

Question : Decision rule -2

- How to select k^* that minimize the prob of error

$$P(E) = P(H_1 \cap \hat{H}_0) + P(H_0 \cap \hat{H}_1)$$

where E : error event.

\hat{H}_0 : decide H_0

\hat{H}_1 : decide H_1

let X_0 : # number of heads of H_0 .

X_1 : number of heads of H_1 .

$$P(H_1 \cap \hat{H}_0) = P(\hat{H}_0 | H_1) P(H_1)$$

$$= P(X_1 \geq k) \cdot P(H_1)$$

$$P(H_0 \cap \hat{H}_1) = P(\hat{H}_1 | H_0) P(H_0)$$

$$= P(X_0 \leq k) P(H_0)$$

(see implementation) using
cdf of Binomial RV.

observation: B_0, B_1, B_2

★ Additionally: $P(\hat{H}_0 | H_1) = P(\hat{H}_0 \cap B_0 | H_1) + P(\hat{H}_0 \cap B_1 | H_1)$

$$+ P(\hat{H}_0 \cap B_2 | H_1) = P(\hat{H}_0 | \underline{B_0 \cap H_1}) \underline{P(B_0 | H_1)}$$

★ $P(A \cap B | C) = P(A | B \cap C) P(B | C) / \underline{P(C)}$ $P(X_0 \geq k)$

Question 2: $E(X) = \sum_{i=1}^3 x_i P(x_i) = (-2) \times \frac{1}{3} + 1 \times \frac{1}{2} + (-1) \times \frac{1}{6}$
 $= -\frac{1}{3}.$

$$E(X^2) = \sum_{i=1}^3 x_i^2 P(x_i) = (-2)^2 \cdot \frac{1}{3} + 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{6}$$

$$= 2$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 2 - \frac{1}{9} = \frac{17}{9}.$$

$$E(2X+5) = 2E(X) + 5 = 2 \times \left(-\frac{1}{3}\right) + 5 = \frac{-13}{5}$$

$$E(3X^2+6) = 3E(X^2) + 6 = 3 \times 2 + 6 = 12$$

$$\text{Var}(2X+5) = 4\text{Var}(X). \quad \text{STD}(2X+5) = 2\text{STD}(X)$$

Question 2. (RV.)

X : the number of cars being repaired.
 range of X : $[0, 3]$

$$\left. \begin{array}{l} P(X=2) = P(X=1) = p_1 \\ P(X=3) = P(X=0) = p_0 \end{array} \right\} \begin{array}{l} 2p_1 + 2p_0 = 1 \\ \Rightarrow p_1 + p_0 = \frac{1}{2} \end{array} \quad (1)$$

$$P(X=1 \vee X=2) = \frac{1}{2} P(X=0 \vee X=3)$$

$$P(X=1) + P(X=2) - \underbrace{P(X=1 \wedge X=2)}_{=0}$$

$$= \frac{1}{2} [P(X=2) + P(X=0) - \underbrace{P(X=3 \wedge X=0)}_{=0}]$$

$$2P_1 = \frac{1}{2} \cdot 2P_0 \Rightarrow 2P_1 = P_0 \quad (2)$$

combine (1) and (2)

$$P_1 = \frac{1}{6}, \quad P_0 = \frac{1}{3}.$$

$$\therefore P(X=1) = P(X=2) = \frac{1}{6}$$

$$P(X=3) = P(X=0) = \frac{1}{3}.$$

Question 3

$$Z = X - Y.$$

Y \ X	1	2	3	4	5	6
6	-5	-4	-3	-2	-1	0
5	-4	-3	-2	-1	0	1
4	-3	-2	-1	0	1	2
3	-2	-1	0	1	2	3
2	-1	0	1	2	3	4
1	0	1	2	3	4	5

$$P(Z=-1) = \frac{5}{36} = P(Z=1)$$

$$P(Z=-2) = \frac{4}{36} = P(Z=2)$$

$$P(Z=-3) = \frac{3}{36} = P(Z=3)$$

$$P(Z=-4) = \frac{2}{36} = P(Z=4)$$

$$P(Z=-5) = \frac{1}{36} = P(Z=5)$$

$$P(Z=0) = \frac{6}{36}.$$

implementation omitted.

Question 7:

$$P(X < 1) = \int_{-5}^1 \frac{1}{10} dx$$
$$= \frac{6}{10}$$

$$P(|X-1| \geq 2) = P(X-1 \geq 2) + P(1-X \geq 2)$$
$$= P(X \geq 3) + P(X < -1)$$
$$= \frac{2}{10} + \frac{4}{10} = \frac{6}{10}$$

Question 8. (Removed. incorrect problem statement)

Q12:

$$\textcircled{1} \quad E(X_1 + \dots + X_n - (Y_1 + \dots + Y_m))$$
$$= n\mu_1 - m\mu_2$$

$$\text{Var}(X_1 + \dots + X_n - (Y_1 + \dots + Y_m))$$
$$= \sum_{i=1}^n \text{Var}(X_i) + \sum_{i=1}^m \text{Var}(Y_i) \cdot (-1)^2$$

$$= n \cdot \sigma_1^2 + m \sigma_2^2$$

$$\textcircled{2}: E(\bar{X} - \bar{Y}) = \mu_1 - \mu_2; \quad \text{Var}(\bar{X} - \bar{Y}) = \text{Var}(\bar{X}) + \text{Var}(\bar{Y})$$

$$= \frac{1}{n^2} \cdot n \sigma_1^2 + \frac{1}{m^2} \cdot m \sigma_2^2$$

$$= \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}$$

Question . CLT:

$$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

λ : expected number of events in a given interval.

$$\lambda = 10 \text{ (per day)} \quad ; \quad E(X) = \lambda = 10$$

$$Y = X_1 + \dots + X_{365} \text{ is the}$$

total number of accidents.

$$Y \sim N(365 \times E(X), 365 \times \text{Var}(X))$$

$$N(3650, 3650)$$

$$\therefore P(Y > 3850) = Q\left(\frac{3850 - 3650}{\sqrt{3650}}\right)$$

$$\approx 0.0004$$

Question 2.

$$X_i \sim \text{Bernoulli}(0.1)$$

$$E(X_i) = 0.1$$

$$\text{Var}(X_i) = 0.09$$

$$Y = X_1 + \dots + X_{1000} \text{ is the total}$$

number of errors.

$$\text{Var}(Y) = n \text{Var}(X_i) = 0.09 \times 1000 = 90$$

$$P(\text{decoding failure}) = P(Y > 125)$$

$$= Q\left(\frac{125 - 100}{\sqrt{90}}\right) = 0.0042.$$

in python simulation. obtain similar results.

Question 3: Coin $X_i \sim \text{Bernoulli}(0.5)$

$$\mathbb{E}(X) = 0.5, \quad \text{Var}(X) = 0.25.$$

$$Y = \frac{X_1 + \dots + X_n}{n}$$

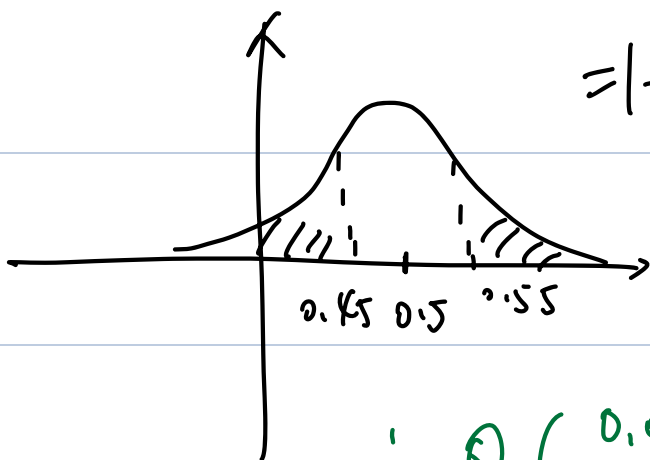
$$\therefore \mathbb{E}(Y) = 0.5.$$

$$\text{Var}(Y) = \frac{\text{Var}(X)}{n}$$

$$P(0.45 \leq Y \leq 0.55) > 0.95$$

$$\text{RHS} = P(Y \geq 0.45) - P(Y \geq 0.55)$$

$$= 1 - 2P(Y \geq 0.55) \quad \text{Due to symmetry.}$$



$$= 1 - 2Q\left(\frac{0.55 - 0.5}{\sqrt{0.25/n}}\right) > 0.95$$

(Q is the survival function of standard Gaussian)

$$\therefore Q\left(\frac{0.05\sqrt{n}}{0.5}\right) < 0.025$$

using inverse of Q .

$$\frac{0.05\sqrt{n}}{0.5} > Q^{-1}(0.025)$$

$$\Rightarrow \sqrt{n} > \frac{Q^{-1}(0.025) \times 0.5}{0.05} \approx 19.5$$

$$\therefore n \geq 19.5^2 \approx 385.$$

CLT-4: $X_i \sim \text{Bernoulli}(0.9)$

$$\mu = E(X_i) = 0.9$$

$$\text{Var}(X_i) = 0.9 \times 0.1 = 0.09$$

$$\begin{aligned} S_{100} = \sum_{i=1}^{100} X_i &\sim N(0.9 \times 100, 0.09 \times 100) \\ &= N(90, 9) \end{aligned}$$

$$P(S_{100} \leq 80) = P\left(\frac{S_{100} - 90}{3} \leq \frac{80 - 90}{3}\right)$$

$$= P(Z \leq -\frac{10}{3})$$

Z : standard normal RV.

$$= Q(-3.33)$$

$$\approx 0.0004$$

$$P(S_{100} \leq 80) < \alpha = 0.01$$

so, reject the company's statement.

programming. 1.

$$Pr(P | Z) = \frac{P(Z | P) P(P)}{P(Z)}$$