IDE

Phase Plane Drawing

Try your hand at drawing a shape in the phase plane and see the component tx and ty graphs unfold in real time.

and that $t = 0$ where each phase plane trajectory crosses the positive $\dot{x}$ axis. The time series graphs always show $x(0)$ and $\dot{x}(0)$ values on their vertical axes, so you know that on the phase plane graph $t = 0$ when $x = 0$ and $\dot{x}$ is positive.

A Systems View of Second-Order Equations

Note that a second-order equation

$$a\ddot{x} + b\dot{x} + cx = 0, \tag{11}$$

where a, b and c are real constants with $a \neq 0$, can also be expressed as a system of two first-order equations. If you introduce a second variable $y = \dot{x}$, then $\dot{y} = \ddot{x}$ and you should confirm that (11) can be written equivalently as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{c}{a}x - \frac{b}{a}y. \end{aligned}$$

This system creates an xy phase plane description that is exactly the $x\dot{x}$ phase plane just described.

Many open-ended DE solvers on computers require second-order equations to be entered in the equivalent form, so you should become facile with the conversion. We will look at this process in more detail in Sec. 4.6.

Modeling Electrical Circuits

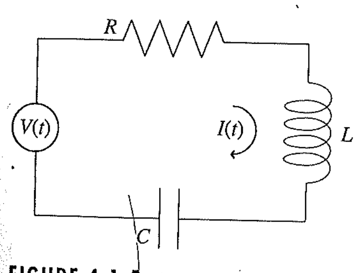


FIGURE 4.1.5 A series LRC-circuit; the arrow for $I(t)$ indicates the positive direction for the current.

The basic law of physics for electrical circuits is **Kirchoff's Voltage Law** instead of Newton's Second Law. First note that the current I in a wire, measured in *amperes* (amps), is a *flow* of charges, negative in the direction of the flow of electrons (negative charges); that is, the current I is the rate of change of the charge Q:

$$I(t) = \dot{Q}(t).$$

For a simple closed electrical series circuit like the one in Fig. 4.1.5, this law tells us that *the input voltage equals the sum of the voltage drops around the circuit.* These voltage drops are of three kinds.

1. **Drop across a Resistor:** The voltage drop $V(t)$ across a resistor is proportional to the current $I(t)$ passing through the resistor **(Ohm's Law)**:[6]

$$V_R(t) = RI(t), \tag{12}$$

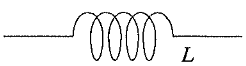
where the constant of proportionality R is the **resistance** of the resistor, measured in *ohms*. A resistor is often a carbon device with resistance of 100 or 200 ohms, while the resistance of a copper wire is generally negligible.

[6]German physicist Georg Simon Ohm (1789–1854) discovered the law that carries his name. Although this work was to be of great influence in developing the theory of electrical circuits, its importance was not recognized by his colleagues for more than a decade.

Resistance

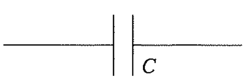
$$V_R = RI$$

Inductance



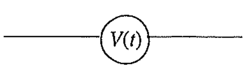
$$V_L = L\dot{I}$$

Capacitance



$$V_C = \frac{1}{C}Q$$

Input Voltage



Current and Charge

$$I(t) = \dot{Q}(t)$$

2. **Drop across an Inductor:** According to **Faraday's Law**,[7] *the voltage drop across an inductor is proportional to the time rate of change of the current passing through it*:
$$V_L(t) = L\dot{I}(t), \tag{13}$$
where the constant of proportionality L is the **inductance**, measured in *henrys*. Inductors are generally coils of wire and are drawn as such.
3. **Drop across a Capacitor:** The voltage drop across a capacitor is proportional to the charge $Q(t)$ on the capacitor. The proportionality constant is written as $1/C$, where C is the **capacitance** of the capacitor, measured in *farads*:
$$V_C(t) = \frac{1}{C}Q(t).$$
A capacitor usually consists of two parallel plates separated by a gap through which no current flows; $Q(t)$ is the charge on one plate relative to the other. Although no current crosses the gap, the (alternating) current surges back and forth from plate to plate through the rest of the circuit. Since $I(t) = \dot{Q}(t)$, this voltage drop can be written
$$V_C(t) = \frac{1}{C}\int I(t)\,dt. \tag{14}$$

We can now apply Kirchoff's Voltage Law to the circuit of Fig. 4.1.5, where the voltage source $V(t)$ is a battery or electric generator:

$$RI + L\dot{I} + \frac{1}{C}\int I(t)\,dt = V(t). \tag{15}$$

The left side of equation (15) is the sum of the voltage drops given by equations (12), (13) and (14) for the elements of the circuit. The result, containing both a derivative and an integral, is called an **integro-differential equation**. The simplest version of this equation for the series circuit equation uses again the fact that $I(t) = \dot{Q}(t)$:

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t).$$

If we assume that there is no voltage source, so that $V(t) \equiv 0$, equation (15) can be written

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0, \tag{16}$$

a linear homogeneous second-order DE with constant coefficients. It is, in fact, just the equation of the harmonic oscillator in disguise. If an initial charge $Q(0) = Q_0$ and an initial current $I(0) = \dot{Q}(0) = I_0$ are given, we have an initial-value problem for equation (16). The solution $Q(t)$ of this IVP and its

[7] Michael Faraday (1791–1867), English chemist and physicist, united electricity and chemistry with his strong work on electrolysis, the process that liberates an element by passing electric current through a molten compound that contains the element. Equation (13) describes the electrical view; Faraday's other law of electrolysis gives the chemical view: the mass liberated by a given quantity of electricity is proportional to the atomic weight of the element liberated and is inversely proportional to its valence.

derivative $\dot{Q}(t) = I(t)$ give the capacitor charge and circuit current for subsequent times.

We can derive a differential equation for the current I by differentiating (15) with respect to t. Since $I(t) = \dot{Q}(t)$, the result is

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}(t).$$

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Series Circuits

Set values of R, L and C for a circuit. Click an initial value on the phase plane and see the electrifying results.

For $\dot{V}(t) \equiv 0$, we get the homogeneous equation

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0. \tag{17}$$

Appropriate initial conditions for (17) would be $I(0) = I_0$ and $\dot{I}(0) = \dot{I}_0$. The various circuit elements an their units are summarized in Table 4.1.2.

Table 4.1.2 Summary of units for electric circuit elements

Quantity		Units
Input voltage	$V(t)$	volt
Resistance	R	ohm
Inductance	L	henry
Capacitance	C	farad
Charge	$Q(t)$	coulomb
Current	$I(t)$	ampere

The Mechanical-Electrical Analog

We have noticed that equations (16) and (17) are just new instances of the harmonic oscillator equation (1). The mass on the table surges back and forth in the same mathematical pattern as the electrical current surging back and forth through the circuit of Fig. 4.1.5 (called the "*LRC*-circuit" because it contains an inductor, a resistor and a capacitor). The analogy enables us to apply methods of electrical engineering to problems in mechanics and vice versa. The basis of simulation of mechanical systems by analog computers is this correspondence between mechanical and electrical elements. Resistance, for example, plays the part of the friction term in the mass-spring system: both slow the process. Table 4.1.3 summarizes these correlations.

Notice that the capacitor stores charge and hence stores potential energy as does a compressed or stretched spring. The inductor produces a "back-voltage" as the current increases through it, which tends to retard the charge, adding inertia to the system as does the mass in the mechanical system. The power of the analogy can be seen in the following example.

Table 4.1.3 Mechanical-electrical analog

Mechanical System $m\ddot{x} + b\dot{x} + kx = f(t)$		Electrical System $L\ddot{Q} + R\dot{Q} + (1/C)Q = V(t)$	
Displacement	x	Q	Charge
Velocity	$\dot{x}$	$\dot{Q} = I$	Current
Mass	m	L	Inductance
Damping constant	b	R	Resistance
Spring constant	k	$1/C$	1/Capacitance
External force	$f(t)$	$V(t)$	Input voltage

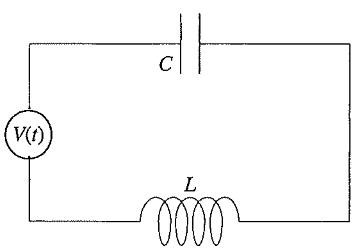

FIGURE 4.1.6 Comparison circuit.

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Series Circuit

Set $R = 0$ and $V = 0$. Find conditions that make the circuit oscillate.

EXAMPLE 3 **Comparison Circuit** Consider a circuit composed of a capacitor with capacitance C and an inductor with inductance L hooked in series (Fig. 4.1.6) and suppose that at time $t = 0$ a charge Q_0 is put on the capacitor. The IVP is

$$L\ddot{Q} + CQ = 0, \qquad Q(0) = Q_0, \quad \dot{Q}(0) = 0.$$

By equation (8) we can see that the solution is

$$Q(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \qquad \omega_0 = \sqrt{\frac{1}{LC}},$$

where c_1 and c_2 can be determined from the initial conditions. The important point here is that we have an oscillating circuit. Charge moves from the capacitor through the coil causing a change in current, which in turn causes a back-voltage, which causes the capacitor to reacquire its charge and so forth, forever. The lack of resistance acts like a lack of damping in a mechanical system. ■

Summary

Formulations of the differential equations for the mass-spring system from mechanics and the LRC-circuit from electrical theory lead to the same linear second-order model called the harmonic oscillator. Solutions in the undamped case are sinusoidal vibrations.

4.1 Problems

The Undamped Oscillator *In Problems 1–6, find the simple harmonic motion described by the initial-value problem. See also Problems 21–32.*

1. $\ddot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0$

2. $\ddot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 1$

3. $\ddot{x} + 9x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 1$

4. $\ddot{x} + 4x = 0, \quad x(0) = 1, \quad \dot{x}(0) = -2$

5. $\ddot{x} + 16x = 0, \quad x(0) = -1, \quad \dot{x}(0) = 0$

6. $\ddot{x} + 16x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 4$