12.1: Show (12.4) is Consistent with (12.1) and (12.2)

(12.1) $u_t + \left(\frac{1}{2}u^2\right)_x = 0$

(12.2) $\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0$

(12.4) $U_j^{n+1} = U_j^n - \frac{k}{h}U_j^n\left(U_j^n - U_{j-1}^n\right)$

To show consistency, we need to show $\|L_k(x,t)\| \to 0$ as $k \to 0$ where

$$L_k(x,t) = \frac{1}{k}\left[u(x,t+k) - \mathcal{H}_k(u(\cdot,t);x)\right]$$

$$L_k(x,t) = \frac{1}{k}\left[u(x,t+k) - u(x,t) + \frac{k}{h}u(x,t)\left(u(x,t) - u(x-h,t)\right)\right]$$

$$= \frac{1}{k}\left[\left(u + k u_t + \frac{k^2}{2}u_{tt} + \dots\right) - u + \frac{k}{h}u\left(u - \left(u - hu_x + \frac{h^2}{2}u_{xx} - \dots\right)\right)\right]$$

$$= \frac{1}{k}u + u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt} - \frac{1}{k}u + uu_x - \frac{h}{2}uu_{xx} + \frac{h^2}{6}uu_{xxx} - \dots$$

$$= u_t + uu_x + \frac{k}{2}u_{tt} - \frac{h}{2}uu_{xx} + \frac{k^2}{6}u_{ttt} + \frac{h^2}{6}uu_{xxx} - \dots$$

from (12.1) $u_t + \left(\frac{1}{2}u^2\right)_x = u_t + uu_x = 0$

from (12.2) $\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 2uu_t + 2u^2u_x = 2u\left(u_t + uu_x\right) = 0$
and assuming $u \neq 0$, we have $u_t + uu_x = 0$ as well. So

$$= \underbrace{\cancel{u_t + uu_x}}_{=0} + \frac{k}{2}u_{tt} - \frac{h}{2}uu_{xx} + \frac{k^2}{6}u_{ttt} + \frac{h^2}{6}uu_{xxx} - \dots$$

$$= k\left[\frac{1}{2}u_{tt} - \frac{h}{2k}uu_{xx} + \frac{k}{6}u_{ttt} + \frac{h^2}{6k}uu_{xxx} - \dots\right]$$

$= O(k)$ as $k \to 0$

So $\|L_k(x,t)\| \to 0$ as $k \to 0$ so (12.4) is consistent for 12.1 and 12.2.

12.2: Verify that the Lax-Friedrichs flux (12.15) is consistent (including Lipschitz continuity).

$$(12.15)\quad U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{k}{2h}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

In conservation form, we have

$$(12.16)\quad F(U_j, U_{j+1}) = \frac{h}{2k}\left(U_j - U_{j+1}\right) + \frac{1}{2}\left(f(U_j) + f(U_{j+1})\right)$$

We first check and see

$$\begin{aligned} F(\bar{u},\bar{u}) &= \frac{h}{2k}\left(\bar{u}-\bar{u}\right) + \frac{1}{2}\left(f(\bar{u}) + f(\bar{u})\right) \\ &= 0 + \frac{1}{2}(2f(\bar{u})) \\ &= f(\bar{u}) \end{aligned}$$

So (12.17) is satisfied.

Now, take v, w

$$\begin{aligned} |F(v,w) - f(\bar{u})| &= \frac{h}{2k}(v-w) + \frac{1}{2}\left(f(v)+f(w)\right) - f(\bar{u}) \\ &= \frac{h}{2k}\left((v-\bar{u}) - (w-\bar{u})\right) + \frac{1}{2}\left(f(v) - f(\bar{u})\right) + \frac{1}{2}\left(f(w) - f(\bar{u})\right) \end{aligned}$$

f is assumed to be smooth, thus Lipschitz. So if $|v-\bar{u}| < \varepsilon$ $|w-\bar{u}| < \delta$ then

$$|F(v,w) - f(\bar{u})| \le \frac{h}{2k}(\varepsilon-\delta) + \frac{1}{2}K\varepsilon + \frac{1}{2}K\delta$$

WLOG, let $\varepsilon > \delta$ $\quad \le \frac{h}{k}\varepsilon + K\varepsilon = \left(\frac{h}{k} + K\right)\varepsilon$

So $F(v,w)$ is Lipschitz and thus (12.15) is consistent.

12.3: (Paraphrased) Show (12.26)

1) reduced to (12.24) in the constant coefficient linear case,

2) is second order accurate to nonlinear problems (smooth solutions),

3) is conservative;

4) write (12.26) in conservation form.

$$(12.24) \quad U_j^{n+1} = U_j^n - \frac{k}{2h} A\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2}{2h^2} A^2\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

$$(12.26) \quad U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{k}{2h}\left[f(U_{j+1}^n) - f(U_j^n)\right]$$

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[f\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) - f\left(U_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right)\right].$$

1) In the constant coefficient linear case, $f(u) = Au$

So, for 12.26

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{k}{2h} A\left(U_{j+1}^n - U_j^n\right)$$

$$U_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{j-1}^n + U_j^n\right) - \frac{k}{2h} A\left(U_j^n - U_{j-1}^n\right)$$

$$U_j^{n+1} = U_j^n - \frac{k}{h} A\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - U_{j-\frac{1}{2}}^{n-\frac{1}{2}}\right)$$

$$= U_j^n - \frac{k}{h} A\Big[\frac{1}{2} U_j^n + \frac{1}{2} U_{j+1}^n - \frac{k}{2h} A U_{j+1}^n + \frac{k}{2h} A U_j^n - \frac{1}{2} U_{j-1}^n - \frac{1}{2} U_j^n + \frac{k}{2h} A U_j^n - \frac{k}{2h} A U_{j-1}^n\Big]$$

$$= U_j^n - \frac{k}{2h} A\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2}{2h^2} A^2\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

$= (2.24)$ as wanted.

2) Using Taylor Series Expansion, recall $u_t + f(u)_x = 0$

$$u_j^{n+1} - u_j^n + \frac{k}{h}\left[f\left(u_{j+1/2}^{n+1/2}\right) - f\left(u_{j-1/2}^{n+1/2}\right)\right] = 0$$

$$= u + k u_t + \frac{k^2}{2}u_{tt} + O(k^3) - u + \frac{k}{h}\Big(f(u) + \frac{h}{2}f(u)_x + \frac{k}{2}f(u)_t + \frac{h^2}{4}f(u)_{xx} + \frac{kh}{4}f(u)_{xt} + \frac{k^2}{4}f(u)_{tt} + \dots$$
$$- f(u) + \frac{h}{2}f(u)_x - \frac{k}{2}f(u)_t - \frac{h^2}{4}f(u)_{xx} + \frac{kh}{4}f(u)_{xt} - \frac{k^2}{4}f(u)_{tt} + \dots\Big)$$

$$= k u_t + \frac{k^2}{2}u_{tt} + \frac{k}{h}\left(hf(u)_x + \frac{kh}{2}f(u)_{xt} + O(k^3)\right) + O(k^3)$$

$$= k\Big[\underbrace{u_t + f(u)_x}_{=0} + \frac{k}{2}\big(\underbrace{u_{tt} + f(u)_{xt}}_{=0}\big) + O(k^3)\Big] + O(k)^3$$

Since $u_t + f(u)_x = 0$ we have $u_{tt} + f(u)_{xt} = 0$ as well by deriving again.

$$= O(k^3)$$

So the method is second order accurate for nonlinear problems

4) In Conservation form

(12.7) $\quad u_j^{n+1} - \frac{k}{h}\left[F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)\right]$

(12.26) has

$$F(u_j, u_{j+1}) = f\left(u_{j+1/2}^{n+1/2}\right)$$

$$\boxed{F(u_j, u_{j+1}) = f\left(\tfrac{1}{2}[u_j + u_{j+1}] - \tfrac{k}{2h}[f(u_{j+1}) - f(u_j)]\right)}$$

3) To show (12.26) is consistent, we first check

$$(12.17) \quad F(\bar{u},\bar{u}) = f\left(\tfrac{1}{2}[\bar{u}+\bar{u}] - \tfrac{k}{2h}[f(\bar{u}) - f(\bar{u})]\right)$$
$$= f\left(\tfrac{1}{2}(2\bar{u}) - \tfrac{k}{2h}(0)\right)$$
$$= f(\bar{u}) \quad \text{as wanted.}$$

Now, to show F is Lipschitz. Let f be smooth and v, w be such that $|v-\bar{u}| < \varepsilon$, $|w-\bar{u}| < \delta$ and WLOG $\varepsilon > \delta$.

$$|F(v,w) - f(\bar{u})| = f\left(\tfrac{1}{2}(v+w) - \tfrac{k}{2h}[f(v)-f(w)]\right) - f(\bar{u})$$

$$= f\left(\tfrac{1}{2}(v-\bar{u}) + \tfrac{1}{2}(w-\bar{u}) + \bar{u} - \tfrac{k}{2h}\left[(f(v)-f(\bar{u})) - (f(w)-f(\bar{u}))\right]\right) - f(\bar{u})$$

$$= f\left(\bar{u} + \tfrac{1}{2}(v-\bar{u}) + \tfrac{1}{2}(w-\bar{u}) - \tfrac{k}{2h}\left[(f(v)-f(\bar{u})) - (f(w)-f(\bar{u}))\right]\right) - f(\bar{u})$$

$$\leq C\left(\tfrac{1}{2}(v-\bar{u}) + \tfrac{1}{2}(w-\bar{u}) - \tfrac{k}{2h}\left[(f(v)-f(\bar{u})) - (f(w)-f(\bar{u}))\right]\right)$$

$$\leq C\left(\tfrac{1}{2}\varepsilon + \tfrac{1}{2}\delta - \tfrac{k}{2h}(C\varepsilon + C\delta)\right)$$

$$\leq C\left(\varepsilon + C\tfrac{k}{h}\varepsilon\right)$$

$$= C^2\left(\tfrac{1}{C} + \tfrac{k}{h}\right)\varepsilon$$

Where C is the Lipschitz constant of f.

So F is Lipschitz and thus (12.26) is consistent.

12.4: Consider Upwind w/ flux (12.51). Take $h/k = 1/2$ and apply it to Burgers' Eqn with

$$u_0(x) = \begin{cases} -1 & x < 1 \\ 1 & x > 1 \end{cases}$$

discretized with cell averages $U_j^0 = u_j^0$ (10.3). Argue, refering to the discussion on pg 134, that

1) $U_\ell(x,t)$ for $k_\ell = \frac{1}{2\ell}$ converges to the correct refraction wave solution

2) $U_\ell(x,t)$ for $k_\ell = \frac{1}{2\ell+1}$ converges to an entropy violating shock.

3) $U_\ell(x,t)$ for $k_\ell = \frac{1}{\ell}$ does not converge

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The main change in each case is what the value of U_j^0 is.

From 10.3,

$$U_j^0 = \bar{u}_j^0 = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,dx \quad , \quad \text{here } x_{j\pm1/2} = (j \pm 1/2)h$$

Since, $h/k = 1/2$. $h = 2k$

Note, for any $h < 1$

If $x_j < 1$, $\hookrightarrow j < 1/h$
$$\frac{1}{h}\int_{(j-1/2)h}^{(j+1/2)h} u_0(x)\,dx = -\frac{1}{h}\int_{(j-1/2)h}^{(j+1/2)h} dx = -\frac{1}{h}\left[(j+1/2)h - (j-1/2)h\right] = -1 \quad , \quad j \in \mathbb{Z}$$

If $x_j > 1$, $\hookrightarrow j \geq \frac{1}{h}$
$$\frac{1}{h}\int_{(j-1/2)h}^{(j+1/2)h} u_0(x)\,dx = \frac{1}{h}\int_{(j-1/2)h}^{(j+1/2)h} dx = \frac{1}{h}\left[(j+1/2)h - (j-1/2)h\right] = 1$$

So for each K_ℓ, we only need to check the case when $x_j = 1$
$\hookrightarrow j = \frac{1}{h} = \ell$

1) If $K_\ell = \frac{1}{2\ell}$, $h_\ell = \frac{1}{\ell}$. then $j = \ell$

$$U_1^0 = \ell \int_{1-\frac{1}{2\ell}}^{1+\frac{1}{2\ell}} u_0(x)\,dx = \ell\left[\int_1^{1+1/2\ell} u_0(x)\,dx + \int_{1-\frac{1}{2\ell}}^{1} u_0(x)\,dx\right]$$

$$= \ell\left[(1)\left(1+\tfrac{1}{2\ell}-1\right) + (-1)\left(1-1+\tfrac{1}{2\ell}\right)\right]$$

$$= \ell(0) = 0$$

So

$$U_j^0 = \begin{cases} -1 & j < \ell \ (x_j < 1) \\ 0 & j = \ell \ (x_j = 1) \text{ for all } \ell \text{ as } \ell \to 0 \\ 1 & j > \ell \ (x_j > 1) \end{cases}$$

as discussed in the text, the upwind method should work well here and converge to the proper refraction wave.

2) If $K_\ell = \frac{1}{2\ell+1}$, $h_\ell = \frac{2}{2\ell+1}$, then $x_j = 1$ isn't even possible as

$$x_j = j \cdot h = 1 \Rightarrow j = \frac{1}{h}$$ but j is an integer and $\frac{1}{h} = \frac{2\ell+1}{2}$ is never an integer.

So,

$$U_j^0 = \begin{cases} -1 & j < \frac{2\ell+1}{2} \ (x_j < 1) \\ 1 & j > \frac{2\ell+1}{2} \ (x_j > 1) \end{cases}$$

as exampled in the text, since

$$F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)$$ since $f(-1) = f(1)$

$$= f(U_j^n) - f(U_{j-1}^n) = 0$$

So $u_j^{n+1} = u_j^n + \frac{k}{h}(0)$ and the solution converges to

$u(x,t) = u_0(x)$

here, the shock speed is 0, and $f'(x) = x$

we have $f'(u_\ell) = -1$ and $f'(u_r) = 1$

So we breaks the entropy condition $f'(u_\ell) > s > f'(u_r)$

3) If, $k_\ell = \frac{1}{\ell}$, $h_\ell = \frac{2}{\ell}$, then we consider $j = \frac{\ell}{2}$

however, this isn't always an integer. That is, $x_j = 1$ only if $\ell \equiv 0 \pmod 2$

So

<u>if ℓ is even</u> <u>ℓ is odd</u>

$$U_j^0 = \begin{cases} -1 & \text{if } j < \frac{\ell}{2} \\ 0 & j = \frac{\ell}{2} \\ 1 & j > \frac{\ell}{2} \end{cases}, \quad U_j^0 = \begin{cases} -1 & j < \frac{\ell}{2} \\ 1 & j > \frac{\ell}{2} \end{cases}$$

If ℓ is even, U_ℓ converges to the refraction wave (case 1)

If ℓ is odd, $U_\ell \to u_0(x)$ (case 2)

U_ℓ cannot converge to multiple solutions, so it <u>does not converge</u> as $\ell \to \infty$.