Exercises from *Matrix Groups for Undergraduates* by Kristopher Tapp

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Contents

Chapter 1

Matrices

Exercise 1.1 [Complete]

Describe a natural 1-to-1 correspondence between elements of SO(3) and elements of

$$T^1S^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |p| = |v| = 1 \text{ and } p \perp q\}$$

Solution. Using the globe analogy from Question 1.2, fix a point r to be the north pole, and a point e that lies on the equator induced by the choice of r, and assert this as the arbitrary 'identity'.

Next, given some $A \in \mathrm{SO}(3)$, identify an element in T^1S^2 via $A \mapsto (Ar, Av)$, as in first where A maps the north pole r, and then how A rotates the globe about the axis induced by r and its antipodal point. $\square$

Exercise 1.2 [Complete]

Prove equation 1.3:

$$(A \cdot B)^T = B^T \cdot A^T$$

Solution. First,

$$\begin{aligned}(A \cdot B)^T_{ij} &= (A \cdot B)_{ji} \\ &= \sum_{k=1}^{n} A_{jk}B_{ki}\end{aligned}$$

Next,

$$\begin{aligned}(B^T \cdot A^T)_{ij} &= \sum_{k=1}^{n} (B^T)_{ik}(A^T)_{kj} \\ &= \sum_{k=1}^{n} B_{ki}A_{jk} \\ &\overset{*}{=} \sum_{k=1}^{n} A_{jk}B_{ki}\end{aligned}$$

$\square$

Note the $\overset{*}{=}$ step uses the commutativity of multiplication, hence the above proof does not work when $\mathbb{K} = \mathbb{H}$.

Exercise 1.3 [Complete]

Prove equation 1.4:

$$\operatorname{trace}(A \cdot B) = \operatorname{trace}(B \cdot A)$$

Solution. First,

$$\operatorname{trace}(A \cdot B)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

Next,

$$\begin{aligned}\operatorname{trace}(B \cdot A)_{ii} &= \sum_{i=1}^{n} (B \cdot A)_{ii} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ik} A_{ki}\end{aligned}$$

Carefully reindexing and resumming gives the result. □

Note that the 'careful reindexing and resumming process' implies trace() is invariant under cyclic permutation, e.x.:

$$\operatorname{trace}(A \cdot B \cdot C) = \operatorname{trace}(C \cdot A \cdot B)$$

Exercise 1.4 [Complete]

Let $A, B \in M_n\mathbb{K}$. Prove that if $A \cdot B = I$ then $B \cdot A = I$.

Solution. Note that

$$A \cdot B = I \iff A = B^{-1}.$$

The result follows. □

Exercise 1.5 [Complete]

Suppose that the determinant of $A \in M_n(\mathbb{H})$ were defined as in Equation 1.5. Show for

$$A = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} \in M_n(\mathbb{H})$$

that $\det(A) \neq 0$ but

$$R_A : H^2 \to H^2$$

is not invertible.

Solution. Given the definition from Equation 1.5:

$$\begin{aligned}\det(A) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} \\ &= \mathbf{ij} - \mathbf{ji} \\ &= (1) - (-1) \\ &= 2 \neq 0\end{aligned}$$

However,

$$R_A((-\mathbf{i}, \mathbf{i})) = (-\mathbf{i}, \mathbf{i}) \cdot \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} = (-\mathbf{i}^2 + \mathbf{i}^2, -\mathbf{ij} + \mathbf{ij}) = (1 - 1, -\mathbf{k} + \mathbf{k}) = (0, 0)$$

Hence, R_A has a non-zero determiant, but is not invertible as the kernel is non-trivial. Similarly, clearly the columns of A are linearly dependent. $\square$

Exercise 1.6 [Complete]

Find $B \in M_2(\mathbb{R})$ such that $R_B : \mathbb{R}^2 \to \mathbb{R}^2$ is a counter-clockwise rotation through an angle θ.

Solution. Note that we can 'represent' both 1 and i in $M_2(\mathbb{R})$ via

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \text{ and } i \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where the latter works since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

capturing the fact that $i^2 = -1$. Building on this, we can represent any $a + bi \in \mathbb{C}$ via

$$\rho : a + bi \mapsto a \cdot I + b \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Next, note that the function $f_\theta(z) = ze^{i\theta}$ rotates elements counter-clockwise in $\mathbb{C}$ by an angle θ. To see this, letting $z = re^{i\varphi}$,

$$f_\theta(z) = ze^{i\theta} = re^{i\varphi}e^{i\theta} = re^{i(\varphi+\theta)}.$$

Applying ρ gives

$$\begin{aligned} \rho_1(e^{i\theta}) &= \rho_1(\cos(\theta) + i\sin(\theta)) \\ &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = B \end{aligned}$$

$\square$

Exercise 1.7 [WIP]

Describe all elements $A \in \mathrm{GL}_n(\mathbb{R})$ with the property $AB = BA$ for all $B \in \mathrm{GL}_n(\mathbb{R})$.

Solution. Matrices A that commute with all matrices in $\mathrm{GL}_n(\mathbb{R})$ are scalar multiples of the identity

$$A = \lambda I$$

where $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. $\square$

Exercise 1.8 [Complete]

Let $\mathrm{SL}_2(\mathbb{Z})$ denote 2 by 2 matrices with integer entries and determinant 1. Prove that $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Is $SL_n(\mathbb{Z})$ a subgroup of $\mathrm{GL}_n(\mathbb{R})$ in general?

Solution. Fixing $A, B \in \mathrm{SL}_2(\mathbb{Z})$, its clear that $A \cdot B$ must have all integer entries. Since

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

we have that $A \cdot B \in \mathrm{SL}_2(\mathbb{Z})$ (closure). Next, fix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

Using Cramer's Rule to compute the inverse of A we get

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $ad-bc = 1$ since $\det A = 1$, so A^{-1} has all integer entries, and is a member of $\mathrm{SL}_2(\mathbb{Z})$. Therefore, $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. □

The critical step in the above proof is discerning that the factor extracted from A^{-1} is $1/\det A = 1$, which ensures the entries of the inverse are all in $\mathbb{Z}$. This factor is the same for any n, so $\mathrm{SL}_n(\mathbb{Z})$ is always a subgroup of $\mathrm{GL}_n(\mathbb{R})$ for all n.

Exercise 1.9 [Complete]

Describe the block matrix blah blabh blabhj TODO write this out

Solution. Suppose A and B are block matrices in $M_n(\mathbb{K})$, given by

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

where $\sum \mathrm{dim} A_i = \mathrm{dim} A = \sum \mathrm{dim} B_i = \mathrm{dim} B$, and $\mathrm{dim} A_i = \mathrm{dim} B_i$ for each i. Then,

$$A \cdot B = \begin{pmatrix} A_1 \cdot B_1 & 0 & \cdots & 0 \\ 0 & A_2 \cdot B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \cdot B_n \end{pmatrix}$$

Which can be applied to the above question to derive a simple answer.

□

Exercise 1.10 [Complete]

If $G_1 \subset \mathrm{GL}_{n_1}(\mathbb{K})$ and $G_2 \subset \mathrm{GL}_{n_2}(\mathbb{K})$ are subgroups, describe a subgroup of $\mathrm{GL}_{n_1+n_2}(\mathbb{K})$ isomorphic to $G_1 \times G_2$.

Solution. Define a map

$$\varphi : G_1 \times G_2 \to \mathrm{GL}_{n_1+n_2}(\mathbb{K})$$

given by

$$(A_1, A_2) \mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

The image of φ is a subset of $\mathrm{GL}_{n_1+n_2}(\mathbb{K})$ since

$$\det \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \det A_1 \cdot \det A_2 \neq 0$$

so $\varphi(A_1, A_2) \in \mathrm{GL}_{n_1+n_2}(\mathbb{K})$. To prove φ is a group homomorphism, observe

$$\begin{aligned}\varphi((A_1, A_2)\cdot(B_1, B_2)) &= \varphi(A_1\cdot B_1, A_2\cdot B_2)\\ &= \begin{pmatrix} A_1\cdot B_1 & 0 \\ 0 & A_2\cdot B_2\end{pmatrix}\\ &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2\end{pmatrix}\cdot\begin{pmatrix} B_1 & 0 \\ 0 & B_2\end{pmatrix}\\ &= \varphi(A_1, A_2)\cdot\varphi(B_1, B_2)\end{aligned}$$

hence the result.

□

Exercise 1.11 [TODO]

Exercise 1.12 [Complete]

Show that for purely imaginary $q_1, q_2 \in \mathbb{H}$, $-\mathrm{Re}(q_1\cdot q_2)$ is the vector dot product in $\mathbb{R}^3 = \mathrm{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathrm{Im}(q_1\cdot q_2)$ is the vector cross-product.

Solution. First,

$$\begin{aligned}-\mathrm{Re}(q_1\cdot q_2) &= -\mathrm{Re}((b_1\mathbf{i}+c_1\mathbf{j}+d_1\mathbf{k})\cdot(b_2\mathbf{i}+c_2\mathbf{j}+d_2\mathbf{k}))\\ &= -\mathrm{Re}((-b_1b_2-c_1c_2-d_1d_2)+\dots)\\ &= b_1b_2+c_1c_2+d_1d_2\end{aligned}$$

Next,

$$\begin{aligned}\mathrm{Im}(q_1\cdot q_2) &= \mathrm{Im}((b_1\mathbf{i}+c_1\mathbf{j}+d_1\mathbf{k})\cdot(b_2\mathbf{i}+c_2\mathbf{j}+d_2\mathbf{k}))\\ &= (c_1d_2-d_1c_2)\mathbf{i}+(d_1b_2-b_1d_2)\mathbf{j}+(b_1c_2-c_1b_2)\mathbf{j}\end{aligned}$$

Mapping $\mathrm{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to the standard basis in $\mathbb{R}^3$ gives both desired results. □

Exercise 1.13 [WIP]

Prove that non-real elements $q_1, q_2 \in \mathbb{H}$ commute if and only if their imaginary parts are parallel; that is, $\mathrm{Im}(q_1) = \lambda\cdot\mathrm{Im}(q_2)$ for some $\lambda \in \mathbb{R}$.

Solution. ($\implies$) Let $\mathrm{Im}(q_1) = \lambda\cdot\mathrm{Im}(q_2)$ for some $\lambda\in\mathbb{R}$, so that

$$\mathrm{Im}(q_1) = b\mathbf{i}+c\mathbf{j}+d\mathbf{k} = \lambda b\mathbf{i}+\lambda c\mathbf{j}+\lambda d\mathbf{k} = \mathrm{Im}(q_2)$$

then,

$$\begin{aligned}q_1\cdot q_2 &= (a_1+b\mathbf{i}+c\mathbf{j}+d\mathbf{k})\cdot(a_2+\lambda b\mathbf{i}+\lambda c\mathbf{j}+\lambda d\mathbf{k})\\ &= \left(a_1a_2-\lambda\left(b^2+c^2+d^2\right)\right)b(a_1\lambda+a_2)\mathbf{i}c(a_1\lambda+a_2)\mathbf{j}d(a_1\lambda+a_2)\mathbf{k}\\ &= (a_2+\lambda\, b\,\mathbf{i}+\lambda\, c\,\mathbf{j}+\lambda\, d\,\mathbf{k})\cdot(a_1+b\,\mathbf{i}+c\,\mathbf{j}+d\,\mathbf{k})\\ &= q_2\cdot q_1.\end{aligned}$$

($\impliedby$) Let $q_1\cdot q_2 = q_2\cdot q_1$ where

$$q_1 = (a_1+b_1\mathbf{i}+c_1\mathbf{j}+d_1\mathbf{k}), q_2 = (a_2+b_2\mathbf{i}+c_2\mathbf{j}+d_2\mathbf{k}).$$

Then the following equalities must hold:

□

Exercise 1.14 [TODO]

Charachterize the pairs $q_1, q_2 \in \mathbb{H}$ which anti-commute, that is $q_1 q_2 = -q_2 q_1$.

Exercise 1.15 [Complete]

If $q \in \mathbb{H}$ satisfies $q\mathbf{i} = \mathbf{i}q$, prove that $q \in \mathbb{C}$.

Solution. Let $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Then,

$$q\mathbf{i} = a\mathbf{i} + b\mathbf{ii} + c\mathbf{ji} + d\mathbf{ki} = -b + a\mathbf{i} + d\mathbf{j} - c\mathbf{k}$$

and

$$\mathbf{i}q = \mathbf{i}a + b\mathbf{ii} + c\mathbf{ij} + d\mathbf{ik} = -b + a\mathbf{i} - d\mathbf{j} + c\mathbf{k}.$$

Identifying terms gives

$$d = -d \implies d = 0$$

$$c = -c \implies c = 0$$

hence $q = a + bi \in \mathbb{C}$.

□

Exercise 1.16 [Complete]

Prove that complex multiplication in $\mathbb{C} \approxeq \mathbb{R}^2$ does not extend to a multiplication operation on $\mathbb{R}^3$ that makes $\mathbb{R}^3$ into a real division algebra.

Solution. Assume such an extension exists. Consider the map analagous to the extension of $\mathbb{R}^2$ given by

$$(a, b, c) \mapsto a + b\mathbf{i} + c\mathbf{j}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = -1$. Then there must exist a linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

with $T^2 = -I$. Represent T by a 3×3 real matrix M. Then -1 is in the spectrum of M^2 since $M^2 = -I$, thus $\pm i$ is in the spectrum of M. Since $\det(M) = \prod \lambda_k$ where λ_k is an eigenvalue of M, there must exist some real value λ such that

$$\det(M) = (i)(-i)(\lambda) = \lambda$$

where λ must be in $\mathbb{R}$ since complex eigenvalues come in pairs. Thus, we must have

$$\det(M^2) = \det(-I) = -1$$

and

$$\det(M^2) = \det(M)^2 = \lambda^2$$

thus $\lambda^2 = -1$, which contradicts λ being real.

□

Exercise 1.17 [Complete]

Describe a subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$ that is isomorphic to $\mathbb{R}^n$ under vector-addition.

Solution. Consider the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R})$$

and the map that takes such matrices to $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

$\square$

Exercise 1.18 [Complete]

If $\lambda \in \mathbb{H}$ commutes with every element of $\mathbb{H}$, prove that $\lambda \in \mathbb{R}$.

Solution. Let λ have the property that $\lambda \cdot w = w \cdot \lambda$ for all $w \in \mathbb{H}$. Letting $w = \mathbf{i}$, $\lambda \in \mathbb{C}$ per Exercise 1.15. Letting $\lambda = a + b\mathbf{i}$ and $w = \mathbf{j}$, we must have

$$(a + b\mathbf{i}) \cdot \mathbf{j} = a\mathbf{j} + b\mathbf{k} = a\mathbf{j} - b\mathbf{k} = \mathbf{j}(a + b\mathbf{i})$$

hence $b = -b \implies b = 0$, therefore $\lambda = a \in \mathbb{R}$.

$\square$

Chapter 2

All matrix groups are real matrix groups

2.1 Exercises

Exercise 2.1 [Complete]

Prove that ρ_n makes the diagram in 2.1 commute.

Solution. First, note the definition of ρ_n, that is

$$\rho_1(a + b\mathbf{i}) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Say $A \in M_n(\mathbb{C})$, then the map $\rho_n : M_n(\mathbb{C}) \to M_{2n}$ is simply given by

$$\rho_n(A) = \rho_n \left(\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \right) := \begin{bmatrix} \rho_1(z)_{11} & \rho_1(z)_{12} & \cdots & \rho_1(z)_{1n} \\ \rho_1(z)_{21} & \rho_1(z)_{22} & \cdots & \rho_1(z)_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1(z)_{n1} & \rho_1(z)_{n2} & \cdots & \rho_1(z)_{nn} \end{bmatrix}$$

where each $\rho_1(z)_{ij}$ is a 2×2 block matrix as defined above. We want to show the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "f_n"] \arrow[d, "R_A"'] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(A)}"] \\ \mathbb{C}^n \arrow[r, "f_n"] & \mathbb{R}^{2n} \end{tikzcd}$$

where

$$f_n(a_1 + b_1\mathbf{i}, \ldots, a_n + b_n\mathbf{i}) := (a_1, b_1, \ldots, a_n, b_n)$$

and $A \in M_n(\mathbb{C})$. Let $z = a + b\mathbf{i} \in \mathbb{C}^1$ and $A = c + d\mathbf{i} \in \mathbb{C}^1$. Then,

$$f_1 \circ R_A(z) = f_1((ac - bd) + (ad + bc)\mathbf{i}) = (ac - bd, ad + bc)$$

and

$$R_{\rho_1(A)} \circ f_1(z) = R_{\rho_1(A)}(a,b) = (ac - bd, ad + bc).$$

Inducting over n using the definition of matrix multiplication and block matrix multiplication produces the result.

$\square$

Exercise 2.2 [TODO]

Exercise 2.3 [TODO]

Prove proposition 2.6.

Exercise 2.4 [TODO]

Prove proposition 2.7.

Exercise 2.5 [Complete]

Prove that for any $A \in M_1(\mathbb{H})$, $\det(A) \in \mathbb{R}$.

Solution. Let $A = z + w\mathbf{j} \in M_n(\mathbb{H})$. Then,

$$\begin{aligned}
\det(A) &= \det \circ \Psi_1(A) \\
&= \det \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \\
&= z \cdot \bar{z} + w \cdot \bar{w} \\
&= \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 + \mathrm{Re}(w)^2 + \mathrm{Im}(z)^2 \in \mathbb{R}
\end{aligned}$$

$\square$

Exercise 2.6 [Complete]

Prove that $\mathrm{SL}_n(\mathbb{H}) = \{A \in \mathrm{GL}_n(\mathbb{H}) : \det(A) = 1\}$ is a subgroup. Describe a natural bijection between elements of $\mathrm{SL}_1(\mathbb{H})$ and elements of the 3-sphere S^3.

Solution. Fix $A, B \in \mathrm{SL}_n(\mathbb{H})$. Then,

$$\begin{aligned}
\det(AB) &= \det \circ \Psi_n(AB) \\
&= \det\left(\Psi_n(A) \cdot \Psi_n(B)\right) \\
&= \det(\Psi_n(A)) \cdot \det(\Psi_n(B)) \\
&= (1) \cdot (1) = 1
\end{aligned}$$

hence, $\mathrm{SL}_n(\mathbb{H})$ is closed. Next,

$$\begin{aligned}
1 &= \det(I) \\
&= \det(AA^{-1}) \\
&= \det \circ \Psi_n(AA^{-1}) \\
&= \det\left(\Psi_n(A) \cdot \Psi_n(A^{-1})\right) \\
&= \det(\Psi_n(A)) \cdot \det(\Psi_n(A^{-1})) \\
&= (1) \cdot \det(\Psi_n(A^{-1})) \\
&\implies \det(A^{-1}) = 1 \implies A^{-1} \in \mathrm{SL}_n(\mathbb{H})
\end{aligned}$$

hence $\mathrm{SL}_n(\mathbb{H})$ is a subgroup. From Exercise 2.6, we have that for any $A = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathrm{SL}_n(\mathbb{H})$ we have that

$$1 = \det(A) = a^2 + b^2 + c^2 + d^2$$

hence, the map

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto (a, b, c, d) \in S^3$$

is obviously a bijection. $\square$

Exercise 2.7 [TODO]

Exercise 2.8 [TODO]

Exercise 2.9 [TODO]

Exercise 2.10 [TODO]

Exercise 2.11 [TODO]

Exercise 2.12 [TODO]

Let $q \in \mathbb{H}$ and define $\mathbb{C} \cdot q := \{\lambda \cdot q : \lambda \in \mathbb{C}\}$ and $q \cdot \mathbb{C} := \{q \cdot \lambda : \lambda \in \mathbb{C}\}$.

1. With $g_1 : \mathbb{H} \to \mathbb{C}^2$ defined as in section Section 2, show that $g_1(\mathbb{C} \cdot q)$ is a one-dimensional $\mathbb{C}-$subspace of $\mathbb{C}^2$.

2. Define a natural identification $\hat{g}_1 : \mathbb{H} \to \mathbb{C}^2$ so that $\hat{g}_1(q \cdot \mathbb{C})$ is a one-dimensional $\mathbb{C}-$subspace of $\mathbb{C}^2$.

Solution. (1). Let $q = z + w\mathbf{j} \in \mathbb{H}$ and $z, w \in \mathbb{C}$. Then,

$$\begin{aligned} g_1(\mathbb{C} \cdot q) &= \{g_1(\lambda \cdot q) : \lambda \in \mathbb{C}\} \\ &= \{g_1(\lambda z + \lambda w\mathbf{j}) : \lambda, z, w \in \mathbb{C}\} \\ &= \{(\lambda z, \lambda w) : \lambda, z, w \in \mathbb{C}\} \\ &= \{\lambda \cdot (z, w) : \lambda, z, w \in \mathbb{C}\} \end{aligned}$$

where z and w are determined by q (hence they are fixed), and so $g_1(\mathbb{C} \cdot q)$ is clearly a one-dimensional $\mathbb{C}-$subspace of $\mathbb{C}^2$.

(2). Let $\iota : \mathbb{H} \to \mathbb{H}$ be given by $\iota(q) = \iota(z + w\mathbf{j}) = \iota(z + \mathbf{j}w)$ with $z, w \in \mathbb{C}$ and $f_1 : \mathbb{H} \to \mathbb{C}^2$ given by

$$f_1(w + \mathbf{j}z) := (w, z).$$

Then, let $\hat{g}_1 = f_1 \circ \iota$. Therefore,

$$\begin{aligned} \hat{g}_1(q \cdot \mathbb{C}) &= \{\hat{g}_1(q \cdot \lambda) : \lambda \in \mathbb{C}\} \\ &= \{f_1 \circ \iota(q \cdot \lambda) : \lambda \in \mathbb{C}\} \\ &= \{f_1(z\lambda + \mathbf{j}w\lambda) : \lambda, z, w \in \mathbb{C}\} \\ &= \{(z\lambda, w\lambda) : \lambda, z, w \in \mathbb{C}\} \\ &= \{\lambda \cdot (z, w) : \lambda, z, w \in \mathbb{C}\} \end{aligned}$$

which is the same subspace from part (1), hence the result. $\square$

Chapter 3

The orthogonal groups

3.1 Exercises

Exercise 3.1 [Complete]

Prove part (4) of Proposition 3.3:

$$\overline{\langle X, Y \rangle} = \langle Y, X \rangle$$

Solution.

$$\begin{aligned}
\overline{\langle X, Y \rangle} &= \overline{\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle} \\
&= \overline{x_1 \bar{y_1} + \cdots + x_n \bar{y_n}} \\
&= \overline{x_1 \bar{y_1}} + \cdots + \overline{x_n \bar{y_n}} \\
&= y_1 \bar{x_1} + \cdots + y_n \bar{x_n} \\
&= \langle (y_1, \ldots, y_n), (x_1, \ldots, x_n) \rangle \\
&= \langle Y, X \rangle
\end{aligned}$$

□

Exercise 3.2 [Complete]

Prove Equations 3.5 and 3.6:

$$(3.5) : \langle X, Y \rangle_{\mathbb{C}} = \langle f(X), f(Y) \rangle_{\mathbb{R}} + \mathbf{i} \langle f(X), f(\mathbf{i}Y) \rangle_{\mathbb{R}}$$

$$(3.6) : |X|_{\mathbb{C}} = |f(X)|_R$$

where

$$f = f_n : \mathbb{C}^n \to \mathbb{R}^{2n}$$

is given by

$$f(a_1 + b_1\mathbf{i}, \ldots, a_n + b_n\mathbf{i}) := (a_1, b_1, \ldots, a_n, b_n).$$

Solution. First, for Equation 3.5,

$$\begin{aligned}
\langle X, Y\rangle_{\mathbb{C}} &= \langle (a_1 + b_1\mathbf{i}, \dots, a_n + b_n\mathbf{i}), (c_1 + d_1\mathbf{i}, \dots, c_n + d_n\mathbf{i})\rangle_{\mathbb{C}} \\
&= (a_1 + b_1\mathbf{i})\overline{(c_1 + d_1\mathbf{i})} + \cdots + (a_n + b_n\mathbf{i})\overline{(c_n + d_n\mathbf{i})} \\
&= (a_1 + b_1\mathbf{i})(c_1 - d_1\mathbf{i}) + \cdots + (a_n + b_n\mathbf{i})(c_n - d_n\mathbf{i}) \\
&= [(a_1c_1 + b_1d_1) + (-a_1d_1 + b_1c_1)\mathbf{i}] + \cdots + [(a_nc_n + b_nd_n) + (-a_nd_n + b_nc_n)\mathbf{i}] \\
&= (a_1c_1 + b_1d_1 + \cdots + a_nc_n + b_nd_n) + (-a_1d_1 + b_1c_1 + \cdots - a_nd_n + b_nc_n)\mathbf{i} \\
&= \langle (a_1, b_1, \dots, a_n, b_n), (c_1, d_1, \dots c_n, d_n)\rangle_{\mathbb{R}} + \mathbf{i}\langle (a_1, b_1, \dots, a_n, b_n), (-d_1, c_1, \dots, -d_n, c_n)\rangle_{\mathbb{R}} \\
&= \langle f(X), f(Y)\rangle_{\mathbb{R}} + \mathbf{i}\langle f(X), \underbrace{f(\mathbf{i}Y)}_{!}\rangle_{\mathbb{R}}
\end{aligned}$$

where the equality of $\underbrace{f(\mathbf{i}Y)}_{!}$ is due to

$$\begin{aligned}
f(\mathbf{i}Y) &= f(\mathbf{i}(c_1 + d_1\mathbf{i}, \dots, c_n + d_n\mathbf{i})) \\
&= f((-d_1 + \mathbf{i}c_1, \dots, -d_n + \mathbf{i}c_n)) \\
&= (-d_1, c_1, \dots, -d_n, c_n)
\end{aligned}$$

Next, for Equation 3.6,

$$\begin{aligned}
|X|_{\mathbb{C}} &= \sqrt{\langle X, X\rangle_{\mathbb{C}}} \\
&= \sqrt{\langle (a_1 + b_1\mathbf{i}, \dots, a_n + b_n\mathbf{i}), (a_1 + b_1\mathbf{i}, \dots, a_n + b_n\mathbf{i})\rangle_{\mathbb{C}}} \\
&= \sqrt{(a_1 + b_1\mathbf{i})(a_1 - b_1\mathbf{i}) + \cdots + (a_n + b_n\mathbf{i})(a_n - b_n\mathbf{i})} \\
&= \sqrt{a_1^2 + b_1^2 + \cdots + a_n^2 + b_n^2} \\
&= \sqrt{\langle (a_1, b_1, \dots, a_n, b_n), (a_1, b_1, \dots, a_n, b_n)\rangle} \\
&= \sqrt{\langle f(X), f(X)\rangle}_{\mathbb{R}} \\
&= |f(X)|_{\mathbb{R}}
\end{aligned}$$

□

Exercise 3.3 [WIP]

Prove Proposition 3.5, that is $\{X_1, \dots X_n\} \in \mathbb{C}^n$ is an orthonormal basis if and only if $\{f(X_1), f(\mathbf{i}X_1), \dots, f(X_n), f(\mathbf{i}X_n)\}$ is an othonormal basis of $\mathbb{R}^{2n}$.

Solution. ($\Longrightarrow$) Let $\{Y_1, \dots Y_{2n}\} = \{f(X_1), f(\mathbf{i}X_1), \dots, f(X_n), f(\mathbf{i}X_n)\}$ be an orthonorml basis of $\mathbb{R}^{2n}$. Then,

$$\langle Y_i, Y_j\rangle_{\mathbb{R}} = \delta_{ij}$$

where δ_{ij} is the Kronecker Delta. Next, consider

$$\langle f^{-1}(Y_i), f^{-1}(Y_j)\rangle_{\mathbb{R}} = f^{-1}()$$

($\Longleftarrow$) Let $X, Y \in \mathbb{C}^n$ be orthogonal. Then,

$$\langle X, Y\rangle_{\mathbb{C}} = \langle f(X), f(Y)\rangle_{\mathbb{R}} + \mathbf{i}\langle f(X), f(\mathbf{i}Y)\rangle_{\mathbb{R}} = 0$$

Since $\langle f(X), f(\mathbf{i}Y)\rangle_{\mathbb{R}} \in \mathbb{R}$, and hence $\mathbf{i}\langle f(X), f(\mathbf{i}Y)\rangle_{\mathbb{R}} \in \mathbb{C}$, both factors must vanish. Hence, f maps $\mathbb{C}^n$ to an orthonormal basis of $\mathbb{R}^{2n}$.

□

Exercise 3.4 [Complete]

Prove Proposition 3.18, that is, for any $X \subset \mathbb{R}^n$, $\mathrm{Symm}^+(X) \subset \mathrm{Symm}(X)$ is a subgroup with index 1 or 2.

Solution. First, to show $\mathrm{Symm}^+(X)$ is a subgroup, let $A, B \in \mathrm{Symm}^+(X)$. Then,

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1 \implies AB \in \mathrm{Symm}^+(X)$$

and

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A^{-1}) \implies A{-}1 \in \mathrm{Symm}^+(X)$$

so $\mathrm{Symm}^+(X)$ is a subgroup. It's clear that $\mathrm{Symm}^+(X)$ has at most 2 cosets given the symmetry of the determinant, and has 1 if $\mathrm{Symm}^+(X) = \mathrm{Symm}(X)$, hence it's index is 1 or 2. □

Exercise 3.5 [Complete]

Let $A \in \mathrm{GL}_n(\mathbb{K})$. Prove that $A \in \mathrm{O}_n(\mathbb{K})$ if and only if the columns of A are an orthonormal basis of $\mathbb{K}^n$.

Solution. ($\implies$) Let the columns of A form an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{K}^n$. Thus,

$$\langle v_i, v_j\rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker Delta. Next,

$$(A^*A)_{ij} = \sum_{k=1}^{n} A^*_{ij}A_{kj} = \sum_{k=1}^{n} A_{ki}A_{kj}$$

where A_{ki} is the kth component of v_i, hence

$$(A^*A)_{ij} = \langle v_i, v_j\rangle = \delta_{ij} \implies A^*A = I$$

($\impliedby$) Fix some $A \in \mathrm{O}_n(\mathbb{K})$ with columns $\{v_1, \dots, v_n\}$. Since A is an element of a group, A^{-1} is defined, and so the columns are linearly independent. Next,

$$\begin{aligned}(I)_{ij} &= (A^*A)_{ij}\\ &= \langle v_i, v_j\rangle\\ &= \delta_{ij}\end{aligned}$$

Hence, the columns are orthonormal.

□

Exercise 3.6 [Complete]

1. Show that for every $A \in \mathrm{O}(2) - \mathrm{SO}(2)$, $R_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a flip about some line through the origin. How is this line determined by the angle of A?

2. Let $B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2)$. Assume θ is not an integer multiple of π. Prove that B does not commute with any $A \in \mathrm{O}(2) - \mathrm{SO}(2)$.

Solution. (1). First, note that given some $X = (x_1, x_2) \in \mathbb{R}^2$,

$$(x_1, x_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (x_1, -x_2)$$

hence, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ corresponds to a flip through the x-axis. Per Equation 3.8, we have

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

where $\theta \in [0, 2\pi)$. Using this, observe that

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where the second factor on the RHS is in SO(2) and corresponds to a counter-clockwise rotation of θ. Hence, A is a rotation flip then rotation.

(2) Let $B \in \mathrm{SO}(2)$. Then

$$R_{AB}((x_1, x_2)) = XAB = (x_{1,\theta}, -x_{2,\theta})B = (x_{1,\theta+\varphi}, -x_{2,\theta+\varphi})$$

is given by the flip of the first factor of A, then the rotations θ and φ of the next two factors. However,

$$R_{BA}((x_1, x_2)) = XBA = (x_{1,\varphi}, x_{2,\varphi})A.$$

In this case, $x_{2,\varphi}$ will first be flipped and then rotated by θ, and so we cannot assert that it is equal to $-x_{2,\varphi+\theta}$, hence the matrices cannot commute.

□

Exercise 3.7 [WIP]

Describe the product of any two elements in $O(2)$ in terms of their angles.

Solution. Let $A, B \in \mathrm{SO}(2)$ and $C, D \in O(2) - \mathrm{SO}(2)$. We need to describe R_{AB}, R_{CD}, and R_{AC}. First,

$$\begin{aligned} R_{AB} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\sin\theta\cos\varphi - \sin\varphi\cos\theta \\ \cos\theta\sin\varphi + \cos\varphi\sin\theta & -\sin\theta\sin\varphi + cos\theta\cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\varphi) & \sin(\theta+\varphi) \\ -\sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \end{aligned}$$

□

Exercise 3.8 [TODO]

Exercise 3.9 [TODO]
Exercise 3.10 [Complete]

Prove that $\mathrm{Trans}(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Isom}(\mathbb{R}^n)$.

Solution. Fix $\begin{pmatrix} A & 0 \\ X & 1 \end{pmatrix} \in \mathrm{Isom}(\mathbb{R}^n)$. Then,

$$\begin{aligned} A \cdot \mathrm{Trans}(\mathbb{R}^n) &= \begin{pmatrix} A & 0 \\ X & 1 \end{pmatrix} \cdot \left\{ \begin{pmatrix} I & 0 \\ Y & 1 \end{pmatrix} : Y \in \mathbb{R}^n \right\} \\ &= \left\{ \begin{pmatrix} A & 0 \\ X + Y & 1 \end{pmatrix} : Y \in \mathbb{R}^n \right\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Trans}(\mathbb{R}^n) \cdot A &= \left\{ \begin{pmatrix} I & 0 \\ Y & 1 \end{pmatrix} : Y \in \mathbb{R}^n \right\} \cdot \begin{pmatrix} A & 0 \\ X & 1 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} A & 0 \\ X + R_A(Y) & 1 \end{pmatrix} : Y \in \mathbb{R}^n \right\} \end{aligned}$$

R_A is one-to-one and onto, so we can always identify $X + Y_1$ with some $X + R_A(Y_2)$ with Y_1, Y_2 unique, hence both cosets are equivalent, and $\mathrm{Trans}(\mathbb{R}^n)$ is normal. □

Exercise 3.11 [Complete]

Prove that the Affine group

$$\mathrm{Aff}_n(\mathbb{K}) = \left\{ \begin{pmatrix} A & 0 \\ V & 1 \end{pmatrix} : A \in \mathrm{GL}_n(\mathbb{K}), V \in \mathbb{K}^n \right\}$$

1. Is a subgroup of $\mathrm{GL}_{n+1}(\mathbb{K})$
2. Prove that $f(X) := R_A(X) + V$ sends translated lines to translated lines

Solution. (1). First, note that

$$\begin{pmatrix} A_1 & 0 \\ V_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & 0 \\ V_2 & 1 \end{pmatrix} = \begin{pmatrix} A_1 \cdot A_2 & 0 \\ R_{A_2}(V_1) + V_2 & 1 \end{pmatrix}$$

so $\mathrm{Aff}_n(\mathbb{K})$ is closed. Next,

$$\begin{pmatrix} A & 0 \\ V & 1 \end{pmatrix} \cdot \begin{pmatrix} A^{-1} & 0 \\ -R_{A^{-1}}(V) & 1 \end{pmatrix} = \begin{pmatrix} AA^{-1} & 0 \\ R_{A^{-1}}(V) - R_{A^{-1}}(V) & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}$$

hence $\mathrm{Aff}_n(\mathbb{K})$ is a subgroup.

(2). This is trivial since $\mathrm{Aff}_n(\mathbb{K})$ is a subgroup. □

Exercise 3.12 [Complete]

Is $\mathrm{Aff}_1(\mathbb{R})$ abelian?

Solution. Think of $\mathrm{Aff}_1(\mathbb{R})$ as maps $f_{\lambda,V}(X) = \lambda X + V$. Then

$$\begin{aligned} f_{\lambda_2,V_2} \circ f_{\lambda_1,V_1}(X) &= f_{\lambda_2,V_2}(\lambda_1 X + V_1) \\ &= \lambda_2 \lambda_1 X + \lambda_2 V_1 + V_2 \end{aligned}$$

$$\begin{aligned} f_{\lambda_1,V_1} \circ f_{\lambda_2,V_2}(X) &= f_{\lambda_1,V_1}(\lambda_2 X + V_2) \\ &= \lambda_1\lambda_2 X + \lambda_1 V_2 + V_1 \end{aligned}$$

which are equal if and only if $\lambda_1 = \lambda_2 = 1$, hence elements in $\mathrm{Aff}_1(\mathbb{R})$ do not commute in general, so $\mathrm{Aff}_1(\mathbb{R})$ is not abelian. $\square$

Exercise 3.13 [Complete]

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1. Calculate $R_A(x, y, z, w)$.
2. Define a subgroup H of $O(4)$ that is isomorphic to S_4.
3. Describe a subgroup H of $O(n)$ that is isomorphic to S_n. What is $H \cap SO(n)$?
4. Prove that every finite group is a subgroup of $O(n)$ for some n.

Solution. (1).

$$R_A(x, y, z, w) = (x, y, z, w) \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = (y, z, w, x).$$

(2). Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$ and $\sigma \in S_4$ a permutation on 4 charachters. Then define a map $f : S_n \to O(4)$ via

$$f(\sigma) := \begin{bmatrix} | & | & | & | \\ e_{\sigma(1)} & e_{\sigma(2)} & e_{\sigma(3)} & e_{\sigma(4)} \\ | & | & | & | \end{bmatrix}.$$

Then,

$$\begin{aligned} f(\sigma_1 \circ \sigma_2) &= \begin{bmatrix} | & | & | & | \\ e_{\sigma_1\circ\sigma_2(1)} & e_{\sigma_1\circ\sigma_2(2)} & e_{\sigma_1\circ\sigma_2(3)} & e_{\sigma_1\circ\sigma_2(4)} \\ | & | & | & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & | & | \\ e_{\sigma_1(1)} & e_{\sigma_1(2)} & e_{\sigma_1(3)} & e_{\sigma_1(4)} \\ | & | & | & | \end{bmatrix} \cdot \begin{bmatrix} | & | & | & | \\ e_{\sigma_2(1)} & e_{\sigma_2(2)} & e_{\sigma_2(3)} & e_{\sigma_2(4)} \\ | & | & | & | \end{bmatrix} \\ &= f(\sigma_1) \circ f(\sigma_2) \end{aligned}$$

Hence f is a group homomorphism. Note that $\ker f$ is trivial, so $H = \mathrm{Image} f \approxeq S_4$ is a subgroup of $O(4)$.

(3). Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and $\sigma \in S_n$ a permutation on n charachters. Then define a map $f : S_n \to O(n)$ via

$$f(\sigma) := \begin{bmatrix} | & | & & | & | \\ e_{\sigma(1)} & e_{\sigma(2)} & \cdots & e_{\sigma(3)} & e_{\sigma(4)} \\ | & | & & | & | \end{bmatrix}.$$

Let $H = \text{image} f \approxeq S_n$, which is a subgroup of $O(n)$ generalizing the logic from step (2). Consider the diagram

$$\begin{tikzcd} S_n \arrow[r, "f"] \arrow[dr, "\text{sgn}"'] & O(n) \arrow[d, "\det"] \\ & \{\pm 1\} \end{tikzcd}$$

f, sgn, and det are all group homomorphisms (where the operation in $\{\pm 1\}$ is multiplication), hence the diagram must commute. Therefore $H \cap SO(n)$ is the permutation matrices in H corresponding to permutations with positive sign in the pre-image of f.

(4). By Cayley's Theorem for finite groups G, $G \leq H \leq O(n)$, and since subgroups of subgroups are subgroups, $G \leq O(n)$.

$\square$

Exercise 3.14 [Complete]

Let $\mathfrak{g}$ be a $\mathbb{K}-$subspace of $\mathbb{K}^n$, with dimension d. Let $\beta = \{X_1, \ldots, X_d\}$ be an orthonormal basis of $\mathfrak{g}$. Let $f : \mathfrak{g} \to \mathfrak{g}$ be $\mathbb{K}-$linear. Let $A \in M_d(\mathbb{K})$ represent f over the basis β. Prove that the following are equivalent:

1. $A \in O_d(\mathbb{K})$
2. $\langle f(X), f(Y) \rangle = \langle X, Y \rangle$ for all $X, Y \in \mathfrak{g}$.

Show by example that this is false when β is not orthogonormal.

Solution. ((1) $\implies$ (2)). Let $A \in O_d(\mathbb{K})$. Then for any $X, Y \in \mathfrak{g}$,

$$\begin{aligned} \langle f(X), f(Y) \rangle &= \langle R_A(X), R_A(Y) \rangle \\ &= \langle X \cdot A, Y \cdot A \rangle \\ &= \langle X, Y \rangle \end{aligned}$$

by the definition of $O_d(\mathbb{K})$.

((2) $\implies$ (1)). Let $\langle f(X), f(Y) \rangle = \langle X, Y \rangle$ for all $X, Y \in \mathfrak{g}$. Then for any $X_i, X_j \in \beta$,

$$\begin{aligned} \langle f(X_i), f(X_j) \rangle &= \langle X_i, X_j \rangle \\ &= \delta_{ij} \end{aligned}$$

Thus, representing f as A must be done in a way so that A is invariant under the inner product. In other words, $A \in O_d(\mathbb{K})$. Hence, (1) $\iff$ (2).

Say that $\beta = \{1, 2\mathbf{i}\}$ is a basis of $\mathbb{C}$. Then let $f(z) = ze^{\mathbf{i}\pi/2}$, hence $A \in U(1) \approxeq O_2(\mathbb{R})$. Then,

$$\begin{aligned}\langle f(1), f(2\mathbf{i})\rangle &= \langle e^{\mathbf{i}\pi/2}, 2\mathbf{i}e^{\mathbf{i}\pi/2}\rangle \\ &= \langle \mathbf{i}, -2\rangle \\ &= -2\mathbf{i} \\ &\neq 2\mathbf{i} \\ &= \langle 1, 2\mathbf{i}\rangle\end{aligned}$$

□

Exercise 3.15 [Complete]

Prove that the symmetries of the tetrahedron form the group S_4 and the proper symmetries form the group A_4.

Solution. Fix a vertex $v \in \{v_1, v_2, v_3, v_4\}$ of the tetrahedron. Consider the symmetries of the face opposite to v, which consist of 3 rotational symmetries and 3 flips. Hence there are $4 \cdot (3 + 3) = 4!$ elements. Since clearly the symmetries act on the set $\{v_1, v_2, v_3, v_4\}$, and there are $4!$ distinct actions, the symmetries of the tetrahedron must be isomorphic to S_4.

Exactly $1/2$ of the elements are the rotational symmetries described above, which are orientation preserving. There are $4!/2$ of these, corresponding directly the subgroup A_4.

□

Exercise 3.16 [Complete]

Think of $\mathrm{Sp}(1)$ as the group of unit length quarternions, that is $\mathrm{Sp}(1) = \{q \in \mathbb{H} : |q| = 1\}$.

1. Show that the conjugation map $C_q : \mathbb{H} \to \mathbb{H}$ given by $C_q(v) := qv\bar{q}$ is an orthogonal linear transformation. Thus, w.r.t the natural basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, C_q can be regaurded as an element of $\mathrm{O}(4)$.

2. For every $q \in \mathrm{Sp}(1)$, show that $C_q(1) = 1$ and therefore that C_q sends $\mathrm{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to itself. Conclude that the restriction $C_q\big|_{\mathrm{Im}(\mathbb{H})}$ can be regaurded as an element of $\mathrm{O}(3)$.

3. Define $\varphi : \mathrm{Sp}(1) \to \mathrm{O}(3)$ given by $\varphi(q) := C_q\big|_{\mathrm{Im}(\mathbb{H})}$. Verify φ is a group homomorphism.

4. Verify that the kernel of φ is $\{1, -1\}$ and therefore φ is two-to-one.

Solution. (1). To show C_q is linear, let $v, w \in \mathbb{H}$, then

$$\begin{aligned}C_q(v + w) &= q(v + w)\bar{q} \\ &= q(v\bar{q} + w\bar{q}) \\ &= qv\bar{q} + qw\bar{q} \\ &= c_q(v) + C_q(w).\end{aligned}$$

Next, fix $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned}C_q(\lambda v) &= q(\lambda v)\bar{q} \\ &= \lambda(qv\bar{q}) \\ &= \lambda C_q(v)\end{aligned}$$

hence C_q is a linear transformation. To show C_q is orthogonal, note that

$$\begin{aligned}
\langle C_q(v), C_q(w)\rangle &= \langle qv\bar{q}, qw\bar{q}\rangle \\
&= \mathrm{Re}(\overline{qv\bar{q}} \cdot qw\bar{q}) \\
&= \mathrm{Re}(\overline{v\bar{q}}\bar{q} \cdot qw\bar{q}) \\
&= \mathrm{Re}(\overline{v\bar{q}}(\bar{q}q)w\bar{q}) \\
&= \mathrm{Re}(\overline{v\bar{q}}w\bar{q}) \\
&= \mathrm{Re}(\overline{v\bar{q}}w\bar{q}) \\
&= \mathrm{Re}(q\bar{v}w\bar{q}) \\
&\overset{*}{=} \mathrm{Re}(\bar{v}w) \\
&= \langle v, w\rangle
\end{aligned}$$

where the $\overset{*}{=}$ step can be verified using

$$\mathrm{Re}(qx\bar{q}) = \frac{1}{2}q(x+\bar{x})\bar{q} = \frac{1}{2}\cdot 2\mathrm{Re}(x)\cdot q\bar{q} = \mathrm{Re}(x)$$

Therefore, C_q can be regaurded as an element of $\mathrm{O}(4)$ by identifying it with a matrix $A \in \mathrm{O}(4)$ so that the following commutes:

$$\begin{tikzcd}
\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} \arrow[r, "C_q"] \arrow[d, "\pi"'] & \mathbb{H} \arrow[d, "\rho"] \\
\{e_1, e_2, e_3, e_4\} \arrow[r, "A"'] & \mathbb{R}^4
\end{tikzcd}$$

where π and ρ are the obvious maps.

(2). Note that

$$C_q(1) = q(1)\bar{q} = q\bar{q} = 1$$

since C_q is linear,

$$C_q(v) = C_q(\mathrm{Re}(v) + \mathrm{Im}(v)) = C_q(\mathrm{Re}(v)) + C_q(\mathrm{Im}(v)) = \mathrm{Re}(v) + C_q(\mathrm{Im}(v))$$

fixing $\mathrm{Re}(v) = 0$, it follows that C_q maps $\mathrm{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to itself. Therefore, $C_q\big|_{\mathrm{Im}(\mathbb{H})}$ can be regaurded as an element of $\mathrm{O}(3)$ by identifying it with a matrix $A \in \mathrm{O}(3)$ so that the following commutes:

$$\begin{tikzcd}
\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \arrow[r, "C_q|_{\mathrm{Im}(\mathbb{H})}"] \arrow[d, "\pi"'] & \mathrm{Im}(\mathbb{H}) \arrow[d, "\rho"] \\
\{e_1, e_2, e_3\} \arrow[r, "A"'] & \mathbb{R}^3
\end{tikzcd}$$

where π and ρ are the obvious maps.

(3). To show φ is a group homomorphism, fix $q, r \in \mathrm{Sp}(1)$ and $v \in \mathrm{Im}(\mathbb{H})$. Then,

$$\begin{aligned}
\varphi(qr)(v) &= C_{qr}\Big|_{\mathrm{Im}(\mathbb{H})}(v) \\
&= (qr)v(\overline{qr}) \\
&= (qr)v(\bar{r}\bar{q}) \\
&= q(rv\bar{r})\bar{q} \\
&= q \cdot C_r\Big|_{\mathrm{Im}(\mathbb{H})}(v) \cdot \bar{q} \\
&= C_q\Big|_{\mathrm{Im}(\mathbb{H})} \circ C_r\Big|_{\mathrm{Im}(\mathbb{H})}(v) \\
&= \varphi(q) \circ \varphi(r)(v)
\end{aligned}$$

(4) The kernel of φ is given by

$$\ker \varphi := \{q \in \mathbb{H} : \varphi(q) = \mathbf{id}\}$$

where **id** is the identity map

$$\varphi(q)(v) = C_q\Big|_{\mathrm{Im}(\mathbb{H})}(v) = qv\bar{q} = v$$

for all v. Hence, if $\varphi(q) \in \ker \varphi$, we must have $qv = vq$, and so by Exercise 1.18, $q \in \mathbb{R}$, and since $q \in \mathrm{Sp}(1)$, $|q| = 1$, hence it must be the case that $\ker \varphi = \{1, -1\}$. Therefore, φ is two-to-one.

□

Exercise 3.17 [Complete]

Think of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ as the group of pairs of unit length quarternions.

1. For every $q = (q_1, q_2) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ the map $F(q) : \mathbb{H} \to \mathbb{H}$ defined by $F(q) := q_1 v \bar{q}_2$ is an orthogonal linear transformation, and can thus be identified with an element of $\mathrm{O}(4)$ w.r.t the natural basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

2. Show that the function $F : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{O}(4)$ is a group homomorphism.

3. Verify $\ker F = \{(1, 1), (-1, -1)\}$ and is therefore two-to-one.

4. How is F related to φ from the previous exercise?

Solution. (1). To show F is a linear transformation, let $v, w \in \mathbb{H}$ and fix $q = (q_1, q_2) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Then,

$$\begin{aligned}
F(q)(v + w) &= q_1(v + w)\bar{q}_2 \\
&= q_1(v\bar{q}_2 + w\bar{q}_2) \\
&= q_1 v\bar{q}_2 + q_1 w\bar{q}_2 \\
&= F(q)(v) + F(q)(w).
\end{aligned}$$

Next, fixing $\lambda \in \mathbb{R}$,

$$\begin{aligned}
F(q)(\lambda v) &= q_1(\lambda v)\bar{q}_2 \\
&= \lambda(q_1 v\bar{q}_2) \\
&= \lambda \cdot F(q)(v)
\end{aligned}$$

hence $F(q)$ is linear. Next,

$$\begin{aligned}
\langle F(q)(v), F(q)(w)\rangle &= \langle q_1 v\bar{q}_2, q_1 w\bar{q}_2\rangle \\
&= \mathrm{Re}(\overline{q_1 v\bar{q}_2} \cdot q_1 w\bar{q}_2) \\
&= \mathrm{Re}(\overline{v\bar{q}_2}\bar{q}_1 \cdot q_1 w\bar{q}_2) \\
&= \mathrm{Re}(\overline{v\bar{q}_2}(\bar{q}_1 q_1) w\bar{q}_2) \\
&= \mathrm{Re}(\overline{v\bar{q}_2} w\bar{q}_2) \\
&= \mathrm{Re}(q_2 \bar{v} w\bar{q}_2) \\
&\overset{*}{=} \mathrm{Re}(\bar{v}w) \\
&= \langle v, w\rangle
\end{aligned}$$

where the $\overset{*}{=}$ is justified in the same manner as Exercise 3.15.1. Hence, $F(q)$ is othogonal and can be identified with an element $A \in \mathrm{O}(4)$ diagramatically via

$$\begin{tikzcd}
\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} \arrow[r, "F(q)"] \arrow[d, "\pi"'] & \mathbb{H} \arrow[d, "\rho"] \\
\{e_1, e_2, e_3, e_4\} \arrow[r, "A"'] & \mathbb{R}^4
\end{tikzcd}$$

where π and ρ are the obvious maps.

(2). To show F is a group homomorphism, fix $q, r \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and $v \in \mathbb{H}$. Then,

$$\begin{aligned}
F(qr)(v) &= F((q_1, q_2) \cdot (r_1, r_2))(v) \\
&= F((q_1 r_1, q_2 r_2))(v) \\
&= (q_1 r_1) v (\overline{q_2 r_2}) \\
&= q_1 (r_1 v \bar{r}_2) \bar{q}_2 \\
&= q_1 \cdot (F(r)(v)) \cdot \bar{q}_2 \\
&= F(q) \circ F(r)(v)
\end{aligned}$$

hence the result.

(3). The kernel of F is given by

$$\ker F := \{q \in \mathbb{H} : F(q) = \mathbf{id}\}$$

where $\mathbf{id}$ is the identity map

$$F(q)(v) = q_1 v \bar{q}_2 = v$$

for all v. Hence, $q_1 v = v q_2$, and using a similar argument to Exercise 3.16.4, the only solutions are $(1, 1)$ and $(-1, -1)$, hence

$$\ker F = \{(1, 1), (-1, -1)\}$$

(4). F is a double cover of $\mathrm{O}(4)$, and φ is a double cover of $\mathrm{O}(3)$. Moreover, we get the following diagram:

$$\begin{tikzcd}
\mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[r, "F"] \arrow[d, "\pi"'] & \mathrm{O}(4) \arrow[d, "?"] \\
\mathrm{Sp}(1) \arrow[r, "\varphi"'] & \mathrm{O}(3)
\end{tikzcd}$$

where ? is an interesting map which maps both connected components of $\mathrm{O}(4)$ to the corresponding connected components of $\mathrm{O}(3)$. □

Exercise 3.18 [WIP]

(Gram-Schmidt). For $m < n$, and $S = \{v_1, \ldots, v_m\} \subset \mathbb{K}^n$ an orthonormal set, show that

1. There exist vectors $v_{m+1}, \ldots, v_n$ that form $\{v_1, \ldots, v_n\}$ into an orthonormal basis of $\mathbb{K}^n$.

Solution. (1). Let $v_i, v_j \in S$, and let x be a vector not in the span of S. Then, let

$$w = x - \sum_{i=1}^{m} \langle x, v_i \rangle v_i$$

which is orthogonal to all the vectors in S, hence $S \cup \{\frac{w}{||w||}\}$ is an orthonormal set. Inducting gives the result.

(2).

□

Chapter 4

The topology of matrix groups

4.1 Exercises

Exercise 4.1 [**TODO**]

Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 5

Lie algebras

5.1 Exercises

Exercise 5.1 [TODO]
Exercise 5.2 [Complete]

Verify $\alpha'(0) = A$ in the proof of Theorem 5.9.

Solution. Note that

$$\gamma(t) := I + tA$$

where $A \in M_n(\mathbb{K})$ has trace 0. Then $\alpha(t)_{ij} = \gamma(t)_{ij}$ for all i, j, except when $i = 0$, in which case

$$\alpha(t)_{0j} = \frac{\gamma(t)}{\det \gamma(t)}$$

Therefore,

$$\begin{aligned}
\alpha'(0)_{0j} &= \frac{d}{dt}_{t=0} \frac{\gamma(t)_{0j}}{\det \gamma(t)} \\
&= \frac{d}{dt}_{t=0} \gamma(t)_{0j} \cdot (\det \gamma(t))^{-1} \\
&= \gamma'(0)_{0j} \cdot (\det \gamma(0))^{-1} - \gamma(0)_{0j} \cdot (\det \gamma(0))^{-1} \cdot (\det \gamma'(0)) \cdot (\det \gamma(0))^{-1} \\
&= \gamma'(0)_{0j} - \gamma(0)_{ij} \cdot \mathrm{trace}(\gamma'(0)) \\
&= \gamma'(0)_{0j}
\end{aligned}$$

hence, $\alpha'(0) = A$, since all other entries are obviously equivalent. □

Exercise 5.3 [TODO]
Exercise 5.4 [TODO]
Exercise 5.5 [Complete]

Describe the Lie algebra of the Affine group.

Solution. Consider the path

$$\gamma(t) = \begin{pmatrix} A(t) & 0 \\ V(t) & 1 \end{pmatrix}$$

then,

$$\frac{d}{dt}_{t=0} \gamma(t) = \begin{pmatrix} A'(0) & 0 \\ V'(0) & 0 \end{pmatrix} = \begin{pmatrix} T & 0 \\ v & 0 \end{pmatrix}$$

with $T \in gl_n(\mathbb{K})$ and $v \in \mathbb{K}^n$. So, the Lie algebra of the Affine group is isomorphic to $gl_n(\mathbb{K}) \oplus \mathbb{R}^n$ as a vector space. $\square$

Exercise 5.6 [Complete]

Describe the Lie algebra of $\text{Isom}(\mathbb{R}^n)$.

Solution. Using the same technique as Exercise 5.6, the Lie algebra is isomorphic to $o_n(\mathbb{R}) \oplus \mathbb{R}^n$. $\square$

Exercise 5.7 [Complete]

Describe the Lie algebra of $UT_n(\mathbb{K})$.

Solution. Let $\gamma(t)$ be given as follows. First, γ is given by the matrix

$$(\gamma(t))_{ij} = a_{ij}(t).$$

If $i < j$, then $a_{ij}(t) = 0$. If $i > j$, then $a_{ij}(0) = 0$. If $i = j$, then $a_{ij}(0) = 1$. Hence, the Lie algebra is again $UT_n(\mathbb{K})$. $\square$

Exercise 5.8 [TODO]

Describe the Lie algebra

Exercise 5.9 [TODO]

Exercise 5.10 [TODO]

Exercise 5.11 [TODO]

Exercise 5.12 [TODO]

Exercise 5.13 [TODO]

Exercise 5.14 [TODO]

Describe the Lie algebra of $SL_n(\mathbb{H})$.

Exercise 5.15 [TODO]

Exercise 5.16 [TODO]

Chapter 6

Matrix exponentiation

6.1 Exercises

Exercise 6.1 [WIP]

Prove Proposition 6.5, that is, suppose that $\sum A_l$ and $\sum B_l$ converge, at least one absolutely. Let $C_l = \sum_{k=0}^{l} A_k B_{l-k}$. Prove that

$$\sum C_l = (\sum A_l)(\sum B_l)$$

Solution.

$$\begin{aligned}(\sum C_l)_{ij} &= (\sum_{k=0}^{l} A_k B_{l-k})_{ij} \\ &= (\sum_{k=0}^{l} \sum_{r=1} (A_k))\end{aligned}$$

□

Exercise 6.2 [Complete]

Prove that $(e^A)^* = e^{A^*}$ for all $A \in M_n(\mathbb{K})$.

Solution. Using the fact that $(XY)^* = Y^*X^*$ and $(X+Y)^* = X^* + Y^*$,

$$\begin{aligned}e^{A^*} &= \sum_{k=0}^{\infty} \frac{(A^*)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\overbrace{A^* \cdot A^* \cdots A^*}^{k-times}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(\overbrace{A \cdot A \cdots A}^{k-times})^*}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(A^k)^*}{k!} \\ &= \left(\sum_{k=0}^{\infty} \frac{(A^k)}{k!}\right)^* \\ &= (e^A)^*\end{aligned}$$

□

Exercise 6.3 [Complete]

1. Let $A = \operatorname{diag}(a_1, \ldots, a_n) \in M_n(\mathbb{R})$. Calculate e^A and give a simple prove that $\det(e^A) = e^{\operatorname{trace}(A)}$ when A is diagonal.

2. Give a simple proof that $\det(e^A) = e^{\operatorname{trace}(A)}$ when A is conjugate to a diagonal matrix.

Solution. (1). Let $A = \operatorname{diag}(a_1, \ldots, a_n) \in M_n(\mathbb{R})$. Then,

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{\operatorname{diag}(a_1, \ldots, a_n)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\operatorname{diag}(a_1^k, \ldots, a_n^k)}{k!} \\ &= \operatorname{diag}\left(\sum_{k=0}^{\infty} \frac{a_1^k}{k!}, \ldots, \sum_{k=0}^{\infty} \frac{a_n^k}{k!}\right) \\ &= \operatorname{diag}(e^{a_1}, \ldots, e^{a_n}). \end{aligned}$$

Thus,

$$\begin{aligned} \det(e^A) &= \det\left(\operatorname{diag}(e^{a_1}, \ldots, e^{a_n})\right) \\ &= \prod_{i=1}^{n} e^{a_i} \\ &= e^{\sum_{j=1}^{n} a_j} \\ &= e^{\operatorname{trace}(A)} \end{aligned}$$

(2). Let $A \in M_n(\mathbb{R})$ be conjugate to a diagonal matrix, that is there exists a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ and an invertible matrix P such that $A = PDP^{-1}$. Then,

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\overbrace{(PDP^{-1}PDP^{-1} \cdots PDP^{-1})}^{k-times}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\overbrace{PD(P^{-1}P)D(P^{-1}P) \cdots (P^{-1}P)DP^{-1})}^{k-times}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} \\ &= P\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right) P^{-1} \\ &= Pe^DP^{-1}. \end{aligned}$$

Finishing it off:

$$\begin{aligned}
\det(e^A) &= \det(Pe^D P^{-1}) \\
&= \det(P) \cdot \det(e^D) \cdot \det(P^{-1}) \\
&= e^{\operatorname{trace}(D)} \\
&= e^{\operatorname{trace}(A)}
\end{aligned}$$

where the last equality holds since similar matrices have the same trace. □

Exercise 6.4 [WIP]

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Compute e^{tA} for arbitrary $t \in \mathbb{R}$.

Solution. First, consider the following table:

k	A^k
0	I
1	A
2	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
3	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
4	I

Hence, A has periodicity 4. Therefore,

$$\begin{aligned}
e^{tA} &= \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \\
&= \sum_{i=0}^{\infty} \frac{t^{4i}}{i!} I + \sum_{j=0}^{\infty} \frac{t^{4j+1}}{j!} A^{4j+1} + \sum_{l=0}^{\infty} \frac{t^{4l+2}}{l!} A^{4l+1} + \sum_{r=0}^{\infty} \frac{t^{4r+3}}{r!} A^{4r+3} \\
&= \sum_{i=0}^{\infty} \frac{t^{4i}}{i!} I + \sum_{j=0}^{\infty} \frac{t^{4j+1}}{j!} A + \sum_{l=0}^{\infty} \frac{t^{4l+2}}{l!} A^2 + \sum_{r=0}^{\infty} \frac{t^{4r+3}}{r!} A^3 \\
&= \sum_{i=0}^{\infty} \frac{(t^4)^i}{i!} I + t\sum_{j=0}^{\infty} \frac{(t^4)^j}{j!} A + t^2 \sum_{l=0}^{\infty} \frac{(t^4)^l}{l!} A^2 + t^3 \sum_{r=0}^{\infty} \frac{(t^4)^r}{r!} A^3 \\
&= e^{t^4} I + te^{t^4} A + t^2 e^{t^4} A^2 + t^3 e^{t^4} A^3 \\
&= e^{t^4} \left(I + tA + t^2 A^2 + t^3 A^3 \right) \\
&= e^{t^4} \begin{pmatrix} 1 - t^2 & t - t^3 \\ -t + t^3 & 1 - t^2 \end{pmatrix}
\end{aligned}$$

Noticing that as $t \to 0$ we have $e^{tA} \to I$ is a good sanity check! □

Exercise 6.5 [Complete]

Can a one-parameter group ever cross itself?

Solution. No - a one-parameter group is differentiable, hence any singularity would contradict its differentiability. Alternatively, by Proposition 6.17, every one-parameter group is described by $\gamma(t) = e^{tA}$ for some $A \in gl_n(\mathbb{K})$, and e^{tA} is injective. □

Exercise 6.6 [WIP]

Describe all one-parameter groups of $\mathrm{GL}_1(\mathbb{C})$. Draw several in the x-y plane.

Solution. Let $z = a + b\mathbf{i} \in \mathrm{GL}_1(\mathbb{C})$. Then, any one-parameter group has the form

$$\begin{aligned}\gamma_z(t) &= e^{tz} \\ &= e^{t(a+b\mathbf{i})} \\ &= e^{at}e^{\mathbf{i}bt}\end{aligned}$$

$e^{\mathbf{i}bt}$ can always be identified with a point on the unit circle, scaled by e^{at}. Hence, $\gamma_z(t)$ makes a spiral that "spirals" exponentially faster as t increases that starts at $\gamma_z(0) = 1$. z determines the initial condition and initial "spiral rate".

□

Exercise 6.7 [WIP]

Let $G = \left\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) : x > 0\right\}$. Describe the one-parameter groups in G, and draw several on the $xy-$plane.

Solution. First, note that given $A \in G$,

$$A = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} x^2 & y(x+1) \\ 0 & 1 \end{pmatrix}$$

and

$$A^3 = \begin{pmatrix} x^3 & y(x^2+x+1) \\ 0 & 1 \end{pmatrix}$$

It can be show inductively that

$$A^k = \begin{pmatrix} x^k & y(x^{k-1}+\cdots+1) \\ 0 & 1 \end{pmatrix}$$

and so,

$$\begin{aligned}e^{tA} &= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} x^k & y(x^{k-1}+\cdots+1) \\ 0 & 1 \end{pmatrix} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} x^k & y\frac{x^k-1}{x-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{tx} & \frac{y}{x-1}\sum_{k=0}^{\infty}(tx)^k/k! - t^k/k! \\ 0 & e^t \end{pmatrix} \\ &= \begin{pmatrix} e^{tx} & \frac{y}{x-1}(e^{tx}-e^t) \\ 0 & e^t \end{pmatrix}\end{aligned}$$

No fucking way am I drawing these. ∎

Exercise 6.8 [WIP]

Visually describe the path $\gamma(t) = e^{t\mathbf{j}}$ in $Sp(1) \approx S^3$.

Solution. First, note that

k	$\mathbf{j}^k$
0	1
1	$\mathbf{j}$
2	-1
3	1

hence, $\mathbf{j}$ has periodicity 3. Next,

$$\begin{aligned}\gamma(t) &= e^{t\mathbf{j}} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{j}^k \\ &= \sum_{i=0}^{\infty} \frac{t^i}{i!}1 + \sum_{j=0}^{\infty} \frac{t^j}{j!} + \sum_{l=0}^{\infty} \frac{t^l}{l!}\end{aligned}$$

∎

Exercise 6.9 [WIP]

Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in gl_n\mathbb{R}$. Compute e^A.

Solution. First, note that

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

is a decomposition of A into two commuting matrices. Thus,

$$\begin{aligned}e^A &= \exp\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ &= \exp\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right) \\ &= \exp\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \\ &= \left(\sum_{k=0}^{\infty} \frac{a^k I^k}{k!}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{b^k}{j!} \exp\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^j\right) \\ &= e^a I \cdot e^{b^4} \begin{pmatrix} 1 - b^2 & b - b^3 \\ -b + b^3 & 1 - b^2 \end{pmatrix} \\ &= e^{b^4 + a} \begin{pmatrix} 1 - b^2 & b - b^3 \\ -b + b^3 & 1 - b^2 \end{pmatrix}\end{aligned}$$

∎

Exercise 6.10 [TODO]

Repeat the previous problem with $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

Exercise 6.11 [Complete]

When A is in the Lie algebra of $UT_n(\mathbb{K})$, prove that $e^A \in UT_n(\mathbb{K})$.

Solution. Let $A_1, A_2 \in UT_n(\mathbb{K})$ and k a positive integer. Then clearly $A_1 + A_2$ and $R_{A_1}(A_2)$ are in $UT_n(\mathbb{K})$. Thus, for any positive integer $k \in \mathbb{Z}$, $A^k \in UT_n(\mathbb{K})$. Further, $\frac{1}{k!}A_1$ is certainly in $UT_n(\mathbb{K})$. Hence, we can assert that

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \in UT_n(\mathbb{K}).$$

□

Exercise 6.12 [Complete]

When A is in the Lie algebra of $\mathrm{Isom}(\mathbb{R}^n)$, prove that $e^A \in \mathrm{Isom}(\mathbb{R}^n)$.

Solution. Let $A_1, A_2 \in \mathrm{Isom}(\mathbb{R}^n)$ and k a positive integer. Then $A_1 + A_2$ and $R_{A_1}(A_2)$ are isometries. Thus, for any positive integer $k \in \mathbb{Z}$, A^k is an isometry. Further, $\frac{1}{k!}A_1$ is an isometry. Hence, we can assert that

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \in \mathrm{Isom}(\mathbb{R}^n).$$

□

Exercise 6.13 [Complete]

Describe the one-parameter groups of $\mathrm{Trans}(\mathbb{R}^n)$.

Solution. Fix $A = \begin{pmatrix} I & 0 \\ X & 1 \end{pmatrix} \in \mathrm{Trans}(\mathbb{R}^n)$. Then it is clear that

$$A^k = \begin{pmatrix} I & 0 \\ X & 1 \end{pmatrix}^k = \begin{pmatrix} I & 0 \\ kX & 1 \end{pmatrix}$$

hence,

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} I & 0 \\ kX & 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} I & 0 \\ tX & 1 \end{pmatrix}. \end{aligned}$$

□

Exercise 6.14 [TODO]

Chapter 7

Matrix groups are manifolds

7.1 Exercises

Exercise 7.1 [Complete]

Define the **stereographic projection** as

$$f : S^2 - \{0,0,1\} \to \mathbb{R}^2$$

via shooting a lazer from the north pole of S^2; the lazer hits a point $p = (x, y, z)$ on S^2 and a subsequent point on R^2 (given by the plane $z = -1$), identifying $p \in S^2 - \{0,0,1\}$ with $f(p) \in \mathbb{R}^2$.

1. Show that
$$f(x, y, z) = \frac{2}{1-z}(x, y)$$

2. Find a formula for f^{-1}

3. Find a formula
$$g : S^2 - \{0,0,-1\} \to \mathbb{R}^2$$
defined analgously to f, but the plane of intersection is $z = 1$.

4. Find an explicit formula for the composition
$$g \circ f^{-1} : \mathbb{R}^2 - \{0,0\} \to \mathbb{R}^2 - \{(0,0)\}$$

Solution. (1). Fix the following lines:

1. ℓ_1 between $(0,0,-1)$ and $(0,0,1)$
2. ℓ_2 between $(0,0,1)$ and $f(p) = f(x,y,z)$
3. ℓ_3 between $(0,0,-1)$ and $f(p) = f(x,y,z)$
4. ℓ_4 between $(0,0,1)$ and $(0,0,1-z)$

5. ℓ_5 between $(0,0,1)$ and $p=(x,y,z)$
6. ℓ_6 between $(0,0,1-z)$ and $p=(x,y,z)$

Its clear that $T_1=\{\ell_1,\ell_2,\ell_3\}$ and $T_2=\{\ell_4,\ell_5,\ell_6\}$ are similar. Morever, $\ell_1=2$ and $\ell_4=1-z$. Therefore,

$$\frac{1-z}{2}=\frac{(x,y)}{f(x,y,z)} \implies f(x,y,z)=\frac{2}{1-z}(x,y).$$

(2). Let $f(x,y,z)=(u,v)\in\mathbb{R}^2$. Then, parameterize the line from $(0,0,1)$ to (u,v) with

$$L(t)=(0,0,1)+t(u,v,-2)=(tu,tv,1-2t)$$

where $t\in[0,1]$. We want to know when $L(t)$ intersects the sphere, as in when

$$\begin{aligned}(tu)^2+(tv)^2+(1-2t)^2=1 &\implies (u^2+v^2)t^2-4t+4t^2=0\\ &\implies (u^2+v^2+4)t^2-4t=0\\ &\implies t\left[(u^2+v^2+4)t-4\right]=0\end{aligned}$$

where $t=\frac{4}{u^2+v^2+4}$ is clearly the relevant solution. Therefore,

$$f^{-1}(u,v)=(tu,tv,1-2t)=\left(\frac{4u}{u^2+v^2+4},\frac{4v}{u^2+v^2+4},1-\frac{8}{u^2+v^2+4}\right)$$

(3). We can recover step (1) by mapping $1-z\to z-1$, hence we get

$$g(x,y,z)=\frac{2}{z-1}(x,y)$$

(4). Let $c=\frac{4}{u^2+v^2+4}$. Composition gives

$$\begin{aligned}g\circ f^{-1}(u,v)&=g\left(cu,cv,1-2c\right)\\ &=\frac{2}{1-1-2c}(cu,cv)\\ &=-(u,v)\end{aligned}$$

which is super chill!

□

Exercise 7.2 [WIP]

Prove that $S^n\subset\mathbb{R}^{n+1}$ is an $n-$dimensional manifold.

Solution. Define

$$V=\{(x_1,\ldots,x_{n+1})\in S^n : x_{n+1}>0\}$$

which is a neigbourhood of $(0,0,\ldots,1)\in S^n$. Next, define

$$U=\{(a_1,\ldots,a_n) : a_1^2+\cdots+a_n^2<1\}$$

and define

$$\varphi : U\to V$$
$$\varphi(a_1,\ldots,a_n) := (a_1,a_2,\ldots,a_n,\sqrt{1-a_n^2-\cdots-a_1^2})$$

□

Chapter 8

The Lie bracket

8.1 Exercises

Exercise 8.1 [Complete]

Prove that SO(3) is not abelian in two ways:

1. Find two elements in $\mathfrak{so}(3)$ that do not commute.

2. Find two elements in SO(3) that do not commute.

Which is easier? Prove that SO(n) is not abelian for $n > 2$.

Solution. (1). Consider the basis given for $\mathfrak{so}(3)$ in Theorem 5.12, that is

$$\mathfrak{so}(3) = \operatorname{span}\left\{ E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

then we have

$$[E_1, E_2] = E_3$$

hence E_1 and E_2 do not commute.

(2). Let

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

then

$$R_z(\theta)R_x(\phi) = \begin{bmatrix} \cos\theta & -\sin\theta\cos\phi & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

but

$$R_x(\phi)R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix}$$

hence SO(()3) is not abelian.

to prove $SO(n)$ is not abelian for $n > 2$, consider the basis $\{E_1, \ldots E_n\}$ as described in Theorem 5.2. Then if $n > 2$, we have that

$$[E_i, E_j] = E_k$$

for some $i \neq j \neq k$, hence $\mathfrak{so}(n)$ is not abelian, and so $SO(3)$ is not abelian.

□

Exercise 8.2 [Complete]

Let G_1, G_2 be matrix groups with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$. Suppose that $f : G_1 \to G_2$ is a smooth homomorphism. If $df_I : \mathfrak{g}_1 \to \mathfrak{g}_2$ is bijective, prove that $df_g : T_gG_1 \to T_{f(g)}G_2$ is bijective for all $g \in G_1$.

Solution. Given a group G with $g \in G$, let $L_g : G \to G$ be the left-multiplication map $L_g(G) := gG$. Then L_g is a group automomorphism. It is straightforward to show that

$$df_g = d\left(L_{f(g)}\right) \circ df_I \circ d\left(L_g^{-1}\right).$$

where the composition

$$T_gG_1 \xrightarrow{d(L_g^{-1})} \mathfrak{g}_1 \xrightarrow{df_I} \mathfrak{g}_1 \xrightarrow{d(L_{f(g)})} T_{f(g)}G_2$$

is a composition of linear maps, and hence linear, and hence bijective.

□

Exercise 8.3 [WIP]

Define $d : \mathrm{Sp}(1) \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ as $d(a) := (a, a)$. Explicitly describe the function $\iota : \mathrm{SO}(3) \to \mathrm{SO}(4)$ that makes the following diagram commute:

$$\begin{tikzcd} \mathrm{Sp}(1) \arrow[r, "d"] \arrow[d, "\mathrm{Ad}"'] & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[d, "F"] \\ \mathrm{SO}(3) \arrow[r, "\iota"'] & \mathrm{SO}(4) \end{tikzcd}$$

Solution. Fix $(a, a) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. If $v \in \mathbb{H}$ is purely imaginary, then F is exactly the action Ad_a. If v is real, then $F(a, a)(v) = v$, hence the real axis is fixed pointwise when viewing F as acting on $\mathbb{R}^4$. Therefore, given $M \in \mathrm{SO}(3)$ where M coincides with the action of Ad_a, ι is the inclusion

$$\iota(M) = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$$

□

Exercise 8.4 [**TODO**]

Another exercise goes here.

Exercise 8.5 [**TODO**]

Solution. □

Exercise 8.6 [**TODO**]

Another exercise goes here.

Exercise 8.7 [**TODO**]

Another exercise goes here.

Chapter 9

Maximal tori

9.1 Exercises

Exercise 9.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 10

Homogeneous manifolds

10.1 Exercises

Exercise 10.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 11

Roots

11.1 Exercises

Exercise 11.1. Another exercise goes here.

Solution. Placeholder for your solution. □