Exercises from *Algebraic Topology* by Allen Hatcher

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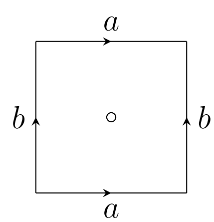
Contents

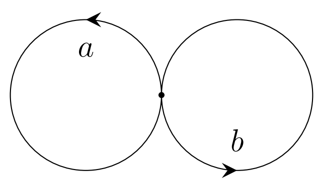
Chapter 0

Some Underlying Geometric Notions

Exercise 0.1 [Complete] Construct an explicit deformation retraction of the torus with one point deleted onto a graph of two circles intersecting at a point, namely, longitude and meridian circles of the torus.
Solution. Consider the fundamental square of the torus with a point removed:



where the deformation retract of the removed point retaracts to the boundary of the square; identifying the sides gives a bouquet of two circles:



where the point at the intersection of the circles is the identification of the four corners.

Alternatively, let $p \in T$ be the removed point. Choose your favorite open set U about p. Retract U to the meridian, so that T is now missing a longitudinal strip, contracted to a line spanning the meridial width of U. Then there are two open copies of S^1 seperated by this width; contract them along the meridian in the opposite direction of eachother. Then, exactly two copies of S^1 are left, connected by a single point.

□

Exercise 0.2 [Complete] Construct an explicit deformation retract from $\mathbb{R}^n - \{0\}$ to S^{n-1}.
Solution. Define a map

$$h : (\mathbb{R}^n - \{0\}) \times I \longrightarrow S^{n-1} \subset \mathbb{R}^n - \{0\}$$

via

$$h(x,t) := (1-t)x + \frac{tx}{||x||}$$

where $||\cdot||$ is the Euclidean norm in $\mathbb{R}^n$. Then, $h(x,0)=x$ and $h(1)=x/||x||$, retracting each point to a point on S^{n-1}.

$\square$

Exercise 0.3 [Complete]

1. Show that the composition of homotopy equivalences $X\to Y$ and $Y\to Z$ is a homotopy equivalence $X\to Z$. Deduce that homotopy equivalence is an equivalence relation.

2. Show that the relation of homotopy among maps $X\to Y$ is an equivalence relation.

3. Show that a map homotopic to a homotopy equivalence is an equivalence relation.

Solution. (1). Define homotopies $f:X\to Y$, $f':Y\to X$, $g:Y\to Z$ and $g':Z\to Y$ s.t.

$$f'\circ f\simeq \mathbb{1}_X, f\circ f'\simeq \mathbb{1}_Y$$
$$g'\circ g\simeq \mathbb{1}_Y, g\circ g'\simeq \mathbb{1}_Z.$$

Then,

$$\begin{aligned}(f'\circ g')\circ(g\circ f)&=f'\circ(g'\circ g)\circ f\\&=f'\circ\mathbb{1}_Y\circ f\\&=f'\circ f\\&=\mathbb{1}_X\end{aligned}$$

and

$$\begin{aligned}(g\circ f)\circ(f'\circ g')&=g\circ(f\circ f')\circ g'\\&=g\circ\mathbb{1}_Y\circ g'\\&=g\circ g'\\&=\mathbb{1}_Z\end{aligned}$$

hence, the composition map induced by $X\to Y\to Z$ induces a homotopy equivalence $X\to Z$. Or, in other words, the following diagram commutes:

$$\begin{tikzcd}
 & Y \arrow[loop above, "\mathbb{1}_Y"] \arrow[dl, bend left, "f'"] \arrow[dr, bend left, "g"] & \\
X \arrow[loop left, "\mathbb{1}_X"] \arrow[ur, bend left, "f"] \arrow[rr, bend left, "g\circ f"] & & Z \arrow[loop right, "\mathbb{1}_Z"] \arrow[ul, bend left, "g'"] \arrow[ll, bend left, "f'\circ g'"]
\end{tikzcd}.$$

Thus, homotopy equivalence is transitive.

Next, let $h:X\to X$ be a homotopy s.t. $h_t=\mathbb{1}_X$ for all $t\in I$. Then X is certainly homotopy equivalent to itself, and so homotopy equivalence is reflexive.

Lastly, let $k : X \to Y$ be a homotopy. Then there exists some k' s.t. $k' \circ k \simeq \mathbb{1}_X$ and $k \circ k' \simeq \mathbb{1}_Y$. Thus, k' induces a homotopy equivalence $Y \to X$, and so homotopy equivalence is symmetric. Therefore, homotopy equivalence is an equivalence relation.

(2). Let $f_0, f_1, f_2 : X \to Y$ be maps so that $f_0 \simeq f_1$ and $f_1 \simeq f_2$. Then there exist homotopies $F, G : X \times I \to Y$ s.t.

$$F(x,0) = f_0(x),\ F(x,1) = f_1(x)$$

$$G(x,0) = f_1(x),\ G(x,1) = f_2(x).$$

Then, define a map $H : X \times I \to Y$ s.t.

$$H(x,t) := \begin{cases} F(x,2t), & 0 \leq t \leq \frac{1}{2}, \\ G(x,2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then certainly H is a homotopy, so $f_0 \simeq f_2$. Therefore, homotopy relations of maps is transitive.

Next, let $f : X \to X$ be a map. Then, consider the homotopy $F : X \times I \to X$ s.t. $F(x,t) = f(x)$ for all t. Thus $f \simeq f$, and so homotopy relations of maps is reflexive.

Lastly, let $f_0, f_1 : X \to Y$ be homotopic. Then there exists a homotopy $F : X \times I \to Y$ s.t.

$$F(x,0) = f_0(x),\ F(x,1) = f_1(x).$$

Consider the map

$$H(x,t) := F(x, 1-t)$$

then,

$$H(x,0) = f_1(x),\ H(x,1) = f_0(x)$$

and so H is a homotopy from f_1 to f_0, hence $f_0 \simeq f_1 \iff f_1 \simeq f_0$. Therefore, homotopy relations of maps is symmetric, and homotopy relations of maps is an equivalence relation.

(3). Let $X \to Y$ be a homotopy equivalence. Then there exist maps $f : X \to Y$ and $g : Y \to X$ s.t. $f \circ g \simeq \mathbb{1}_Y$ and $g \circ f \simeq id_X$. Let $h \simeq g \circ f$ be a map homotopic to the homotopy equivalence. Then by (1) and (2), h is an equivalence relation.

□

Exercise 0.4 [WIP] A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 \simeq \mathbb{1}_X$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Solution. Let $i : A \hookrightarrow X$ be such an inclusion. Then, note that

$$(f_t \circ i)_{t=0} = f_0|_A \simeq \mathbb{1}_X|_A = \mathbb{1}_A$$

Next, let $f_1 = r$ be the resulting deformation retract in the weak sense, so that $r(X) \subset A$. Then,

$$(i \circ f_t)_{t=1} = i \circ r = i|_{r(X)} = \mathbb{1}_{r(X)}$$

where the final equality holds given $r(X) \subset A$. Finally, its clear that A must deformation retract to $r(X)$. Hence, i is a homotopy equivalence. □

Exercise 0.5 [TODO]
Exercise 0.6 [TODO]
Exercise 0.7 [TODO]
Exercise 0.8 [TODO]
Exercise 0.9 [TODO]
Exercise 0.10 [TODO]
Exercise 0.11 [WIP] Show that $f : X \to Y$ is a hopomotopy equivalence if there exist maps $g, h : Y \to X$ such that $f \circ g \simeq \mathbb{1}_Y$ and $h \circ f \simeq \mathbb{1}_X$. More generally, show that f is a homotopy equivalence if $f \circ g$ and $h \circ f$ are homotopy equivalences.
Solution. Note that

$$g \simeq (h \circ f) \circ g = h \circ (f \circ g) \simeq h.$$

Therefore, both h and g serve as a two-sided homotopic inverse of f, as in

$$g \circ f \simeq \mathbb{1}_X, \quad f \circ g \simeq \mathbb{1}_Y$$

$$h \circ f \simeq \mathbb{1}_X, \quad f \circ h \simeq \mathbb{1}_Y$$

Choosing either shows that f is a homotopy equivalence. More generally, let $f \circ g$ and $h \circ f$ be homotopy equivalences. Then there exist maps $u : X \to X$ and $v : Y \to Y$ s.t.

$$(h \circ f) \circ u \simeq u \circ (h \circ f) \simeq \mathbb{1}_X$$

$$(f \circ g) \circ v \simeq v \circ (f \circ g) \simeq \mathbb{1}_Y$$

Then,

$$g \circ u \circ h \circ f$$

□

Exercise 0.12 [TODO]

Chapter 1

The Fundamental Group

Exercise 1.1 [Complete] Show that composition of paths satisfies the following cancellation property: if $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.
Solution. Consider two homotopies $H_1, H_2 : I \times I \to X$ s.t.

$$H_1(s,0) = (f_0 \cdot g_0)(s), \quad H_1(s,1) = (f_1 \cdot g_1)(s)$$

$$H_2(s,0) = g_0(s), \quad H_2(s,1) = g_1(s).$$

Then, consider

$$H_2^*(s,t) = f_0 \cdot H_2(s,t)$$

which is clearly also a homotopy given by

$$H_2^*(s,0) = f_0 \cdot g_0, \quad H_2^*(s,1) = f_0 \cdot g_1.$$

Therefore, by transitivity, $f_1 \cdot g_1 \simeq f_0 \cdot g_1$, where cancelling g_1 on the right gives $f_0 \simeq f_1$.

□

Exercise 1.2 [Complete] Show that the change-of-basepoint homomoprhism β_h depends only on the homotopy class of h.
Solution.

Let $h_1, h_2 \in [h]$ with $h_1(0) = h_2(0) = x_1$ and $h_1(1) = h_2(1) = x_0$. Then consider the maps

$$\beta_{h_1}, \beta_{h_2} : \pi_1(X, x_1) \to \pi_1(X, x_0).$$

We want to show that for any $[f] \in \pi_1(X, x_1)$, we have that

$$\beta_{h_1}([f]) = [h_1 \cdot f \cdot \overline{h_1}] = [h_2 \cdot f \cdot \overline{h_2}] = \beta_{h_2}([f]).$$

This is equivalent to showing that $h_1 \cdot f \cdot \overline{h_1} \simeq h_2 \cdot f \cdot \overline{h_2}$ for any $f \in [f]$. Since $\overline{h_1} \simeq \overline{h_2}$ and $h_1 \simeq h_2$, this follows immediately from **Exercise 1.1** (while additionally using a similar argument for $g_0 \cdot f_0 \simeq g_1 \cdot f_1$, $g_0 \simeq g_1 \implies f_0 \simeq f_1$), hence the result.

□

Exercise 1.3 [Complete] For a path-connected space X, show that $\pi_1(X)$ is abelian if and only if all change-of-basepoint homomorphisms β_h depend only on the endpoints of h.
Solution.

($\Longleftarrow$) Assume $\pi_1(X)$ is not abelian. Then, there exist $[f],[g] \in \pi_1(X)$ with $[g]\cdot[f]\cdot[g]^{-1} \neq [f]$, and thus

$$\beta_g([f]) = [g \cdot f \cdot \bar{g}] = [g] \cdot [f] \cdot [g]^{-1} \neq [f]$$

where the second equality holds due to **Exercise 1.2**. Let $\{x_0\}$ be the constant loop, then

$$\beta_{\{x_0\}}([f]) = [\{x_0\}] \cdot [f] \cdot [\{x_0\}] = [f]$$

and so $\beta_g \neq \beta_{\{x_0\}}$, a contradiciton.

($\Longrightarrow$) Let $\pi_1(X)$ be abelian. Let $h_1, h_2 \in [h]$. Then,

$$\beta_{h_1}([f]) = [h_1 \cdot f \cdot \bar{h_1}] = [h_1 \cdot f \cdot \bar{h_2}] \cdot [h_2 \cdot \bar{h_1}]$$

where the last equality holds since $[h_2 \cdot \bar{h}]$ is trivial as they are in the same homotopy class. Since $\pi_1(X)$ is abelian, we get that

$$\beta_{h_1}([f]) = [h_1 \cdot f \cdot \bar{h_2}] \cdot [h_2 \cdot \bar{h_1}] = [h_2 \cdot \bar{h_1}] \cdot [h_1 \cdot f \cdot \bar{h_2}] = [h_2 \cdot f \cdot \bar{h_2}] = \beta_{h_2}([f])$$

hence the result.

$\square$

Exercise 1.4 [TODO]

Exercise 1.5 [Complete] Show the following are equivalent:

1. Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

2. Every map $S^1 \to X$ extends to a map $D^2 \to X$.

3. $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space is simply-connected iff all maps $S^1 \to X$ are homotopic.

Solution.

($1 \implies 2$). Let $f : S^1 \to X$. Let $H : S^1 \times I \to X$ be a homotopy from (1) s.t.

$$H(x,0) = f(x), \quad H(s,1) = \{x_0\}$$

with the restraint that $H(s,t) \subset f(S^1)$ for all $t > 0$. Embed $S^1 \times I$ into D^2 as an annulus with center c; extend H over all of D^2 by mapping it to c.

($2 \implies 3$). Think of $\pi_1(X, x_0)$ as basepoint-preserving homotopy classes of maps $f : (S^1, s_0) \to (X, x_0)$. By hypothesis f extends to a map $\tilde{f} : (D^2, s_0) \to X$, and so there is a deformation retraction r of D^2 onto s_0. Let $H : S^1 \to X$ be a homotopy s.t.

$$H(s,t) := f \circ r_t(s)$$

which preserves the basepoint s_0, and is a homotopy of f to $\{f(s_0)\} = \{x_0\}$, hence the result.

($3 \implies 1$). Again, think of $\pi_1(X, x_0)$ as basepoint-preserving homotopy classes of maps $f : (S^1, s_0) \to (X, x_0)$. By hypothesis, there must exist a homotopy taking f to $\{f(s_0)\} = \{x_0\}$, hence the result.

If all maps $S^1 \to X$ are homotopic, then $\pi_1(X, x_0)$ has exactly one element, which must be the identity, and so X must be simply-connected. Conversely, if X is simply-connected, then by the above, all maps $S^1 \to X$ are homotopic to a constant map at some basepoint.

□

Exercise 1.6 [**TODO**]

We can regard $\pi_1(X, x_0)$ as all basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no condition on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ is a one-to-one correspondence between $[S^1, X]$ and $\pi_1(X)$ if X is path-connected.

Solution.

Surjectivity. Let $[h] \in [S^1, X]$ be a homotopy class of maps $S^1 \to X$. Since X is path-connected, certainly there is some $h \in [h]$ with $h(t) = x_0$ for some $t \in I$. Reperamaterize so that $h(0) = h(1) = x_0$. Then there must be some $f \in [f] \in \pi_1(X, x_0)$ s.t. h is homotopic to f, hence $\Phi([f]) = [h]$, and so Φ is onto.

Injectivity. ($\Longleftarrow$) Let $[f]$ and $[g]$ be conjugate in $\pi_1(X, x_0)$. Then there exists a homotopy H mapping $f \to g$ for any $f \in [f]$ and $g \in [g]$.

□

Exercise 1.7 [**TODO**]
Exercise 1.8 [**TODO**]
Exercise 1.9 [**TODO**]
Exercise 1.10 [**TODO**]
Exercise 1.11 [**TODO**]
Exercise 1.12 [**TODO**]
Exercise 1.13 [**TODO**]
Exercise 1.14 [**TODO**]
Exercise 1.15 [**TODO**]
Exercise 1.16 [**TODO**]

Show that there are no retractions $r : X \to A$ in the following cases:

1. $X = \mathbb{R}^3$ and A any subspace homeomorphic to S^1.
2. $X = S^1 \times D^2$ and A its boundary torus $S^1 \times S^1$.
3. $X = S^1 \times D^2$ and A the circle shown in the figure.
4.

Solution.

(1). The fundamental group of the subspace of a retraction is a subgroup of the space. $\pi_1(\mathbb{R}^3)$ is trivial, and $\pi_1(S^1) = \mathbb{Z}$, and so no retraction can exist as obviously $\mathbb{Z}$ is not a subgroup of the trivial group.

(2) The fundamental group of the subspace of a retraction is a subgroup of the space. $\pi_1(S^1 \times D^2) = \mathbb{Z}$ as $S^1 \times D^2$ deformation retracts to S^1, whereas $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$. $\mathbb{Z} \times \mathbb{Z}$ cannot be a subgroup of $\mathbb{Z}$, hence the result.

(3)

□

Exercise 1.17 [**TODO**]

Exercise 1.18 [**TODO**]
Exercise 1.19 [**TODO**]
Exercise 1.20 [**TODO**]
Exercise 1.21 [**TODO**]
Exercise 1.22 [**TODO**]
Exercise 1.23 [**TODO**]