# Modelling and Optimizing a Golf Swing with Lagrangian Analysis

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PHYS 350, Applications of Classical Mechanics

April 10, 2022

## 1 Abstract

The purpose of this project is to analyze the dynamics of a golf swing with Lagrangian mechanics. We model a golf swing with a rigid rod for the golfer's arms, a rigid rod for the golf shaft, and a point mass for the golf clubhead. Through various numerical calculations and simulations, we perform a rigorous analysis to find the optimal parameters for a golf swing with this model to maximize the speed of the clubhead at the point of contact with the ball. We find that for this model, the optimal wrist flick angle changes with the golfer's arms' angular acceleration, with a higher acceleration requiring a delayed wrist flick time. We also find that simply passively dropping the arms results in a faster clubhead speed for small values of angular acceleration from the golfer.

## 2 Background and Motivation

Golf is an individual sport in which an athlete hits a ball with a club towards a target. A major component of this sport involves optimizing the golf club and golf swing in hopes of maximizing the speed of the clubhead at the point of contact with the ball to drive the ball as far as possible. The goal of this project is to model the dynamics of a golf swing with Lagrangian mechanics, and then analyze the system to identify how to optimize the golf swing under this model.

## 3 Problem and Model Definition

Modeling the full complexity of the dynamics of a golf swing is very challenging, as it would involve analysis of biomechanics, material science, nonlinear dynamics, and more. We will need to make simplifications and assumptions about the problem in order to analyze it, and this comes with a number of limitations, which we describe later in this report. In order to meaningfully model the dynamics of a golf swing, it is important to first understand what a typical golf swing looks like. Figure 1 shows a high-level overview of a typical golf swing. We have a golfer holding a golf club with their arms, and we have a golf ball directly below the golfer sitting on a golf tee that allows us to change the height of the golf ball depending on the situation. The golfer starts in a stationary backswing position, with the arms raised and wrist bent such that there is a $90°$ angle between the arms and the club. Then, the golfer begins to swing the arms while keeping the wrist angle fixed at $90°$. At some point on the downswing, the golfer releases their wrist tension, allowing the wrist angle to change as the club swings. This is an important aspect of golfing that allows golfers to achieve higher club speeds, and conventional wisdom suggests that the wrist angle be released as late as possible to maximize clubhead speed. Finally, the club makes contact with the ball when the clubhead is directly under the golfer. Note that the height of the clubhead does not matter because we can vary the golf ball's height with the tee in advance.

Figure 2 shows how we will be modelling a golf swing. We define our IRF to be at position P, which is the position of the golfer's shoulder, with X to the right and Y to the top. We assume that the golfer's

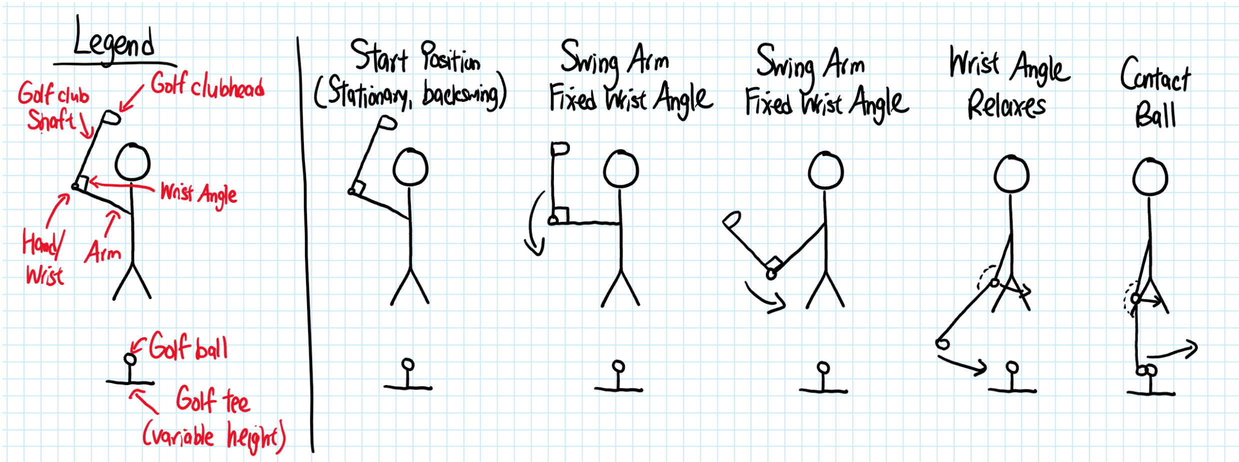


Figure 1: High-level overview of a typical golf swing.

shoulder will stay fixed in place at a height $y_{ball}$ above the golf ball, and the golfer's arms will rotate around this point. We define point R to be the position of the golfer's hands/wrist and point Q to be the position of the golf clubhead. We model the arms as a uniform, rigid bar with mass $M_A$ and length $L_A$, the club shaft as a uniform, rigid bar with mass $M_S$ and length $L_S$, and the clubhead as a point mass $M_C$ at the tip of the club shaft. Thus, our goal is the maximize the speed of the clubhead $v_Q$ at the point of contact with the golf ball at $(X = 0, Y = -y_{ball})$, where we can vary $y_{ball}$ with a tee. Thus, we simply want to maximize $v_Q$ when $x_Q = 0$ and then we can adjust $y_{ball}$ to match $y_Q$ to make contact.

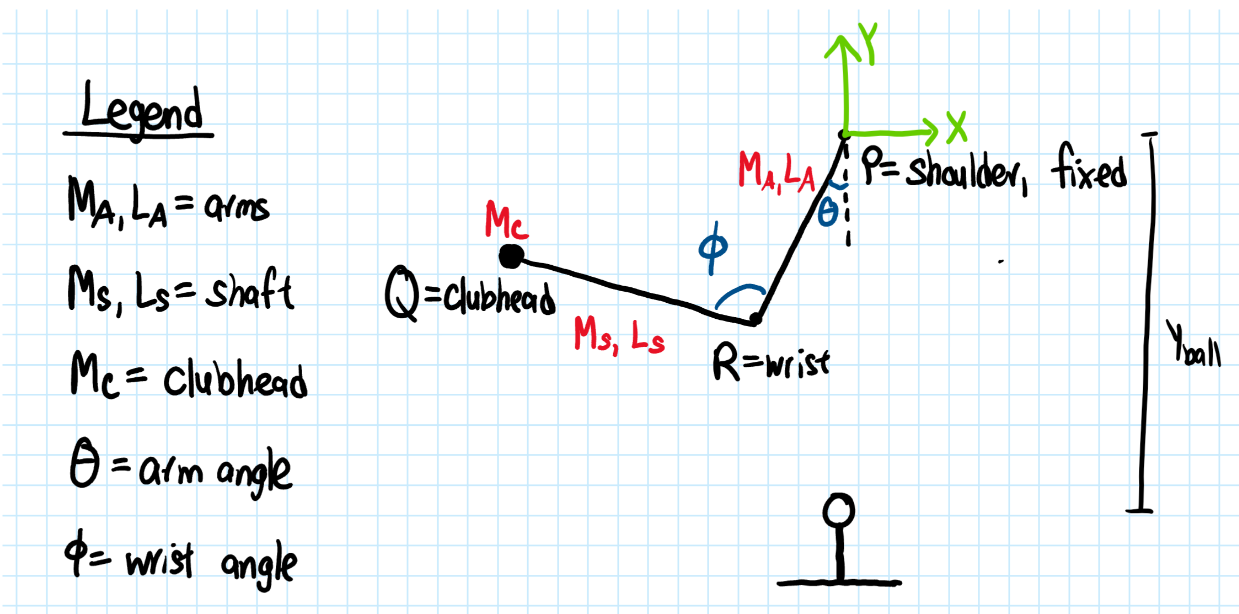


Figure 2: Physics model of a golf swing used in this project.

From here, we still need to model how the golfer acts on the system over time. There are two main components we will model. First, we need to model the action on the arms that the golfer makes. We will model this passively (golfer simply lets the arms drop to swing) or with an external field (golfer controls the arms with a given $\dot{\theta}(t)$). We will model both of these systems to compare their behavior. As a simplification of the external field case, we will assume that the golfer will put a constant angular acceleration on their arms so that $\ddot{\theta}(t) = -a$ and $\dot{\theta}(t) = -at$. Second, we need to model the wrist angle control from the golfer. We model this by breaking up the model into two stages. From $\theta(t=0) = \theta_0$ to $\theta(t = t_{fixed}) = \theta_{fixed}$, we will have a fixed wrist angle $\phi = 90°$ throughout the motion. From $\theta(t = t_{fixed}) = \theta_{fixed}$ onwards, we will have a passively varying wrist angle $\phi$ (the wrist point R will act as a free revolute joint).

In summary, we will be modeling two distinct systems. We will call the first system the passive arm system, as the arms are passively controlled by the golfer, and we will call the second system the controlled arm system, as the arms are actively controlled by the golfer. In each system, we have two stages. The wrist angle is first fixed at 90° and then the wrist angle is allowed to change freely/passively.

# 4 Model Analysis

With this model, we analyze its dynamics using Lagrangian mechanics. We will write out the Lagrangian for the system and use the Euler-Lagrange equations to give us differential equations of motion, which we can use to run simulations. First, we write out the equations of motion for the system in the general case as functions of $\phi$ and $\theta$:

$$
\begin{aligned}
x_R &= -L_A \sin\theta \\
y_R &= -L_A \cos\theta \\
x_Q &= -L_A \sin\theta - L_S \sin(\phi - \theta) \\
y_Q &= -L_A \cos\theta + L_S \cos(\phi - \theta) \\
\dot{x}_R &= -L_A \cos\theta\dot{\theta} \\
\dot{y}_R &= L_A \sin\theta\dot{\theta} \\
\dot{x}_Q &= -L_A \cos\theta\dot{\theta} - L_S \cos(\phi - \theta)\left(\dot{\phi} - \dot{\theta}\right) \\
\dot{y}_Q &= L_A \sin\theta\dot{\theta} - L_S \sin(\phi - \theta)\left(\dot{\phi} - \dot{\theta}\right) \\
v_R^2 &= \dot{x}_R^2 + \dot{y}_R^2 \\
&= \left(L_A \cos\theta\dot{\theta}\right)^2 + \left(L_A \sin\theta\dot{\theta}\right)^2 \\
&= L_A^2\dot{\theta}^2 \\
v_Q^2 &= \dot{x}_Q^2 + \dot{y}_Q^2 \\
&= \left(-L_A \cos\theta\dot{\theta} - L_S \cos(\phi - \theta)\left(\dot{\phi} - \dot{\theta}\right)\right)^2 + \left(L_A \sin\theta\dot{\theta} - L_S \sin(\phi - \theta)\left(\dot{\phi} - \dot{\theta}\right)\right)^2 \\
&= \left(-L_A \cos\theta\dot{\theta}\right)^2 + \left(L_S \cos(\phi - \theta)\left(\dot{\phi} - \dot{\theta}\right)\right)^2 + 2L_A L_S \cos\theta\cos(\phi - \theta)\dot{\theta}\left(\dot{\phi} - \dot{\theta}\right) \\
&\quad + \left(L_A \sin\theta\dot{\theta}\right)^2 + \left(L_S \sin(\phi - \theta)\left(\dot{\phi} - \dot{\theta}\right)\right)^2 - 2L_A L_S \sin\theta\sin(\phi - \theta)\dot{\theta}\left(\dot{\phi} - \dot{\theta}\right) \\
&= L_A^2\dot{\theta}^2 + L_S^2\left(\dot{\phi} - \dot{\theta}\right)^2 + 2L_A L_S\dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)(\cos\theta\cos(\phi - \theta) - \sin\theta\sin(\phi - \theta)) \\
&= L_A^2\dot{\theta}^2 + L_S^2\left(\dot{\phi} - \dot{\theta}\right)^2 + 2L_A L_S\dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\cos(\theta + (\phi - \theta)) \\
&= L_A^2\dot{\theta}^2 + L_S^2\left(\dot{\phi} - \dot{\theta}\right)^2 + 2L_A L_S\dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\cos\phi
\end{aligned}
$$

With this, we can write out the kinetic energy and potential energy for the arm, shaft, and clubhead as functions of *phi* and $\theta$. The arm has zero speed at point P, so we use $O = P$ for the kinetic energy of the arm. The shaft does not have zero speed anywhere in the general case, so we use the center of mass

$O = CM - S$.

$$\begin{aligned}
T_A &= \frac{1}{2}\vec{\Omega}_A \hat{I}_A^{(P)} \vec{\Omega}_A \\
&= \frac{1}{2}\dot{\theta}^2 \left( I_{CM-A} + M_A \left(\frac{L_A}{2}\right)^2 \right) \\
&= \frac{1}{2}\dot{\theta}^2 \left( \frac{M_A L_A^2}{12} + \frac{M_A L_A^2}{4} \right) \\
&= \frac{1}{6} M_A L_A^2 \dot{\theta}^2 \\
x_{CM-S} &= -L_A \sin\theta - \frac{L_S}{2}\sin(\phi - \theta) \\
y_{CM-S} &= -L_A \cos\theta + \frac{L_S}{2}\cos(\phi - \theta) \\
v_{CM-S}^2 &= L_A^2\dot{\theta}^2 + \left(\frac{L_S}{2}\right)^2 \left(\dot{\phi} - \dot{\theta}\right)^2 + 2L_A \left(\frac{L_S}{2}\right) \dot{\theta} \left(\dot{\phi} - \dot{\theta}\right)\cos\phi \\
&= L_A^2\dot{\theta}^2 + \frac{L_S^2}{4}\left(\dot{\phi} - \dot{\theta}\right)^2 + L_A L_S \dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\cos\phi \\
T_S &= \frac{1}{2} M_S v_{CM-S}^2 + \frac{1}{2}\vec{\Omega}_S \hat{I}_S^{(CM-S)} \vec{\Omega}_S \\
&= \frac{1}{2} M_S \left( L_A^2\dot{\theta}^2 + \frac{L_S^2}{4}\left(\dot{\phi} - \dot{\theta}\right)^2 + L_A L_S \dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\cos\phi \right) + \frac{1}{2}\left(\dot{\phi} - \dot{\theta}\right)^2 \left(\frac{M_S L_S^2}{12}\right) \\
&= \frac{1}{2} M_S \left( L_A^2\dot{\theta}^2 + \frac{L_S^2}{4}\left(\dot{\phi} - \dot{\theta}\right)^2 + L_A L_S \dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\cos\phi + \frac{L_S^2}{12}\left(\dot{\phi} - \dot{\theta}\right)^2 \right) \\
&= \frac{1}{2} M_S \left( L_A^2\dot{\theta}^2 + \frac{L_S^2}{3}\left(\dot{\phi} - \dot{\theta}\right)^2 + L_A L_S \dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\cos\phi \right) \\
T_C &= \frac{1}{2} M_C v_Q^2 \\
&= \frac{1}{2} M_C \left( L_A^2\dot{\theta}^2 + L_S^2\left(\dot{\phi} - \dot{\theta}\right)^2 + 2L_A L_S \dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\cos\phi \right) \\
U_A &= M_A g y_{CM-A} = M_A g \left( -\frac{L_A}{2}\cos\theta \right) \\
U_S &= M_S g y_{CM-S} = M_S g \left( -L_A\cos\theta + \frac{L_S}{2}\cos(\phi - \theta) \right) \\
U_C &= M_C g y_Q = M_C g \left( -L_A\cos\theta + L_S\cos(\phi - \theta) \right)
\end{aligned}$$

Putting this together, we get:

$$
\begin{aligned}
T &= T_A + T_S + T_C \\
&= \left(\frac{1}{6}M_A L_A^2 \dot{\theta}^2\right) + \left(\frac{1}{2}M_S\left(L_A^2\dot{\theta}^2 + \frac{L_S^2}{3}\left(\dot{\phi}-\dot{\theta}\right)^2 + L_A L_S \dot{\theta}\left(\dot{\phi}-\dot{\theta}\right)\cos\phi\right)\right) \\
&\quad + \left(\frac{1}{2}M_C\left(L_A^2\dot{\theta}^2 + L_S^2\left(\dot{\phi}-\dot{\theta}\right)^2 + 2L_A L_S\dot{\theta}\left(\dot{\phi}-\dot{\theta}\right)\cos\phi\right)\right) \\
&= \left(\frac{M_A}{6}+\frac{M_S}{2}+\frac{M_C}{2}\right)L_A^2\dot{\theta}^2 + \left(\frac{M_S}{6}+\frac{M_C}{2}\right)L_S^2\left(\dot{\phi}-\dot{\theta}\right)^2 + \left(\frac{M_S}{2}+M_C\right)L_A L_S\dot{\theta}\left(\dot{\phi}-\dot{\theta}\right)\cos\phi \\
U &= U_A + U_S + U_C \\
&= \left(M_A g\left(-\frac{L_A}{2}\cos\theta\right)\right) + \left(M_S g\left(-L_A\cos\theta + \frac{L_S}{2}\cos(\phi-\theta)\right)\right) + \left(M_C g\left(-L_A\cos\theta + L_S\cos(\phi-\theta)\right)\right) \\
&= \left(-\frac{M_A}{2} - M_S - M_C\right)gL_A\cos\theta + \left(\frac{M_S}{2}+M_C\right)gL_S\cos(\phi-\theta)
\end{aligned}
$$

Next, we solve for the Euler-Lagrange equations:

$$
\begin{aligned}
\mathcal{L} &= T - U \\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= 2\left(\frac{M_A}{6}+\frac{M_S}{2}+\frac{M_C}{2}\right)L_A^2\dot{\theta} - 2\left(\frac{M_S}{6}+\frac{M_C}{2}\right)L_S^2\left(\dot{\phi}-\dot{\theta}\right) + \left(\frac{M_S}{2}+M_C\right)L_A L_S\left(\dot{\phi}-2\dot{\theta}\right)\cos\phi \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) &= 2\left(\frac{M_A}{6}+\frac{M_S}{2}+\frac{M_C}{2}\right)L_A^2\ddot{\theta} - 2\left(\frac{M_S}{6}+\frac{M_C}{2}\right)L_S^2\left(\ddot{\phi}-\ddot{\theta}\right) + \left(\frac{M_S}{2}+M_C\right)L_A L_S\left(\ddot{\phi}-2\ddot{\theta}\right)\cos\phi \\
&\qquad - \left(\frac{M_S}{2}+M_C\right)L_A L_S\left(\dot{\phi}-2\dot{\theta}\right)\sin\phi\dot{\phi} \\
\frac{\partial \mathcal{L}}{\partial \theta} &= -\left(\frac{M_A}{2}+M_S+M_C\right)gL_A\sin\theta - \left(\frac{M_S}{2}+M_C\right)gL_S\sin(\phi-\theta) \\
\frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= 2\left(\frac{M_S}{6}+\frac{M_C}{2}\right)L_S^2\left(\dot{\phi}-\dot{\theta}\right) + \left(\frac{M_S}{2}+M_C\right)L_A L_S\dot{\theta}\cos\phi \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) &= 2\left(\frac{M_S}{6}+\frac{M_C}{2}\right)L_S^2\left(\ddot{\phi}-\ddot{\theta}\right) + \left(\frac{M_S}{2}+M_C\right)L_A L_S\ddot{\theta}\cos\phi \\
\frac{\partial \mathcal{L}}{\partial \phi} &= -\left(\frac{M_S}{2}+M_C\right)L_A L_S\dot{\theta}\left(\dot{\phi}-\dot{\theta}\right)\sin\phi + \left(\frac{M_S}{2}+M_C\right)gL_S\sin(\phi-\theta)
\end{aligned}
$$

From here, we need to apply system and timing specific boundary conditions.

## 4.1 Passive Arm System

In this system, we have the following initial, boundary, and external conditions:

- $\phi(t) = 90^\circ$ for $0 \le t \le t_{fixed}$
- $\phi(t)$ is a degree of freedom for $t > t_{fixed}$
- $\phi(t_{fixed}) = 90^\circ$ because $\phi(t)$ is continuous
- $\theta(t)$ is a degree of freedom for both $0 \le t \le t_{fixed}$ and $t > t_{fixed}$
- $\theta(0) = \theta_0$, which is a backswing angle parameter
- $\theta(t_{fixed}^+) = \theta(t_{fixed}^-)$ because $\theta(t)$ is continuous

Next, we need to look at the two stages of the swing.

### 4.1.1 Fixed Wrist: $0 \leq t \leq t_{fixed}$

In this case, the problem is completely defined by $\theta(t)$, so $s = 1$ and $q = (\theta)$.

$$
\begin{aligned}
\phi(t) &= 90^\circ \\
\dot{\phi}(t) &= \ddot{\phi}(t) = 0 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) &= 2\left(\frac{M_A}{6} + \frac{M_S}{2} + \frac{M_C}{2}\right) L_A^2 \ddot{\theta} - 2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2 \left((0) - \ddot{\theta}\right) + \left(\frac{M_S}{2} + M_C\right) L_A L_S \left((0) - 2\ddot{\theta}\right) \cos(90^\circ) \\
&= \left(2\left(\frac{M_A}{6} + \frac{M_S}{2} + \frac{M_C}{2}\right) L_A^2 + 2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2\right) \ddot{\theta} \\
&= \left(\left(\frac{M_A}{3} + M_S + M_C\right) L_A^2 + \left(\frac{M_S}{3} + M_C\right) L_S^2\right) \ddot{\theta} \\
\frac{\partial \mathcal{L}}{\partial \theta} &= -\left(\frac{M_A}{2} + M_S + M_C\right) g L_A \sin\theta - \left(\frac{M_S}{2} + M_C\right) g L_S \sin((0) - \theta) \\
&= -\left(\frac{M_A}{2} + M_S + M_C\right) g L_A \sin\theta + \left(\frac{M_S}{2} + M_C\right) g L_S \sin(\theta) \\
&= \left(-\left(\frac{M_A}{2} + M_S + M_C\right) L_A + \left(\frac{M_S}{2} + M_C\right) L_S\right) g \sin(\theta)
\end{aligned}
$$

Putting this together into the Euler-Lagrange equations, we get:

$$
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}
$$

$$
\left(\left(\frac{M_A}{3} + M_S + M_C\right) L_A^2 + \left(\frac{M_S}{3} + M_C\right) L_S^2\right) \ddot{\theta} = \left(-\left(\frac{M_A}{2} + M_S + M_C\right) L_A + \left(\frac{M_S}{2} + M_C\right) L_S\right) g \sin(\theta)
$$

We cannot solve this ODE analytically, but we can solve it numerically, with initial condition $\theta(0) = \theta_0$.

### 4.1.2 Loose Wrist: $t > t_{fixed}$

In this case, both $\theta$ and $\phi$ are free to change, so $s = 2$ and $q = (\theta, \phi)$.

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) &= 2\left(\frac{M_A}{6} + \frac{M_S}{2} + \frac{M_C}{2}\right) L_A^2 \ddot{\theta} - 2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2 \left(\ddot{\phi} - \ddot{\theta}\right) + \left(\frac{M_S}{2} + M_C\right) L_A L_S \left(\ddot{\phi} - 2\ddot{\theta}\right) \cos\phi \\
&\quad - \left(\frac{M_S}{2} + M_C\right) L_A L_S \left(\dot{\phi} - 2\dot{\theta}\right) \sin\phi \dot{\phi} \\
&= E\ddot{\theta} + F\ddot{\phi} + G\dot{\phi}\dot{\theta} + H\dot{\phi}^2 \\
E &= 2\left(\frac{M_A}{6} + \frac{M_S}{2} + \frac{M_C}{2}\right) L_A^2 + 2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2 - 2\left(\frac{M_S}{2} + M_C\right) L_A L_S \cos\phi \\
&= \left(\frac{M_A}{3} + M_S + M_C\right) L_A^2 + \left(\frac{M_S}{3} + M_C\right) L_S^2 - \left(M_S + 2M_C\right) L_A L_S \cos\phi \\
F &= -2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2 + \left(\frac{M_S}{2} + M_C\right) L_A L_S \cos\phi \\
&= -\left(\frac{M_S}{3} + M_C\right) L_S^2 + \left(\frac{M_S}{2} + M_C\right) L_A L_S \cos\phi \\
G &= 2\left(\frac{M_S}{2} + M_C\right) L_A L_S \sin\phi \\
&= \left(M_S + 2M_C\right) L_A L_S \sin\phi \\
H &= -\left(\frac{M_S}{2} + M_C\right) L_A L_S \sin\phi \\
\frac{\partial \mathcal{L}}{\partial \theta} &= -\left(\frac{M_A}{2} + M_S + M_C\right) g L_A \sin\theta - \left(\frac{M_S}{2} + M_C\right) g L_S \sin(\phi - \theta) \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) &= 2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2 \left(\ddot{\phi} - \ddot{\theta}\right) + \left(\frac{M_S}{2} + M_C\right) L_A L_S \ddot{\theta} \cos\phi \\
&= \left(\left(\frac{M_S}{2} + M_C\right) L_A L_S \cos\phi - 2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2\right) \ddot{\theta} + \left(2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2\right) \ddot{\phi} \\
&= \left(\left(\frac{M_S}{2} + M_C\right) L_A L_S \cos\phi - \left(\frac{M_S}{3} + M_C\right) L_S^2\right) \ddot{\theta} + \left(\frac{M_S}{3} + M_C\right) L_S^2 \ddot{\phi} \\
\frac{\partial \mathcal{L}}{\partial \phi} &= -\left(\frac{M_S}{2} + M_C\right) L_A L_S \dot{\theta} \left(\dot{\phi} - \dot{\theta}\right) \sin\phi + \left(\frac{M_S}{2} + M_C\right) g L_S \sin(\phi - \theta)
\end{aligned}
$$

From the Euler-Lagrange equations, we know that:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) &= \frac{\partial \mathcal{L}}{\partial \theta} \\
E\ddot{\theta} + F\ddot{\phi} + G\dot{\phi}\dot{\theta} + H\dot{\phi}^2 &= -\left(\frac{M_A}{2} + M_S + M_C\right) g L_A \sin\theta - \left(\frac{M_S}{2} + M_C\right) g L_S \sin(\phi - \theta) \\
E\ddot{\theta} + F\ddot{\phi} &= Z \\
Z &= -\left(\frac{M_A}{2} + M_S + M_C\right) g L_A \sin\theta - \left(\frac{M_S}{2} + M_C\right) g L_S \sin(\phi - \theta) - G\dot{\phi}\dot{\theta} - H\dot{\phi}^2
\end{aligned}
$$

We also know that:

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi} \\
&J\ddot{\theta} + K\ddot{\phi} = -\left(\frac{M_S}{2} + M_C\right) L_A L_S \dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\sin\phi + \left(\frac{M_S}{2} + M_C\right) g L_S \sin(\phi - \theta) \\
&J\ddot{\theta} + K\ddot{\phi} = P \\
&\qquad J = \left(\frac{M_S}{2} + M_C\right) L_A L_S \cos\phi - \left(\frac{M_S}{3} + M_C\right) L_S^2 \\
&\qquad K = \left(\frac{M_S}{3} + M_C\right) L_S^2 \\
&\qquad P = -\left(\frac{M_S}{2} + M_C\right) L_A L_S \dot{\theta}\left(\dot{\phi} - \dot{\theta}\right)\sin\phi + \left(\frac{M_S}{2} + M_C\right) g L_S \sin(\phi - \theta)
\end{aligned}
$$

We cannot solve these coupled ODEs analytically, but we can solve them numerically, with initial conditions $\theta(t_{fixed}^{+}) = \theta(t_{fixed}^{-})$ and $\phi(t_{fixed}) = 90°$.

## 4.2 Controlled Arm System

In this system, we have the following initial, boundary, and external conditions:

- $\phi(t) = 90°$ for $0 \leq t \leq t_{fixed}$
- $\phi(t)$ is a degree of freedom for $t > t_{fixed}$
- $\phi(t_{fixed}) = 90°$ because $\phi(t)$ is continuous
- $\dot{\theta}(t) = -at$ with a constant angular acceleration of $a$ for all $t$
- $\theta(0) = \theta_0$ which is a backswing angle parameter

Next, we need to look at the two stages of the swing.

### 4.2.1 Fixed Wrist: $0 \leq t \leq t_{fixed}$

In this case, the problem is completely defined by the external field $\theta(t)$, so $s = 0$.

$$
\begin{aligned}
\dot{\theta}(t) &= -at \\
\theta(t) &= -\frac{1}{2}at^2 + \theta_0 \\
\dot{\phi}(t) &= 0 \\
\phi(t) &= 90°
\end{aligned}
$$

#### 4.2.2 Loose Wrist: $t > t_{fixed}$

In this case, $\theta(t)$ is still controlled by the external field, but $\phi(t)$ is free to change, so we have $s = 1$, $q = (\phi)$.

$$\begin{aligned}
\theta(t) &= -\frac{1}{2}at^2 + \theta_0 \\
\dot{\theta}(t) &= -at \\
\ddot{\theta}(t) &= -a \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) &= 2\left(\frac{M_S}{6} + \frac{M_C}{2}\right) L_S^2 \left(\ddot{\phi} - \ddot{\theta}\right) + \left(\frac{M_S}{2} + M_C\right) L_A L_S \ddot{\theta} \cos\phi \\
&= \left(\frac{M_S}{3} + M_C\right) L_S^2 \ddot{\phi} - \left(\frac{M_S}{3} + M_C\right) L_S^2 \ddot{\theta} + \left(\frac{M_S}{2} + M_C\right) L_A L_S \ddot{\theta} \cos\phi \\
&= \left(\frac{M_S}{3} + M_C\right) L_S^2 \ddot{\phi} - \left(\frac{M_S}{3} + M_C\right) L_S^2 (-a) + \left(\frac{M_S}{2} + M_C\right) L_A L_S (-a) \cos\phi \\
\frac{\partial \mathcal{L}}{\partial \phi} &= -\left(\frac{M_S}{2} + M_C\right) L_A L_S \dot{\theta} \left(\dot{\phi} - \dot{\theta}\right) \sin\phi + \left(\frac{M_S}{2} + M_C\right) g L_S \sin(\phi - \theta) \\
&= -\left(\frac{M_S}{2} + M_C\right) L_A L_S (-at) \left(\dot{\phi} - (-at)\right) \sin\phi + \left(\frac{M_S}{2} + M_C\right) g L_S \sin\left(\phi - \left(-\frac{1}{2}at^2 + \theta_0\right)\right)
\end{aligned}$$

Putting this together, we get:

$$\begin{aligned}
\ddot{\phi} &= \frac{W + \left(\frac{M_S}{3} + M_C\right) L_S^2 (-a) - \left(\frac{M_S}{2} + M_C\right) L_A L_S (-a) \cos\phi}{\left(\frac{M_S}{3} + M_C\right) L_S^2} \\
W &= -\left(\frac{M_S}{2} + M_C\right) L_A L_S (-at) \left(\dot{\phi} - (-at)\right) \sin\phi + \left(\frac{M_S}{2} + M_C\right) g L_S \sin\left(\phi - \left(-\frac{1}{2}at^2 + \theta_0\right)\right)
\end{aligned}$$

We cannot solve this ODE analytically, but we can solve it numerically, with initial condition $\phi(t_{fixed}) = 90°$.

## 5 Simulation

In this section, we describe how we can model the system dynamics numerically to simulate the system forward to gain insight about the motion. One approach to solve the ODEs numerically is with a simple forward Euler method. Let:

$$\begin{aligned}
\ddot{\theta}_t &= f(\theta_t, \dot{\theta}_t, \phi_t, \dot{\phi}_t, t) \\
\ddot{\phi}_t &= g(\theta_t, \dot{\theta}_t, \phi_t, \dot{\phi}_t, t)
\end{aligned}$$

Then:

$$\begin{aligned}
\dot{\theta}_{t+dt} &= \dot{\theta}_t + dt f(\theta_t, \dot{\theta}_t, \phi_t, \dot{\phi}_t, t) \\
\theta_{t+dt} &= \theta_t + dt\dot{\theta}_t \\
\dot{\phi}_{t+dt} &= \dot{\phi}_t + dt g(\theta_t, \dot{\theta}_t, \phi_t, \dot{\phi}_t, t) \\
\phi_{t+dt} &= \phi_t + dt\dot{\phi}_t
\end{aligned}$$

We can potentially improve the accuracy of the numerical solution with higher-order methods such as an improved Euler method that uses variables with a $*$ superscript to denote intermediate variables:

$$\begin{aligned}
\dot{\theta}^*_{t+dt} &= \dot{\theta}_t + dt f(\theta_t, \dot{\theta}_t, \phi_t, \dot{\phi}_t, t) \\
\dot{\phi}^*_{t+dt} &= \dot{\phi}_t + dt g(\theta_t, \dot{\theta}_t, \phi_t, \dot{\phi}_t, t) \\
\theta^*_{t+dt} &= \theta_t + dt\dot{\theta}_t \\
\phi^*_{t+dt} &= \phi_t + dt\dot{\phi}_t \\
\dot{\theta}_{t+dt} &= \dot{\theta}_t + dt\frac{f(\theta_t, \dot{\theta}_t, \phi_t, \dot{\phi}_t, t) + f(\theta^*_{t+dt}, \dot{\theta}^*_{t+dt}, \phi^*_{t+dt}, \dot{\phi}^*_{t+dt}, t+dt)}{2} \\
\dot{\phi}_{t+dt} &= \dot{\phi}_t + dt\frac{g(\theta_t, \dot{\theta}_t, \phi_t, \dot{\phi}_t, t) + g(\theta^*_{t+dt}, \dot{\theta}^*_{t+dt}, \phi^*_{t+dt}, \dot{\phi}^*_{t+dt}, t+dt)}{2} \\
\theta_{t+dt} &= \theta_t + dt\frac{\dot{\theta}_t + \dot{\theta}_{t+dt}}{2} \\
\phi_{t+dt} &= \phi_t + dt\frac{\dot{\phi}_t + \dot{\phi}_{t+dt}}{2}
\end{aligned}$$

# 6 Experimental Setup and Results

In this section, we will describe how we setup our experiments and show some key results from the numerical simulations.

## 6.1 Experiment Details

We implement this simulation in Python using the NumPy and Matplotlib libraries. First, we describe the parameters we use that are fixed across all simulations. We assign $M_A = 2 \times 5.4kg$ (typical arm mass) and $L_A = 0.5m$ (typical arm length). We also align $M_S = 0.115kg$, $L_S = 1.14m$, and $M_C = 0.2kg$ (typical golf club parameters). Everything starts at rest at $t = 0$, so $\dot{\theta}(0) = \dot{\phi}(0) = 0$. The wrist angle $\phi$ is fixed at $90°$ up until $\theta(t = t_{fixed}) = \theta_{fixed}$. We simulate for a total of $T = 3s$, and then truncate the visualizations up to 0.5 seconds after the $t_{collision}$ with the golf ball (when $x_Q(t_{collision}) = 0$). We use a fixed backswing angle of $\theta_0 = 120°$. We use a simulation time interval of $dt = 0.001s$ to achieve a good balance between runtime and accuracy.

Next, we describe the parameters we vary across simulations. We experiment with different wrist flick angles $\theta_{fixed} = [0, \theta_0]$. We also vary the arm type between passive arms and controlled arms, as well as the controlled arm angular acceleration $a$. We test out both the forward Euler method and the improved Euler method, and do not notice a significant difference between the two. Figure 3 shows some example snapshots of the golfclub swing visualizations, with associated links to videos.

## 6.2 Passive Arm System vs. Controlled Arm System

First, we qualitatively compare how the passive arm system and controlled arm systems compare for various values of wrist flick angle $\theta_{fixed}$, backswing angle $\theta_0$, and arm angular acceleration $a$. Figure 4 shows graphs of $\theta$ vs $t$ and $\phi$ vs $t$ for both passive arm and controlled arm systems, which shows that the general shape of the curves are similar up the collision time.

## 6.3 Optimizing wrist flick angle $\theta_{fixed}$

Next, we optimize for the wrist flick angle $\theta_{fixed}$ that maximizes the clubhead speed. Figure 5 shows how the clubhead speed varies with $\theta_{fixed}$, for various systems. First, we look at the passive arm system. We find that for the passive arm system, the optimal wrist flick angle is about $80°$. For the controlled arm system with a small angular acceleration of $a = -2rad/s^2$, the optimal wrist flick angle is about $105°$. For the controlled arm system with a medium angular acceleration of $a = -5rad/s^2$, the optimal wrist flick angle is about $95°$. For the controlled arm system with a large angular acceleration of $a = -20rad/s^2$, the optimal

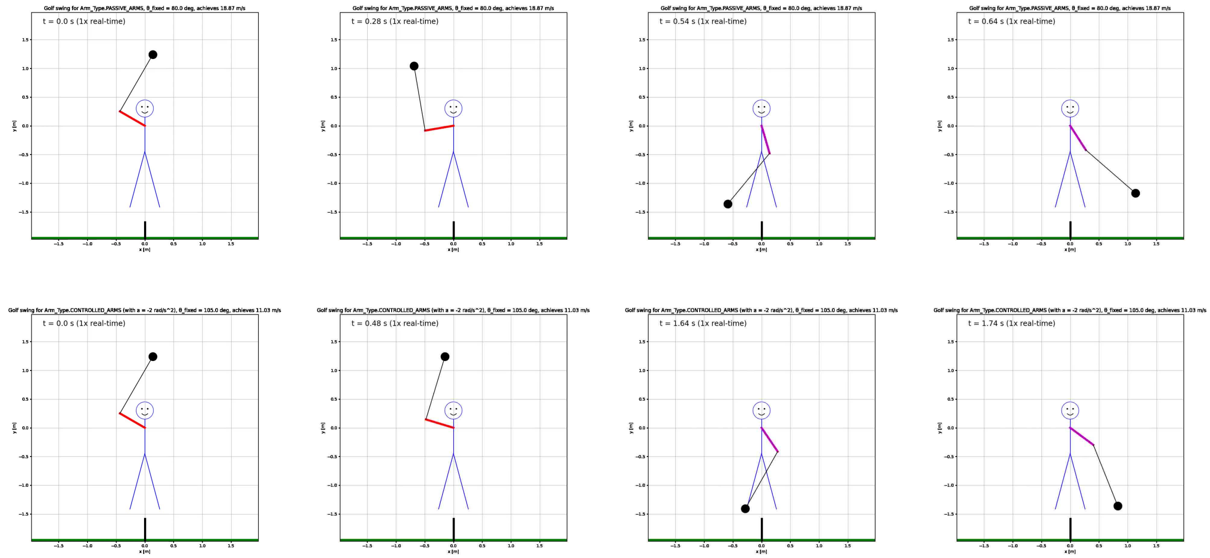

Figure 3: Examples of golfclub swing visualizations. The top row shows snapshots of a passive arm swing and the bottom row shows snapshots of a controlled arm swing with a low angular acceleration. Note that for the controlled arm case, the optimal wrist flick angle is much larger and the follow through wrist angle is much smaller.

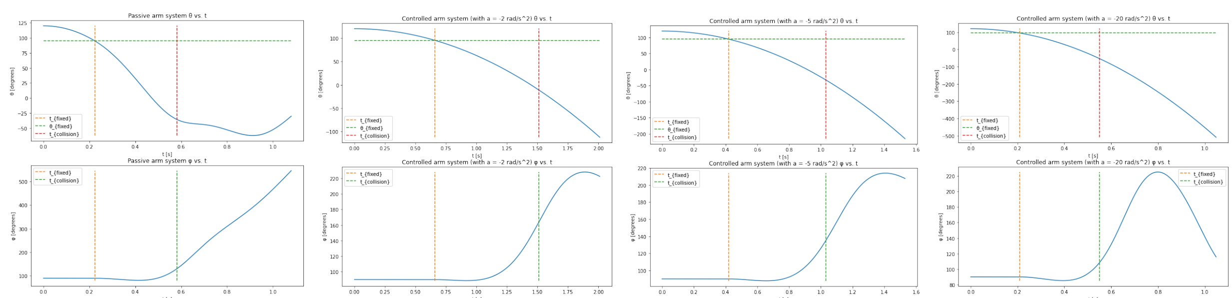

Figure 4: Examples of golfclub swing $\theta$ and $\phi$ values over time for various passive and controlled arm systems, which shows that the general shape of the curves are quite consistent up to the collision time.

wrist flick angle is about 70°. These results suggest that under this model, a golfer that can exert a greater angular acceleration on their arms should flick their wrist at a lower angle $\theta_{fixed}$ (later time) than golfers that output a smaller angular acceleration.

# 7 Limitations

There are a number of limitations to this simplified golf swing model. For example, we are only optimizing for clubhead speed at the point of contact with the golf ball, while the true goal in golf is to maximize the distance that the ball travels. We do not model the collision dynamics when the clubhead meets the ball, nor do we model how the loft angle (angle of the golf club contact surface with respect to the ground) and collision angle (angle at which the club contacts the ball) affect the ball's travel distance. We also have a very simple model of the golfer's wrists, which does not model realistic torques generated from the athlete's wrist at point R. The swing behavior could also vary with more accurate details of the arm and club (rigidity, mass distribution, biomechanical torque generation, etc). The question remains about how the addition of these details changes the dynamics and conclusions of this analysis.

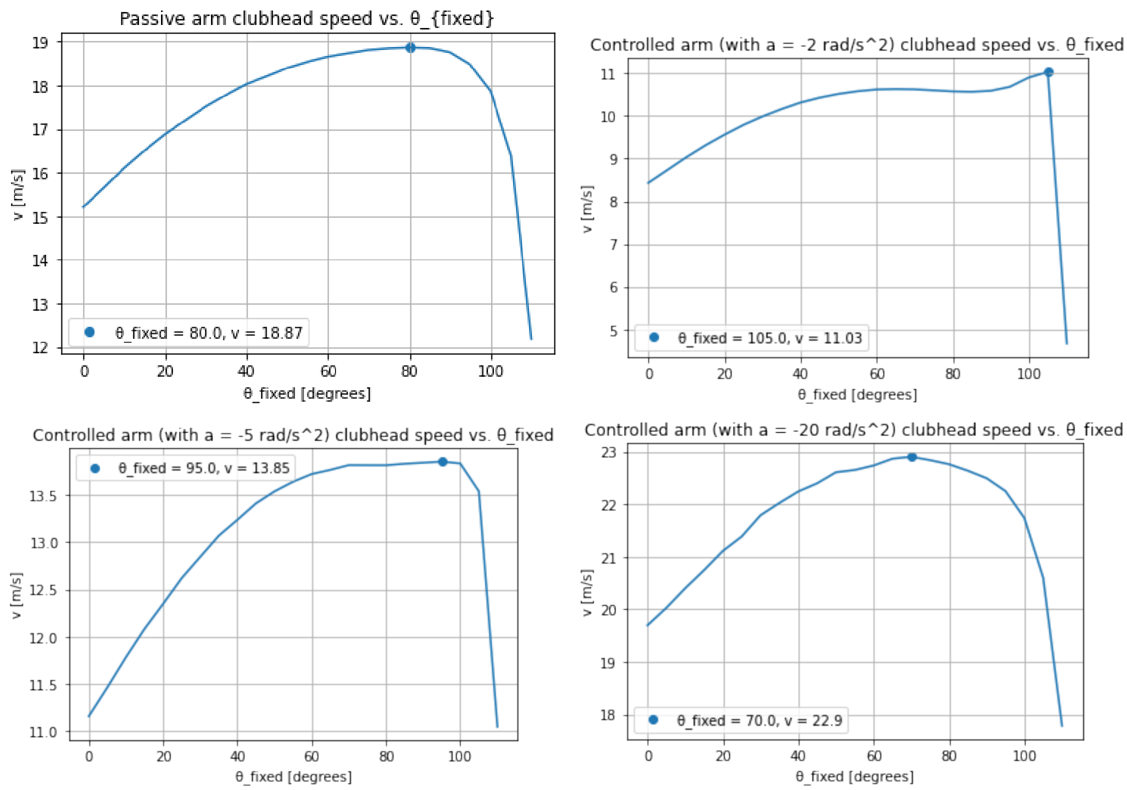


Figure 5: Clubhead speed as a function of $\theta_{fixed}$ for various systems, which shows the optimal wrist flick angle for each system.

There are many exciting directions we could take in future work. In golf, the accuracy and consistency of a swing is very important, so we could analyze the model's sensitivity to parameters and conditions. We could also analyze different conditions for the wrist angle, such as different initial wrist conditions and wrist angle speeds. The golfer does not need to be at the same x position as the ball, so we could explore moving the golfer left and right with respect to the ball. A true golf ball's trajectory will be affected by the aerodynamics of the golf ball, including how its surface properties and how it spins. We could also expand the complexity of the model to include 3D geometry, the golfer's elbow, and the golf clubhead geometry.

# 8 Conclusion

In this project, we developed a simplified model of a golf swing and identified optimal parameters of wrist flick angle and angular acceleration for maximum clubhead speed at the point of impact with a ball. We found that golfer's with higher arm angular acceleration can create higher clubhead speeds if they delay their wrist flick. We also find that passively dropping the club from the backswing position can result in higher clubhead speeds if the golfer's arms' angular acceleration is not sufficiently high (on the order of $a = -20rad/s^2$). This suggests that golfers should aim to maximize their arms' angular acceleration and progressively delay their wrist flick as their acceleration increases. However, these conclusions hold for this specific simplified model, which misses many aspects of the complexity of the problem. For example, the external field $\dot{\theta}(t)$ is controlled by the complicated biomechanics system of the athlete, which likely creates complicated torque patterns on the system along the path, as the torque output of a person depends highly on their skill, muscle dynamics, and physical traits. In future work, we hope to explore more complex golf dynamics models that capture additional nuances about the sport to validate this simplified model's applicability.