Weibull distribution $(\beta \geq 1, \theta > 0)$

$$f(x) = \begin{cases} \left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

To show f is a probability density function

see that

① $f(x) \geq 0 \quad \forall x$ because $\theta > 0$
$\beta \geq 1$
and $e^{-\left(\frac{x}{\theta}\right)^{\beta}} > 0 \quad \forall x$

② we want to show $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}} dx$$

let $u = \left(\frac{x}{\theta}\right)^{\beta}$ $\quad du = \beta\left(\frac{x}{\theta}\right)^{\beta-1}\left(\frac{1}{\theta}\right)dx$

limits $x = 0 \Rightarrow u = 0$
$x \to \infty \Rightarrow u \to \infty$

$$\int_0^{\infty} e^{-u}\,du = \lim_{b\to\infty} \int_0^b e^{-u}\,du = \lim_{b\to\infty} -e^{-u}\Big|_0^b$$

$$= \lim_{b\to\infty} 1 - e^{-b} = 1$$