Exercises from *Matrix Groups for Undergraduates* by Kristopher Tapp

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Contents

Chapter 1

Matrices

Exercise 1.1. Describe a natural 1-to-1 correspondence between elements of SO(3) and elements of

$$T^1S^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |p| = |v| = 1 \text{ and } p \perp q\}$$

Solution. Using the globe analogy from Question 1.2, fix a point r to be the north pole, and a point e that lies on the equator induced by the choice of r, and assert this as the arbitrary 'identity'.

Next, given some $A \in \mathrm{SO}(3)$, identify an element in T^1S^2 via $A \mapsto (Ar, Av)$, as in first where A maps the north pole r, and then how A rotates the globe about the axis induced by r and its antipodal point. □

Exercise 1.2. Prove equation 1.3:

$$(A \cdot B)^T = B^T \cdot A^T$$

Solution. First,

$$\begin{aligned}(A \cdot B)^T_{ij} &= (A \cdot B)_{ji} \\ &= \sum_{k=1}^{n} A_{jk}B_{ki}\end{aligned}$$

Next,

$$\begin{aligned}(B^T \cdot A^T)_{ij} &= \sum_{k=1}^{n} (B^T)_{ik}(A^T)_{kj} \\ &= \sum_{k=1}^{n} B_{ki}A_{jk} \\ &\overset{*}{=} \sum_{k=1}^{n} A_{jk}B_{ki}\end{aligned}$$

□

Note the $\overset{*}{=}$ step uses the commutativity of multiplication, hence the above proof does not work when $\mathbb{K} = \mathbb{H}$.

Exercise 1.3. Prove equation 1.4:

$$\mathrm{trace}(A \cdot B) = \mathrm{trace}(B \cdot A)$$

Solution. First,

$$\mathrm{trace}(A \cdot B)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

Next,

$$\begin{aligned} \mathrm{trace}(B \cdot A)_{ii} &= \sum_{i=1}^{n} (B \cdot A)_{ii} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ik} A_{ki} \end{aligned}$$

Carefully reindexing and resumming gives the result. □

Note that the 'careful reindexing and resumming process' implies trace() is invariant under cyclic permutation, e.x.:

$$\mathrm{trace}(A \cdot B \cdot C) = \mathrm{trace}(C \cdot A \cdot B)$$

Exercise 1.4. Let $A, B \in M_n\mathbb{K}$. Prove that if $A \cdot B = I$ then $B \cdot A = I$.

Solution. Note that

$$A \cdot B = I \iff A = B^{-1}.$$

The result follows. □

Exercise 1.5. Suppose that the determinant of $A \in M_n(\mathbb{H})$ were defined as in Equation 1.5. Show for

$$A = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} \in M_n(\mathbb{H})$$

that $\det(A) \neq 0$ but

$$R_A : H^2 \to H^2$$

is not invertible.

Solution. Given the definition from Equation 1.5:

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} \\ &= \mathbf{ij} - \mathbf{ji} \\ &= (1) - (-1) \\ &= 2 \neq 0 \end{aligned}$$

However,

$$R_A((-\mathbf{i}, \mathbf{i})) = (-\mathbf{i}, \mathbf{i}) \cdot \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} = (-\mathbf{i}^2 + \mathbf{i}^2, -\mathbf{ij} + \mathbf{ij}) = (1 - 1, -\mathbf{k} + \mathbf{k}) = (0, 0)$$

Hence, R_A has a non-zero determiant, but is not invertible as the kernel is non-trivial. Similarly, clearly the columns of A are linearly dependent. □

Exercise 1.6. Find $B \in M_2(\mathbb{R})$ such that $R_B : \mathbb{R}^2 \to \mathbb{R}^2$ is a counter-clockwise rotation through an angle θ.

Solution. Note that we can 'represent' both 1 and i in $M_2(\mathbb{R})$ via

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \text{ and } i \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where the latter works since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

capturing the fact that $i^2 = -1$. Building on this, we can represent any $a + bi \in \mathbb{C}$ via

$$\rho : a + bi \mapsto a \cdot I + b \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Next, note that the function $f_\theta(z) = ze^{i\theta}$ rotates elements counter-clockwise in $\mathbb{C}$ by an angle θ. To see this, letting $z = re^{i\varphi}$,

$$f_\theta(z) = ze^{i\theta} = re^{i\varphi}e^{i\theta} = re^{i(\varphi+\theta)}.$$

Applying ρ gives

$$\begin{aligned} \rho_1(e^{i\theta}) &= \rho_1(\cos(\theta) + i\sin(\theta)) \\ &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = B \end{aligned}$$

□

Exercise 1.7. Describe all elements $A \in \mathrm{GL}_n(\mathbb{R})$ with the property $AB = BA$ for all $B \in \mathrm{GL}_n(\mathbb{R})$.

Solution. Matrices A that commute with all matrices in $\mathrm{GL}_n(\mathbb{R})$ are scalar multiples of the identity

$$A = \lambda I$$

where $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. □

Exercise 1.8. Let $\mathrm{SL}_2(\mathbb{Z})$ denote 2 by 2 matrices with integer entries and determinant 1. Prove that $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Is $SL_n(\mathbb{Z})$ a subgroup of $\mathrm{GL}_n(\mathbb{R})$ in general?

Solution. Fixing $A, B \in \mathrm{SL}_2(\mathbb{Z})$, its clear that $A \cdot B$ must have all integer entries. Since

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

we have that $A \cdot B \in \mathrm{SL}_2(\mathbb{Z})$ (closure). Next, fix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

Using Cramer's Rule to compute the inverse of A we get

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $ad-bc = 1$ since $\det A = 1$, so A^{-1} has all integer entries, and is a member of $\mathrm{SL}_2(\mathbb{Z})$. Therefore, $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. $\square$

The critical step in the above proof is discerning that the factor extracted from A^{-1} is $1/\det A = 1$, which ensures the entries of the inverse are all in $\mathbb{Z}$. This factor is the same for any n, so $\mathrm{SL}_n(\mathbb{Z})$ is always a subgroup of $\mathrm{GL}_n(\mathbb{R})$ for all n.

Exercise 1.9. Describe the block matrix blah blabh blabhj TODO write this out

Solution. Suppose A and B are block matrices in $M_n(\mathbb{K})$, given by

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

where $\sum \dim A_i = \dim A = \sum \dim B_i = \dim B$, and $\dim A_i = \dim B_i$ for each i. Then,

$$A \cdot B = \begin{pmatrix} A_1 \cdot B_1 & 0 & \cdots & 0 \\ 0 & A_2 \cdot B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \cdot B_n \end{pmatrix}$$

Which can be applied to the above question to derive a simple answer.

$\square$

Exercise 1.10. If $G_1 \subset \mathrm{GL}_{n_1}(\mathbb{K})$ and $G_2 \subset \mathrm{GL}_{n_2}(\mathbb{K})$ are subgroups, describe a subgroup of $\mathrm{GL}_{n_1+n_2}(\mathbb{K})$ isomorphic to $G_1 \times G_2$.

Solution. Define a map

$$\varphi : G_1 \times G_2 \to \mathrm{GL}_{n_1+n_2}(\mathbb{K})$$

given by

$$(A_1, A_2) \mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

The image of φ is a subset of $\mathrm{GL}_{n_1+n_2}(\mathbb{K})$ since

$$\det \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \det A_1 \cdot \det A_2 \neq 0$$

so $\varphi(A_1, A_2) \in \mathrm{GL}_{n_1+n_2}(\mathbb{K})$. To prove φ is a group homomorphism, observe

$$\begin{aligned} \varphi((A_1, A_2) \cdot (B_1, B_2)) &= \varphi(A_1 \cdot B_1, A_2 \cdot B_2) \\ &= \begin{pmatrix} A_1 \cdot B_1 & 0 \\ 0 & A_2 \cdot B_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \cdot \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \\ &= \varphi(A_1, A_2) \cdot \varphi(B_1, B_2) \end{aligned}$$

hence the result.

$\square$

Exercise 1.11.

Exercise 1.12. Show that for purely imaginary $q_1, q_2 \in \mathbb{H}$, $-\Re(q_1 \cdot q_2)$ is the vector dot product in $\mathbb{R}^3 = \text{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\Im(q_1 \cdot q_2)$ is the vector cross-product.

Solution. First,

$$\begin{aligned} -\Re(q_1 \cdot q_2) &= -\Re((b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \cdot (b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})) \\ &= -\Re((-b_1b_2 - c_1c_2 - d_1d_2) + \dots) \\ &= b_1b_2 + c_1c_2 + d_1d_2 \end{aligned}$$

Next,

$$\begin{aligned} \Im(q_1 \cdot q_2) &= \Im((b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \cdot (b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})) \\ &= (c_1d_2 - d_1c_2)\mathbf{i} + (d_1b_2 - b_1d_2)\mathbf{j} + (b_1c_2 - c_1b_2)\mathbf{j} \end{aligned}$$

Mapping $\text{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to the standard basis in $\mathbb{R}^3$ gives both desired results. $\square$

Exercise 1.13. Prove that non-real elements $q_1, q_2 \in \mathbb{H}$ commute if and only if their imaginary parts are parallel; that is, $\Im(q_1) = \lambda \cdot \Im(q_2)$ for some $\lambda \in \mathbb{R}$.

Solution. ($\implies$) Let $\Im(q_1) = \lambda \cdot \Im(q_2)$ for some $\lambda \in \mathbb{R}$, so that

$$\Im(q_1) = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \lambda b\mathbf{i} + \lambda c\mathbf{j} + \lambda d\mathbf{k} = \Im(q_2)$$

then,

$$\begin{aligned} q_1 \cdot q_2 &= (a_1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \cdot (a_2 + \lambda b\mathbf{i} + \lambda c\mathbf{j} + \lambda d\mathbf{k}) \\ &= \Big(a_1a_2 - \lambda\left(b^2 + c^2 + d^2\right)\Big) b(a_1\lambda + a_2)\mathbf{i}c(a_1\lambda + a_2)\mathbf{j}d(a_1\lambda + a_2)\mathbf{k} \\ &= (a_2 + \lambda\, b\,\mathbf{i} + \lambda\, c\,\mathbf{j} + \lambda\, d\,\mathbf{k}) \cdot (a_1 + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}) \\ &= q_2 \cdot q_1. \end{aligned}$$

($\impliedby$) Let $q_1 \cdot q_2 = q_2 \cdot q_1$ where

$$q_1 = (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}), q_2 = (a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}).$$

Then the following equalities must hold:

$\square$

Exercise 1.14. Charachterize the pairs $q_1, q_2 \in \mathbb{H}$ which anti-commute, that is $q_1q_2 = -q_2q_1$.

Exercise 1.15. If $q \in \mathbb{H}$ satisfies $q\mathbf{i} = \mathbf{i}q$, prove that $q \in \mathbb{C}$.

Solution. Let $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Then,

$$q\mathbf{i} = a\mathbf{i} + b\mathbf{i}\mathbf{i} + c\mathbf{j}\mathbf{i} + d\mathbf{k}\mathbf{i} = -b + a\mathbf{i} + d\mathbf{j} - c\mathbf{k}$$

and

$$\mathbf{i}q = \mathbf{i}a + b\mathbf{i}\mathbf{i} + c\mathbf{i}\mathbf{j} + d\mathbf{i}\mathbf{k} = -b + a\mathbf{i} - d\mathbf{j} + c\mathbf{k}.$$

Identifying terms gives

$$d = -d \implies d = 0$$
$$c = -c \implies c = 0$$

hence $q = a + bi \in \mathbb{C}$.

$\square$

Exercise 1.16. Prove that complex multiplication in $\mathbb{C} \approxeq \mathbb{R}^2$ does not extend to a multiplication operation on $\mathbb{R}^3$ that makes $\mathbb{R}^3$ into a real division algebra.

Solution. Assume such an extension exists. Consider the map analagous to the extension of $\mathbb{R}^2$ given by

$$(a, b, c) \mapsto a + b\mathbf{i} + c\mathbf{j}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = -1$. Then there must exist a linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

with $T^2 = -I$. Represent T by a 3×3 real matrix M. Then -1 is in the spectrum of M^2 since $M^2 = -I$, thus $\pm i$ is in the spectrum of M. Since $\det(M) = \prod \lambda_k$ where λ_k is an eigenvalue of M, there must exist some real value λ such that

$$\det(M) = (i)(-i)(\lambda) = \lambda$$

where λ must be in $\mathbb{R}$ since complex eigenvalues come in pairs. Thus, we must have

$$\det(M^2) = \det(-I) = -1$$

and

$$\det(M^2) = \det(M)^2 = \lambda^2$$

thus $\lambda^2 = -1$, which contradicts λ being real.

$\square$

Exercise 1.17. Describe a subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$ that is isomorphic to $\mathbb{R}^n$ under vector-addition.

Solution. Consider the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R})$$

and the map that takes such matrices to $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

$\square$

Exercise 1.18. If $\lambda \in \mathbb{H}$ commutes with every element of $\mathbb{H}$, prove that $\lambda \in \mathbb{R}$.

Solution. Let λ have the property that $\lambda \cdot w = w \cdot \lambda$ for all $w \in \mathbb{H}$. Letting $w = \mathbf{i}$, $\lambda \in \mathbb{C}$ per Exercise 1.15. Letting $\lambda = a + b\mathbf{i}$ and $w = \mathbf{j}$, we must have

$$(a + b\mathbf{i}) \cdot \mathbf{j} = a\mathbf{j} + b\mathbf{k} = a\mathbf{j} - b\mathbf{k} = \mathbf{j}(a + b\mathbf{i})$$

hence $b = -b \implies b = 0$, therefore $\lambda = a \in \mathbb{R}$.

$\square$

Chapter 2

All matrix groups are real matrix groups

2.1 Exercises

Exercise 2.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 3

The orthogonal groups

3.1 Exercises

Exercise 3.1. Another exercise goes here.

Solution. Placeholder for your solution. $\square$

Chapter 4

The topology of matrix groups

4.1 Exercises

Exercise 4.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 5

Lie algebras

5.1 Exercises

Exercise 5.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 6

Matrix exponentiation

6.1 Exercises

Exercise 6.1. Another exercise goes here.

Solution. Placeholder for your solution. $\square$

Chapter 7

Matrix groups are manifolds

7.1 Exercises

Exercise 7.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 8

The Lie bracket

8.1 Exercises

Exercise 8.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 9

Maximal tori

9.1 Exercises

Exercise 9.1. Another exercise goes here.

Solution. Placeholder for your solution. ☐

Chapter 10

Homogeneous manifolds

10.1 Exercises

Exercise 10.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 11

Roots

11.1 Exercises

Exercise 11.1. Another exercise goes here.

Solution. Placeholder for your solution. □