

Exercises from *Matrix Groups for Undergraduates*  
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# Chapter 1

## Matrices

### Exercise 1.1 [Complete]

Describe a natural 1-to-1 correspondence between elements of  $\text{SO}(3)$  and elements of

$$T^1S^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |p| = |v| = 1 \text{ and } p \perp v\}$$

**Solution.** Using the globe analogy from Question 1.2, fix a point  $r$  to be the north pole, and a point  $e$  that lies on the equator induced by the choice of  $r$ , and assert this as the arbitrary ‘identity’.

Next, given some  $A \in \text{SO}(3)$ , identify an element in  $T^1S^2$  via  $A \mapsto (Ar, Av)$ , as in first where  $A$  maps the north pole  $r$ , and then how  $A$  rotates the globe about the axis induced by  $r$  and its antipodal point.  $\square$

### Exercise 1.2 [Complete]

Prove equation 1.3:

$$(A \cdot B)^T = B^T \cdot A^T$$

**Solution.** First,

$$\begin{aligned} (A \cdot B)_{ij}^T &= (A \cdot B)_{ji} \\ &= \sum_{k=1}^n A_{jk} B_{ki} \end{aligned}$$

Next,

$$\begin{aligned} (B^T \cdot A^T)_{ij} &= \sum_{k=1}^n (B^T)_{ik} (A^T)_{kj} \\ &= \sum_{k=1}^n B_{ki} A_{jk} \\ &\stackrel{*}{=} \sum_{k=1}^n A_{jk} B_{ki} \end{aligned}$$

$\square$

Note the  $\stackrel{*}{=}$  step uses the commutativity of multiplication, hence the above proof does not work when  $\mathbb{K} = \mathbb{H}$ .

### Exercise 1.3 [Complete]

Prove equation 1.4:

$$\text{trace}(A \cdot B) = \text{trace}(B \cdot A)$$

**Solution.** First,

$$\text{trace}(A \cdot B)_{ii} = \sum_{i=1}^n \sum_{k=1}^n A_{ik} B_{ki}$$

Next,

$$\begin{aligned} \text{trace}(B \cdot A)_{ii} &= \sum_{i=1}^n (B \cdot A)_{ii} \\ &= \sum_{i=1}^n \sum_{k=1}^n B_{ik} A_{ki} \end{aligned}$$

Carefully reindexing and resumming gives the result.  $\square$

Note that the ‘careful reindexing and resumming process’ implies  $\text{trace}()$  is invariant under cyclic permutation, e.x.:

$$\text{trace}(A \cdot B \cdot C) = \text{trace}(C \cdot A \cdot B)$$

**Exercise 1.4 [Complete]**

Let  $A, B \in M_n \mathbb{K}$ . Prove that if  $A \cdot B = I$  then  $B \cdot A = I$ .

**Solution.** Note that

$$A \cdot B = I \iff A = B^{-1}.$$

The result follows.  $\square$

**Exercise 1.5 [Complete]**

Suppose that the determinant of  $A \in M_n(\mathbb{H})$  were defined as in Equation 1.5. Show for

$$A = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} \in M_2(\mathbb{H})$$

that  $\det(A) \neq 0$  but

$$R_A : H^2 \rightarrow H^2$$

is not invertible.

**Solution.** Given the definition from Equation 1.5:

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} \\ &= \mathbf{i}\mathbf{j} - \mathbf{j}\mathbf{i} \\ &= (1) - (-1) \\ &= 2 \neq 0 \end{aligned}$$

However,

$$R_A((- \mathbf{i}, \mathbf{i})) = (- \mathbf{i}, \mathbf{i}) \cdot \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} = (- \mathbf{i}^2 + \mathbf{i}^2, - \mathbf{i}\mathbf{j} + \mathbf{i}\mathbf{j}) = (1 - 1, - \mathbf{k} + \mathbf{k}) = (0, 0)$$



Hence,  $R_A$  has a non-zero determinant, but is not invertible as the kernel is non-trivial. Similarly, clearly the columns of  $A$  are linearly dependent.  $\square$

**Exercise 1.6 [Complete]**

Find  $B \in M_2(\mathbb{R})$  such that  $R_B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a counter-clockwise rotation through an angle  $\theta$ .

**Solution.** Note that we can ‘represent’ both 1 and  $i$  in  $M_2(\mathbb{R})$  via

$$1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \text{ and } i \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where the latter works since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

capturing the fact that  $i^2 = -1$ . Building on this, we can represent any  $a + bi \in \mathbb{C}$  via

$$\rho : a + bi \mapsto a \cdot I + b \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Next, note that the function  $f_\theta(z) = ze^{i\theta}$  rotates elements counter-clockwise in  $\mathbb{C}$  by an angle  $\theta$ . To see this, letting  $z = re^{i\varphi}$ ,

$$f_\theta(z) = ze^{i\theta} = re^{i\varphi}e^{i\theta} = re^{i(\varphi+\theta)}.$$

Applying  $\rho$  gives

$$\begin{aligned} \rho_1(e^{i\theta}) &= \rho_1(\cos(\theta) + i\sin(\theta)) \\ &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = B \end{aligned}$$

$\square$

**Exercise 1.7 [WIP]**

Describe all elements  $A \in \text{GL}_n(\mathbb{R})$  with the property  $AB = BA$  for all  $B \in \text{GL}_n(\mathbb{R})$ .

**Solution.** Matrices  $A$  that commute with all matrices in  $\text{GL}_n(\mathbb{R})$  are scalar multiples of the identity

$$A = \lambda I$$

where  $\lambda \in \mathbb{R}$  and  $\lambda \neq 0$ .  $\square$

**Exercise 1.8 [Complete]**

Let  $\text{SL}_2(\mathbb{Z})$  denote 2 by 2 matrices with integer entries and determinant 1. Prove that  $\text{SL}_2(\mathbb{Z})$  is a subgroup of  $\text{GL}_n(\mathbb{Z})$ . Is  $\text{SL}_n(\mathbb{Z})$  a subgroup of  $\text{GL}_n(\mathbb{R})$  in general?

**Solution.** Fixing  $A, B \in \text{SL}_2(\mathbb{Z})$ , it's clear that  $A \cdot B$  must have all integer entries. Since

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

we have that  $A \cdot B \in \text{SL}_2(\mathbb{Z})$  (closure). Next, fix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

Using Cramer's Rule to compute the inverse of  $A$  we get

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where  $ad - bc = 1$  since  $\det A = 1$ , so  $A^{-1}$  has all integer entries, and is a member of  $\mathrm{SL}_2(\mathbb{Z})$ . Therefore,  $\mathrm{SL}_2(\mathbb{Z})$  is a subgroup of  $\mathrm{GL}_2(\mathbb{R})$ .  $\square$

The critical step in the above proof is discerning that the factor extracted from  $A^{-1}$  is  $1/\det A = 1$ , which ensures the entries of the inverse are all in  $\mathbb{Z}$ . This factor is the same for any  $n$ , so  $\mathrm{SL}_n(\mathbb{Z})$  is always a subgroup of  $\mathrm{GL}_n(\mathbb{R})$  for all  $n$ .

**Exercise 1.9 [Complete]**

Describe the block matrix blah blabh blabhj TODO write this out

**Solution.** Suppose  $A$  and  $B$  are block matrices in  $M_n(\mathbb{K})$ , given by

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

where  $\sum \dim A_i = \dim A = \sum \dim B_i = \dim B$ , and  $\dim A_i = \dim B_i$  for each  $i$ . Then,

$$A \cdot B = \begin{pmatrix} A_1 \cdot B_1 & 0 & \cdots & 0 \\ 0 & A_2 \cdot B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \cdot B_n \end{pmatrix}$$

Which can be applied to the above question to derive a simple answer.  $\square$

**Exercise 1.10 [Complete]**

If  $G_1 \subset \mathrm{GL}_{n_1}(\mathbb{K})$  and  $G_2 \subset \mathrm{GL}_{n_2}(\mathbb{K})$  are subgroups, describe a subgroup of  $\mathrm{GL}_{n_1+n_2}(\mathbb{K})$  isomorphic to  $G_1 \times G_2$ .

**Solution.** Define a map

$$\varphi : G_1 \times G_2 \rightarrow \mathrm{GL}_{n_1+n_2}(\mathbb{K})$$

given by

$$(A_1, A_2) \mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

The image of  $\varphi$  is a subset of  $\mathrm{GL}_{n_1+n_2}(\mathbb{K})$  since

$$\det \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \det A_1 \cdot \det A_2 \neq 0$$

so  $\varphi(A_1, A_2) \in \text{GL}_{n_1+n_2}(\mathbb{K})$ . To prove  $\varphi$  is a group homomorphism, observe

$$\begin{aligned}\varphi((A_1, A_2) \cdot (B_1, B_2)) &= \varphi(A_1 \cdot B_1, A_2 \cdot B_2) \\ &= \begin{pmatrix} A_1 \cdot B_1 & 0 \\ 0 & A_2 \cdot B_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \cdot \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \\ &= \varphi(A_1, A_2) \cdot \varphi(B_1, B_2)\end{aligned}$$

hence the result. □

**Exercise 1.11 [TODO]**

**Exercise 1.12 [Complete]**

Show that for purely imaginary  $q_1, q_2 \in \mathbb{H}$ ,  $-\Re(q_1 \cdot q_2)$  is the vector dot product in  $\mathbb{R}^3 = \text{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$  and  $\Im(q_1 \cdot q_2)$  is the vector cross-product.

**Solution.** First,

$$\begin{aligned}-\Re(q_1 \cdot q_2) &= -\Re((b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \cdot (b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})) \\ &= -\Re((-b_1b_2 - c_1c_2 - d_1d_2) + \dots) \\ &= b_1b_2 + c_1c_2 + d_1d_2\end{aligned}$$

Next,

$$\begin{aligned}\Im(q_1 \cdot q_2) &= \Im((b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \cdot (b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})) \\ &= (c_1d_2 - d_1c_2)\mathbf{i} + (d_1b_2 - b_1d_2)\mathbf{j} + (b_1c_2 - c_1b_2)\mathbf{k}\end{aligned}$$

Mapping  $\text{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$  to the standard basis in  $\mathbb{R}^3$  gives both desired results. □

**Exercise 1.13 [WIP]**

Prove that non-real elements  $q_1, q_2 \in \mathbb{H}$  commute if and only if their imaginary parts are parallel; that is,  $\Im(q_1) = \lambda \cdot \Im(q_2)$  for some  $\lambda \in \mathbb{R}$ .

**Solution.** ( $\implies$ ) Let  $\Im(q_1) = \lambda \cdot \Im(q_2)$  for some  $\lambda \in \mathbb{R}$ , so that

$$\Im(q_1) = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \lambda b\mathbf{i} + \lambda c\mathbf{j} + \lambda d\mathbf{k} = \Im(q_2)$$

then,

$$\begin{aligned}q_1 \cdot q_2 &= (a_1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \cdot (a_2 + \lambda b\mathbf{i} + \lambda c\mathbf{j} + \lambda d\mathbf{k}) \\ &= (a_1a_2 - \lambda(b^2 + c^2 + d^2))b(a_1\lambda + a_2)\mathbf{i}c(a_1\lambda + a_2)\mathbf{j}d(a_1\lambda + a_2)\mathbf{k} \\ &= (a_2 + \lambda b\mathbf{i} + \lambda c\mathbf{j} + \lambda d\mathbf{k}) \cdot (a_1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \\ &= q_2 \cdot q_1.\end{aligned}$$

( $\impliedby$ ) Let  $q_1 \cdot q_2 = q_2 \cdot q_1$  where

$$q_1 = (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}), q_2 = (a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}).$$

Then the following equalities must hold: □

**Exercise 1.14 [TODO]**

Characterize the pairs  $q_1, q_2 \in \mathbb{H}$  which anti-commute, that is  $q_1 q_2 = -q_2 q_1$ .

**Exercise 1.15 [Complete]**

If  $q \in \mathbb{H}$  satisfies  $qi = iq$ , prove that  $q \in \mathbb{C}$ .

**Solution.** Let  $q = a + bi + cj + dk$ . Then,

$$qi = ai + bii + cji + dki = -b + ai + dj - ck$$

and

$$iq = ia + bii + cij + dik = -b + ai - dj + ck.$$

Identifying terms gives

$$d = -d \implies d = 0$$

$$c = -c \implies c = 0$$

hence  $q = a + bi \in \mathbb{C}$ . □

**Exercise 1.16 [Complete]**

Prove that complex multiplication in  $\mathbb{C} \cong \mathbb{R}^2$  does not extend to a multiplication operation on  $\mathbb{R}^3$  that makes  $\mathbb{R}^3$  into a real division algebra.

**Solution.** Assume such an extension exists. Consider the map analagous to the extension of  $\mathbb{R}^2$  given by

$$(a, b, c) \mapsto a + bi + cj$$

with  $i^2 = j^2 = -1$ . Then there must exist a linear map

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

with  $T^2 = -I$ . Represent  $T$  by a  $3 \times 3$  real matrix  $M$ . Then  $-1$  is in the spectrum of  $M^2$  since  $M^2 = -I$ , thus  $\pm i$  is in the spectrum of  $M$ . Since  $\det(M) = \prod \lambda_k$  where  $\lambda_k$  is an eigenvalue of  $M$ , there must exist some real value  $\lambda$  such that

$$\det(M) = (i)(-i)(\lambda) = \lambda$$

where  $\lambda$  must be in  $\mathbb{R}$  since complex eigenvalues come in pairs. Thus, we must have

$$\det(M^2) = \det(-I) = -1$$

and

$$\det(M^2) = \det(M)^2 = \lambda^2$$

thus  $\lambda^2 = -1$ , which contradicts  $\lambda$  being real. □

**Exercise 1.17 [Complete]**

Describe a subgroup of  $\text{GL}_{n+1}(\mathbb{R})$  that is isomorphic to  $\mathbb{R}^n$  under vector-addition.

**Solution.** Consider the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \text{GL}_{n+1}(\mathbb{R})$$

and the map that takes such matrices to  $(x_1, \dots, x_n) \in \mathbb{R}^n$ .

□

**Exercise 1.18 [Complete]**

If  $\lambda \in \mathbb{H}$  commutes with every element of  $\mathbb{H}$ , prove that  $\lambda \in \mathbb{R}$ .

**Solution.** Let  $\lambda$  have the property that  $\lambda \cdot w = w \cdot \lambda$  for all  $w \in \mathbb{H}$ . Letting  $w = \mathbf{i}$ ,  $\lambda \in \mathbb{C}$  per Exercise 1.15. Letting  $\lambda = a + b\mathbf{i}$  and  $w = \mathbf{j}$ , we must have

$$(a + b\mathbf{i}) \cdot \mathbf{j} = a\mathbf{j} + b\mathbf{k} = a\mathbf{j} - b\mathbf{k} = \mathbf{j}(a + b\mathbf{i})$$

hence  $b = -b \implies b = 0$ , therefore  $\lambda = a \in \mathbb{R}$ .

□



## Chapter 2

# All matrix groups are real matrix groups

### 2.1 Exercises





# Chapter 3

## The orthogonal groups

### 3.1 Exercises

#### Exercise 3.1 [Complete]

Prove part (4) of Proposition 3.3:

$$\overline{\langle X, Y \rangle} = \langle Y, X \rangle$$

**Solution.**

$$\begin{aligned}\overline{\langle X, Y \rangle} &= \overline{\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle} \\ &= \overline{x_1 \bar{y}_1 + \dots + x_n \bar{y}_n} \\ &= \overline{x_1 \bar{y}_1} + \dots + \overline{x_n \bar{y}_n} \\ &= y_1 \bar{x}_1 + \dots + y_n \bar{x}_n \\ &= \langle (y_1, \dots, y_n), (x_1, \dots, x_n) \rangle \\ &= \langle Y, X \rangle\end{aligned}$$

□

#### Exercise 3.2 [Complete]

Prove Equations 3.5 and 3.6:

$$(3.5) : \langle X, Y \rangle_{\mathbb{C}} = \langle f(X), f(Y) \rangle_{\mathbb{R}} + \mathbf{i} \langle f(X), f(\mathbf{i}Y) \rangle_{\mathbb{R}}$$

$$(3.6) : |X|_{\mathbb{C}} = |f(X)|_{\mathbb{R}}$$

where

$$f = f_n : \mathbb{C}^n \rightarrow \mathbb{R}^{2n}$$

is given by

$$f(a_1 + b_1 \mathbf{i}, \dots, a_n + b_n \mathbf{i}) := (a_1, b_1, \dots, a_n, b_n).$$

**Solution.** First, for Equation 3.5,

$$\begin{aligned}
\langle X, Y \rangle_{\mathbb{C}} &= \langle (a_1 + b_1 \mathbf{i}, \dots, a_n + b_n \mathbf{i}), (c_1 + d_1 \mathbf{i}, \dots, c_n + d_n \mathbf{i}) \rangle_{\mathbb{C}} \\
&= (a_1 + b_1 \mathbf{i}) \overline{(c_1 + d_1 \mathbf{i})} + \dots + (a_n + b_n \mathbf{i}) \overline{(c_n + d_n \mathbf{i})} \\
&= (a_1 + b_1 \mathbf{i})(c_1 - d_1 \mathbf{i}) + \dots + (a_n + b_n \mathbf{i})(c_n - d_n \mathbf{i}) \\
&= [(a_1 c_1 + b_1 d_1) + (-a_1 d_1 + b_1 c_1) \mathbf{i}] + \dots + [(a_n c_n + b_n d_n) + (-a_n d_n + b_n c_n) \mathbf{i}] \\
&= (a_1 c_1 + b_1 d_1 + \dots + a_n c_n + b_n d_n) + (-a_1 d_1 + b_1 c_1 + \dots - a_n d_n + b_n c_n) \mathbf{i} \\
&= \langle (a_1, b_1, \dots, a_n, b_n), (c_1, d_1, \dots, c_n, d_n) \rangle_{\mathbb{R}} + \mathbf{i} \langle (a_1, b_1, \dots, a_n, b_n), (-d_1, c_1, \dots, -d_n, c_n) \rangle_{\mathbb{R}} \\
&= \langle f(X), f(Y) \rangle_{\mathbb{R}} + \mathbf{i} \underbrace{\langle f(X), f(\mathbf{i}Y) \rangle_{\mathbb{R}}}_{!}
\end{aligned}$$

where the equality of  $\underbrace{f(\mathbf{i}Y)}_{!}$  is due to

$$\begin{aligned}
f(\mathbf{i}Y) &= f(\mathbf{i}(c_1 + d_1 \mathbf{i}, \dots, c_n + d_n \mathbf{i})) \\
&= f((-d_1 + \mathbf{i}c_1, \dots, -d_n + \mathbf{i}c_n)) \\
&= (-d_1, c_1, \dots, -d_n, c_n)
\end{aligned}$$

Next, for Equation 3.6,

$$\begin{aligned}
|X|_{\mathbb{C}} &= \sqrt{\langle X, X \rangle_{\mathbb{C}}} \\
&= \sqrt{\langle (a_1 + b_1 \mathbf{i}, \dots, a_n + b_n \mathbf{i}), (a_1 + b_1 \mathbf{i}, \dots, a_n + b_n \mathbf{i}) \rangle_{\mathbb{C}}} \\
&= \sqrt{(a_1 + b_1 \mathbf{i})(a_1 - b_1 \mathbf{i}) + \dots + (a_n + b_n \mathbf{i})(a_n - b_n \mathbf{i})} \\
&= \sqrt{a_1^2 + b_1^2 + \dots + a_n^2 + b_n^2} \\
&= \sqrt{\langle (a_1, b_1, \dots, a_n, b_n), (a_1, b_1, \dots, a_n, b_n) \rangle} \\
&= \sqrt{\langle f(X), f(X) \rangle_{\mathbb{R}}} \\
&= |f(X)|_{\mathbb{R}}
\end{aligned}$$

□

### Exercise 3.3 [WIP]

Prove Proposition 3.5, that is  $\{X_1, \dots, X_n\} \in \mathbb{C}^n$  is an orthonormal basis if and only if  $\{f(X_1), f(\mathbf{i}X_1), \dots, f(X_n), f(\mathbf{i}X_n)\}$  is an orthonormal basis of  $\mathbb{R}^{2n}$ .

**Solution.** ( $\implies$ ) Let  $\{Y_1, \dots, Y_{2n}\} = \{f(X_1), f(\mathbf{i}X_1), \dots, f(X_n), f(\mathbf{i}X_n)\}$  be an orthonormal basis of  $\mathbb{R}^{2n}$ . Then,

$$\langle Y_i, Y_j \rangle_{\mathbb{R}} = \delta_{ij}$$

where  $\delta_{ij}$  is the Kronecker Delta. Next, consider

$$\langle f^{-1}(Y_i), f^{-1}(Y_j) \rangle_{\mathbb{R}} = f^{-1}()$$

( $\impliedby$ ) Let  $X, Y \in \mathbb{C}^n$  be orthogonal. Then,

$$\langle X, Y \rangle_{\mathbb{C}} = \langle f(X), f(Y) \rangle_{\mathbb{R}} + \mathbf{i} \langle f(X), f(\mathbf{i}Y) \rangle_{\mathbb{R}} = 0$$

Since  $\langle f(X), f(iY) \rangle_{\mathbb{R}} \in \mathbb{R}$ , and hence  $i\langle f(X), f(iY) \rangle_{\mathbb{R}} \in \mathbb{C}$ , both factors must vanish. Hence,  $f$  maps  $\mathbb{C}^n$  to an orthonormal basis of  $\mathbb{R}^{2n}$ .  $\square$

### Exercise 3.4 [Complete]

Prove Proposition 3.18, that is, for any  $X \subset \mathbb{R}^n$ ,  $\text{Symm}^+(X) \subset \text{Symm}(X)$  is a subgroup with index 1 or 2.

**Solution.** First, to show  $\text{Symm}^+(X)$  is a subgroup, let  $A, B \in \text{Symm}^+(X)$ . Then,

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1 \implies AB \in \text{Symm}^+(X)$$

and

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A^{-1}) \implies A^{-1} \in \text{Symm}^+(X)$$

so  $\text{Symm}^+(X)$  is a subgroup. It's clear that  $\text{Symm}^+(X)$  has at most 2 cosets given the symmetry of the determinant, and has 1 if  $\text{Symm}^+(X) = \text{Symm}(X)$ , hence it's index is 1 or 2.  $\square$

### Exercise 3.5 [Complete]

Let  $A \in \text{GL}_n(\mathbb{K})$ . Prove that  $A \in \text{O}_n(\mathbb{K})$  if and only if the columns of  $A$  are an orthonormal basis of  $\mathbb{K}^n$ .

**Solution.** ( $\implies$ ) Let the columns of  $A$  form an orthonormal basis  $\{v_1, \dots, v_n\}$  of  $\mathbb{K}^n$ . Thus,

$$\langle v_i, v_j \rangle = \delta_{ij}$$

where  $\delta_{ij}$  is the Kronecker Delta. Next,

$$(A^*A)_{ij} = \sum_{k=1}^n A_{ij}^* A_{kj} = \sum_{k=1}^n A_{ki} A_{kj}$$

where  $A_{ki}$  is the  $k$ th component of  $v_i$ , hence

$$(A^*A)_{ij} = \langle v_i, v_j \rangle = \delta_{ij} \implies A^*A = I$$

( $\impliedby$ ) Fix some  $A \in \text{O}_n(\mathbb{K})$  with columns  $\{v_1, \dots, v_n\}$ . Since  $A$  is an element of a group,  $A^{-1}$  is defined, and so the columns are linearly independent. Next,

$$\begin{aligned} (I)_{ij} &= (A^*A)_{ij} \\ &= \langle v_i, v_j \rangle \\ &= \delta_{ij} \end{aligned}$$

Hence, the columns are orthonormal.  $\square$

### Exercise 3.6 [Complete]

1. Show that for every  $A \in \text{O}(2) - \text{SO}(2)$ ,  $R_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a flip about some line through the origin. How is this line determined by the angle of  $A$ ?

2. Let  $B = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \text{SO}(2)$ . Assume  $\theta$  is not an integer multiple of  $\pi$ . Prove that  $B$  does not commute with any  $A \in \text{O}(2) - \text{SO}(2)$ .

**Solution.** (1). First, note that given some  $X = (x_1, x_2) \in \mathbb{R}^2$ ,

$$(x_1, x_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (x_1, -x_2)$$

hence,  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  corresponds to a flip through the  $x$ -axis. Per Equation 3.8, we have

$$A = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

where  $\theta \in [0, 2\pi)$ . Using this, observe that

$$A = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

where the second factor on the RHS is in  $\text{SO}(2)$  and corresponds to a counter-clockwise rotation of  $\theta$ . Hence,  $A$  is a rotation flip then rotation.

(2) Let  $B \in \text{SO}(2)$ . Then

$$R_{AB}((x_1, x_2)) = XAB = (x_{1,\theta}, -x_{2,\theta})B = (x_{1,\theta+\varphi}, -x_{2,\theta+\varphi})$$

is given by the flip of the first factor of  $A$ , then the rotations  $\theta$  and  $\varphi$  of the next two factors. However,

$$R_{BA}((x_1, x_2)) = XBA = (x_{1,\varphi}, x_{2,\varphi})A.$$

In this case,  $x_{2,\varphi}$  will first be flipped and then rotated by  $\theta$ , and so we cannot assert that it is equal to  $-x_{2,\varphi+\theta}$ , hence the matrices cannot commute. □

### Exercise 3.7 [WIP]

Describe the product of any two elements in  $\text{O}(2)$  in terms of their angles.

**Solution.** Let  $A, B \in \text{SO}(2)$  and  $C, D \in \text{O}(2) - \text{SO}(2)$ . We need to describe  $R_{AB}$ ,  $R_{CD}$ , and  $R_{AC}$ . First,

$$\begin{aligned} R_{AB} &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta \cos \varphi - \sin \theta \sin \varphi & -\sin \theta \cos \varphi - \sin \varphi \cos \theta \\ \cos \theta \sin \varphi + \cos \varphi \sin \theta & -\sin \theta \sin \varphi + \cos \theta \cos \varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta + \varphi) & \sin(\theta + \varphi) \\ -\sin(\theta + \varphi) & \cos(\theta + \varphi) \end{pmatrix} \end{aligned}$$

□

### Exercise 3.8 [TODO]

**Exercise 3.9 [TODO]****Exercise 3.10 [TODO]****Exercise 3.11 [TODO]****Exercise 3.12 [TODO]****Exercise 3.13 [TODO]****Exercise 3.14 [TODO]****Exercise 3.15 [TODO]**

Prove that the symmetries of the tetrahedron form the group  $S_4$  and the proper symmetries form the group  $A_4$ .

**Solution.**

□

**Exercise 3.16 [Complete]**

Think of  $\text{Sp}(1)$  as the group of unit length quaternions, that is  $\text{Sp}(1) = \{q \in \mathbb{H} : |q| = 1\}$ .

1. Show that the conjugation map  $C_q : \mathbb{H} \rightarrow \mathbb{H}$  given by  $C_q(v) := qv\bar{q}$  is an orthogonal linear transformation. Thus, w.r.t the natural basis  $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ ,  $C_q$  can be regarded as an element of  $\text{O}(4)$ .
2. For every  $q \in \text{Sp}(1)$ , show that  $C_q(1) = 1$  and therefore that  $C_q$  sends  $\text{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  to itself. Conclude that the restriction  $C_q|_{\mathfrak{S}(\mathbb{H})}$  can be regarded as an element of  $\text{O}(3)$ .
3. Define  $\varphi : \text{Sp}(1) \rightarrow \text{O}(3)$  given by  $\varphi(q) := C_q|_{\mathfrak{S}(\mathbb{H})}$ . Verify  $\varphi$  is a group homomorphism.
4. Verify that the kernel of  $\varphi$  is  $\{1, -1\}$  and therefore  $\varphi$  is two-to-one.

**Solution.** (1). To show  $C_q$  is linear, let  $v, w \in \mathbb{H}$ , then

$$\begin{aligned}
 C_q(v + w) &= q(v + w)\bar{q} \\
 &= q(v\bar{q} + w\bar{q}) \\
 &= qv\bar{q} + qw\bar{q} \\
 &= C_q(v) + C_q(w).
 \end{aligned}$$

Next, fix  $\lambda \in \mathbb{R}$ . Then,

$$\begin{aligned}
 C_q(\lambda v) &= q(\lambda v)\bar{q} \\
 &= \lambda(qv\bar{q}) \\
 &= \lambda C_q(v)
 \end{aligned}$$

hence  $C_q$  is a linear transformation. To show  $C_q$  is orthogonal, note that

$$\begin{aligned}
 \langle C_q(v), C_q(w) \rangle &= \langle qv\bar{q}, qw\bar{q} \rangle \\
 &= \Re(\overline{qv\bar{q}} \cdot qw\bar{q}) \\
 &= \Re(\overline{v\bar{q}\bar{q}} \cdot qw\bar{q}) \\
 &= \Re(\overline{v\bar{q}}(\bar{q}q)w\bar{q}) \\
 &= \Re(\overline{v\bar{q}}w\bar{q}) \\
 &= \Re(\overline{v\bar{q}}w\bar{q}) \\
 &= \Re(q\bar{v}w\bar{q}) \\
 &\stackrel{*}{=} \Re(\bar{v}w) \\
 &= \langle v, w \rangle
 \end{aligned}$$

where the  $\stackrel{*}{=}$  step can be verified using

$$\Re(qx\bar{q}) = \frac{1}{2}q(x + \bar{x})\bar{q} = \frac{1}{2} \cdot 2\Re(x) \cdot q\bar{q} = \Re(x)$$

Therefore,  $C_q$  can be regarded as an element of  $O(4)$  by identifying it with a matrix  $A \in O(4)$  so that the following commutes:

$$\begin{array}{ccc}
 \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} & \xrightarrow{C_q} & \mathbb{H} \\
 \pi \downarrow & & \downarrow \rho \\
 \{e_1, e_2, e_3, e_4\} & \xrightarrow{A} & \mathbb{R}^4
 \end{array}$$

where  $\pi$  and  $\rho$  are the obvious maps.

(2). Note that

$$C_q(1) = q(1)\bar{q} = q\bar{q} = 1$$

since  $C_q$  is linear,

$$C_q(v) = C_q(\Re(v) + \Im(v)) = C_q(\Re(v)) + C_q(\Im(v)) = \Re(v) + C_q(\Im(v))$$

fixing  $\Re(v) = 0$ , it follows that  $C_q$  maps  $\text{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  to itself. Therefore,  $C_q|_{\Im(\mathbb{H})}$  can be regarded as an element of  $O(3)$  by identifying it with a matrix  $A \in O(3)$  so that the following commutes:

$$\begin{array}{ccc}
 \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} & \xrightarrow{C_q|_{\Im(\mathbb{H})}} & \Im(\mathbb{H}) \\
 \pi \downarrow & & \downarrow \rho \\
 \{e_1, e_2, e_3\} & \xrightarrow{A} & \mathbb{R}^3
 \end{array}$$

where  $\pi$  and  $\rho$  are the obvious maps.

(3). To show  $\varphi$  is a group homomorphism, fix  $q, r \in \text{Sp}(1)$  and  $v \in \mathfrak{S}(\mathbb{H})$ . Then,

$$\begin{aligned}\varphi(qr)(v) &= C_{qr}|_{\mathfrak{S}(\mathbb{H})}(v) \\ &= (qr)v(\overline{qr}) \\ &= (qr)v(\bar{r}\bar{q}) \\ &= q(rv\bar{r})\bar{q} \\ &= q \cdot C_r|_{\mathfrak{S}(\mathbb{H})}(v) \cdot \bar{q} \\ &= C_q|_{\mathfrak{S}(\mathbb{H})} \circ C_r|_{\mathfrak{S}(\mathbb{H})}(v) \\ &= \varphi(q) \circ \varphi(r)(v)\end{aligned}$$

(4) The kernel of  $\varphi$  is given by

$$\ker \varphi := \{q \in \mathbb{H} : \varphi(q) = \mathbf{id}\}$$

where  $\mathbf{id}$  is the identity map

$$\varphi(q)(v) = C_q|_{\mathfrak{S}(\mathbb{H})}(v) = qv\bar{q} = v$$

for all  $v$ . Hence, if  $\varphi(q) \in \ker \varphi$ , we must have  $qv = vq$ , and so by Exercise 1.18,  $q \in \mathbb{R}$ , and since  $q \in \text{Sp}(1)$ ,  $|q| = 1$ , hence it must be the case that  $\ker \varphi = \{1, -1\}$ . Therefore,  $\varphi$  is two-to-one. □

### Exercise 3.17 [Complete]

Think of  $\text{Sp}(1) \times \text{Sp}(1)$  as the group of pairs of unit length quaternions.

1. For every  $q = (q_1, q_2) \in \text{Sp}(1) \times \text{Sp}(1)$  the map  $F(q) : \mathbb{H} \rightarrow \mathbb{H}$  defined by  $F(q) := q_1 v \bar{q}_2$  is an orthogonal linear transformation, and can thus be identified with an element of  $O(4)$  w.r.t the natural basis  $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ .
2. Show that the function  $F : \text{Sp}(1) \times \text{Sp}(1) \rightarrow O(4)$  is a group homomorphism.
3. Verify  $\ker F = \{(1, 1), (-1, -1)\}$  and is therefore two-to-one.
4. How is  $F$  related to  $\varphi$  from the previous exercise?

**Solution.** (1). To show  $F$  is a linear transformation, let  $v, w \in \mathbb{H}$  and fix  $q = (q_1, q_2) \in \text{Sp}(1) \times \text{Sp}(1)$ . Then,

$$\begin{aligned}F(q)(v + w) &= q_1(v + w)\bar{q}_2 \\ &= q_1(v\bar{q}_2 + w\bar{q}_2) \\ &= q_1v\bar{q}_2 + q_1w\bar{q}_2 \\ &= F(q)(v) + F(q)(w).\end{aligned}$$

Next, fixing  $\lambda \in \mathbb{R}$ ,

$$\begin{aligned}F(q)(\lambda v) &= q_1(\lambda v)\bar{q}_2 \\ &= \lambda(q_1v\bar{q}_2) \\ &= \lambda \cdot F(q)(v)\end{aligned}$$

hence  $F(q)$  is linear. Next,

$$\begin{aligned}
 \langle F(q)(v), F(q)(w) \rangle &= \langle q_1 v \bar{q}_2, q_1 w \bar{q}_2 \rangle \\
 &= \Re(\overline{q_1 v \bar{q}_2} \cdot q_1 w \bar{q}_2) \\
 &= \Re(\overline{v \bar{q}_2} \bar{q}_1 \cdot q_1 w \bar{q}_2) \\
 &= \Re(\overline{v \bar{q}_2} (\bar{q}_1 q_1) w \bar{q}_2) \\
 &= \Re(\overline{v \bar{q}_2} w \bar{q}_2) \\
 &= \Re(q_2 \bar{v} w \bar{q}_2) \\
 &\stackrel{*}{=} \Re(\bar{v} w) \\
 &= \langle v, w \rangle
 \end{aligned}$$

where the  $\stackrel{*}{=}$  is justified in the same manner as Exercise 3.15.1. Hence,  $F(q)$  is othogonal and can be identified with an element  $A \in \text{O}(4)$  diagrammatically via

$$\begin{array}{ccc}
 \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} & \xrightarrow{F(q)} & \mathbb{H} \\
 \downarrow \pi & & \downarrow \rho \\
 \{e_1, e_2, e_3, e_4\} & \xrightarrow{A} & \mathbb{R}^4
 \end{array}$$

where  $\pi$  and  $\rho$  are the obvious maps.

(2). To show  $F$  is a group homomorphism, fix  $q, r \in \text{Sp}(1) \times \text{Sp}(1)$  and  $v \in \mathbb{H}$ . Then,

$$\begin{aligned}
 F(qr)(v) &= F((q_1, q_2) \cdot (r_1, r_2))(v) \\
 &= F((q_1 r_1, q_2 r_2))(v) \\
 &= (q_1 r_1) v (\overline{q_2 r_2}) \\
 &= q_1 (r_1 v \bar{r}_2) \bar{q}_2 \\
 &= q_1 \cdot (F(r)(v)) \cdot \bar{q}_2 \\
 &= F(q) \circ F(r)(v)
 \end{aligned}$$

hence the result.

(3). The kernel of  $F$  is given by

$$\ker F := \{q \in \mathbb{H} : F(q) = \mathbf{id}\}$$

where  $\mathbf{id}$  is the identity map

$$F(q)(v) = q_1 v \bar{q}_2 = v$$

for all  $v$ . Hence,  $q_1 v = v q_2$ , and using a similar argument to Exercise 3.16.4, the only solutions are  $(1, 1)$  and  $(-1, -1)$ , hence

$$\ker F = \{(1, 1), (-1, -1)\}$$



(4).  $F$  is a double cover of  $O(4)$ , and  $\varphi$  is a double cover of  $O(3)$ . Moreover, we get the following diagram:

$$\begin{array}{ccc} \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \xrightarrow{F} & O(4) \\ \downarrow \pi & & \downarrow ? \\ \mathrm{Sp}(1) & \xrightarrow{\varphi} & O(3) \end{array}$$

where  $?$  is an interesting map which maps both connected components of  $O(4)$  to the corresponding connected components of  $O(3)$ .  $\square$

**Exercise 3.18 [TODO]**



# Chapter 4

## The topology of matrix groups

### 4.1 Exercises

**Exercise 4.1 [TODO]**

Another exercise goes here.

**Solution.** Placeholder for your solution.

□



# Chapter 5

## Lie algebras

### 5.1 Exercises

- Exercise 5.1 [TODO]
- Exercise 5.2 [TODO]
- Exercise 5.3 [TODO]
- Exercise 5.4 [TODO]
- Exercise 5.5 [TODO]
- Exercise 5.6 [TODO]
- Exercise 5.7 [TODO]
- Exercise 5.8 [TODO]
- Exercise 5.9 [TODO]
- Exercise 5.10 [TODO]
- Exercise 5.11 [TODO]
- Exercise 5.12 [TODO]
- Exercise 5.13 [TODO]
- Exercise 5.14 [TODO]
- Exercise 5.15 [TODO]
- Exercise 5.16 [TODO]



# Chapter 6

## Matrix exponentiation

### 6.1 Exercises

#### Exercise 6.1 [WIP]

Prove Proposition 6.5, that is, suppose that  $\sum A_l$  and  $\sum B_l$  converge, at least one absolutely. Let  $C_l = \sum_{k=0}^l A_k B_{l-k}$ . Prove that

$$\sum C_l = (\sum A_l)(\sum B_l)$$

**Solution.**

$$\begin{aligned} (\sum C_l)_{ij} &= (\sum_{k=0}^l A_k B_{l-k})_{ij} \\ &= (\sum_{k=0}^l \sum_{r=1}^l (A_k))_{ij} \end{aligned}$$

□

#### Exercise 6.2 [Complete]

Prove that  $(e^A)^* = e^{A^*}$  for all  $A \in M_n(\mathbb{K})$ .

**Solution.** Using the fact that  $(XY)^* = Y^* X^*$  and  $(X + Y)^* = X^* + Y^*$ ,

$$\begin{aligned} e^{A^*} &= \sum_{k=0}^{\infty} \frac{(A^*)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\overbrace{A^* \cdot A^* \cdots A^*}^{k\text{-times}}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\overbrace{(A \cdot A \cdots A)^*}^{k\text{-times}}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(A^k)^*}{k!} \\ &= \left( \sum_{k=0}^{\infty} \frac{(A^k)}{k!} \right)^* \\ &= (e^A)^* \end{aligned}$$

□

**Exercise 6.3 [Complete]**

1. Let  $A = \text{diag}(a_1, \dots, a_n) \in M_n(\mathbb{R})$ . Calculate  $e^A$  and give a simple prove that  $\det(e^A) = e^{\text{trace}(A)}$  when  $A$  is diagonal.
2. Give a simple proof that  $\det(e^A) = e^{\text{trace}(A)}$  when  $A$  is conjugate to a diagonal matrix.

**Solution.** (1). Let  $A = \text{diag}(a_1, \dots, a_n) \in M_n(\mathbb{R})$ . Then,

$$\begin{aligned}
 e^A &= \sum_{k=0}^{\infty} \frac{\text{diag}(a_1, \dots, a_n)^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{\text{diag}(a_1^k, \dots, a_n^k)}{k!} \\
 &= \text{diag} \left( \sum_{k=0}^{\infty} \frac{a_1^k}{k!}, \dots, \sum_{k=0}^{\infty} \frac{a_n^k}{k!} \right) \\
 &= \text{diag}(e^{a_1}, \dots, e^{a_n}).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \det(e^A) &= \det(\text{diag}(e^{a_1}, \dots, e^{a_n})) \\
 &= \prod_{i=1}^n e^{a_i} \\
 &= e^{\sum_{j=1}^n a_j} \\
 &= e^{\text{trace}(A)}
 \end{aligned}$$

(2). Let  $A \in M_n(\mathbb{R})$  be conjugate to a diagonal matrix, that is there exists a diagonal matrix  $D = \text{diag}(d_1, \dots, d_n)$  and an invertible matrix  $P$  such that  $A = PDP^{-1}$ . Then,

$$\begin{aligned}
 e^A &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{\overbrace{(PDP^{-1}PDP^{-1} \dots PDP^{-1})}^{k\text{-times}}}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{\overbrace{PD(P^{-1}P)D(P^{-1}P) \dots (P^{-1}P)DP^{-1}}^{k\text{-times}}}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} \\
 &= P \left( \sum_{k=0}^{\infty} \frac{D^k}{k!} \right) P^{-1} \\
 &= Pe^D P^{-1}.
 \end{aligned}$$



Finishing it off:

$$\begin{aligned}
 \det(e^A) &= \det(Pe^D P^{-1}) \\
 &= \det(P) \cdot \det(e^D) \cdot \det(P^{-1}) \\
 &= e^{\text{trace}(D)} \\
 &= e^{\text{trace}(A)}
 \end{aligned}$$

where the last equality holds since similar matrices have the same trace. □

**Exercise 6.4 [WIP]**

Let  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Compute  $e^{tA}$  for arbitrary  $t \in \mathbb{R}$ .

**Solution.** First, consider the following table:

$k$	$A^k$
0	$I$
1	$A$
2	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
3	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
4	$I$

Hence,  $A$  has periodicity 4. Therefore,

$$\begin{aligned}
 e^{tA} &= \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \\
 &= \sum_{i=0}^{\infty} \frac{t^{4i}}{i!} I + \sum_{j=0}^{\infty} \frac{t^{4j+1}}{j!} A^{4j+1} + \sum_{l=0}^{\infty} \frac{t^{4l+2}}{l!} A^{4l+2} + \sum_{r=0}^{\infty} \frac{t^{4r+3}}{r!} A^{4r+3} \\
 &= \sum_{i=0}^{\infty} \frac{t^{4i}}{i!} I + \sum_{j=0}^{\infty} \frac{t^{4j+1}}{j!} A + \sum_{l=0}^{\infty} \frac{t^{4l+2}}{l!} A^2 + \sum_{r=0}^{\infty} \frac{t^{4r+3}}{r!} A^3 \\
 &= \sum_{i=0}^{\infty} \frac{(t^4)^i}{i!} I + t \sum_{j=0}^{\infty} \frac{(t^4)^j}{j!} A + t^2 \sum_{l=0}^{\infty} \frac{(t^4)^l}{l!} A^2 + t^3 \sum_{r=0}^{\infty} \frac{(t^4)^r}{r!} A^3 \\
 &= e^{t^4} I + t e^{t^4} A + t^2 e^{t^4} A^2 + t^3 e^{t^4} A^3 \\
 &= e^{t^4} (I + tA + t^2 A^2 + t^3 A^3) \\
 &= e^{t^4} \begin{pmatrix} 1 - t^2 & t - t^3 \\ -t + t^3 & 1 - t^2 \end{pmatrix}
 \end{aligned}$$

Noticing that as  $t \rightarrow 0$  we have  $e^{tA} \rightarrow I$  is a good sanity check! □

**Exercise 6.5 [Complete]**

Can a one-parameter group ever cross itself?

**Solution.** No - a one-parameter group is differentiable, hence any singularity would contradict its differentiability. Alternatively, by Proposition 6.17, every one-parameter group is described by  $\gamma(t) = e^{tA}$  for some  $A \in gl_n(\mathbb{K})$ , and  $e^{tA}$  is injective.  $\square$

**Exercise 6.6 [WIP]**

Describe all one-parameter groups of  $GL_1(\mathbb{C})$ . Draw several in the x-y plane.

**Solution.** Let  $z = a + bi \in GL_1(\mathbb{C})$ . Then, any one-parameter group has the form

$$\begin{aligned}\gamma_z(t) &= e^{tz} \\ &= e^{t(a+bi)} \\ &= e^{at}e^{ibt}\end{aligned}$$

$e^{ibt}$  can always be identified with a point on the unit circle, scaled by  $e^{at}$ . Hence,  $\gamma_z(t)$  makes a spiral that "spirals" exponentially faster as  $t$  increases that starts at  $\gamma_z(0) = 1$ .  $z$  determines the initial condition and initial "spiral rate".  $\square$

**Exercise 6.7 [WIP]**

Let  $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R}) : x > 0 \right\}$ . Describe the one-parameter groups in  $G$ , and draw several on the  $xy$ -plane.

**Solution.** First, note that given  $A \in G$ ,

$$\begin{aligned}A &= \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \\ A^2 &= \begin{pmatrix} x^2 & y(x+1) \\ 0 & 1 \end{pmatrix}\end{aligned}$$

and

$$A^3 = \begin{pmatrix} x^3 & y(x^2 + x + 1) \\ 0 & 1 \end{pmatrix}$$

It can be show inductively that

$$A^k = \begin{pmatrix} x^k & y(x^{k-1} + \dots + 1) \\ 0 & 1 \end{pmatrix}$$

and so,

$$\begin{aligned}e^{tA} &= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} x^k & y(x^{k-1} + \dots + 1) \\ 0 & 1 \end{pmatrix} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} x^k & y \frac{x^k - 1}{x - 1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{tx} & \frac{y}{x-1} \sum_{k=0}^{\infty} (tx)^k / k! - t^k / k! \\ 0 & e^t \end{pmatrix} \\ &= \begin{pmatrix} e^{tx} & \frac{y}{x-1} (e^{tx} - e^t) \\ 0 & e^t \end{pmatrix}\end{aligned}$$

No fucking way am I drawing these. □

**Exercise 6.8 [WIP]**

Visually describe the path  $\gamma(t) = e^{t\mathbf{j}}$  in  $Sp(1) \cong S^3$ .

**Solution.** First, note that

$$\begin{array}{c|c} k & \mathbf{j}^k \\ \hline 0 & 1 \\ 1 & \mathbf{j} \\ 2 & -1 \\ 3 & 1 \end{array}$$

hence,  $\mathbf{j}$  has periodicity 3. Next,

$$\begin{aligned} \gamma(t) &= e^{t\mathbf{j}} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{j}^k \\ &= \sum_{i=0}^{\infty} \frac{t^i}{i!} 1 + \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathbf{j} + \sum_{l=0}^{\infty} \frac{t^l}{l!} (-1)^l \end{aligned}$$

□

**Exercise 6.9 [WIP]**

Let  $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in gl_n \mathbb{R}$ . Compute  $e^A$ .

**Solution.** First, note that

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

is a decomposition of  $A$  into two commuting matrices. Thus,

$$\begin{aligned} e^A &= \exp \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ &= \exp \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right) \\ &= \exp \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \exp \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \\ &= \left( \sum_{k=0}^{\infty} \frac{a^k I^k}{k!} \right) \cdot \left( \sum_{j=0}^{\infty} \frac{b^j}{j!} \exp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^j \right) \\ &= e^a I \cdot e^{b^4} \begin{pmatrix} 1 - b^2 & b - b^3 \\ -b + b^3 & 1 - b^2 \end{pmatrix} \\ &= e^{b^4+a} \begin{pmatrix} 1 - b^2 & b - b^3 \\ -b + b^3 & 1 - b^2 \end{pmatrix} \end{aligned}$$

□

**Exercise 6.10 [TODO]**

Repeat the previous problem with  $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

**Exercise 6.11 [TODO]****Exercise 6.12 [TODO]****Exercise 6.13 [TODO]****Exercise 6.14 [TODO]**

# Chapter 7

## Matrix groups are manifolds

### 7.1 Exercises

#### Exercise 7.1 [Complete]

Define the **stereographic projection** as

$$f : S^2 - \{0, 0, 1\} \rightarrow \mathbb{R}^2$$

via shooting a lazer from the north pole of  $S^2$ ; the lazer hits a point  $p = (x, y, z)$  on  $S^2$  and a subsequent point on  $\mathbb{R}^2$  (given by the plane  $z = -1$ ), identifying  $p \in S^2 - \{0, 0, 1\}$  with  $f(p) \in \mathbb{R}^2$ .

1. Show that

$$f(x, y, z) = \frac{2}{1 - z}(x, y)$$

2. Find a formula for  $f^{-1}$

3. Find a formula

$$g : S^2 - \{0, 0, -1\} \rightarrow \mathbb{R}^2$$

defined analogously to  $f$ , but the plane of intersection is  $z = 1$ .

4. Find an explicit formula for the composition

$$g \circ f^{-1} : \mathbb{R}^2 - \{0, 0\} \rightarrow \mathbb{R}^2 - \{(0, 0)\}$$

**Solution.** (1). Fix the following lines:

1.  $\ell_1$  between  $(0, 0, -1)$  and  $(0, 0, 1)$
2.  $\ell_2$  between  $(0, 0, 1)$  and  $f(p) = f(x, y, z)$
3.  $\ell_3$  between  $(0, 0, -1)$  and  $f(p) = f(x, y, z)$
4.  $\ell_4$  between  $(0, 0, 1)$  and  $(0, 0, 1 - z)$

5.  $\ell_5$  between  $(0, 0, 1)$  and  $p = (x, y, z)$

6.  $\ell_6$  between  $(0, 0, 1 - z)$  and  $p = (x, y, z)$

It's clear that  $T_1 = \{\ell_1, \ell_2, \ell_3\}$  and  $T_2 = \{\ell_4, \ell_5, \ell_6\}$  are similar. Moreover,  $\ell_1 = 2$  and  $\ell_4 = 1 - z$ . Therefore,

$$\frac{1 - z}{2} = \frac{(x, y)}{f(x, y, z)} \implies f(x, y, z) = \frac{2}{1 - z}(x, y).$$

(2). Let  $f(x, y, z) = (u, v) \in \mathbb{R}^2$ . Then, parameterize the line from  $(0, 0, 1)$  to  $(u, v)$  with

$$L(t) = (0, 0, 1) + t(u, v, -2) = (tu, tv, 1 - 2t)$$

where  $t \in [0, 1]$ . We want to know when  $L(t)$  intersects the sphere, as in when

$$\begin{aligned} (tu)^2 + (tv)^2 + (1 - 2t)^2 &= 1 \implies (u^2 + v^2)t^2 - 4t + 4t^2 = 0 \\ &\implies (u^2 + v^2 + 4)t^2 - 4t = 0 \\ &\implies t[(u^2 + v^2 + 4)t - 4] = 0 \end{aligned}$$

where  $t = \frac{4}{u^2 + v^2 + 4}$  is clearly the relevant solution. Therefore,

$$f^{-1}(u, v) = (tu, tv, 1 - 2t) = \left( \frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, 1 - \frac{8}{u^2 + v^2 + 4} \right)$$

(3). We can recover step (1) by mapping  $1 - z \rightarrow z - 1$ , hence we get

$$g(x, y, z) = \frac{2}{z - 1}(x, y)$$

(4). Let  $c = \frac{4}{u^2 + v^2 + 4}$ . Composition gives

$$\begin{aligned} g \circ f^{-1}(u, v) &= g(cu, cv, 1 - 2c) \\ &= \frac{2}{1 - 1 - 2c}(cu, cv) \\ &= -(u, v) \end{aligned}$$

which is super chill!

□

### Exercise 7.2 [WIP]

Prove that  $S^n \subset \mathbb{R}^{n+1}$  is an  $n$ -dimensional manifold.

**Solution.** Define

$$V = \{(x_1, \dots, x_{n+1}) \in S^n : x_{n+1} > 0\}$$

which is a neighbourhood of  $(0, 0, \dots, 1) \in S^n$ . Next, define

$$U = \{(a_1, \dots, a_n) : a_1^2 + \dots + a_n^2 < 1\}$$

and define

$$\begin{aligned} \varphi : U &\rightarrow V \\ \varphi(a_1, \dots, a_n) &:= (a_1, a_2, \dots, a_n, \sqrt{1 - a_n^2 - \dots - a_1^2}) \end{aligned}$$

□

# Chapter 8

## The Lie bracket

### 8.1 Exercises

**Exercise 8.1 [TODO]**

Another exercise goes here.

**Exercise 8.2 [TODO]**

Let  $G_1, G_2$  be matrix groups with Lie algebras  $\mathfrak{g}_1, \mathfrak{g}_2$ . Suppose that  $f : G_1 \rightarrow G_2$  is a smooth homomorphism. If  $df_I : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is bijective, prove that  $df_g : T_g G_1 \rightarrow T_{f(g)} G_2$  is bijective for all  $g \in G_1$ .

**Solution.** Let  $a(0) = b(0) = I_{G_1}$  with  $a'(0) = A, b'(0) = B$  for some  $A, B \in \mathfrak{g}_1$ .

□

**Exercise 8.3 [TODO]**

Another exercise goes here.

**Exercise 8.4 [TODO]**

Another exercise goes here.

**Exercise 8.5 [TODO]**

Another exercise goes here.

**Exercise 8.6 [TODO]**

Another exercise goes here.

**Exercise 8.7 [TODO]**

Another exercise goes here.





# Chapter 9

## Maximal tori

### 9.1 Exercises

**Exercise 9.1.** Another exercise goes here.

**Solution.** Placeholder for your solution.

□



# Chapter 10

## Homogeneous manifolds

### 10.1 Exercises

**Exercise 10.1.** Another exercise goes here.

**Solution.** Placeholder for your solution.

□



# Chapter 11

## Roots

### 11.1 Exercises

**Exercise 11.1.** Another exercise goes here.

**Solution.** Placeholder for your solution.

□