Exercises from *Matrix Groups for Undergraduates* by Kristopher Tapp

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Contents

Chapter 1

Matrices

Exercise 1.1. Describe a natural 1-to-1 correspondence between elements of SO(3) and elements of

$$T^1S^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |p| = |v| = 1 \text{ and } p \perp q\}$$

Solution. Using the globe analogy from Question 1.2, fix a point r to be the north pole, and a point e that lies on the equator induced by the choice of r, and assert this as the arbitrary 'identity'.

Next, given some $A \in \mathrm{SO}(3)$, identify an element in T^1S^2 via $A \mapsto (Ar, Av)$, as in first where A maps the north pole r, and then how A rotates the globe about the axis induced by r and its antipodal point. □

Exercise 1.2. Prove equation 1.3:

$$(A \cdot B)^T = B^T \cdot A^T$$

Solution. First,

$$\begin{aligned}(A \cdot B)^T_{ij} &= (A \cdot B)_{ji} \\ &= \sum_{k=1}^{n} A_{jk} B_{ki}\end{aligned}$$

Next,

$$\begin{aligned}(B^T \cdot A^T)_{ij} &= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} \\ &= \sum_{k=1}^{n} B_{ki} A_{jk} \\ &\stackrel{*}{=} \sum_{k=1}^{n} A_{jk} B_{ki}\end{aligned}$$

□

Note the $\stackrel{*}{=}$ step uses the commutativity of multiplication, hence the above proof does not work when $\mathbb{K} = \mathbb{H}$.

Exercise 1.3. Prove equation 1.4:

$$\mathrm{trace}(A \cdot B) = \mathrm{trace}(B \cdot A)$$

Solution. First,

$$\mathrm{trace}(A \cdot B)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

Next,

$$\begin{aligned} \mathrm{trace}(B \cdot A)_{ii} &= \sum_{i=1}^{n} (B \cdot A)_{ii} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ik} A_{ki} \end{aligned}$$

Carefully reindexing and resumming gives the result. $\square$

Note that the 'careful reindexing and resumming process' implies trace() is invariant under cyclic permutation, e.x.:

$$\mathrm{trace}(A \cdot B \cdot C) = \mathrm{trace}(C \cdot A \cdot B)$$

Exercise 1.4. Let $A, B \in M_n\mathbb{K}$. Prove that if $A \cdot B = I$ then $B \cdot A = I$.

Solution. Note that

$$A \cdot B = I \iff A = B^{-1}.$$

The result follows. $\square$

Exercise 1.5. Suppose that the determinant of $A \in M_n(\mathbb{H})$ were defined as in Equation 1.5. Show for

$$A = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} \in M_n(\mathbb{H})$$

that $\det(A) \neq 0$ but

$$R_A : H^2 \to H^2$$

is not invertible.

Solution. Given the definition from Equation 1.5:

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} \\ &= \mathbf{ij} - \mathbf{ji} \\ &= (1) - (-1) \\ &= 2 \neq 0 \end{aligned}$$

However,

$$R_A((-\mathbf{i}, \mathbf{i})) = (-\mathbf{i}, \mathbf{i}) \cdot \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix} = (-\mathbf{i}^2 + \mathbf{i}^2, -\mathbf{ij} + \mathbf{ij}) = (1 - 1, -\mathbf{k} + \mathbf{k}) = (0, 0)$$

Hence, R_A has a non-zero determiant, but is not invertible as the kernel is non-trivial. Similarly, clearly the columns of A are linearly dependent. $\square$

Exercise 1.6. Find $B \in M_2(\mathbb{R})$ such that $R_B : \mathbb{R}^2 \to \mathbb{R}^2$ is a counter-clockwise rotation through an angle θ.

Solution. Note that we can 'represent' both 1 and i in $M_2(\mathbb{R})$ via

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \text{ and } i \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where the latter works since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

capturing the fact that $i^2 = -1$. Building on this, we can represent any $a + bi \in \mathbb{C}$ via

$$\rho : a + bi \mapsto a \cdot I + b \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Next, note that the function $f_\theta(z) = ze^{i\theta}$ rotates elements counter-clockwise in $\mathbb{C}$ by an angle θ. To see this, letting $z = re^{i\varphi}$,

$$f_\theta(z) = ze^{i\theta} = re^{i\varphi}e^{i\theta} = re^{i(\varphi+\theta)}.$$

Applying ρ gives

$$\begin{aligned} \rho_1(e^{i\theta}) &= \rho_1(\cos(\theta) + i\sin(\theta)) \\ &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = B \end{aligned}$$

□

Exercise 1.7. Describe all elements $A \in \mathrm{GL}_n(\mathbb{R})$ with the property $AB = BA$ for all $B \in \mathrm{GL}_n(\mathbb{R})$.

Solution. Matrices A that commute with all matrices in $\mathrm{GL}_n(\mathbb{R})$ are scalar multiples of the identity

$$A = \lambda I$$

where $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. □

Exercise 1.8. Let $\mathrm{SL}_2(\mathbb{Z})$ denote 2 by 2 matrices with integer entries and determinant 1. Prove that $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Is $SL_n(\mathbb{Z})$ a subgroup of $\mathrm{GL}_n(\mathbb{R})$ in general?

Solution. Fixing $A, B \in \mathrm{SL}_2(\mathbb{Z})$, its clear that $A \cdot B$ must have all integer entries. Since

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

we have that $A \cdot B \in \mathrm{SL}_2(\mathbb{Z})$ (closure). Next, fix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

Using Cramer's Rule to compute the inverse of A we get

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $ad-bc = 1$ since $\det A = 1$, so A^{-1} has all integer entries, and is a member of $\mathrm{SL}_2(\mathbb{Z})$. Therefore, $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. $\square$

The critical step in the above proof is discerning that the factor extracted from A^{-1} is $1/\det A = 1$, which ensures the entries of the inverse are all in $\mathbb{Z}$. This factor is the same for any n, so $\mathrm{SL}_n(\mathbb{Z})$ is always a subgroup of $\mathrm{GL}_n(\mathbb{R})$ for all n.

Exercise 1.9. Describe the block matrix blah blabh blabhj TODO write this out

Solution. Suppose A and B are block matrices in $M_n(\mathbb{K})$, given by

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$$

where $\sum \dim A_i = \dim A = \sum \dim B_i = \dim B$, and $\dim A_i = \dim B_i$ for each i. Then,

$$A \cdot B = \begin{pmatrix} A_1 \cdot B_1 & 0 & \cdots & 0 \\ 0 & A_2 \cdot B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \cdot B_n \end{pmatrix}$$

Which can be applied to the above question to derive a simple answer.

$\square$

Exercise 1.10. If $G_1 \subset \mathrm{GL}_{n_1}(\mathbb{K})$ and $G_2 \subset \mathrm{GL}_{n_2}(\mathbb{K})$ are subgroups, describe a subgroup of $\mathrm{GL}_{n_1+n_2}(\mathbb{K})$ isomorphic to $G_1 \times G_2$.

Solution. Define a map

$$\varphi : G_1 \times G_2 \to \mathrm{GL}_{n_1+n_2}(\mathbb{K})$$

given by

$$(A_1, A_2) \mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

The image of φ is a subset of $\mathrm{GL}_{n_1+n_2}(\mathbb{K})$ since

$$\det \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \det A_1 \cdot \det A_2 \neq 0$$

so $\varphi(A_1, A_2) \in \mathrm{GL}_{n_1+n_2}(\mathbb{K})$. To prove φ is a group homomorphism, observe

$$\begin{aligned} \varphi((A_1, A_2) \cdot (B_1, B_2)) &= \varphi(A_1 \cdot B_1, A_2 \cdot B_2) \\ &= \begin{pmatrix} A_1 \cdot B_1 & 0 \\ 0 & A_2 \cdot B_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \cdot \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \\ &= \varphi(A_1, A_2) \cdot \varphi(B_1, B_2) \end{aligned}$$

hence the result.

□

Exercise 1.11.

Exercise 1.12. Show that for purely imaginary $q_1, q_2 \in \mathbb{H}$, $-\Re(q_1 \cdot q_2)$ is the vector dot product in $\mathbb{R}^3 = \text{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\Im(q_1 \cdot q_2)$ is the vector cross-product.

Solution. First,

$$\begin{aligned} -\Re(q_1 \cdot q_2) &= -\Re((b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \cdot (b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})) \\ &= -\Re((-b_1b_2 - c_1c_2 - d_1d_2) + \dots) \\ &= b_1b_2 + c_1c_2 + d_1d_2 \end{aligned}$$

Next,

$$\begin{aligned} \Im(q_1 \cdot q_2) &= \Im((b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \cdot (b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})) \\ &= (c_1d_2 - d_1c_2)\mathbf{i} + (d_1b_2 - b_1d_2)\mathbf{j} + (b_1c_2 - c_1b_2)\mathbf{j} \end{aligned}$$

Mapping $\text{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to the standard basis in $\mathbb{R}^3$ gives both desired results. □

Exercise 1.13. Prove that non-real elements $q_1, q_2 \in \mathbb{H}$ commute if and only if their imaginary parts are parallel; that is, $\Im(q_1) = \lambda \cdot \Im(q_2)$ for some $\lambda \in \mathbb{R}$.

Solution. ($\implies$) Let $\Im(q_1) = \lambda \cdot \Im(q_2)$ for some $\lambda \in \mathbb{R}$, so that

$$\Im(q_1) = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \lambda b\mathbf{i} + \lambda c\mathbf{j} + \lambda d\mathbf{k} = \Im(q_2)$$

then,

$$\begin{aligned} q_1 \cdot q_2 &= (a_1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \cdot (a_2 + \lambda b\mathbf{i} + \lambda c\mathbf{j} + \lambda d\mathbf{k}) \\ &= \Big(a_1a_2 - \lambda\left(b^2 + c^2 + d^2\right)\Big) b(a_1\lambda + a_2)\mathbf{i} c(a_1\lambda + a_2)\mathbf{j} d(a_1\lambda + a_2)\mathbf{k} \\ &= (a_2 + \lambda\, b\,\mathbf{i} + \lambda\, c\,\mathbf{j} + \lambda\, d\,\mathbf{k}) \cdot (a_1 + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}) \\ &= q_2 \cdot q_1. \end{aligned}$$

($\impliedby$) Let $q_1 \cdot q_2 = q_2 \cdot q_1$ where

$$q_1 = (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}), q_2 = (a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}).$$

Then the following equalities must hold:

□

Exercise 1.14. Charachterize the pairs $q_1, q_2 \in \mathbb{H}$ which anti-commute, that is $q_1q_2 = -q_2q_1$.

Exercise 1.15. If $q \in \mathbb{H}$ satisfies $q\mathbf{i} = \mathbf{i}q$, prove that $q \in \mathbb{C}$.

Solution. Let $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Then,

$$q\mathbf{i} = a\mathbf{i} + b\mathbf{i}\mathbf{i} + c\mathbf{j}\mathbf{i} + d\mathbf{k}\mathbf{i} = -b + a\mathbf{i} + d\mathbf{j} - c\mathbf{k}$$

and

$$\mathbf{i}q = \mathbf{i}a + b\mathbf{i}\mathbf{i} + c\mathbf{i}\mathbf{j} + d\mathbf{i}\mathbf{k} = -b + a\mathbf{i} - d\mathbf{j} + c\mathbf{k}.$$

Identifying terms gives

$$d = -d \implies d = 0$$
$$c = -c \implies c = 0$$

hence $q = a + bi \in \mathbb{C}$.

□

Exercise 1.16. Prove that complex multiplication in $\mathbb{C} \approxeq \mathbb{R}^2$ does not extend to a multiplication operation on $\mathbb{R}^3$ that makes $\mathbb{R}^3$ into a real division algebra.

Solution. Assume such an extension exists. Consider the map analagous to the extension of $\mathbb{R}^2$ given by

$$(a, b, c) \mapsto a + b\mathbf{i} + c\mathbf{j}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = -1$. Then there must exist a linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

with $T^2 = -I$. Represent T by a 3×3 real matrix M. Then -1 is in the spectrum of M^2 since $M^2 = -I$, thus $\pm i$ is in the spectrum of M. Since $\det(M) = \prod \lambda_k$ where λ_k is an eigenvalue of M, there must exist some real value λ such that

$$\det(M) = (i)(-i)(\lambda) = \lambda$$

where λ must be in $\mathbb{R}$ since complex eigenvalues come in pairs. Thus, we must have

$$\det(M^2) = \det(-I) = -1$$

and

$$\det(M^2) = \det(M)^2 = \lambda^2$$

thus $\lambda^2 = -1$, which contradicts λ being real.

□

Exercise 1.17. Describe a subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$ that is isomorphic to $\mathbb{R}^n$ under vector-addition.

Solution. Consider the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R})$$

and the map that takes such matrices to $(x_1, \dots, x_n) \in \mathbb{R}^n$.

□

Exercise 1.18. If $\lambda \in \mathbb{H}$ commutes with every element of $\mathbb{H}$, prove that $\lambda \in \mathbb{R}$.

Solution. Let λ have the property that $\lambda \cdot w = w \cdot \lambda$ for all $w \in \mathbb{H}$. Letting $w = \mathbf{i}$, $\lambda \in \mathbb{C}$ per Exercise 1.15. Letting $\lambda = a + b\mathbf{i}$ and $w = \mathbf{j}$, we must have

$$(a + b\mathbf{i}) \cdot \mathbf{j} = a\mathbf{j} + b\mathbf{k} = a\mathbf{j} - b\mathbf{k} = \mathbf{j}(a + b\mathbf{i})$$

hence $b = -b \implies b = 0$, therefore $\lambda = a \in \mathbb{R}$.

□

Chapter 2

All matrix groups are real matrix groups

2.1 Exercises

Chapter 3

The orthogonal groups

3.1 Exercises

Exercise 3.1. Prove part (4) of Proposition 3.3:

$$\overline{\langle X, Y\rangle} = \langle Y, X\rangle$$

Solution.

$$\begin{aligned}
\overline{\langle X, Y\rangle} &= \overline{\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle} \\
&= \overline{x_1\bar{y_1} + \cdots + x_n\bar{y_n}} \\
&= \overline{x_1\bar{y_1}} + \cdots + \overline{x_n\bar{y_n}} \\
&= y_1\bar{x_1} + \cdots + y_n\bar{x_n} \\
&= \langle (y_1, \ldots, y_n), (x_1, \ldots, x_n)\rangle \\
&= \langle Y, X\rangle
\end{aligned}$$

□

Exercise 3.2. Prove Equations 3.5 and 3.6:

$$(3.5) : \langle X, Y\rangle_{\mathbb{C}} = \langle f(X), f(Y)\rangle_{\mathbb{R}} + \mathbf{i}\langle f(X), f(\mathbf{i}Y)\rangle_{\mathbb{R}}$$

$$(3.6) : |X|_{\mathbb{C}} = |f(X)|_R$$

where

$$f = f_n : \mathbb{C}^n \to \mathbb{R}^{2n}$$

is given by

$$f(a_1 + b_1\mathbf{i}, \ldots, a_n + b_n\mathbf{i}) := (a_1, b_1, \ldots, a_n, b_n).$$

Solution. First, for Equation 3.5,

$$\begin{aligned}
\langle X, Y\rangle_{\mathbb{C}} &= \langle(a_1 + b_1\mathbf{i}, \dots, a_n + b_n\mathbf{i}), (c_1 + d_1\mathbf{i}, \dots, c_n + d_n\mathbf{i})\rangle_{\mathbb{C}} \\
&= (a_1 + b_1\mathbf{i})\overline{(c_1 + d_1\mathbf{i})} + \cdots + (a_n + b_n\mathbf{i})\overline{(c_n + d_n\mathbf{i})} \\
&= (a_1 + b_1\mathbf{i})(c_1 - d_1\mathbf{i}) + \cdots + (a_n + b_n\mathbf{i})(c_n - d_n\mathbf{i}) \\
&= [(a_1c_1 + b_1d_1) + (-a_1d_1 + b_1c_1)\mathbf{i}] + \cdots + [(a_nc_n + b_nd_n) + (-a_nd_n + b_nc_n)\mathbf{i}] \\
&= (a_1c_1 + b_1d_1 + \cdots + a_nc_n + b_nd_n) + (-a_1d_1 + b_1c_1 + \cdots - a_nd_n + b_nc_n)\mathbf{i} \\
&= \langle(a_1, b_1, \dots, a_n, b_n), (c_1, d_1, \dots c_n, d_n)\rangle_{\mathbb{R}} + \mathbf{i}\langle(a_1, b_1, \dots, a_n, b_n), (-d_1, c_1, \dots, -d_n, c_n)\rangle_{\mathbb{R}} \\
&= \langle f(X), f(Y)\rangle_{\mathbb{R}} + \mathbf{i}\langle f(X), \underbrace{f(\mathbf{i}Y)}_{!}\rangle_{\mathbb{R}}
\end{aligned}$$

where the equality of $\underbrace{f(\mathbf{i}Y)}_{!}$ is due to

$$\begin{aligned}
f(\mathbf{i}Y) &= f(\mathbf{i}(c_1 + d_1\mathbf{i}, \dots, c_n + d_n\mathbf{i})) \\
&= f((-d_1 + \mathbf{i}c_1, \dots, -d_n + \mathbf{i}c_n)) \\
&= (-d_1, c_1, \dots, -d_n, c_n)
\end{aligned}$$

Next, for Equation 3.6,

$$\begin{aligned}
|X|_{\mathbb{C}} &= \sqrt{\langle X, X\rangle_{\mathbb{C}}} \\
&= \sqrt{\langle(a_1 + b_1\mathbf{i}, \dots, a_n + b_n\mathbf{i}), (a_1 + b_1\mathbf{i}, \dots, a_n + b_n\mathbf{i})\rangle_{\mathbb{C}}} \\
&= \sqrt{(a_1 + b_1\mathbf{i})(a_1 - b_1\mathbf{i}) + \cdots + (a_n + b_n\mathbf{i})(a_n - b_n\mathbf{i})} \\
&= \sqrt{a_1^2 + b_1^2 + \cdots + a_n^2 + b_n^2} \\
&= \sqrt{\langle(a_1, b_1, \dots, a_n, b_n), (a_1, b_1, \dots, a_n, b_n)\rangle} \\
&= \sqrt{\langle f(X), f(X)\rangle}_{\mathbb{R}} \\
&= |f(X)|_{\mathbb{R}}
\end{aligned}$$

□

Exercise 3.3. Prove Proposition 3.5, that is $\{X_1, \dots X_n\} \in \mathbb{C}^n$ is an orthonormal basis if and only if $\{f(X_1), f(\mathbf{i}X_1), \dots, f(X_n), f(\mathbf{i}X_n)\}$ is an othonormal basis of $\mathbb{R}^{2n}$.

Solution. ($\implies$) Let $\{Y_1, \dots Y_{2n}\} = \{f(X_1), f(\mathbf{i}X_1), \dots, f(X_n), f(\mathbf{i}X_n)\}$ be an orthonorml basis of $\mathbb{R}^{2n}$. Then,

$$\langle Y_i, Y_j\rangle_{\mathbb{R}} = \delta_{ij}$$

where δ_{ij} is the Kronecker Delta. Next, consider

$$\langle f^{-1}(Y_i), f^{-1}(Y_j)\rangle_{\mathbb{R}} = f^{-1}()$$

($\impliedby$) Let $X, Y \in \mathbb{C}^n$ be orthogonal. Then,

$$\langle X, Y\rangle_{\mathbb{C}} = \langle f(X), f(Y)\rangle_{\mathbb{R}} + \mathbf{i}\langle f(X), f(\mathbf{i}Y)\rangle_{\mathbb{R}} = 0$$

Since $\langle f(X), f(\mathbf{i}Y)\rangle_{\mathbb{R}} \in \mathbb{R}$, and hence $\mathbf{i}\langle f(X), f(\mathbf{i}Y)\rangle_{\mathbb{R}} \in \mathbb{C}$, both factors must vanish. Hence, f maps $\mathbb{C}^n$ to an orthonormal basis of $\mathbb{R}^{2n}$.

□

Exercise 3.4. Prove Proposition 3.18, that is, for any $X \subset \mathbb{R}^n$, $\text{Symm}^+(X) \subset \text{Symm}(X)$ is a subgroup with index 1 or 2.

Solution. First, to show $\text{Symm}^+(X)$ is a subgroup, let $A, B \in \text{Symm}^+(X)$. Then,

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1 \implies AB \in \text{Symm}^+(X)$$

and

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A^{-1}) \implies A{-}1 \in \text{Symm}^+(X)$$

so $\text{Symm}^+(X)$ is a subgroup. It's clear that $\text{Symm}^+(X)$ has at most 2 cosets given the symmetry of the determinant, and has 1 if $\text{Symm}^+(X) = \text{Symm}(X)$, hence it's index is 1 or 2. □

Exercise 3.5. Let $A \in \text{GL}_n(\mathbb{K})$. Prove that $A \in \text{O}_n(\mathbb{K})$ if and only if the columns of A are an orthonormal basis of $\mathbb{K}^n$.

Solution. ($\implies$) Let the columns of A form an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{K}^n$. Thus,

$$\langle v_i, v_j\rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker Delta. Next,

$$(A^*A)_{ij} = \sum_{k=1}^{n} A^*_{ij}A_{kj} = \sum_{k=1}^{n} A_{ki}A_{kj}$$

where A_{ki} is the kth component of v_i, hence

$$(A^*A)_{ij} = \langle v_i, v_j\rangle = \delta_{ij} \implies A * A = I$$

($\impliedby$) Fix some $A \in \text{O}_n(\mathbb{K})$ with columns $\{v_1, \ldots, v_n\}$. Since A is an element of a group, A^{-1} is defined, and so the columns are linearly independent. Next,

$$\begin{aligned}(I)_{ij} &= (A^*A)_{ij} \\ &= \langle v_i, v_j\rangle \\ &= \delta_{ij}\end{aligned}$$

Hence, the columns are orthonormal.

□

Exercise 3.6.

1. Show that for every $A \in \text{O}(2) - \text{SO}(2)$, $R_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a flip about some line through the origin. How is this line determined by the angle of A?

2. Let $B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2)$. Assume θ is not an integer multiple of π. Prove that B does not commute with any $A \in \mathrm{O}(2) - \mathrm{SO}(2)$.

Solution. (1). First, note that given some $X = (x_1, x_2) \in \mathbb{R}^2$,

$$(x_1, x_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (x_1, -x_2)$$

hence, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ corresponds to a flip through the x-axis. Per Equation 3.8, we have

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

where $\theta \in [0, 2\pi)$. Using this, observe that

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where the second factor on the RHS is in SO(2) and corresponds to a counter-clockwise rotation of θ. Hence, A is a rotation flip then rotation.

(2) Let $B \in \mathrm{SO}(2)$. Then

$$R_{AB}((x_1, x_2)) = XAB = (x_{1,\theta}, -x_{2,\theta})B = (x_{1,\theta+\varphi}, -x_{2,\theta+\varphi})$$

is given by the flip of the first factor of A, then the rotations θ and φ of the next two factors. However,

$$R_{BA}((x_1, x_2)) = XBA = (x_{1,\varphi}, x_{2,\varphi})A.$$

In this case, $x_{2,\varphi}$ will first be flipped and then rotated by θ, and so we cannot assert that it is equal to $-x_{2,\varphi+\theta}$, hence the matrices cannot commute.

□

Exercise 3.7. Describe the product of any two elements in $O(2)$ in terms of their angles.

Solution. Let $A, B \in \mathrm{SO}(2)$ and $C, D \in O(2) - \mathrm{SO}(2)$. We need to describe R_{AB}, R_{CD}, and R_{AC}. First,

$$\begin{aligned} R_{AB} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\sin\theta\cos\varphi - \sin\varphi\cos\theta \\ \cos\theta\sin\varphi + \cos\varphi\sin\theta & -\sin\theta\sin\varphi + cos\theta\cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\varphi) & \sin(\theta+\varphi) \\ -\sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \end{aligned}$$

□

Chapter 4

The topology of matrix groups

4.1 Exercises

Exercise 4.1. Another exercise goes here.

Solution. Placeholder for your solution. $\square$

Chapter 5

Lie algebras

5.1 Exercises

Exercise 5.1. Another exercise goes here.

Solution. Placeholder for your solution. $\square$

Chapter 6

Matrix exponentiation

6.1 Exercises

Exercise 6.1. Prove Proposition 6.5, that is, suppose that $\sum A_l$ and $\sum B_l$ converge, at least one absolutely. Let $C_l = \sum_{k=0}^{l} A_k B_{l-k}$. Prove that

$$\sum C_l = (\sum A_l)(\sum B_l)$$

Solution.

$$\begin{aligned}
(\sum C_l)_{ij} &= (\sum_{k=0}^{l} A_k B_{l-k})_{ij} \\
&= (\sum_{k=0}^{l} \sum_{r=1} (A_k))
\end{aligned}$$

$\square$

Exercise 6.2. Prove that $(e^A)^* = e^{A^*}$ for all $A \in M_n(\mathbb{K})$.

Solution. Using the fact that $(XY)^* = Y^*X^*$ and $(X+Y)^* = X^* + Y^*$,

$$\begin{aligned}
e^{A^*} &= \sum_{k=0}^{\infty} \frac{(A^*)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\overbrace{A^* \cdot A^* \cdots A^*}^{k-times}}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(\overbrace{A \cdot A \cdots A}^{k-times})^*}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(A^k)^*}{k!} \\
&= \left(\sum_{k=0}^{\infty} \frac{(A^k)}{k!} \right)^* \\
&= (e^A)^*
\end{aligned}$$

□

Exercise 6.3.

1. Let $A = \operatorname{diag}(a_1, \ldots, a_n) \in M_n(\mathbb{R})$. Calculate e^A and give a simple prove that $\det(e^A) = e^{\operatorname{trace}(A)}$ when A is diagonal.

2. Give a simple proof that $\det(e^A) = e^{\operatorname{trace}(A)}$ when A is conjugate to a diagonal matrix.

Solution. (1). Let $A = \operatorname{diag}(a_1, \ldots, a_n) \in M_n(\mathbb{R})$. Then,

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{\operatorname{diag}(a_1, \ldots, a_n)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\operatorname{diag}(a_1^k, \ldots, a_n^k)}{k!} \\ &= \operatorname{diag}\left(\sum_{k=0}^{\infty} \frac{a_1^k}{k!}, \ldots, \sum_{k=0}^{\infty} \frac{a_n^k}{k!} \right) \\ &= \operatorname{diag}(e^{a_1}, \ldots, e^{a_n}). \end{aligned}$$

Thus,

$$\begin{aligned} \det(e^A) &= \det\left(\operatorname{diag}(e^{a_1}, \ldots, e^{a_n})\right) \\ &= \prod_{i=1}^{n} e^{a_i} \\ &= e^{\sum_{j=1}^{n} a_j} \\ &= e^{\operatorname{trace}(A)} \end{aligned}$$

(2). Let $A \in M_n(\mathbb{R})$ be conjugate to a diagonal matrix, that is there exists a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ and an invertible matrix P such that $A = PDP^{-1}$. Then,

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\overbrace{(PDP^{-1}PDP^{-1} \cdots PDP^{-1})}^{k-times}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\overbrace{PD(P^{-1}P)D(P^{-1}P) \cdots (P^{-1}P)DP^{-1})}^{k-times}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} \\ &= P\left(\sum_{k=0}^{\infty} \frac{D^k}{k!} \right) P^{-1} \\ &= Pe^DP^{-1}. \end{aligned}$$

Finishing it off:

$$\begin{aligned}\det(e^A) &= \det(Pe^D P^{-1}) \\ &= \det(P)\cdot\det(e^D)\cdot\det(P^{-1}) \\ &= e^{\mathrm{trace}(D)} \\ &= e^{\mathrm{trace}(A)}\end{aligned}$$

where the last equality holds since similar matrices have the same trace. □

Exercise 6.4. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Compute e^{tA} for arbitrary $t \in \mathbb{R}$.

Solution. First, consider the following table:

k	A^k
0	I
1	A
2	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
3	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
4	I

Hence, A has periodicity 4. Therefore,

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \\ &= \sum_{i=0}^{\infty} \frac{t^{4i}}{i!} I + \sum_{j=0}^{\infty} \frac{t^{4j+1}}{j!} A^{4j+1} + \sum_{l=0}^{\infty} \frac{t^{4l+2}}{l!} A^{4l+1} + \sum_{r=0}^{\infty} \frac{t^{4r+3}}{r!} A^{4r+3} \\ &= \sum_{i=0}^{\infty} \frac{t^{4i}}{i!} I + \sum_{j=0}^{\infty} \frac{t^{4j+1}}{j!} A + \sum_{l=0}^{\infty} \frac{t^{4l+2}}{l!} A^2 + \sum_{r=0}^{\infty} \frac{t^{4r+3}}{r!} A^3 \\ &= \sum_{i=0}^{\infty} \frac{(t^4)^i}{i!} I + t\sum_{j=0}^{\infty} \frac{(t^4)^j}{j!} A + t^2\sum_{l=0}^{\infty} \frac{(t^4)^l}{l!} A^2 + t^3\sum_{r=0}^{\infty} \frac{(t^4)^r}{r!} A^3 \\ &= e^{t^4} I + te^{t^4} A + t^2 e^{t^4} A^2 + t^3 e^{t^4} A^3 \\ &= e^{t^4}\left(I + tA + t^2A^2 + t^3A^3\right) \\ &= e^{t^4}\begin{pmatrix} 1-t^2 & t-t^3 \\ -t+t^3 & 1-t^2 \end{pmatrix} \end{aligned}$$

Noticing that as $t \to 0$ we have $e^{tA} \to I$ is a good sanity check! □

Exercise 6.5. Can a one-parameter group ever cross itself?

Solution. No - a one-parameter group is differentiable, hence any singularity would contradict its differentiability. Alternatively, by Proposition 6.17, every one-parameter group is described by $\gamma(t) = e^{tA}$ for some $A \in gl_n(\mathbb{K})$, and e^{tA} is injective. $\square$

Exercise 6.6. Describe all one-parameter groups of $\mathrm{GL}_1(\mathbb{C})$. Draw several in the x-y plane.

Solution. Let $z = a + b\mathbf{i} \in \mathrm{GL}_1(\mathbb{C})$. Then, any one-parameter group has the form

$$\begin{aligned}\gamma_z(t) &= e^{tz} \\ &= e^{t(a+b\mathbf{i})} \\ &= e^{at}e^{\mathbf{i}bt}\end{aligned}$$

$e^{\mathbf{i}bt}$ can always be identified with a point on the unit circle, scaled by e^{at}. Hence, $\gamma_z(t)$ makes a spiral that "spirals" exponentially faster as t increases that starts at $\gamma_z(0) = 1$. z determines the initial condition and initial "spiral rate".

$\square$

Exercise 6.7. Let $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) : x > 0 \right\}$. Describe the one-parameter groups in G, and draw several on the $xy-$plane.

Solution. First, note that given $A \in G$,

$$A = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} x^2 & y(x+1) \\ 0 & 1 \end{pmatrix}$$

and

$$A^3 = \begin{pmatrix} x^3 & y(x^2+x+1) \\ 0 & 1 \end{pmatrix}$$

It can be show inductively that

$$A^k = \begin{pmatrix} x^k & y(x^{k-1} + \cdots + 1) \\ 0 & 1 \end{pmatrix}$$

and so,

$$\begin{aligned}e^{tA} &= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} x^k & y(x^{k-1} + \cdots + 1) \\ 0 & 1 \end{pmatrix} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} x^k & y\frac{x^k-1}{x-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{tx} & \frac{y}{x-1}\sum_{k=0}^{\infty}(tx)^k/k! - t^k/k! \\ 0 & e^t \end{pmatrix} \\ &= \begin{pmatrix} e^{tx} & \frac{y}{x-1}(e^{tx} - e^t) \\ 0 & e^t \end{pmatrix}\end{aligned}$$

No fucking way am I drawing these. $\square$

Exercise 6.8. Visually describe the path $\gamma(t) = e^{t\mathbf{j}}$ in $Sp(1) \approx S^3$.

Solution. First, note that

k	$\mathbf{j}^k$
0	1
1	$\mathbf{j}$
2	-1
3	1

hence, $\mathbf{j}$ has periodicity 3. Next,

$$\begin{aligned}\gamma(t) &= e^{t\mathbf{j}} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{j}^k \\ &= \sum_{i=0}^{\infty} \frac{t^i}{i!}1 + \sum_{j=0}^{\infty} \frac{t^j}{j!} + \sum_{l=0}^{\infty} \frac{t^l}{l!}\end{aligned}$$

□

Exercise 6.9. Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in gl_n\mathbb{R}$. Compute e^A.

Solution. First, note that

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

is a decomposition of A into two commuting matrices. Thus,

$$\begin{aligned}e^A &= \exp\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ &= \exp\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right) \\ &= \exp\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \\ &= \left(\sum_{k=0}^{\infty} \frac{a^k I^k}{k!}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{b^k}{j!} \exp\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^j\right) \\ &= e^a I \cdot e^{b^4} \begin{pmatrix} 1-b^2 & b-b^3 \\ -b+b^3 & 1-b^2 \end{pmatrix} \\ &= e^{b^4+a} \begin{pmatrix} 1-b^2 & b-b^3 \\ -b+b^3 & 1-b^2 \end{pmatrix}\end{aligned}$$

□

Exercise 6.10. Repeat the previous problem with $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

Chapter 7

Matrix groups are manifolds

7.1 Exercises

Exercise 7.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 8

The Lie bracket

8.1 Exercises

Exercise 8.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 9

Maximal tori

9.1 Exercises

Exercise 9.1. Another exercise goes here.

Solution. Placeholder for your solution. □

Chapter 10

Homogeneous manifolds

10.1 Exercises

Exercise 10.1. Another exercise goes here.

Solution. Placeholder for your solution. $\square$

Chapter 11

Roots

11.1 Exercises

Exercise 11.1. Another exercise goes here.

Solution. Placeholder for your solution. □