MAT133 Learning Objectives

2 Financial Mathematics (~ 4.5 hours)

The mathematics of this entire chapter is straightforward. Overall, the learning objective of this section is *not* the ability to plug things into a formula. Our objective is to have students recognize when to use the appropriate formula; for example, future versus present value, continuously compounding interest, annuities versus simple compounding interest.

2.1 Compounding Interest (~ 2 hours)

Students should leave this section with a coarse understanding of the time value of money [Sec 2.1.1-1]. They should be introduced to appropriate terminology [Sec 2.1.1-2], and be able to compute the future/present value of simple compounding interest [Sec 2.1.1-3], [Sec 2.1.2-1]. They should be able to apply the concepts of Future/Present Value in the context of continuously compounding interest [Sec 2.1.3-1].

All previous concepts should be used to solve intermediate questions involving different compounding periods, multiple investment accounts, et cetera.

2.1.1 Compounding Interest (~ 1 hour)

1. Students should understand the time value of money, and in particular that through inflation and investment, money today is worth more than money in the future. Use an example of an investment with a high interest rate to make this point.

2. Students should be introduced to the following words:

   - Principal
   - Compounding
   - Bonds
   - Equity
   - Term Rate
   - Annual Percentage Rate (APR)
   - Nominal Rate
   - Future Value
   - Present Value

3. Students should know Equation (2.1) and how it is derived. **Important:** They should understand the difference between Equations (2.1) and (2.2):

$$P(1+r)^N \quad \text{for term rate, and} \qquad P\left(1+\frac{r}{t}\right)^{nt} \quad \text{for APR.}$$

   As such, they should be able to convert from an APR to a term rate.

4. Students should be able to apply the future value of simple compounding interest to solve intermediate level questions, such as those involving multiple investment accounts (Example 2.3) or different compounding periods.

5. Students should understand the Effective Annual Rate (EAR) Equation (2.3), how its formula is derived, and how to apply it to compare different APR scenarios. Given two investment schemes, students should be able to determine which has a higher annual return, based of EAR.

6. Students should understand that the more often an investment compounds, the faster it grows. This can be illustrated either using the future value, or the effective annual rate (Table 2). Emphasis should be placed on the relationship between the EAR and the APR. (If $t > 1$ then the EAR is greater than the APR). They should be able to answer simple True/False questions, such as "If quoted an APR of $r\%$ compounded monthly, with r_a the effective annual rate, then $r < r_a$".

2.1.2 Present Value (~ 30 minutes)

1. Students should understand the present value formula (Equation (2.4)) $P = S(1+r)^{-N}$, and how it is derived. As with the future value, they should be able to differentiate between $P = S(1+r)^{-N}$ and the corresponding formula for a given APR.

2. Students should understand that the present value can be thought of as the amount of money one must invest today to achieve the future value S, and use this to determine the amount of money that must be invested to achieve a goal (Example 2.4).

3. Students should be able to apply the concept of present value to compare two investments (Example 2.5), and solve intermediate level questions involving multiple accounts (Example 2.6).

   Tip: Emphasize bonds, as they are a good pedagogical tool.

2.1.3 Continuous Compounding Interest (~ 30 minutes)

1. Students should understand what "continuously compounding means." This section is difficult to motivate in the absence of a discussion of limits.

2. It is prudent to discuss the advantages of using continuously compounding interest. Fewer variables, easier to solve for r and n.

3. Students should understand what Effective Continuous Rate (Equation (2.5)) means and be able to compute it given an APR. Emphasis should be placed on how effective the effective continuous rates is related to the APR (it's smaller).

4. As per the chapter learning objectives, these concepts should be combined whenever possible to create more complicated questions. Choose exotic compounding periods, multiple accounts (Example 2.8), and change between present and future value. Bonds are a great security to use for multiple examples.

2.2 Annuities (~ 2 hours)

Students should understand what an annuity is, and how it is different from simple compounding interest [Sec 2.2.1-1]. The derivation of the future and present values formula for an annuity should be given, with emphasis on when this formula applies. In both future and present value cases, students should be able to distinguish the term-rate formula from the APR formula. Students will understand why amortizations are modelled using the present value of an annuity, and should understand how to derive the "interest/principal data" after k-periods of an amortization [Sec 2.2.2-2].

This should all be combined with the ability to apply these concepts to solve intermediate problems, involving different APRs, and multiple accounts.

2.2.1 Future and Present Value of an Annuity (~ 1.5 hours)

1. Introduce annuities as those investments with recurring deposits. Explain the difference between ordinary annuities and annuities due. We will only deal with ordinary annuities in this class.
2. Build up the derivation of the annuities formula (Equation (2.6)), and how the formula for the sum of a finite geometric series allows us to simplify this. Give the same formula for a quoted APR (Equation (2.7)), but emphasize their difference. Students should be able to rederive this formula if different compounding or deposit terms are given.
3. Introduce the Present Value of an Annuity (Equation (2.8)), complete with derivation. Emphasize the paradigm that this is the amount of money that must be invested today to cover the annuity payments (Example 2.10).
4. Emphasize in both the future and present value cases, the importance that the compounding period and deposit period coincide.
5. Students should be able to use future and present values of annuities to solve intermediate level questions, involving multiple accounts (Example 2.12), or to compare the return on investment for two different strategies (Example 2.11).

2.2.2 Amortization (~ 30 minutes)

1. Motivate the use of the present value formula; namely, that the bank is purchasing an annuity from you, whose present value is the value of the loan.
2. Go through the derivation of the various formulas provided at the end of the section, such as "The principal remaining at the beginning of the kth statement".

   Tip: The $k-1$ term is confusing for many students, but occurs because we're looking at the principal at the *beginning* of the kth term.
3. Students should be able to derive these formulas given a basic understanding of what everything means.

Note: Amortization will not be heavily examined outside of present value. It is included as an important financial instrument, and it's worth pointing out why the Present Value of an Annuity formula is used.

2.3 Perpetuities ($\sim$ 30 minutes)

Students should leave this sections with knowledge of what defines a perpetuity; namely, what separates it from an annuity. They should be familiar with the derivation of the perpetuity formula, and how this formula can be applied to solve straightforward questions.

1. Students will understand that a perpetuity has no maturity, and consists of infinitely many recurring deposits. Emphasis should be placed on why we would expect perpetuities to have a finite present value. Students should understand that the present value corresponds to the amount of money that must be invested today to cover all future payments.

2. The perpetuity formula (2.10) should be derived, with explanation. Students should understand how to modify this if the compounding or deposit periods change.

3. Students should be able to apply the perpetuity formula to find the present value in simple cases.

3 Linear Algebra

Here we'll introduce students to linear systems, their matrix representations, and algorithms for solving those linear systems. We'll introduce some advanced tools, such as determinants and eigenvectors, but will avoid discussing linear transformations, linear independence, span, bases, or dimension.

Students should leave this unit with an understanding of how linear systems are related to matrices and matrix equations, and how to convert between them. They should be able to solve relatively small systems (no larger than 5×5), but know how to generalize this to larger systems as necessary. Students should be able to use Gaussian elimination to convert a matrix into Reduced Row Echelon Form, and from this infer the rank of the system. Students should be able to use the rank to determine the number of solutions/number of parameters in the solution set.

Students should be able to add matrices of relatively small size, as well as perform scalar multiplication, matrix multiplication, and the transpose. Students should understand how column vectors arise from matrices, and determine whether one vector is a linear combination of others. Students will know why the inverse is useful, and be able to find the inverse for relatively small systems. Students will be able to compute determinants for relatively small matrices, understand how it relates to invertibility, and use this to compute eigenvectors.

Notes:

1. The geometry of linear algebra is useful, but we won't emphasize it too much. Bring it up as a motivational tool, but don't spend too much time on it.

2. Students like to resort to their high-school methods for solving linear systems, versus learning new tools like Gaussian Elimination. Dissuade them of this early, as it won't help them when it comes time to compute rank, inverses, eigenvectors, etc.

3.1 Linear Systems and Equations

1. Students should be introduced to the following terminology:

   - Linear equation
   - Coefficient
   - Variable
   - Constant term
   - Linear system (of equations)
   - Solution (to a linear system)

2. Students should be able to identify a homogeneous system, and recognize that all such systems admit the trivial solution. Such systems will be revisited in greater detail later. **??**

3. Discuss the geometry of solution sets. Students should be able to identify the three possibilities: No solution, A unique solution, infinitely many solutions, and how these manifest in two and three dimensions.

4. (NTS: Move Parameterization into this subsection). Motivate why we use parameterizations to encode infinitely many solutions. Students should recognize the difference between a parameterized solution and an implicit solution. Emphasize this using $\{(x, y) : y = mx\}$ vs $\{(t, mt), t \in \mathbb{R}\}$, then generalize.

   Tip: A good remark here is to ask students "How do you describe a line in three dimensions?" (Intersection of two planes, vs parametrization) and let them come up with the solution.

3.2 Matrix Representations

Students should be able to convert between a linear system and its matrix representation. Students should be able to apply the Elementary Row Operations to perform Gaussian Elimination, putting a relatively small matrix into (Reduced) Row Echelon Form. From RREF, students should be able to determine the rank of a matrix, and from the rank determine the size of the solution set.

3.2.1 Matrix Representations

1. Motivate that matrices are (at first) a book-keeping tool. Students should be able to translate between a linear system, and its matrix representation. Introduce here both the augmented matrix, and the coefficient matrix (NTS: Introduce coefficient matrix to notes)

2. Discuss the Elementary Row Operations (EROs). Emphasize that row operations do not affect the solution. Students should be able to perform the elementary row operations to a relatively small matrix, and clearly indicate what operations they have performed.

   Tip: As per the notes, have a simple linear system nearby whose solution is known. Demonstrate the row operations to the linear system, to convince students that the solutions are not affected.

   Warning: Students often want to "Add a number to a row." It may not warrant explicit mention, but it appears often enough to be wary.

3. Define Reduced Echelon Form (REF) and Reduced Row Echelon Form (RREF). Given example matrices, students should be able to identify which matrices are in REF vs RREF vs neither. Motivate that (R)REF represents variable elimination, and therefore simplification of the linear system without changing the solutions.

3.2.2 Gaussian Elimination

1. Give them the rough Gaussian Elimination Algorithm (GEA) (NTS: The algorithm in the notes needs a major overhaul). The GEA is difficult to understand when written, so be sure to do several examples to re-enforce how it's done.
2. Show how REF/RREF can be used to determine the solutions to a linear system. Be sure to do an example with infinitely many solutions, and emphasize that parameters should be chosen as those columns lacking a leading one (NTS: Add to notes).

Tip: The GEA is really best designed for computers, and strict adherence can lead to terrible fractions. Create an example wherein flexible use of the GEA leads to nicer computations. Emphasize that the GEA is not the only path to RREF, and as long as the EROs are correctly used, any path to RREF is fine (REFERENCE, NTS: Add to notes).

3.2.3 Rank

1. Define the rank. Note that REF vs RREF does not change the rank, so either could be used. Students should be able to find the rank of a relatively small matrix.
2. State how the rank can be compared to the number of variables in a system to determine the number of solutions. (NTS: This also needs to be changed in the notes, to make it clear that the rank is of the coefficient matrix, not the augmented matrix).

3.3 Matrix Operations

Here we will step back and treat matrices independently of linear systems (though keeping linear systems in the background for examples). Students will learn to add, scalar multiply, multiply, and transpose vectors. Students should be able to recognize the requirements on the dimensions of the matrices to perform each operation. Matrix multiplication in particular should be used to emphasize the relationship between linear systems and matrix equations.

Column vectors are also discussed here, with a return to homogeneous systems. Students should be able to identify when one vector is a linear combination of others. Students should be able to make the connection between a linear system, matrix equation, and linear combinations.

3.3.1 Addition and Scalar Multiplication

1. Students should be introduced to the notation $A_{i,j}$, indicating the element of A in the ith row and jth column of A. Given a matrix, students should be able to identify these elements.

   Note: Students have trouble with double indices. You should take extra time to explain this.

2. Students should be able to add two matrices and multiply a scalar into a matrix. Emphasis should be placed on the dimensions of the matrices matching when adding.

3. Demonstrate the properties of addition and scalar multiplication, but do not prove them. Emphasize that generally the rules of real addition/multiplication do not have to apply to matrices, and that unless they've been told otherwise, they cannot assume they hold.

3.3.2 Column Vectors

1. Define a column vector. Students should identify that a column vector is just a special type of matrix. Define $\mathbb{R}^n$ to be the set of column vectors.

   Note: Here there is some obvious ambiguity, as $\mathbb{R}^n$ is typically defined as the collection of n-tuples, and how do we distinguish between the n-tuples and row/column vectors? This topic is too subtle for the majority of students, and is best avoided unless the subject is specifically asked during class.

2. Briefly give a geometric representation of vector addition and scalar multiplication.

3. Define linear combinations and briefly discuss the geometry. Importantly, show that determining whether a vector is a linear combination of other vectors is equivalent to solving a linear system. Students should be able to determine whether a vector is a linear combination of others.

4. Re-address homogeneous systems, rephrasing the "Number of Solutions" result in the context of homogeneous systems. Do a multi-parameter example showing that all solutions are linear combinations of a set of basic solutions.

(NTS: Distinguish the revisiting of homogeneous systems in Column Vectors)

3.3.3 Transpose

1. Define the transpose. Students should be able to identify that the rows/columns of the transpose matrix are the columns/rows of the original matrix. Emphasize that the dimension of the matrix switches.

2. Demonstrate the properties of the transpose.

   Tip: Good examples for working with the transpose are to work with (anti-)symmetric matrices.

3.3.4 Matrix Multiplication

1. Introduce the dot product. Students should be able to compute the dot product between two arbitrary vectors.

2. Define matrix multiplication in terms of the dot product. Do several examples. Make sure you give students an opportunity to try this one on their own. Students should be able to perform

matrix multiplication on matrices of size up to 3×3 to begin with, with the expectation that proficiency will improve to larger matrices with time.

3. Emphasize the importance of dimension matching. Given abstract matrices, students should be able to identify when a matrix product is not defined.

4. Define the identity matrix. Demonstrate that it plays the role of unity in matrix multiplication.

5. Give the properties of matrix multiplication. Emphasize here especially non-commutativity, and the existence of zero-divisors. Students should be able to answer relatively simple True/False statements about these concepts.

   Tip: The failure of commutativity and existence of zero-divisors are good exercises to leave as open questions, and have the class resolve on their own.

6. Introduce matrix-vector multiplication as a special case of matrix-matrix multiplication. Show that $A\mathbf{x}$ is the same thing as constructing a linear combination of the columns of A.

7. Emphasize how the matrix equation $A\mathbf{x} = \mathbf{b}$ is precisely the same thing as a linear system. Also emphasize how $A\mathbf{x} = \mathbf{b}$ says that $\mathbf{b}$ is a linear combination of the columns of A. Students should be able to convert a linear system of equations into a matrix equation.

8. State and prove **??** (NTS: Make if and only if). Students should be able to identify the particular and homogeneous solutions of a linear system.

   Note: It may be worthwhile to reprove that the sum of homogeneous systems is homogeneous, now that we have a convenient notation.

3.3.5 Matrix Inversion

1. Motivate why we might want an inverse from $A\mathbf{x} = \mathbf{b}$, then define the inverse. Students should be able to demonstrate that a matrix is an inverse (through multiplication). Emphasize that not all matrices have inverses, give an example.

2. Give the inversion formula for a 2×2 matrix. Students should be able to apply this formula to find the inverse of a 2×2 matrix, and to determine when a 2×2 matrix is non-invertible.

3. Demonstrate how to find the inverse of a general $n \times n$ matrix. Students should be able to find the inverse of relatively small matrices (no larger than 5×5), and use these inverses to solve linear systems.

   Note: It is not unusual to see the following happen on a test:

   Part (a) Find the inverse of this $n \times n$ matrix A
   Part (b) Find a solution to the linear system $A\mathbf{x} = \mathbf{b}$.
   Wherein students will do Part (a) correctly, then row reduce the augmented matrix $[A \,|\, \mathbf{b}]$ to answer part (b). They struggle to understand why we'd want A^{-1} other than as a computation they're expected to know.

4. Give the properties of the inverse. Students should be able to use these properties to do some basic algebra (see Example 3.33)

3.4 Determinants

Students will be able to compute the determinant for relatively small systems (4×4 being an extreme case, 3×3 being more common). Students will know how the determinant relates to invertibility, know that it is multiplicative, and use these properties to answer straightforward True/False questions.

3.4.1 Definition

1. Motivate the determinant, in our case mostly for the purpose of determining whether a matrix is invertible. Students should be able to compute the 1×1 and 2×2 determinant immediately.
2. Define submatrices and cofactors. Students should be able to (i, j)-submatrix and -cofactor for relatively small matrices (no larger than 4×4). Emphasize how cofactors are recursive, and require a large number of operations to compute.
3. Define the determinant. Students should be able to find the determinant of a 3×3 matrix without much trouble, with a 4×4 matrix coming with more maturity.

   Tip: Compute the determinant of an upper/lower triangular matrix and see students can infer the pattern.

3.4.2 Properties of the Determinant

1. Demonstrate that determinants are multiplicative, and use this to find the determinant of the inverse matrix. Use this to motivate that a matrix is invertible precisely when its determinant is non-zero (NTS: Be more clear about this in the notes).
2. Demonstrate that the determinant is preserved by the transpose.
3. Students should be able to answer straightforward True/False questions about determinants. For example, "If $A^2 = I$ then $\det(A) = 1$ " is false.
4. Student should understand the relationship between invertibility and the determinant.

3.5 Eigenvalues and Eigenvectors

Students should be able to find the eigenvalues/vectors for relatively small systems, and answer straightforward True/False questions about eigenvalues.

1. Define eigenvalues/vectors. Given an eigenvalue/eigenvector pair, students should be able to confirm they are in fact an eigenvalue/vector pair for a given matrix.

   Note: It's difficult to motivate this in the absence of linear transformations, but you can allude to Section 4.4.
2. Motivate how eigenvectors are computed using the determinant. Students should be able to compute the eigenvalues/vectors of a 2×2 and 3×3 matrix handily, with knowledge of how to do the 4×4 and higher cases were it necessary.

3. Students should be able to answer simple True/False questions, such as "If $\mathbf{v}, \mathbf{w}$ are eigenvectors of A corresponding to the eigenvalue λ, then $\mathbf{v} + \mathbf{w}$ is an eigenvector of λ."

4 Probability and Counting

Students should leave this section with a basic but fundamental understanding of simple counting arguments and simple probability computations. For example, given simple word problems, students should be able to identify whether order matters, and then identify whether the problem is one to which the Basic Counting problem, a permutation, or a combination.

Students should be able to use their counting skills to create the event spaces for simple probabilities. They should then be able to compute the probability of an event happening, given a probability distribution. Students will learn about conditional probabilities, Bayes' theorem, and applications as well.

4.1 Counting (~ 2 hours)

Here we'll introduce the students to combinatorial arguments. In particular, students will become acquainted with the Basic Counting Principle, (repeated) permutations, and combinations. Students should understand from where the corresponding formulas are derived, and be able to use them to solve straightforward problems.

Note: This is a hard section, since counting arguments are often ad hoc and not formulaic. Our examples will not be too sophisticated, but will require students to exercise their brains rather than resort to rote memorization.

1. Students should understand the statement of the Basic Counting Principle (BCP), and be able to apply it to straightforward word problems, such as Example 4.1. While trickier, Example 4.3 is a good example of how changing the perspective of a problem leads to a BCP solution.

2. Students should be able to recognize a permutation problem, and what separates it from a BCP problem. In particular, emphasize the importance of order in a permutation problem. Students should understand from where the ${}_nP_k$ formula comes, and use it to be able to solve problems. Example 4.4 is a good example of a straightforward question. Spend some time discussing why ${}_nP_n = n!$, and why we expect this.

3. Students should understand why repeated permutations are different from permutations. In particular, emphasize that if the items are indistinguishable, we over count with a traditional permutation. Emphasize how we fix the over counting problem, as this will factor heavily in the future. Students should be able to apply repeated permutations, either in the classic sense: "Number of words formed by the letters AREA" versus the 'ways of placing in a box' paradigm, such as "Placing 10 items into a box of size 2, a box of size 3, and a box of size 5."

4. Students should be able to identify what distinguishes a combination from a permutation: Emphasize how order does not matter. Show how a permutation over counts a combination, and therefore the derivation of the binomial coefficients ${}_nC_k$.

4.2 First Principles of Probability (~ 2 hours)

Students will apply their counting skills to create sample spaces and events. Given a probability distribution, they will compute the probability of events. Here we'll use the set-theoretic arguments we introduced in Chapter 1, and discuss their relationship to probability computations. We'll also touch on the Law of Total Probability, before discussing conditional probability.

1. Students should be able to define the following words

   - Sample Space
   - Probability distribution
   - Equiprobable
   - Event
   - Mutually Exclusive

   and comment on how they are related.

2. Students should be able to construct a sample space for simple situations, using their counting skills. Similarly, students should be able to construct sets representing events. Both of these are well illustrated in Example 4.10.

3. Given a sample space, event, and distribution, students should be able to compute the probability of an event. Examples should not be much more sophisticated than Example 4.14.

4. Discuss relation between probability and set-theoretic operations. Students should conceptually understand the notion of disjunction, conjunction, and complements, how these to probabilities, and how these are computed. Emphasize in particular the use of the union and complement: Example 4.16 is a good example of the corresponding difficulty level we're looking for.

5. Discuss the Law of Total Probability (LoTP). LoTP problems will often be given in the form of a table, so students should be able to read this information from a table, and infer the appropriate probabilities.

4.3 Conditional Probability (~ 2 hours)

Here we'll discuss conditional probability, with a view to Bayes' theorem and eventually Markov chains. Conceptually, this is a difficult topic for students. We'll motivate the concept and formula for computing conditional probabilities, and rephrase the LoTP in this language. We'll touch on more advanced probability examples than those of the previous section, discuss independence in greater detail, and introduce Bayes' theorem.

1. Give the formula for the conditional probability $P(E|F)$. Motivate it, but don't worry too much about it. Use Example 4.18 to highlight the importance of language in determining the probability. Students should be able to replicate the computation of a similar example.

2. Give the special cases of conditional probability between independent and mutually exclusive events. Discuss why this should be intuitively true.

3. Restate the Law of Total Probability using conditional probabilities. Example 4.20 represents the most difficult problem students can expect to face. Students should be able to construct a similar probability tree to help solve these problems.

4. Motivate and state Bayes' theorem. Students should be able to apply Bayes' theorem to solve very straightforward problems to compute $P(F|E)$ where $P(E)$ and $P(F)$ are both given.

4.4 Applications ($\sim$ 3 hours)

Here we'll discuss some applications of probability: expected values, Markov chains, options pricing, and binomial evolution. Expected value and binomial evolution are testable subjects, with very straightforward Markov chain problems also being accessible. Options pricing is interesting, but generally will not be examined.

4.4.1 Expected Value

1. Define a random variable. Make clear how random variables are different than probability distributions, as this is often a point of confusion.

2. Define the expected value of a random variable. Given a simple experiment, students should be able to compute the expected value of random variable.

4.4.2 Markov Chains

1. Motivate Markov chains, their restrictions, and why we are interested in them. Given the description of conditional data, students should be able to construct the matrix describing a Markov chain.

2. Students should recognize the role of eigenvalues/vectors in Markov chains, and should be able to compute these in the 3×3 case.

4.4.3 Binomial Evolution

1. Motivate binomial evolution as a simplification of more complex processes, then build up to the model of the binomial distribution. Students should be able to compute the binomial distribution for any value of n and k.

2. Show how the binomial distribution can be used to model binomial evolution. Students should be able to compute expected values on binomial evolution trees.

4.4.4 Options Pricing

1. Relay the importance of options pricing in finance, and discuss the notion of a replicating portfolio in determining the price of the security.

2. Give the derivation. Students have been asked to do this in the past as an assignment question, but it's not the sort of thing we'd ask them to replicate in a test. This is just a "general interest" section.

5 Limits

Here we'll introduce student to a *non-rigorous* formulation of limits. Students often miss the point of limits, which are used to characterize the behaviour of a function near a point, so it's worthwhile to spend some time motivating this.

By the end of this section, we expect students to be able to compute intermediate difficulty limits, and to have mastered several techniques. These include factoring, limit laws, and rationalization. Students should be able to recognize which techniques to use, and apply them to find an appropriate answer.

Students will need to be familiar with the notion of one-sided limits, limits as $x \to \pm\infty$ (including horizontal asymptotes), and the notion of vertical asymptotes. They will need to be able to compute these limits for relatively simple functions, and answer some simple true/false questions relating these limits.

Finally, students will be introduced to the notion of continuity. They should be able to recognize a continuous function, or the points of discontinuity of a function. This includes the classification of points of discontinuity. Notably, they should understand that continuous functions "allow the limit to pass inside the function."

5.1 Motivation

1. Some time should be spent motivating limits. Students often miss the point that a limit can be used to determine the character of a function near a point. Emphasize why this is useful, making note that the function might not be defined at the limiting point.

2. Students receive their first exposure to some computational techniques, such as factoring for rational functions, and rationalization for functions with radicals. They should be able to use these techniques to solve problems similar to Example 5.1 and Example 5.3.

3. Emphasize that the limit must exist in both directions. This can be used as a segue into the next section.

4. Emphasize that equality of limits is not equality of functions; namely, why we can write

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} [x+1]$$

even though $(x^2-1)/(x-1) \neq x+1$.

Note: While the notes use a table of values to justify the existence of a limit, this is blasphemy. If you also choose to use the table, clearly emphasize that this is not how they should compute limits.

(NTS: Restructure OSL and Motivation into a single section, differentiate the first computations from the intuition.)

5.2 One Sided Limits

1. Introduce functions with different one-sided behaviour. This will be used later on in the classification of dis-continuity and is important in discussing vertical asymptotes.

2. Students should be able to recognize when they are being asked to calculate a one-sided limit, and how it differs from a two-sided limit. They should be comfortable restricting the domain of a function to the left/right of a point in order to apply their calculation techniques.

5.3 Limit Laws

1. Introduce the limit laws. Emphasize that the limits can only be applied if we know *a priori*. Students then should be able to recognize when they can apply the limit laws, and use those laws to evaluate intermediate difficulty limits.

2. Students should recognize how one can inductively use the Limit Laws to find the limits of polynomials (do not use the word induct though). This should translate to rational functions as well. We will discuss continuity of such functions later, but this is a good precursor.

5.4 Infinite Limits

Here we'll discuss limits as $x \to \pm\infty$ and vertical asymptotes $f(x) \to \pm\infty$. Students are expected to be able to compute these limits, and to recognize when they exist. There are important conceptual obstacles in this section.

Notes: When $f(x) \to \pm\infty$, the limit does not exist, but we're prescribing precisely how it fails to exist. Students see the equal sign – an abuse of notation – and assume this means the limit exists.

5.4.1 Vertical Asymptotes

1. Students should be able to identify the vertical asymptotes of a function, and determine the asymptotic behaviour. Functions will typically be rational (Example 5.11) but they should understand how to generalize this to other functions, such as $1/(e^x - 1)$.

2. Students should be able to make cogent arguments about the sign of a function near a singular point to determine whether $f(x) \to \infty$ or $f(x) \to -\infty$.

5.4.2 Horizontal Asymptotes

1. Define horizontal asymptotes (NTS: Add divergent behaviour, and non-divergent failure of the limit). Students should be able to recognize when they are being asked to compute a horizontal asymptote.

2. Students should understand the notion of "renormalizing" by the fastest growing function. They should be able to apply this to find the asymptotes of any rational function (Example 5.13).

3. Spend time on radical functions, and how to find the asymptotes of these functions. Students should be able to find the horizontal asymptotes of any function similar to Example 5.16.

Notes: Students struggle with several important concepts:

- That $e^x \xrightarrow{x\to-\infty} 0$.
- That $x = -\sqrt{x^2}$ when $x < 0$.

You should pay particular attention to these points when they arise.

5.5 Continuity

5.5.1 Definition

1. Define continuity and emphasize why it is special. Give examples of continuous and non-continuous functions. List of the students the functions they are allowed to assume are continuous.

2. Students should recognize how the Limit Laws translate to continuous functions, and therefore be able to identify a continuous function. Alternatively, they should be able to recognize when a function fails to be continuous (Example 5.20). Students should be able to answer simple true/false questions based off this.

3. Students should recognize that continuous functions allow us to "take the limit inside" (Theorem 5.22). This is a critical concept, which will be used often. Students should be able to use this to solve questions like Example 5.23 (NTS: Add more sophisticated example).

4. Students should recognize that the composition of continuous functions is continuous.

5.5.2 Failures of Continuity

1. Define one-sided continuity. Explain why this requires a special definition (continuity is an open condition). We won't worry about this much, so don't spend much time on it.

2. Show students how to classify a discontinuity. Given a intermediate difficulty function, students show now be able to find and classify the points of discontinuity.

6 Differentiation

Students will be introduced to the notion of the derivative and its various interpretations, both geometric and algebraic. They will learn the properties of the derivative, such as linearity and the relationship to continuity. Students should know the derivatives of the basic functions (polynomials, logarithms, exponentials, radicals), and be able to combine them with the product, quotient, and chain rules to facilitate computation of almost any function.

6.1 First Principles

Here we will motivate the limit definition of the derivative as the instantaneous rate of change, and include several examples of how to compute derivatives at a point. Students should in turn be able to use either limit definition to compute the derivatives of straightforward functions. We will introduce the notion of a derivative function, and students should be able to use the limit definition to compute a closed form expression for the derivative function in simple cases. We will associate the derivative with the slope of the tangent line, and students should be able to find the equation of the tangent line at any point. Finally, students will be introduced to Leibniz notation and the derivative operator.

6.1.1 Definition

1. Motivate the definition of the derivative as the instantaneous rate of change, using your favourite example. Students should be able to generalize the motivation to any other dynamic example.

2. Define differentiability and the derivative at a using Equation (6.1):

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

   and do several examples. Students in turn should be able to compute the derivative of straightforward functions (quadratic and cubic polynomials, square roots, rational functions) using the limit definition, with an expectation that they eventually mature to more difficult questions (higher order polynomials and other radicals).

   Note: Here you are free to start with Equation (6.3) if you so choose. It may be advantageous to start with this definition, as it makes the secant/tangent line conversation somewhat easier.

   Note: Students often conflate the notion of differentiable and derivative, failing to understand that differentiability is a property of the function, while the derivative is a number. You will want to make this point clear.

3. Introduce the derivative as a function $f' : x \mapsto f'(x)$. By using Equation (6.1), students should be able to find a closed form expression for f' for the same straightforward functions mentioned above.

6.1.2 Geometry of the Derivative

1. Define the secant line and compute its slope. Argue that the tangent line is the limit of the secant lines, and hence the derivative is the slope of the tangent.

2. Compute the equation of a tangent line, emphasizing that the only point you know that the tangent line passes through is the point on the curve. Introduce the point slope formula. Students should be able to compute the equation of the tangent line for straightforward (see above) functions now, and any differentiable function as they develop more sophisticated tools for computing derivatives.

6.1.3 A Different Parameterization

1. Introduce Equation 6.3. This can be motivated in two ways: The first is to use Figure 6.1 and set $b = x_0 + h$, while the other is to do a "limit substitution."

   Note: The limit substitution technique is nice, but often poorly motivated. It is commonly employed, but the theorem used to justify limit substitution is rarely taught even in advanced courses. It may be more confusing than arguing geometrically.

2. Do several examples using this new definition of the limit. Try to repeat a previous example to demonstrate that the answer has not changed. Students should be able to apply this definition to straightforward functions (as above), to find the derivative.

   Note: If you started with Equation 6.3 you'll naturally have to introduce Equation 6.1. Students will never be required to use a particular definition.

6.1.4 Leibniz Notation

1. Introduce Leibniz notation $\frac{\mathrm{d}y}{\mathrm{d}x}$. It's worth emphasizing the subtle difference a simple function f, and writing $y = f(x)$. Caution the students about dropping the argument, and that this is **not** a fraction.

2. Introduce the student to the derivative operator $\frac{\mathrm{d}}{\mathrm{d}x}$. Students should recognize any of the following and know what it means:

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}y, \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^2+1).$$

6.2 Some Derivative Results

Despite its name, this section consists of a lot of derivative results, including linearity, the power rule, the product and quotient rules, differentiating exponentials, and higher order derivatives. While we will cover the chain rule next, by the end of this section, students should be able to compute the derivatives of almost any function which does not explicitly require a composition. This includes recognizing how threefold and higher products are differentiated. They should be able to compute second, third, and higher order derivatives, know how to denote these derivatives, and to guess patterns for nth order derivatives when appropriate.

6.2.1 Linearity and the Power Rule

1. State Proposition 6.9, stating that the derivative is linear. Students should be able to apply this differentiate sums/scalar multiples of functions they've already seen.

Note: The proof of linearity is not too hard, but we generally won't ask students to do anything like this. Nonetheless, it can be a good argument in using the Limit Laws. Do it at your discretion, but if you do, do so very slowly.

2. State the Power Rule (Proposition 6.10), and demonstrate how it can be used to differentiate monomials. Combine this with linearity to differentiate arbitrary polynomials. Students in turn should be able to differentiate any polynomial.

6.2.2 The Natural Exponent

1. Recall that e is defined as
$$1 = \lim_{h\to 0} \frac{e^h - 1}{h}.$$
It is worth pointing out that this is the value of the derivative at 0.

2. Show that $f(x) = e^x$ is everywhere differentiable, and $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$. This proof is worth doing (slowly), as it shows how the derivative of $f(x) = e^x$ is related to $f'(0)$.

   Note: You may wish to show that if $f(x) = a^x$ for some $a > 0$, then $f'(x) = a^x f'(0)$. We are not in a position to find $f'(0)$, but this is a nice result and allows you to demonstrate that e is the special value for which $f'(0) = 1$.

6.2.3 The Product and Quotient Rule

1. State (but do not prove) the product rule. Give several concrete examples of using the product rule. (NTS: Need to add some of these)

2. Demonstrate how an n-fold product of functions can be differentiated. *Note:* This is a good example to let them work out on their own, if you have time.

3. State (but do not prove) the quotient rule. Give several concrete examples of using the quotient rule.

4. Students should be able to differentiate the product/quotient of any two of the basic functions with which they are acquainted (polynomials, exponentials, radicals, simple rational functions like $1/x$). Students should also be able to perform relatively simple product rule computations in abstract settings, such as Example 6.15.

5. Emphasize that many quotients can be thought of as products, and that the answer is identical regardless of how the computation is done (See Example 6.18 in relation to Example 6.15).

6.2.4 Higher Order Derivatives

1. Introduce the notion of higher order derivatives, including the notation $f^{(k)}$ and $\frac{\mathrm{d}^k y}{\mathrm{d}x^k}$. Students should be able to compute $f^{(k)}$ for most functions and relatively small k ($k = 2, 3, 4, 5$).

2. Students should be able to guess patterns for finding general $f^{(k)}$ when appropriate (Examples 6.19 and 6.20).

6.3 Smoothness of Differentiable Functions

Here we'll discuss the smoothness of differentiable functions, including how all differentiable functions are continuous, and how a function can fail to be differentiable. We expect students to recognize that differentiable functions are continuous, to answer straightforward true/false questions about this relationship, and to be able to write down a function which is continuous but not differentiable. Students should be able to examine a function (either algebraically or through a graph) to determine where it fails to be differentiable, and explain why.

6.3.1 Differentiable Implies Continuous

1. State Proposition 6.21. We will not ask them to do anything like the proof; however, you may wish to comment on the intuition. Students should understand that the converse of this is not true, and be able to produce a counter-example. In more extreme cases (such as their assignment), they will be asked to construct particular counter-examples, such as functions which are everywhere continuous, but fail to be differentiable at precisely two points.

6.3.2 Failures of Differentiability

1. Determine all the ways in which a function could be fail to be differentiable (the limit fails to exist) and translate these into their graphical representations. Do several examples. Students should be able to use the algebraic description of the function, or the graph of a function, to determine where the function fails to be differentiable.

2. Students should be able to construction functions which fail to be differentiable in particular ways, and at particular points. This will be done in the exercises, but you may wish to spend some time constructing examples.

6.4 Chains and Inverses

Here we will complete the toolbox for derivative calculations by introducing the chain rule, differentiating the logarithm, differentiating general exponential functions, and generalizing the power rule. We'll also introduce logarithmic differentiation. Students will be expected to be able to know the derivatives of all the elementary functions, and from this use the tools provided to differentiate practically any function given. Students will be expected to know logarithmic differentiation, and how to use to this differentiate large products/quotients of functions. More importantly, students should be able to apply this technique to differentiate functions for the form $f(x)^{g(x)}$.

6.4.1 The Chain Rule

1. State Theorem 6.25. You should not prove it, though you can provide some motivation/intuition for the result if you so choose. Give the Leibniz notation statement. Students should be able to apply the Chain Rule to differentiate any composition of the basic functions whose derivatives we've seen so far (Examples 6.26 and 6.27).

2. Discuss how the Chain Rule can be extended to a composition of three or more functions. Students are expected to be able to apply the chain rule to arbitrary compositions, though normally questions will persist of no more than 4 compositions (Examples 6.26 and 6.29). There is an expectation that students should be able to work with the Chain Rule in some level of abstraction (Example 6.30 and Exercise 6-25. Example 6.31 is on the harder side).

6.4.2 Derivatives of Inverse Functions

1. State Theorem 6.32. Students should understand from where Equation 6.7 comes, and how to use it. Unfortunately, we will not have many examples of how this theorem is used.

2. State and prove Proposition 6.33. In particular, we'll use this as an example of how to use Equation 6.7.

3. State and prove Theorem 6.34. Emphasize the previously, we only knew the result for $n \in \mathbb{Z}$, and now we know the result for all real numbers. Emphasize the technique of writing $x^n = e^{n\ln(x)}$ as this will be used a fair amount in the upcoming sessions.

4. State and prove Theorem 6.35. Again emphasize that we can write $a^x = e^{x\ln(a)}$. Also make clear connections to the fact that $\ln(e) = 1$, making this result consistent with what we know about the exponential function $f(x) = e^x$.

5. Students should now be able to differentiate almost any function consisting entirely of elementary functions. Do several examples (NTS: Add more examples).

6.4.3 Logarithmic Differentiation

Note: Currently, this section uses notation similar to that of implicit differentiation (Section 7.1). This is an artifact of having moved sections around. It can still be done with the tools at our disposal, but the notation must be changed.

1. Discuss how logarithms move down the arithmetic complexity ladder (Products to Sums, Exponents to products), and motivate how we can use this to simplify differentiation. The discussion in the book is somewhat general, so it may be worthwhile to limit your discussion to a product of three functions.

2. Do an example of a complicated product/quotient of functions we know how to differentiate. Students in turn should be able to replicate this procedure for similar problems.

   Note: At this point, I like to have the students create the function for me. I encourage them to make the problem as difficult as possible, then differentiate logarithmically. It usually works well to serve the point.

3. Use the fact that $f(x)^{g(x)} = e^{g(x)\ln(f(x))}$ to show how to differentiate functions of this form. Do an example or two. Students should be able to compute similar derivatives, such as $x^{\ln(x)}$ or x^{xe^x}.

7 Applications of Derivatives

Here we'll use the tools developed in the previous section to discuss possible applications. These include implicit differentiation, rates of change, curve sketching, (un)constrained optimization, and polynomial approximation (Taylor polynomials). Students will be responsible for determining the correct approach to each question, in addition to solving the question itself.

7.1 Implicit Differentiation

Students will be introduced to implicit differentiation. This is a topic with which many students struggle, so it's important to emphasize what's happening from an intuitive viewpoint.

1. Discuss explicit versus implicit functions, and do several examples of how one variable can be determined given knowledge of another (NTS: Give examples of this in the notes). For example, the equation $e^x + xy + y = 1$ is transcendental in x, but given $y = 0$ one can still find a corresponding value of x. This makes x an implicit function of y. Given a value for one of x or y, students should be able to solve such an equation for the other variable in reasonable cases.

2. Demonstrate how implicit differentiation is performed. Be sure to emphasize how the chain rule is employed. For example, in differentiating y with respect to x, why $\frac{\mathrm{d}}{\mathrm{d}x}y^2 = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$. Do examples where numbers must be substituted in (Exercises 7-4 and 7-5) (NTS: Add examples like this to the notes). Students should be able to perform similar computations.

3. We expect students to be able to differentiate implicitly with respect to any variable, to be able to find second or even third derivatives of implicit functions, and to understand the relationship between $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}x}{\mathrm{d}y}$.

   Note: Be sure to do examples of both $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}x}{\mathrm{d}y}$. Students will often jump to the former without thinking if not presented with examples of the latter. It is prudent to do an example where you compute both, to emphasize their relationship through the Inverse Function Theorem.

7.2 Rates of Change

This section will not be heavily examined, but instead is used to demonstrate how derivatives appear in concrete application. However, our learning objectives do not (currently) include the ability for students to apply these ideas in novel ways, and so examining this section would consist largely of memorization. If you have time, you may choose your favourite application and present it. Ensure that you announce that this material is not examinable.

7.3 Derivatives and the Shape of a Graph

Here we'll discuss how the first and second derivative influence the shape of a graph, and what that means in terms of the function. In the absence of the Mean Value Theorem, everything must be motivated heuristically. Students should be able to determine where a suitably differentiable function has critical points, is increasing/decreasing, and is concave up/concave down.

7.3.1 First Derivative Information

1. State Theorem 7.12. Emphasize the difference between being identically zero on an interval, and zero at a point. Students are expected to know this theorem, and answer true/false questions based off it.

2. Define (strictly) increasing/decreasing and non-decreasing/increasing functions, together with Theorem 7.14. Emphasize that this definition holds for functions which are not necessarily differentiable. Students should be able to identify where a function is increasing and where it is decreasing.

3. Define critical points. Students should be able to find the critical points of a suitably nice function (low order polynomials, compositions of polynomial, exponentials, logarithms, and roots).

   Note: You may wish to define critical points prior to increasing/decreasing, as a tool by which to determine the intervals of increasing/decreasing.

4. We expect students to be able to translate this information to and from a graph. In particular, given the graph of a function f, they should be able to identify the critical points and intervals of increasing/decreasing.

7.3.2 Second Derivative Information

1. Define concavity for general functions, noting that the definition holds for not necessarily differentiable functions. Given the graph of a function f, Students should be able to identify the intervals where the function is concave up/down.

2. Define an inflection point. Relate to this the derivative condition $f''(x) = 0$, and state Proposition 7.21. For any suitably nice function (and especially simple rational functions), students should be able to compute prospective inflections points and use this to identify the intervals of concavity (Example 7.22).

7.4 Maxima and Minima

Here we will introduce both unconstrained and constrained optimization. For unconstrained optimization, we will discuss necessary conditions for being an extreme value (critical points) and the First and Second Derivative Tests. Students should be able to use these tools to find local and global minima for any reasonable function. For constrained optimization, we not only expect students to recognize how to solve the problem, but to parse word problems into their corresponding mathematical framework.

7.4.1 Introduction

1. Define maxima and minima, both local and global. Students should recognize that this definition holds independent of any differentiability condition on a function. Given a graph, they should be able to identify any local and global extreme values.

2. State the Extreme Value Theorem. Students should recognize that the Extreme Value Theorem guarantees the existence of extreme values. They should be able to answer simple True/False statements about the Extreme Value Theorem.

3. State Lagrange's Theorem (critical points are a necessary condition for extreme values at interior points). Students should readily understand that this is not a sufficient condition, and be able to provide a counter-example (a critical point which is not an extreme value)

4. State the First Derivative Test, ensuring that you give strong motivation as to why it's intuitively true. Do several examples. Students should be able to use the First Derivative Test to find and classify extreme values for any reasonable function.

5. Discuss boundary critical points. Here it is essential to show students why boundary critical points are not detected by the differentiability condition. This will be alluded to later when we cover Lagrange multipliers, so invest the time to make this clear now. Do an example where at least one of the global extreme values is a boundary point. Students should be able to solve most extreme value problems now, including those which occur on closed intervals.

6. State the Second Derivative Test, again emphasizing its intuition against concavity. Be sure to give a (simple) example in which the second derivative fails to conclusively identify the extreme value, requiring higher order derivatives. Discuss the pros and cons of employing the second derivative test, and do several examples. Students should be able to use *either* the First or Second Derivative Tests to classify extreme values.

7.4.2 (Constrained) Optimization

1. Discuss the differences between constrained and unconstrained optimization, and demonstrate the strategy for solving such optimization problems. Do several examples to demonstrate this strategy, including word problems like Examples 7.36 and 7.37.

2. Students should be able to take an intermediate difficulty word problem, assess its mathematical components, and solve the problem.

7.5 Curve Sketching

Here students will use the tools of differentiation to determine the qualitative properties of a function. Using this information, they will then create a sketch of the plot. Given any reasonable function, students should be able to create a relatively strong semblance between their sketch and the actual graph of the function.

1. Define what it means for f to behave asymptotically like g, and use this to define oblique asymptotes. Students should be to use polynomial long division to find the oblique asymptotes

of an appropriate rational function. It is worth pointing out that there are non-rational functions with oblique asymptotes ($f(x) = xe^{1/x}$ has an oblique asymptote of $y = x + 1$ as $x \to \infty$, but we lack the tools to show this).

2. Enumerate the data we can use to determine the shape of the graph of a function, and how we compute that data. Do several examples. Students should be able to use this data to sketch the graphs of reasonable functions.

   Note: Much of the curve sketching data is redundant, but this can help error-correct against mistakes. Nonetheless, we expect students to be to determine when they've found sufficient information to sketch the graph – It is not necessary that they compute everything.

7.6 Polynomial Approximation

In lieu of a full discussion of Taylor series, we'll treat a small section on using derivatives to create polynomial approximations. Students should leave this section with an understating of why this is desirable, and the ability to compute low-order approximations for most functions. Given an approximation, students should be able to use this information to estimate functions at particular values.

Note: In this section, there are two main paradigms for how to approach the estimation problem. For example, to find $\sqrt{80}$, you can use the function $f(x) = \sqrt{x}$ approximated near $x = 81$, or $g(x) = \sqrt{81 - x}$ approximated near $x = 0$. Students don't like choice, and have trouble understanding why these give the same result. You may wish to spend some time discussing this.

7.6.1 Linear Approximation

1. Demonstrate to students that the tangent line represents the best linear approximation to a function, and use this to estimate the value of some non-polynomial functions near known values (Example 7.42). Students should be able to compute the linear approximation for any reasonable function, and similarly employ it to find such values. Students should understand that the further away from the approximating point one goes, the worse the approximation becomes.

7.6.2 Quadratic and Higher Approximations

1. Give the formula for a general nth order polynomial approximation at the point a. Use this to find higher order polynomial approximations, and to give estimate values for non-polynomial functions near a known point. We expect students to be able to compute up to a fourth-order approximation for any reasonable function. Students should be able to compute values of a function near a point, though this will usually be limited to no higher than quadratic approximations.

8 Integration

Here we will introduce the definite integral through Riemann sums, and connect it to anti-differentiation through the Fundamental Theorem of Calculus. Students should be able to compute relatively simple Riemann sums, know the fundamental anti-derivatives by heart, and understand how the Fundamental Theorem is used to compute definite integrals. We'll introduce them to indefinite integrals, and students should be able to recognize the difference between an indefinite and definite integral. Substitution and Integration by Parts are the computational tools we'll be introducing. Students should be able to take a relatively sophisticated integral and identify the correct integration technique, possibly applying one or both technique multiple times.

8.1 The Definite Integral

Here we'll motivate the definition of the definite integral, and introduce Riemann sums. All Riemann sums will occur on uniform subintervals, and like limits we will avoid the rigorous definition of the integral. However, we expect students to understand that we are approximating the area under the graph, and to be able to compute simple Riemann sums (uniform subintervals, no more than 5 subintervals, polynomial functions).

8.1.1 Intuition

1. Motivate the definite integral, using both the idea of determining the area under the graph, and reversing the process of differentiation (given velocity $v(t)$, determine how far you've travelled). It helps you start with one motivation, but then draw attention to the second. This will make the FTC less surprising.

2. You can if you so choose work out an estimate. This process will be made more formal in the next section.

8.1.2 Estimating Area

1. Cover how to break the interval $[a, b]$ into precisely n subintervals. Students have a lot of trouble with this, so extra time should be taken on this subject. Students in turn should be able to take an intervals $[a, b]$ (either abstractly or with actual numbers), break it into n subintervals (either given an explicit n, or left abstract), and identify the ith subinterval.

2. Define the left, right, and midpoint Riemann sums. Note that there are many more than just these three, but these are the "standard" choices. We will not use the midpoint sum very often. Do one or two explicit examples of this computation, using numbers and an actual function (usually a polynomial). Students should similarly be able to compute the left and right Riemann sums when numbers are given and the function is no more complicated than a quadratic polynomial, with the expectation that they will eventually need to do it in general (Examples 8.1 and 8.2).

   Note: If you choose to use a different function than that in the notes, this is a good time to scaffold your examples. In the next section, you will compute the definite integral by evaluating a limit. Choose a function for which this limit can be computed, and save yourself some time in presenting the example in that section [Sec 8.1.3-1].

8.1.3 Defining the Definite Integral

1. Define the definite integral (Definition 8.3), and explain the intuition that $n \to \infty$ corresponds to taking better approximations. Do an explicit example, such as Example 8.4, though you may choose a different function. If you choose a different function, this is a good opportunity to scaffold an example from the previous section.

8.2 Anti-Derivatives

Seemingly non-sequitur, this section will discuss the process of anti-differentiation. Here we will introduce the fundamental anti-derivatives, discuss uniqueness of anti-derivatives, and talk about initial value problems. It is expected that students will know the fundamental anti-derivatives by heart. They should understand the difference between a general anti-derivative, and one specified through an initial value. We expect students to generalize this process somewhat, and be able to find a function f given its higher order derivatives (no worse than third order).

8.2.1 Anti-Differentiation

1. Define an anti-derivative, and build a table to fundamental anti-derivatives. Demonstrate that anti-derivatives are linear, and non-unique. In particular, that anti-derivatives are only unique up to an additive constant.

2. Compute several anti-derivatives, paying particular attention to how scalar multiples are handled. Students should be able to compute the anti-derivative any linear combination of the fundamental derivatives.

   Note: Have the students help you build the table of anti-derivatives.

3. Do an example with a second derivative. Emphasize how two constants appear. Students should also be able to emulate this process and similarly produce a general f given f'' or f'''.

8.2.2 Initial Value Problems

1. Show that given an initial condition, anti-derivatives become unique. We expect students to be able to apply an initial condition to determine the unique anti-derivative.

2. Generalize this discussion to the case where f'' and f''' is given. Emphasize that multiple initial values must now be given (equal to the order of the derivative given). Students should be able to apply an initial condition to find the unique f given f'' or f'''.

8.3 The Fundamental Theorem of Calculus

We'll introduce the Fundamental Theorem of Calculus as both a computational and theoretical tool. Students will learn to use the FTC to compute definite integrals without computing Riemann sums, thereby joining the concept of anti-differentiation with integration. We'll also introduce additional properties of the integral, such as additivity of domain and monotonicity. We'll conclude with a discussion of indefinite integrals representing the entire set of anti-derivatives, and a brief discussion regarding notation.

We expect students to compute integrals of functions which are linear combinations of our fundamental anti-derivatives. We expect students to recognize and use linearity of the integral, understand additivity of domain, and to be able to answer simple True/False questions regarding monotonicity and subnormality of the integral. Students should be able to differentiate between the definite and indefinite integral.

8.3.1 The FTC

1. State the Fundamental Theorem (both parts). Remark 8.14 has a few points which students find confusing, so you should go over these as well. Students should recognize how the FTC can be used to compute integrals, and be able to do as such with simple functions.

2. Students should be comfortable with some of the more abstract applications of the FTC, such as Examples 8.17 and 8.18. In particular, they should be able to differentiate integral functions, applying the Chain Rule as necessary.

   Note: Either version of the FTC can be used to solve Example 8.18. Some students find one way easier to conceptualize than the other.

8.3.2 Properties of the Definite Integral

1. Give the six properties of the definite integral. Students will be expected to use the linearity properties and Additivity of Domain extensively. In particular, you should use additivity of domain to justify that $\int_a^b f = -\int_b^a f$. Monotonicity and subnormality will be less used, but students should still be able to answer simple True/False questions about these properties, and understand them from an intuitive point of view.

8.3.3 Indefinite Integrals

1. Introduce the indefinite integral as representing the entire class of anti-derivatives. Demonstrate that we enumerate this list by appending "$+C$" to our answer, and why. Students should be able to compute the indefinite integral for functions of basic difficulty, with the expectation that the level of sophistication will expand once substitution and integration by parts have been discussed.

8.3.4 Integral Notation

1. Here we want to summarize the three uses of the integral sign: The number representing the area under the graph, the function used in the FTC, and the set of all anti-derivatives. Students confuse and conflate the three notations; for example, students will add a "$+C$" when asked for a definite integral. Students should be able to identify what they are being asked for, and respond appropriately.

8.4 Integration Techniques

We'll discuss substitution and integration by parts, both for definite and indefinite integrals. Students will be expected to know how and when to apply these two techniques, and use them to correctly compute intermediate level integrals.

8.4.1 Substitution

1. Introduce substitution as the reverse of the chain rule, and give the technique for identifying the correct substitution. I recommend starting with the indefinite integral, and doing several examples.

2. Introduce the definite integral, and emphasize the change in the bounds of integration. Do several examples. It may be worthwhile introducing two approaches: One the bounds of integration are explicitly evaluated, and one where the bounds of integration are substituted with placeholders, before reverting to the original variable and original bounds of integration (as one sometimes does with inverse substitution).

   Note: Students have a lot of trouble understanding why the bounds of integration change. It is worth dedicating a lot of time to trying to resolve this issue right away, as it will only become worse as the term progresses.

3. Students should be able to compute simple substitutions, and advance to more sophisticated substitutions with time and practice.

8.4.2 Integration by Parts

1. Introduce Integration by Parts as the reverse process of the product rule. Students do not have trouble with the definite integral here, so introducing them in either order or simultaneously should be fine.

2. Students should be able to identify an IBP problem versus a substitution problem, and make the appropriate choice of the parts. There is a well used mnemonic here, which you are free to employ if you wish. I prefer using Equation (8.7), but I'll leave that to your discretion.

3. Students should be able to solve simple IBP problems, with the understanding that the problems can and will become more sophisticated with practice.

   Note: I strongly suggest giving an example here which uses both substitution and IBP.

9 Applications of Integration

This section is an assorted collection of results related to integration. This includes computing areas between curves, improper integrals, and some applications to Economics and Finance. In the first section, we'll emphasize the difference between signed and unsigned areas, and compute the areas between curves. Here it is essential to explore the geometric intuition behind the integral formulas. Following this we'll talk of improper integrals. This is a fairly theoretical and technical section, and is often done poorly. Finally, we finish with some fairly contrived applications to Finance and Economics. These examples do appear in real life, but for example, Consumer/Producer surplus can often be done without the use of integrals.

9.1 Area Computations

Integration was motivated as finding the (signed) area under the graph of a function. Here we'll expand on this, and use it to compute the unsigned area, the area between two curves, and explore alternative techniques to finding the area. It is essential to stress the geometric insight of this section. We expect students to leave this section with the ability to to compute the areas listed above for relatively nice functions.

9.1.1 What we already know

1. We expect students to be able to set up and derive the classical area formulas for rectangles and triangles using an integral. While we lack the tools to do the circle (trigonometric substitution), this example should be shown to the student. *Note:* This is a good opportunity to have students participate, and determine the functions and their domains.

9.1.2 More complicated shapes

1. Re-iterate how the integral computes the area under the graph of a function. This is a good opportunity to really emphasize that the integral computes signed area. Give a few examples where the integral is zero despite the fact that the integrand is not identically zero.

2. We expect the student to leave this section with the ability to parse an area problem into mathematics (Examples 9.1 and 9.2). Perhaps more importantly, we expect the student to exercise his/her common sense. If presented with the picture of an odd function on a symmetric interval, they should be able to recognize that the integral is zero without any computation.

9.1.3 Unsigned Area

1. Introduce the unsigned area using the absolute value. Students should understand why the absolute value is present, which may necessitate an brief review of how the graph of $y = f(x)$ relates to $y = |f(x)|$. Given a problem statement, students should be able to recognize whether we are asking for the signed or unsigned area, and set up the integral correctly. This should be doable regardless of the function given, and should also work for abstract functions.

2. Students should be able to solve the integral $\int_a^b |f(x)|\,\mathrm{d}x$ for easily factored quadratic polynomials, with the understanding that they could do higher level polynomials given sufficient time and resources (such as the factorization of the polynomial). Simple functions like $f(x) = e^x - 1$, whose roots are easy to determine, are also fair game.

9.1.4 Integrating along the y-axis

1. Introduce the student to the idea of Riemann sums, with the y-axis partitioned into intervals, and lengths sampled from the function $x = g(y)$. Students should be able to discern the different areas depending on how the integral is being done (Figure 9.3).

2. Given a function $y = f(x)$ or $x = g(y)$, students should be able to set up the y-axis integral and find the signed or unsigned area under the graph. Students should know how the bounds of integration change in the y-axis integration, and how the integrand should be a function of y.

Note: Feel free to discuss the fact that we're just "turning our head to the side" and integrating, reformulating everything in terms of $y = f^{-1}(x)$ when necessary. However, be careful about the change of orientation in the axes: Turning your head results in the positive y-axis going to the left, rather than the right as per usual.

9.1.5 Area between curves

1. Set up the integral for the area between two curves, $y = f(x)$ and $y = g(x)$, when $f(x) \leq g(x)$ on the domain of integration. Students should be comfortable recognizing this as

$$\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x = \int_a^b [f(x) - g(x)]\,\mathrm{d}x.$$

   If the domain is not explicitly given, students should be able to find the given domain. In these cases, very simple functions will be given (low degree polynomials, radicals, or simple exponential/logarithmic functions).

2. Students should recognize that we are always asking for unsigned area in this case, and therefore the order of f and g matter above, but only up to sign. If they get a negative number, they should recognize that they only need to flip the sign to get the correct answer. Given two functions f and g whose intersection is easy to find, students should be able to determine whether $f(x) \leq g(x)$ or $f(x) \geq g(x)$ for all x in the domain.

3. In the more advanced regime, students should be able to recognize when two functions have multiple crossings, and how to adapt the integral in that case.. You can introduce this as $\int_a^b |f(x) - g(x)|\,\mathrm{d}x$, or split the integral into its appropriate domains. Students should be able to compute this for very simple functions. (NTS: Add a crossing example to the notes)

9.2 Improper Integral

Here we'll extend the notion of improper integrals to unbounded domains and unbounded functions. We'll use the Basic and Limit Comparison Tests to help determine convergence. By the end of this section, students should be able to recognize an indefinite integral, assess whether they are being asked for convergence or an explicit value, and apply the appropriate technique. Explicit convergence will be limited to exponential, radical, and rational functions; while convergence questions will effectively look like these classes of functions, with additional distractions added.

Improper integrals on unbounded domains will be our primary study, though unbounded functions will be briefly mentioned.

9.2.1 Unbounded Intervals

1. Discuss (or have students explain) why integration cannot be done on an interval like (a, ∞). Give Definition 9.8, and explain why it's the reasonable thing to do. Students should know the words *converge* and *diverge*. Given an improper integral and a simple function (radical, exponential, rational), students should be able to expand the integral using the limit definition, and evaluate the integral.

2. Give Theorem **??** (the p-test). While we do not do proofs, this one is computational and worth doing. Students should be able to replicate this proof at will. Give some intuition into why the result is true. Students should be able to recognize immediately whether $\int_1^\infty x^{-p}\,\mathrm{d}x$ converges, given some value of p.

3. Define the two-sided improper integral (Definition 9.12). Explain why this cannot be done as a single integral. Note only should students be able to compute the improper integral for the class of functions listed above, they should also be able to show that integrals like $\int_{-\infty}^{\infty} e^{-x}\,\mathrm{d}x$ or $\int_{-\infty}^{\infty} x\,\mathrm{d}x$ (requiring the proper definition) don't converge.

9.2.2 Unbounded Functions

1. Discuss why an unbounded function cannot be integrated. This one is trickier than an unbounded domain, so you may need to help the student realize this. Demonstrate that the FTC cannot be applied in these cases (Example 9.14).

2. Give the definition of the improper integral (Definition 9.15). Students should be able to solve straightforward examples of these improper integrals, including exponential, radical, and rational functions.

3. Give the p-test for unbounded functions (Theorem **??**). Draw the comparison to the p-test for unbounded domains, and that there is never any convergence for $p = 1$. Students should be able to immediately identify whether $\int_0^1 x^{-p}\,\mathrm{d}x$ converges for a given value of p.

9.2.3 The Basic Comparison Test

1. Discuss determining convergence over precise values, with an example of a difficult function. Give Theorem 9.19. Here it is essential to point out which versions of the theorem do not

hold (If $\int_a^\infty g(t)\,\mathrm{d}t$ diverges, we know nothing about $\int_a^\infty f(t)\,\mathrm{d}t$). Appeal to the geometric intuition of the result. Students should be able to state the result, and recognize the versions which don't hold.

2. Do some straightforward examples of how to apply the BCT (Examples 9.20 and 9.21). Students in turn should be able to apply the Basic Comparison Test to integrands of similar difficulty. In particular, they should be able to identify when to apply the BCT, ascertain the dominant terms of the integrand to find an comparison function, then construct the appropriate inequality to apply the theorem.

9.2.4 The Limit Comparison Test

1. Provide a shortcoming of the BCT (where the inequality goes the wrong direction), but motivate the LCT by saying the comparison function should "grow at the same rate" and so have an equivalent convergence. Introduce Theorem 9.22 with this in mind. Students should recognize that the order of the functions $f(x)/g(x)$ does not matter, as taking the reciprocal still results in a positive number.

2. Examples 9.23 and 9.24 are good examples of the level of difficulty of question we might ask of a student. Students should be able to recognize when we're asking for convergence over a particular result, and when to apply the LCT over the BCT.

   Note: Students have a lot of trouble with the LCT. In particular, there are two limits: The one which shows equivalence of convergence, and the that actually demonstrates convergence. Many students wrongly assume that if they show $f(x)/g(x) \to L$ for some positive L, then both integral converge. Be sure to do an example where this is not the case.

9.3 Applications in Economics and Finance

Three applications will be covered: Continuous compounding annuities, marginality in economics, and consumer/producer surplus. This section is a general interest section: Nothing here will be examined.

1. Derive the equation for the future value of a continuously invested annuity. Like showing students that exponential growth comes from a simple differential equation, thence ignore the differential equation, the same is true of this equation. Students are only expected to be able to compute the integral, we would give them the formula.

2. Show students that "marginal" means derivative in economics, and give several examples of marginality. We do not expect students to remember anything from this, and they will not be tested on this.

3. Define and derive consumer/producer surplus. Give a graphical representation of these quantities, and how we can use an integral to solve for them. Again, there is no expectation that the student be able to reproduce any of this material.

10 Differential Equations

(NTS: Add modelling, systems, and analysis (currently in assignments))

Here we'll very briefly cover differential equations, including separable, linear, and second order differential equations. Our focus is on solving these systems, both in general, and for specific initial value problems. By the end of this section, we expect the student to be able to classify a differential equation into one of the three types listed above, solve the corresponding system in general, and evaluate to find initial values. Some integration techniques may be needed, such as integration by parts and substitution. In these cases, the examples will be no harder than those presented in Section 8.4.

Notes:

1. In every section, it is worth taking time to show students how the constants 'absorb' numbers and functions. For example, when $\ln(y) = x + C_1$, we might write $y = C_2 e^x$ where $C_2 = e^{C_1}$. We will not punish students for reusing a single letter, but we should enforce the practice of using iterated constants $(C_1, C_2, \ldots,)$ and of indicating their relationships $(C_2 = e^{C_1})$.

2. In initial value problems, the substitution can be made at any time. It is often more practical to do this before solving the equation for the dependent variable, and showing students how this is done strengthens the point we're trying to make in Remark 1 above.

10.1 Basic Differential Equations

This section serves as a jumping point into non-trivial ODE's. Here we will define a differential equation, motivate their use, start with basic ODEs (those that can be solved via integration), and recall the role of initial values.

1. Students should be give a rough definition of a differential equation; namely, as any equation involving both an dependent variable (say y) and its derivatives (y', y'', etc). Student should be able to recognize when an equation is a differential equation.

2. Introduce students to "trivial" differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$. Students should immediately recognize that this can be solved for integrating. Use this to re-iterate general solutions, versus specific solutions given by initial values. Students should be able to solve any such problem where f is a function typically seen as an integrand in Section 8.4.

3. (NTS: Need to put this in notes) Given an explicit form of the solution, students should be able to verify a solution to a differential equation. This applies to any reasonably differentiable function.

Note: As a segue to the rest of the chapter, it's worthwhile to note how difficult it is to solve real differential equations, with one possibility to give students simple examples and have them try to guess solutions. For example,

ODE	Solution
$\frac{\mathrm{d}y}{\mathrm{d}x} = y$	$y = Ce^x$
$\frac{\mathrm{d}y}{\mathrm{d}x} = yx$	$y = Ce^{x^2/2}$
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$	$y = Cx$
$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$	$y = \frac{1}{C - x}$

10.2 Separable Differential Equations

Here we'll introduce the class of ODE's of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$ and discuss their solution. By the end of this section, students should be able to solve such differential equations when f and g are no more complicated than the types of integrands found in Section 8.4. Students should be able to identify when a differential equation is separable. Students should be able to find general solutions with arbitrary constants, and unique solutions given an initial value.

1. Introduce the general form of a separable differential equation. Given a list of arbitrary differential equations, students should be able to identify which equations are separable versus those which are not.

2. Introduce the algorithm for solving separable differential equations, and cover examples of both general solution, and initial value problems. We expect that students should in turn be able to solve such questions when the integrands are no more complicated than those found in Section 8.4.

10.3 Linear Differential Equations

Students will be introduced to the notion of the integrating factor, and how it can be used to solve linear differential equations. Given a differential equation, students should be able to find the integrating factor, and find the solution for the differential equation, both in the general and unique sense. After applying the integrating factor, students my be asked to use advanced integration techniques, but nothing more complicated than that found in Section 8.4.

1. Give the motivation and derivation for the integrating factor. Students should be able to repeat this process if asked, and may be asked simple conceptual problems based on the integrating factor. For example, a question of the form:

   What function $I(x)$ satisfies

   $$\frac{\mathrm{d}y}{\mathrm{d}x} I(x) + 3I(x)y = \frac{\mathrm{d}}{\mathrm{d}x}\left[yI(x)\right]?$$

might be asked.

2. Students should be able to identify a linear differential equation versus a separable differential equation (note that some ODEs are both separable and linear).

3. Students should be able to solve simple linear differential equations, both in general terms and unique solutions given an initial value problem. While more advanced integration techniques might be used, Example **??** or Exercise **??** represent upper bounds to the difficulty of questions they might reasonably be asked.

Note: It is worthwhile choosing an ODE which is both linear and separable, and solving it both ways to show equivalence of solutions.

10.4 Second Order Differential Equations

Students will see their first – and only – second order differential equation. Several points need to be made in this section which are different than those of separable and linear first order equations. These include the requirement of two initial values to specify a unique solution, and the degenerate case of repeated roots. We expect students to solve differential equations of the form $ay''+by'+cy = 0$ for any a, b, c which give real roots.

1. Motivate and derive the characteristic equation for second order differential equations. Students should be able to repeat this process and answer simple conceptual problems about the derivative. Use this to give the general form of the solution set.

2. Note the degenerate case of repeated roots. Demonstrate, by means of explicit calculation, that this is indeed a solution of the differential equation.

3. Given any triple (a, b, c) for which $ax^2 + bx + c$ admits real roots, students should be able to solve $ay'' + by' + cy = 0$, both in the general and initial value cases.

4. Students should recognize that two initial values need to be given to specify a unique solution. Be sure to give examples of the form $y(a) = b, y(c) = d$ and $y(a) = b, y'(c) = d$. (NTS: Add such an example)

11 Multivariable Calculus

Students will be quickly introduced to a myriad of multivariable topics, including partial derivative, optimization with Lagrange multipliers, and iterated integrals. The goal here is proficiency of computation, with some conceptual understanding of partial derivatives.

Notes:

- Our students have trouble with multivariable functions, and they tend to think of functions in terms of their graphs. It is worthwhile to spend some time discussing multivariable functions. Try grounding them with a real-life example, such as unit-cost of a good as a function of its material costs, production run, and demand. This will tie in nicely with partial derivatives latter.

- Your definitions should hold for functions of n-variables. However, be sure to recall the difference in notation between $f(x,y,z)$ and $f(x_1,x_2,x_3)$, as this is a confusing point for many students.

11.1 Partial Derivatives

Students will see the limit definition of a partial derivative, and should be able to compute these with either a limit or by keeping the other variables constant. Students should recognize that $\partial y/\partial x$ gives the instantaneous rate of change of y with respect to x.

1. Motivate the problem of finding the instantaneous rate of change of a multivariable function. Use this to define the limit definition of the partial derivative at a particular point, and eventually at a general point. Explain how this is the instantaneous rate of change of the function with respect to that variable. Students should be able to compute this limit definition for polynomial functions of multiple variables, such as $f(x,y) = 2x + x^2 y$.

2. Introduce alternate notations for the partial derivative, including

$$\frac{\partial f}{\partial x}, \quad \partial_x f, \quad \partial_{x_1} f, \quad \partial_1 f, \quad f_x.$$

   While some of these do appear in the literature, as a rule **we will only use the first three**, and you can tell the students this.

3. Argue that the partial derivative $\partial f/\partial x_i$ can be computed by keeping all other variables constant and differentiating (but do not prove this). Students should be able to compute the partial derivative of any function composed of functions they have seen before, including polynomials, rational functions, radical, exponentials, and logarithms.

   Note: Computing the partials of functions like $f(x,y) = e^{xy}$ is confusing for them. Take the time to do several 'non-separable' examples of this sort.

4. Define the gradient as the row vector of partial derivatives. Do not worry about giving the geometric intuition for the gradient. As an extension of objective [Sec 11.1.0-3], they should be able to compute the gradient of any function

11.2 Applications of Partial Derivatives

This section will not be heavily tested, but is useful to establish context for how applicable partial derivatives can be. Students will learn about marginality, and complementary/competitive products.

1. Discuss the word *marginal* in an economics framework. Students should be able to recognize that the *marginal A of B* is $\partial A/\partial B$, and hence compute it in very simple frameworks.

2. Discuss competitive versus complementary products. It's easy to create intuitive examples of such products, so use this to ground the definitions in reality. Given a simply analyzed demand function $q(x, y)$, students should be able to determine whether x and y are complementary or competitive.

11.3 Higher-Order Partial Derivatives

Students will learn about second and higher order partial derivatives, how they are computed, the Hessian, and Clairut's Theorem. Returning to Section 4.1, we expect that students will be able to count the number of ways of computing a particular derivative. We will conclude this session with a heuristic for computing the chain rule (NTS: Move chain rule).

1. Define the second order derivatives for a function of n-variables. Students should recognize that there are n^2 such derivatives, and be able to compute the second derivatives for any function of at most four (4) variables, composed of elementary functions they have seen, such as polynomials, radical, exponentials, and logarithms.

2. State Clairut's theorem, and demonstrate it with an example. We will only work with C^1 functions in the section, so for all intents and purposes, this theorem is universal in our context. Students should be able to apply Clairut's theorem to simplify the process of computing second order partial derivatives, and answer simple True/False questions based on the theorem.

   Note: A good exercise is to give students a function and ask them to compute the second partials. Ask them if they see a pattern. You can do this with several function, or a single three variable function might work nicely.

3. Define the Hessian of f, $H_f(\mathbf{x})$, as the $n \times n$ matrix of second order partial derivatives. By extension to Objective [Sec 11.3.0-2], students should be able to compute the Hessian for any elementary function of at most four (4) variables, with the understanding that they could do a higher number of variables given sufficient time.

4. Motivate the chain rule (NTS: Add something to this effect), and give a heuristic for how to compute partial derivatives by means of constructing a dependence tree. We will not state the Chain Rule proper, as it involves vector-valued functions.

5. Students should be able to compute chain-rule based partial derivatives given explicit functions. They should understand how to evaluate these derivatives at particular points (See Example **??**). In addition, abstract chain rule computations, such as those **??** are fair game.

6. Using the chain rule to compute second order derivatives can be done time-permitting. For example, if $u = f(x, y)$ with $x = g(s, t)$ and $y = h(s, t)$, our only objective with this sort of computation is to have students recognize that

   $$\frac{\partial}{\partial x}\frac{\partial u}{\partial s}$$

   requires yet another chain rule, as $\partial_s u$ itself is a function of x and y, and hence of s and t through the chain rule. Do not spend much time on this.

11.4 Optimization

As with optimization in a single variable, we will discuss finding extreme values at interior points (using critical points and a second derivative test) and finding extreme values on the boundary (Lagrange multipliers). Students may be asked to solve an optimization requiring both techniques, or just one.

1. Define a critical point of a function of n-variables. Draw the analogy between $\nabla f(\mathbf{x}) = \mathbf{0}$ and $f'(x) = 0$. However, do not try to explain the geometric intuition of the horizontal tangent plane, as we are ignoring the geometry of the gradient.

2. Demonstrate that solving for critical points generally results in a non-linear system of equations, which can be solve. Demonstrate that we can have critical sets; namely, continuous connected sets of critical points. Students should be able to find critical points for relatively simple functions, which admit relatively simple systems.

3. State Theorem **??**, and motivate the classification of critical points. Point out that minima and maxima still exist, but now we also have saddle points.

4. Give the second derivative test (Proposition **??**). Students should be able to classify the critical points of a function of at most three variables, using their knowledge of eigenvalues/vectors from Chapter 3. Proposition **??** represents a special case of Proposition **??** in the 2-dimensional regime, and can optionally be given, though is not necessary.

5. State Theorem **??** (Lagrange Multipliers). It is important that you motivate why we must introduce this condition. Draw analogy to the boundary extreme value problem in one-variable calculus.

   Note: Even advanced students have trouble identifying why Lagrange multipliers is necessary. Take some time to properly motivate this, and show that it is different from the critical point approach taken above.

   Note: There are several approaches to Lagrange multipliers. The book uses the condition that $\nabla f = \lambda \nabla g$. Of course, you can set up the Lagrangian $\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$ if you prefer, for which $\nabla \mathbf{L} = \mathbf{0}$.

6. Students should be able to solve relatively simple Lagrange multiplier problems (See Examples **??** and **??**). Some effort should be made to emphasize that these are non-linear systems of equations, and generally have to be solved in an *ad hoc* manner.

Note: We will not ask students to solve multi-constraint problems, though you may wish to point out how this is done if you have some time.

11.5 Iterated Integrals

Without discussing integration in $\mathbb{R}^n$, we will show students how to compute multivariate integrals. We will being by doing this on rectangular domains, then move to non-rectangular domains.

1. Discuss how to compute the integral of a function of at most three variables on a rectangular domain. Students should be able to compute this for any sufficiently nice function (Example **??**).
2. Discuss iterated integrals on non-rectangular domains, and do many examples. Show students how to find the bounding functions. Given a sufficiently nice integrand, and a set bounded by a piecewise smooth curve, students should be able to find the value of the iterated integral.

   Note: Finding the values of the bounding functions is difficult for students. You should do this slowly, and spend a lot of time reinforcing how this done. This is best done with many examples.