Lecture 2: Conditional Probability, Total Probability

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EEL 3850 S25

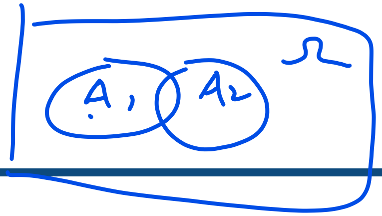
Discrete sequential model

- A sequential model is a type of experiment that has an inherent sequential character.

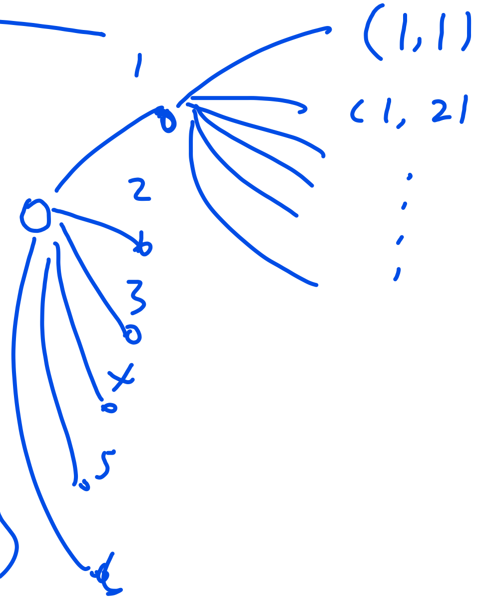
Example:

- Flipping a coin 3 times
- Receiving eight successive digits at a communication receiver
- Observing the value of a stock on five successive days

Discrete sequential model



- Consider the experiment where we roll a 6-sided fair die 2 times and the event E ≡ observing a 1 or 2 on either roll.
  - What is the sample space?
  - What are the subset of sample space for event E?
  - What is the probability of event E?



A_1: observe 1 or 2 on a die roll. (1st)

$$P(A_1) = \frac{2}{6} \; ; \; P(A_1) = P(\text{roll } 1 \text{ or roll}(2))$$

$$= P(\text{roll}(1)) + P(\text{roll}(2)) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

A_2: observe 1 or 2 on 2nd roll.

"independent"

$$P(A_1 \cup A_2) = P(E) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1)P(A_2)$$

$$= \frac{2}{6} + \frac{2}{6} - \left(\frac{2}{6}\right)^2 = \frac{20}{36} = \frac{5}{9}.$$

Discrete sequential model

- Consider the experiment where we roll a 6-sided fair die 2 times and the event E ≡ at least one roll is 4.
  - What is the probability of event E?

A_1: First roll is 4.

A_2: second roll is 4.

$P(A_1) = P(A_2) = \frac{1}{6}$

$$P(E) = P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= P(A_1) + P(A_2) - P(A_1)P(A_2)$$

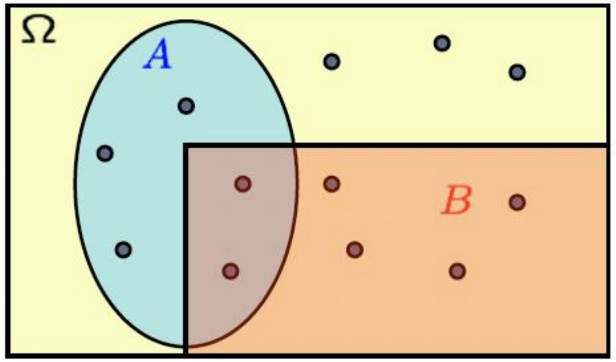
$$= \frac{1}{6} + \frac{1}{6} - \left(\frac{1}{6}\right)^2 = \frac{11}{36}$$

Conditioning

Assume 12 equally likely events, without any information, what is the probability of event A? Event B?

P(A)= $\frac{5}{12}$

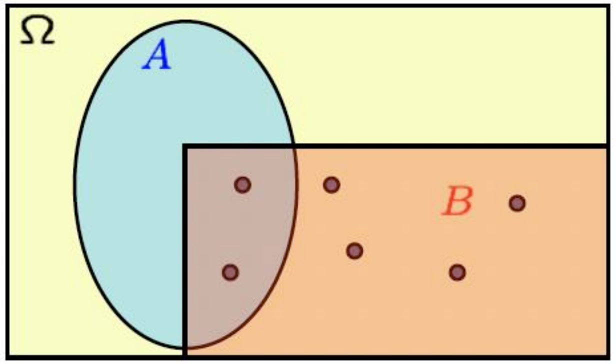
P(B)= $\frac{6}{12}$



Received information that B occurred, what is the probability of event A given this information?

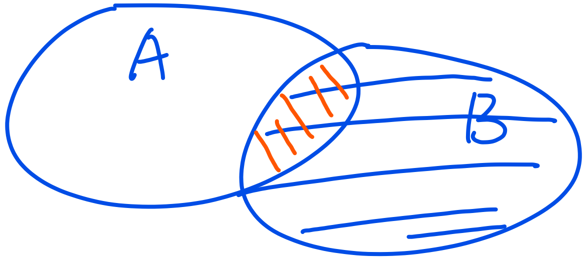
P(A|B)= $\frac{2}{6}$

P(B|B)= $\frac{6}{6} = 1$

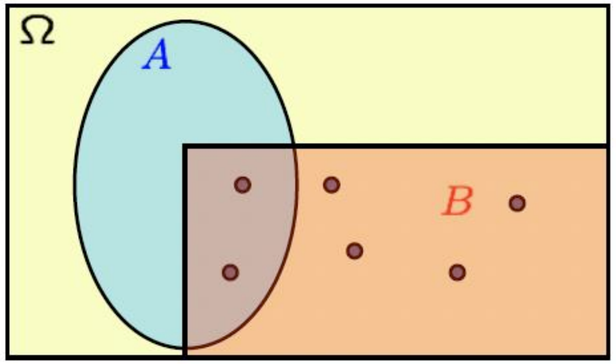


Definition of conditional probability

ϕ: empty set.



- Notation: The probability of A given B: P(A|B)



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
only defined when $P(B) > 0$.

P(A|B) =

Fair

$P(A \cap B) = \frac{|A \cap B|}{|\Omega|}$, $P(B) = \frac{|B|}{|\Omega|}$ $\Rightarrow P(A|B) = \frac{|A \cap B|}{|B|}$

Special cases:

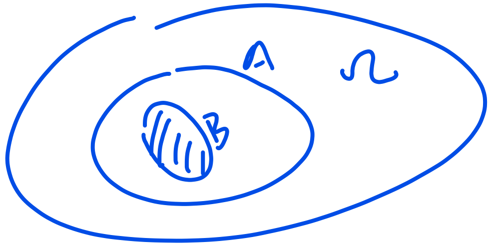
$P(A \cap B) = P(\phi) = 0$

- $A \cap B = \emptyset$: The two events are exclusive. $P(A|B) = \frac{P(A \cap B)}{P(B)} = 0$ $\hookrightarrow \frac{P(A \cap B)}{P(B)}$

- $B \subseteq A$

A: "Even face"

B: either 2 or 6 occurs

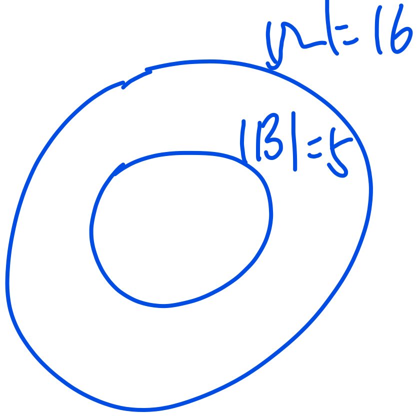


$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

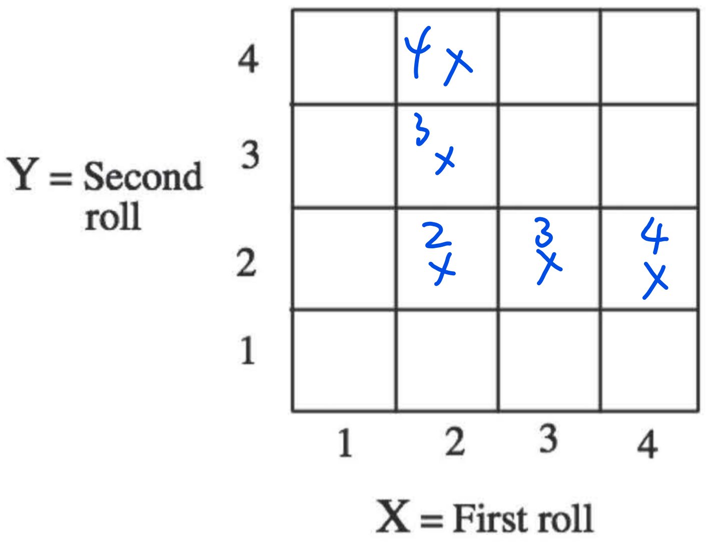
Example

- Consider roll of 4-sided die twice. If I tell you the minimal number of two rolls is a 2, what is the probability of that the maximal number of the two rolls is 1 (or 2, or 3, or 4)?

$|\Omega| = 16$ outcomes.



Determine the conditional probability $P(A|B)$ where
$A = \max(X,Y) = m,\ B = \{\min(X,Y) = 2\}$, where $m = 1, 2, 3, 4$.



$m=1$: $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0$

$m=2$: $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1}{5}$

$m=3$ " " $= \frac{2}{5}$

$m=4$ " " $= \frac{2}{5}$

Chain Rule

- Using Conditional Probability to Decompose Events

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

and

$$P(B \cap A) = P(B|A)\,P(A)$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(B|A)P(A)$$

Relation: $P(B|A)\,P(A) = P(A|B)\,P(B)$

independence:

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

A, B, indep.

$= \frac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}}$

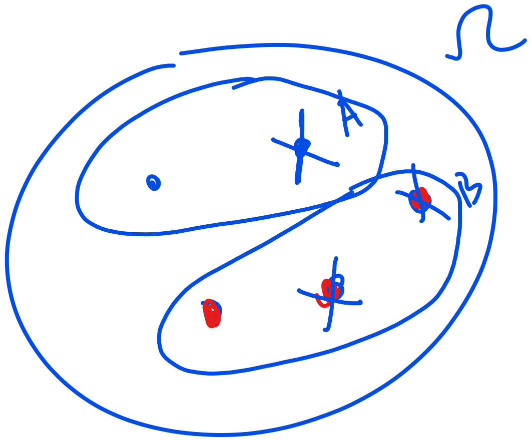
$= P(A)$

Example

- A computer lab contains: 5 computers.
  - two computers from manufacturer A, one of which is defective
  - three computers from manufacturer B, two of which are defective

  A user sits down at a computer at random. Let the properties of the computer she sits down at be denoted by a two letter code, where the first letter is the manufacturer and the second letter is D for a defective computer or N for a non-defective computer.

Let E_A be the event that the selected computer is from manufacturer A
E_B be the event that the selected computer is from manufacturer B
E_D be the event that the selected computer is defective
Let's find



$P(E_A) = \frac{2}{5}$

$P(E_B) = \frac{3}{5}$

$P(E_D) = \frac{3}{5}$

Example

- I tell you the computer is from manufacturer A. Then what is the probability that it is defective?

$$P(E_D|E_A) = \frac{1}{2}$$

$$P(E_D|E_B) = \frac{2}{3}$$

$$P(E_D|E_A) = \frac{P(E_D \cap E_A)}{P(E_A)}$$

$$\begin{cases} P(E_D \cap E_A) = P(E_A|E_D) \cdot P(E_D) \\ P(E_D \cap E_A) = P(E_D|E_A) \cdot P(E_A) \end{cases}$$

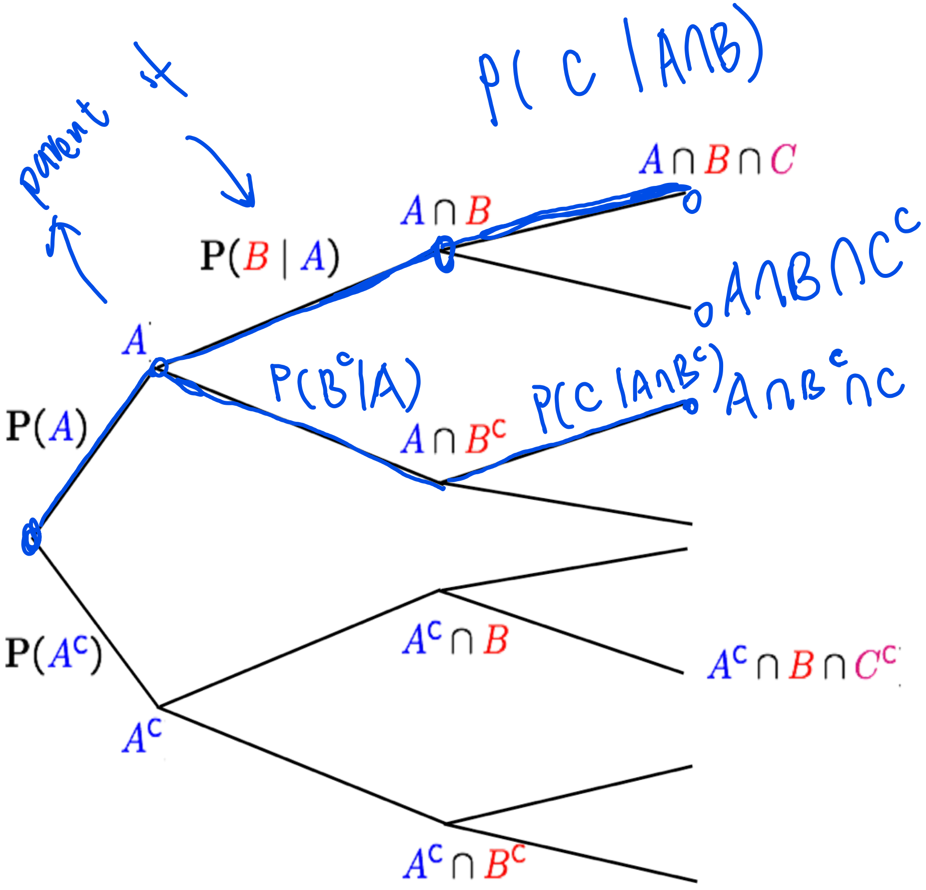
? $P(E_A|E_D) = \frac{1}{3}$

$$= \frac{P(E_D|E_A)\,P(E_A)}{P(E_D)} = \frac{\frac{1}{2} \times \frac{2}{5}}{\frac{3}{5}} = \frac{1}{3}$$

Chain Rule

$P(A \cap B) = P(B|A)P(A)$

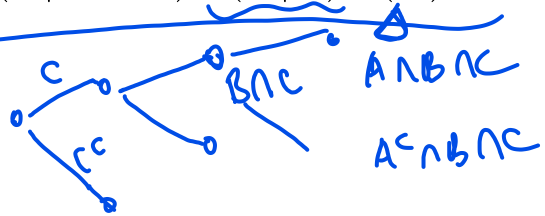
- Generalization



$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$
$$= \cancel{P(A)} \cdot \frac{P(\cancel{A} \cap B)}{\cancel{P(A)}} \cdot \frac{P(A \cap B \cap C)}{\cancel{P(A \cap B)}}$$

Similarly,

$$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$$

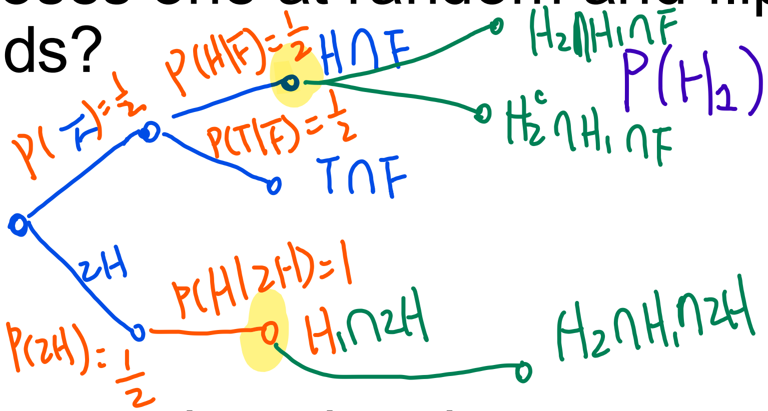


Generalization $P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)\ldots P\left(A_n| \cap_{i=1}^{n-1} A_i\right)$

Example

$$P(H_2 \cap H_1) = P(H_2 \cap H_1 \cap F) + P(H_2 \cap H_1 \cap 2H)$$
$$= P(H_2 \mid H_1 \cap F) \cdot P(H_1 \mid F)\, P(F) + P(H_2 \mid H_1 \cap 2H)\, P(H_1 \mid 2H)\, P(2H)$$
$$= \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} + 1 \times 1 \times \tfrac{1}{2} =$$

- A magician has in her pocket a fair coin and a two-headed coin. She chooses one at random and flips it. What is the probability that the result is heads?



$$P(H_2) = P(H \cap F) + P(H \cap F^c)$$
$$= P(H \mid F)\, P(F) + P(H \mid 2H)\, P(2H)$$
$$= \tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{2} \times 1 = \tfrac{3}{4}$$

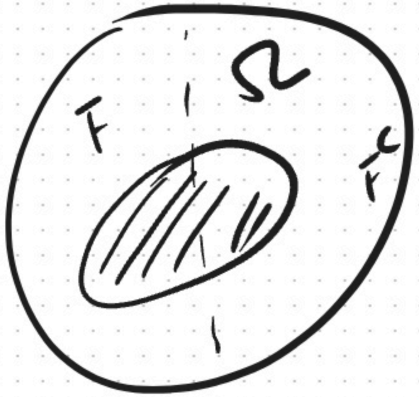
- Suppose that she chooses a coin at random. Using that coin, she flips it once and observes heads. What is the probability of observing heads in the second flip (using the same coin) if we observed heads in the first flip?

H_1: observe heads in the first flip.

$$P(H_2 \mid H_1) = P(H_2 \mid H_1 \cap F)\, \underbrace{\cancel{P(F)}}_{P(F \mid H_1)} + P(H_2 \mid H_1 \cap 2H)\, \underbrace{\cancel{P(2H)}}_{P(2H \mid H_1)}$$

$$P(H_2 \cap H_1) = P(H_2 \cap H_1 \mid F)\,P(F)$$

$$+ P(H_2 \cap H_1 \mid 2H)\,P(2H)$$



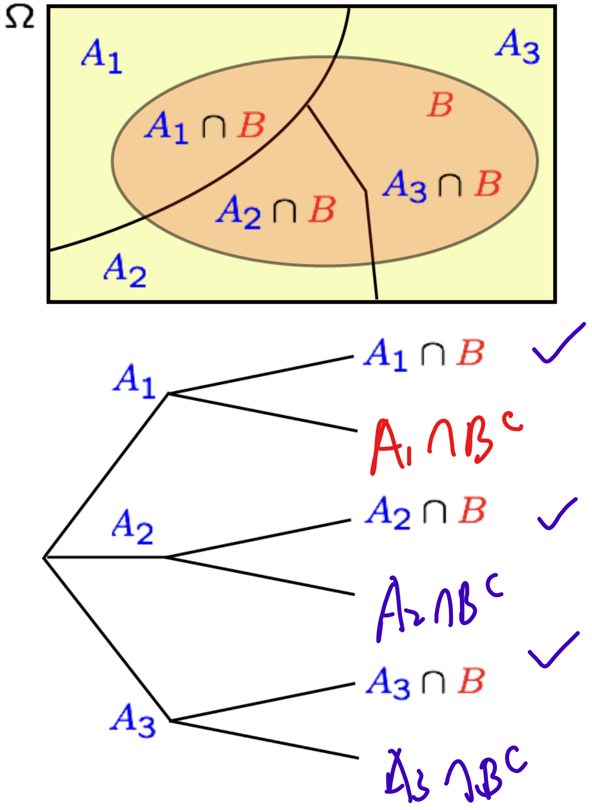
$$= \left(\frac{1}{2}\right)^2 \times \frac{1}{2} + 1 \times \frac{1}{2}$$

$$= 0.125 + 0.5 = 0.625$$

$$P(H_2 \mid H_1) = \frac{P(H_1 \cap H_2)}{P(H_1)}$$

$$= \frac{0.625}{0.75}$$

Total Probability Theorem (the Law of Total Probability)



- A set of events partitioning the sample space.
  - Having $P(A_i), i = 1, 2, 3.$
  - Having $P(B|A_i),\ i = 1, 2, 3.$

$$P(B)? = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

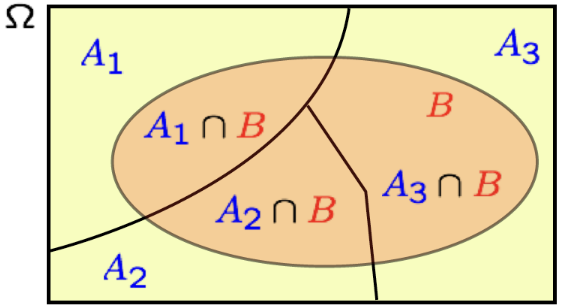
$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$$

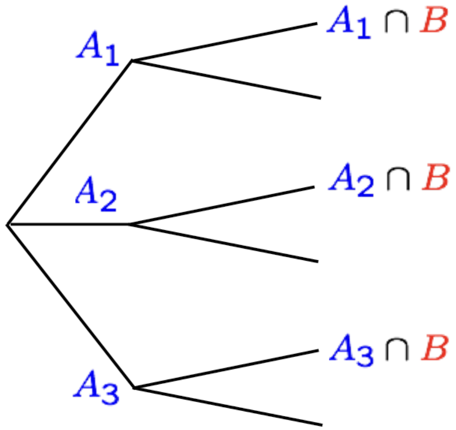
$$+ P(B|A_3)P(A_3)$$

Total Probability Theorem (the Law of Total Probability)

- A set of events partitioning the sample space.
  - Having $P(A_i), i = 1, 2, 3.$
  - Having $P(B|A_i),\ i = 1, 2, 3.$

$P(B)$?





Total Probability Law:
if the set of events $\{A_i\}$ partitions Ω, then

$$P(B) = \sum_i P(B|A_i)P(A_i)$$

Example

You have **two urns**:

1. **Urn A** contains 3 red balls and 2 blue balls.
2. **Urn B** contains 1 red ball and 4 blue balls.

A fair coin is flipped to decide which urn to choose:

• If heads, you select **Urn A**.

• If tails, you select **Urn B**.

After choosing the urn, a ball is drawn at random.

Question:

- What is the probability of drawing a red ball?