Central Limit Theorem

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- e.g. medians, OLS regression coefficients, MLE parameter estimates, ...

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- Market structure: largest firm, highest productivity when winner takes all

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But the limiting distribution may not exist! Depends on distribution underlying X_i's

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Key condition: tail behavior of $F(x)$ as $x \to \infty$

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The T1EV assumption also gives us closed-form choice probabilities and $\mathbb{E}(U)$

J alternatives, $\epsilon_{ij} \overset{iid}{\sim}$ Type I extreme value; $U_{ij} = u_{ij} + \epsilon_{ij}$; $F(\epsilon) = e^{-e^{-\epsilon}}$ (CDF)

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$$P_{ij} = \Pr(u_{ij} + \epsilon_{ij} > u_{ik} + \epsilon_{ik} \;\; \forall \;\; k \neq j)$$

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We also want to know the formula for expected utility:

$$\mathbb{E}_\epsilon \max_k \left\{u_{ik} + \epsilon_{ik}\right\} = \log\left(\sum_k \exp\left(u_{ik}\right)\right) + \gamma$$

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Depends on	Finite variance	Tail behavior of $F(x)$
Economic applications	Sampling distributions, inference	Choice models, auctions, extremes