

Lecture 11: Vectors, matrices, and tensors

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EEL 3850

Vector: An $n \times 1$ matrix.

$$y = \begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}$$

$y_i = i^{th}$ element

1-indexed vs 0-indexed:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Matrix: Rectangular array of numbers:

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

Dimension of matrix: number of rows x number of columns

$$\text{Dim}(A) = 4 \times 2$$

Matrix Elements (entries of matrix)

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

$A_{ij} =$ “ i, j entry” in the i^{th} row, j^{th} column.

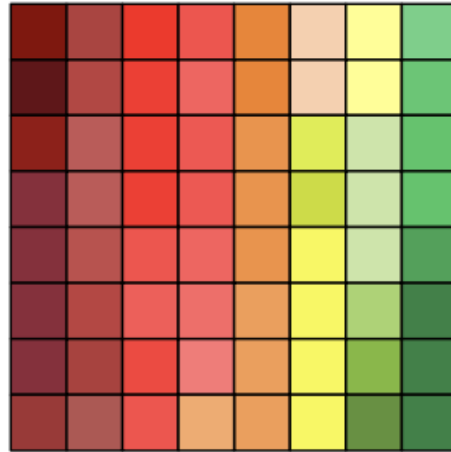
Tensor: High-dimensional array

Vector



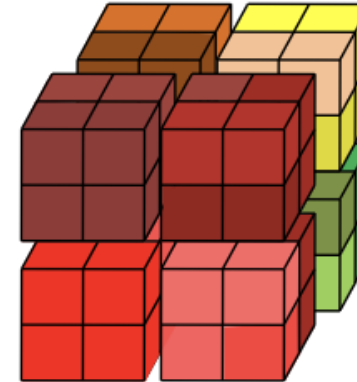
$$\mathbb{R}^{64}$$

Matrix



$$\mathbb{R}^{8 \times 8}$$

Tensor



$$\mathbb{R}^{4 \times 4 \times 4}$$

Basic algebra

Transposing a column vector results in a row vector, and vice versa

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow V^T =$$

Vector-Vector Product

“Inner product” or “dot product”

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^n x_i y_i.$$

$$\left\langle \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} e \\ f \\ g \end{bmatrix} \right\rangle = [a \quad b \quad c] \cdot \begin{bmatrix} e \\ f \\ g \end{bmatrix} = a \cdot e + b \cdot f + c \cdot g$$

$$\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 2 \\ 8 \end{bmatrix} = 2 \cdot 8 + 7 \cdot 2 + 1 \cdot 8 = 38$$

Dot product

Usages of inner products

0.1 L2-norm: the intuitive notion of length of the vector

The L_2 norm (also known as the Euclidean norm) of a vector $\mathbf{x} \in \mathbb{R}^n$ is defined as:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

0.2 Squared error computation

The squared error between a sample $\{x_1, x_2, \dots, x_n\}$ and its mean is given by:

$$SE = \sum_{i=1}^n (x_i - \mu_x)^2 = |\vec{x} - \mu_x \mathbf{1}|^2.$$

Geometric interpretation of inner product

An alternative expression using vector norms and the angle θ between the two vectors is:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

where:

- $\|\mathbf{a}\| = \sqrt{\sum_{i=1}^n a_i^2}$ is the norm of \mathbf{a} .
- $\|\mathbf{b}\| = \sqrt{\sum_{i=1}^n b_i^2}$ is the norm of \mathbf{b} .
- θ is the angle between the two vectors.

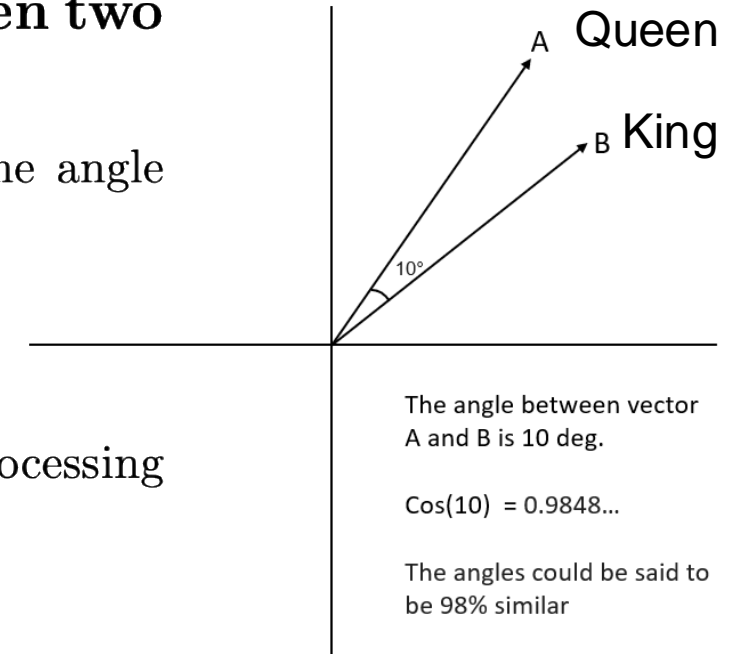
Usage of inner product

0.1 Cosine Similarity: measure the similarity between two vectors

Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the cosine similarity is the cosine of the angle between the vectors, and can be computed as:

$$\text{cosine similarity} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

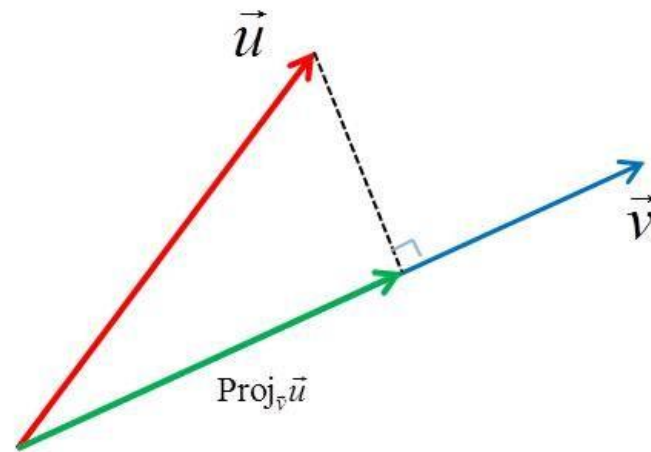
This is commonly used in text mining and natural language processing (NLP) to measure the similarity between document embeddings.



Projection

- To “project” a vector \mathbf{u} onto \mathbf{v} (basically, rotate a vector and place it on a second vector);

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Cauchy-Schwarz inequality

Cauchy-Schwarz Inequality:

For any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the inner product satisfies:

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

In component form, for vectors $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$:

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2}$$

Scalar Multiplication

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

then for any scalar 'k'

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}_{m \times n}$$

$$3 \times \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} =$$

Matrix Addition is element-wise

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \end{bmatrix} =$$

Matrix transpose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3 \times 2}$$

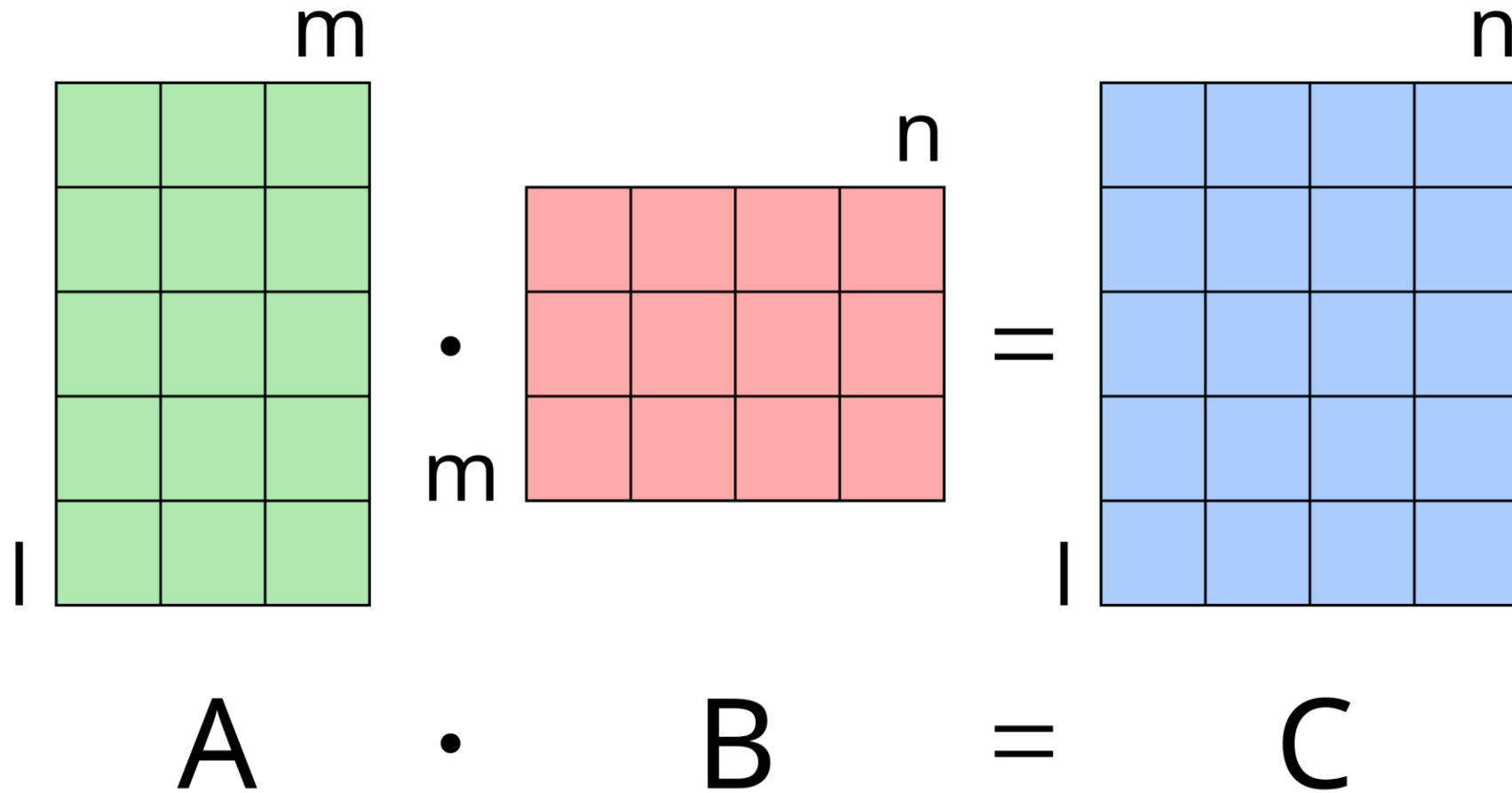
$$(A^T)_{ij} = A_{ji}$$

Matrix-vector multiplication

$$\begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \times \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} A \times G + B \times H \\ C \times G + D \times H \\ E \times G + F \times H \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} =$$

Matrix-Matrix Product

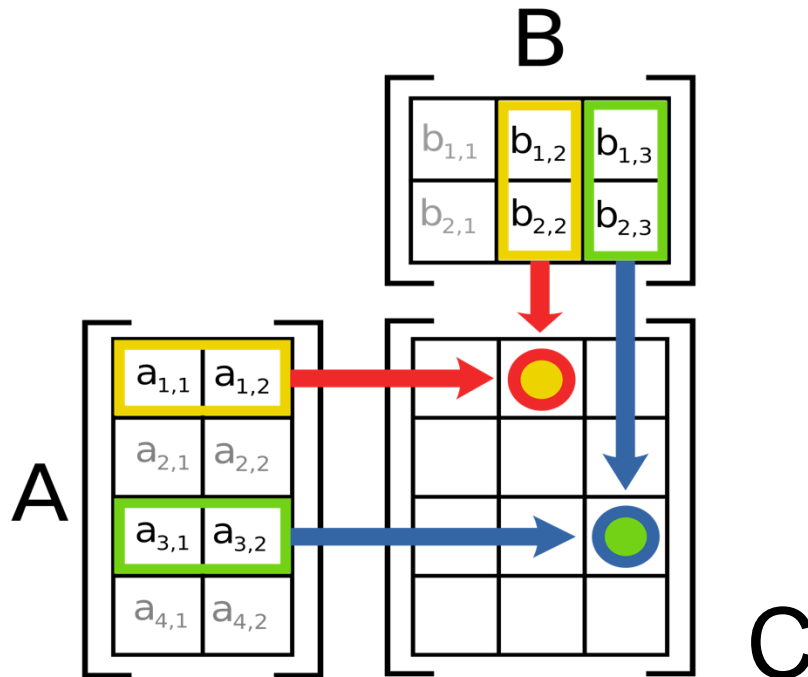


Matrix-Matrix Product

$$\begin{array}{c} A \quad B \quad C \\ \begin{array}{|c|c|c|} \hline \text{red} & \text{green} & \text{blue} \\ \hline \text{red} & \text{green} & \text{blue} \\ \hline \text{red} & \text{green} & \text{blue} \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline a & d & g \\ \hline b & e & h \\ \hline c & f & i \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \text{light pink} & \text{pink} & \text{dark pink} \\ \hline \text{light pink} & \text{pink} & \text{dark pink} \\ \hline \text{light pink} & \text{pink} & \text{dark pink} \\ \hline \end{array} \\ \\ \begin{array}{|c|} \hline \text{light pink} \\ \hline \text{light pink} \\ \hline \text{light pink} \\ \hline \end{array} = \begin{array}{|c|} \hline a \\ \hline b \\ \hline c \\ \hline \end{array} \begin{array}{|c|} \hline \text{red} \\ \hline \text{red} \\ \hline \text{red} \\ \hline \end{array} + \begin{array}{|c|} \hline b \\ \hline e \\ \hline f \\ \hline \end{array} \begin{array}{|c|} \hline \text{green} \\ \hline \text{green} \\ \hline \text{green} \\ \hline \end{array} + \begin{array}{|c|} \hline c \\ \hline \text{blue} \\ \hline \text{blue} \\ \hline \end{array} \\ \\ \begin{array}{|c|} \hline \text{pink} \\ \hline \text{pink} \\ \hline \text{pink} \\ \hline \end{array} = \begin{array}{|c|} \hline d \\ \hline e \\ \hline f \\ \hline \end{array} \begin{array}{|c|} \hline \text{red} \\ \hline \text{red} \\ \hline \text{red} \\ \hline \end{array} + \begin{array}{|c|} \hline e \\ \hline \text{green} \\ \hline \text{green} \\ \hline \end{array} + \begin{array}{|c|} \hline f \\ \hline \text{blue} \\ \hline \text{blue} \\ \hline \end{array} \\ \\ \begin{array}{|c|} \hline \text{dark pink} \\ \hline \text{dark pink} \\ \hline \text{dark pink} \\ \hline \end{array} = \begin{array}{|c|} \hline g \\ \hline h \\ \hline i \\ \hline \end{array} \begin{array}{|c|} \hline \text{red} \\ \hline \text{red} \\ \hline \text{red} \\ \hline \end{array} + \begin{array}{|c|} \hline h \\ \hline \text{green} \\ \hline \text{green} \\ \hline \end{array} + \begin{array}{|c|} \hline i \\ \hline \text{blue} \\ \hline \text{blue} \\ \hline \end{array}\end{array}$$

The i -th column of the matrix \mathbf{C} is obtained by multiplying \mathbf{A} with the i column of \mathbf{B}

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \text{---} & \mathbf{a}_1^\top & \text{---} \\ \text{---} & \mathbf{a}_2^\top & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_m^\top & \text{---} \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{b}_1 & \mathbf{a}_1^\top \mathbf{b}_2 & \cdots & \mathbf{a}_1^\top \mathbf{b}_p \\ \mathbf{a}_2^\top \mathbf{b}_1 & \mathbf{a}_2^\top \mathbf{b}_2 & \cdots & \mathbf{a}_2^\top \mathbf{b}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_m^\top \mathbf{b}_1 & \mathbf{a}_m^\top \mathbf{b}_2 & \cdots & \mathbf{a}_m^\top \mathbf{b}_p \end{bmatrix}$$



The i, j -th element of the matrix \mathbf{C} is the inner product of the i -th row of \mathbf{A} with the j -th column of \mathbf{B}

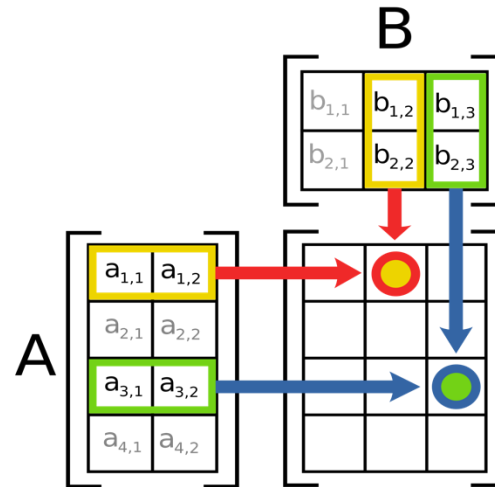
Example

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

A and ***B*** must have conforming dimensions!



$$\text{Dim}(A) = m \times n$$

$$\text{Dim}(B) = n \times p$$

$$\text{Dim}(AB) = m \times p$$