11.1:

Compute the modified equation for the method

$$U_j^{n+1} - U_j^n + \frac{k}{2h} A\left(U_{j+1}^n - U_{j-1}^n\right) = 0$$

Explain why this method might be unstable for all k/h.

The Taylor Series expansions are

$$U_j^{n+1} = u(x_j, t_n + k) = u(x_j, t_n) + k\partial_t u(x_j, t_n) + \frac{k^2}{2}\partial_{tt} u(x_j, t_n) + O(k^3)$$

$$U_{j+1}^n = u(x_j + h, t_n) = u(x_j, t_n) + h\partial_x u(x_j, t_n) + \frac{h^2}{2}\partial_{xx} u(x_j, t_n)_{xx} + O(h^3)$$

$$U_{j-1}^n = u(x_j - h, t_n) = u(x_j, t_n) - h\partial_x u(x_j, t_n) + \frac{h^2}{2}\partial_{xx} u(x_j, t_n) + O(h^3)$$

For ease of writing, denote $\partial_x u(x_j, t_n)$ by $(\bar{u}_j^n)_x$, likewise for the others

The method above is then (excluding terms of $O(k^3)$)

$$0 = \bar{u}_j^n + k(\bar{u}_j^n)_t + \frac{k^2}{2}(\bar{u}_j^n)_t - \bar{u}_j^n$$
$$+ \frac{k}{2h} A\left(\bar{u}_j^n + h(\bar{u}_j^n)_x + \frac{h^2}{2}(\bar{u}_j^n)_{xx} - \bar{u}_j^n + h(\bar{u}_j^n)_x - \frac{h^2}{2}(\bar{u}_j^n)_{xx}\right)$$

$$= k(\bar{u}_j^n)_t + \frac{k^2}{2}(\bar{u}_j^n)_{tt} + \frac{k}{2h} A\left[2h(\bar{u}_j^n)_x\right]$$

$$= k\left[(\bar{u}_j^n)_t + A(\bar{u}_j^n)_x + \frac{k}{2}(\bar{u}_j^n)_{tt}\right]$$

$$= (\bar{u}_j^n)_t + A(u_j^n)_x + \frac{k}{2}(\bar{u}_j^n)_{tt}$$

Since u satisfies $u_t + Au_x = 0$, we just have

$\frac{k}{2}(\bar{u}_j^n)_{tt}$ as our local truncation error.

This gives the modified equation

$$u_t + Au_x + \frac{k}{2}u_{tt} = 0$$

which we can simplify

$$u_t = -Au_x - \frac{k}{2}u_{tt}$$

$$u_{tt} = -Au_{xt} - \frac{k}{2}u_{tt}$$

$$= -A[-Au_{xx} + O(k)]$$
$$= A^2 u_{xx} + O(k)$$

So the modified equation is

$$\boxed{u_t + Au_x = -\frac{k}{2}A^2 u_{xx}} \quad (*)$$

Notice, the eigenvalues of $D = -\frac{k}{2}A^2$ are $-\frac{k}{2}(\lambda_p)^2$

which are all negative. This makes $(*)$ ill-posed and thus the numerical method will not be stable for any k/h.