Hamiltonian Path Problem

Define: 給定一圖為 $G=(V,E)$, G中是否具 Hamiltonian path?

i.e. 是否含一個 path 經每個點恰一次

另一等價問題為:

G上是否存在一 min-degree spanning tree T of maximum degree two

Formal definition:

$HP = \{\langle G \rangle \mid G$ 為具HP之圖$\}$

Theorem: HP Problem is NP-Complete

①. claim: $HP \in NP$

給定一 $G=(V,E)$ 和一 certificate V', 驗證V'中是否含V中所有點恰一次

會為 polynomial-solvable

②. claim: Hamiltonian Cycle $\leq_p$ Hamiltonian path

給定 G為HC上之 instance, 而 $G=(V,E)$, 建構一 $G'=(V',E')$為HP之 instance

令 $v\in V$, v為G上之一點, 加入 $v', s, t \notin V$

其中: v'為v之 copy

s 為 $\deg(s)=1$ 且 $(s,v)\in E'$

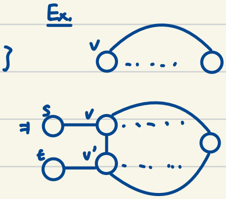
t 為 $\deg(t)=1$ 且 $(t,v')\in E'$

則: $V' = V \cup \{v', s, t\}$

$E' = E \cup \{(u,v') \mid (u,v)\in E\} \cup \{(s,v), (t,v'), (v,v')\}$

$G'=(V',E')$為 HP上之 instance

Ex.



claim: G上具HC ⇔ G'上具HP

(⇒) 設G上具HC為 $\langle v,u,\ldots,u',v\rangle$, 表示$\langle v,u,\ldots,u',v\rangle$走訪了所有G上 vertex

則: $\langle s,v,u,\ldots,u',v',t\rangle$ 為G'上之HP

(⇐) 設G'上具HP為 $\langle s,v,u,\ldots,u',v',t\rangle$ 為一HP

則 $\langle v,u,\ldots,u',v\rangle$ 為G上之一HC

3. †† (13%) A Hamiltonian cycle of a graph G is a simple cycle that visits all nodes in G. Suppose there exists an $O(n^7)$-time algorithm that decides HamC(G) for any n-node graph G.

> HamC(G)
>
> Input: a simple undirected graph G.
>
> Output: "true," if G has a Hamiltonian cycle; "false," otherwise.

Complete Algorithm 1, an $O(n^7)$-time algorithm that uses HamC at most once to decide HamP$_{2\times 3}$ for any n-node graph G, for any distinct nodes $a_1, a_2, x_1, x_2, x_3 \in G$.

> HamP$_{2\times 3}$($G = (V, E), a_1, a_2, x_1, x_2, x_3$)
>
> Input: a simple undirected graph $G = (V, E)$ that contains at least the five distinct nodes a_1, a_2, x_1, x_2, x_3.
>
> Output: "true," if G has a simple path of length $|V| - 4$ that starts at a_i for some $i \in \{1, 2\}$, visits every node in $V \setminus \{a_1, a_2, x_1, x_2, x_3\}$ exactly once, and finally stops at x_j for some $j \in \{1, 2, 3\}$; otherwise, "output false."

Algorithm 1: HamP$_{2\times 3}$($G = (V, E), a_1, a_2, x_1, x_2, x_3$)

1 $U \leftarrow V \cup \{\ell_1, \ell_2, \ell_3, \ell_4\}$;
2 $F \leftarrow E$;
  `/* Add some edges incident to node` ℓ_1 `to` F. `*/`
3 $F \leftarrow F \cup \{\underline{\qquad\quad}\}$;
  `/* Add some edges incident to node` ℓ_2 `to` F. `*/`
4 $F \leftarrow F \cup \{\underline{\qquad\quad}\}$;
  `/* Add some edges incident to node` ℓ_3 `to` F. `*/`
5 $F \leftarrow F \cup \{\underline{\qquad\quad}\}$;
  `/* Add some edges incident to node` ℓ_4 `to` F. `*/`
6 $F \leftarrow F \cup \{\underline{\qquad\quad}\}$;
7 **return** HamC($H = (U, F)$);

- Which of the following undirected edges shall be placed in the missing part ___⑥___ of Line 3?
  (A) (ℓ_1, ℓ_2) (B) (ℓ_1, a_1) (C) (ℓ_1, a_2) (D) (ℓ_1, x_2)

BC

- Which of the following undirected edges shall be placed in the missing part ___⑦___ of Line 4?
  ABCD (A) (ℓ_2, a_1) (B) (ℓ_2, a_2) (C) (ℓ_2, x_2) (D) (ℓ_2, x_3)

- Which of the following undirected edges shall be placed in the missing part ___⑧___ of Line 5?
  CD (A) (ℓ_3, ℓ_1) (B) (ℓ_3, ℓ_2) (C) (ℓ_3, x_1) (D) (ℓ_3, x_2)

- Which of the following undirected edges shall be placed in the missing part ___⑨___ of Line 6?
  (A) (ℓ_4, ℓ_3) (B) (ℓ_4, x_1) (C) (ℓ_4, x_2) (D) (ℓ_4, x_3)

BCD

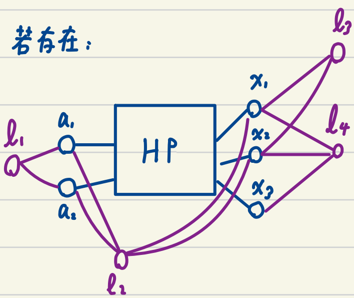
$HamP_{2\times3} \le_p HamC$

給定一 $\langle G, a_1, a_2, x_1, x_2, x_3 \rangle$ 為 $HamP_{2\times3}$ 上之 instance

建構一 $\langle G' \rangle$ 為 HamC 之一 instance

使得 G上存在一HP由 a_1 或 a_2 開始, x_1, x_2 或 x_3 結束 $\Leftrightarrow$ G'上有HC

idea: 若存在:



必有一 $G-\{a_1, a_2, x_1, x_2, x_3\}$ 上之 $|V|-4$ 長度的HP

若存在 $HP_{2\times3}$, 則有HP和 a_1, a_2 相連

x_1, x_2, x_3 相連