HW1 – Part A

Question 1

Model Setup:
Indices:
- Product Type:
    - $i \in \{roll, croissant, bread\}$

    The index *i* represents the set of products produced by Benny's bakery: rolls, croissants, and bread loaves.

- Scenario Index:
    - $s \in \{ref, cf\}$

    The index *s* represents alternative modeling scenarios. The reference scenario *(ref)* corresponds to the baseline problem with no bundling requirement, while the counterfactual scenario *(cf)* imposes the additional constraint that each croissant must be paired with at least one roll. The scenario index is used to evaluate how the optimal production plan and profits change under this counterfactual restriction.

Parameters:
For each product *i*:
- Selling Price:
    - *rev(i) = revenue ($/item)*
    - Roll = 2.25, Croissant = 5.50, and Bread = 10.00

- Variable Cost:
    - $cost(i)$ = cost ($/item)
    - Roll = 1.50, Croissant = 2.00, and Bread = 5.00

- Labor Time Per Item:
    - time(i) = labor time (hours/item)
    - Roll = 1.50, Croissant = 2.25, and Bread = 5.00

- Profit
    - $profit_i = rev_i - cost_i$ = unit profit ($/item)

System- wide:
- Total Available Labor Time:
    - *H = available labor (hours/week)*

- ▪ *H = 40*

Switching:
- - Switch for Bundling Constraint
    - ○ $sw_s \in \{0,1\}$
        - ▪ $sw_s = 0$ means "bundling off"
        - ▪ $sw_s = 1$ means "bundling on"

Each product is described by how much money it brings in, how much it costs to make, and how much labor it requires. Revenue shows the selling price per item, cost reflects the expense of producing one item, and labor time measures the hours needed to make each unit. Profit per item is simply revenue minus cost and indicates how much each product contributes to total profit. The bakery has a fixed amount of labor available, limited to 40 hours per week, which constrains how much it can produce overall. The model also includes a simple on–off switch that determines whether a special sales rule applies. When the switch is off, products can be sold independently. When the switch is on, the bakery must produce at least one roll for every croissant sold. Based on the data, croissants earn the most profit per item, bread earns a moderate amount, and rolls earn the least, while bread requires the most labor time per unit and rolls the least.

Decision Variables:
Production decision variable
- - x_i = production of product *i* (items/week)
- - $x_i \geq 0$

Profit outcome variable
- - *z* = total profit ($/week)

The variable x_i represents the weekly production level of product i, measured in items per week, and captures the bakery's production decisions subject to labor and policy constraints. These variables determine how limited labor hours are allocated across products, with the nonnegativity condition reflecting that negative production is not possible. The variable *z* represents total weekly profit, measured in dollars per week, and aggregates the profit generated from all products based on the chosen production levels. Maximizing *z* is the objective of the model and provides a single summary measure of the bakery's economic performance under each scenario.

Objective function:
Maximize Weekly Profit:
- - $max\ z = \sum_i profit_i * x_i$

This objective function directs the model to choose the combination of weekly production quantities that generates the highest possible total profit. Total profit is calculated as the sum of each product's unit profit multiplied by the number of units produced. In doing so, the model

implicitly weighs the profitability of each product against the constraints on labor and any policy rules in place.

Constraints:
- Labor capacity (hours/week)
  - $\sum_i \text{time}_i \ x_i \leq H$

- Bundling constraint (switch-controlled)
  - $x_{\text{roll}} \geq \text{sw} \cdot x_{\text{croissant}}$

       The model includes two constraints that limit and shape the bakery's production decisions. The labor capacity constraint ensures that the total labor required to produce all items does not exceed the 40 hours of labor available each week, which reflects the bakery's primary physical limitation and forces tradeoffs across products with different labor intensities. The second constraint is a switch-controlled bundling rule that links the production of rolls and croissants. In the reference scenario, the switch is off, making the constraint inactive and allowing products to be produced independently, while in the counterfactual scenario the switch is on, requiring the bakery to produce at least one roll for every croissant, thereby restricting specialization and altering the optimal production mix.

Counterfactual Constraint:
Bundling requirement
- $x_{roll} \geq x_{croissant}$

      Under the counterfactual policy, Benny requires that one roll be sold with every croissant, while rolls may still be sold on their own. This requirement is represented by a simple inequality linking the production of rolls and croissants. The constraint ensures that weekly roll production is at least as large as weekly croissant production, so each croissant is effectively paired with a roll.

Switch-controlled formulation
- $x_{\text{roll}} \geq \text{sw} \cdot x_{\text{croissant}}$, with sw $\in \{0,1\}$

- Every croissant must be sold with at least one roll. Rolls may be sold individually:
  - $x_{roll} \geq x_{croissant}$

      This formulation allows the reference and counterfactual scenarios to be solved within a single model. When the switch is off (sw=0), the constraint is inactive and does not affect production decisions. When the switch is on (sw=1), the bundling requirement is enforced, restricting the bakery's ability to specialize in croissants and changing the optimal production mix.

<u>GAMS Solution</u>:
The GAMS implementation solves the same linear programming model for both the reference and counterfactual cases by looping over the scenario index s. Within this loop, a scalar switch variable, sw, is set equal to the scenario-specific parameter, sw_s. In the reference scenario $sw = 0$, which deactivates the bundling constraint, while in the counterfactual scenario $sw = 1$, which enforces the requirement that rolls must accompany croissants. This approach allows the counterfactual to be evaluated without rewriting the model and directly reflects the instruction to use a switch to enable the counterfactual. Because the model is a continuous linear program with no integer restrictions, the optimal solution may involve fractional production levels, which can be interpreted as average weekly output.

Reference Scenario (Bundling Off: $sw = 0$)
In the reference case, no bundling requirement is imposed, so products can be produced independently. Comparing profit per labor hour shows that croissants provide the highest return on labor, followed by bread, and then rolls. As a result, all available labor is optimally allocated to croissant production.

The optimal solution is:
- $x_{\text{croissant}} = \frac{40}{2.25} = 17.7778$ items/week
- $x_{\text{roll}} = 0$
- $x_{\text{bread}} = 0$

Total weekly profit is:
- $z = 3.50 \times 17.7778 = 62.2222$ dollars/week

All 40 hours of labor are fully utilized, and the labor constraint is binding.

Counterfactual Scenario (Bundling On: $sw = 1$)
In the counterfactual scenario, the bundling constraint requires that roll production be at least as large as croissant production. Since rolls earn a lower profit per labor hour than croissants, the optimal solution satisfies the constraint at equality, meaning rolls are produced only to the extent required by croissant sales.

Each croissant–roll pair uses 3.75 hours of labor and generates $4.25 in profit, yielding a profit per labor hour that remains higher than bread. As a result, bread is not produced in the optimal solution.

The optimal solution is:
- $x_{\text{croissant}} = 40/3.75 = 10.6667$ (items/week)
- $x_{\text{roll}} = 10.6667$ (items/week)
- $x_{\text{bread}} = 0$

Total weekly profit is:
- $z = 4.25 \times 10.6667 = 45.3333$ dollars/week

Again, all 40 hours of labor are fully utilized, and the labor constraint is binding.

Results Summary

Scenario	x_roll (items/week)	x_croissant (items/week)	x_bread (items/week)	Profit z ($/week)	Labor Used (hours/ week)
Counterfactual (sw = 1)	0.0000	17.7778	0.0000	62.2222	40.0000
Counterfactual (sw = 1)	10.6667	10.6667	0.0000	45.3333	40.0000

Imposing the roll-per-croissant requirement reduces the bakery's ability to specialize in the product with the highest profit per labor hour. Labor must be diverted toward lower margin rolls in order to satisfy the bundling rule, which lowers total profit from approximately $62.22 per week in the reference case to $45.33 per week in the counterfactual case, even though total labor utilization remains unchanged.

Question 2

Model Formulation:
Indices/Sets:
- Machines:
  - $m \in \{X1, X2\}$

- Jellybean Colors:
  - $c \in \{yellow, blue, green, orange, purple\}$

      The model uses two indices to organize production decisions. The index m represents the two machines in June's factory, *X1* and *X2*. Although the machines have identical operating capabilities in the baseline model, they are tracked separately so that production limits and allocation decisions can be applied at the machine level. The index c represents the set of jellybean colors that can be produced. Each color earns a different net revenue per bean, which creates differences in profitability and drives how the factory's limited production capacity is allocated across colors.

Parameters:
For each color *c*:
- Net Revenue per Bean
  - rev(c) = net revenue ($/bean)
  - Yellow = 1, Blue = 1.05, Green = 1.07, Orange = 0.95, Purple = 0.9

For each machine *m*:
- Production Rate
  - rate(m) = production rate (beans/hour)
  - $rate(m) = 100 \forall m$

- Available Operating Time:
  - $H(m) =$ available machine time (hours/week)
  - $H(m) = 40 \forall m$

- Weekly Production Capacity:
  - $cap(m) = rate(m) \cdot H(m)$(beans/week)
  - $cap(m) = 100 \times 40 = 4,000 \forall m$

      Net revenue varies slightly across jellybean colors, creating incentives to favor higher-value products. Production, however, is constrained by physical machine limits. Each machine can operate for only 40 hours per week and produces jellybeans at a fixed rate of 100 beans per hour. Together, these parameters imply a weekly production capacity of 4,000 beans per machine, which places a hard upper bound on total output and forces tradeoffs across colors.

Decision variables:
- Production decision variable:
  - $x_{m,c}$ = production of color c on machine m (beans/week)
  - $x_{m,c} \geq 0$

- Profit outcome variable
  - z = total net revenue ($/week)

The variable $x_{m,c}$ represents the weekly number of jellybeans of color c produced on machine m. These variables capture how June allocates each machine's limited weekly production capacity across different jellybean colors, subject to machine capacity constraints. The nonnegativity condition reflects that production quantities cannot be negative. The variable z represents total weekly net revenue, measured in dollars per week, and aggregates the net revenue generated from producing all jellybean colors across both machines. Maximizing z is the objective of the model and provides a single summary measure of the factory's economic performance under the given production constraints.

Objective Function:
Maximize Weekly Net Revenue
- Max $Z = \sum_{m \in M} \sum_{c \in C} r_c \, x_{m,c}$ ($/week)

The objective of the model is to maximize June's total net revenue earned in a week. Total revenue is calculated by multiplying the net revenue per bean for each color by the number of beans produced and summing across all colors and machines. Because revenue is measured in dollars per bean and production is measured in beans per week, the objective value Z represents total weekly net revenue in dollars per week.

Constraints:
Machine capacity (beans/week), for each machine:
- $\sum_{c \in C} x_{m,c} \leq rate(m) * H(m)$

Nonnegativity:
- $x_{m,c} \geq 0 \; \forall \; m, c$

These constraints ensure that June's production decisions are physically feasible. Each machine can operate for at most 40 hours per week and produces jellybeans at a fixed rate of 100 beans per hour, which limits weekly output on each machine to 4,000 beans and forces tradeoffs in how production capacity is allocated across colors with different revenues. In addition, production quantities are required to be nonnegative, ruling out infeasible solutions with negative output.

June's profit-maximizing production profile:

June's profit-maximizing production profile is to specialize entirely in the highest revenue jellybean color, green. Because both machines have identical capacities of 4,000 beans per week and there are no restrictions on color production, the optimal solution allocates all available production capacity on both machines to green jellybeans. This results in a total weekly output of **8,000 green jellybeans** and generates **maximum net revenue of $8,560 per week**.

How June's production profile changes under 5% constraint:

Imposing the ±5 percent deviation constraint eliminates June's ability to specialize in the highest-margin product. Instead, production must be tightly clustered across all five jellybean colors, with no color allowed to differ from another by more than 5 percent. The profit-maximizing response is to place the two highest-revenue colors, green and blue, at the upper bound of the allowable range, while pushing the remaining three colors to the lower bound. Given total weekly capacity of 8,000 beans, this results in approximately 1,647 beans each of green and blue and 1,569 beans each of yellow, orange, and purple. This constraint forces output to be spread across all colors rather than concentrated in green, sharply reducing the value of production flexibility. Weekly profit falls from $8,560 in the unconstrained case to about **$7,962.72** under the near-equality requirement, **a decline of roughly $597 per week**, or about 7 percent.

How June's production profile changes under machine color restriction:

Imposing machine–color restrictions prevents June from producing only green jellybeans across both machines. Under these constraints, the profit-maximizing strategy is for Machine X1, which can produce yellow, blue, and green, to specialize entirely in green jellybeans, the highest-revenue option in its feasible set. Machine X2, which is limited to yellow, orange, and purple, optimally specializes in yellow jellybeans, since yellow dominates the other available colors in net revenue. This results in production of 4,000 green jellybeans on X1 and 4,000 yellow jellybeans on X2 each week, with no production of the remaining colors. Weekly profit under this restriction is **$8,280**, representing a **reduction of $280 relative to the unconstrained baseline profit of $8,560**. The loss arises because one machine is forced to substitute from green to a lower-margin product, illustrating how technological or operational constraints reduce the value of specialization even when total capacity is unchanged..