CS 294, Fa 2023 – HW 2

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1 Problem 1

I'm completely lost on this problem. I tried writing a counterexample checker, which enumerates over all possible schemas, tries randomly generated (boolean) tables, and compares the result of $(A \bowtie B) \ltimes Q$ and $(A \ltimes Q) \bowtie (B \ltimes Q)$. `https://gist.github.com/tylerhou/f308121e88817b46562149c7501ef164`.

The checker cannot find counterexamples when A and B don't separately share variables with Q; i.e. the above equality holds if

$$\text{Vars}(A) \cap \text{Vars}(Q) = \text{Vars}(B) \cap \text{Vars}(Q)$$

.

But this feels too strong: since Q contains all variables, this implies that $\text{Vars}(A) = \text{Vars}(B)$.

The simplest such counterexample when the above does not hold is the database $A(aq) = \{(0,)(1,)\}$, $B(bq) = \{(0,),(1,)\}$, and $Q(aq,bq) = \{(0,1),(1,0),(1,1)\}$. Then:

$$\begin{aligned}
(A \bowtie B) \ltimes Q &= \{(0,0),(0,1),(1,0),(1,1)\} \ltimes Q \\
&= \{(0,1),(1,0),(1,1)\} = Q \\
&\neq \{(0,0),(0,1)(1,0),(1,1)\} \\
&= \{(0,),(1,)\}_{(aq,)} \bowtie \{(0,),(1,)\}_{(bq,)} \\
&= (A \ltimes Q) \bowtie (B \ltimes Q)
\end{aligned}$$

2 Problem 2

2.1 Part A

1) Q_1 Cyclic. Triangle join shown in lecture.
2) Q_2 Acyclic.

$$\begin{aligned}
Q_2 &= R(X,Y,Z) \wedge S(Y,Z,U) \wedge T(Z,U,V) \\
&= R(Y,Z) \wedge S(Y,Z,U) \wedge (T(Z,U) && \text{(Remove isolated variables } X \text{ and } V) \\
&= S(Y,Z,U) && \text{(Remove ears } R(Y,Z) \text{ and } T(Z,U)) \\
&= - && \text{(Remove isolated } Y,Z,U)
\end{aligned}$$

3) Q_3 Acyclic.

$$\begin{aligned}
Q_3 &= A(X,Y,Z) \wedge R(X,Y) \wedge S(Y,Z) \wedge T(Z,X) \\
&= A(X,Y,Z) && \text{(Remove ears } (R,S,T \text{ contained in } A)) \\
&= - && \text{(Remove isolated)}
\end{aligned}$$

4) Q_4 There are no isolated variables; we can only remove ears A,B,C. But this gives us Q_1, which is cyclic.

2.2 Part B

1. 3. Number a cycle's vertices $1 \to 2 \to 3 \to \ldots \to n \to 1$. Then $(0,1,n) \to (1,2,n) \to (2,3,n) \to \ldots \to (n-2, n-1, n)$ is a tree decomposition with tree width 3.

2. Julia claims that it has tree width n. I have no idea how to prove.

3. I have no idea.

3 Problem 3

3.1 Part A

We have:

$$
\begin{aligned}
&Q_1 \subseteq Q_4 && (h = (x,y) \to (x,y)) \\
\\
&Q_2 \subseteq Q_1 && (h = (x,y,z,u) \to (z,u,z,u)) \\
&Q_2 \subseteq Q_4 && (h = (x,y,z,u) \to (x,y,x,x)) \\
\\
&Q_3 \subseteq Q_1 && (h = (x,y,z,u) \to (x,y,x,y)) \\
&Q_3 \subseteq Q_2 && (h = (x,y,z,u) \to (x,y,z,x)) \\
&Q_3 \subseteq Q_4 && (h = (x,y,z,u) \to (x,y,x,x)) \\
\\
&Q_4 \subseteq Q_1 && (h = (x,y,z,u) \to (x,y,x,y))
\end{aligned}
$$

Note that $Q_1 \equiv Q_4$.
Homomorphisms found via brute force. `https://gist.github.com/tylerhou/7a67cee962214de408a9ec88334f25ce`

3.2 Part B

1. $Q_1 \equiv Q_2$. Q_1 is a query that evaluates to true on a directed triangle. Q_2 is a query that evaluates to true on a directed triangle with some vertex $x \geq y$, where x has an edge to y. It is clear that $Q_2 \subseteq Q_1$.

   To show the other direction, consider any directed triangle on three vertices. Let x be a maximal vertex, and consider its outgoing edge to y. Either $x = y$ or $x > y$ by assumption of maximality.

2. $Q_3 \subseteq Q_1$. This is clear.

3. $Q_1 \nsubseteq Q_3$, and $Q_2 \nsubseteq Q_3$. Q_3 is true for all directed triangles that have a path with increasing vertex values. The triangle $0 \to 2 \to 1 \to 0$ is a counter example for both cases: it only has three paths, and none are increasing. However, it is a directed triangle, and there is one increasing edge $(0 \to 2)$.

4. $Q_3 \subseteq Q_2$. If a triangle has an increasing path, it also has an increasing edge.

3.3 Part C

$Q_1 \subseteq Q_2$. Suppose that a graph has a (possibly non-simple) cycle of length 5 such that the first and second vertices are not equal. We show that it must also have a cycle of length 5 where the first and third vertices are not equal.

Let $a \to b \to c \to d \to e \to a$ be the cycle of length 5 where $a \neq b$. There are two cases. If $a \neq c$, then we are done. Otherwise, $a = c$. Then, the graph has a cycle of length 3: $(c =)\ a \to d \to e \to a$. Again, there are two cases. If $a \neq e$, we have the cycle $a \to d \to e \to a \to b \to c\ (= a)$ with the desired property. Otherwise, $a = e$ which implies $R(a, a)$:

there is a self edge from a to itself. Hence, the path $a \to a \to b \to a \to a \to a$ is a cycle of length 5 with the desired property.

The other direction: $Q_2 \subseteq Q_1$. Suppose a graph has a (again possibly non-simple) cycle of length 5 such that the first and third vertices are not equal.

Again, call such a cycle $a \to b \to c \to d \to e \to a$ such that $a \neq c$. There are two cases; if $a \neq b$, we are done. Otherwise, $a = b$, so again a has a self loop, and a has an edge to c. We can create a cycle $a \to c \to d \to e \to a \to a$ of length 5 where the first and second vertices are different.