Computational Geometry

給定一n維歐氏空間，求關於該空間的資訊

Line Segment (Intersegment)

Define: 給定空間中兩個線段 S_1, S_2 之左右端點 判斷兩線段是否相交。

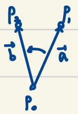
$\Longrightarrow$ 用線代方法：

等同兩聯立方程式是否有解問題

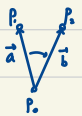
但电腦硬体有極限，能表示的數範圍有限（除，根号皆不精確）

無法用代數方法解決

Note:



(left-turn)
counterclockwise



(right-turn)
clockwise

①. 若 $\det[\vec{a}\ \vec{b}] > 0$，則 $\vec{a}$ 至 $\vec{b}$ 為 counterclockwise

②. 若 $\det[\vec{a}\ \vec{b}] < 0$，則 $\vec{a}$ 至 $\vec{b}$ 為 clockwise

③. 若 $\det[\vec{a}\ \vec{b}] = 0$，則 $\vec{a}, \vec{b}$ 為 parallel

Example:

①. $P_0 = (0,0)$ $P_1 = (1,0)$ $P_2 = (0,1)$

$\Rightarrow \vec{a} = [1, 0]$, $\vec{b} = [0, 1]$ $\det[\vec{a}\ \vec{b}] = 1 > 0$

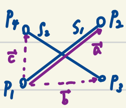
$\Rightarrow$ left turn

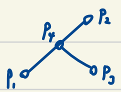
②. $P_0 = (1,1)$ $P_1 = (3,3)$ $P_2 = (4,2)$

$\Rightarrow \vec{a} = [2, 2]$, $\vec{b} = [3, 1]$ $\det[\vec{a}\ \vec{b}] = -4 < 0$

$\Rightarrow$ right turn

Note: 如何在 O(1) time 判斷兩線段相交?





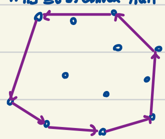
①. S_1 和 S_2 相交 $\Leftrightarrow$ S_1 straddles S_2 且 S_2 straddles S_1

或其中一線段之 end point 位於另一線段上

②. 設 $\vec{a} = \overrightarrow{P_1P_2}$, $\vec{b} = \overrightarrow{P_1P_3}$, $\vec{c} = \overrightarrow{P_1P_4}$

$\overrightarrow{P_1P_2}$ straddles $\overrightarrow{P_3P_4}$ $\Leftrightarrow$ $\det[\vec{a}\ \vec{b}]$ 和 $\det[\vec{a}\ \vec{c}]$ 之正負相反

Convex Hull

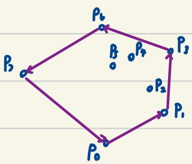


Problem: 給定n個點之set Q，則包含Q之最小凸多邊形

即Q之convex hull，記作CH(Q)

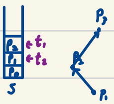
求CH(Q)之頂點，並以counterclockwise順序輸出

Example: 利用Gram's scan 求CH(Q)

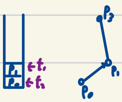


1. 找y坐標最小的點為P_0
2. 根據P_0之左旋角度排序$P_1 \sim P_7$
3. 排序後，依序確認是否左旋，是時，push至stack S
4. 若否，則pop當前元素
5. 確認下一點是否可push至stack
6. return stack 反向輸出

$i=3$:

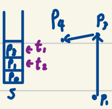


$\Rightarrow$ 右旋，pop t_1



$\Rightarrow$ 左旋，push P_3

$i=4$:

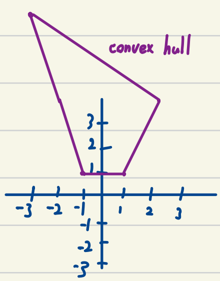


$\Rightarrow$ 左旋，push P_4

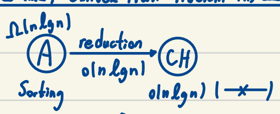
Time Complexity:
① 排序polar angle: $\theta(n\lg n)$
② 利用Amortized Cost來算：每個點最多pop一次 $\Rightarrow O(\frac{n^2}{n}) = O(n)$
$\Rightarrow O(n\lg n) + O(n) = O(n\lg n)$

Theorem:

給定n個點，Convex Hull Problem需至少$\Omega(n\lg n)$



<Pf>:



Input: $-1, -3, 2, -2, 1 \xrightarrow[(x, x^2)]{\text{転換}} (-1,1), (-3,9), (2,4), (-2,4), (1,1)$

$\because x^2$為凸函數 $\therefore (x, x^2)$皆在convex hull上

且照逆時針順序之convex hull，即為排序問題結果

$\therefore$ 該reduction為有效的

又該転換為$O(n)$

∵ Sorting 至少需 $\Omega(n \lg n)$ 時間才能被解決

∴ CH 也至少需 $\Omega(n \lg n)$ time