3.1: Let $f(u) = au$, with a constant, and let $u_0(x)$ be any integrable function. Verify that $u(x,t) = u_0(x-at)$ satisfies eqn 2.16 for any x_1, x_2, t_1, t_2

Sol/ Eqn 2.16:

$$\underbrace{\int_{x_1}^{x_2} u(x,t_2)\,dx}_{(1)} = \underbrace{\int_{x_1}^{x_2} u(x,t_1)\,dx}_{(2)} + \underbrace{\int_{t_1}^{t_2} f(u(x_1,t))\,dt}_{(3)} - \underbrace{\int_{t_1}^{t_2} f(u(x_2,t))\,dt}_{(4)}$$

$u(x,t) = u_0(x-at)$, $f(u(x,t)) = au(x,t) = a\,u_0(x-at)$

Say u_0 has antiderivative U,

(1) $$\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u_0(x-at_2)\,dx = U(x_2-at_2) - U(x_1-at_2)$$

(2) $$\int_{x_1}^{x_2} u(x,t_1)\,dx = U(x_2-at_1) - U(x_1-at_1)$$

(3) $$\int_{t_1}^{t_2} f(u(x_1,t))\,dt = a\int_{t_1}^{t_2} u_0(x_1-at)\,dt = \frac{a}{-a}\int_{x_1-at_1}^{x_1-at_2} u_0(v)\,dt = -U(x_1-at_2) + U(x_1-at_1)$$

(4) $$\int_{t_1}^{t_2} f(u(x_2,t))\,dt = -U(x_2-at_2) + U(x_2-at_1)$$

The RHS of eqn 2.16

$$U(x_2-at_1) - U(x_1-at_1) + \left(-U(x_1-at_2) + U(x_1-at_1)\right) - \left(-U(x_2-at_2) + U(x_2-at_1)\right)$$

$$= U(x_2-at_2) - U(x_1-at_2)$$

(1) $$= \int_{x_1}^{x_2} u(x,t_2)\,dx \quad \text{from above work.}$$

So eqn 2.16 is satisfied.

3.2: Show the Vanishing viscosity Solution $\lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ is equal to $u_0(x-at)$.

Sol/ As noted in the text, a Change of Variables gives

$$V^\varepsilon(x,t) = u^\varepsilon(x+at,t)$$

and

(3.12) $V_t^\varepsilon(x,t) = \varepsilon V_{xx}^\varepsilon(x,t)$ is satisfied.

$$V^\varepsilon(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4\varepsilon t} u_0(y)\,dy$$ is a solution to (3.12)

$$u_0(x) = V^\varepsilon(x,0) = u^\varepsilon(x,0)$$

Specifically $Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4\varepsilon t}} e^{-p^2}\,dp$ and

$$V^\varepsilon(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\,u_0(y)\,dy$$

Found these results in "Partial Differential Equations An Introduction" by Walter Strauss

Notice in $Q(x,t)$. $\lim_{t\to 0} Q(x,t) = 1$ if $x>0$
$= 0$ if $x<0$

we look at $Q(x-at-y,t)$ here

$$Q(x-at-y,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{(x-at-y)/\sqrt{4\varepsilon t}} e^{-p^2}\,dp$$

and notice

$$\lim_{\varepsilon\to 0} Q(x-at-y,t) = \begin{cases} 1 & \text{if } y < x-at \\ 0 & \text{if } y > x-at \end{cases}$$?

$$\lim_{y\to\infty} Q(x-at-y,t) = 0, \quad \lim_{y\to-\infty} Q(x-at-y,t) = 1$$

$$\lim_{\varepsilon\to 0} u(x,t) = \lim_{\varepsilon\to 0} V(x-at,t) = \underbrace{\lim_{\varepsilon\to 0} \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-at-y,t)\, u_0(y)\,dy}_{(1)}$$

* Integrate by parts

$$(1) = \lim_{\varepsilon\to 0}\left[-u_0(y)\,Q(x-at-y,t)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} Q(x-at-y,t)\,u_0'(y)\,dy \right]$$

$$= \lim_{\varepsilon\to 0}\left(-u_0(y)\,\frac{1}{\sqrt{\pi}}\int_0^{\frac{x-at-y}{\sqrt{4\varepsilon t}}} e^{-p^2}\,dp \,\Bigg|_{-\infty}^{\infty} \right) + \int_{-\infty}^{\infty} \lim_{\varepsilon\to 0} Q(x-at-y,t)\,u_0'(y)\,dy$$

$$= u_0(-\infty) + \int_{-\infty}^{x-at} u_0'(y)\,dy$$

$$= u_0(-\infty) + u_0(x-at) - u_0(-\infty)$$

$$= u_0(x-at)$$

So the vanishing viscosity solution is $u_0(x-at)$

3.3: Show that if we Solve (3.14) with Smooth initial data $u_0(x)$ for which $u'(x)$ is Somewhere negative, then the wave will break at time

$$T_b = \frac{-1}{\min u_0'(x)}$$

Generalize this to arbitrary Convex Scalar equations.

Sol/

eqn (3.18) is $x = \xi + u(\xi,0)t$

(3.19) is $u(x,t) = u(\xi,0)$

So

$$u(x,t) = u(\xi,0) = u_0(\xi) = u_0(x - u(x,t)t)$$

$$\frac{\partial u}{\partial x} = u_0'(\xi)\left(1 - t\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x}\left(1 + u_0'(\xi)t\right) = u_0'(\xi)$$

$$\frac{\partial u}{\partial x} = \frac{u_0'(\xi)}{1 + u_0'(\xi)t}$$

If $u_0' > 0$, there are no issues.

If $u_0' < 0$ at some point. The derivative is infinite when

$$t = \frac{-1}{u_0'(\xi)}$$

As written in the text, when u_0 is Smooth, this appears first when u_0' is most negative. So

$$T_b = \frac{-1}{\min(u_0'(x))}$$

3.4:

Verify that

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases}$$

is a weak solution to Burgers' equations by showing

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x f(u)]\,dx\,dt = -\int_{-\infty}^\infty \phi(x,0)\,u(x,0)\,dx \quad -(3)$$

is satisfied for all $\phi \in C_0^1$.

Sol/ For Burgers' equation $f(u) = \frac{1}{2}u^2$. let $\phi \in C_0^1$.

We'll check $(1)+(2)+(3) = 0$

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x f(u)]\,dx\,dt = \overbrace{\int_{-\infty}^\infty\int_0^\infty \phi_t u \,dt\,dx}^{(1)} + \overbrace{\frac{1}{2}\int_0^\infty\int_{-\infty}^\infty \phi_x u^2\,dx\,dt}^{(2)}$$

Splitting up the integrals based on u:

$$(1) = \int_{-\infty}^\infty \int_0^{x/s} \phi_t u_r\,dt\,dx + \int_{-\infty}^\infty \int_{x/s}^\infty \phi_t u_l\,dt\,dx$$

$\phi(x,\text{"}\infty\text{"}) = 0$

$$= u_r \int_{-\infty}^\infty [\phi(x,x/s) - \phi(x,0)]\,dx + u_l \int_{-\infty}^\infty [0 - \phi(x,x/s)]\,dx$$

$$= (u_r - u_l)\int_{-\infty}^\infty \phi(x,x/s)\,dx - u_r\int_{-\infty}^\infty \phi(x,0)\,dx$$

$$(2) = \frac{1}{2}\int_0^\infty \int_{-\infty}^{st} u_l^2 \phi_x\,dx\,dt + \frac{1}{2}\int_0^\infty\int_{st}^\infty u_r^2\phi_x\,dx\,dt$$

$$= \frac{1}{2}u_l^2 \int_0^\infty [\phi(st,t) - 0]\,dt + \frac{1}{2}u_r^2\int_0^\infty [0 - \phi(st,t)]\,dt$$

$$= \frac{1}{2}(u_l^2 - u_r^2)\int_0^\infty \phi(st,t)\,dt = s(u_l - u_r)\int_0^\infty \phi(st,t)\,dt$$

$x = st,\ t = x/s,\ dt = \frac{1}{s}dx$

$$= (u_l - u_r)\int_0^\infty \phi(x, x/s)\,dx$$

$$(3) = \int_{-\infty}^{\infty} \phi(x,0)\, u(x,0)\, dx = u_l \int_{-\infty}^{0} \phi(x,0)\, dx + u_r \int_{0}^{\infty} \phi(x,0)\, dx$$

$$(1)+(2)+(3) = (u_r - u_l) \int_{-\infty}^{\infty} \phi(x, x/s)\, dx - u_r \int_{-\infty}^{\infty} \phi(x,0)\, dx + \overset{-(u_r - u_l)}{(u_l - u_r)} \int_{0}^{\infty} \phi(x, x/s)\, dx$$

$$+ u_l \int_{-\infty}^{0} \phi(x,0)\, dx + u_r \int_{0}^{\infty} \phi(x,0)\, dx$$

$$= (u_r - u_l) \int_{-\infty}^{0} \phi(x, x/s)\, dx - u_r \int_{-\infty}^{\infty} \phi(x,0)\, dx + u_l \int_{-\infty}^{0} \phi(x,0)\, dx + u_r \int_{0}^{\infty} \phi(x,0)\, dx$$

$$= (u_r - u_l) \int_{-\infty}^{0} \phi(x, x/s)\, dx - (u_r - u_l) \int_{-\infty}^{0} \phi(x,0)\, dx$$

$$\downarrow *$$

$$= (u_r - u_l) \int_{-\infty}^{0} \phi(x, 0)\, dx - (u_r - u_l) \int_{-\infty}^{0} \phi(x,0)\, dx$$

$= 0$ as wanted.

* I made this step as $t \in [0,\infty)$ but we are integrating on $[-\infty, 0]$, so $t = \frac{x}{s} < 0$ doesn't make sense. I'm not very confident in the validity of that step.

3.5: Show that the viscous equation (3.15) has traveling wave solution of the form $u^{\varepsilon}(x,t) = w(x-st)$ by deriving an ODE for w and verifying that this ODE has solutions of the form

$$w(y) = u_r + \frac{1}{2}(u_l - u_r)\left[1 - \tanh\left((u_l-u_r)\, y/4\varepsilon\right)\right]$$

with s given by $s = (u_l + u_r)/2$.

· Note that $w(y) \to u_l$ as $y \to -\infty$ and $w(y) \to u_r$ as $y \to \infty$.

Sketch this solution and indicate how it varies as $\varepsilon \to 0$

Sol/ (3.15) is

$$u_t + u u_x = \varepsilon u_{xx}$$

$$u^{\varepsilon}(x,t) = w(x-st)$$

$$-\frac{1}{2}(u_l+u_r)w'(x-st) + ww'(x-st) = \varepsilon w''(x-st)$$

Potential ODE:

$$\boxed{-\frac{1}{2}(u_l+u_r)\, w' + w w' = \varepsilon w''}$$

let's check with $w(y)$

$$w(y) = u_r + \frac{1}{2}(u_l - u_r)\left[1 - \tanh\left((u_l-u_r)\, y/4\varepsilon\right)\right]$$

$$w'(y) = -\frac{1}{8\varepsilon}(u_l-u_r)^2 \operatorname{sech}^2\left(\left(u_l-u_r\right) y/4\varepsilon\right)\Big)\Big]$$

$$w''(y) = \frac{1}{16\varepsilon^2}(u_l-u_r)^3 \operatorname{sech}^2\left((u_l-u_r)\, y/4\varepsilon\right)\tanh\left((u_l-u_r)\, y/4\varepsilon\right)$$

$$-\frac{1}{2}(u_l+u_r)\, w' = \frac{1}{16\varepsilon}(u_l+u_r)(u_l-u_r)^2 \operatorname{sech}^2(\sim)$$

$$w w' = \frac{-u_r}{8\varepsilon}(u_l-u_r)^2 \operatorname{sech}^2(\sim) - \frac{1}{16\varepsilon}(u_l-u_r)^3 \operatorname{sech}^2(\sim)$$
$$+ \frac{1}{16\varepsilon}(u_l-u_r)^3 \operatorname{sech}^2(\sim)\tanh(\sim)$$

$$\varepsilon w'' = \frac{1}{16\varepsilon}(u_l - u_r)^3 \operatorname{sech}^2(\sim)\tanh(\sim)$$

$$-\tfrac{1}{2}(u_l + u_r)w' + ww' - \varepsilon w''$$

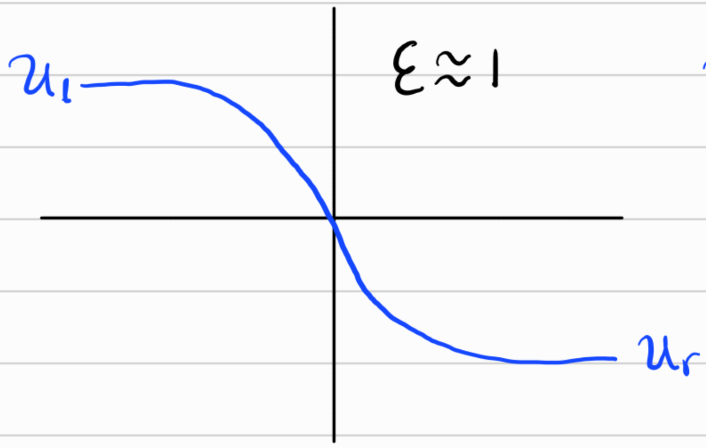
$$= \frac{1}{16\varepsilon}\left(\left((u_l-u_r)+2u_r\right)(u_l-u_r)^2 \operatorname{sech}^2(\sim) - \frac{u_r}{8\varepsilon}(u_l-u_r)^2\operatorname{sech}^2(\sim)\right.$$
$$- \frac{1}{16\varepsilon}(u_l-u_r)^3\operatorname{sech}^2(\sim)$$

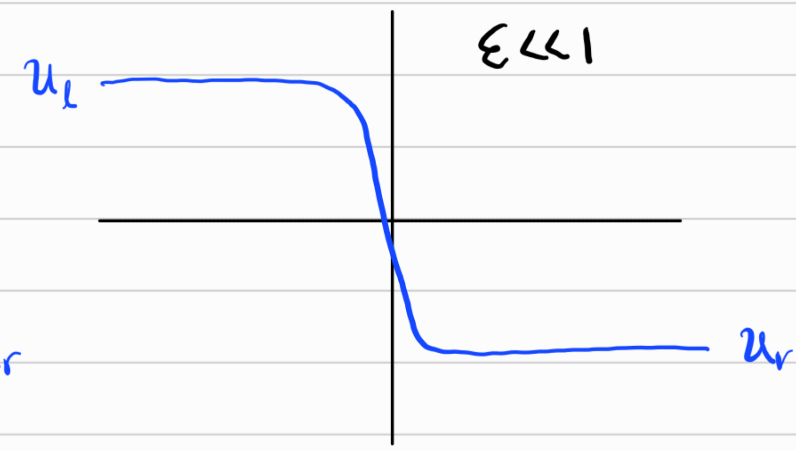
$$= \frac{1}{16\varepsilon}(u_l-u_r)^3\operatorname{sech}^2(\sim) + \frac{u_r}{8\varepsilon}(u_l-u_r)^2\operatorname{sech}^2(\sim)$$
$$- \frac{u_r}{8\varepsilon}(u_l-u_r)^2\operatorname{sech}^2(\sim) - \frac{1}{16\varepsilon}(u_l-u_r)^3\operatorname{sech}^2(\sim)$$

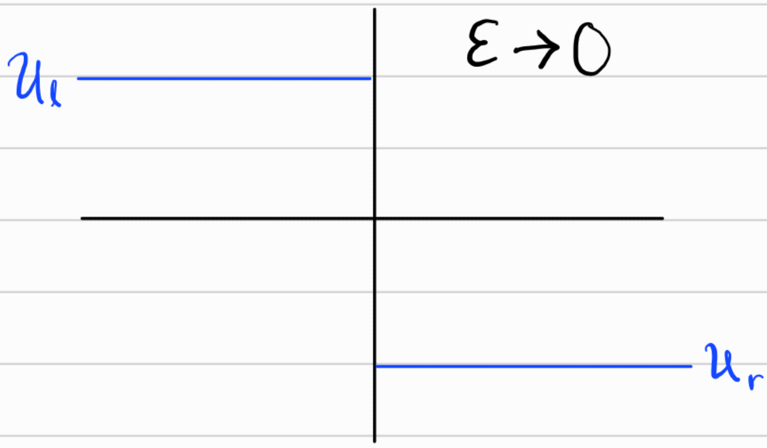
$$= 0$$

So $w(y)$ is a solution to $-\tfrac{1}{2}(u_l+u_r)w' + ww' = \varepsilon w''$

Graphs of $w(y)$







3.6: There are infinitely many other weak solutions to (3.14) when $u_l < u_r$. Show

$$u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t \le x \le u_m t \\ x/t & u_m t \le x \le u_r t \\ u_r & u_r > u_r t \end{cases}$$

is a weak solution for any u_m with $u_l \le u_m \le u_r$, and $s_m = (u_l + u_m)/2$.
Sketch the characteristics for this solution. Find a class of weak solutions with three discontinuities.

Sol/ (3.14) is Burgers eqn with $u_t + uu_x = 0$, as $f(u) = \frac{1}{2}u^2$.

Let $\phi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$. We will work this problem similar to 3.4.
That is, we want to show

$$\underbrace{\int_{-\infty}^{\infty}\int_0^{\infty} \phi_t u \, dt\, dx}_{(1)} + \underbrace{\frac{1}{2}\int_0^{\infty}\int_{-\infty}^{\infty} \phi_x u^2 \, dx\, dt}_{(2)} + \underbrace{\int_{-\infty}^{\infty} \phi(x,0)\, u(x,0)\, dx}_{(3)} = 0$$

$$(1) = u_r \int_{-\infty}^{\infty}\int_0^{x/u_r} \phi_t \, dt\, dx + \int_{-\infty}^{\infty}\int_{x/u_r}^{x/u_m} \frac{x}{t}\phi_t \, dt\, dx + u_m \int_{-\infty}^{\infty}\int_{x/u_m}^{x/s_m} \phi_t \, dt\, dx + u_l \int_{-\infty}^{\infty}\int_{x/s_m}^{\infty} \phi_t \, dt\, dx$$

$$= u_r \int_{-\infty}^{\infty} \phi\left(x, \frac{x}{u_r}\right) dx - u_r \int_{-\infty}^{\infty} \phi(x,0)\, dx + \quad + u_m \int_{-\infty}^{\infty} \phi(x, x/s_m)\, dx - u_m \int_{-\infty}^{\infty} \phi(x, x/u_m) - u_l \int_{-\infty}^{\infty} \phi(x, x/s_m)$$

$$(2) = \left[u_l^2 \int_0^{\infty} \phi(s_m t, t)\, dt + u_m^2 \int_0^{\infty} \phi(u_m t, t) - \phi(s_m t, t)\, dt + \int_0^{\infty}\int_{u_m t}^{u_r t} \frac{x^2}{t^2}\phi_x \, dx\, dt - u_r^2 \int_0^{\infty} \phi(u_r t, t)\, dt \right] \cdot \frac{1}{2}$$

$$(3) = u_l \int_{-\infty}^{0} \phi(x,0)\, dx + u_r \int_0^{\infty} \phi(x,0)\, dx$$

$$(1)+(2)+(3) = (u_m - u_l)\int_{-\infty}^{\infty}\phi(x, x/s_m)\,dx + u_l\int_{-\infty}^{\infty}\phi(x, x/u_r)\,dx - u_m\int_{-\infty}^{\infty}\phi(x, x/u_m)\,dx$$

$$- u_r\int_{-\infty}^{\infty}\phi(x,0)\,dx + \int_{-\infty}^{\infty}\int_{x/u_r}^{x/u_m}\frac{x}{t}\phi_t\,dt\,dx + \underbrace{\frac{1}{2}(u_l^2 - u_m^2)}_{S_m(u_l-u_m)}\int_0^{\infty}\phi(s_m t, t)\,dt \qquad x = s_m t,\ t = x/s_m,\ dt = \frac{1}{s_m}dx$$

$$+ \frac{1}{2}u_m^2\int_0^{\infty}\phi(u_m t, t)\,dt - \frac{1}{2}u_r^2\int_0^{\infty}\phi(u_r t, t)\,dt + \frac{1}{2}\int_0^{\infty}\int_{u_m t}^{u_r t}\frac{x^2}{t^2}\phi_x\,dx\,dt$$

$$+ u_l\int_{-\infty}^{0}\phi(x,0)\,dx + u_r\int_0^{\infty}\phi(x,0)\,dx$$

$$= (u_m - u_l)\int_{-\infty}^{\infty}\phi(x, x/s_m)\,dx + u_r\int_{-\infty}^{\infty}\phi(x, \tfrac{x}{u_r})\,dx - u_m\int_{-\infty}^{\infty}\phi(x, x/u_m)\,dx$$

$$-(u_r - u_l)\int_{-\infty}^{0}\phi(x,0)\,dx + \overset{-(u_m-u_l)}{(u_l - u_m)}\int_0^{\infty}\phi(x, \tfrac{x}{s_m})\,dx + \frac{1}{2}u_m\int_0^{\infty}\phi(x, \tfrac{x}{u_m})\,dx$$

$$- \frac{1}{2}u_r\int_0^{\infty}\phi(x, x/u_r)\,dx + \underbrace{\frac{1}{2}\int_0^{\infty}\int_{u_l t}^{u_r t}\frac{x^2}{t^2}\phi_x\,dx\,dt}_{(1)} + \underbrace{\int_{-\infty}^{\infty}\int_{x/u_r}^{x/u_l}\frac{x}{t}\phi_t\,dt\,dx}_{(2)}$$

$$(1) = \frac{1}{2}\int_0^{\infty}\left[\frac{x^2}{t^2}\phi\Big|_{u_l t}^{u_r t} - \frac{2}{t^2}\int_{u_l t}^{u_r t} x\phi\,dx\right]dt = \frac{1}{2}u_r^2\int_0^{\infty}\phi(u_r t, t)\,dt - \frac{1}{2}u_l^2\int_0^{\infty}\phi(u_l t, t) - \sim$$

$$(2)\ \int_{-\infty}^{\infty}\left[\frac{x}{t}\phi\Big|_{x/u_r}^{x/u_l} + \int_{x/u_r}^{x/u_l}\frac{x}{t^2}\phi\,dt\right]dx = u_l\int_{-\infty}^{\infty}\phi(x, \tfrac{x}{u_l})\,dx - u_r\int_{-\infty}^{\infty}\phi(x, \tfrac{x}{u_r})\,dr + \int_{-\infty}^{\infty}\int_{x/u_r}^{x/u_l}\phi\,dt\,dx$$

3.7: Show that for a general convex scalar problem (3.13) with data (3.24) and $u_l < u_r$, the rarefaction wave solution is given by

$$u(x,t) = \begin{cases} u_l & x \le f'(u_l)t \\ v(x/t) & f'(u_l)t \le x \le f'(u_r)t \\ u_r & x > f'(u_r)t \end{cases}$$

where $v(\xi)$ is the solution to $f'(v(\xi)) = \xi$

Sol/

$$(3.13)\quad u_t + f(u)_x = 0, \qquad (3.24)\quad u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

Let $\phi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$. As usual, we want to show

$$\underbrace{\int_{-\infty}^{\infty}\int_0^{\infty} \phi_t u \, dt\, dx}_{(1)} + \underbrace{\int_0^{\infty}\int_{-\infty}^{\infty} \phi_x f(u) \, dx\, dt}_{(2)} + \underbrace{\int_{-\infty}^{\infty} \phi(x,0)\, u(x,0)\, dx}_{(3)} = 0$$

(1) $$\int_{-\infty}^{0}\int_0^{\infty} u_l \phi_t \, dt\, dx + \int_0^{\infty}\int_0^{\infty} u \phi_t \, dt\, dx$$

$$= -u_l \int_{-\infty}^{0} \phi(x,0)\, dx + u_r \int_0^{\infty}\int_0^{x/f'(u_r)} \phi_t \, dt\, dx + \int_0^{\infty}\int_{x/f'(u_r)}^{x/f'(u_l)} \phi_t V(x/t)\, dt\, dx + u_l \int_0^{\infty}\int_{x/f'(u_l)}^{\infty} \phi_t \, dt\, dx$$

$$= -u_l \int_{-\infty}^{0} \phi(x,0)\, dx - u_r \int_0^{\infty} \phi(x,0)\, dx + u_r \int_0^{\infty} \phi\left(x, \frac{x}{f'(u_r)}\right) - u_l \int_0^{\infty} \phi\left(x, \frac{x}{f'(u_l)}\right) dx + \underbrace{\int_0^{\infty}\int_{x/f'(u_r)}^{x/f'(u_l)} \phi_t V(x/t)\, dt\, dx}_{I_1}$$

(2) $$f(u_l)\int_0^{\infty}\int_{-\infty}^{f'(u_l)t} \phi_x \, dx\, dt + \int_0^{\infty}\int_{f'(u_l)t}^{f'(u_r)t} \phi_x f(v(x/t))\, dx\, dt + f(u_r)\int_0^{\infty}\int_{f'(u_r)t}^{\infty} \phi_x \, dx\, dt$$

$$= f(u_l)\int_0^{\infty} \phi(f'(u_l)t, t)\, dt - f(u_r)\int_0^{\infty} \phi(f'(u_r)t, t)\, dt + \underbrace{\int_0^{\infty}\int_{f'(u_l)t}^{f'(u_r)t} \phi_x f(v(x/t))\, dx\, dt}_{I_2}$$

(3) $$u_l \int_{-\infty}^{0} \phi(x,0)\, dx + u_r \int_0^{\infty} \phi(x,0)\, dx$$

$$I_1 = \int_{-\infty}^{\infty}\left[\phi(x,t)V(x/t)\Big|_{x/f'(u_r)}^{x/f'(u_l)} + \int_{x/f'(u_r)}^{x/f'(u_l)} \phi(x,t)\,V'(x/t)\,\frac{x}{t^2}\,dt \right]dx$$

$$= \int_{-\infty}^{\infty}\left[\phi\left(x, \frac{x}{f'(u_l)}\right)V(f'(u_l)) - \phi\left(x, \frac{x}{f'(u_r)}\right)V(f'(u_r)) + \int_{x/f'(u_r)}^{x/f'(u_l)} \phi(x,t)\,V'(x/t)\,\frac{x}{t^2}\,dt \right]dx$$

$$= \underline{u_l\int_{-\infty}^{\infty} \phi\left(x, \frac{x}{f'(u_l)}\right)dx} - \underline{\underline{u_r\int_{-\infty}^{\infty}\phi\left(x,\frac{x}{f'(u_r)}\right)dx}} + \int_{-\infty}^{\infty}\int_{x/f'(u_r)}^{x/f'(u_l)} \phi(x,t)\,V'(x/t)\,\frac{x}{t^2}\,dt\;dx$$

$\star$ Since $f''>0$, f' has an inverse. We are given $f'(v(\xi)) = \xi$, so $v = (f')^{-1}$ or $v^{-1} = f'$.

$$I_2 = \int_0^{\infty}\left[\phi(x,t)f(v(x/t))\Big|_{f'(u_l)t}^{f'(u_r)t} - \int_{f'(u_l)t}^{f'(u_r)t} \phi(x,t)\,f'(v(x/t))\,v'(x/t)\,\frac{1}{t}\,dx \right]dt$$

$f(v(f'(u_r)))$, $v=(f')^{-1}$

$$= \underline{f(u_r)\int_0^{\infty}\phi(f'(u_r)t,t)\,dt} - \underline{\underline{f(u_l)\int_0^{\infty}\phi(f'(u_l)t,t)\,dt}} - \int_0^{\infty}\int_{f'(u_l)t}^{f'(u_r)t} \phi(x,t)\,V'(x/t)\,\frac{x}{t^2}\,dx\,dt$$

After cancelation (Marked in Green and Orange) we get

$$(1)+(2)+(3) = \int_{-\infty}^{\infty}\int_{x/f'(u_r)}^{x/f'(u_l)} \phi(x,t)\,V'(x,t)\,\frac{x}{t^2}\,dt\,dx - \int_0^{\infty}\int_{f'(u_l)t}^{f'(u_r)t} \phi(x,t)\,V'(x/t)\,\frac{x}{t^2}\,dx\,dt$$

Notice $\left\{(x,t) : -\infty \le x \le \infty,\ 0 \le \frac{x}{f'(u_r)} \le t \le \frac{x}{f'(u_l)}\right\} = \left\{(x,t) : f'(u_l)t \le x \le f'(u_r)t,\ 0 \le t \le \infty\right\}$

So the integrals above are over the same domain, call it D.
Then

$$(1)+(2)+(3) = \iint_D \phi(x,t)\,V'(x,t)\,\frac{x}{t^2}\,dD - \iint_D \phi(x,t)\,V'(x/t)\,\frac{x}{t^2}\,dD$$

$$= 0 \quad \text{as wanted.}$$

So, $u(x,t)$ is a weak solution to the convex scalar problem.

3.8: Solve Burgers' equation with initial data

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & 0 < x < 2 \\ 0 & 2 < x \end{cases}$$

Sketch the Characteristics and shock paths in the x-t plane.

Hint: the two shocks merge into one at some point.

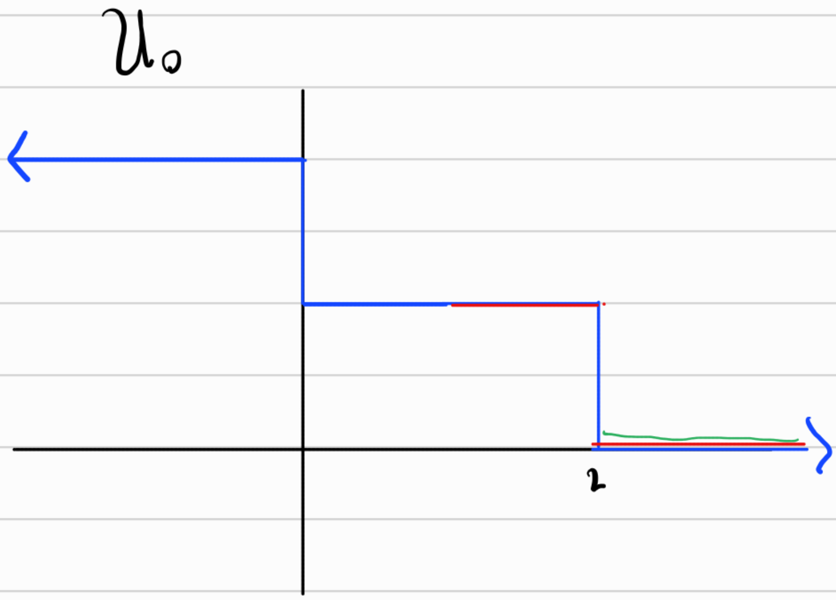
Sol/ Burger's equation:

$$u_t + u u_x = 0$$

Characteristics lines

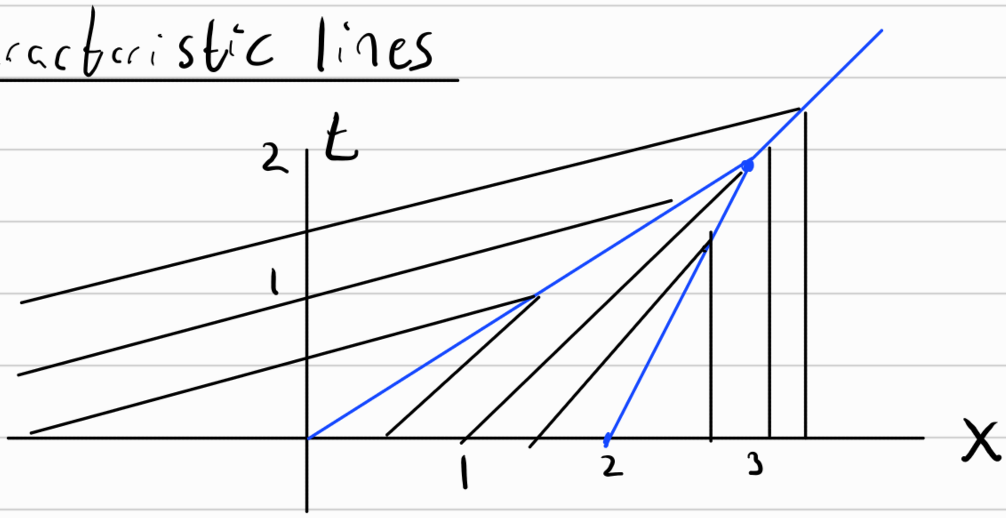
$$x = \begin{cases} \xi + 2t & \xi < 0 \\ \xi + t & 0 < \xi < 2 \\ \xi & 2 < \xi \end{cases}$$

ξ is x-int in x-t plane.



$x = \frac{1}{2}t + k$

Characteristic lines



$\frac{1}{2}t + 2 = \frac{3}{2}t$

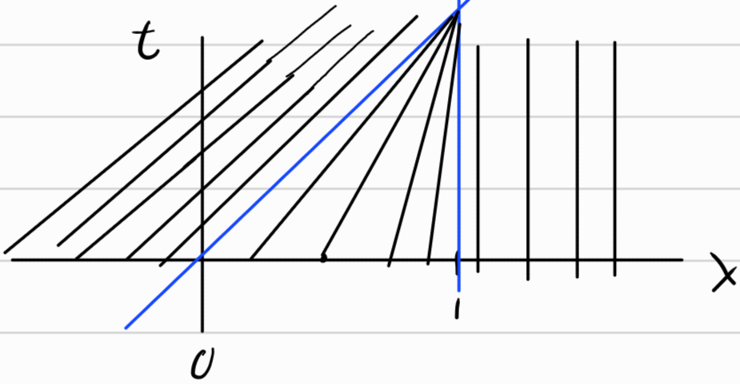
$2 = t$

$t < 2$

$$u(x,t) = \begin{cases} 2 & x \le \frac{3}{2}t \\ 1 & \frac{3}{2}t < x < \frac{1}{2}t + 2 \\ 0 & x > \frac{1}{2}t + 2 \end{cases}$$

$t > 2$

$$u(x,t) \begin{cases} 2 & x < t + 1 \\ 0 & x > t + 1 \end{cases}$$



$$u(x,t) = \begin{cases} 1 & x < t \\ \dfrac{1-x}{1-t} & t < x \le 1 \\ 0 & 1 < x \end{cases}$$