Problem Set 04

Friday, October 28, 2022 2:54 PM

(a) $\Psi(R) = bR[1+\gamma\cos(2\theta)]$

$$\Psi(x,y) = bR[1+\gamma(\cos^2\theta - \sin^2\theta)]$$
$$= bR[1+\gamma\tfrac{1}{R^2}(x^2-y^2)]$$
$$= b[R+\gamma\tfrac{1}{R}(x^2-y^2)]$$
$$= b\left[(x^2+y^2)^{1/2} + \frac{\gamma(x^2-y^2)}{(x^2+y^2)^{1/2}}\right]$$

From 10/06 Notes: $\vec{\nabla}\Psi = \vec{\alpha}$

$\hookrightarrow$ $\partial_x \Psi = \alpha_x$

$\partial_y \Psi = \alpha_y$

$$\alpha_x = b\left[\partial_x (x^2+y^2)^{1/2} + \gamma\,\partial_x\left(\frac{(x^2-y^2)}{(x^2+y^2)^{1/2}}\right)\right]$$

$$= b\left[\frac{x}{(x^2+y^2)^{1/2}} + \gamma\,\frac{(x^2+y^2)^{1/2}\partial_x(x^2-y^2) - (x^2-y^2)\partial_x(x^2+y^2)^{1/2}}{x^2+y^2}\right]$$

$$= b\left[\frac{x}{(x^2+y^2)^{1/2}} + \gamma\cdot\left(\frac{2x}{(x^2+y^2)^{1/2}} - \frac{x(x^2-y^2)}{(x^2+y^2)^{3/2}}\right)\right]$$

$$= \frac{b}{(x^2+y^2)^{1/2}}\left[x + \gamma\left(2x - \frac{x(x^2-y^2)}{(x^2+y^2)}\right)\right]$$

$$= \frac{bx}{(x^2+y^2)^{1/2}}\left[1 + \gamma\left(2 - \frac{x^2-y^2}{x^2+y^2}\right)\right]$$

$$\boxed{\alpha_x = \frac{bx}{(x^2+y^2)^{1/2}}\left[1+\gamma\left(\frac{x^2+3y^2}{x^2+y^2}\right)\right]}$$

$$\alpha_y = b\left[\partial_y (x^2+y^2)^{1/2} + \gamma\,\partial_y\left(\frac{(x^2-y^2)}{(x^2+y^2)^{1/2}}\right)\right]$$

$$= b\left[\frac{y}{(x^2+y^2)^{1/2}} + \gamma\,\frac{(x^2+y^2)^{1/2}\,\partial_y(x^2-y^2) - (x^2-y^2)\,\partial_y(x^2+y^2)^{1/2}}{x^2+{}^2\quad y}\right]$$

$$= b\left[\frac{y}{(x^2+y^2)^{1/2}} + \gamma\left(\frac{-2y}{(x^2+y^2)^{1/2}} + \frac{y(y^2-x^2)}{(x^2+y^2)^{3/2}}\right)\right]$$

$$= \frac{by}{(x^2+y^2)^{1/2}}\left[1+\gamma\left(-2+\frac{y^2-x^2}{x^2+y^2}\right)\right]$$

$$\boxed{\alpha_y = \frac{by}{(x^2+y^2)^{1/2}}\left[1-\gamma\left(\frac{3x^2+y^2}{x^2+y^2}\right)\right]}$$

(b.)

From 10/04 Notes:

$$K = \tfrac{1}{2}(\psi_{xx}+\psi_{yy})$$

$$= \tfrac{1}{2}(\partial_x\alpha_x + \partial_y\alpha_y)$$

$$\psi_{xx} = \partial_x\alpha_x = \partial_x\left(\frac{bx}{}\left[1+\gamma\left(\frac{x^2+3y^2}{}\right)\right]\right)$$

$$\psi_{xx} = \partial_x \alpha_x = \partial_x \left(\frac{bx}{(x^2+y^2)^{1/2}} \left[1 + \gamma \left(\frac{x^2+3y^2}{x^2+y^2} \right) \right] \right)$$

$$= \left[1 + \gamma \left(\frac{x^2+3y^2}{x^2+y^2} \right) \right] \cdot \partial_x \left(\frac{bx}{(x^2+y^2)^{1/2}} \right)$$

$$+ \left(\frac{bx}{(x^2+y^2)^{1/2}} \right) \cdot \partial_x \left[1 + \gamma \left(\frac{x^2+3y^2}{x^2+y^2} \right) \right]$$

$$= \left[1 + \gamma \left(\frac{x^2+3y^2}{x^2+y^2} \right) \right] \cdot \left(\frac{by^2}{(x^2+y^2)^{3/2}} \right)$$

$$+ \left(\frac{bx}{(x^2+y^2)^{1/2}} \right) \cdot \left[\frac{-4\gamma x \cdot y^2}{(x^2+y^2)^2} \right]$$

$$= \frac{by^2}{(x^2+y^2)^{3/2}} + \frac{\gamma b y^2 (x^2+3y^2)}{(x^2+y^2)^{5/2}}$$

$$- \frac{4\gamma b x^2 y^2}{(x^2+y^2)^{5/2}}$$

$$= \frac{by^2}{(x^2+y^2)^{3/2}} + \frac{\gamma b y^2 (x^2+3y^2-4x^2)}{(x^2+y^2)^{5/2}}$$

$$\psi_{xx} = \frac{by^2}{(x^2+y^2)^{3/2}} \left[1 + \frac{3\gamma (y^2-x^2)}{x^2+y^2} \right]$$

$$\circ\ \psi_{yy} = \partial_y \alpha_y = \partial_y \left[\frac{by}{(x^2+y^2)^{1/2}} \left[1 - \gamma \left(\frac{3x^2+y^2}{x^2+y^2} \right) \right] \right.$$

$$= \left[1 - \gamma \left(\frac{3x^2+y^2}{x^2+y^2} \right) \right] \cdot \partial_y \left(\frac{by}{(x^2+y^2)^{1/2}} \right)$$

$$+ \left(\frac{by}{(x^2+y^2)^{1/2}} \right) \cdot \partial_y \left[1 - \gamma \left(\frac{3x^2+y^2}{x^2+y^2} \right) \right]$$

$$= \left[1 - \gamma \left(\frac{3x^2+y^2}{x^2+y^2} \right) \right] \left(\frac{bx^2}{(x^2+y^2)^{3/2}} \right)$$

$$+ \left(\frac{by}{(x^2+y^2)^{1/2}} \right) \cdot \left[\frac{4\gamma x^2 y}{(x^2+y^2)^2} \right]$$

$$= \frac{bx^2}{(x^2+y^2)^{3/2}} - \frac{\gamma b x^2 (3x^2+y^2)}{(x^2+y^2)^{5/2}}$$

$$+ \frac{4\gamma b x^2 y^2}{(x^2+y^2)^{5/2}}$$

$$= \frac{bx^2}{(x^2+y^2)^{3/2}} + \frac{\gamma b x^2 (4y^2 - 3x^2 - y^2)}{(x^2+y^2)^{5/2}}$$

$$\psi_{zy} = \frac{bx^2}{(x^2+y^2)^{3/2}} \left[1 + \frac{3\gamma (y^2 - x^2)}{x^2+y^2} \right]$$

So for $K = \frac{1}{2}(\psi_{xx} + \psi_{yy})$

$$K = \frac{by^2}{2(x^2+y^2)^{3/2}}\left[1 + \frac{3\gamma(y^2-x^2)}{x^2+y^2}\right]$$

$$+ \frac{bx^2}{2(x^2+y^2)^{3/2}}\left[1 + \frac{3\gamma(y^2-x^2)}{x^2+y^2}\right]$$

$$= \frac{b(x^2+y^2)}{2(x^2+y^2)^{3/2}}\left[1 + \frac{3\gamma(y^2-x^2)}{x^2+y^2}\right]$$

$$\boxed{K = \frac{b}{2(x^2+y^2)^{3/2}}\left[1 + \frac{3\gamma(y^2-x^2)}{x^2+y^2}\right]}$$

From 10/04 Notes: $\gamma_1 = \frac{1}{2}(\psi_{xx} - \psi_{yy})$

$$\boxed{\gamma_1 = \frac{b(y^2-x^2)}{2(x^2+y^2)^{3/2}}\left[1 + \frac{3\gamma(y^2-x^2)}{x^2+y^2}\right]}$$

From 10/04 Notes: $\gamma_2 = \psi_{xy}$

$$\psi_{xy} = \partial_y \psi_x = \partial_y\left(\frac{bx}{(x^2+y^2)^{1/2}}\left[1 + \gamma\left(\frac{x^2+3y^2}{x^2+y^2}\right)\right]\right)$$

$$= \left[1 + \gamma\left(\frac{x^2+3y^2}{x^2+y^2}\right)\right] \cdot \partial_y\left(\frac{bx}{(x^2+y^2)^{1/2}}\right)$$

$$+\left(\frac{bx}{(x^2+y^2)^{1/2}}\right)\cdot\partial_y\left[1+\gamma\left(\frac{x^2+3y^2}{x^2+y^2}\right)\right]$$

$$=\left[1+\gamma\left(\frac{x^2+3y^2}{x^2+y^2}\right)\right]\cdot\left(\frac{-bxy}{(x^2+y^2)^{3/2}}\right)$$

$$+\left(\frac{bx}{(x^2+y^2)^{1/2}}\right)\cdot\left(\frac{4\gamma x^2 y}{(x^2+y^2)^2}\right)$$

$$=-\frac{bxy}{(x^2+y^2)^{3/2}}-\frac{\gamma bxy(x^2+3y^2)}{(x^2+y^2)^{5/2}}$$

$$+\frac{4\gamma b x^3 y}{(x^2+y^2)^{5/2}}$$

$$=\frac{-bxy}{(x^2+y^2)^{3/2}}+\frac{\gamma bxy(4x^2-x^2-3y^2)}{(x^2+y^2)^{5/2}}$$

$$=\frac{-bxy}{(x^2+y^2)^{3/2}}\left[1+\frac{3\gamma(y^2-x^2)}{x^2+y^2}\right]$$

So

$$\boxed{\gamma_2=\frac{-bxy}{(x^2+y^2)^{3/2}}\left[1+\frac{3\gamma(y^2-x^2)}{x^2+y^2}\right]}$$

Relationship b/w γ & K

Relationship b/w γ & K

$$\gamma = \sqrt{\gamma_1^2+\gamma_2^2} = \left[\left(\frac{b\,(y^2-x^2)}{2\,(x^2+y^2)^{3/2}} \left[1 + \frac{3\gamma\,(y^2-x^2)}{x^2+y^2} \right] \right)^2 + \left(\frac{-bxy}{(x^2+y^2)^{3/2}} \left[1 + \frac{3\gamma\,(y^2-x^2)}{x^2+y^2} \right] \right)^2 \right]^{1/2}$$

$$= \left[1 + \frac{3\gamma\,(y^2-x^2)}{x^2+y^2} \right] \left[\frac{b^2(y^2-x^2)^2}{4(x^2+y^2)^3} + \frac{b^2x^2y^2}{(x^2+y^2)^3} \right]^{1/2}$$

$$= \left[1 + \frac{3\gamma\,(y^2-x^2)}{x^2+y^2} \right] \left[\left(\frac{b^2}{4(x^2+y^2)^3} \right) \left(\underbrace{(y^2-x^2)^2 + 4x^2y^2}_{= x^4+y^4+2x^2y^2 = (x^2+y^2)^2} \right) \right]^{1/2}$$

$$= \frac{b}{2(x^2+y^2)^{3/2}} \left[1 + \frac{3\gamma\,(y^2-x^2)}{x^2+y^2} \right] \left((x^2+y^2)^2 \right)^{1/2}$$

$$= \frac{b\,(x^2+y^2)}{2(x^2+y^2)^{3/2}} \left[1 + \frac{3\gamma\,(y^2-x^2)}{x^2+y^2} \right]$$

$$= \frac{b}{2(x^2+y^2)^{3/2}} \left[1 + \frac{3\gamma\,(y^2-x^2)}{x^2+y^2} \right] = K$$

$$\boxed{\gamma = K}$$

(c.)

From 10/04 Notes:

$A = [\,1 \;\; 0 \;\; x \qquad\qquad x \;\;]$

$$A = \begin{bmatrix} 1-K-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-K+\gamma_1 \end{bmatrix}$$

$$\det[A] = (1-K-\gamma_1)(1-K+\gamma_1) - \gamma_2^2$$

$$= K^2 - 2K - \gamma_1^2 + 1 - \gamma_2^2$$

$$= K^2 - 2K + 1 - \gamma_1^2 - \gamma_2^2$$

$$= (K-1)^2 - (\gamma_1^2 + \gamma_2^2)$$

$$= (K-1)^2 - \gamma^2$$

Plug in $K = \gamma$:

$$\det[A] = (K-1)^2 - K^2$$

$$= -2K + 1$$

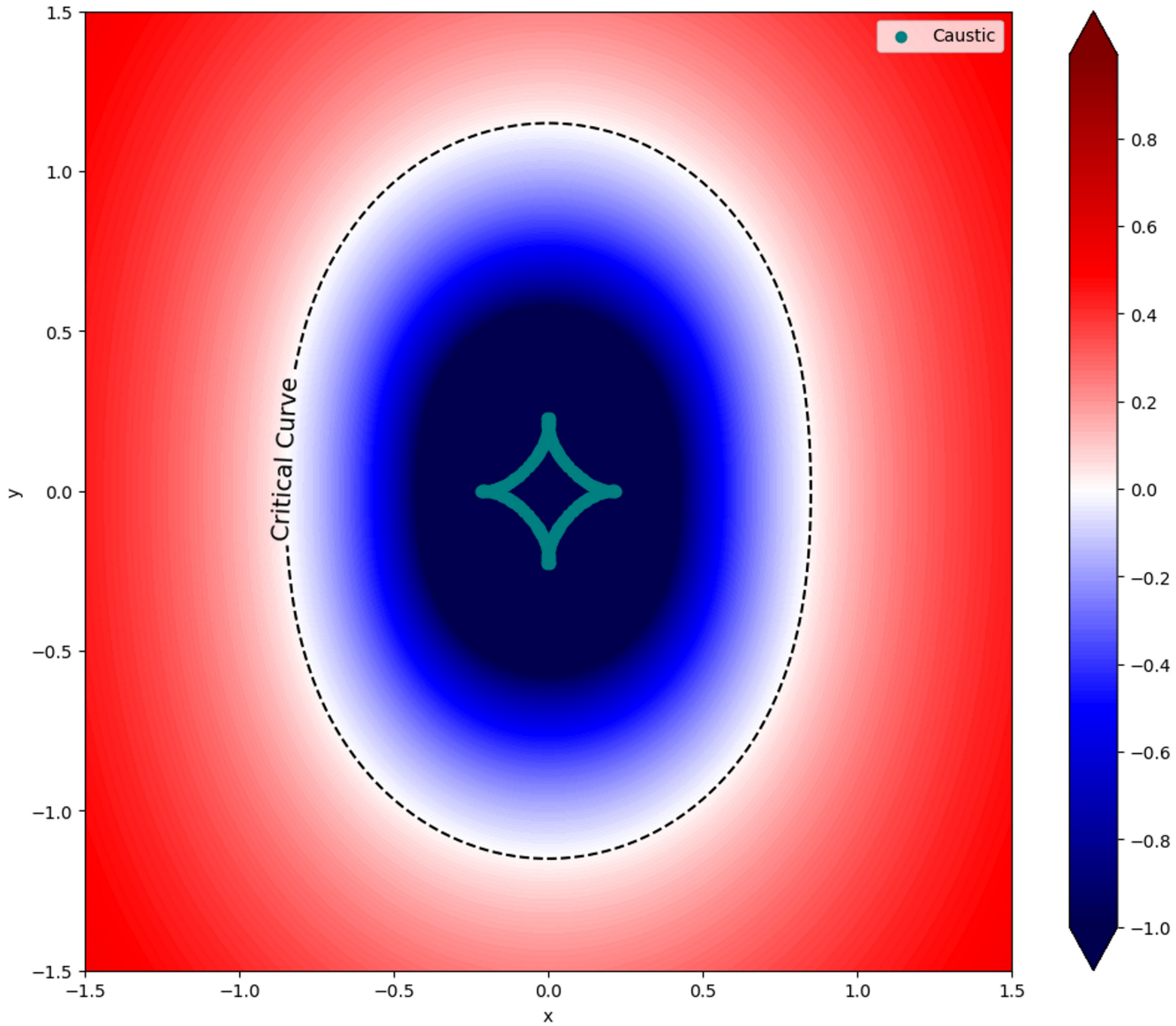
Plug in K:

$$\det[A] = 1 - \frac{b}{(x^2+y^2)^{3/2}} \left[1 + \frac{3\gamma(y^2-x^2)}{x^2+y^2} \right]$$

K can be distributed to one term:

$$\det[A] = 1 - \frac{(bx^2+by^2)\left((1-3\gamma)x^2 + (1+3\gamma)y^2\right)}{2(x^2+y^2)^{5/2}}$$

(d.)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Critical Curve
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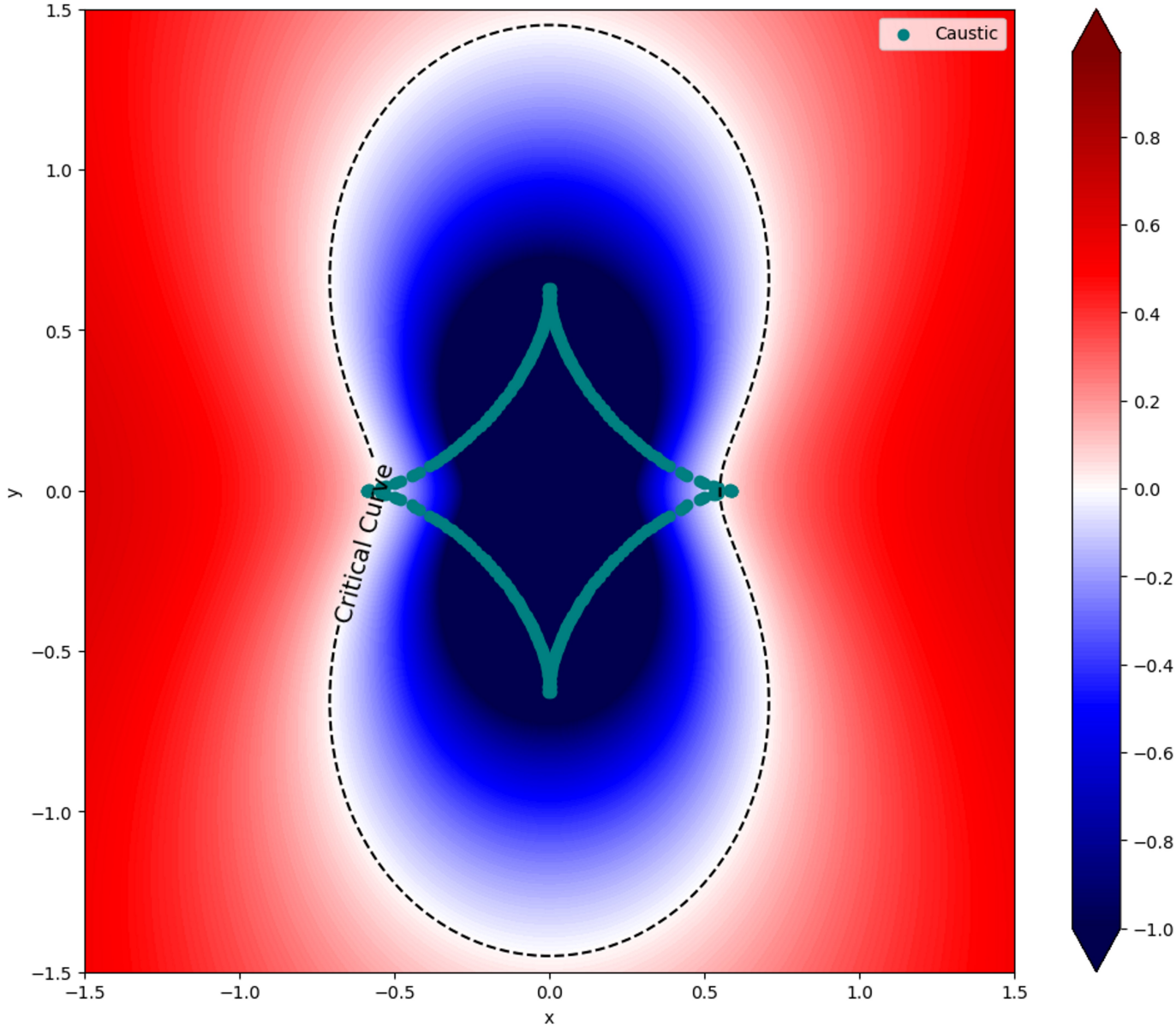
(e.)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