__Ex__ Solve the IVP

$$x' = 3x + 4y \qquad x(0) = 1$$
$$y' = 3x + 2y \qquad y(0) = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find eigenvalues

$$\begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 12 = 0$$
$$= \lambda^2 - 5\lambda + 6 - 12 = 0$$
$$= \lambda^2 - 5\lambda - 6 = 0$$
$$= (\lambda - 6)(\lambda + 1) = 0$$
$$\lambda = 6, -1$$

Find eigen vectors

$\underline{\lambda_1 = 6}$

$$(A - 6I)\bar{v}_1 = \bar{0} \qquad \bar{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3a + 4b = 0$$
$$a = \frac{4}{3}b$$

$$\bar{v}_1 = \begin{bmatrix} \frac{4}{3}b \\ b \end{bmatrix} = b \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} = b_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\bar{V}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$\underline{\lambda_2 = -1}$

$$(A + I)\bar{v}_2 = \bar{0}$$

$$\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$4a + 4b = 0 \qquad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a = -b$$

$$b = -a$$

general soln

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{6t}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Use IC to find particular soln

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$4c_1 + c_2 = 1$$

$$3c_1 - c_2 = 1 \qquad c_1 = \frac{2}{7} \quad c_2 = -\frac{1}{7}$$

$$\overline{7c_1 + 0c_2 = 2}$$

$$c_2 = 3c_1 - 1 = \frac{6}{7} - 1 = -\frac{1}{7}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{2}{7}e^{6t}\begin{bmatrix} 4 \\ 3 \end{bmatrix} - \frac{1}{7}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Ex Solve the IVP

$$x' = 9x + 5y \qquad x(0) = 1$$
$$y' = -6x - 2y \qquad y(0) = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find eigen values

$$\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = (9-\lambda)(-2-\lambda) + 30 = 0$$
$$= \lambda^2 - 7\lambda - 18 + 30 = 0$$
$$= \lambda^2 - 7\lambda + 12 = 0$$
$$= (\lambda - 4)(\lambda - 3) = 0$$
$$\lambda = 4, \ \lambda = 3$$

Find eigen vectors

$\underline{\lambda_1 = 4}$

$$(A - 4I)\bar{v}_1 = \bar{0}$$

$$\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow a = -b$$

$$\bar{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix} = b \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\bar{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 3}$

$$(A - 3I)\bar{v}_2 = \bar{0} \qquad \bar{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{v}_2 = \begin{bmatrix} -\frac{5}{6} b \\ b \end{bmatrix} = b \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix} = b_1 \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$$6a + 5b = 0$$

$$a = -\frac{5}{6} b$$

$$\bar{v}_2 = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

general soln

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

use IC to find particular soln $\quad x(0) = 1$, $y(0) = 0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$$-c_1 - 5c_2 = 1$$

$$\underline{c_1 + 6c_2 = 0}$$

$$c_2 = 1 \Rightarrow c_2 = -6$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -6 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

EX Solve the IVP

$$x' = x - 2y \qquad x(0) = 0$$
$$y' = 2x + y \qquad y(0) = 4$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find eigenvalues

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) + 4 = 0$$
$$= \lambda^2 - 2\lambda + 1 + 4 = 0$$
$$= \lambda^2 - 2\lambda + 5 = 0$$
$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

Find one eigenvector

$\underline{\lambda_1 = 1+2i}$

$$(A - (1+2i)I)\bar{v}_1 = \bar{0} \qquad \bar{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2ia - 2b = 0$$
$$b = -ia \quad \Rightarrow \quad \bar{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Break soln into real and imaginary parts

$$\bar{x}_1 = e^{(1+2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^{t} \begin{bmatrix} 1 \\ -i \end{bmatrix} \left(\cos(2t) + i \sin(2t) \right)$$

$$= e^{t} \begin{bmatrix} \cos(2t) + i \sin(2t) \\ -i\cos(2t) + \sin(2t) \end{bmatrix}$$

$$= e^{t} \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + i e^{t} \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix}$$

write real-valued general soln

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{t} \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + c_2 e^{t} \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix}$$

use IC to find particular soln $\quad x(0) = 0$, $y(0) = 4$

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} c_1 = 0 \\ c_2 = -4 \end{array}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -4 e^{t} \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix}$$

Ex Solve the IVP

$$x' = 2x - 5y \qquad x(0) = 2$$
$$y' = 4x - 2y \qquad y(0) = 3$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find eigenvalues

$$\begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 20 = 0$$
$$= \lambda^2 - 4 + 20 = 0$$
$$= \lambda^2 + 16 = 0$$
$$\lambda = \pm 4i$$

Find one eigenvector

$\underline{\lambda = 4i}$

$$(A - 4iI)\bar{v}_1 = \bar{0}$$

$$\begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \bar{v}_1 = \begin{bmatrix} a \\ \frac{2-4i}{5} a \end{bmatrix} = \frac{a}{5} \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$$

$$(2-4i)a - 5b = 0$$
$$\Rightarrow b = \left(\frac{2-4i}{5}\right) a \qquad \bar{v}_1 = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$$

Find soln and break into real and imaginary parts

$$\bar{x}_1 = e^{4it}\begin{bmatrix}5\\2-4i\end{bmatrix} = \begin{bmatrix}5\\2-4i\end{bmatrix}(\cos 4t + i\sin 4t)$$

$$= \begin{bmatrix}5\cos(4t) + 5i\sin(4t)\\(2-4i)\cos(4t)+(2i+4)\sin(4t)\end{bmatrix}$$

$$= \begin{bmatrix}5\cos(4t)\\2\cos 4t + 4\sin 4t\end{bmatrix} + i\begin{bmatrix}5\sin(4t)\\-4\cos(4t)+2\sin(4t)\end{bmatrix}$$

Real valued general soln

$$\begin{bmatrix}x\\y\end{bmatrix} = c_1\begin{bmatrix}5\cos(4t)\\2\cos(4t)+4\sin(4t)\end{bmatrix} + c_2\begin{bmatrix}5\sin(4t)\\-4\cos(4t)+2\sin(4t)\end{bmatrix}$$

Use IC to find particular soln $\quad x(0)=2$
$\quad y(0)=3$

$$\begin{bmatrix}2\\3\end{bmatrix} = c_1\begin{bmatrix}5\\2\end{bmatrix} + c_2\begin{bmatrix}0\\-4\end{bmatrix}$$

$5c_1 + 0c_2 = 2 \qquad c_1 = \frac{2}{5}$

$2c_1 - 4c_2 = 3 \qquad c_2 = \frac{2c_1-3}{4} = \frac{\frac{4}{5}-3}{4} = \frac{-11}{20}$

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{2}{5}\begin{bmatrix}5\cos(4t)\\2\cos(4t)+4\sin(4t)\end{bmatrix} - \frac{11}{20}\begin{bmatrix}5\sin(4t)\\-4\cos(4t)+2\sin(4t)\end{bmatrix}$$