Topic 3 - Manipulating Power Series

1) $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

If $|x| < 1$ this is a convergent geometric series. $f(x) = \frac{1}{1-x}$ domain $(-1, 1)$

2) $g(x) = \frac{x}{1-x}$ Note $g(x) = x\, f(x)$

$$g(x) = x\,(1 + x + x^2 + x^3 + \cdots)$$

$$= x + x^2 + x^3 + x^4 + \cdots$$

$$= \sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

also converges on interval $(-1, 1)$

3) $h(x) = \frac{1}{1+x^2}$ looks like $f(x)$ where x is replaced with $-x^2$

$$\frac{1}{1+x^2} = \frac{1}{1-(-(x^2))}$$

use same power series of part 1 but with $-x^2$

$$h(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}} \right| = |x^2| < 1 \Rightarrow -1 < x < 1$$

Check endpoints

$x=-1$ $\quad \sum_{n=0}^{\infty} (-1)^n (-1)^{2n}$ diverges

$x=1)$ $\quad \sum_{n=0}^{\infty} (-1)^n \, 1^{2n}$ diverges

interval of convergence $(-1, 1)$

4) Note $\quad \tan^{-1} x + C = \int \frac{1}{1+x^2}\,dx$

integrate power series for $h(x)$ above

$$\int h(x)\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C = \tan^{-1} x + C$$

$x=0 \Rightarrow C=0$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

interval of convergence?

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)} \frac{(2n+1)}{(-1)^n x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)\, x^2 \,(2n+1)}{(2n+3)} \right|$$

$$= |x^2| = x^2 < 1$$

$-1 < x < 1$

both endpoints converge

interval of convergence $[-1, 1]$