanical [stability]
  - Acoustics
  - Tides
  - Astrophysics….

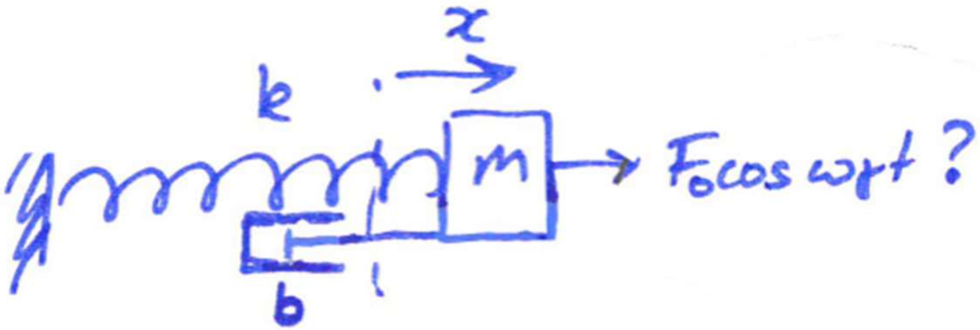


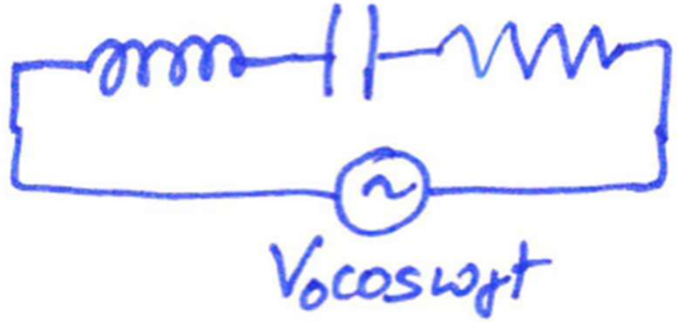
https://xkcd.com/228/

Forced oscillations and resonance



Add a driving term





$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega_f t$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = a_0 \cos \omega_f t$$

$$\gamma = b/m$$

$$\omega_0^2 = k/m$$

$$f_0 = a_0 = F_0/m$$

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = f_0 e^{i\omega_f t}$$

where $z = x + iy$ and we take $\mathrm{Re}[z]$ to get x

How does the response of the system depend on ω_f?

How to solve the ODE….

- $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = f_0 e^{i\omega_f t}$
  - Inhomogeneous second-order linear differential equation

- To solve:
  - Find any solution to $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = f_0 e^{i\omega_f t}$
    - $z = z_P(t)$ "particular solution"
  - Find the general solution to the complementary homogeneous equation $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0$
    - $z = z_C(t)$
    - We did this! Over/critical/underdamped motion depending on γ, ω_0 and initial conditions
  - General solution is $z = z_P(t) + z_C(t)$

Solving the ODE (continued)

- *Any* particular solution will do
  - the difference of any 2 particular solutions is a complementary solution
    (choice of z_P is ironed out by change in the complementary function)
- Proof: two particular solutions z_1 and z_2 walked into a bar:

$$\ddot{z}_2 + \gamma \dot{z}_2 + \omega_0^2 z_2 = f_0 e^{i\omega_f t}$$

$$\ddot{z}_1 + \gamma \dot{z}_1 + \omega_0^2 z_1 = f_0 e^{i\omega_f t}$$

$$\Rightarrow \quad \ddot{z}_2 - \ddot{z}_1 + \gamma(\dot{z}_2 - \dot{z}_1) + \omega_0^2(z_2 - z_1) = 0$$

$$\therefore \quad \ddot{z}_d + \gamma \dot{z}_d + \omega_0^2 z_d = 0 \quad \text{where} \quad z_d = z_2 - z_1$$

Complementary functions are damped

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Underdamped: $\gamma < 2\omega_0$

$$x = e^{-\frac{\gamma}{2}t}(A\, e^{\mathrm{i}\omega_d t} + B\, e^{-\mathrm{i}\omega_d t})$$

$\omega_d = (\omega_0^2 - \gamma^2/4)^{1/2}$ is real

Overdamped: $\gamma > 2\omega_0$

$$x = e^{-\frac{\gamma}{2}t}(A\, e^{\Omega t} + B\, e^{-\Omega t})$$

$$\Omega = (\gamma^2/4 - \omega_0^2)^{1/2}$$

$\Omega < \gamma/2$ so both terms decay with time

Critical damping: $\gamma = 2\omega_0$

$$x = (A + Bt)\, e^{-\frac{\gamma}{2}t}$$

Damping $\Longrightarrow$ complementary function dies away over time: a *transient*

Long-term behaviour is determined by response to the driving force, not by the initial conditions

Particular solution

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = f_0 e^{i\omega_f t}$$

Let's try $z_P = Ae^{\alpha t}$

$$\underbrace{(\alpha^2 + \gamma\alpha + \omega_0^2)A}_{\text{constant}} e^{\alpha t} = f_0 e^{i\omega_f t}$$

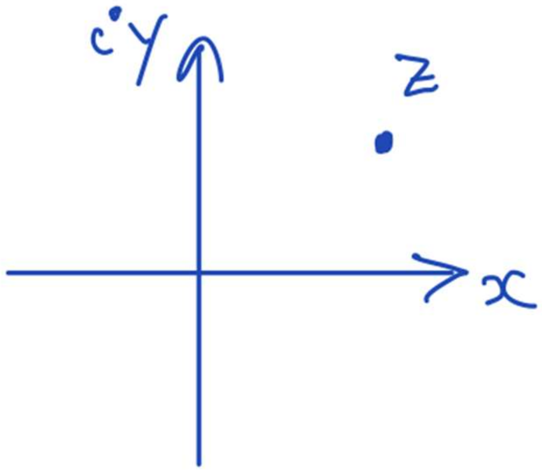
functions of t

$\Longrightarrow \quad \alpha = i\omega_f$ system oscillates at the driving frequency

and $\quad A = \dfrac{f_0}{(\omega_0^2 - \omega_f^2) + i\omega_f \gamma}$ note: contains amplitude and phase

$$\therefore \quad z_P = \frac{f_0}{(\omega_0^2 - \omega_f^2) + i\omega_f \gamma} e^{i\omega_f t}$$

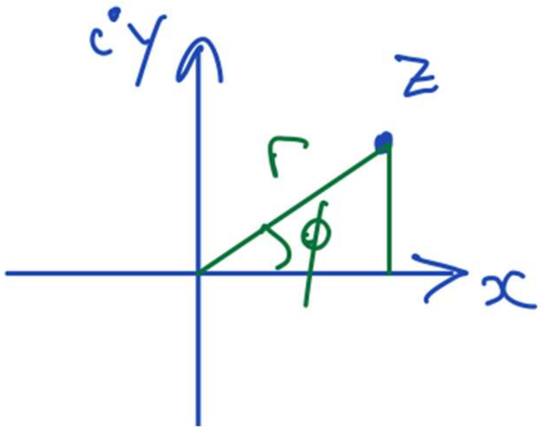
Reminder: complex numbers…..



$$i^2 = -1 \qquad \mathrm{Re}(z) = x$$

$$z = x + iy \qquad \mathrm{Im}(z) = y$$

$$\left.\begin{array}{l} z_1 = x_1 + iy_1 \\ z_2 = x_2 + iy_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2) \\ z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \end{array}\right.$$



$$|z| = \sqrt{x^2+y^2} = r$$

$$\phi = \tan^{-1}(y/x)$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

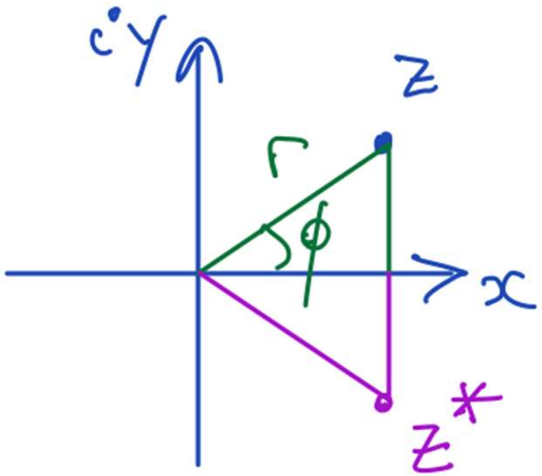
$$|e^{i\phi}| = 1$$

$$z = re^{i\phi}$$

$$z_1 z_2 = |z_1|\,|z_2|\, e^{i(\phi_1+\phi_2)}$$

$$e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y)$$

Complex conjugate



$$z^* = x - iy = re^{-i\phi}$$

$$(z_1 + z_2)^* = z_1^* + z_2^*$$

$$(z_1 z_2)^* = z_1^* z_2^*$$

$$\mathrm{Re}(z) = \frac{1}{2}(z + z^*)$$

$$\mathrm{Im}(z) = \frac{1}{2i}(z - z^*)$$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{z^*}{|z|^2} = \frac{1}{r}e^{-i\phi}$$

Take the real part…

$$z_P = Ae^{i\omega_f t}, \quad A = \frac{f_0}{\left(\omega_0^2 - \omega_f^2\right) + i\omega_f \gamma}$$

As usual, its best to first express *A* in terms of its modulus $A_P = |A|$ and phase ϕ_P, i.e. $A = A_P e^{i\phi_P}$

$$x_P = \mathrm{Re}[z_P] = \mathrm{Re}\left[A_P e^{i\phi_P} e^{i\omega_f t}\right] = A_P \mathrm{Re}\left[e^{i(\omega_f t + \phi_P)}\right]$$

$$\therefore x_P = A_P \cos\left(\omega_f t + \phi_P\right)$$

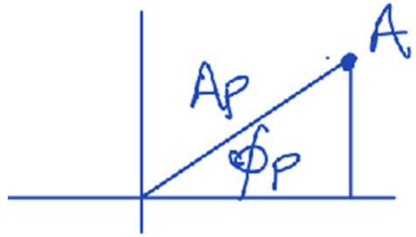
now to find A_P and ϕ_P …..

$$z_P = Ae^{i\omega_f t}, \quad A = \frac{f_0}{\left(\omega_0^2 - \omega_f^2\right) + i\omega_f \gamma}$$

$$A = \frac{f_0}{\omega_0^2 - \omega_f^2 + i\omega_f \gamma}$$

$$= \frac{f_0}{\omega_0^2 - \omega_f^2 + i\omega_f \gamma} \frac{\omega_0^2 - \omega_f^2 - i\omega_f \gamma}{\omega_0^2 - \omega_f^2 - i\omega_f \gamma}$$

$$= f_0 \frac{\omega_0^2 - \omega_f^2 - i\omega_f \gamma}{(\omega_0^2 - \omega_f^2)^2 + \omega_f^2 \gamma^2}$$



$$A_P = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + \omega_f^2 \gamma^2}}$$

$$\phi_P = \tan^{-1}\left(\frac{-\omega_f \gamma}{\omega_0^2 - \omega_f^2}\right)$$

Complete solution (for underdamped...)

$$x(t) = x_P(t) + x_C(t)$$

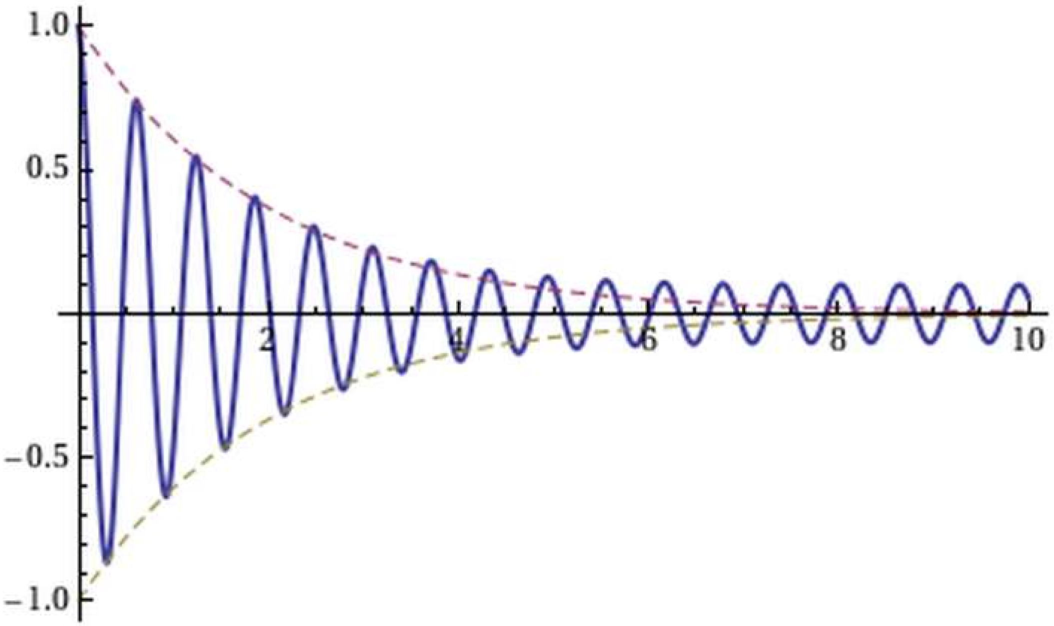
determined by driving force $f_0 \cos \omega_f t$

determined by x_0, v_0

$$\therefore\ x = A_P \cos\left(\omega_f t + \phi_P\right) + A_C\, e^{-\gamma t/2} \cos\left(\omega_d t + \phi_C\right)$$

transient

steady state



'Transient' phase while the complementary solution is significant. Sensitive to initial conditions.

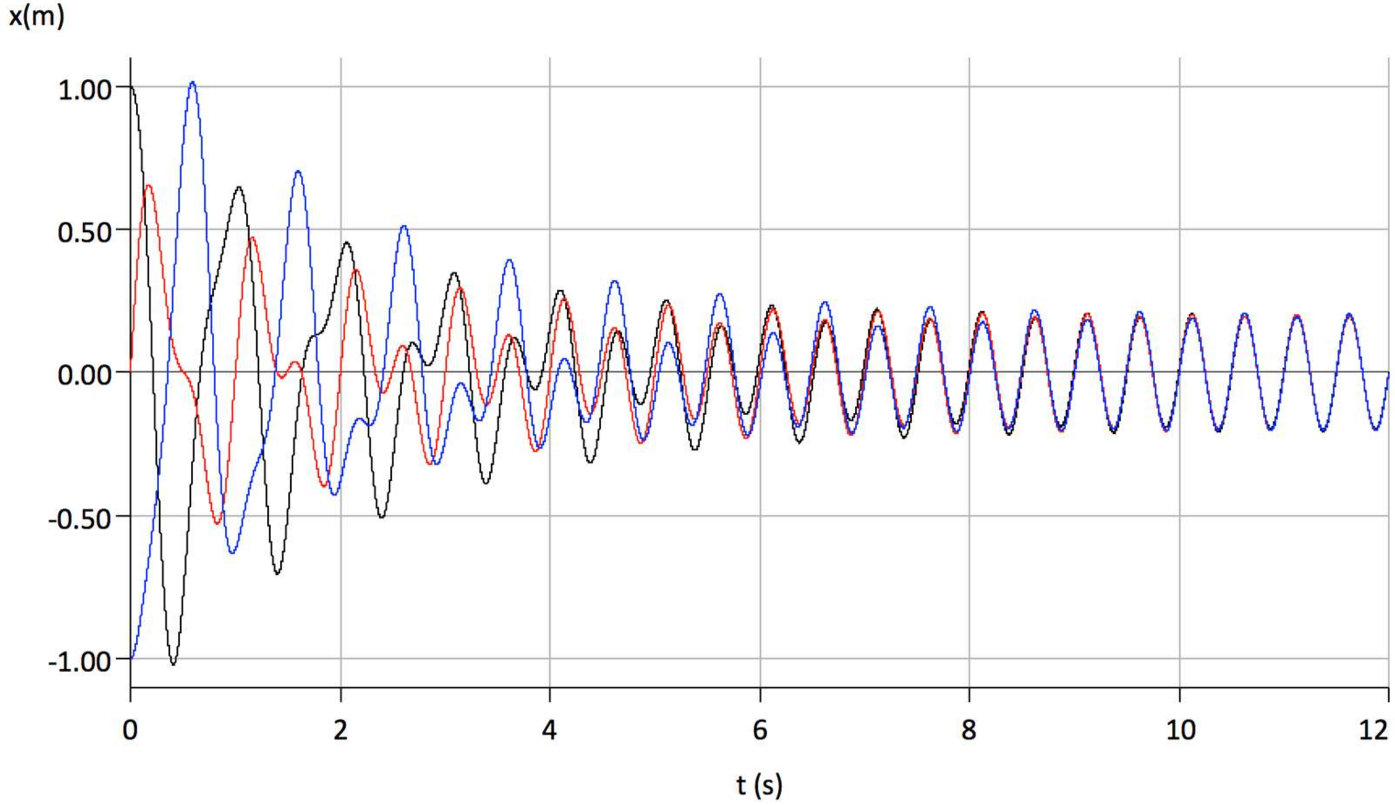
'Steady state' for $t \gg \gamma/2$, independent of initial conditions.

Complete solution (for underdamped…)

$$x = A_C\, e^{-\gamma t/2} \cos\left(\omega_d t + \phi_C\right) + A_P \cos\left(\omega_f t + \phi_P\right)$$

'Transient' phase while the complementary solution is significant. Sensitive to initial conditions.

'Steady state' for $t \gg \gamma/2$, independent of initial conditions.



Complete solution (for underdamped…)

$$x = A_C\, e^{-\gamma t/2} \cos(\omega_d t + \phi_C) + A_P \cos(\omega_f t + \phi_P)$$

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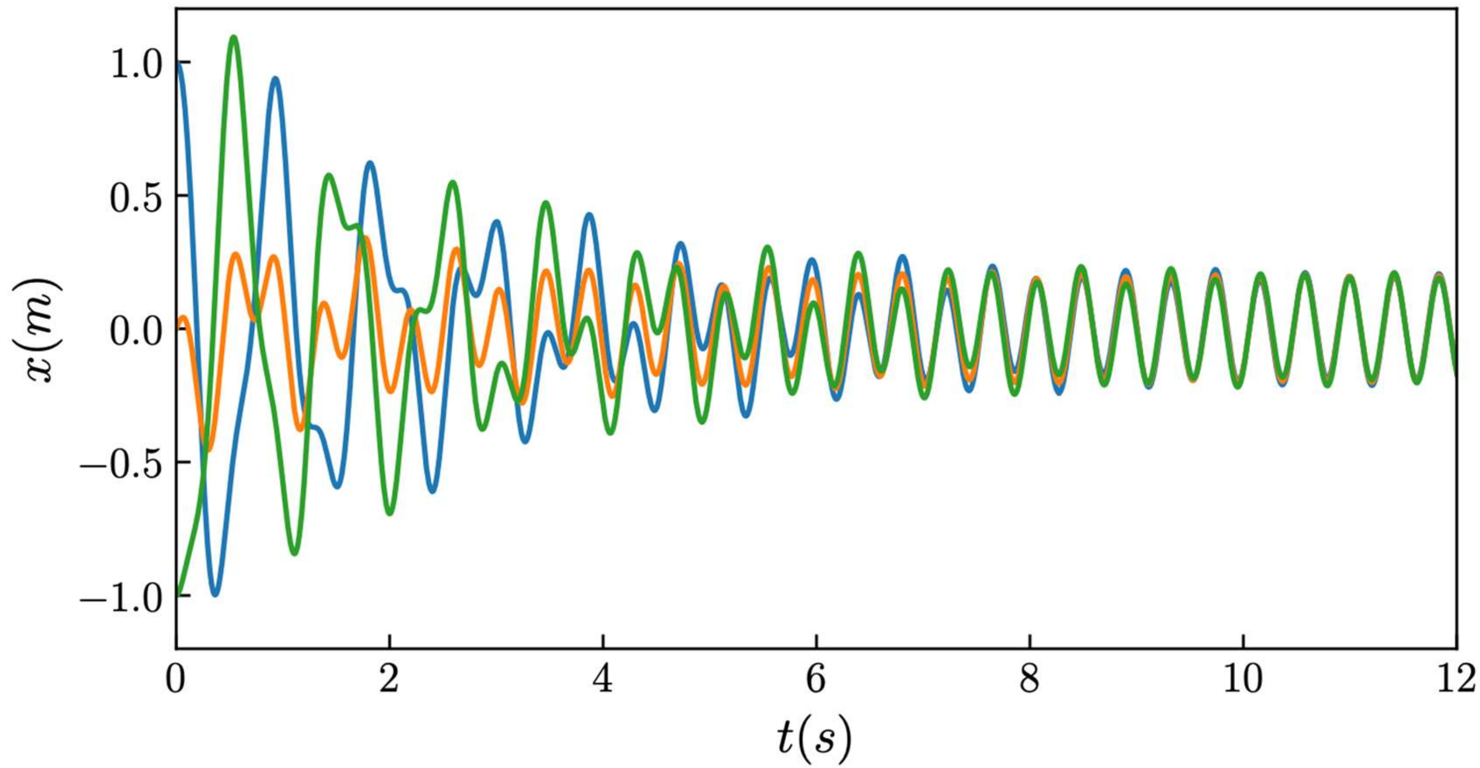


Transient behaviour

- Remember that steady state amplitude and phase is entirely defined by the driving
  - If initial conditions are not consistent with this motion, then there must be an initial period of adjustment
  - This is described by the decaying 'complementary function'

$$x = A_P \cos\left(\omega_f t + \phi_P\right) + A_C\, e^{-\gamma t/2} \cos\left(\omega_d t + \phi_C\right)$$

- Transient behaviour may be complicated
  - $x_C(t)$ superimposes on $x_P(t)$
  - $x_C(t)$ oscillates at ω_d not ω_f, so we can get beating
  - Significant only for a time of order $\gamma/2$

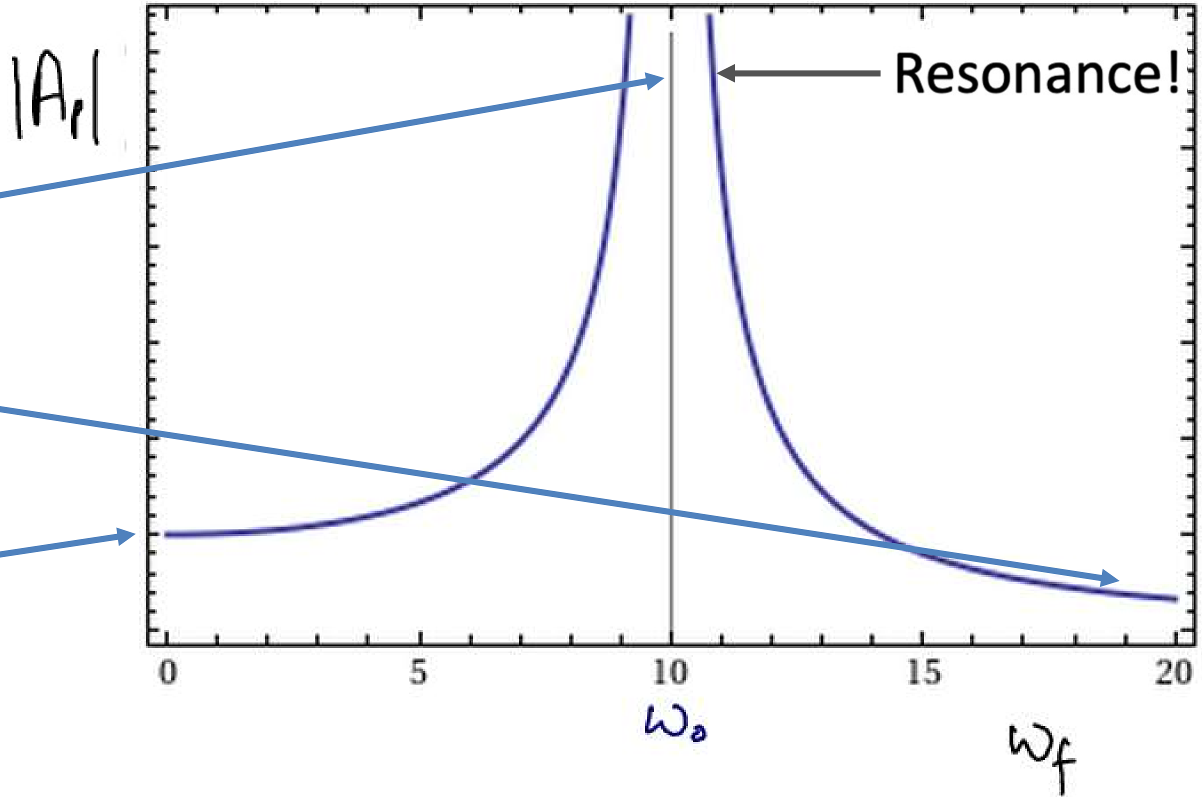
Steady state behaviour - undamped

$$x_P = A_P\cos(\omega_f t + \phi_P) \qquad A_P = \frac{f_0}{\sqrt{\left(\omega_0^2-\omega_f^2\right)^2+\omega_f^2\gamma^2}}, \qquad \phi_P = \tan^{-1}\frac{-\omega_f\gamma}{\omega_0^2-\omega_f^2}$$

$$\gamma = 0 \implies A_P = \frac{f_0}{|\omega_0^2 - \omega_f^2|}$$

natural frequency of oscillator — driving frequency

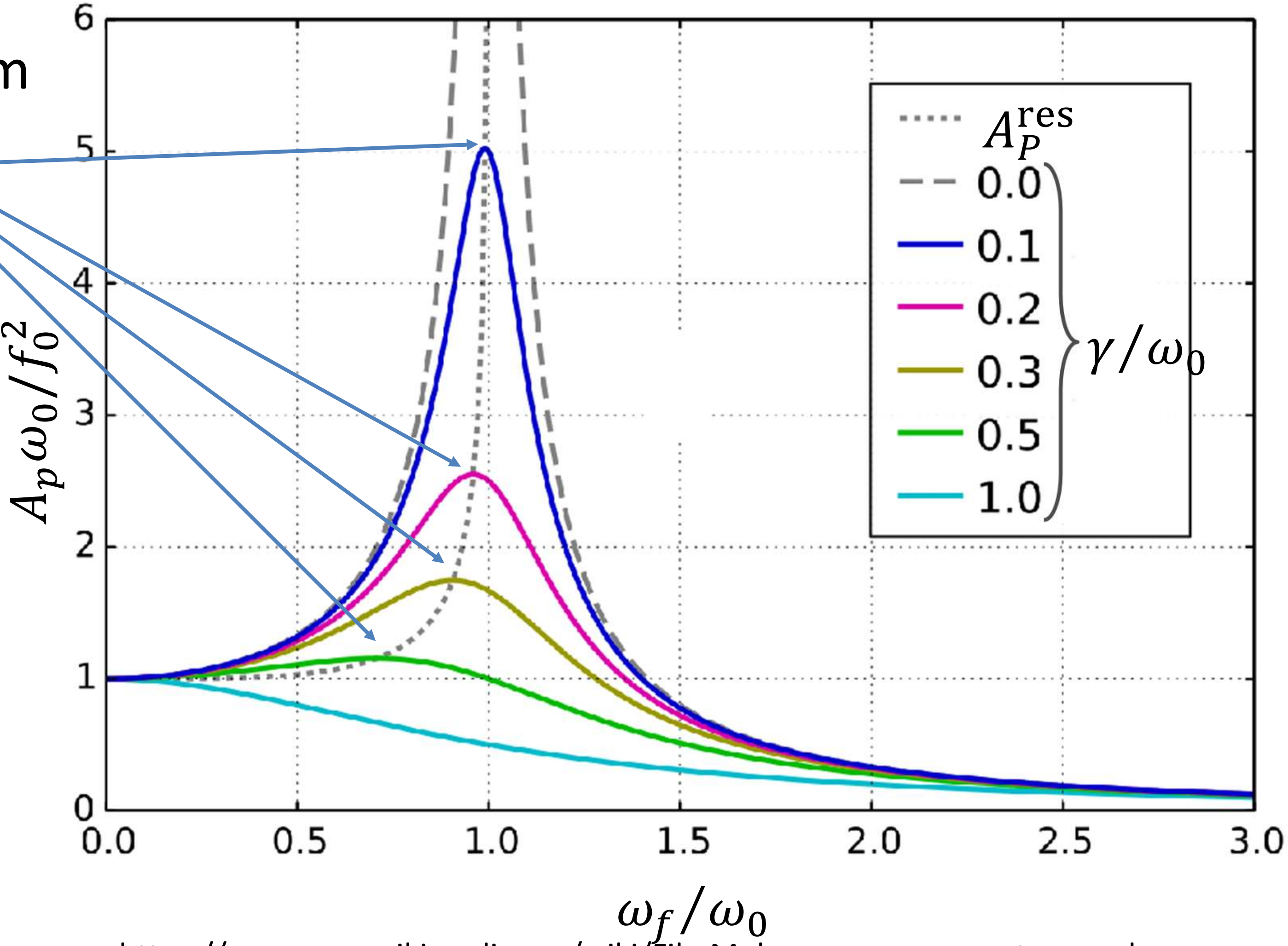
- A_P^{res} infinite at $\omega_f = \omega_0$
- $A_P^{\infty} = 0$ as $\omega_f \to \infty$
- $A_P^0 = f_0/\omega_0^2$ at $\omega_f = 0$



Steady state behaviour - damped

$$x_P = A_P \cos(\omega_f t + \phi_P) \qquad A_P = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + \omega_f^2 \gamma^2}},$$

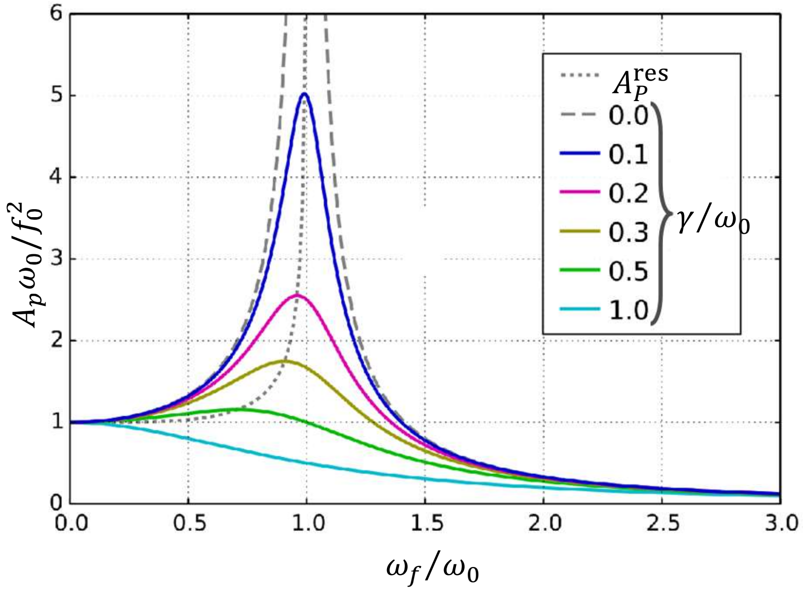
Amplitude has maximum A_P^{res} at $\omega_f = \omega_{\text{res}}$



https://commons.wikimedia.org/wiki/File:Mplwp_resonance_zeta_envelope.svg

Finding ω_{res} and A_P^{res}

$$A_p = \frac{f_0}{\sqrt{\underbrace{(\omega_0^2-\omega_f^2)^2 + \omega_f^2\gamma^2}_{D}}} = \frac{f_0}{\sqrt{D}}$$



maximum when $\dfrac{dA_p}{d\omega_f} = \dfrac{f_0}{2D^{3/2}} \cdot \dfrac{dD}{d\omega_f} = 0$

$$\Rightarrow 0 = \frac{dD}{d\omega_f} = 2(\omega_0^2-\omega_f^2)\cdot(-2\omega_f) + 2\omega_f\gamma^2 = 2\omega_f\left[\gamma^2 - 2(\omega_0^2-\omega_f^2)\right]$$

$$\Rightarrow \boxed{\omega_{\text{res}} = \sqrt{\omega_0^2 - \gamma^2/2}}$$

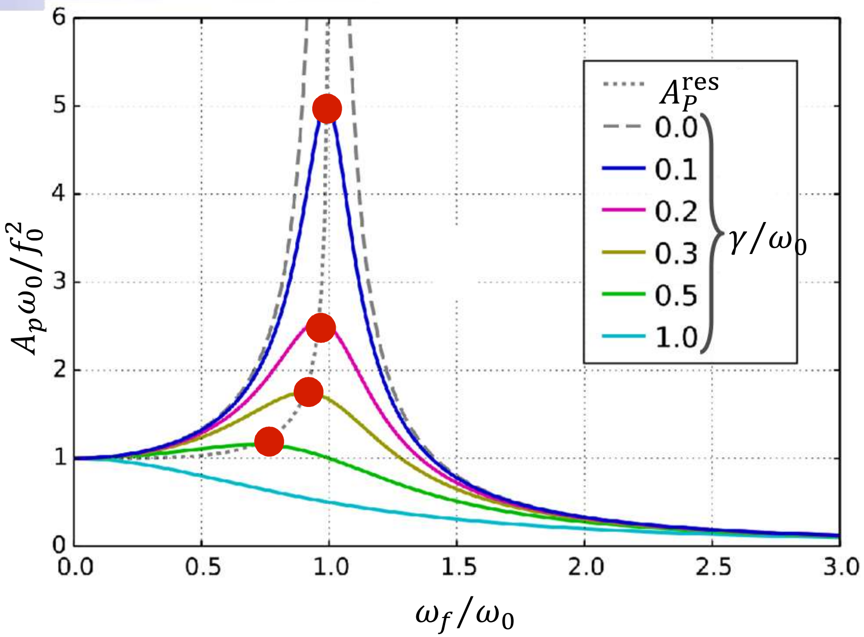
cf $\omega_d = \sqrt{\omega_0^2 - \gamma^2/4}$

$$\boxed{\omega_{\text{res}} < \omega_d < \omega_0}$$

NB/ $\omega_0^2 - \omega_{\text{res}}^2 = \gamma^2/2$

$\omega_0^2 - \omega_d^2 = \gamma^2/4$

Finding ω_{res} and A_P^{res}



$$A_p = \frac{f_0}{\sqrt{(\omega_0^2-\omega_f^2)^2+\omega_f^2\gamma^2}}$$

maximum when $\omega_{res} = \sqrt{\omega_0^2-\gamma^2/2}$

evaluate A_P for $\omega_f = \omega_{\text{res}}$:

$$(\omega_0^2-\omega_{res}^2)^2+\gamma^2\omega_{res}^2 = \frac{\gamma^4}{4}+\gamma^2(\omega_0^2-\gamma^2/2) = \gamma^2\left[\omega_0^2-\gamma^2/4\right] = \gamma^2\omega_d^2$$

$$\Rightarrow \boxed{A_p^{res} = \frac{f_0}{\gamma\omega_d}}$$

Steady state: sub-critical damping

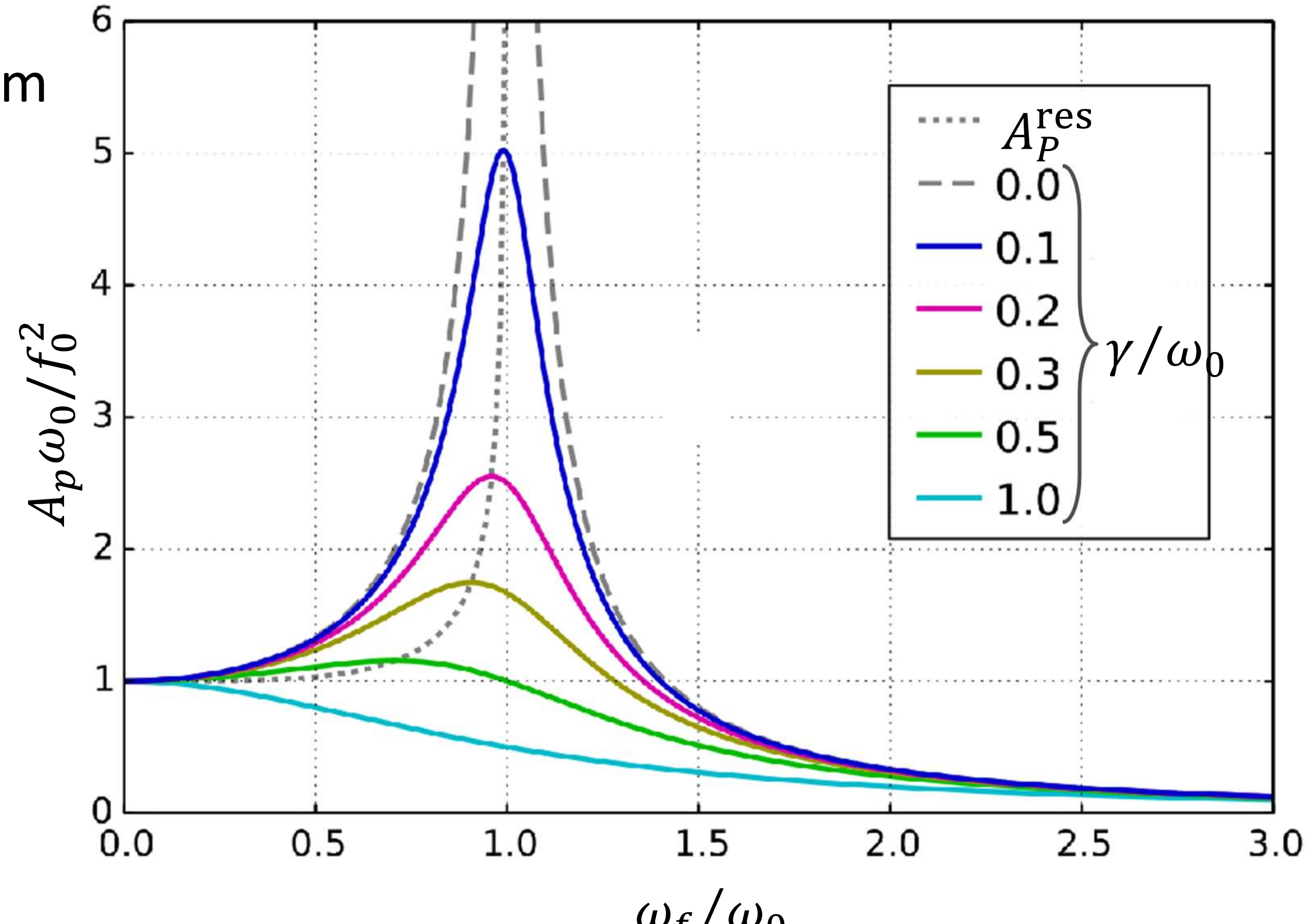
$$x_P = A_P \cos(\omega_f t + \phi_P) \qquad A_P = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + \omega_f^2 \gamma^2}},$$

Amplitude has maximum A_P^{res} at $\omega_f = \omega_{\mathrm{res}}$

$$A_P^{res} = f_0/(\gamma\omega_d)$$
$$A_P^0 = f_0/\omega_0^2$$
$$A_P^{res}/A_P^0 \approx Q!$$

$$\omega_{res} = (\omega_0^2 - \gamma^2/2)^{1/2}$$
$$\omega_d = (\omega_0^2 - \gamma^2/4)^{1/2}$$

$$\omega_{\mathrm{res}} < \omega_d < \omega_0$$



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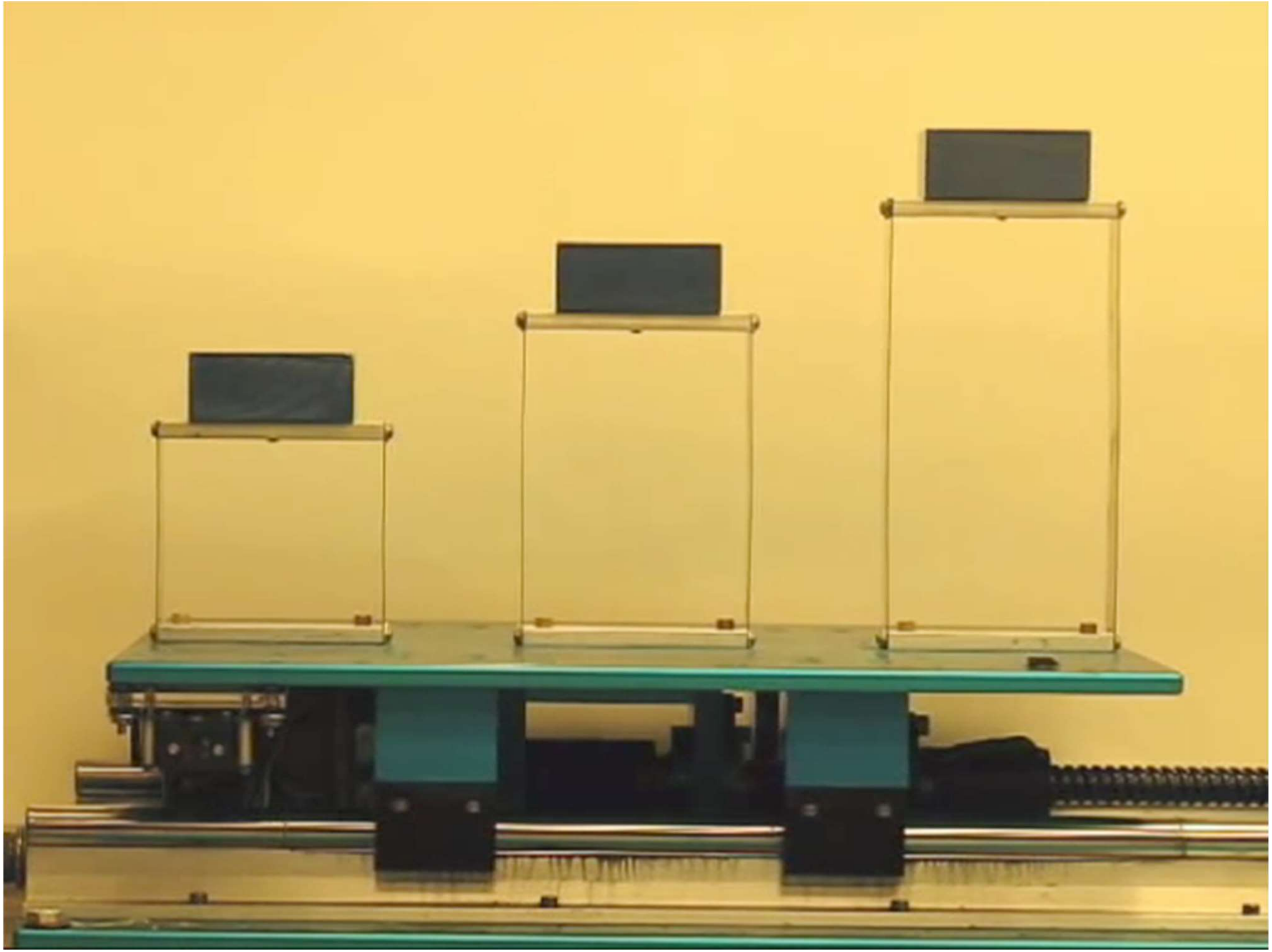
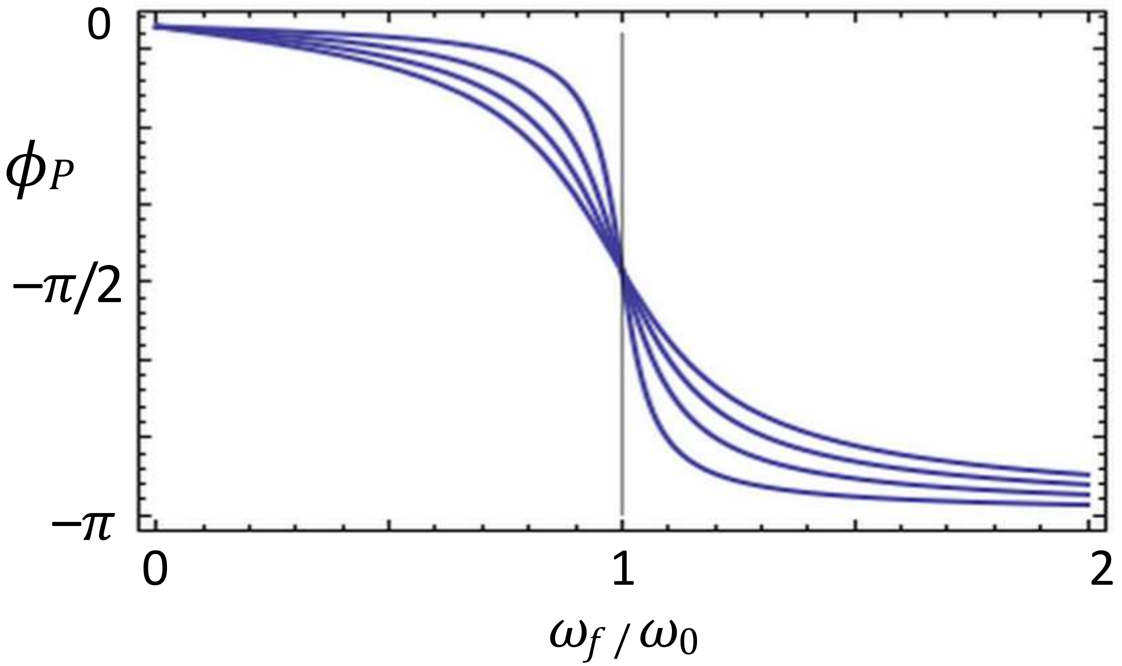
Forced oscillations and resonance



Steady state behaviour - damped

$$x_P = A_P \cos(\omega_f t + \phi_P) \qquad \phi_P = \tan^{-1}\frac{-\omega_f \gamma}{\omega_0^2 - \omega_f^2}$$

ϕ_P ranges from 0 to $-\pi$



Phase vs frequency

https://youtu.be/aZNnwQ8HJHU

Wine glass resonance in slow motion