4.1: Sketch the Particle Path and Characteristics for a case with $\rho_\ell + \rho_r < \rho_{max}$.

The initial data is $\rho(x,0) = \begin{cases} \rho_\ell & x<0 \\ \rho_r & x>0 \end{cases}$ w/ $0<\rho_\ell<\rho_r<\rho_{max}$

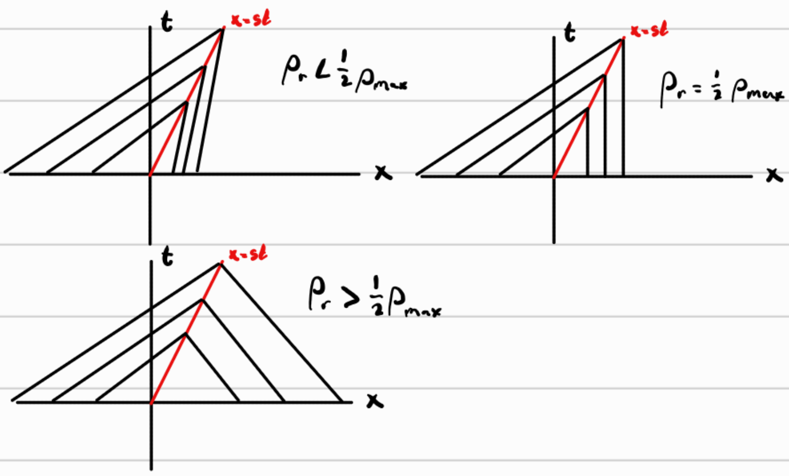
The Shock Speed is $S = u_{max}\left(1 - \frac{\rho_\ell+\rho_r}{\rho_{max}}\right)$, Since $\rho_\ell+\rho_r < \rho_{max}$ we have $S>0$. So the traffic will still go forward. This makes sense as the density of ρ_ℓ or ρ_ℓ is always less than ρ_{max}, So traffic is never stand-still.

Since the density on the right of the Shock is larger, the Velocity will be less. That is $u(\rho_\ell) > u(\rho_r)$ So the cars travel more distance in the Same time to the left of the Shock (See the trajectory graph).

Since $\rho_\ell < \rho_r$ and $\rho_\ell + \rho_r < \rho_{max}$, We must have $\rho_\ell < \frac{1}{2}\rho_{max}$. In this case, $f'(\rho_\ell) = u_{max}\left(1 - \frac{2\rho_\ell}{\rho_{max}}\right) > 0$, So the Characteristics to the left of the shock will move right.

However, for ρ_r, we can have $\rho_r < \frac{1}{2}\rho_{max}$, $\rho_r = \frac{1}{2}\rho_{max}$ or $\rho_r > \frac{1}{2}\rho_{max}$ all Possible. Each gives a different value of $f'(\rho_r)$: $f'(\rho_r)>0$, $f'(\rho_r)=0$, $f'(\rho_r)<0$

Characteristics

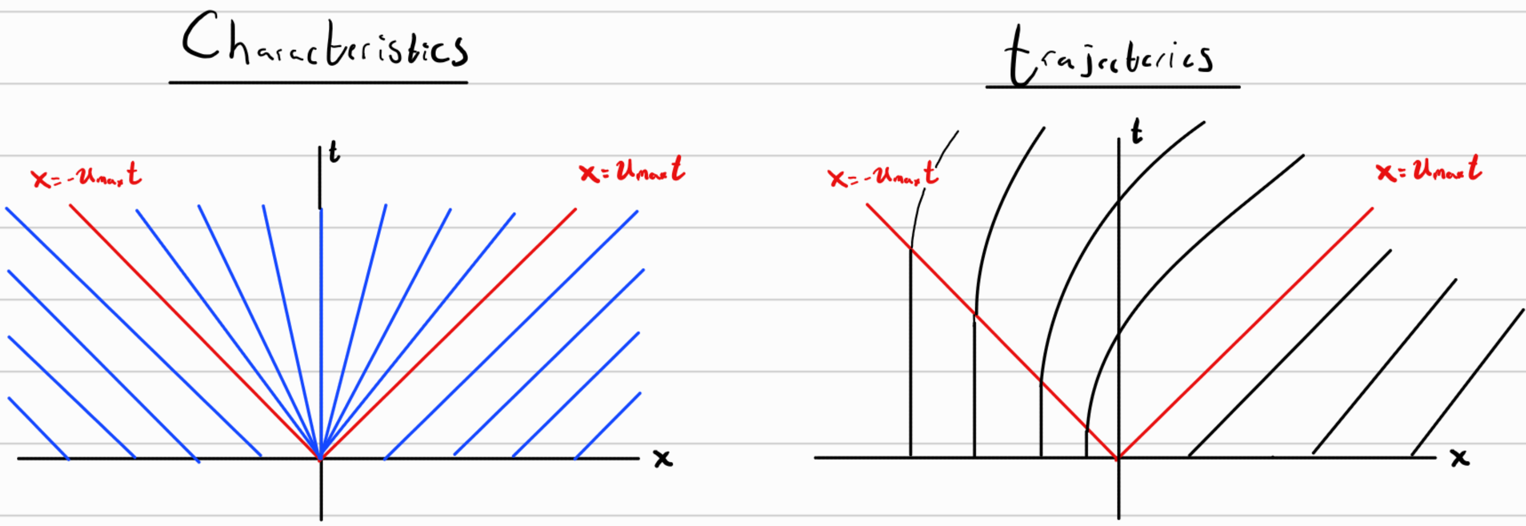


Vehicle Trajectories

4.2: Take Initial condition

$$\rho(x,0) = \begin{cases} \rho_{max} & x<0 \\ 0 & x>0 \end{cases}$$

Solve this Riemann Problem and sketch the characteristics and trajectories.



$$\boxed{\rho(x,t) = \begin{cases} \rho_{max} & x< -u_{max}t \\ \frac{\rho_{max}}{2}\left(1-\frac{x}{u_{max}t}\right) & -u_{max}t < x < u_{max}t \\ 0 & u_{max}t < x \end{cases}}$$

To find this solution, recall eqn (3.29)

$$\rho(x,t) = \begin{cases} \rho_l & x < f'(\rho_l)t \\ v(x/t) & f'(\rho_l)t < x < f'(\rho_r)t \\ \rho_r & f'(\rho_r)t < x \end{cases} \qquad \text{where } f'(v(\xi)) = \xi$$

here, $\rho_l = \rho_{max}$, $\rho_r = 0$ and $f'(\rho_{max}) = -u_{max}$, $f'(0) = u_{max}$.

So, we just need to find $v(\xi)$.

$$f'(\rho) = u_{max}\left(1-\frac{2\rho}{\rho_{max}}\right)$$

$$f'(v(\xi)) = \xi = u_{max}\left(1 - \frac{2v(\xi)}{\rho_{max}}\right)$$

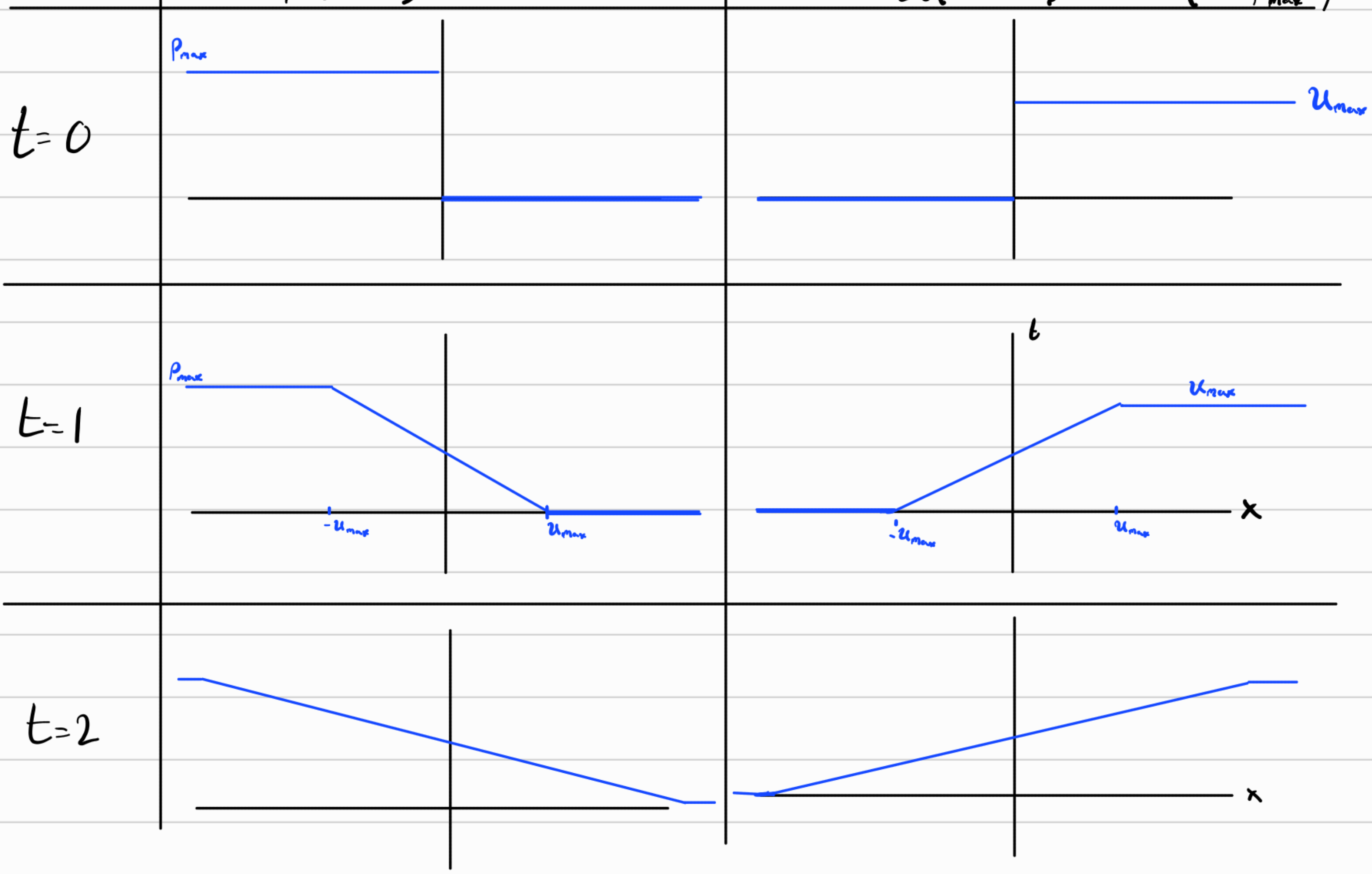
$$1 - \frac{\xi}{u_{max}} = \frac{2v(\xi)}{\rho_{max}}$$

$$\frac{\rho_{max}}{2}\left(1 - \frac{\xi}{u_{max}}\right) = v(\xi)$$

So, $v\left(\frac{x}{t}\right) = \frac{\rho_{max}}{2}\left(1 - \frac{x}{t\,u_{max}}\right)$ as written above.

4.3: Sketch the distribution of ρ and u at some fixed $t > 0$ for exercise 4.2

	$\rho(x, t)$	$u(\rho(x,t)) = u_{max}\left(1 - \frac{\rho(x,t)}{\rho_{max}}\right)$



4.4: Determine the manner in which a given car accelerates in the solution to 4.2, i.e. determine $V(t)$ where V is velocity along same particle path.

For fixed x^*,

$x^* < 0$ | $x^* > 0$

$$V(t) = \begin{cases} 0 & t < -\frac{x^*}{u_{max}} \\ u_{max} + \frac{x^*}{t} & t > -\frac{x^*}{u_{max}} \end{cases} \qquad V(t) = u_{max}$$

When $V(t) = 0$ or $V(t) = u_{max}$ is clear. For the third case $\rho(x,t) = \frac{\rho_{max}}{2}\left(1 - \frac{x}{u_{max} t}\right)$

update position

$$V(t) = u(\rho(x^* + tV(t), t)) = u\left(\frac{\rho_{max}}{2} - \frac{\rho_{max} x^* + \rho_{max} t V(t)}{2 t u_{max}}\right)$$

$$= u\left(\frac{\rho_{max}}{2} - \frac{\rho_{max} x^*}{2 t u_{max}} - \frac{\rho_{max} V(t)}{2 u_{max}}\right)$$

$$= u_{max}\left(1 - \frac{\frac{\rho_{max}}{2} - \frac{\rho_{max} x^*}{2 t u_{max}} - \frac{\rho_{max} V(t)}{2 u_{max}}}{\rho_{max}}\right)$$

$$= u_{max}\left(1 - \frac{1}{2} + \frac{x^*}{2 t u_{max}} + \frac{V(t)}{2 u_{max}}\right)$$

$$= \frac{1}{2}\left(u_{max} + \frac{x^*}{t}\right) + \frac{1}{2} V(t)$$

So,

$$\frac{1}{2} V(t) = \frac{1}{2}\left(u_{max} + \frac{x^*}{t}\right) \Rightarrow V(t) = u_{max} + \frac{x^*}{t}$$

4.5: For $c = f'(\rho_0) - u(\rho_0) = -u_{max}\frac{\rho_0}{\rho_{max}}$, what is the the Physical significance of the fact that $c<0$?

$f'(\rho_0)$ is the speed at which the variations propagate. Since $c<0$, we have

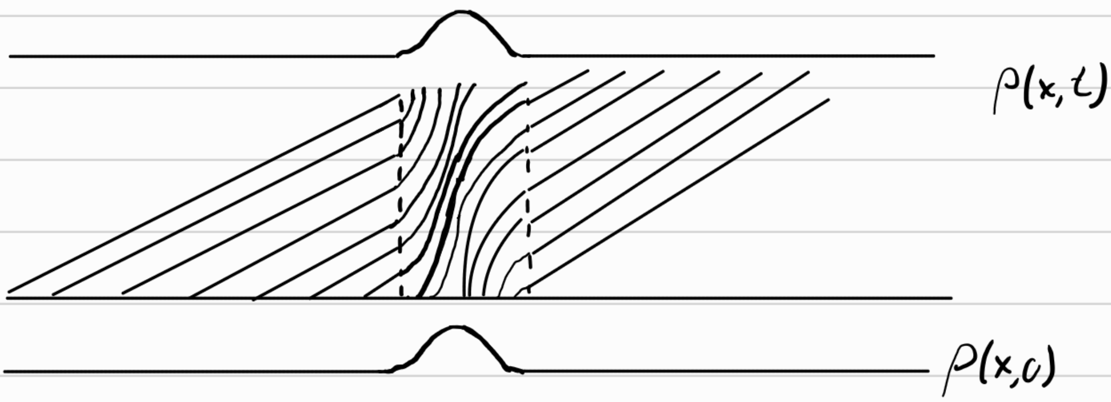
$$f'(\rho_0) < u(\rho_0)$$

which ensures that the speed of the disturbance cannot propagate faster than the flow of traffic.

4.6: Sketch Particle Paths for the case $\rho_0 = \frac{1}{2}\rho_{max}$.

Here, $c = -u(\rho_0)$ so the disturbance propagates at the same speed as traffic (just in the opposite direction). (or note $f'(\rho_0)=0$)

Relative to the flow of traffic, the disturbance doesn't move, and locally, the density is always the same.



4.7: Consider a shock wave with left and right states ρ_l and ρ_r, and let the shock strength approach zero, by $\rho_l \to \rho_r$.

Show the shock speed for these weak shock approaches the linearized propagation speed $f'(\rho_r)$.

$$S = \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} = U_{max}\left(1 - \frac{\rho_l + \rho_r}{\rho_{max}}\right)$$

if $\rho_l \to \rho_r$

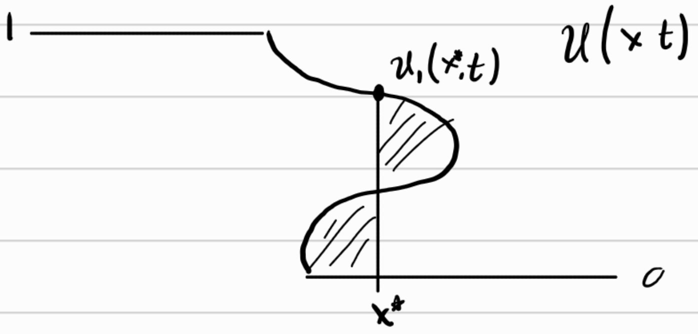
$$S = \lim_{\rho_l \to \rho_r} \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} = f'(\rho_r) \text{ by definition}$$

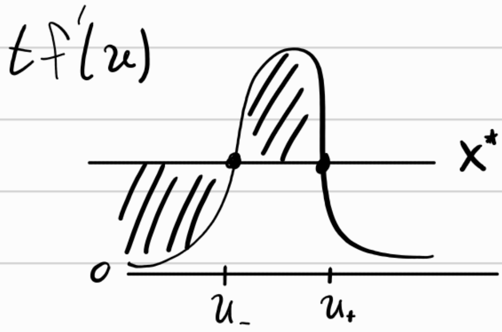
alternatively,

$$S = U_{max}\left(1 - \frac{\rho_l + \rho_r}{\rho_{max}}\right)$$

$$\lim_{\rho_l \to \rho_r} U_{max}\left(1 - \frac{\rho_l + \rho_r}{\rho_{max}}\right) = U_{max}\left(1 - \frac{2\rho_r}{\rho_{max}}\right) = f'(\rho_r) \text{ as well.}$$

4.8: Use the equal area rule to find an expression for the shock location as a function of t and verify that the Rankine-Hugoniot condition is always satisfied.





$$\int_0^{u_-} x^* - tf'(u)\,du = \int_{u_-}^{u_+} tf'(u) - x^*\,du$$

$$= u_-x^* - tf(u_-) + \underbrace{tf(0)}_{=0} = tf(u_+) - tf(u_-) - u_+x^* + u_-x^*$$

$$tf(u_+) = u_+x^*$$

$$x^* = \frac{tu_+}{u_+^2 + a(1-u_+)^2}, \quad tf'(u_+) = x^* = t\,\frac{2u_+(u_+^2 + a(1-u_+)^2) - u_+^2(2u_+ - 2a + 2au_+)}{(u_+^2 + a(1-u)^2)^2}$$

$$= t\,\frac{2u_+^3 + 2au_+ - 4au_+^2 + 2au_+^3 - 2u_+^3 + 2au_+^2 - 2au_+^3}{(u_+^2 + a(1-u_+)^2)^2}$$

$$\frac{tu_+}{u_+^2 + a(1-u_+)^2} = t \cdot 2au_+\,\frac{1-u_+}{(u_+^2 + a(1-u_+)^2)^2}$$

$$u_+^2 + a(1-u_+)^2 = 2a - 2au_+$$

$$u_+^2 + a - 2au_+ + au_+^2 = 2a - 2au_+$$

$$(1+a)\,u_+^2 = a$$

$$u_+^2 = \frac{a}{1+a} \rightarrow u_+ = \sqrt{\frac{a}{1+a}}$$

Ignore negative as $u_+ > 0$

Then, $\boxed{x^* = tf'\left(\sqrt{\tfrac{a}{1+a}}\right)}$

$$= tf\left(\sqrt{\frac{a}{1+a}}\right) \cdot \sqrt{\frac{1+a}{a}}$$

The Speed of the Shock is $f'(u_\ell)$ with $u_\ell = \sqrt{\frac{a}{1+a}}$ and $u_r = 0$

then $\dfrac{f(u_\ell) - f(0)}{u_\ell - 0} = \dfrac{f(u_\ell)}{u_\ell}$, from above we know $\dfrac{f(u_\ell)}{u_\ell} = f'(u_\ell)$

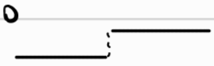
So the Rankine-Hugoniot Condition is satisfied.

4.9: Explain why its impossible to have a Riemann Solution involving both a shock and rarefaction wave when f is convex or concave.

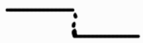
We will consider the two cases of concavity and convexity separate.

In Riemann problem

u_0



or

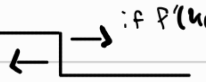


Convex $(f'' > 0)$

If $u_l > u_r$, then by convexity

$$f'(u_l) > f'(u_r)$$

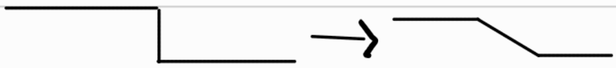
So we will have a shock moving



in the direction of $f'(u_l)$. The only two values are u_l or u_r, so this will remain a shock.

Concave $(f'' < 0)$

If $u_l > u_r$, then $f'(u_l) < f'(u_r)$



as the right gets pulled away from the left. Since $f'' < 0$, we have f' decreasing.

So, for values of u^* s.t. $u_l > u^* > u_r$ we have $f'(u_l) < f'(u^*) < f'(u_r)$ so the characteristics cannot converge as u^* cannot "catch up" to any point around it. So only a rarefaction wave is possible.

Convex: If $u_l < u_r$, the idea is the same as the concave case for $u_l > u_r$.

We only have a rarefaction wave.

Concave: If $u_l < u_r$, the idea is the same as the convex case when $u_l > u_r$.

We only have a shock.

4.10: Show that (3.46) is violated if the Shock goes above u^*.

(3.46)

$$\frac{f(u)-f(u_l)}{u-u_l} \geq s \geq \frac{f(u)-f(u_r)}{u-u_r}$$ for all u between u_l and u_r

If the Shock was connected to some point $\hat{u} > u^*$,

Since $u_l = \hat{u}$ here, we have

$$\frac{f(u)-f(\hat{u})}{u-\hat{u}} \geq s \geq \frac{f(u)-f(u_r)}{u-u_r} \text{ for all } u \in [0,\hat{u})$$

However, this certainly cannot be true for all u as $u^* \in [0,\hat{u})$

and here, $\frac{f(u^*)-f(u_r)}{u^*-u_r} = s$ is the slope of the secant from u^* to u^r
$= f'(u^*)$

and $$\frac{f(u^*)-f(\hat{u})}{u^*-\hat{u}} < f'(u^*) = \frac{f(u^*)-f(u_r)}{u^*-u_r}$$

as with u^*, we had a convex hull, so all secant lines on the hull are contained in the hull, so $f'(u^*)$ will be larger than any secant on the hull. Since $\hat{u} > u^*$, both points are on the hull as u^* is minimal to create a convex hull. So (3.46) is violated if the shock goes above u^*