Lecture 11: Vectors, matrices, and tensors

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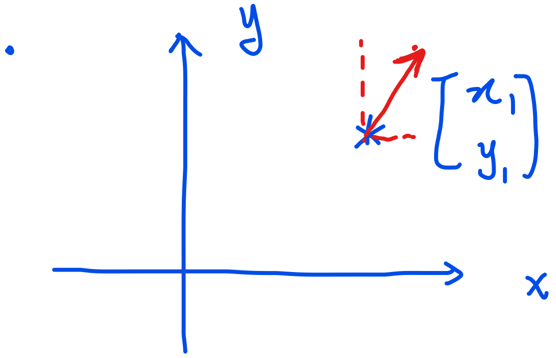
EEL 3850

Vector: An n x 1 matrix.

$$y = \begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}$$

$y_i = i^{th}$ element



velocity of $*$ is $\begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$

Acc: $*$ $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

1-indexed vs 0-indexed:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Zero, ones, and unit vectors

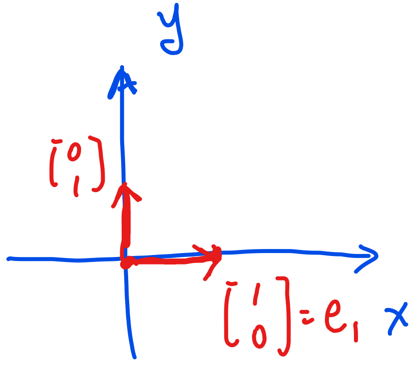
- n-vector with all entries 0 is denoted 0_n or just 0
- n-vector with all entries 1 is denoted $\mathbf{1}_n$ or just $\mathbf{1}$
- a *unit vector* has one entry 1 and all others 0
- denoted e_i where i is entry that is 1
- unit vectors of length 3:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$1_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad 1_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$x = \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2e_1 - 3e_2$$

Matrix: Rectangular array of numbers:

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

4 rows. 2 columns

Dimension of matrix: number of rows x number of columns

Dim(A) = 4 x 2

by

Matrix Elements (entries of matrix)

$$A = \begin{array}{cc} & \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \boxed{1402} & 191 \\ 1371 & \boxed{821} \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix} \end{array}$$

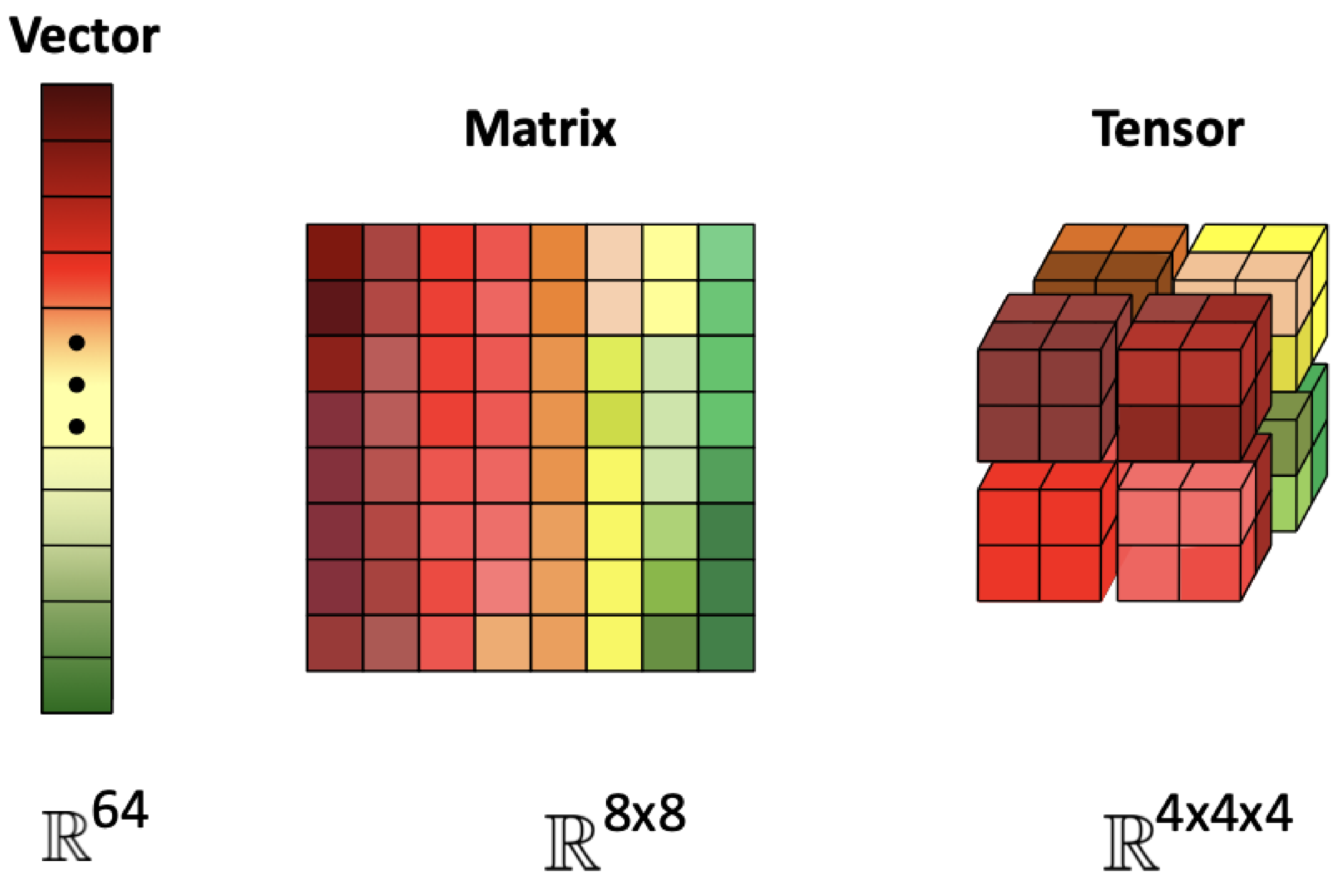
$A_{11} = 1402$

$A_{22} = 821$

$A_{31} = 949$

$A_{ij} =$ "i ,j entry" in the i^{th} row, j^{th} column.

Tensor: High-dimensional array



Basic algebra

Vector addition

- n-vectors a and b can be added, with sum denoted $a + b$
- to get sum, add corresponding entries:

$$\underset{a}{\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix}} + \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}} = \underset{c}{\begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}}$$

$c_i = a_i + b_i$ for $i = 1, \ldots n$.

- subtraction is similar

$c = a - b \Rightarrow c_i = a_i - b_i$ for $i = 1, \ldots, n$.

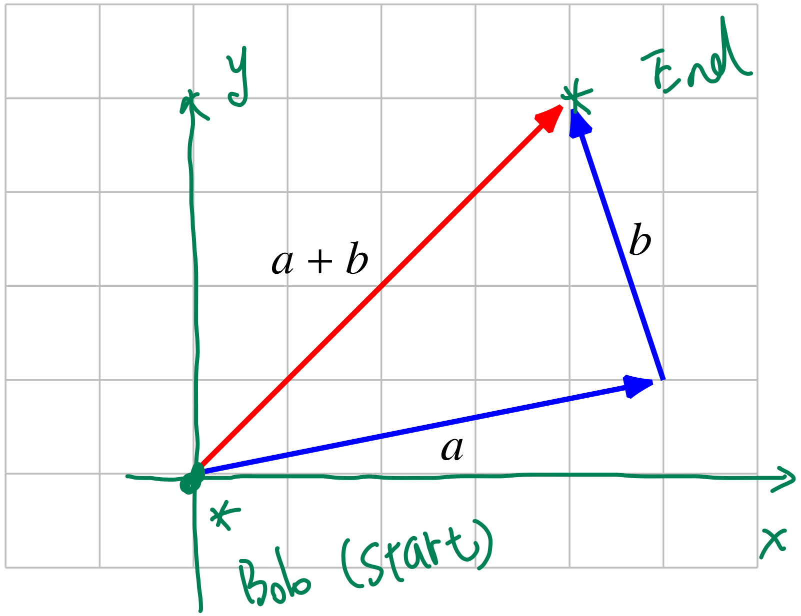
Properties of vector addition

- *commutative*: $a + b = b + a$
- *associative*: $(a + b) + c = a + (b + c)$
  (so we can write both as $a + b + c$)
- $a + 0 = 0 + a = a$
- $a - a = 0$

these are easy and boring to verify

Adding displacements

if 3-vectors a and b are displacements, $a + b$ is the sum displacement



Scalar-vector multiplication

- scalar β and n-vector a can be multiplied

$$\beta a = (\beta a_1, \ldots, \beta a_n)$$

- also denoted $a\beta$
- example:

$$(-2)\begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ -12 \end{bmatrix}$$

Transposing a column vector results in a row vector, and vice versa

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow V^T = [a \quad b \quad c]$$

1×3

$n \times 1$ $\quad$ 3×1

Vector-Vector Product

"Inner product" or "dot product"

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

$\triangle$

$$\left\langle \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} e \\ f \\ g \end{bmatrix} \right\rangle = [a \quad b \quad c] \cdot \begin{bmatrix} e \\ f \\ g \end{bmatrix} = a \cdot e + b \cdot f + c \cdot g$$

$= [2 \quad 7 \quad 1] \cdot \begin{bmatrix} 8 \\ 2 \\ 8 \end{bmatrix} = 2 \times 8 + 7 \times 2 + 1 \times 8$

$$\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 2 \\ 8 \end{bmatrix} = 2 \cdot 8 + 7 \cdot 2 + 1 \cdot 8 = 38$$

Dot product

Usages of inner products

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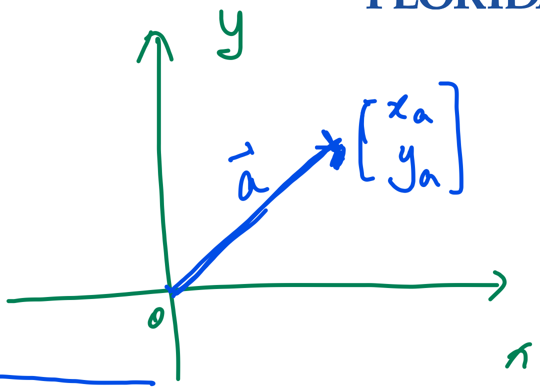
0.1 L2-norm: the intuitive notion of length of the vector

The L_2 norm (also known as the Euclidean norm) of a vector $\mathbf{x} \in \mathbb{R}^n$ is defined as:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$



$$\|\vec{a}\|_2 = \sqrt{x_a^2 + y_a^2}$$

0.2 Squared error computation

The squared error between a sample $\{x_1, x_2, \ldots, x_n\}$ and its mean is given by:

Recall var.

$x_1, x_2 \dots x_n$. $\hat{\mu}_x$: estimated mean.

$$\hat{\sigma}_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \hat{\mu}_x)^2$$

constant

$$SE = \sum_{i=1}^{n} (x_i - \mu_x)^2 = \|\vec{x} - \underset{\Delta}{\mu_x} \mathbf{1}\|_2^2.$$

$\rightarrow$ $\boxed{\text{L2-norm squared}}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

n×1 vector

$$\vec{x} - \mu_x \mathbf{1}_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \underset{\Delta}{\mu_x} \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} \mu_x \\ \mu_x \\ \vdots \\ \mu_x \end{bmatrix} = \begin{bmatrix} x_1 - \mu_x \\ x_2 - \mu_x \\ \vdots \\ x_n - \mu_x \end{bmatrix}$$

Geometric interpretation of inner product

$\cos(\alpha - \beta)$

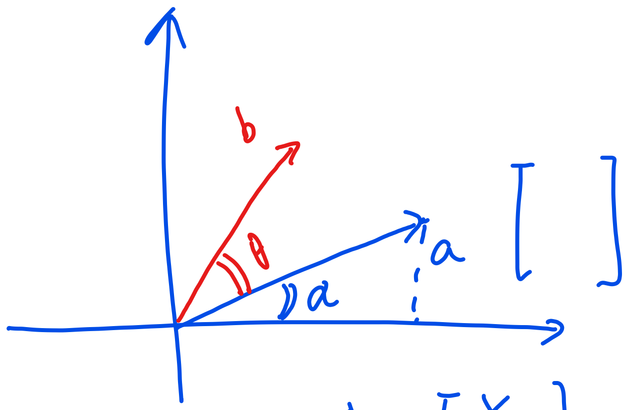
$= \cos\alpha \cos\beta + \sin\alpha \sin\beta$

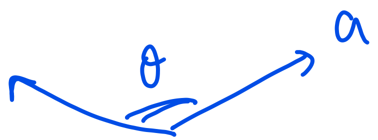
An alternative expression using vector norms and the angle θ between the two vectors is:

$$\mathbf{a} \cdot \mathbf{b} = \underbrace{\|\mathbf{a}\|}_{L_2\text{-norm}}\|\mathbf{b}\| \cos\theta$$

where:

- $\|\mathbf{a}\| = \sqrt{\sum_{i=1}^{n} a_i^2}$ is the norm of $\mathbf{a}$.
- $\|\mathbf{b}\| = \sqrt{\sum_{i=1}^{n} b_i^2}$ is the norm of $\mathbf{b}$.
- θ is the angle between the two vectors.





$$\vec{a} = \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} \|a\| \cos\alpha \\ \|a\| \sin\alpha \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} \|b\| \cos(\alpha+\theta) \\ \|b\| \sin(\alpha+\theta) \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} x_a \\ y_a \end{bmatrix} \cdot \begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} x_a & y_a \end{bmatrix} \begin{bmatrix} x_b \\ y_b \end{bmatrix}$$

$$= x_a x_b + y_a y_b = \|a\| \|b\| \underline{\cos\alpha \cos(\alpha+\theta)} + \|a\| \|b\| \underline{\sin\alpha \cdot \sin(\alpha+\theta)}$$

$$= \|a\| \|b\| \cos\theta$$

Usage of inner product

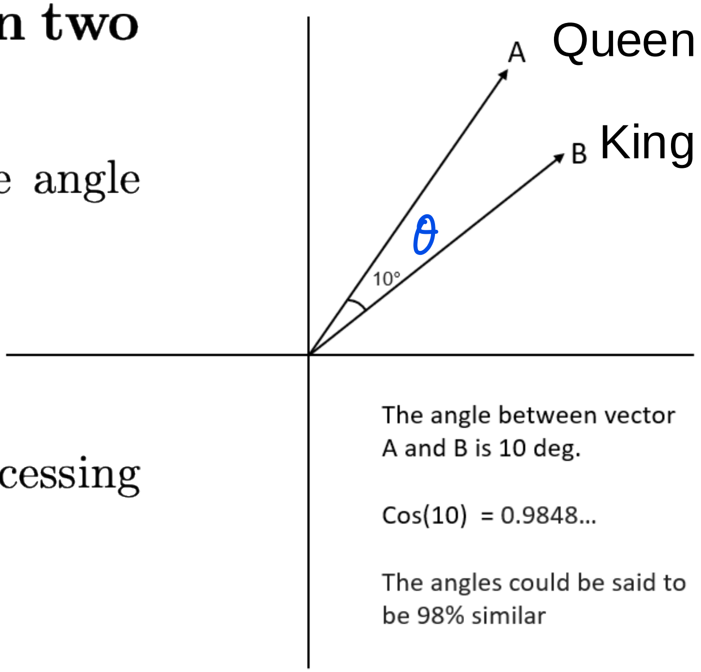
0.1 Cosine Similarity: measure the similarity between two vectors

Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the cosine similarity is the cosine of the angle between the vectors, and can be computed as:

$$\cos\theta = \text{cosine similarity} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

This is commonly used in text mining and natural language processing (NLP) to measure the similarity between document embeddings.

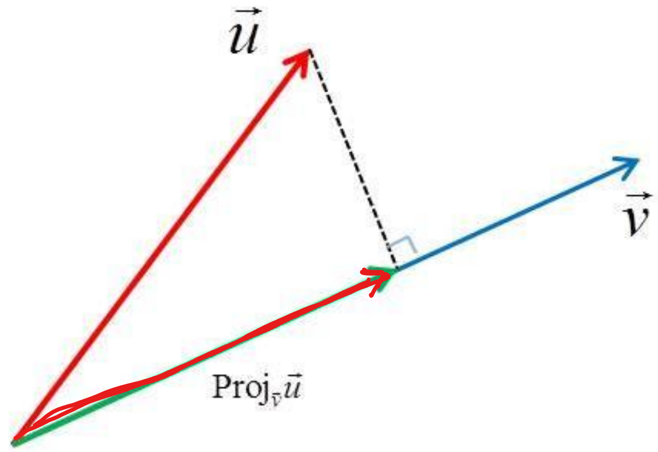
$$a \cdot b = \|a\| \|b\| \cos\theta$$

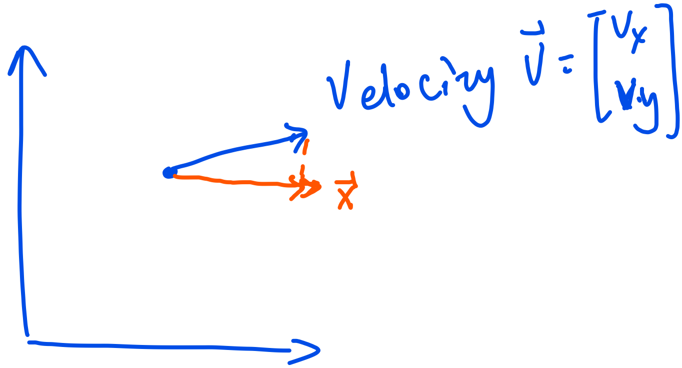


Projection

- To "project" a vector **u** onto **v** (basically, rotate a vector and place it on a second vector);

$$proj_v u = \left(\frac{u \cdot v}{\|v\|^2}\right) v$$





$$\frac{u \cdot v}{\|v\|^2} \cdot \vec{v} = \frac{\|u\|\|v\|\cos\theta}{\|v\|^2} = \frac{\|u\|}{\|v\|}\cos\theta \cdot \vec{v} = \|u\|\cos\theta \cdot \frac{\vec{v}}{\|v\|}$$

unit vector with direction as $\vec{v}$

Cauchy-Schwarz inequality

Cauchy-Schwarz Inequality:
For any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the inner product satisfies:

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

$|a \cdot b| = \big| \|a\| \|b\| \cos\theta \big| = \|a\| \|b\| |\cos\theta| \leq \|a\| \|b\|$

$|\cos\theta| \leq 1$

In component form, for vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$:

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}$$

$a_1 b_1 + a_2 b_2 \leq \sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2}$

Equality: $\theta = ?$

Scalar Multiplication

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

then for any scaler 'k'

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}_{m \times n}$$

$$3 \times \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} =$$

Matrix Addition is element-wise

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \\ 0 & 1 \end{bmatrix} =$$

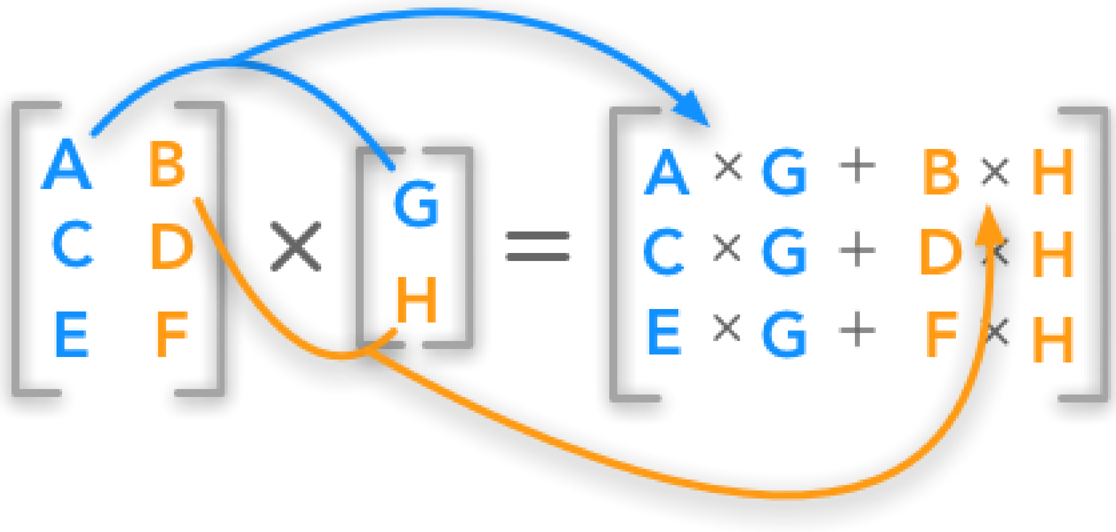
$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \end{bmatrix} =$$

Matrix transpose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \qquad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3 \times 2}$$

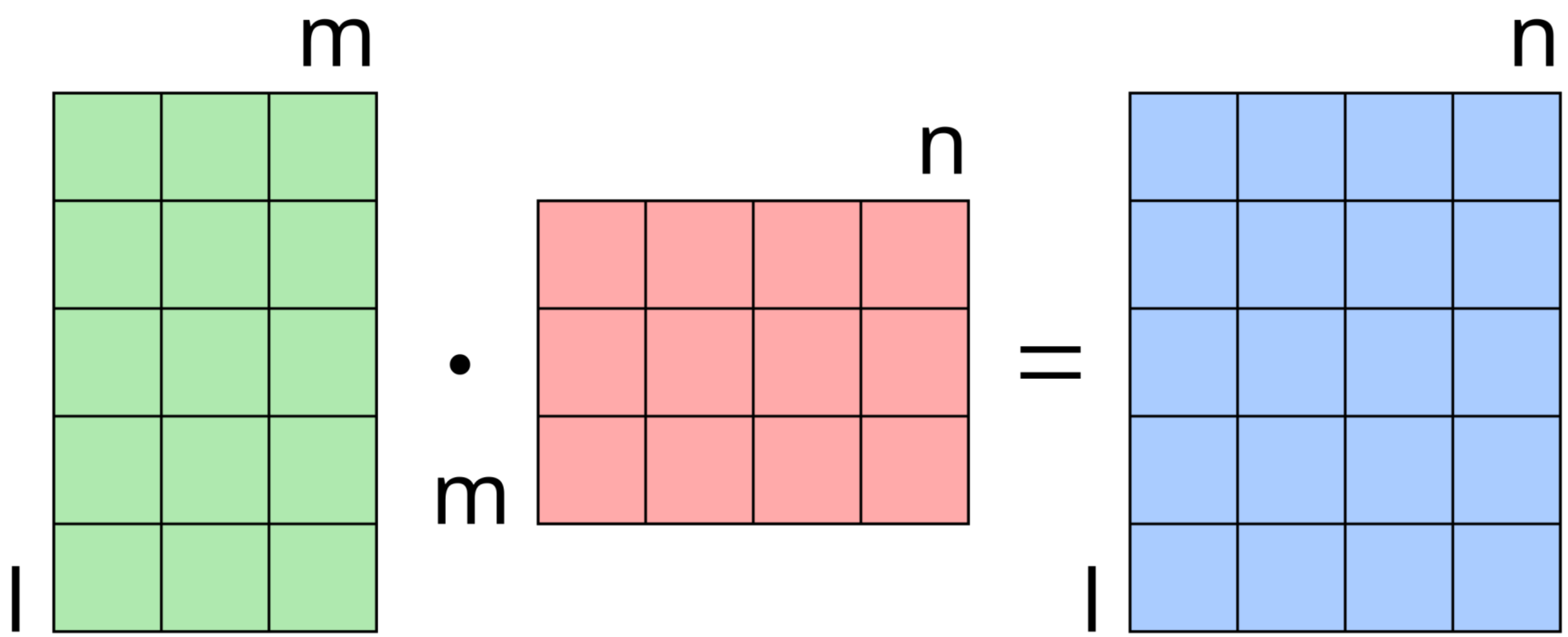
$$(\mathbf{A}^\top)_{ij} = A_{ji}$$

Matrix-vector multiplication



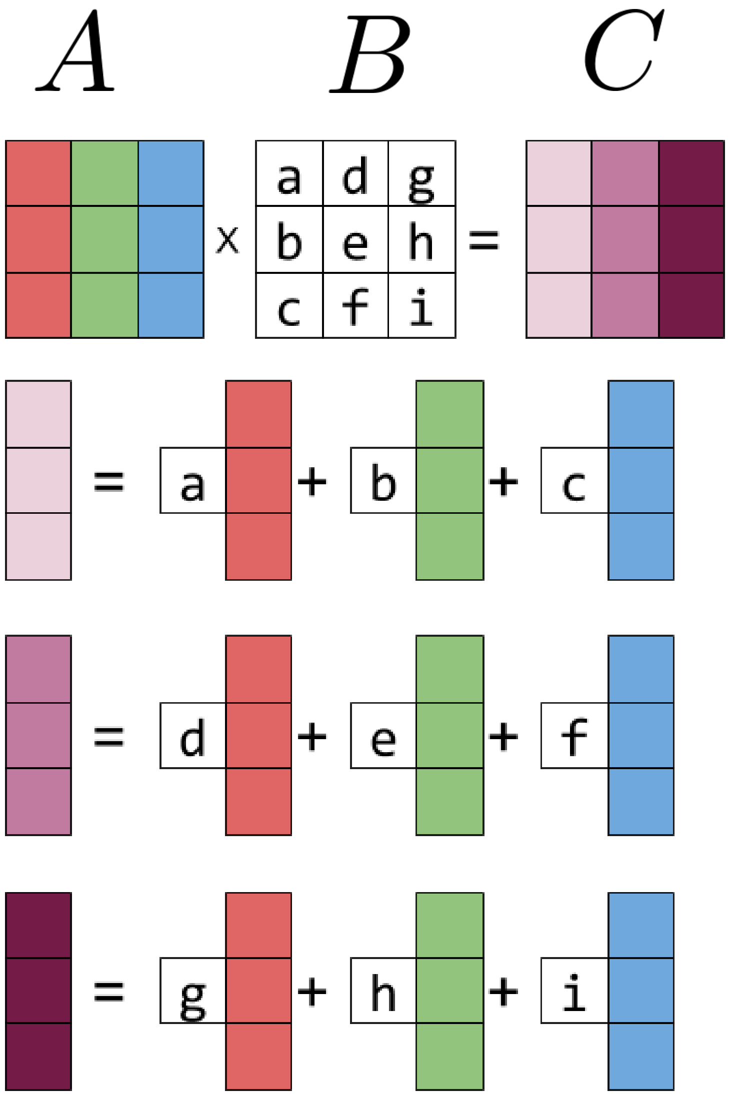
$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} =$$

Matrix-Matrix Product



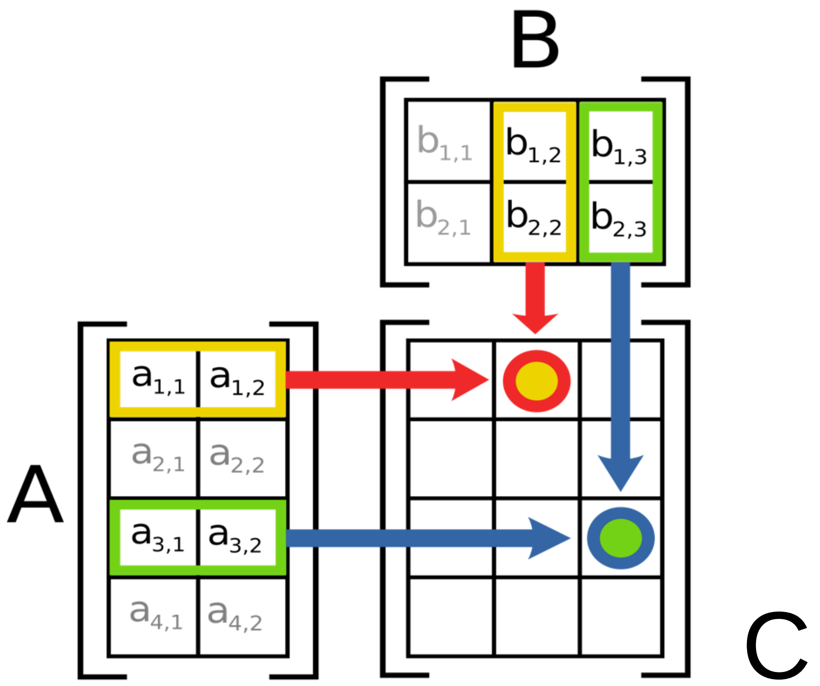
$$A \cdot B = C$$

Matrix-Matrix Product



The ***i***-th column of the matrix ***C*** is obtained by multiplying ***A*** with the ***i*** column of ***B***

$$\mathbf{C} = \mathbf{A}\mathbf{B} = \begin{bmatrix} — & \mathbf{a}_1^\top & — \\ — & \mathbf{a}_2^\top & — \\ & \vdots & \\ — & \mathbf{a}_m^\top & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{b}_1 & \mathbf{a}_1^\top \mathbf{b}_2 & \cdots & \mathbf{a}_1^\top \mathbf{b}_p \\ \mathbf{a}_2^\top \mathbf{b}_1 & \mathbf{a}_2^\top \mathbf{b}_2 & \cdots & \mathbf{a}_2^\top \mathbf{b}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_m^\top \mathbf{b}_1 & \mathbf{a}_m^\top \mathbf{b}_2 & \cdots & \mathbf{a}_m^\top \mathbf{b}_p \end{bmatrix}$$



The ***i,j*-th** element of the matrix ***C*** is the inner product of the ***i***-th row of ***A*** with the ***j***-th column of ***B***

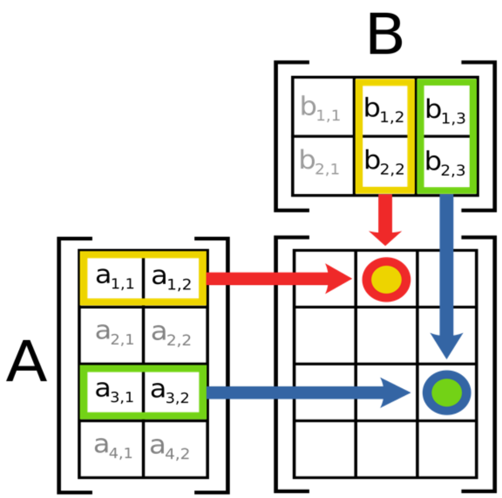
Example

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

A and *B* must have conforming dimensions!



Dim(A) = m x n
Dim(B) = n x p
Dim(AB) = m x p