Assignment #2 Part 1: MATLAB Software Design

Due Date: Wednesday, February 2 at 5:00 PM

Type of Assignment: Group

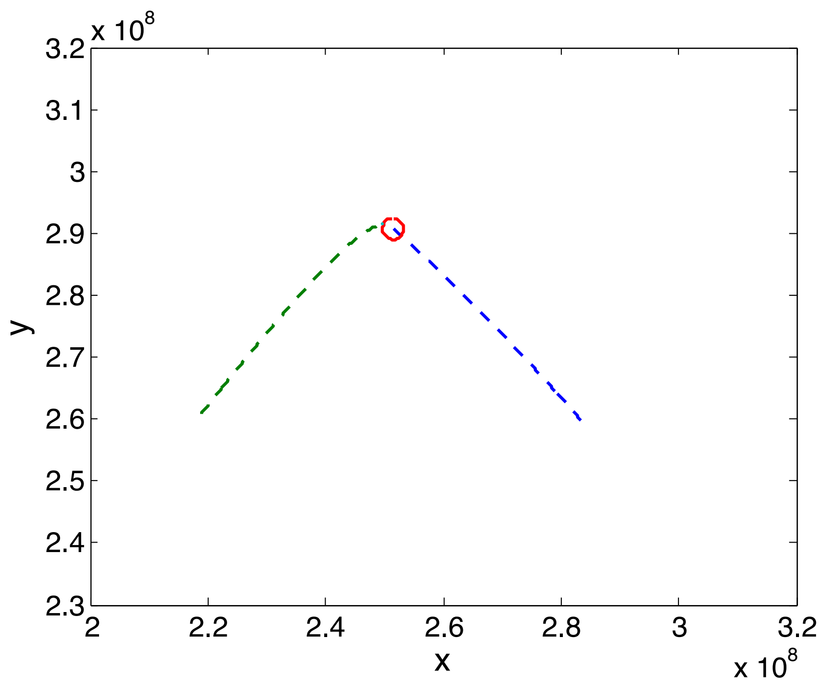
Submission Procedure: Please submit your assignment via Canvas. We request you send your files in a single zip archive. *Only one group member needs to submit the assignment.*

Instructions:

This assignment involves developing a program for solving the spacecraft-Earth-moon three-body problem. There are several components required in such a program, and these components are described in detail below. You will need to implement your program in MATLAB. Be sure to document your code well.

Background:

Three of our best astronauts were on their way to the moon when a series of explosions knocked out the main power of their spacecraft and wrecked havoc on life support. Now, the spacecraft is on a prime collision course with the moon, as illustrated below.



In the above figure, the green dashed line denotes the projected path of the spacecraft, the blue dashed line denotes the projected path of the moon, and the red circle denotes the final position of the moon.

Luckily, there is a chance to use onboard thrusters to change the current velocity of the spacecraft and hence its projected trajectory. Your objective is to determine the velocity change required in order to guarantee the spacecraft's successful return to Earth. Ideally, you would also like to get the astronauts home as fast as possible due to the failing life support system. However, it is not known just how much power can be supplied to the thrusters and how much longer the thrusters will be operational. It is possible to use the moon's gravity field in order to slingshot the spacecraft around the moon and back to Earth. This will allow for a small change in current velocity to be used to get the astronauts safely back home. This is also what was used to get the Apollo 13 astronauts back home.

Gravitational Forces:

In order to determine the required change in current velocity, you will first need to understand how to simulate the moon-Earth-spacecraft system. The combined gravitational pulls of the Earth and the moon alter the path of the spacecraft as a function of time, and, similarly, the combined gravitational pulls of the Earth and the spacecraft alter the path of the moon.

Consider two bodies A and B with masses m_A and m_B respectively. Assume that body A is centered at a position (x_A, y_A) and that body B is centered at (x_B, y_B). Then, the distance between bodies, denoted by d_{AB} or d_{BA}, is given by:

$$d_{AB} = d_{BA} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}. \tag{1}$$

The body A exerts a gravitational force on body B of magnitude

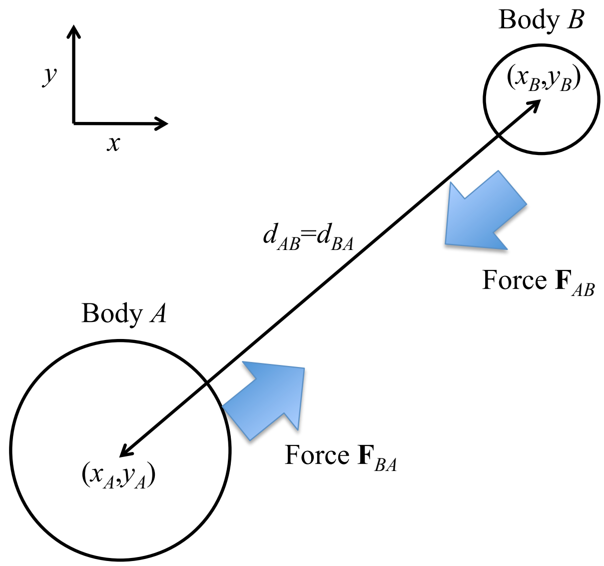
$$F_{AB} = \frac{G m_A m_B}{d_{AB}^2} \tag{2}$$

where G is the gravitational constant

$$G = 6.674 \times 10^{-11} \mathrm{N(m/kg)}^2. \tag{3}$$

The force exerted on B by A is pointed directly in the direction of A. Similarly, the body B exerts a gravitational force on body A of equal magnitude and opposite direction. This situation is depicted in the figure given below.

Note that the force exerted on B by A and the force exerted on A by B are both vectors. We can explicitly state the x- and $y-$ components of the vectors in terms of the positions



of the two bodies. Notably, we have:

$$(F_{AB})_x = \frac{Gm_A m_B(x_A - x_B)}{d_{AB}^3} \tag{4}$$

$$(F_{AB})_y = \frac{Gm_A m_B(y_A - y_B)}{d_{AB}^3} \tag{5}$$

$$(F_{BA})_x = \frac{Gm_B m_A(x_B - x_A)}{d_{BA}^3} \tag{6}$$

$$(F_{BA})_y = \frac{Gm_B m_A(y_B - y_A)}{d_{BA}^3}. \tag{7}$$

Note immediately that $(F_{AB})_x = -(F_{BA})_x$ and $(F_{AB})_y = -(F_{BA})_y$. This is because the force vectors $\mathbf{F}_{AB}$ and $\mathbf{F}_{BA}$ are perfectly balanced.

We can repeat the exercises above to precisely find the forces exerted on the spacecraft, moon, or Earth by the other two respective bodies. In what follows, we will use the notation S to denote the spacecraft, M to denote the moon, and E to denote the Earth. For example, x_S will denote the x-coordinate of the center of the spacecraft, d_{SE} will denote the distance between the spacecraft and the Earth, and $(F_{EM})_x$ will denote the x-component of the force exerted on the moon by the Earth.

Equations of Motion:

Now that we know exactly what the gravitational forces are in our moon-Earth-spacecraft system, we can discuss the equations governing the motion of the system. The basis of this discussion will be precisely Newton's second law: $F = MA$. That is, force equals mass times acceleration. Consider the acceleration of the spacecraft in the x-direction which we will denote by a_{S_x}. To determine such an acceleration, we simply need to know the forces exerted on the spacecraft in the x-direction. These are precisely the gravitational forces exerted on the spacecraft by the moon and the Earth. Hence, Newton's second law gives us:

$$m_S a_{S_x} = (F_{MS})_x + (F_{ES})_x. \tag{8}$$

We can repeat this line of thought to obtain equations for the acceleration of the spacecraft in the y-direction as well as the accelerations of the moon and the Earth. ***In what follows, we make the following assumption: the gravitational pulls of the moon and the spacecraft do not alter the path of the Earth.*** The basis behind this assumption is that the mass of the Earth is so great that the gravitational forces exerted on the Earth by the other two bodies is negligible. With this assumption in mind, we obtain the following six equations for the accelerations of the moon-Earth-spacecraft system:

$$a_{S_x} = \frac{(F_{MS})_x + (F_{ES})_x}{m_S} \tag{9}$$

$$a_{S_y} = \frac{(F_{MS})_y + (F_{ES})_y}{m_S} \tag{10}$$

$$a_{M_x} = \frac{(F_{EM})_x + (F_{SM})_x}{m_M} \tag{11}$$

$$a_{M_y} = \frac{(F_{EM})_y + (F_{SM})_y}{m_M} \tag{12}$$

$$a_{E_x} = 0 \tag{13}$$

$$a_{E_y} = 0. \tag{14}$$

The above accelerations are related to the velocities of our bodies through the following derivative relations:

$$a_{S_x} = \frac{dv_{S_x}}{dt} \tag{15}$$

$$a_{S_y} = \frac{dv_{S_y}}{dt} \tag{16}$$

$$a_{M_x} = \frac{dv_{M_x}}{dt} \tag{17}$$

$$a_{M_y} = \frac{dv_{M_y}}{dt} \tag{18}$$

$$a_{E_x} = \frac{dv_{E_x}}{dt} \tag{19}$$

$$a_{E_y} = \frac{dv_{E_y}}{dt} \tag{20}$$

where v_{S_x}, for example, is the velocity of the spacecraft in the x-direction. Finally, the above velocities are related to the center positions of our bodies through the following derivative relations:

$$v_{S_x} = \frac{dx_S}{dt} \tag{21}$$

$$v_{S_y} = \frac{dy_S}{dt} \tag{22}$$

$$v_{M_x} = \frac{dx_M}{dt} \tag{23}$$

$$v_{M_y} = \frac{dy_M}{dt} \tag{24}$$

$$v_{E_x} = \frac{dx_E}{dt} \tag{25}$$

$$v_{E_y} = \frac{dy_E}{dt}. \tag{26}$$

Euler's Method:

We now have equations governing the positions, velocities, and accelerations of our bodies in terms of derivative relations. So, how do we solve them? Unfortuntely, the given equations are rather nasty nonlinear ordinary differential equations that are not susceptible to arithmetic or analytical manipulation. However, we can use a very basic ***numerical method*** to ***integrate*** the given equations in time. The idea is as follows. Suppose we have an equation of the form:

$$\frac{dz}{dt}(t) = g(z(t), t). \tag{27}$$

The function g can be any function of two variables. Now, by definition, we know that:

$$\frac{dz}{dt}(t) = \lim_{\delta \to 0} \frac{z(t+\delta) - z(t)}{\delta}. \tag{28}$$

Hence, a reasonable approximation might be to replace δ in the above relation by some small but finite ***time-step size*** Δt. This yields the following approximation:

$$\frac{dz}{dt}(t) \approx \frac{z(t+\Delta t) - z(t)}{\Delta t}. \tag{29}$$

By combining the above equation with the equation given in (27), we obtain:

$$z(t+\Delta t) \approx z(t) + \Delta t g(z(t), t). \tag{30}$$

The above is Euler's approximation of the value of z at time $t + \Delta t$, and the resulting scheme of approximation is called Euler's method. It is the most basic numerical method for solving ordinary differential equations on a computer.

If we employ Euler's method to find the velocities of the moon, Earth, and spacecraft at time $t + \Delta t$ given the corresponding velocities and gravitational forces at time t by solving the equations given by (9)-(20), we end up with the following approximations:

$$v_{S_x}(t+\Delta t) \approx v_{S_x}(t) + \Delta t \frac{(F_{MS})_x(t) + (F_{ES})_x(t)}{m_S} \tag{31}$$

$$v_{S_y}(t+\Delta t) \approx v_{S_y}(t) + \Delta t \frac{(F_{MS})_y(t) + (F_{ES})_y(t)}{m_S} \tag{32}$$

$$v_{M_x}(t+\Delta t) \approx v_{M_x}(t) + \Delta t \frac{(F_{EM})_x(t) + (F_{SM})_x(t)}{m_M} \tag{33}$$

$$v_{M_y}(t+\Delta t) \approx v_{M_y}(t) + \Delta t \frac{(F_{EM})_y(t) + (F_{SM})_y(t)}{m_M} \tag{34}$$

$$v_{E_x}(t+\Delta t) \approx v_{E_x}(t) \tag{35}$$

$$v_{E_y}(t+\Delta t) \approx v_{E_y}(t). \tag{36}$$

It should be noted that in order to evaluate the gravitational forces listed above, the center positions of the moon, Earth, and spacecraft must be known at time t.

Finally, if we know the positions and velocities of the moon, Earth, and spacecraft at time t, we can employ Euler's method to find the corresponding positions at time $t + \Delta t$. The resulting approximations are:

$$x_S(t+\Delta t) \approx x_S(t) + \Delta t v_{S_x}(t) \tag{37}$$

$$y_S(t+\Delta t) \approx y_S(t) + \Delta t v_{S_y}(t) \tag{38}$$

$$x_M(t+\Delta t) \approx x_M(t) + \Delta t v_{M_x}(t) \tag{39}$$

$$y_M(t+\Delta t) \approx y_M(t) + \Delta t v_{M_y}(t) \tag{40}$$

$$x_E(t+\Delta t) \approx x_E(t) + \Delta t v_{E_x}(t) \tag{41}$$

$$y_E(t+\Delta t) \approx y_E(t) + \Delta t v_{E_y}(t). \tag{42}$$

The above discussion suggests the following methodology for simulating the moon-Earth-spacecraft system. Given the velocities and positions of the three bodies at time $t = 0$, we find the gravitational forces between the bodies at time $t = 0$ using the procedure outlined in **Gravitational Forces**. Then, we use the equations given by (31)-(36) to find the velocities at time Δt and the equations given by (37)-(42) to find the positions at time Δt. Once we know those positions and velocities, we can update the time to $t = \Delta t$ and find the gravitational forces at time $t = \Delta t$ and iterate. In this fashion, we can obtain the positions and velocities of the moon-Earth-spacecraft system at time instances $t = 0, \Delta t, 2\Delta t, 3\Delta t, \ldots, n\Delta t, (n+1)\Delta t, \ldots$.

Higher-Order Time Integration Methods:

Euler's method is quite simple to understand and implement, but it is not very accurate. In fact, it can be shown that is only first-order accurate in time. However, there are several alternative time integration methods with higher-order accuracy. The simplest of these is the explicit midpoint method, which employs the following time update scheme:

$$z(t+\Delta t) \approx z(t) + \Delta t g\left(z(t) + \frac{\Delta t}{2} g\left(z(t), t\right), t + \frac{\Delta t}{2}\right). \tag{43}$$

The explicit midpoint method is second-order accurate in time. Therefore, larger time steps can be taken with the explicit midpoint method while maintaining the same accuracy of the Euler method. Both the Euler's method and the explicit midpoint method are examples of Runge-Kutta methods. For this project, you may employ any time integration method you like with the caveat that you must ensure your final solution is sufficiently accurate.

Termination Conditions:

The only question remaining is when to stop the simulation. In this project, we will consider three possible termination conditions:

1. The spacecraft has crashed into the moon.
2. The spacecraft has returned to Earth.
3. The spacecraft is lost to the vast reaches of outer space.

Termination condition #1 will occur if the distance between the center of the spacecraft and the center of the moon dips below the radius of the moon. That is, the spacecraft will have crashed into the moon at time t if

$$d_{MS}(t) \leq r_M \tag{44}$$

where r_M meters is the radius of the moon. Obviously, if termination condition #1 occurs, the mission will be considered a complete and utter failure.

Termination condition #2 will occur if the distance between the center of the spacecraft and the center of the Earth dips below the radius of the Earth. That is, the spacecraft will have crashed into the Earth at time t if

$$d_{ES}(t) \leq r_E \tag{45}$$

where r_E is the radius of the Earth. If termination condition #2 occurs, the mission will be considered a success.

There is some uncertainty as to when termination condition #3 will occur. For the sake of this project, assume that termination condition #3 will occur if the distance between the center of the spacecraft and the center of the Earth exceeds twice the distance between the center of the moon and the center of the Earth. That is, we assume the spacecraft is lost to space at time t is

$$d_{ES}(t) \geq 2d_{EM}(t). \tag{46}$$

If the spacecraft is lost to space, the mission will be considered a failure.

Physical Property Details:

In order to conduct the simulation depicted in the flow chart, you will need to know various physical properties of the moon, Earth, and spacecraft. These are listed below.

Mass of moon: $m_M = 7.34767309 \times 10^{22}$ kg
Mass of Earth: $m_E = 5.97219 \times 10^{24}$ kg
Mass of spacecraft: $m_S = 28,833$ kg
Radius of moon: $r_M = 1,737,100$ m
Radius of Earth: $r_E = 6,371,000$ m.

Simulation Details:

Before employing thrusters, the initial conditions (the positions and velocities at time $t = 0$) of the spacecraft are as follows:

$$x_S(0) = d_{ES}(0)\cos(\theta_S) \tag{47}$$
$$y_S(0) = d_{ES}(0)\sin(\theta_S) \tag{48}$$
$$v_{S_x}(0) = v_S(0)\cos(\theta_S) \tag{49}$$
$$v_{S_y}(0) = v_S(0)\sin(\theta_S) \tag{50}$$
$$d_{ES}(0) = 340,000,000 \text{ m} \tag{51}$$
$$v_S(0) = 1,000 \text{ m/s} \tag{52}$$
$$\theta_S = 50^\circ \tag{53}$$
$$x_M(0) = d_{EM}(0)\cos(\theta_M) \tag{54}$$
$$y_M(0) = d_{EM}(0)\sin(\theta_M) \tag{55}$$
$$v_{M_x}(0) = -v_M(0)\sin(\theta_M) \tag{56}$$
$$v_{M_y}(0) = v_M(0)\cos(\theta_M) \tag{57}$$
$$d_{EM}(0) = 384,403,000 \text{ m} \tag{58}$$
$$v_M(0) = \sqrt{\frac{Gm_E^2}{(m_E + m_M)d_{EM}(0)}} \tag{59}$$

$$\theta_M = 42.5^\circ \tag{60}$$
$$x_E(0) = 0 \text{ m} \tag{61}$$
$$y_E(0) = 0 \text{ m} \tag{62}$$
$$v_{E_x}(0) = 0 \text{ m/s} \tag{63}$$
$$v_{E_y}(0) = 0 \text{ m/s} \tag{64}$$

Your job will be to find a perturbed velocity initial condition for the spacecraft:

$$v_{S_x}(0) = v_S(0)\cos(\theta_S) + \Delta v_{S_x} \tag{65}$$
$$v_{S_y}(0) = v_S(0)\sin(\theta_S) + \Delta v_{S_y}. \tag{66}$$

The size of this change is measured as:

$$|\Delta v_S| = \sqrt{\Delta v_{S_x}^2 + \Delta v_{S_y}^2}. \tag{67}$$

Assignment Objectives:

For this assignment, you must complete one of two objectives. Both of these objectives will require you design a computer program to achieve the associated objective and to complete the following tasks:

- Construct flowcharts which map the overall program control flow and the control flow of individual subroutines.
- Write a MATLAB script to execute the program and associated functions to execute subroutines.
- Explain the design process you utilized in arriving at your final program.

You are not required to use any particular MATLAB functionality. However, certain functionalities will lead to more efficient code.

Objective 1: Find the smallest change in initial velocity which will guarantee the spacecraft will make it safely back to Earth. In addition, plot the resulting spacecraft trajectory as it slingshots around the moon and returns to Earth, and explain the procedure as to how you arrived at the smallest change in initial velocity.

Objective 2: Find the change in initial velocity which will result in the fastest return to Earth subject to the constraint:

$$|\Delta v_S| \leq 100 \text{ m/s}. \tag{68}$$

In addition, plot the resulting spacecraft trajectory as it slingshots around the moon and returns to Earth, and explain the procedure as to how you arrived at your optimal change in initial velocity.

Note that in order to complete both objectives, you will have to run many MATLAB simulations. I would suggest first running your code for the original initial conditions to see if you can replicate the spacecraft impacting the Earth. You will need to choose a sufficiently small time-step size in order to get a stable solution. However, if you choose too small a time-step size, your simulation may never finish. I would suggest initially trying time-step sizes on the order of 1-100 seconds before adjusting the time-step size to improve accuracy or runtime.

In order to get full credit for this assignment, you must obtain a solution that is "sufficiently accurate". For Objective 1, your solution may be considered sufficiently accurate if your obtained velocity perturbation magnitude is within 0.5% of the optimal solution. For Objective 2, your solution may be considered sufficiently accurate if your obtained return time to Earth is within 0.5% of the optimal solution.