Question : Decision rule -2

- How to select k^* that minimize the prob of error

$P(E) = P(H_1 \cap \hat{H}_0) + P(H_0 \cap \hat{H}_1)$

where E: error event.

$\hat{H}_0$: decide $\hat{H}_0$

$\hat{H}_1$: decide $\hat{H}_1$

let X_0: # number of heads of H_0.

X_1: number of heads of H_1.

$P(H_1 \cap \hat{H}_0) = P(\hat{H}_0 | H_1) P(H_1)$

$= P(X_1 \geq K) \cdot P(H_1)$

$P(H_0 \cap \hat{H}_1) = P(\hat{H}_1 | H_0) P(H_0)$

$= P(X_0 < K) P(H_0)$

(see implementation) using cdf of Binomial RV.

observation: B_0, B_1, B_2

☆ Additionally: $P(\hat{H}_0 | H_1) = P(\hat{H}_0 \cap B_0 | H_1) + P(\hat{H}_0 \cap B_1 | H_1) + P(\hat{H}_0 \cap B_2 | H_1) = P(\hat{H}_0 | \underline{B_0} \cap H_1) \underline{P(B_0 | H_1)}$

☆ $P(A \cap B | C) = P(A | B \cap C) P(B \cap C) / P(C)$ $\quad P(X_1 \geq K)$

Question 2: $\mathbb{E}(X) = \sum_{i=1}^{3} x_i P(x_i) = (-2) \times \frac{1}{3} + 1 \times \frac{1}{2} + (-1) \times \frac{1}{6}$

$= -\frac{1}{3}.$

$\mathbb{E}(X^2) = \sum_{i=1}^{3} x_i^2 P(x_i) = (-2)^2 \cdot \frac{1}{3} + 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{6}$

$= 2$

$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2 - \frac{1}{9} = \frac{17}{9}.$

$\mathbb{E}(2X+5) = 2\mathbb{E}(X) + 5 = 2 \times (-\frac{1}{3}) + 5 = \frac{13}{3}$

$\mathbb{E}(3X^2+6) = 3\,\mathbb{E}(X^2) + 6 = 3 \times 2 + 6 = 12$

$Var(2X+5) = 4Var(X).$ $STD(2X+5) = 2\,STD(X)$

Question 2. (RV.)

X: the number of cars being repaired.

range of X: $[0, 3]$

$\left.\begin{array}{l} P(X=2) = P(X=1) = P_1 \\ P(X=3) = P(X=0) = P_0 \end{array}\right\} \; 2P_1 + 2P_0 = 1$

$\Rightarrow P_1 + P_0 = \frac{1}{2}$ ①

$P(X=1 \vee X=2) = \frac{1}{2} P(X=0 \vee X=3)$

$P(X=1) + P(X=2) - \underbrace{P(X=1 \wedge X=2)}^{=0}$

$= \frac{1}{2}[P(X=2) + P(X=0) - \underbrace{P(X=3 \wedge X=0)}^{=0}$

$$2P_1 = \frac{1}{2}\cdot 2P_0 \Rightarrow 2P_1 = P_0 \quad ②$$

combine ① and ②

$$P_1 = \frac{1}{6}. \qquad P_0 = \frac{1}{3}.$$

$$\therefore P(X=1) = P(X=2) = \frac{1}{6}$$

$$P(X=3) = P(X=0) = \frac{1}{3}.$$

Question 3

$$Z = X - Y.$$

Y \ X	1	2	3	4	5	6
6	-5	-4	-3	-2	-1	0
5	-4	-3	-2	-1	0	1
4	-3	-2	-1	0	1	2
3	-2	-1	0	1	2	3
2	-1	0	1	2	3	4
1	0	1	2	3	4	5

$$P(Z=-1) = \frac{5}{36} = P(Z=1)$$

$$P(Z=-2) = \frac{4}{36} = P(Z=2)$$

$$P(Z=-3) = \frac{3}{36} = P(Z=3)$$

$$P(Z=-4) = \frac{2}{36} = P(Z=4)$$

$$P(Z=5) = \frac{1}{36} = P(Z=-5)$$

$$P(Z=0) = \frac{6}{36}.$$

implementation omitted.

Question 7:

$$P(X<1) = \int_{-5}^{1} \frac{1}{10}\,dx$$

$$= \frac{6}{10}$$

$$P(|X-1| \geq 2) = P(X-1 > 2) + P(1-X \geq 2)$$

$$= P(X>3) + P(X<-1)$$

$$= \frac{2}{10} + \frac{4}{10} = \frac{6}{10}$$

Question 8. (Removed, incorrect problem statement)

Q12:

① $E(X_1 + \cdots X_n - (Y_1 + \cdots + Y_m))$

$$= n\mu_1 - m\mu_2$$

$$Var(X_1 + \cdots + X_n - (Y_1 + \cdots + Y_m))$$

$$= \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{m} Var(Y_i)\cdot(-1)^2$$

$= n \cdot \sigma_1^2 + m \sigma_2^2$

②: $\mathbb{E}(\bar{X}-\bar{Y}) = \mu_1 - \mu_2$; $Var(\bar{X}-\bar{Y}) = Var(\bar{X}) + Var(\bar{Y})$

$= \frac{1}{n^2} \cdot n\sigma_1^2 + \frac{1}{m^2} \cdot m\sigma_2^2$

$= \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}$.

Question. CLT:

$$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

λ: expected number of events in a given interval.

$\lambda = 10$ per day $\therefore \mathbb{E}(X) = \lambda = Var$

$Y = X_1 + \cdots + X_{365}$ is the total number of accident.

$Y \sim N(365 \times \mathbb{E}(X); 365 \times Var(X))$

$N(3650; 3650)$

$\therefore P(Y > 3850) = Q\left(\frac{3850-3650}{\sqrt{3650}}\right)$

$\doteq 0.0004$

Question 2.

$$X_i \sim \text{Bernoulli}(0.1) \qquad \mathbb{E}(X_i) = 0.1$$
$$\text{Var}(X_i) = 0.09$$

$Y = X_1 + \cdots + X_{1000}$ is the total number of errors.

$$\text{Var}(Y) = n\,\text{var}(X_i) = 0.09 \times 1000 = 90$$

$$P(\text{decoding failure}) = P(Y > 125)$$

$$= Q\left(\frac{125 - 100}{\sqrt{90}}\right) = 0.0042.$$

in pythod simulation. obtain similar results.

Question 3: Coin $X_i \sim \text{Bernoulli}(0.5)$

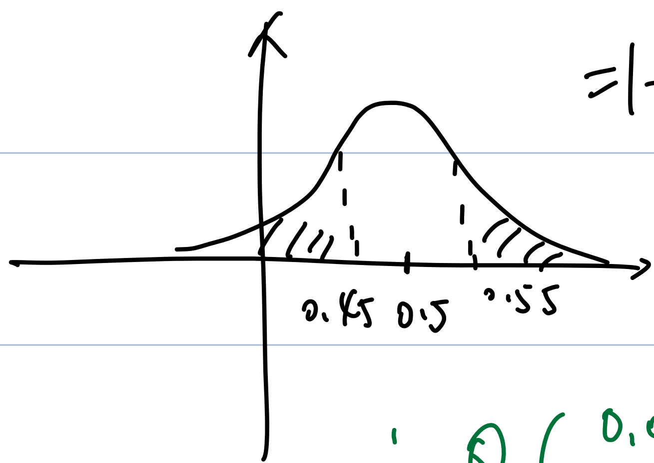
$\mathbb{E}(X) = 0.5$. $\quad Var(X) = 0.25$.

$$Y = \frac{X_1 + \cdots + X_n}{n} \quad : \mathbb{E}(Y) = 0.5. \quad Var(Y) = \frac{Var(X)}{n}$$

$$P(0.45 \le Y \le 0.55) > 0.95$$

$$RHS = P(Y \geq 0.45) - P(Y \geq 0.55)$$

$$= 1 - 2 \times P(Y \geq 0.55) \quad \text{Due to symmetry.}$$



$$= 1 - 2 \times Q\left(\frac{0.55 - 0.5}{\sqrt{0.25/n}}\right) > 0.95$$

(Q is the survival function of standard Gaussian)

$$\therefore Q\left(\frac{0.05\sqrt{n}}{0.5}\right) < 0.025$$

using inverse of Q.

$$\frac{0.05\sqrt{n}}{0.5} > Q^{-1}(0.025)$$

$$\Rightarrow \sqrt{n} > \frac{Q^{-1}(0.025) \times 0.5}{0.05} \approx 19.5$$

$$\therefore n \geq 19.5^2 \approx 385.$$

CLT-4: $X_i \sim \text{Bernoulli}(0.9)$

$$\mu = E(X_i) = 0.9$$

$$\text{var}(X_i) = 0.9 \times 0.1 = 0.09.$$

$$S_{100} = \sum_{i=1}^{100} X_i \sim N(0.9 \times 100,\ 0.09 \times 100)$$

$$= N(90, 9)$$

$$P(S_{100} \le 80) = P\left(\frac{S_{100} - 90}{3} \le \frac{80 - 90}{3}\right)$$

$$= P\left(Z \le -\frac{10}{3}\right)$$

Z : standard normal RV.

$$= Q(-3.33)$$

$$\approx 0.0004$$

$P(S_{100} \le 80) < \alpha = 0.01$

So. reject the company's statement.

programming. 1,

$$Pr(P \mid Z) = \frac{P(Z \mid P)\, P(P)}{P(Z)}$$