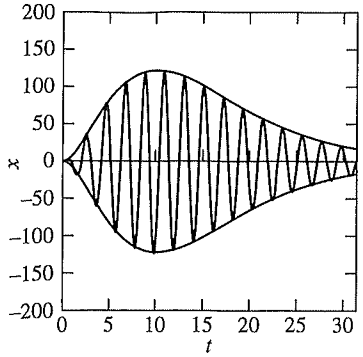

FIGURE 3.6.15. The solution $x(t) = e^{-t/5}[3t\cos t + (9t^2-1)\sin 3t]$ and the envelope curves $x(t) = \pm e^{-t/5}\sqrt{(3t)^2 + (9t^2-1)^2}$ with external force $F(t) = 2700te^{-t/5}\cos 3t$.

INVESTIGATION 3: With damped oscillatory external force

$$F(t) = 2700te^{-t/5}\cos 3t,$$

we have a still more complicated resonance situation. The *Mathematica* commands

```
de3 = 25 x''[t] + 10 x'[t] + 226 x[t] ==
        2700 t Exp[-t/5] Cos[3t]
soln = DSolve[{de3, x[0] == 0, x'[0] == 0}, x[t], t]
x = First[x[t]  /.   soln]
amp = Exp[-t/5] Sqrt[(3t)^2 + (9t^2 - 1)^2]
Plot[{x, amp, -amp}, {t, 0, 10 Pi}]
```

produce the plot shown in Fig. 3.6.15. We see the solution

$$x(t) = e^{-t/5}\left[3t\cos 3t + (9t^2-1)\sin 3t\right]$$

oscillating up and down between the envelope curves

$$x = \pm e^{-t/5}\sqrt{(3t)^2 + (9t^2-1)^2}.$$

3.7 Electrical Circuits

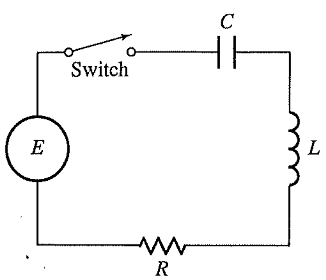

FIGURE 3.7.1. The series *RLC* circuit.

Here we examine the *RLC* circuit that is a basic building block in more complicated electrical circuits and networks. As shown in Fig. 3.7.1, it consists of

A **resistor** with a resistance of R *ohms*,
An **inductor** with an inductance of L *henries*, and
A **capacitor** with a capacitance of C *farads*

in series with a source of electromotive force (such as a battery or a generator) that supplies a voltage of $E(t)$ *volts* at time t. If the switch shown in the circuit of Fig. 3.7.1 is closed, this results in a current of $I(t)$ *amperes* in the circuit and a charge of $Q(t)$ *coulombs* on the capacitor at time t. The relation between the functions Q and I is

$$\frac{dQ}{dt} = I(t). \tag{1}$$

We will always use mks electric units, in which time is measured in seconds.

According to elementary principles of electricity, the **voltage drops** across the three circuit elements are those shown in the table in Fig. 3.7.2. We can analyze the behavior of the series circuit of Fig. 3.7.1 with the aid of this table and one of Kirchhoff's laws:

Circuit Element	Voltage Drop
Inductor	$L\dfrac{dI}{dt}$
Resistor	RI
Capacitor	$\dfrac{1}{C}Q$

FIGURE 3.7.2. Table of voltage drops.

> The (algebraic) sum of the voltage drops across the elements in a simple loop of an electrical circuit is equal to the applied voltage.

As a consequence, the current and charge in the simple *RLC* circuit of Fig. 3.7.1 satisfy the basic circuit equation

➤ $$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t). \tag{2}$$

If we substitute (1) in Eq. (2), we get the second-order linear differential equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{3}$$

for the charge $Q(t)$, under the assumption that the voltage $E(t)$ is known.

In most practical problems it is the current I rather than the charge Q that is of primary interest, so we differentiate both sides of Eq. (3) and substitute I for Q' to obtain

➤
$$LI'' + RI' + \frac{1}{C}I = E'(t). \tag{4}$$

We do *not* assume here a prior familiarity with electrical circuits. It suffices to regard the resistor, inductor, and capacitor in an electrical circuit as "black boxes" that are calibrated by the constants R, L, and C. A battery or generator is described by the voltage $E(t)$ that it supplies. When the switch is open, no current flows in the circuit; when the switch is closed, there is a current $I(t)$ in the circuit and a charge $Q(t)$ on the capacitor. All we need to know about these constants and functions is that they satisfy Eqs. (1) through (4), our mathematical model for the *RLC* circuit. We can then learn a good deal about electricity by studying this mathematical model.

The Mechanical–Electrical Analogy

It is striking that Eqs. (3) and (4) have precisely the same form as the equation

➤
$$mx'' + cx' + kx = F(t) \tag{5}$$

of a mass–spring–dashpot system with external force $F(t)$. The table in Fig. 3.7.3 details this important **mechanical–electrical analogy**. As a consequence, most of the results derived in Section 3.6 for mechanical systems can be applied at once to electrical circuits. The fact that the same differential equation serves as a mathematical model for such different physical systems is a powerful illustration of the unifying role of mathematics in the investigation of natural phenomena. More concretely, the correspondences in Fig. 3.7.3 can be used to construct an electrical model of a given mechanical system, using inexpensive and readily available circuit elements. The performance of the mechanical system can then be predicted by means of accurate but simple measurements in the electrical model. This is especially useful when the actual mechanical system would be expensive to construct or when measurements of displacements and velocities would be inconvenient, inaccurate, or even dangerous. This idea is the basis of *analog computers*—electrical models of mechanical systems. Analog computers modeled the first nuclear reactors for commercial power and submarine propulsion before the reactors themselves were built.

Mechanical System	**Electrical System**
Mass m	Inductance L
Damping constant c	Resistance R
Spring constant k	Reciprocal capacitance $1/C$
Position x	Charge Q (using (3) (or current I using (4)))
Force F	Electromotive force E (or its derivative E')

FIGURE 3.7.3. Mechanical–electrical analogies.

In the typical case of an alternating current voltage $E(t) = E_0 \sin \omega t$, Eq. (4) takes the form

$$LI'' + RI' + \frac{1}{C}I = \omega E_0 \cos \omega t. \tag{6}$$

As in a mass–spring–dashpot system with a simple harmonic external force, the solution of Eq. (6) is the sum of a **transient current** I_{tr} that approaches zero as $t \to +\infty$ (under the assumption that the coefficients in Eq. (6) are all positive, so the roots of the characteristic equation have negative real parts), and a **steady periodic current** I_{sp}; thus

$$I = I_{\text{tr}} + I_{\text{sp}}. \tag{7}$$

Recall from Section 3.6 (Eqs. (19) through (22) there) that the steady periodic solution of Eq. (5) with $F(t) = F_0 \cos \omega t$ is

$$x_{\text{sp}}(t) = \frac{F_0 \cos(\omega t - \alpha)}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}},$$

where

$$\alpha = \tan^{-1} \frac{c\omega}{k - m\omega^2}, \quad 0 \leqq \alpha \leqq \pi.$$

If we make the substitutions L for m, R for c, $1/C$ for k, and ωE_0 for F_0, we get the steady periodic current

$$I_{\text{sp}}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \tag{8}$$

with the phase angle

$$\alpha = \tan^{-1} \frac{\omega RC}{1 - LC\omega^2}, \quad 0 \leqq \alpha \leqq \pi. \tag{9}$$

Reactance and Impedance

The quantity in the denominator in (8),

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad \text{(ohms)}, \tag{10}$$

is called the **impedance** of the circuit. Then the steady periodic current

$$I_{\text{sp}}(t) = \frac{E_0}{Z} \cos(\omega t - \alpha) \tag{11}$$

has amplitude

$$I_0 = \frac{E_0}{Z}, \tag{12}$$

reminiscent of Ohm's law, $I = E/R$.

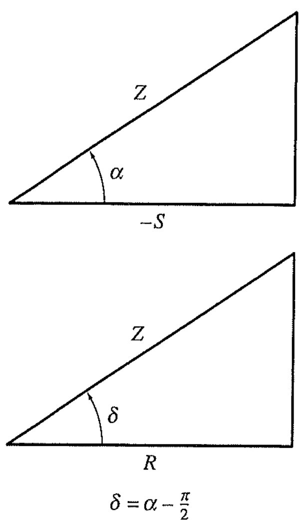

FIGURE 3.7.4. Reactance and delay angle.

Equation (11) gives the steady periodic current as a cosine function, whereas the input voltage $E(t) = E_0 \sin \omega t$ was a sine function. To convert I_{sp} to a sine function, we first introduce the **reactance**

$$S = \omega L - \frac{1}{\omega C}. \tag{13}$$

Then $Z = \sqrt{R^2 + S^2}$, and we see from Eq. (9) that α is as in Fig. 3.7.4, with delay angle $\delta = \alpha - \frac{1}{2}\pi$. Equation (11) now yields

$$\begin{aligned} I_{sp}(t) &= \frac{E_0}{Z}(\cos\alpha\cos\omega t + \sin\alpha\sin\omega t) \\ &= \frac{E_0}{Z}\left(-\frac{S}{Z}\cos\omega t + \frac{R}{Z}\sin\omega t\right) \\ &= \frac{E_0}{Z}(\cos\delta\sin\omega t - \sin\delta\cos\omega t). \end{aligned}$$

Therefore,

$$I_{sp}(t) = \frac{E_0}{Z}\sin(\omega t - \delta), \tag{14}$$

where

$$\delta = \tan^{-1}\frac{S}{R} = \tan^{-1}\frac{LC\omega^2 - 1}{\omega RC}. \tag{15}$$

This finally gives the **time lag** δ/ω (in seconds) of the steady periodic current I_{sp} behind the input voltage (Fig. 3.7.5).

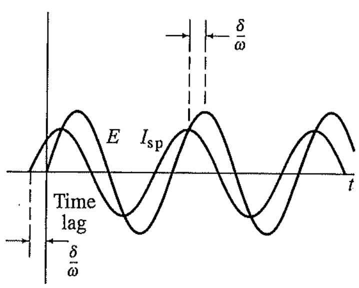

FIGURE 3.7.5. Time lag of current behind imposed voltage.

Initial Value Problems

When we want to find the transient current, we are usually given the initial values $I(0)$ and $Q(0)$. So we must first find $I'(0)$. To do so, we substitute $t = 0$ in Eq. (2) to obtain the equation

$$LI'(0) + RI(0) + \frac{1}{C}Q(0) = E(0) \tag{16}$$

to determine $I'(0)$ in terms of the initial values of current, charge, and voltage.

Example 1 Consider an RLC circuit with $R = 50$ ohms (Ω), $L = 0.1$ henry (H), and $C = 5 \times 10^{-4}$ farad (F). At time $t = 0$, when both $I(0)$ and $Q(0)$ are zero, the circuit is connected to a 110-V, 60-Hz alternating current generator. Find the current in the circuit and the time lag of the steady periodic current behind the voltage.

Solution A frequency of 60 Hz means that $\omega = (2\pi)(60)$ rad/s, approximately 377 rad/s. So we take $E(t) = 110 \sin 377t$ and use equality in place of the symbol for approximate equality in this discussion. The differential equation in (6) takes the form

$$(0.1)I'' + 50I' + 2000I = (377)(110)\cos 377t.$$

We substitute the given values of R, L, C, and $\omega = 377$ in Eq. (10) to find that the impedance is $Z = 59.58\ \Omega$, so the steady periodic amplitude is

$$I_0 = \frac{110 \text{ (volts)}}{59.58 \text{ (ohms)}} = 1.846 \text{ amperes (A)}.$$

With the same data, Eq. (15) gives the sine phase angle:

$$\delta = \tan^{-1}(0.648) = 0.575.$$

Thus the time lag of current behind voltage is

$$\frac{\delta}{\omega} = \frac{0.575}{377} = 0.0015 \text{ s},$$

and the steady periodic current is

$$I_{\text{sp}} = (1.846)\sin(377t - 0.575).$$

The characteristic equation $(0.1)r^2 + 50r + 2000 = 0$ has the two roots $r_1 \approx -44$ and $r_2 \approx -456$. With these approximations, the general solution is

$$I(t) = c_1 e^{-44t} + c_2 e^{-456t} + (1.846)\sin(377t - 0.575),$$

with derivative

$$I'(t) = -44c_1 e^{-44t} - 456c_2 e^{-456t} + 696\cos(377t - 0.575).$$

Because $I(0) = Q(0) = 0$, Eq. (16) gives $I'(0) = 0$ as well. With these initial values substituted, we obtain the equations

$$\begin{aligned} I(0) &= c_1 + c_2 - 1.004 = 0, \\ I'(0) &= -44c_1 - 456c_2 + 584 = 0; \end{aligned}$$

their solution is $c_1 = -0.307$, $c_2 = 1.311$. Thus the transient solution is

$$I_{\text{tr}}(t) = (-0.307)e^{-44t} + (1.311)e^{-456t}.$$

The observation that after one-fifth of a second we have $|I_{\text{tr}}(0.2)| < 0.000047$ A (comparable to the current in a single human nerve fiber) indicates that the transient solution dies out very rapidly, indeed. ■

Example 2 Suppose that the *RLC* circuit of Example 1, still with $I(0) = Q(0) = 0$, is connected at time $t = 0$ to a battery supplying a constant 110 V. Now find the current in the circuit.

Solution We now have $E(t) \equiv 110$, so Eq. (16) gives

$$I'(0) = \frac{E(0)}{L} = \frac{110}{0.1} = 1100 \text{ (A/s)},$$

and the differential equation is

$$(0.1)I'' + 50I' + 2000I = E'(t) = 0.$$

Its general solution is the complementary function we found in Example 1:

$$I(t) = c_1 e^{-44t} + c_2 e^{-456t}.$$

We solve the equations

$$\begin{aligned} I(0) &= c_1 + c_2 = 0, \\ I'(0) &= -44c_1 - 456c_2 = 1100 \end{aligned}$$

for $c_1 = -c_2 = 2.670$. Therefore,

$$I(t) = (2.670)(e^{-44t} - e^{-456t}).$$

Note that $I(t) \to 0$ as $t \to +\infty$ even though the voltage is constant. ■

Electrical Resonance

Consider again the current differential equation in (6) corresponding to a sinusoidal input voltage $E(t) = E_0 \sin \omega t$. We have seen that the amplitude of its steady periodic current is

$$I_0 = \frac{E_0}{Z} = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}. \tag{17}$$

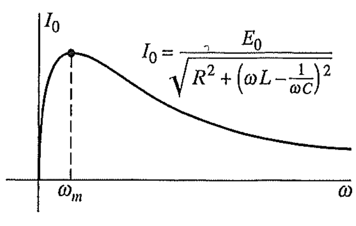

FIGURE 3.7.6. The effect of frequency on I_0.

For typical values of the constants R, L, C, and E_0, the graph of I_0 as a function of ω resembles the one shown in Fig. 3.7.6. It reaches a maximum value at $\omega_m = 1/\sqrt{LC}$ and then approaches zero as $\omega \to +\infty$; the critical frequency ω_m is the **resonance frequency** of the circuit.

In Section 3.6 we emphasized the importance of avoiding resonance in most mechanical systems (the cello is an example of a mechanical system in which resonance is *sought*). By contrast, many common electrical devices could not function properly without taking advantage of the phenomenon of resonance. The radio is a familiar example. A highly simplified model of its tuning circuit is the *RLC* circuit we have discussed. Its inductance L and resistance R are constant, but its capacitance C is varied as one operates the tuning dial.

Suppose that we wanted to pick up a particular radio station that is broadcasting at frequency ω, and thereby (in effect) provides an input voltage $E(t) = E_0 \sin \omega t$ to the tuning circuit of the radio. The resulting steady periodic current I_{sp} in the tuning circuit drives its amplifier, and in turn its loudspeaker, with the volume of sound we hear roughly proportional to the amplitude I_0 of I_{sp}. To hear our preferred station (of frequency ω) the loudest—and simultaneously tune out stations broadcasting at other frequencies—we therefore want to choose C to maximize I_0. But examine Eq. (17), thinking of ω as a constant with C the only variable. We see at a glance—no calculus required—that I_0 is maximal when

$$\omega L - \frac{1}{\omega C} = 0;$$

that is, when

$$C = \frac{1}{L\omega^2}. \tag{18}$$

So we merely turn the dial to set the capacitance to this value.

This is the way that old crystal radios worked, but modern AM radios have a more sophisticated design. A *pair* of variable capacitors are used. The first controls the frequency selected as described earlier; the second controls the frequency of a signal that the radio itself generates, kept close to 455 kilohertz (kHz) above the desired frequency. The resulting *beat* frequency of 455 kHz, known as the *intermediate frequency*, is then amplified in several stages. This technique has the advantage that the several *RLC* circuits used in the amplification stages easily can be designed to resonate at 455 kHz and reject other frequencies, resulting in far more selectivity of the receiver as well as better amplification of the desired signal.