Lecture 9: Estimation

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EEL 3850

Motivating problem

X : height of 3rd students in population

$\mathbb{E}(X) = \mu_X$

- Estimating the average height of 3rd grade students in schools

Data sample 1:

53.49	51.59	53.94	56.57	51.30	51.30
56.74	54.30	50.59	53.63	50.61	50.60
52.73	46.26	46.83	50.31	48.96	52.94
49.28	47.76	56.40	51.32	52.20	47.73

Average = 51.51 inches

$6 \times 4 = 24$ students.

Data sample 2:

52.33	48.55	53.13	50.20	51.12	50.19
57.56	51.96	48.83	54.47	48.34	52.63
46.12	48.02	52.59	54.22	52.51	51.65
51.10	47.56	49.84	50.62	55.17	53.03

Average = 52.56 inches

X_i: ith 3rd grader's height. $\mu_X = \mathbb{E}(X_i) \quad \forall i$

$$M_n = M_{24} = \frac{1}{24}\sum_{i=1}^{24} X_i$$

Classical inference

- unknown parameter θ as a deterministic (not random!) but unknown quantity.
  - Average height.

X_i $\qquad \mu_X$ = Average height.

"constant" that we don't know.

- The observation X is random and its distribution
  - $p_X(x;\theta)$ if X is discrete
  - $f_X(x;\theta)$ if X is continuous
    - depends on the value of θ.

$X_i \sim N(\mu_X, \sigma_X^2)$: $\theta := (\mu_X, \sigma_X)$

parameters used to define the distribution.

$B \sim \text{Bernoulli}(p)$. $\qquad \theta := p$

Classical inference

$$\underbrace{\theta \to f_X(x;\theta)}_{\text{discrete R.V. / continous R.V. / complex}} \xrightarrow{x_1, x_2, \ldots, x_n} \text{Estimator} \to \underbrace{\hat{\theta}}_{\text{estimated value for } \theta}$$

- Given observations $X = (X_1, \ldots, X_n)$, an estimator $\widehat{\Theta} = g(X)$ is function of X.
- Thus, $\widehat{\Theta}$ is a random variable.
- Let n be the number of observations, the mean and variance of $\widehat{\Theta}_n$ are denoted $E_\theta[\widehat{\Theta}_n]$ and $var_\theta[\widehat{\Theta}_n]$, respectively.

X: mean, variance, [interval estimate] that contains the mean

Terminology regarding estimators

output of estimator

Random variable (e.g. M_n)

- The underlying parameter θ to be estimated is a constant.
- Estimation error: $\tilde{\Theta}_n = \hat{\Theta}_n - \theta$.
- Bias of the estimator: $b_\theta(\hat{\Theta}_n) = E_\theta[\hat{\Theta}_n] - \theta$, is the expected value of the estimation error. $\| \; \mathbb{E}(\tilde{\Theta}_n)$

e.g. $\mathbb{E}(M_n) - M_X = 0$

Estimation of the Mean

unknown R.V. X ; μ_X mean

"sample Average"

- Suppose that the observations $X_1, \ldots, X_n$ are i.i.d., with an unknown common mean μ_X.
- M_n = $\hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$ is unbiased estimator
  - for any n, the expected value of the average is equal to the true mean.

```
heights= np.array([121.92, 132.64, 113.31, 97.20, 140.94, 139.04, 115.98, 128.27, 121.84, 97.73])
```

The average height estimate

```
average = np.mean(heights)= 122.185
```

$$M_n \sim N\left(\mu_X, \frac{\sigma_X^2}{n}\right)$$

$$M_n = \hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$(X_1, X_2, \ldots X_n) \longrightarrow \boxed{\text{Estimator}} \longrightarrow \hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Properties of the Estimator of the mean

- $\hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$ is unbiased estimator
  - for any n, the expected value of the average is equal to the true mean.

$$\text{bias}(\hat{\mu}_X) = \mathbb{E}(\hat{\mu}_X - \mu_X)$$

constant.

$$= \mathbb{E}(\hat{\mu}_X) - \underbrace{\mathbb{E}(\mu_X)}$$

$$= \mathbb{E}(\hat{\mu}_X) - \mu_X = \mu_X - \mu_X = 0$$

$$\hat{\mu}_X \sim N\left(\mu_X, \frac{\sigma_X^2}{n}\right)$$

Average error $\Rightarrow 0$

Estimating the variance $X.$ $\mathrm{Var}(X) = \mathbb{E}[(X-\mu_X)^2]$

Let σ_X^2 denote the variance of the random variables. Then there are two cases that should be considered for estimating the variance.

Known mean: If the mean of the random variables, μ_X, is known. Let the sample-variance estimator for this case be defined by

$\mathbb{E}(g(X)) = \frac{1}{N}\sum_{i=1}^{N} g(x_i)$

$$\hat{\sigma_X^2} = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu_X)^2.$$

Constant

sample Approximation

$$\mathbb{E}[(X-\mu_X)^2] \approx \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_X)^2$$

sample set $\{x_1, x_2, \ldots x_N\}$.

Estimating the variance

- Let's first determine if the sample variance estimator is biased when the true mean is known: (*experiment validate*)

error

$$\text{bias}(\hat{\sigma}_x^2) = \underline{\mathbb{E}(\hat{\sigma}_x^2} - \sigma_x^2) = 0$$

proof.

$$\mathbb{E}(\hat{\sigma}_x^2) = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}(X_i - \mu_x)^2\right) \longrightarrow \sigma_x^2$$

Estimating the variance

Known mean: If the mean of the random variables, μ_X, is known. Let the sample-variance estimator for this case be defined by

$$\hat{\sigma}_X^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu_X)^2.$$

Unknown mean: it is natural to replace μ_X with its sample estimate $\hat{\mu}_X$:

$$\hat{\sigma}_X^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \hat{\mu}_X)^2.$$

See the ADDITIONAL PDF for a proof why the variance estimator above is biased.

Estimating the varaince

- Let's first determine if the sample variance estimator is biased when we replace the true mean with its sample estimate: (*experiment validate*)

Estimating the varaince

Known mean: If the mean of the random variables, μ_X, is known. Let the sample-variance estimator for this case be defined by

$$\hat{\sigma_X^2} = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu_X)^2.$$

Unknown mean: unbiased estimator:

$$\hat{\sigma_X^2} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \hat{\mu}_X)^2.$$

The change in denominator is often referred to as a *degrees-of-freedom (dof) correction*.

Example

- Suppose we have a sample of student scores from an exam, and we want to estimate the population mean score.
- Sample data: array([84, 78, 71, 74, 60, 76, 50, 86, 67, 82, 89, 93, 79, 72, 78, 76, 71, 85, 86, 52, 61, 63, 92, 71, 80, 60, 76, 81, 57, 88])
- Total 30 samples
- Using the same sample data, we want to estimate the population variance.

Point estimate vs interval estimate

- Instead of estimating a single value, an interval estimate is also used:

- For an unknown parameter

$$\theta \rightarrow f_X(x;\theta) \xrightarrow{x_1, x_2, \ldots, x_n} \text{Interval Estimator} \rightarrow [\hat{\theta}^-, \hat{\theta}^+]$$

An interval contain the unknown parameter with high probability

Confidence intervals (CIs)

- The value of an estimator may not be informative enough
- Let us first fix a desired confidence level, $1-\alpha$, where α is typically a small number.
- We then replace the point estimator $\widehat{\Theta}_n$ by a lower estimator $\widehat{\Theta}_n^-$ and an upper estimator $\widehat{\Theta}_n^+$, s.t.

$$P\left(\widehat{\Theta}_n^- \leq \theta \leq \widehat{\Theta}_n^+\right) \geq 1-\alpha$$

  for every possible value of θ.

- We call $\left[\widehat{\Theta}_n^-, \widehat{\Theta}_n^+\right]$ a $(1-\alpha)$ confidence interval.

Confidence intervals (CIs)

- $\hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i$
- Recall: the observations $X_1, \ldots, X_n$ are i.i.d., with an unknown common mean μ_X

$$\hat{\mu}_X \sim \mathcal{N}(\mu_X, \frac{\sigma^2}{n})$$

- Recall CLT:

$$\text{We call } [\hat{\mu}_X^-, \hat{\mu}_X^+] \text{ a } (1-\alpha) \text{ confidence interval if}$$

$$P(\hat{\mu}_X^- \leq \mu_X \leq \hat{\mu}_X^+) > 1 - \alpha$$

Confidence intervals (CIs)

- Suppose $\alpha = 0.05$
- Let's compute the 95% confidence interval about the mean of unknown RV using the samples.

https://www.mathsisfun.com/data/standard-normal-distribution-table.html

Confidence interval for mean estimate with unknown variance

- Recall if the variance is unknown, we have an unbiased estimate for the variance

Unknown mean: unbiased estimator:

$$\hat{\sigma_X^2} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \hat{\mu}_X)^2.$$

$$\frac{\hat{\mu}_X - \mu_X}{\sigma_X/\sqrt{n}}$$

has a Student's t-distribution with $\nu = n - 1$ degrees of freedom (dof) t_ν.
(Like a Gaussian, but more spread out!)

Summary

1. Point Estimation for Mean with prior knowledge of the population variance

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

2. Standard Error of the Mean with known population variance (not an estimate but can be computed based on the property of variance.)

$$SE = \frac{\sigma}{\sqrt{n}}$$

3. Point Estimation for Mean without prior knowledge of the population variance

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

4. Point Estimation for Variance without prior knowledge of the population variance

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

summary

- Formula to compute confidence interval

For a population mean μ when the sample size is large ($n \geq 30$), the confidence interval is given by:

$$\left(\hat{\mu}_X - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad \hat{\mu}_X + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \tag{1}$$

where:

- $\hat{\mu}_X$ = sample mean
- σ = population standard deviation (or sample standard deviation if unknown)
- n = sample size
- $z_{\alpha/2}$ = critical value from the standard normal table for a given confidence level