1. Mark each of the following statements true or false:

(a) Every orthonormal set of vectors is linearly independent.

(b) Every nonzero subspace of $\mathbb{R}^n$ has an orthogonal basis.

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(c) If A is a square matrix with orthonormal rows, then A is an orthogonal matrix.

(d) Every orthogonal matrix is invertible.

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(e) If A is a matrix with $\det\ A = 1$, then A is an orthogonal matrix.

(f) If A is an $m \times n$ matrix such that $(\text{row}(A))^{\perp} = \mathbb{R}^n$, then A must be the zero matrix.

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(g) If W is a subspace of $\mathbb{R}^n$ and v is a vector in $\mathbb{R}^n$ such that $\text{proj}_W(v) = 0$, then v must be the zero vector.

(h) If A is a symmetric, orthogonal matrix, then $A^2 = I$.

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(i) Every orthogonally diagonalizable matrix is invertible.

(j) Given any n real numbers $\lambda_1, \ldots, \lambda_n$, there exists a symmetric $n \times n$ matrix with $\lambda_1, \ldots, \lambda_n$ as its eigenvalues.

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2. Find all values of a and b such that

$$\begin{bmatrix} 1 & 4 & a \\ 2 & 1 & b \\ 3 & -2 & 3 \end{bmatrix}$$

is an orthogonal set of vectors.

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3. Find the coordinate vector $[v]_B$ of $v = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$ with respect to the orthogonal basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

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4. The coordinate vector of a vector v with respect to an orthonormal basis $B = \{v_1, v_2\}$ of $\mathbb{R}^2$ is $[v]_B = \begin{bmatrix} -3 \\ 1/2 \end{bmatrix}$. If $v_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, find all possible vectors v.

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5. Show that

$$\begin{bmatrix} 6/7 & 2/\sqrt{5} & 3/7 \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 4/7\sqrt{5} & -15/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

is an orthogonal matrix.

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6. If

$$\begin{bmatrix} 1/2 & a \\ b & c \end{bmatrix}$$

is an orthogonal matrix, find all possible values of a, b, and c.

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7. If Q is an orthogonal $n \times n$ matrix and $\{v_1, \ldots, v_k\}$ is an orthonormal set in $\mathbb{R}^n$, prove that $\{Qv_1, \ldots, Qv_k\}$ is an orthonormal set.

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8. If Q is an $n \times n$ matrix such that the angles $\angle(Qx, Qy)$ and $\angle(x, y)$ are equal for all vectors x and y in $\mathbb{R}^n$, prove that Q is an orthogonal matrix.

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9. In Questions 9-12, find a basis for $W^\perp$.

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10. W is the line in $\mathbb{R}^2$ with general equation $2x - 5y = 0$.

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11. $W = \operatorname{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix}\right\}$

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12. $W = \operatorname{span}\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}\right\}$

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13. Find bases for each of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ -1 & 2 & -2 & 1 & -2 \\ 2 & 1 & 4 & 8 & 9 \\ 3 & -5 & 6 & -1 & 7 \end{bmatrix}$$

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14. Find the orthogonal decomposition of

$$v = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

with respect to

$$W = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

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15. (a) Apply the Gram-Schmidt Process to

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

to find an orthogonal basis for $W = \text{span}\{x_1, x_2, x_3\}$.

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(b) Use the result of part (a) to find a QR factorization of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

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16. Find an orthogonal basis for $\mathbb{R}^4$ that contains the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

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17. Find an orthogonal basis for the subspace

$$W = \text{span}\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\} \text{ of } \mathbb{R}^4$$

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18. Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

(a) Orthogonally diagonalize A.

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(b) Give the spectral decomposition of A.

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19. Find a symmetric matrix with eigenvalues $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -2$ and eigenspaces

$$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right), E_{-2} = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right)$$

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20. If $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and

$$A = c_1 v_1 v_1^T + c_2 v_2 v_2^T + \cdots + c_n v_n v_n^T$$

prove that A is a symmetric matrix with eigenvalues $c_1, c_2, \ldots, c_n$ and corresponding eigenvectors $v_1, v_2, \ldots, v_n$.