1. *Spirals*
Consider the parameterized curve given by $x(t) = e^t \cos(t)$ and $y(t) = e^t \sin(t)$ for $t \in \mathbb{R}$.

(a) Find the length of the curve between the times $t = 0$ and $t = 5$.

(b) For this curve, prove that at any point the vector joining that point to the origin meets the tangent line at that point in a constant angle (i.e., an angle independent of t). Also find the angle.

(c) (BONUS QUESTION) Suppose that we have a parameterization of a curve of the form $x(t) = r(t)\cos(t)$ and $y(t) = r(t)\sin(t)$, with the property similar to the property in part (b): for any point of the curve the vector joining the origin to that point meets the tangent line to that point at a constant angle θ. Show that $r(t)$ must be of the form $r(t) = a\,e^{kt}$, and find the constant k in terms of θ.

Solutions.

(a) To find the arclength, we integrate the speed from $t = 0$ to $t = 5$. The position of the point on the curve at time t is given by $p(t) = (e^t \cos(t),\, e^t \sin(t))$, which has velocity vector $\vec{v}(t) = (e^t(\cos(t) - \sin(t)),\, e^t(\sin(t) + \cos(t)))$. The length of $\vec{v}(t)$ is :

$$\|\vec{v}(t)\| = e^t\sqrt{\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)} = \sqrt{2}e^t.$$

Therefore the arclength is

$$\int_0^5 \sqrt{2}\,e^t\ dt\ =\ \sqrt{2}\,e^t\Big|_{t=0}^{t=5}\ =\ \sqrt{2}(e^5 - 1).$$

(b) Using the position vector $p(t)$ and velocity vector $\vec{v}(t)$ from part (a), we compute that their dot product is

$$p(t)\cdot\vec{v}(t) = e^{2t}(\cos^t(t) - \cos(t)\sin(t) + \cos(t)\sin(t) + \sin^2(t)) = e^{2t}.$$

We computed the length of $\vec{v}(t)$ in part (a). The length of $p(t)$ is :

$$\|p(t)\| = e^t\sqrt{\cos^2(t) + \sin^2(t)} = e^t$$

Therefore, by the dot product formula, the cosine of the angle $\theta(t)$ between them, as a function of t is

$$\cos(\theta(t)) = \frac{p(t) \cdot \vec{v}(t)}{\|p(t)\| \, \|\vec{v}(t)\|} = \frac{e^{2t}}{e^t \sqrt{2} e^t} = \frac{1}{\sqrt{2}}.$$

Since this is constant, the angle $\theta(t)$ is constant too. In fact from the cosine we see that the constant angle is $\theta = \pi/4$.

(c) (Bonus question) If our parameterization is of the form $p(t) = (r(t)\cos(t),\, r(t)\sin(t))$ then the velocity vector at any point is

$$\vec{v}(t) = (r'(t)\cos(t) - r(t)\sin(t),\, r'(t)\sin(t) + r(t)\cos(t))$$

with length

$$\|\vec{v}(t)\| = \sqrt{(r'(t))^2 + (r(t))^2}$$

The dot product of the position vector and the velocity vector is

$$p(t) \cdot \vec{v}(t) = r'(t)r(t).$$

The dot product formula now tells us that the cosine of the angle $\theta(t)$ between $p(t)$ and $\vec{v}(t)$ at time t is

$$\cos\theta(t) = \frac{p(t) \cdot \vec{v}(t)}{\|p(t)\| \, \|\vec{v}(t)\|} = \frac{r'(t)r(t)}{r(t)\sqrt{(r'(t))^2 + (r(t))^2}} = \frac{r'(t)}{\sqrt{(r'(t))^2 + (r(t))^2}}$$

We're supposing that the angle $\theta(t)$ is constant, say equal to θ. If we let $c = \cos(\theta)$, then the equation we need to satisfy is

$$\frac{r'(t)}{\sqrt{(r'(t))^2 + (r(t))^2}} = c,$$

or squaring, that

$$\frac{(r'(t))^2}{(r'(t))^2 + (r(t))^2} = c^2$$

Solving for $r'(t)$, we get the differential equation

$$r'(t) = \left(\sqrt{\frac{c^2}{1 - c^2}} \right) r(t)$$

which has general solution $r(t) = ae^{kt}$ with $k = \sqrt{\frac{c^2}{1-c^2}}$. Remembering that $c = \cos(\theta)$, we compute that $k = \operatorname{cotan}(\theta)$.

This matches the answer from part (a), since for $\theta = \pi/4$, $\operatorname{cotan}(\theta) = 1$.

2. *Velocity and acceleration*

(a) Let $\vec{u}(t) = (u_1(t), u_2(t))$ be a differentiable function $\mathbb{R} \longrightarrow \mathbb{R}^2$. Prove the formula $\frac{d}{dt}\|u(t)\|^2 = 2\,u(t) \cdot u'(t)$.

(b) If $(x(t), y(t))$ is a parameterized curve in $\mathbb{R}^2$, show that the speed of this parameterization is constant if and only if the acceleration is always perpendicular to the velocity vector.

(c) If an object (like a planet) orbits around a more massive object (like the sun) the orbit will be an ellipse with the massive object at one of the two foci of the ellipse.

The parameterization $x(t) = 2\cos(t)$ and $y(t) = \sin(t)$ is a parameterization of the ellipse $\frac{x^2}{4} + y^2 = 1$, which has foci at the points $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. Could this parameterization be a parameterization of an object in orbit? Explain why or why not. (NOTE: Part (c) is unconnected with (a) or (b).)

Solutions.

(a) If $\vec{u}(t) = (u_1(t),\, u_2(t))$, then $\|\vec{u}(t)\|^2 = u_1(t)^2 + u_2(t)^2$, so

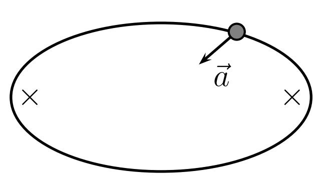
$$\frac{d}{dt}\|\vec{u}(t)\|^2 = 2u_1(t)u_1'(t) + 2u_2(t)u_2'(t) = 2\,\vec{u}(t) \cdot \vec{u}'(t)$$

(b) Suppose that $\vec{v}(t)$ is the velocity vector. Then the derivative $\vec{v}'(t)$ is the acceleration vector $\vec{a}(t)$. The formula from part (a) gives us that

$$\frac{d}{dt}\|\vec{v}(t)\|^2 = 2\,\vec{v}(t) \cdot \vec{a}(t).$$

But the speed is constant if and only if $\|\vec{v}(t)\|^2$ is constant, and that is true if and only if the derivative of $\|\vec{v}(t)\|^2$ with respect to t is zero. By the above formula, that happens if and only if $\vec{v}(t) \cdot \vec{a}(t) = 0$, i.e., if and only if the velocity and acceleration vectors are perpendicular.

(c) The parameterization $x(t) = 2\cos(t)$ and $y = \sin(t)$ does lie on the ellipse $\frac{x^2}{4} + y^2 = 1$, but it does not describe the motion of an object in orbit around a heavier body.



For instance, if we calculate the acceleration vector at time t:

$$\vec{a}(t) = (-2\cos(t),\, -\sin(t))$$

we see that it always points towards the center of the ellipse, while under gravitational acceleration it should always point to one of the two foci (shown with $\times$'s in the picture).

There are many other reasons why this can't be the parameterization of an object in orbit. I.e, many other ways in which this parameterization fails to obey the laws of motion of an object in orbit. Among the other possibilities are

- The length of the acceleration vector is not proportional to one over the square of the distance between the object and one of the foci.
- The speed is symmetric with respect to the natural reflectional symmetries of the ellipse. An object in orbit should move faster near the object it is orbiting around, i.e., near one of the foci but not the other.
- The parameterization doesn't sweep out equal areas in equal time.

3. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $f(x, y, z) = xy + z^2$ and C the parameterized curve given by $x(t) = 3t^2$, $y(t) = t\sin(t)$, and $z(t) = e^{2t}$.

(a) Let p be the point on C corresponding to $t = \pi$, and $\vec{v}$ the velocity vector at that point. Find p and $\vec{v}$.

(b) Find the gradient ∇f at the point p.

(c) Compute $\nabla f(p) \cdot \vec{v}$.

(d) Write out the composite function $f(x(t), y(t), z(t))$ and compute its derivative when $t = \pi$.

(e) The answers to (c) and (d) are of course the same. Explain why this is a consequence of the chain rule and the definition of the gradient.

Solutions. Our parameterization is given by $p(t) = (3t^2,\, t\sin(t),\, e^{2t})$.

(a) When $t = \pi$, $p = p(\pi) = (3\pi^2,\, 0,\, e^{2\pi})$. The velocity vector at time t is

$$p'(t) = \vec{v}(t) = (6t,\, \sin(t) + t\cos(t),\, 2e^{2t}),$$

so the velocity when $t = \pi$ is $\vec{v} = \vec{v}(\pi) = (6\pi,\, -\pi,\, 2e^{2\pi})$.

(b) For $f(x, y, z) = xy + z^2$, the gradient is $\nabla f(x, y, z) = (y, x, 2z)$, so at the point $(3\pi^2,\, 0,\, e^{2\pi})$ we have $\nabla f(3\pi^2,\, 0,\, e^{2\pi}) = (0, 3\pi^2,\, 2e^{2\pi})$.

(c) The dot product is

$$\nabla f(p) \cdot \vec{v} = (0, 3\pi^2, 2e^{2\pi}) \cdot (6\pi, -\pi, 2e^{2\pi}) = 4e^{4\pi} - 3\pi^3$$

(d) The composition is $f(x(t),\, y(t),\, z(t)) = 3t^3 \sin(t) + e^{4t}$, with derivative $\frac{d}{dt} f(x(t), y(t), z(t)) = 9t^2 \sin(t) + 3t^3 \cos(t) + 4e^{4t}$. When $t = \pi$ this is $9\pi^2 \cdot 0 + 9\pi^3(-1) + 4e^{4\pi} = 4e^{4\pi} - 3\pi^3$.

(e) By the chain rule,

$$\frac{d}{dt} f(p(t)) = \mathbf{D}f(p(t))\, \mathbf{D}p(t).$$

The derivative of the parameterization $p : \mathbb{R} \longrightarrow \mathbb{R}^3$ is the velocity vector $\vec{v}(t)$, so that the chain rule then reads

$$\frac{d}{dt} f(p(t)) = \mathbf{D}f(p(t))\, \vec{v}(t).$$

The definition of the gradient is that it is the derivative matrix of f, $\mathbf{D}f$, thought of as a vector. If we compare the formula for multiplying a 1×3 matrix $\begin{bmatrix} a & b & c \end{bmatrix}$ by a vector $\vec{v} = (v_x, v_y, v_z)$, and the formula for taking the dot product of $\vec{v}$ with that matrix converted to a vector, we get the same thing :

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = a\, v_x + b\, v_y + c\, v_z = (a, b, c) \cdot (v_x, v_y, v_z).$$

Therefore, $\mathbf{D}(p(t)\vec{v} = \nabla f(p(t) \cdot \vec{v}$.

Combining these two ideas (the chain rule and the formula telling us that $\mathbf{D}f\vec{v} = \nabla f \cdot \vec{v}$) we get

So, the chain rule tells us that

$$\frac{d}{dt} f(p(t))(\pi) = \mathbf{D}f(3\pi^2, 0, 2e^{2\pi})\, \mathbf{D}p(\pi) = \nabla f(3\pi^2, 0, 2e^{2\pi}) \cdot \vec{v}(\pi),$$

in other words, that the computations in (c) and (d) give the same answer.

4. For the following vector fields, identify those which are conservative, and those which are not conservative. For those which are conservative, find a potential function $\mathbf{F}$. For those which are not conservative, explain how you know this.

(a) $\mathbf{G}(x, y, z) = (y^2,\, 2xy + y^2,\, 2yz)$.

(b) $\mathbf{G}(x, y, z) = (x^2 + y^2,\, 2x^3 + 1,\, z^2)$.

(c) $\mathbf{G}(x, y, z) = (\cos(y) - z\cos(x),\ -x\sin(y),\ -\sin(x))$.

(d) $\mathbf{G}(x, y, z) = (\sin(z) - y\sin(x),\ \cos(x),\ x\sin(z))$.

Solution. The only test we know for vector fields to check that they're not conservative is to check the partial derivatives, and see if they match. As we learned, this is the same as checking if the curl is zero. If the curl *is* zero, and if $\mathbf{G}$ is defined on all of $\mathbb{R}^3$, then by the theorem from the class of Monday, October 16th, we know that there is a function $\mathbf{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ with $\nabla\mathbf{F} = \mathbf{G}$.

(a) $\mathrm{Curl}(\mathbf{G}) = (2z, 0, 0)$, so $\mathbf{G}$ is not conservative.

(b) $\mathrm{Curl}(\mathbf{G}) = (0, 0, 6x^2 - 2y)$, so this $\mathbf{G}$ is not conservative either.

(c) $\mathrm{Curl}(\mathbf{G}) = (0, 0, 0)$, so since $\mathbf{G}$ is defined on all of $\mathbb{R}^3$, it must be the gradient of some function. One of the possibilities is $\mathbf{F}(x, y, z) = x\cos(y) - z\sin(x)$.

One way to find such a function is to follow the procedure outline in the tutorials, namely, follow these steps:

(1) We are looking for a function $\mathbf{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ so that $\frac{\partial\mathbf{F}}{\partial x} = \cos(y) - z\cos(x)$, $\frac{\partial\mathbf{F}}{\partial y} = -x\sin(y)$, and $\frac{\partial\mathbf{F}}{\partial z} = -\sin(x)$.

(2) We start by solving the equation $\frac{\partial\mathbf{F}}{\partial x} = \cos(y) - z\cos(x)$. Integrating with respect to x, we find that one possibility is $\mathbf{F} = x\cos(y) - z\sin(x) + \mathbf{H}(y, z)$, where $\mathbf{H}$ is a function which only depends on y and z (i.e., is a "constant" with respect to x).

(3) Differentiating this possibility with respect to y, we get

$$\frac{\partial\mathbf{F}}{\partial y} = -x\sin(y) + \frac{\partial\mathbf{H}}{\partial y},$$

which we want to equal $-x\sin(y)$. This will happen as long as $\frac{\partial\mathbf{H}}{\partial y} = 0$, i.e., as long as $\mathbf{H}$ doesn't depend on y, and so for instance only depends on z. That is, our solution now looks like $f = x\cos(y) - z\sin(x) + \mathbf{H}(z)$, with $\mathbf{H}$ only depending on z.

(4) Finally, differentiating with respect to z, we get

$$\frac{\partial\mathbf{F}}{\partial z} = -\sin(x) + \frac{\partial\mathbf{H}}{\partial z},$$

which we want to equal to $-\sin(x)$. This clearly happens when $\frac{\partial\mathbf{H}}{\partial z} = 0$. One possibility is to pick $\mathbf{H} = 0$ (or, for $\mathbf{H}$ to be any constant). With the choice $\mathbf{H} = 0$ we arrive at the solution $\mathbf{F} = x\cos(y) - z\sin(x)$ given.

(d) $\mathrm{Curl}(\mathbf{G}) = (0, \cos(z) - \sin(z), 0)$, so $\mathbf{G}$ also isn't a conservative vector field.