A Simple 3D Puzzle

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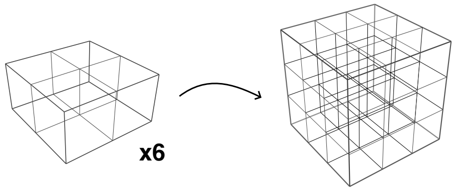
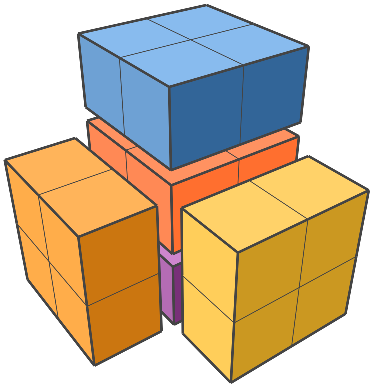
[Formats: html | pdf]

1 The Puzzle

Here's a fun puzzle: Take six boxes, each 1 × 2 × 2 in size, and find a way to pack them into a 3 × 3 × 3 cube.

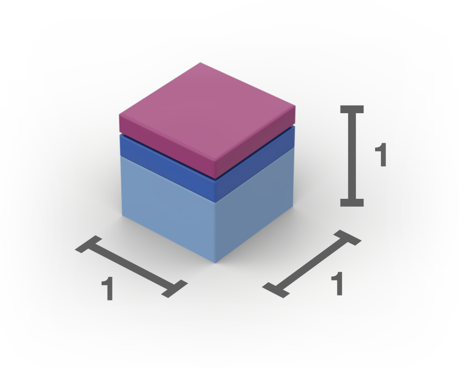


I learned about this puzzle through Donald Knuth's *The Art of Computer Programming,* §7.2.2.1. The six boxes have a total volume of 24 cubies (I'll call a 1 × 1 × 1 unit a "cubie," as Knuth does). They certainly have a chance of fitting into the 27 cubie spaces of the larger 3 × 3 × 3 volume. But the initial configurations I tried failed to fit more than five boxes in the space allowed:

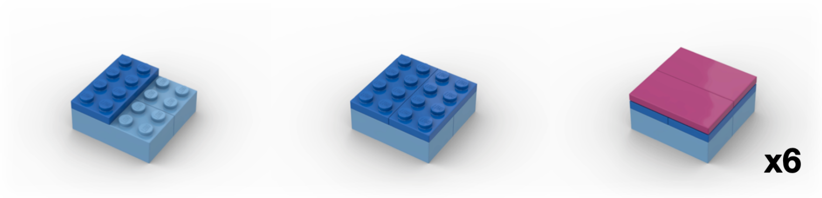


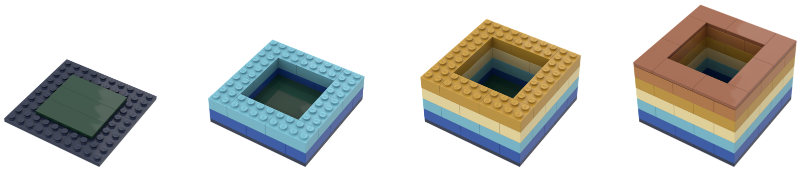
You might be able to solve this by simply thinking about it. But it's even more fun to play with a physical model.

Did you know that a 2 × 2 Lego brick with 2 tile-heights on top forms a perfect cube?

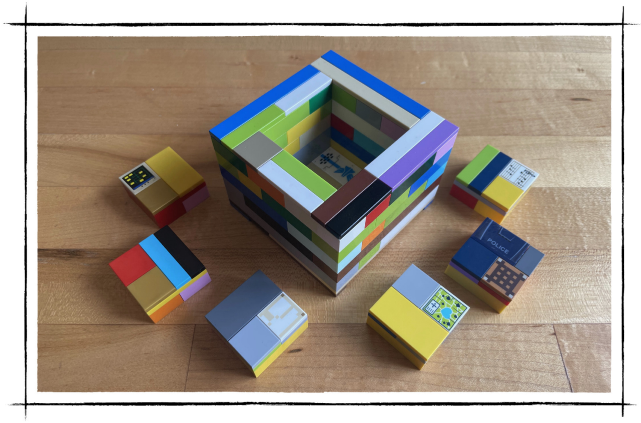


This allows us to construct the puzzle like so:





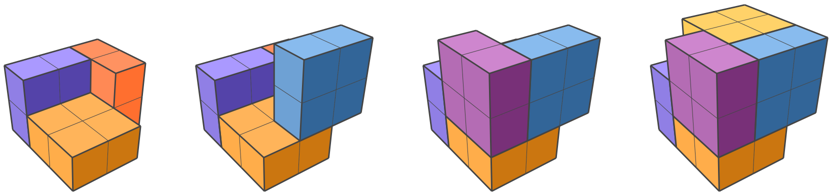
Here's the hodgepodge model I built with my kids' Legos:



I'll write a little about the math behind this puzzle below, but for now I'll give you a vertical break so you don't accidentally see the solution. Try out the puzzle first!

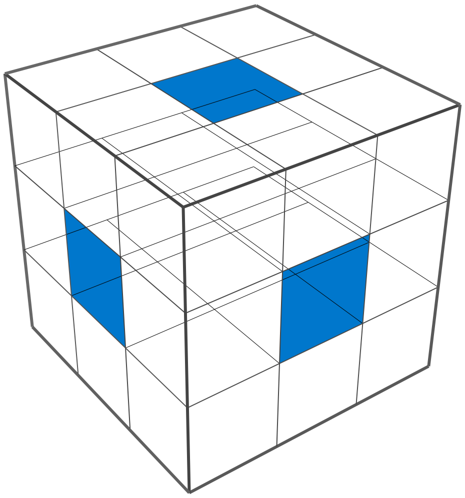
2 The Solution

Here's the solution:



The simplest way to characterize it is to notice that we've left the central cubie unoccupied.

As pointed out by Knuth, there's a nice observation that can help you find this solution. Define a "face cubie" as one that's adjacent to the center cubie. There are 6 of these:



Observation 1 Every piece must occupy at least one face cubie.

This can be confirmed by listing all piece placements within the 3 × 3 × 3 space.

An immediate consequence of this observation is that — since there are 6 face cubies and 6 pieces — each piece must occupy *exactly* 1 face cubie. This alone can give you the solution, but things get even easier when you notice:

Observation 2
Any piece occupying the center cubie must also occupy 2 face cubies.

The consequence of *this* is that we must leave the center cubie vacant.

If you're curious about how algorithms can solve puzzles like this one, I recommend reading Knuth's §7.2.2.1, which is relatively recent (in *The Art of Computer Programming*, volume 4B, published in 2022). He gives credit to Jan Slothouber and William Graatsma for inventing this puzzle, as referenced in their apparently out-of-print book *Cubics* from 1970 (pp. 108–109).

3 Notes on the Images

I wrote custom javascript tools to create the vector images in this article. If I have time, I may one day clean up that code and release it as a library. For now the code is quite messy, and can be found in this directory hosted on github.

I rendered the Lego images with Studio by BrickLink, which is awesome, and I'm grateful it's free.

The dragon image (Here be Spoilers) was created with dall · e 2.