Weibull distribution $(\beta \geq 1, \theta > 0)$

$$f(x) = \begin{cases} \left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

To show f is a probability density function

see that

① $f(x) \geq 0 \quad \forall x$ because $\theta > 0$, $\beta \geq 1$ and $e^{-\left(\frac{x}{\theta}\right)^{\beta}} > 0 \quad \forall x$

② we want to show $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}} dx$$

let $u = \left(\frac{x}{\theta}\right)^{\beta}$ $\quad du = \beta\left(\frac{x}{\theta}\right)^{\beta-1}\left(\frac{1}{\theta}\right)dx$

limits $x = 0 \Rightarrow u = 0$

$x \to \infty \Rightarrow u \to \infty$

$$\int_{0}^{\infty} e^{-u}\,du = \lim_{b\to\infty} \int_{0}^{b} e^{-u}\,du = \lim_{b\to\infty} -e^{-u}\Big|_{0}^{b}$$

$$= \lim_{b\to\infty} 1 - e^{-b} = 1$$

Example 2

$$\int \frac{e^x}{e^{2x}+2e^x-8}\,dx \qquad \begin{array}{l} \text{let } u = e^x \\ du = e^x\,dx \end{array}$$

$$\int \frac{du}{u^2+2u-8} = \int \frac{du}{(u+4)(u-2)}$$

use partial fractions $\frac{1}{(u+4)(u-2)} = \frac{A}{u+4} + \frac{B}{u-2}$

$$1 = A(u-2) + B(u+4)$$

$$u=2 \Rightarrow B = 1/6$$

$$u=-4 \Rightarrow A = -1/6$$

$$\int \frac{du}{u^2+2u-8} = \int \frac{-1}{6}\frac{1}{u+4}\,du + \int \frac{1}{6}\frac{1}{u-2}\,du$$

$$= -\frac{1}{6}\ln|u+4| + \frac{1}{6}\ln|u-2| + C$$

$$= -\frac{1}{6}\ln|e^x+4| + \frac{1}{6}\ln|e^x-2| + C$$

Example 3

$$\int \frac{8-3x}{(x+1)(x^2-4x+6)}\,dx = \int \frac{8-3x}{(x+1)^2(x-5)}\,dx$$

use partial fractions

$$\frac{8-3x}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-5}$$

$$8-3x = A(x+1)(x-5) + B(x-5) + C(x+1)^2$$

$$x=5 \Rightarrow -7 = 36C$$

$$C = -7/36$$

$$x=-1 \Rightarrow 11 = -6B \Rightarrow B = -11/6$$

$$x=0 \Rightarrow 8 = -5A + -5B + C$$

$$8 = -5A + \frac{55}{6} - \frac{7}{36}$$

$$A = \frac{\frac{55}{6} + \frac{7}{36} - 8}{5} = \frac{35/36}{5} = \frac{7}{36}$$

$$\int \frac{8-3x}{(x+1)(x^2-4x+6)}\,dx = \frac{7}{36}\int \frac{dx}{x+1} - \frac{11}{6}\int \frac{dx}{(x+1)^2} - \frac{7}{36}\int \frac{dx}{x-5}$$

$$= \frac{7}{36}\ln|x+1| + \frac{11}{6}\,\frac{1}{x+1} - \frac{7}{36}\ln|x-5| + C$$

Example 4 Joule-Lenz Law

$$Q = .24R \int_0^T I_0^2 \sin^2\left(\frac{2\pi t}{T} - \phi\right) dt$$

$$= .24 R I_0^2 \int_0^T \sin^2\left(\frac{2\pi t}{T} - \phi\right) dt$$

$$= .24 R I_0^2 \int_0^T \frac{1 - \cos\left(\frac{4\pi t}{T} - 2\phi\right)}{2} dt$$

$$= .24 R I_0^2 \left[\frac{1}{2}T - \frac{1}{2}\int_0^T \cos\left(\frac{4\pi t}{T} - 2\phi\right) dt\right]$$

$$= .24 R I_0^2 \left[\frac{1}{2}T - \frac{1}{2} \left.\frac{\sin\left(\frac{4\pi t}{T} - 2\phi\right)}{4\pi/T}\right|_0^T\right]$$

$$= .24 R I_0^2 \left[\frac{1}{2}T - \frac{1}{8\pi}T\left(\sin(4\pi - 2\phi) - \sin(-2\phi)\right)\right]$$

$$= \frac{.24 R I_0^2 T}{2}$$

Example 5 Radioactive Decay $(k<0)$

$$\mu = -k\int_0^\infty t e^{kt}\,dt$$

$$= \lim_{b\to\infty} -k\int_0^b t e^{kt}\,dt$$ IBP $u=t$ $dv=e^{kt}dt$

$du=dt$ $v=\frac{1}{k}e^{kt}$

$$= \lim_{b\to\infty} -k\left[\frac{1}{k}te^{kt}\Big|_0^b - \frac{1}{k}\int_0^b e^{kt}\,dt\right]$$

$$= \lim_{b\to\infty} -k\left[\frac{b}{k}e^{kb} - 0 - \frac{1}{k^2}e^{kt}\Big|_0^b\right]$$

$$= \lim_{b\to\infty} -k\left[\frac{b}{k}e^{kb} - \frac{1}{k^2}e^{kb} + \frac{1}{k^2}\right]$$

as $b\to\infty$ $e^{kb}\to 0$ since $k<0$

$$= -k\left(\frac{1}{k^2}\right) = -\frac{1}{k}$$ Note $-\frac{1}{k}>0$