Topic 2 - Power Series Convergence

A) $\sum_{n=1}^{\infty} \frac{1}{2^n} x^n$

ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}} \frac{2^n}{x^n} \right| = \left|\frac{x}{2}\right| < 1$$

$$\left|\frac{x}{2}\right| < 1 \Rightarrow -2 < x < 2$$

check endpoints

$x=2$ $\quad \sum_{n=1}^{\infty} \frac{1}{2^n} 2^n = \sum_{n=1}^{\infty} 1$ diverges

$x=-2$ $\quad \sum_{n=1}^{\infty} \frac{1}{2^n} (-2)^n = \sum_{n=1}^{\infty} (-1)^n$ alternating series diverges

interval of convergence $(-2, 2)$

center of power series $x = 0$

radius of convergence 2

B) $f(x) = 1 + x + 4x^2 + 9x^3 + 16x^4 + \dots$

$\Rightarrow 1 + \sum_{n=1}^{\infty} n^2 x^n$

$\lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right| = |x| < 1 \qquad -1 < x < 1$

check end points

$x = 1) \quad 1 + \sum_{n=1}^{\infty} n^2 (1^n)$ diverges

$x = -1) \quad 1 + \sum_{n=1}^{\infty} n^2 (-1)^n$ diverges

interval of convergence $(-1, 1)$
center of series $x = 0$
radius of convergence 1

C) $\sum_{n=1}^{\infty} \frac{1}{n+2} x^n$

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+3} \frac{n+2}{x^n} \right| = \lim_{n\to\infty} \left| x \frac{n+2}{n+3} \right| = |x| < 1$

$-1 < x < 1$

check endpoints

$x = 1) \quad \sum_{n=1}^{\infty} \frac{1}{n+2} (1^n)$ diverges b/c limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

$x = -1) \quad \sum_{n=1}^{\infty} \frac{1}{n+2} (-1)^n$ converges by alternating series test

interval of convergence $[-1, 1)$
center of series $x = 0$
radius of convergence 1

D) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+3)!} \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\, x}{(2n+3)(2n+2)} \right| = 0 \quad \forall x$$

interval of convergence is all real #s
$(-\infty, \infty) = \mathbb{R}$

center of series $x=0$
radius is infinite

E) $2 \sum_{n=1}^{\infty} \frac{3^n}{2n} x^n = \sum_{n=1}^{\infty} \frac{3^n}{n} x^n$

$$\lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{n+1} \frac{n}{3^n x^n} \right| = \lim_{n\to\infty} \left| 3x \frac{n}{n+1} \right| = |3x|$$

$$|3x| < 1 \quad \Rightarrow \quad -\frac{1}{3} < x < \frac{1}{3}$$

check end points

$x = -\frac{1}{3}$) $\sum_{n=1}^{\infty} \frac{3^n}{n} \left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges b/c of alternating series test

$x = 1/3$) $\sum_{n=1}^{\infty} \frac{3^n}{n} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges harmonic series

interval of convergence $\left[-\frac{1}{3}, \frac{1}{3}\right)$

center $x=0$

radius of convergence $\frac{1}{3}$

F) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n2^n} (x-3)^n$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2} (x-3)^{n+1}}{(n+1)\, 2^{n+1}} \; \frac{n\, 2^n}{(-1)^{n+1} (x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\,(x-3)}{2} \; \frac{n}{n+1} \right| = \left| \frac{x-3}{2} \right| < 1$$

$$\left| \frac{x-3}{2} \right| < 1 \qquad \Rightarrow \qquad 1 < x < 5$$

check endpoints

$x=1$) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n2^n} (-2)^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$$

$\Rightarrow$ diverges
harmonic series

$x=5$) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n2^n} 2^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

converges
alternating harmonic series

interval of convergence $[1, 5]$
center of power series $x=3$
radius of convergence 2